CONTEMPORARY MATHEMATICS

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Contemporary Mathematics (OpenStax)

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TABLE OF CONTENTS

Licensing

1: Sets

- 1.0: Introduction
- 1.1: Basic Set Concepts
  - 1.1.0: Exercises
- 1.2: Subsets
  - 1.2.0: Exercises
- 1.3: Understanding Venn Diagrams
  - 1.3.0: Exercises
- 1.4: Set Operations with Two Sets
  - 1.4.0: Exercises
- 1.5: Set Operations with Three Sets
  - 1.5.0: Exercises
- 1.6: Chapter Summary
  - 1.6.0: Key Terms
  - 1.6.1: Key Concepts
  - 1.6.2: Videos
  - 1.6.3: Formula Review
  - 1.6.4: Projects
  - 1.6.5: Chapter Review
  - 1.6.6: Chapter Test

2: Logic

- 2.0: Introduction
- 2.1: Statements and Quantifiers
  - 2.1.0: Exercises
- 2.2: Compound Statements
  - 2.2.0: New Page
- 2.3: Constructing Truth Tables
  - 2.3.0: Exercises
- 2.4: Truth Tables for the Conditional and Biconditional
  - 2.4.0: Exercises
- 2.5: Equivalent Statements
  - 2.5.0: Exercises
- 2.6: De Morgan’s Laws
  - 2.6.0: Exercises
- 2.7: Logical Arguments
  - 2.7.0: Exercises
- 2.8: Chapter Summary
  - 2.8.0: Key Terms
  - 2.8.1: Key Concepts

    - 2.8.2: Videos
    - 2.8.3: Projects
    - 2.8.4: Chapter Review
    - 2.8.5: Chapter Test

3: Real Number Systems and Number Theory

- 3.0: Introduction
- 3.1: Prime and Composite Numbers
    - 3.1.0: Exercises
- 3.2: The Integers
    - 3.2.0: Exercises
- 3.3: Order of Operations
    - 3.3.0: Exercises
- 3.4: Rational Numbers
    - 3.4.0: Exercises
- 3.5: Irrational Numbers
    - 3.5.0: Exercises
- 3.6: Real Numbers
    - 3.6.0: Exercises
- 3.7: Clock Arithmetic
    - 3.7.0: Exercises
- 3.8: Exponents
    - 3.8.0: Exercises
- 3.9: Scientific Notation
    - 3.9.0: Exercises
- 3.10: Arithmetic Sequences
    - 3.10.0: Exercises
- 3.11: Geometric Sequences
    - 3.11.0: Exercises
- 3.12: Chapter Summary
    - 3.12.0: Key Terms
    - 3.12.1: Key Concepts
    - 3.12.2: Videos
    - 3.12.3: Formula Review
    - 3.12.4: Projects
    - 3.12.5: Chapter Review
    - 3.12.6: Chapter Test

4: Number Representation and Calculation

- 4.0: Introduction
- 4.0: Hindu-Arabic Positional System
    - 4.0.0: Exercises
- 4.1: Early Numeration Systems
    - 4.1.0: Exercises
- 4.2: Converting with Base Systems
    - 4.2.0: Exercises

- 4.3: Addition and Subtraction in Base Systems
    - 4.3.0: Exercises
- 4.4: Multiplication and Division in Base Systems
    - 4.4.0: Exercises
- 4.5: Chapter Summary
    - 4.5.0: Key Terms
    - 4.5.1: Key Concepts
    - 4.5.2: Videos
    - 4.5.3: Projects
    - 4.5.4: Chapter Review
    - 4.5.5: Chapter Test

5: Algebra

- 5.0: Introduction
- 5.1: Algebraic Expressions
    - 5.1.1: Exercises
- 5.2: Linear Equations in One Variable with Applications
    - 5.2.1: Exercises
- 5.3: Linear Inequalities in One Variable with Applications
    - 5.3.1: Exercises
- 5.4: Ratios and Proportions
    - 5.4.1: Exercises
- 5.5: Graphing Linear Equations and Inequalities
    - 5.5.1: Exercises
- 5.6: Quadratic Equations in One Variable With Applications
    - 5.6.0: Exercises
- 5.7: Functions
    - 5.7.1: Exercises
- 5.8: Graphing Functions
    - 5.8.1: Exercises
- 5.9: Systems of Linear Equations in Two Variables
    - 5.9.1: Exercises
- 5.10: Systems of Linear Inequalities in Two Variables
    - 5.10.0: Exercises
- 5.11: Linear Programming
    - 5.11.1: Exercises
- 5.12: Chapter Summary
    - 5.12.0: Key Terms
    - 5.12.1: Key Concepts
    - 5.12.2: Videos
    - 5.12.3: Formula Review
    - 5.12.4: Projects
    - 5.12.5: Chapter Review
    - 5.12.6: Chapter Test

6: Money Management

- 6.0: Introduction
- 6.1: Understanding Percent
  - 6.1.0: Exercises
- 6.2: Discounts, Markups, and Sales Tax
  - 6.2.0: Exercises
- 6.3: Simple Interest
  - 6.3.0: Exercises
- 6.4: Compound Interest
  - 6.4.0: Exercises
- 6.5: Making a Personal Budget
  - 6.5.0: Exercises
- 6.6: Methods of Savings
  - 6.6.0: Exercises
- 6.7: Investments
  - 6.7.0: Exercises
- 6.8: The Basics of Loans
  - 6.8.0: Exercises
- 6.9: Understanding Student Loans
  - 6.9.0: Exercises
- 6.10: Credit Cards
  - 6.10.0: Exercises
- 6.11: Buying or Leasing a Car
  - 6.11.0: Exercises
- 6.12: Renting and Homeownership
  - 6.12.0: Exercises
- 6.13: Income Tax
  - 6.13.0: Exercises
- 6.14: Chapter Summary
  - 6.14.0: Key Terms
  - 6.14.1: Key Concepts
  - 6.14.2: Videos
  - 6.14.3: Formula Review
  - 6.14.4: Projects
  - 6.14.5: Chapter Review
  - 6.14.6: Chapter Test

7: Probability

- 7.0: Introduction
- 7.1: The Multiplication Rule for Counting
  - 7.1.0: Exercises
- 7.2: Permutations
  - 7.2.0: Exercises
- 7.3: Combinations
  - 7.3.0: Exercises

- 7.4: Tree Diagrams, Tables, and Outcomes
  - 7.4.0: Exercises
- 7.5: Basic Concepts of Probability
  - 7.5.0: Exercises
- 7.6: Probability with Permutations and Combinations
  - 7.6.0: Exercises
- 7.7: What Are the Odds?
  - 7.7.0: Exercises
- 7.8: The Addition Rule for Probability
  - 7.8.0: Exercises
- 7.9: Conditional Probability and the Multiplication Rule
  - 7.9.0: Exercises
- 7.10: The Binomial Distribution
  - 7.10.0: Exercises
- 7.11: Expected Value
  - 7.11.0: Exercises
- 7.12: Chapter Summary
  - 7.12.0: Key Terms
  - 7.12.1: Key Concepts
  - 7.12.2: Formula Review
  - 7.12.3: Projects
  - 7.12.4: Chapter Review
  - 7.12.5: Chapter Test

8: Statistics

- 8.0: Introduction
- 8.1: Gathering and Organizing Data
  - 8.1.0: Exercises
- 8.2: Visualizing Data
  - 8.2.0: Exercises
- 8.3: Mean, Median and Mode
  - 8.3.0: Exercises
- 8.4: Range and Standard Deviation
  - 8.4.0: Exercises
- 8.5: Percentiles
  - 8.5.0: Exercises
- 8.6: The Normal Distribution
  - 8.6.0: Exercises
- 8.7: Applications of the Normal Distribution
  - 8.7.0: Exercises
- 8.8: Scatter Plots, Correlation, and Regression Lines
  - 8.8.0: Exercises
- 8.9: Chapter Summary
  - 8.9.0: Key Terms
  - 8.9.1: Key Concepts

  - 8.9.2: Videos
  - 8.9.3: Formula Review
  - 8.9.4: Projects
  - 8.9.5: Chapter Review
  - 8.9.6: Chapter Test

9: Metric Measurement

- 9.0: Introduction
- 9.1: The Metric System
  - 9.1.0: Exercises
- 9.2: Measuring Area
  - 9.2.0: Exercises
- 9.3: Measuring Volume
  - 9.3.0: Exercises
- 9.4: Measuring Weight
  - 9.4.0: Exercises
- 9.5: Measuring Temperature
  - 9.5.0: Exercises
- 9.6: Chapter Summary
  - 9.6.0: Key Terms
  - 9.6.1: Key Concepts
  - 9.6.2: Videos
  - 9.6.3: Formula Review
  - 9.6.4: Projects
  - 9.6.5: Chapter Review
  - 9.6.6: Chapter Test

10: Geometry

- 10.0: Introduction
- 10.1: Points, Lines, and Planes
  - 10.1.0: Exercises
- 10.2: Angles
  - 10.2.0: Exercises
- 10.3: Triangles
  - 10.3.0: Exercises
- 10.4: Polygons, Perimeter, and Circumference
  - 10.4.0: Exercises
- 10.5: Tessellations
  - 10.5.0: Exercises
- 10.6: Area
  - 10.6.0: Exercises
- 10.7: Volume and Surface Area
  - 10.7.0: Exercises
- 10.8: Right Triangle Trigonometry
  - 10.8.0: Exercises
- 10.9: Chapter Summary

  - 10.9.0: Key Terms
  - 10.9.1: Key Concepts
  - 10.9.2: Videos
  - 10.9.3: Formula Review
  - 10.9.4: Projects
  - 10.9.5: Chapter Review
  - 10.9.6: Chapter Test

11: Voting and Apportionment

- 11.0: Introduction
- 11.1: Voting Methods
  - 11.1.0: Exercises
- 11.2: Fairness in Voting Methods
  - 11.2.0: Exercises
- 11.3: Standard Divisors, Standard Quotas, and the Apportionment Problem
  - 11.3.0: Exercises
- 11.4: Apportionment Methods
  - 11.4.0: Exercises
- 11.5: Fairness in Apportionment Methods
  - 11.5.0: Exercises
- 11.6: Chapter Summary
  - 11.6.0: Key Terms
  - 11.6.1: Key Concepts
  - 11.6.2: Videos
  - 11.6.3: Formula Review
  - 11.6.4: Projects
  - 11.6.5: Chapter Review
  - 11.6.6: Chapter Test

12: Graph Theory

- 12.0: Introduction
- 12.1: Graph Basics
  - 12.1.0: Exercises
- 12.2: Graph Structures
  - 12.2.0: Exercises
- 12.3: Comparing Graphs
  - 12.3.0: Exercises
- 12.4: Navigating Graphs
  - 12.4.0: Exercises
- 12.5: Euler Circuits
  - 12.5.0: Exercises
- 12.6: Euler Trails
  - 12.6.0: Exercises
- 12.7: Hamilton Cycles
  - 12.7.0: Exercises
- 12.8: Hamilton Paths

  - 12.8.1: Exercises
- 12.9: Traveling Salesperson Problem
  - 12.9.1: Exercises
- 12.10: Trees
  - 12.10.1: Exercises
- 12.11: Chapter Summary
  - 12.11.0: Key Terms
  - 12.11.1: Key Concepts
  - 12.11.2: Videos
  - 12.11.3: Formula Review
  - 12.11.4: Projects
  - 12.11.5: Chapter Review
  - 12.11.6: Chapter Test

13: Math and...

- 13.0: Introduction
- 13.1: Math and Art
  - 13.1.0: Exercises
- 13.2: Math and the Environment
  - 13.2.0: Exercises
- 13.3: Math and Medicine
  - 13.3.0: Exercises
- 13.4: Math and Music
  - 13.4.0: Exercises
- 13.5: Math and Sports
  - 13.5.0: Exercises
- 13.6: Chapter Summary
  - 13.6.0: Key Terms
  - 13.6.1: Key Concepts
  - 13.6.2: Formula Review
  - 13.6.3: Projects
  - 13.6.4: Chapter Review
  - 13.6.5: Chapter Test

14: Appendix

- 14.0: Integer Powers of 10

15: Answer Key

- 15.0: Chapter 1
- 15.1: Chapter 2
- 15.2: Chapter 3
- 15.3: Chapter 4
- 15.4: Chapter 5
- 15.5: Chapter 6
- 15.6: Chapter 7
- 15.7: Chapter 8
- 15.8: Chapter 9
- 15.9: Chapter 10

  - 15.10: Chapter 11
  - 15.11: Chapter 12
  - 15.12: Chapter 13

Index

Glossary

Detailed Licensing

Licensing

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CHAPTER OVERVIEW

1: Sets

1.0: Introduction
1.1: Basic Set Concepts
  1.1.0: Exercises
1.2: Subsets
  1.2.0: Exercises
1.3: Understanding Venn Diagrams
  1.3.0: Exercises
1.4: Set Operations with Two Sets
  1.4.0: Exercises
1.5: Set Operations with Three Sets
  1.5.0: Exercises
1.6: Chapter Summary
  1.6.0: Key Terms
  1.6.1: Key Concepts
  1.6.2: Videos
  1.6.3: Formula Review
  1.6.4: Projects
  1.6.5: Chapter Review
  1.6.6: Chapter Test

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1.0: Introduction

Think of a drawer in your kitchen used to store flatware. This drawer likely holds forks, spoons, and knives, and possibly other items such as a meat thermometer and a can opener. The drawer in this case represents a tool used to group a collection of objects. The members of the group are the individual items in the drawer, such as a fork or a spoon.

Figure 1.1 A flatware drawer is like a set in that it contains distinct objects. (credit: modification of work “silverware” by Jo Naylor/Flickr, CC BY 2.0)

The members of a set can be anything, such as people, numbers, or letters of the alphabet. In statistical studies, a set is a well-defined collection of objects used to identify an entire population of interest. For example, in a research study examining the effects of a new medication, there can be two sets of people: one set that is given the medication and a different set that is given a placebo (control group). In this chapter, we will discuss sets and Venn diagrams, which are graphical ways to show relationships between different groups.

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1.1: Basic Set Concepts

Figure 1.1.1: A spoon, fork, and knife are elements of the set of flatware. (credit: modification of work "Cupofjoy" Wikimedia CC0 1.0 Public Domain)

Learning Objectives

After completing this section, you should be able to:

- Represent sets in a variety of ways.
- Represent well-defined sets and the empty set with proper set notation.
- Compute the cardinal value of a set.
- Differentiate between finite and infinite sets.
- Differentiate between equal and equivalent sets.

Sets and Ways to Represent Them

Think back to your kitchen organization. If the drawer is the set, then the forks and knives are elements in the set. Sets can be described in a number of different ways: by roster, by set-builder notation, by interval notation, by graphing on a number line, and by Venn diagrams. Sets are typically designated with capital letters. The simplest way to represent a set with only a few members is the **roster** (or *listing*) **method**, in which the elements in a set are listed, enclosed by curly braces and separated by commas. For example, if F represents our set of flatware, we can represent F by using the following set notation with the roster method:

$F = \{\text{fork, spoon, knife, meat thermometer, can opener}\}$

Example 1.1: Writing a Set Using the Roster or Listing Method

Write a set consisting of your three favorite sports and label it with a capital S.

Answer

There are multiple possible answers depending on what your three favorite sports are, but any answer must list three different sports separated by commas, such as the following:

$$S = \{\text{hockey, basketball, soccer}\}$$

Your Turn 1.1

Write a set consisting of four small hand tools that might be in a toolbox and label it with a capital T.

All the sets we have considered so far have been well-defined sets. A well-defined set clearly communicates whether an element is a member of the set or not. The members of a well-defined set are fixed and do not change over time. Consider the following question. What are your top 10 songs of 2021? You could create a list of your top 10 favorite songs from 2021, but the list your friend creates will not necessarily contain the same 10 songs. So, the set of your top 10 songs of 2021 is not a well-defined set. On the other hand, the set of the letters in your name is a well-defined set because it does not vary (unless of course you change your name). The NFL wide receiver, Chad Johnson, famously changed his name to Chad Ochocinco to match his jersey number of 85.

✓ Example 1.2: Identifying Well-Defined Sets

For each of the following collections, determine if it represents a well-defined set.

1. The group of all past vice presidents of the United States.
2. A group of old cats.

Answer

1. The group of all past vice presidents of the United States is a well-defined set, because you can clearly identify if any individual was or was not a member of that group. For example, Britney Spears is not a member of this set, but Joe Biden is a member of this set.
2. A group of old cats is not a well-defined set because the word old is ambiguous. Some people might consider a seven-year-old cat to be old, while others might think a cat is not old until it is 13 years old. Because people can disagree on what is and what is not a member of this group, the set is not well-defined.

Your Turn 1.2

For each of the following collections, determine if it represents a well-defined set.

A collection of medium-sized potatoes.

The original members of the Black Eyed Peas musical group.

On January 20, 2021, Kamala Harris was sworn in as the first woman vice president of the United States of America. If we were to consider the set of all women vice presidents of the United States of America prior to January 20, 2021, this set would be known as an empty set; the number of people in this set is 0, since there were no women vice presidents before Harris. The empty set, also called the null set, is written symbolically using a pair of braces, $\{\ \}$, or a zero with a slash through it, $\emptyset$.

Checkpoint

The set containing the number $0, \{0\}$, *is a set with one element in it. It is not the same as the empty set,* $\{\ \}$, *which does not have any elements in it. Symbolically:* $\{0\} \neq \{\ \}$.

✓ Example 1.3: Representing the Empty Set Symbolically

Represent each of the following sets symbolically.

1. The set of prime numbers less than 2.
2. The set of birds that are also mammals.

Answer

1. A prime number is a natural number greater than 1 that is only divisible by one and itself. Since there are no prime numbers less than 2, this set is empty, and we can represent it symbolically as follows: $\{\ \}\text{ or }\emptyset$. These two different symbols for the empty set can be used interchangeably.
2. The set of birds and the set of mammals do not intersect, so the set of birds that are also mammals is empty, and we can represent it symbolically as $\emptyset\ \text{ or }\{\ \}$.

Your Turn 1.3

Represent the set of all numbers divisible by 0 symbolically.

Who Knew?: The Number Zero

We use the number zero to represent the concept of nothing every day. The machine language of computers is binary, consisting only of zeros and ones, and even way before that, the number zero was a powerful invention that allowed our understanding of mathematics and science to develop. The historical record shows the Babylonians first used zeros around 300 B.C., while the Mayans developed and began using zero separately around 350 A.D. What is considered the first formal use of zero in arithmetic operations was developed by the Indian mathematician Brahmagupta around 650 A.D.

Brahmagupta, Mathematician and Astronomer

Another interesting feature of the number zero is that although it is an even number, it is the only number that is neither negative nor positive.

For larger sets that have a natural ordering, sometimes an ellipsis is used to indicate that the pattern continues. It is common practice to list the first three elements of a set to establish a pattern, write the ellipsis, and then provide the last element. Consider the set of all lowercase letters of the English alphabet, A. This set can be written symbolically as $A = \{a,\ b,\ c, \ldots ,\ z\}$.

The sets we have been discussing so far are finite sets. They all have a limited or fixed number of elements. We also use an ellipsis for infinite sets, which have an unlimited number of elements, to indicate that the pattern continues. For example, in set theory, the set of natural numbers, which is the set of all positive counting numbers, is represented as $\mathbb{N} = \{1, 2, 3, \ldots\}$.

Notice that for this set, there is no element following the ellipsis. This is because there is no largest natural number; you can always add one more to get to the next natural number. Because the set of natural numbers grows without bound, it is an infinite set.

✓ Example 1.4: Writing a Finite Set Using the Roster Method and an Ellipsis

Write the set of even natural numbers including and between 2 and 100, and label it with a capital E. Include an ellipsis.

Answer

Write the label, E, followed by an equal sign and then a bracket. Write the first three even numbers separated by commas, beginning with the number two to establish a pattern. Next, write the ellipsis followed by a comma and the last number in the list, 100. Finally, write the closing bracket to complete the set.

Write the label, E, followed by an equal sign and then a bracket.

$$E = \{$$

Write the first three even numbers separated by commas, beginning with the number 2 to establish a pattern.

$$E = \{2, 4, 6,$$

Next, write the ellipsis followed by a comma and the last number in the list, 100.

$$E = \{2, 4, 6, \ldots , 100$$

Finally, write the closing bracket to complete the set.

$$E = \{2, 4, 6, \ldots , 100\}$$

Your Turn 1.4

Use an ellipsis to write the set of single digit numbers greater than or equal to zero and label it with a capital\(D.

Our number system is made up of several different infinite sets of numbers. The set of integers, $\mathbb{Z}$, is another infinite set of numbers. It includes all the positive and negative counting numbers and the number zero. There is no largest or smallest integer.

Example 1.5: Writing an Infinite Set Using the Roster Method and Ellipses

Write the set of integers using the roster method, and label it with a $\mathbb{Z}$.

Answer

Step 1: As always, we write the label and then the opening bracket. Because the negative counting numbers are infinite, to represent that the pattern continues without bound to the left, we must use an ellipsis as the first element in our list.

Step 2: We place a comma and follow it with at least three consecutive integers separated by commas to establish a pattern.

Step 3: Add an ellipsis to the end of the list to show that the set of integers continues without bound to the right.

Complete the list with a closing bracket. The set of integers may be represented as follows: $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

Your Turn 1.5

Write the set of odd numbers greater than 0 and label it with a capital\(M.

A shorthand way to write sets is with the use of set builder notation, which is a verbal description or formula for the set. For example, the set of all lowercase letters of the English alphabet, A, written in set builder notation is:

$A = \{x | x \text{ is a lowercase letter of the English alphabet.}\}$

This is read as, “Set A is the set of all elements x such that x is a lowercase letter of the English alphabet.”

Example 1.6: Writing a Set Using Set Builder Notation

Using set builder notation, write the set B of all types of balls. Explain what the notation means.

Answer

The verbal description of the set is, “Set B is the set of all elements b such that b is a ball.” This set can be written in set builder notation as follows: $B = \{b | b \text{ is a ball.}\}$

Your Turn 1.6

Using set builder notation, write the set\(C of all types of cars.

Example 1.7: Writing Sets Using Various Methods

Consider the set of letters in the word “happy.” Determine the best way to represent this set, and then write the set using either the roster method or set builder notation, whichever is more appropriate.

Answer

Because the letters in the word “happy” consist of a small finite set, the best way to represent this set is with the roster method. Choose a label to represent the set, such as H.

$H = \{\text{h}, \text{a}, \text{p}, \text{y}\}$.

Notice that the letter “p” is only represented one time. This occurs because when representing members of a set, each unique element is only listed once no matter how many times it occurs. Duplicate elements are never repeated when representing members of a set.

Your Turn 1.7

Use the roster method or set builder notation to represent the collection of all musical instruments.

Computing the Cardinal Value of a Set

Almost all the sets most people work with outside of pure mathematics are finite sets. For these sets, the cardinal value or cardinality of the set is the number of elements in the set. For finite set A, the cardinality is denoted symbolically as $n\left(A\right)$. For example, a set that contains four elements has a cardinality of 4.

How do we measure the cardinality of infinite sets? The ‘smallest’ infinite set is the set of natural numbers, or counting numbers, $\mathbb{N}=\{1,2,3,\ldots\}$. This set has a cardinality of $\aleph_0$ (pronounced "aleph-null"). All sets that have the same cardinality as the set of natural numbers are countably infinite. This concept, as well as notation using aleph, was introduced by mathematician Georg Cantor who once said, “A set is a Many that allows itself to be thought of as a One.”

Example 1.8: Computing the Cardinal Value of a Set

Write the cardinal value of each of the following sets in symbolic form.

1. $F=\{\text{fork, spoon, knife, meat thermometer, can opener}\}$
2. The empty set.

Answer

1. There are 5 distinct elements in set F: a fork, a spoon, a knife, a meat thermometer, and a can opener. Therefore, the cardinal value of set F is 5 and written symbolically as $n\left(F\right)=5$.
2. Because the empty set does not have any elements in it, the cardinality of the empty set is zero. Symbolically we write this as: $n\left(\emptyset\right)=0$.

Your Turn 1.8

Write the cardinal value of each of the following sets in symbolic form.

Set P is the set of prime numbers less than 2 .

2. Set A is the set of lowercase letters of the English alphabet, $A=\{\text{a},\text{b},\text{c},\ldots,\text{z}\}$.

Now that we have learned to represent finite and infinite sets using both the roster method and set builder notation, we should also be able to determine if a set is finite or infinite based on its verbal or symbolic description. One way to determine if a set is finite or not is to determine the cardinality of the set. If the cardinality of a set is a natural number, then the set is finite.

Example 1.9: Differentiating Between Finite and Infinite Sets

Classify each of the following sets as infinite or finite.

1. $E=\{2,4,6,8,10\}$
2. A is the set of lowercase letters of the English Alphabet, $A=\{\text{a},\text{b},\text{c},\ldots,\text{z}\}$.
3. $\mathbb{Q}=\left\{\frac{p}{q}|p\text{ and }q\text{ are integers and }q\neq 0\right\}$

Answer

1. $n\left(E\right)=5$. Since 5 is a natural number, the set is finite.
2. $n\left(A\right)$ =26n(A)=26. Since 26 is a natural number, the set is finite.
3. Set $\mathbb{Q}$ is the set of rational numbers or fractions. Because the set of integers is a subset of the set of rational numbers, and the set of integers is infinite, the set of rational numbers is also infinite. There is no smallest or largest rational number.

Your Turn 1.9

Classify each of the following sets as infinite or finite.

$B = \{b, a, k, e\}$

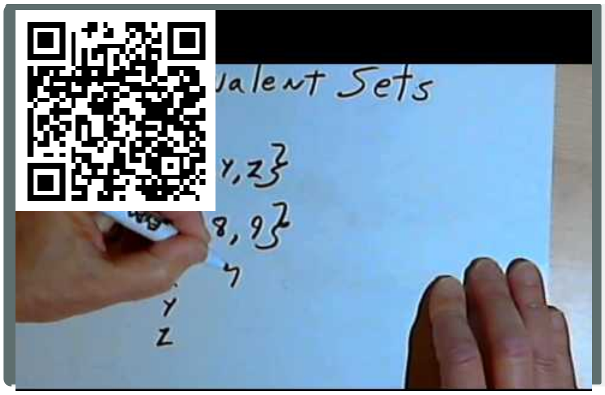
2. $\mathbb{R} = \{x \mid x \text{ is a real number }\}$

Equal versus Equivalent Sets

When speaking or writing we tend to use equal and equivalent interchangeably, but there is an important distinction between their meanings. Consider a new Ford Escape Hybrid and a new Toyota Rav4 Hybrid. Both cars are hybrid electric sport utility vehicles; in that sense, they are equivalent. They will both get you from place to place in a relatively fuel-efficient way. In this example we are comparing the single member set {Toyota Rav4 Hybrid} to the single member set {Ford Escape Hybrid}. Since these two sets have the same number of elements, they are also equivalent mathematically, meaning they have the same cardinality. But they are not equal, because the two cars have different looks and features, and probably even handle differently. Each manufacturer will emphasize the features unique to their vehicle to persuade you to buy it; if the SUVs were truly equal, there would be no reason to choose one over the other.

Now consider two Honda CR-Vs that are made with exactly the same parts, on the same assembly line within a few minutes of each other—these SUVs are equal. They are identical to each other, containing the same elements without regard to order, and the only differentiator when making a purchasing decision would be varied pricing at different dealerships. The set {Honda CR-V} is equal to the set {Honda CR-V}. Symbolically, we represent equal sets as $A = B$ and equivalent sets as $A \sim B$.

Now, let us consider a Toyota dealership that has 10 RAV4s on the lot, 8 Prii, 7 Highlanders, and 12 Camrys. There is a one-to-one relationship between the set of vehicles on the lot and the set consisting of the number of each type of vehicle on the lot. Therefore, these two sets are equivalent, but not equal. The set {RAV4, Prius, Highlander, Camry} is equivalent to the set {10, 8, 7, 12} because they have the same number of elements.



Checkpoint

If two sets are equal, they are also equivalent, because equal sets also have the same cardinality.

Example 1.10: Differentiating Between Equivalent and Equal Sets

Determine if the following pairs of sets are equal, equivalent, or neither.

1. $E = \{2, 4, 6, 8, 10\}$ and $F = \{\text{fork, spoon, knife, meat thermometer, can opener}\}$
2. The empty set and the set of prime numbers less than 2.
3. The set of vowels in the word happiness and the set of consonants in the word happiness.

Answer

1. Sets E and F both have a cardinal value of 5, but the elements in these sets are different. So, the two sets are equivalent, but they are not equal: $E \sim F$.
2. The set of prime numbers consists of the set of counting numbers greater than one that can only be divided evenly by one and itself. The set of prime numbers less than 2 is an empty set, since there are no prime numbers less than 2. Therefore, these two sets are equal (and equivalent).

3. The set of vowels in the word happiness is $\{\mathrm{a},\ \mathrm{i},\ \mathrm{e}\}$ and the set of consonants in the word happiness is $\{\mathrm{h},\ \mathrm{p},\ \mathrm{n},\ \mathrm{s}\}$. The cardinal value of these sets two sets is $n(\{\mathrm{a},\ \mathrm{i},\ \mathrm{e}\}) = 3$ and $n(\{\mathrm{h},\ \mathrm{p},\ \mathrm{n},\ \mathrm{s}\}) = 4$, respectively. Because the cardinality of the two sets differs, they are not equivalent. Further, their elements are not identical, so they are also not equal.

Your Turn 1.10

Determine if the following pairs of sets are equal, equivalent, or neither.

Set $B = \{b, a, k, e\}$ and set $A = \{a, b, e, k\}$

2. Set $B = \{b, a, k, e\}$ and set $F = \{f, l, a, k, e\}$

3. Set $B = \{b, a, k, e\}$ and set $C = \{c, a, k, e\}$

People in Mathematics: Georg Cantor

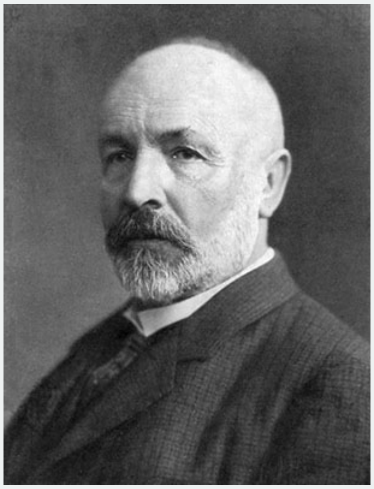


Figure 1.1.2: Georg Cantor (credit: Wikimedia, public domain)

Georg Cantor, the father of modern set theory, was born during the year 1845 in Saint Petersburg, Russa and later moved to Germany as a youth. Besides being an accomplished mathematician, he also played the violin. Cantor received his doctoral degree in Mathematics at the age of 22.

In 1870, at the age of 25 he established the uniqueness theorem for trigonometric series. His most significant work happened between 1874 and 1884, when he established the existence of transcendental numbers (also called irrational numbers) and proved that the set of real numbers are uncountably infinite—despite the objections of his former professor Leopold Kronecker.

Cantor published his final treatise on set theory in 1897 at the age of 52, and was awarded the Sylvester Medial from the Royal Society of London in 1904 for his contributions to the field. At the heart of Cantor's work was his goal to solve the continuum problem, which later influenced the works of David Hilbert and Ernst Zermelo.

References:

Wikipedia contributors. "Cantor." Wikipedia, The Free Encyclopedia, 23 Mar. 2021. Web. 20 Jul. 2021.

Akihiro Kanamori, "Set Theory from Cantor to Cohen," Editor(s): Dov M. Gabbay, Akihiro Kanamori, John Woods, *Handbook of the History of Logic*, North-Holland, Volume 6, 2012.

Check Your Understanding

1. A _____________ is a well-defined collection of objects.
2. The ___________ of a finite set A, denoted $n(A)$, is the number of elements in set A.
3. Determine if the following description describes a well-defined set: "The top 5 pizza restaurants in Chicago."

4. The United States is the only country to have landed people on the moon as of March 21, 2021. What is the cardinality of the set of all people who have walked on the moon prior to this date?
5. Set A is a set of a dozen distinct donuts, and set B is a set of a dozen different types of apples. Is set A equal to set B, equivalent to set B, or neither?
6. Is the set of all butterflies in the world a finite set or an infinite set?
7. Represent the set of all upper-case letters of the English alphabet using both the roster method and set builder notation.

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1.1.0: Exercises

For the following exercises, represent each set using the roster method.

1. The set of primary colors: red, yellow, and blue.
2. A set of the following flowers: rose, tulip, marigold, iris, and lily.
3. The set of natural numbers between 50 and 100.
4. The set of natural numbers greater than 17.
5. The set of different pieces in a game of chess.
6. The set of natural numbers less than 21.

For the following exercises, represent each set using set builder notation.

7. The set of all types of lizards.
8. The set of all stars in the universe.
9. The set of all integer multiples of 3 that are greater than zero.
10. The set of all integer multiples of 4 that are greater than zero.
11. The set of all plants that are edible.
12. The set of all even numbers.

For the following exercises, represent each set using the method of your choice.

13. The set of all squares that are also circles.
14. The set of natural numbers divisible by zero.
15. The set of Mike and Carol's children on the TV show, *The Brady Bunch*.
16. The set of all real numbers.
17. The set of polar bears that live in Antarctica.
18. The set of songs written by Prince.
19. The set of children's books written and illustrated by Mo Willems.
20. The set of seven colors commonly listed in a rainbow.

For the following exercises, determine if the collection of objects represents a well-defined set or not.

21. The names of all the characters in the book, *The Fault in Our Stars* by John Green.
22. The five greatest soccer players of all time.
23. A group of old dogs that are able to learn new tricks.
24. A list of all the movies directed by Spike Lee as of 2021.
25. The group of all zebras that can fly an airplane.
26. The group of National Baseball League Hall of Fame members who have hit over 700 career home runs.

For the following exercises, compute the cardinal value of each set.

27. $P = \{\text{Snuzzle, Butterscotch, Blue Belle, Minty, Blossom, Cotton Candy}\}$
28. $T = \{\text{pepperoni, sausage, bacon, ham, mushrooms, olives, bell pepper, pineapple}\}$
29. $\emptyset$
30. $B = \{5, 6, 7, \ldots, 20\}$
31. $F = \left\{\frac{1}{9}, \frac{2}{9}, \frac{3}{9}, \frac{4}{9}, \frac{5}{9}, \frac{6}{9}, \frac{7}{9}, \frac{8}{9}, \frac{9}{9}\right\}$
32. $\{\ \}$
33. $C = \{n^3 \mid n \text{ is a member of } \mathbb{N}\}$
34. $S = \{7n | n \text{ is an element of } \mathbb{N}\}$
35. $L = \{l, m, n, \ldots, y\}$
36. The set of numbers on a standard 6-sided die.

For the following exercises, determine whether set \(A and set \(B are equal, equivalent or neither.

37. $A = \{\text{ right, acute, obtuse }\}; B = \{\text{ equilateral, scalene, isoceles }\}$.
38. $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right\}; B = \left\{\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1\right\}$.
39. $A = \{\text{ red, orange, yellow }\}; B = \{\text{ green, blue, indigo, violet }\}$.
40. $A = \{5n \mid n \in \mathbb{N}\}; B = \mathbb{N}$.

41. $A=\{-2,-1,0,\ldots\}; B=\{2,3,5,\ldots\}$.
42. $A=\{$ John, Paul, George, Ringo $\}; B=\{$ Bono, Larry, The Edge, Adam $\}$.
43. $A=\emptyset; B=\{\}$.
44. $A=\{$ lemon, lime, orange $\}; B=\{$ orange, lemon, lime, grape $\}$.

For the following exercises, determine if the set described is finite or infinite.

45. The set of natural numbers.
46. The empty set.
47. The set consisting of all jazz venues in New Orleans, Louisiana.
48. The set of all real numbers.
49. The set of all different types of cheeses.
50. The set of all words in *Merriam-Webster's Collegiate Dictionary*, Eleventh Edition, published in 2020.

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1.2: Subsets

Figure 1.4 The players on a soccer team who are actively participating in a game are a subset of the greater set of team members. (Credit: “PAFC-Mezokovesd-108” by Puskás Akadémia/Flickr, Public Domain Mark 1.0)

Learning Objectives

After completing this section, you should be able to:

- Represent subsets and proper subsets symbolically.
- Compute the number of subsets of a set.
- Apply concepts of subsets and equivalent sets to finite and infinite sets.

The rules of Major League Soccer (MLS) allow each team to have up to 30 players on their team. However, only 18 of these players can be listed on the game day roster, and of the 18 listed, 11 players must be selected to start the game. How the coaches and general managers form the team and choose the starters for each game will determine the success of the team in any given year.

The entire group of 30 players is each team’s set. The group of game day players is a subset of the team set, and the group of 11 starters is a subset of both the team set and the set of players on the game day roster.

Set A is a subset of set B if every member of set A is also a member of set B. Symbolically, this relationship is written as $A \subseteq B$.

Sets can be related to each other in several different ways: they may not share any members in common, they may share some members in common, or they may share all members in common. In this section, we will explore the way we can select a group of members from the whole set.

Checkpoint

Every set is also a subset of itself, $B \subseteq B$

Recall the set of flatware in our kitchen drawer from Section 1.1,

$F = \{\text{fork}, \text{spoon}, \text{knife}, \text{meat thermometer}, \text{can opener}\}$. Suppose you are preparing to eat dinner, so you pull a fork and a knife from the drawer to set the table. The set $D = \{\text{knife}, \text{fork}\}$ is a subset of set F, because every member or element of set D is also a member of set F. More specifically, set D is a proper subset of set F, because there are other members of set F not in set D. This is written as $D \subset$F. The only subset of a set that is not a proper subset of the set would be the set itself.

Checkpoint

The empty set or null set, $\emptyset$, is a proper subset of every set, except itself.

Graphically, sets are often represented as circles. In the following graphic, set A is represented as a circle completely enclosed inside the circle representing set B, showing that set A is a proper subset of set B. The element x represents an element that is in both set A and set B.

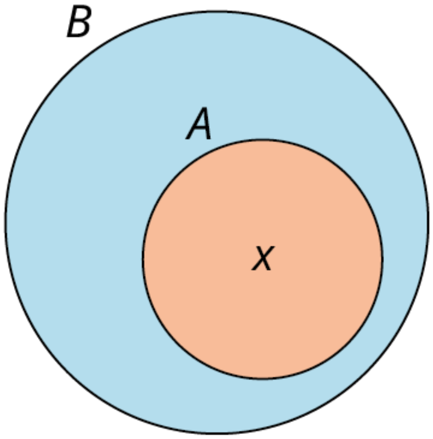

Figure 1.5

Checkpoint

While we can list all the subsets of a finite set, it is not possible to list all the possible subsets of an infinite set, as it would take an infinitely long time.

Example 1.11

Listing All the Proper Subsets of a Finite Set

Set L is a set of reading materials available in a shop at the airport, $L = \{\text{newspaper}, \text{magazine}, \text{book}\}$. List all the subsets of set L.

Answer

Step 1: It is best to begin with the set itself, as every set is a subset of itself. In our example, the cardinality of set L is $n(L) = 3$. There is only one subset of set L that has the same number of elements of set L: $\{\text{newspaper}, \text{magazine}, \text{book}\}$.

Step 2: Next, list all the proper subsets of the set containing $n(L) - 1$ elements. In this case, $3 - 1 = 2$. There are three subsets that each contain two elements: $\{\text{newspaper}, \text{magazine}\}$, $\{\text{newspaper}, \text{book}\}$, and $\{\text{magazine}, \text{book}\}$.

Step 3: Continue this process by listing all the proper subsets of the set containing $n(L) - 2$ elements. In this case, $3 - 2 = 1$. There are three subsets that contain one element: $\{\text{newspaper}\}$, $\{\text{magazine}\}$, and $\{\text{book}\}$.

Step 4: Finally, list the subset containing 0 elements, or the empty set: { }.

Your Turn 1.11

Consider the set of possible outcomes when you flip a coin, /**/S = \{ {\text{heads, tails}}\} /**/. List all the possible subsets of set /**/S./**/

Example 1.12

Determining Whether a Set Is a Proper Subset

Consider the set of common political parties in the United States, $P = \{\text{Democratic}, \text{Green}, \text{Libertarian}, \text{Republican}\}$. Determine if the following sets are proper subsets of P.

1. $M = \{\text{Democratic, Republican}\}$
2. $G = \{\text{Green}\}$
3. $V = \{\text{Republican, Libertarian, Green, Democratic}\}$

Answer

1. M is a proper subset of P, written symbolically as $M \subset P$ because every member of M is a member of set P, but P also contains at least one element that is not in M.
2. G is a single member proper subset of P, written symbolically as $G \subset P$, because Green is a member of set P, but P also contains other members (such as Democratic) that are not in G.
3. V is subset of P because every member of V is also a member of P, but it is not a proper subset of P because there are no members of V that are not also in set P. We can represent the relationship symbolically as $V \subseteq P$, or more precisely, set V is equal to set P, $V = P$.

Your Turn 1.12

Consider the set of generation I legendary Pokémon, /**/L = \{ {\text{Articuno, Zapdos, Moltres, Mewtwo}}\} /**/. Give an example of a proper subset containing:

1. one member.
2. three members.
3. no members.

Example 1.13

Expressing the Relationship between Sets Symbolically

Consider the subsets of a standard deck of cards: $S = \{\text{spades, hearts, diamonds, clubs}\}$; $R = \{\text{hearts, diamonds}\}$; $B = \{\text{spades, clubs}\}$; and $C = \{\text{clubs}\}$.

Express the relationship between the following sets symbolically.

1. Set S and set B.
2. Set C and set B.
3. Set R and R.

Answer

1. $B \subset S$. B is a proper subset of set S.
2. $C \subset B$. C is a proper subset of set B.
3. R ⊆RorR=RR⊆RorR=R. R is subset of itself, but not a proper subset of itself because R is equal to itself.

Your Turn 1.13

Express the relationship between the set of natural numbers, $\mathbb{N} = \{1, 2, 3, \ldots\}$, and the set of even numbers, \(/E = \{ 2,4,6, \ldots \}\).

Exponential Notation

So far, we have figured out how many subsets exist in a finite set by listing them. Recall that in Example 1.11, when we listed all the subsets of the three-element set $L = \{$ newspaper, magazine, book\} we saw that there are eight subsets. In Your Turn 1.11, we discovered that there are four subsets of the two-element subset, $S = \{$ heads, tails $\}$. A one-element set has two subsets, the empty set and itself. The only subset of the empty set is the empty set itself. But how can we easily figure out the number of subsets in a very large finite set? It turns out that the number of subsets can be found by raising 2 to the number of elements in the set, using exponential notation to represent repeated multiplication. For example, the number of subsets of the set $L = \{$ newspaper, magazine, book $\}$ is equal to $2^3 = 2 \cdot 2 \cdot 2 = 8$. Exponential notation is used to represent repeated multiplication, $b^n = b \cdot b \cdot b \cdot \ldots \cdot b$, where b appears as a factor n times.

FORMULA

The number of subsets of a finite set A is equal to 2 raised to the power of $n(A)$, where $n(A)$ is the number of elements in set A: $\text{Number of Subsets of Set } A = 2^{n(A)}$.

Checkpoint

Note that $2^0 = 1$, so this formula works for the empty set, also.

Example 1.14

Computing the Number of Subsets of a Set

Find the number of subsets of each of the following sets.

1. The set of top five scorers of all time in the NBA:
   $S = \{$ LeBron James, Kareem Abdul-Jabbar, Karl Malone, Kobe Bryant, Michael Jordan
2. The set of the top four bestselling albums of all time:
   $A = \{$ Thriller, Hotel California, The Beatles White Album, Led Zepplin IV $\}$.
3. $R = \{$ Snap, Crackle, Pop $\}$.

Answer

1. $n(S) = 5$. So, the total number of subsets of Sis25=2·2·2·2·2=32Sis25=2·2·2·2·2=32.
2. $n(A) = 4$. Therefore, the total number of subsets of Ais24=16Ais24=16.
3. $n(R) = 3$. So, the total number of subsets of Ris23=8Ris23=8.

Your Turn 1.14

Compute the total number of subsets in the set of the top nine tennis grand slam singles winners,
$T = \{$ Margaret Court, Serena Williams, Steffi Graff, Rafael Nadal, Novak Djokovic, Roger Federer, Helen Wills, Martina Navratilova, Chris Everett\}.

Equivalent Subsets

In the early 17th century, the famous astronomer Galileo Galilei found that the set of natural numbers and the subset of the natural numbers consisting of the set of square numbers, semanticsn2n2, are equivalent. Upon making this discovery, he conjectured that the concepts of less than, greater than, and equal to did not apply to infinite sets.

Sequences and series are defined as infinite subsets of the set of natural numbers by forming a relationship between the sequence or series in terms of a natural number, n. For example, the set of even numbers can be defined using set builder notation as $\{a|a = 2n \text{ where } n \text{ is a natural number}\}$. The formula in this case replaces every natural number with two times the number, resulting in the set of even numbers, $\{2, 4, 6, ...\}$. The set of even numbers is also equivalent to the set of natural numbers.

Who Knew?

Employment Opportunities

You can make a career out of working with sets. Applications of equivalent sets include relational database design and analysis.

Relational databases that store data are tables of related information. Each row of a table has the same number of columns as every other row in the table; in this way, relational databases are examples of set equivalences for finite sets. In a relational database, a primary key is set up to identify all related information. There is a one-to-one relationship between the primary key and any other information associated with it.

Database design and analysis is a high demand career with a median entry-level salary of about $85,000 per year, according to salary.com.

Example 1.15

Writing Equivalent Subsets of an Infinite Set

Using natural numbers, multiples of 3 are given by the sequence $\{3, 6, 9, ...\}$. Write this set using set builder notation by expressing each multiple of 3 using a formula in terms of a natural number, n.

Answer

> $\{m|m = 3n \text{ where } n \text{ is a natural number}\}$ or $\{m|m = 3n \text{ where } n \in \mathrm{N}\}$. In this example, m is a multiple of 3 and n is a natural number. The symbol $\in$ is read as “is a member or element of.” Because there is a one-to-one correspondence between the set of multiples of 3 and the natural numbers, the set of multiples of 3 is an equivalent subset of the natural numbers.

Your Turn 1.15

Using natural numbers, multiples of 5 are given by the sequence $\{5, 10, 15, ...\}$. Write this set using set builder notation by associating each multiple of 5 in terms of a natural number, n.

Example 1.16

Creating Equivalent Subsets of a Finite Set That Are Not Equal

A fast-food restaurant offers a deal where you can select two options from the following set of four menu items for $6: a chicken sandwich, a fish sandwich, a cheeseburger, or 10 chicken nuggets. Javier and his friend Michael are each purchasing lunch using this deal. Create two equivalent, but not equal, subsets that Javier and Michael could choose to have for lunch.

Answer

> The possible two-element subsets are: {chicken sandwich, fish sandwich}, {chicken sandwich, cheeseburger}, {chicken sandwich, chicken nuggets}, {fish sandwich, cheeseburger}, {fish sandwich, chicken nuggets}, and {cheeseburger, chicken nuggets}. One possible solution is that Javier picked the set {chicken sandwich, chicken nuggets}, while Michael chose the {cheeseburger, chicken nuggets}. Because Javier and Michael both picked two items, but not exactly the same two items, these sets are equivalent, but not equal.

Your Turn 1.16

Serena and Venus Williams walk into the same restaurant as Javier and Michael, but they order the same pair of items, resulting in equal sets of choices. If Venus ordered a fish sandwich and chicken nuggets, what did Serena order?

Example 1.17

Creating Equivalent Subsets of a Finite Set

A high school volleyball team at a small school consists of the following players: {Angie, Brenda, Colleen, Estella, Maya, Maria, Penny, Shantelle}. Create two possible equivalent starting line-ups of six players that the coach could select for the next game.

Answer

> There are actually 28 possible ways that the coach could choose his starting line-up. Two such equivalent subsets are {Angie, Brenda, Maya, Maria, Penny, Shantelle} and {Angie, Brenda, Colleen, Estella, Maria, Shantelle}. Each subset has six members, but they are not identical, so the two sets are equivalent but not equal.

Your Turn 1.17

Consider the same group of volleyball players from above: {Angie, Brenda, Colleen, Estella, Maya, Maria, Penny, Shantelle}. The team needs to select a captain and an assistant captain from their members. List two possible equivalent subsets that they could select.

Check Your Understanding

8. Every member of a ________ of a set is also a member of the set.

9. Explain what distinguishes a proper subset of a set from a subset of a set.

10. The ________ set is a proper subset of every set except itself.

11. Is the following statement true or false? $A \subseteq A$.

12. If the cardinality of set A is $n(A) = 10$, then set A has a total of ________ subsets.

13. Set A is ________ to set B if $n(A) = n(B)$.

14. If every member of set A is a member of set B and every member of set B is also a member set A, then set A is ________ to set B.

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1.2.0: Exercises

For the following exercises, list all the proper subsets of each set.

1. {chocolate, vanilla, strawberry}
2. {true, false }
3. {mother, father, daughter, son }
4. $\{7\}$

For the following exercises, determine the relationship between the two sets and write the relationship symbolically.

$D=\{0,1,2,\ldots,9\}, A=\{0,2,4,6,8\}, B=\{1,3,5,7,9\}, C=\{8,6,4,2,0\}, Z=\{0\}$, and $\emptyset$

5. D and A

6. B and D

7. C and D

8. Z and C

9. Z and $\emptyset$

10. A and B

11. A and C

12. $\emptyset$ and D

13. B and C

14. A and Z

For the following exercises, calculate the total number of subsets of each set.

15. {Adele, Beyonce, Cher, Madonna, Shakira }

16. { Art, Paul }

17. {Peter, Paul, Mary}

18. $\emptyset$

19. $\{3\}$

20. $\{l, o, v, e\}$

21. $\}$

22. \{football, baseball, basketball, soccer, hockey, tennis, golf\}

23. Set A, if $n(A)=12$.

24. Set B, if $n(B)=9$.

For the following exercises, use the set of letters in the word largest as the set, $U=\{\mathrm{l},\ \mathrm{a},\ \mathrm{r},\ \mathrm{g},\ \mathrm{e},\ \mathrm{s},\ \mathrm{t}\}$.

25. Find a subset of U that is equivalent, but not equal, to the set: $\{l, \mathrm{a}, \mathrm{s}, \mathrm{t}\}$.

26. Find a subset of U that is equal to the set: $\{l, \mathrm{a}, \mathrm{s}, \mathrm{t}\}$.

27. Find a subset of U that is equal to the set: $\{\mathrm{a}, \mathrm{r}, \mathrm{t}\}$.

28. Find a subset of U that is equivalent, but not equal, to the set $\{\mathrm{a}, \mathrm{r}, \mathrm{t}, \mathrm{s}\}$.

29. Find a subset of U that is equivalent, but not equal, to the set: $\{\mathrm{r}, \mathrm{a}, \mathrm{t}, \mathrm{e}, \mathrm{s}\}$.

30. Find a subset of U that is equal to the set: $\{\mathrm{r}, \mathrm{a}, \mathrm{t}, \mathrm{e}, \mathrm{s}\}$.

31. Find two three-character subsets of set U that are equivalent, but not equal, to each other.

32. Find two three-character subsets of set U that are equal to each other.

33. Find two five-character subsets of set U that are equal to each other.

34. Find two five-character subsets of set U that are equivalent, but not equal, to each other.

Find two equal subsets of /**/U/**/ with a cardinality of 4.

35. Find two equivalent subset of U with a cardinality of 7 .
36. Find two equal subsets of U with a cardinality of 4 .
37. Find a subset of U that is equivalent, but not equal to, $\{0, 3, 6, 9, \ldots\}$.
38. Find a subset of U that is equivalent, but not equal to, $\{-1, -4, -9, -16, -25, \ldots\}$.
39. True or False. The set of natural numbers, $\mathbb{N} = \{1, 2, 3, \ldots\}$, is equivalent to set U.
40. True or False. Set U is an equivalent subset of the set of rational numbers, $\mathbb{Q} = \left\{ \frac{p}{q} \middle| \; p \text{ and } q \text{ are integers and } q \neq 0. \right\}$.

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1.3: Understanding Venn Diagrams

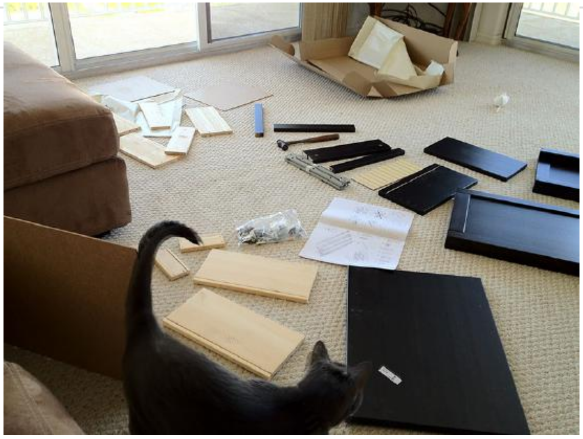

Figure 1.6: When assembling furniture, instructions with images are easier to follow, just like how set relationships are easier to understand when depicted graphically. (credit: "Time to assemble more Ikea furniture!" by Rod Herrea/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

- Utilize a universal set with two sets to interpret a Venn diagram.
- Utilize a universal set with two sets to create a Venn diagram.
- Determine the complement of a set.

Have you ever ordered a new dresser or bookcase that required assembly? When your package arrives you excitedly open it and spread out the pieces. Then you check the assembly guide and verify that you have all the parts required to assemble your new dresser. Now, the work begins. Luckily for you, the assembly guide includes step-by-step instructions with images that show you how to put together your product. If you are really lucky, the manufacturer may even provide a URL or QR code connecting you to an online video that demonstrates the complete assembly process. We can likely all agree that assembly instructions are much easier to follow when they include images or videos, rather than just written directions. The same goes for the relationships between sets.

Interpreting Venn Diagrams

Venn diagrams are the graphical tools or pictures that we use to visualize and understand relationships between sets. Venn diagrams are named after the mathematician John Venn, who first popularized their use in the 1880s. When we use a Venn diagram to visualize the relationships between sets, the entire set of data under consideration is drawn as a rectangle, and subsets of this set are drawn as circles completely contained within the rectangle. The entire set of data under consideration is known as the universal set.

Consider the statement: All trees are plants. This statement expresses the relationship between the set of all plants and the set of all trees. Because every tree is a plant, the set of trees is a subset of the set of plants. To represent this relationship using a Venn diagram, the set of plants will be our universal set and the set of trees will be the subset. Recall that this relationship is expressed symbolically as: $\text{Trees} \subset \text{Plants}$. To create a Venn diagram, first we draw a rectangle and label the universal set “$U = \text{Plants}$.” Then we draw a circle within the universal set and label it with the word “Trees.”

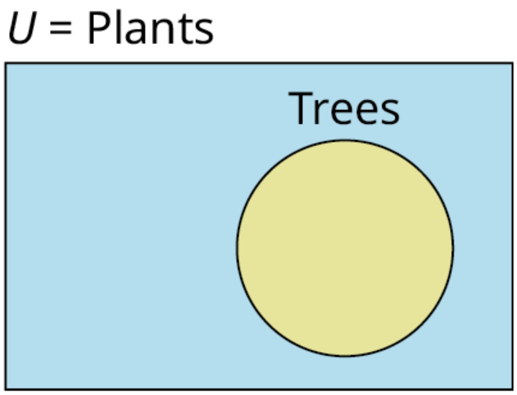

Figure 1.7:

This section will introduce how to interpret and construct Venn diagrams. In future sections, as we expand our knowledge of relationships between sets, we will also develop our knowledge and use of Venn diagrams to explore how multiple sets can be combined to form new sets.

✓ Example 1.18: Interpreting the Relationship between Sets in a Venn Diagram

Write the relationship between the sets in the following Venn diagram, in words and symbolically.

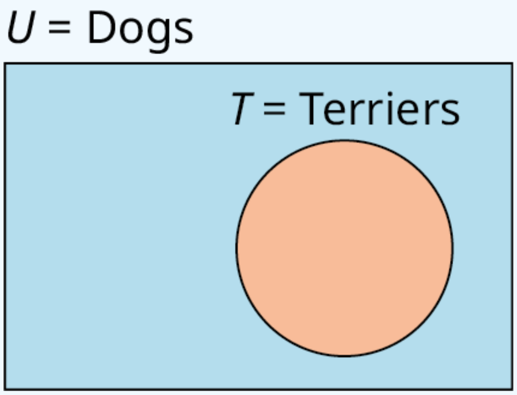

Figure 1.8:

Answer

The set of terriers is a subset of the universal set of dogs. In other words, the Venn diagram depicts the relationship that all terriers are dogs. This is expressed symbolically as $T \subset U$.

Your Turn 1.18

Write the relationship between the sets in the following Venn diagram, in words and symbolically.

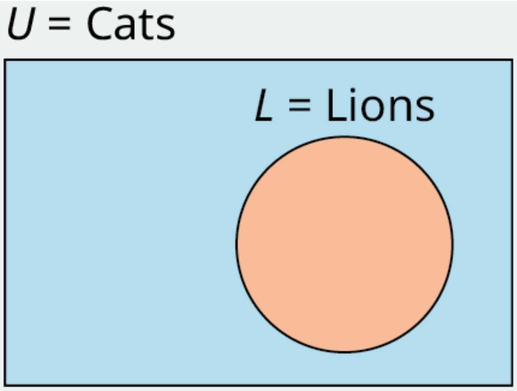

Figure 1.9

So far, the only relationship we have been considering between two sets is the subset relationship, but sets can be related in other ways. Lions and tigers are both different types of cats, but no lions are tigers, and no tigers are lions. Because the set of all lions and the set of all tigers do not have any members in common, we call these two sets disjoint sets, or non-overlapping sets.

Two sets A and B are disjoint sets if they do not share any elements in common. That is, if a is a member of set A, then a is not a member of set B. If b is a member of set B, then b is not a member of set A. To represent the relationship between the set of all cats and the sets of lions and tigers using a Venn diagram, we draw the universal set of cats as a rectangle and then draw a circle for the set of lions and a separate circle for the set of tigers within the rectangle, ensuring that the two circles representing the set of lions and the set of tigers do not touch or overlap in any way.

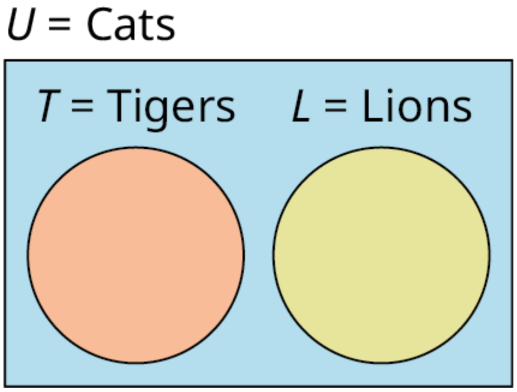

Figure 1.10:

✓ Example 1.19: Describing the Relationship between Sets

Describe the relationship between the sets in the following Venn diagram.

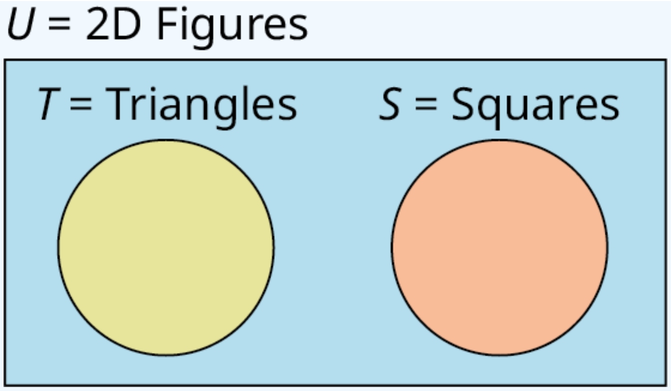

Figure 1.11:

Answer

The set of triangles and the set of squares are two disjoint subsets of the universal set of two-dimensional figures. The set of triangles does not share any elements in common with the set of squares. No triangles are squares and no squares are triangles, but both squares and triangles are 2D figures.

Your Turn 1.19

Describe the relationship between the sets in the following Venn diagram.

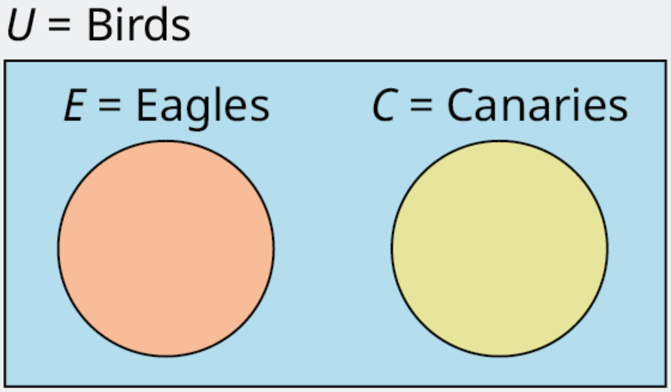

Figure 1.12

Creating Venn Diagrams

The main purpose of a Venn diagram is to help you visualize the relationship between sets. As such, it is necessary to be able to draw Venn diagrams from a written or symbolic description of the relationship between sets.

Procedure

To create a Venn diagram:

1. Draw a rectangle to represent the universal set, and label it $U = \text{set}$nameU=set name.
2. Draw a circle within the rectangle to represent a subset of the universal set and label it with the set name.

Checkpoint

If there are multiple disjoint subsets of the universal set, their separate circles should not touch or overlap.

✓ Example 1.20: Drawing a Venn Diagram to Represent the Relationship Between Two Sets

Draw a Venn diagram to represent the relationship between each of the sets.

1. All rectangles are parallelograms.
2. All women are people.

Answer

1. The set of rectangles is a subset of the set of parallelograms.
   First, draw a rectangle to represent the universal set and label it with $U = \text{Parallelograms}$, then draw a circle completely within the rectangle, and label it with the name of the set it represents, $R = \text{Rectangles}$.

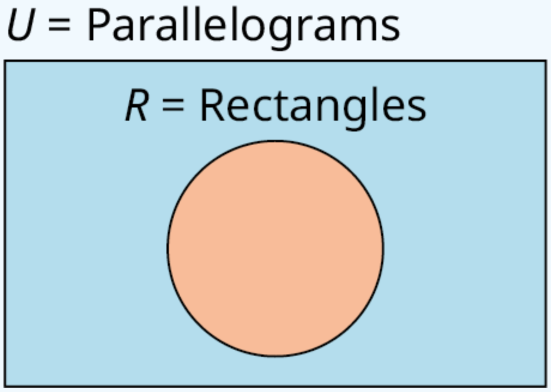

Figure 1.13:

In this example, both letters and names are used to represent the sets involved, but this is not necessary. You may use either letters or names alone, as long as the relationship is clearly depicted in the diagram, as shown below.

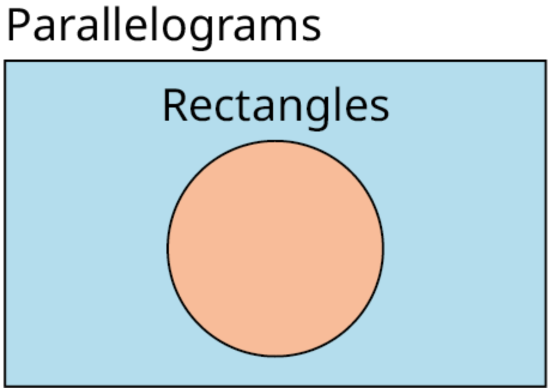

Figure 1.14:

or

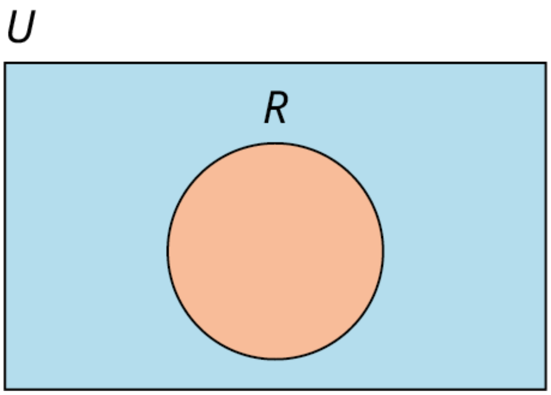

Figure 1.15:

2. The universal set is the set of people, and the set of all women is a subset of the set of people.

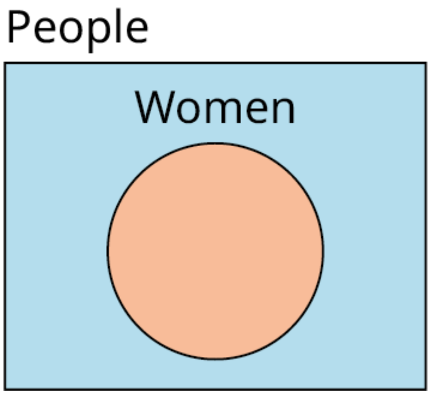

Figure 1.16:

Your Turn 1.20

Draw a Venn diagram to represent the relationship between each of the sets. All natural numbers are integers.

$A \subset U$. Draw a Venn diagram to represent this relationship.

Example 1.21: Drawing a Venn Diagram to Represent the Relationship Between Three Sets

All bicycles and all cars have wheels, but no bicycle is a car. Draw a Venn diagram to represent this relationship.

Answer

Step 1: The set of bicycles and the set of cars are both subsets of the set of things with wheels. The universal set is the set of things with wheels, so we first draw a rectangle and label it with $U = \text{Things with} \text{Wheels}$U=Things with Wheels.

Step 2: Because the set of bicycles and the set of cars do not share any elements in common, these two sets are disjoint and must be drawn as two circles that do not touch or overlap with the universal set.

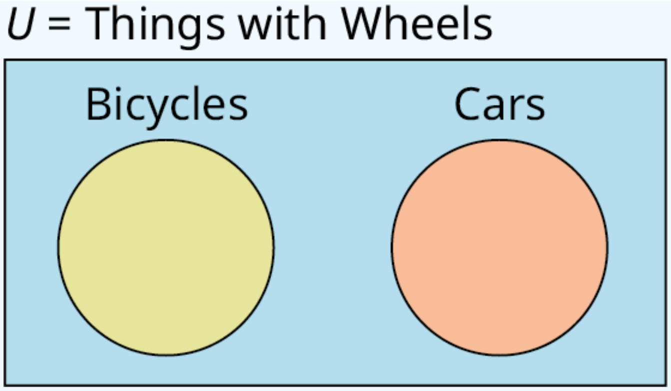

Figure 1.17:

Your Turn 1.21

Airplanes and birds can fly, but no birds are airplanes. Draw a Venn diagram to represent this relationship.

The Complement of a Set

Recall that if set A is a proper subset of set U, the universal set (written symbolically as $A \subset U$), then there is at least one element in set U that is not in set A. The set of all the elements in the universal set U that are not in the subset A is called the complement of set A, A'. In set builder notation this is written symbolically as: $A' = \{x \in U | x \notin A\}$. The symbol $\in$ is used to represent the phrase, "is a member of," and the symbol $\notin$ is used to represent the phrase, "is not a member of." In the Venn diagram below, the complement of set A is the region that lies outside the circle and inside the rectangle. The universal set U includes all of the elements in set A and all of the elements in the complement of set A, and nothing else.

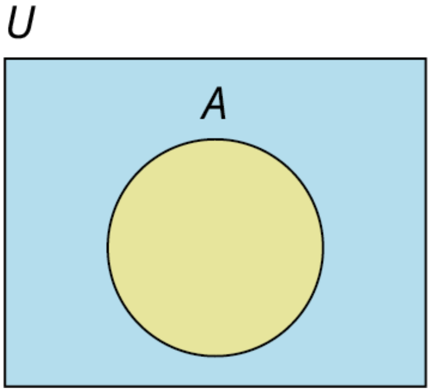

Figure 1.18:

Consider the set of digit numbers. Let this be our universal set, $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Now, let set A be the subset of U consisting of all the prime numbers in set U, $A = \{2, 3, 5, 7\}$. The complement of set A is $A' = \{0, 1, 4, 6, 8, 9\}$. The following Venn diagram represents this relationship graphically.

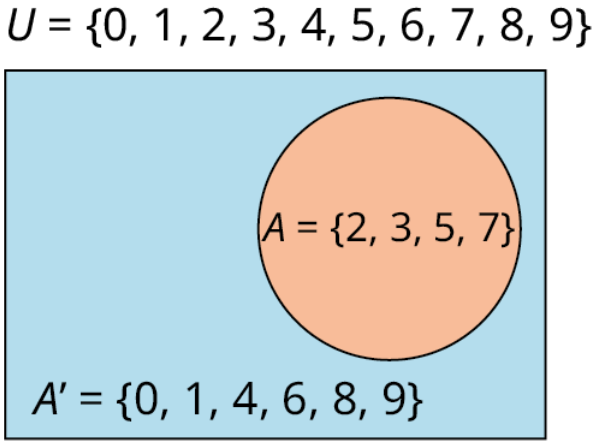

Figure 1.19:

✓ Example 1.22: Finding the Complement of a Set

For both of the questions below, A is a proper subset of U.

1. Given the universal set $U = \{$ Billie Eilish, Donald Glover, Bruno Mars, Adele, Ed Sheeran\} and set $A = \{$ Donald Glover, Bruno Mars, Ed Sheeran $\}$, find A'.

2. Given the universal set $U = \{\mathrm{d} \mid \mathrm{d} \text{ is a } \mathrm{dog}\}$ and $B = \{\mathrm{b} \in \mathrm{U} \mid \mathrm{b}$ is a beagle $\}$, find B'.

Answer

1. The complement of set A is the set of all elements in the universal set U that are not in set A. $A' = \{$ Billie Eilish, Adele $\}$.

2. The complement of set B is the set of all dogs that are not beagles. All members of set B' are in the universal set because they are dogs, but they are not in set B, because they are not beagles. This relationship can be expressed in set build notation as follows:
$B' = \{$ All dogs that are not beagles. $\}$, $B' = \{d \in U \mid d$ is not a beagle. $\}$, or $B' = \{d \in U \mid d \notin B\}$.

Your Turn 1.22

For both of the questions below, A is a proper subset of U. 1. Given the universal set $U = \{$ red, orange, yellow, green, blue, indigo, violet $\}$ and set $A = \{$ yellow, red, blue $\}$, find A'. 2. Given the universal set $U = \{c \mid c$ is a cat $\}$ and set $A = \{c \in U \mid c$ is not a lion $\}$, find A'.

Check Your Understanding

1. A Venn diagram is a graphical representation of the __________ between sets.
2. In a Venn diagram, the set of all data under consideration, the __________ set, is drawn as a rectangle.
3. Two sets that do not share any elements in common are __________ sets.
4. The __________ of a subset A or the universal set, U, is the set of all members of U that are not in A.
5. The sets A and A' are __________ subsets of the universal set.

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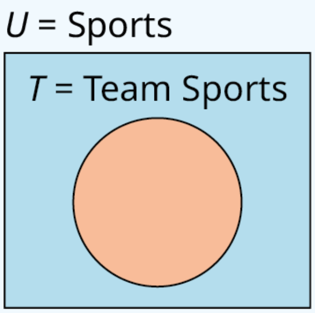
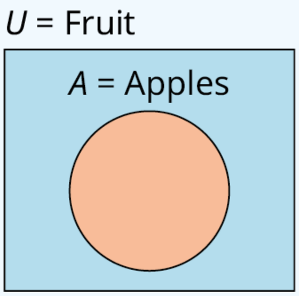
1.3.0: Exercises

For the following exercises, interpret each Venn diagram and describe the relationship between the sets, symbolically and in words.

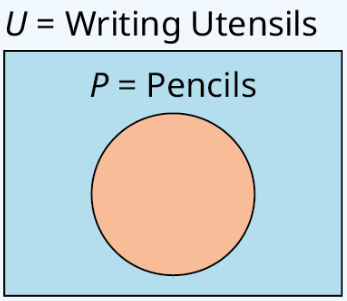
Exercise $1.3.0.1$


U = Sports
T = Team Sports

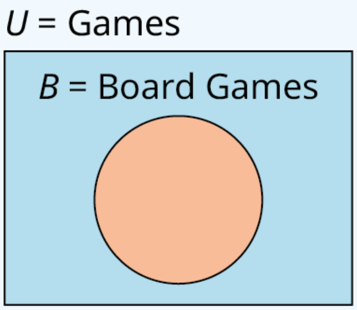
Exercise $1.3.0.2$


U = Fruit
A = Apples

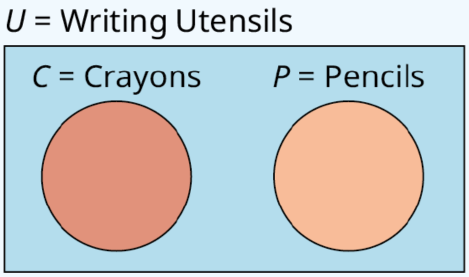
Exercise $1.3.0.3$


U = Writing Utensils
P = Pencils

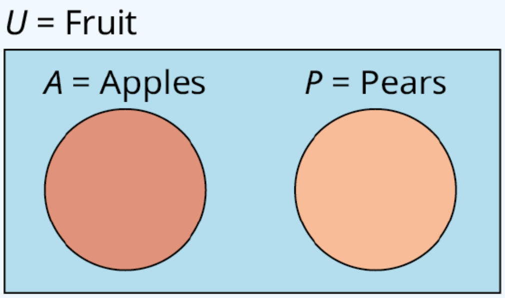
Exercise $1.3.0.4$


U = Games
B = Board Games

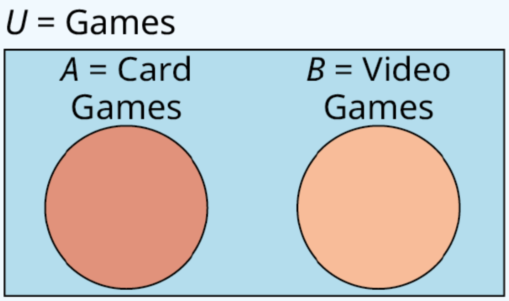
Exercise 1.3.0.5


U = Writing Utensils
C = Crayons
P = Pencils

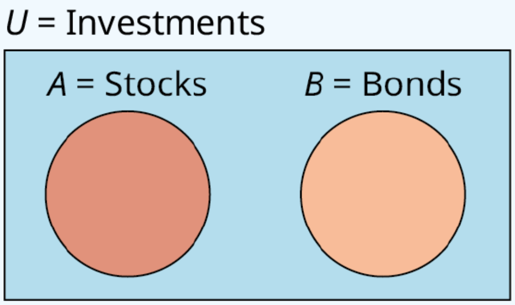
Exercise 1.3.0.6


U = Fruit
A = Apples
P = Pears

Exercise 1.3.0.7


U = Games
A = Card Games
B = Video Games

Exercise 1.3.0.8


U = Investments
A = Stocks
B = Bonds

For the following exercises, create a Venn diagram to represent the relationships between the sets.

Exercise 1.3.0.9

All birds have wings.

? Exercise 1.3.0.10

All cats are animals.

? Exercise 1.3.0.11

All almonds are nuts, and all pecans are nuts, but no almonds are pecans.

? Exercise 1.3.0.12

All rectangles are quadrilaterals, and all trapezoids are quadrilaterals, but no rectangles are trapezoids.

? Exercise 1.3.0.13

Lizards $\subset$ Reptiles.

? Exercise 1.3.0.14

Ladybugs $\subset$ Insects.

? Exercise 1.3.0.15

Ladybugs $\subset$ Insects and Ants $\subset$ Insects, but no Ants are Ladybugs.

? Exercise 1.3.0.16

Lizards $\subset$ Reptiles and Snakes $\subset$ Reptiles, but no Lizards are Snakes.

? Exercise 1.3.0.17

A and B are disjoint subsets of U.

? Exercise 1.3.0.18

C and D are disjoint subsets of U.

? Exercise 1.3.0.19

T is a subset of U.

? Exercise 1.3.0.20

S is a subset of U.

? Exercise 1.3.0.21

$J =$ Jazz, $M =$ Music, and $J \subset M$.

? Exercise 1.3.0.22

$R =$ Reggae, $M =$ Music, and $R \subset M$.

Exercise 1.3.0.23

$J=$ Jazz, $R=$ Reggae, and $M=$ Music are sets with the following relationships: $J\subset M, R\subset M$, and R is disjoint from J.

Exercise 1.3.0.24

$J=$ Jazz, $B=$ Bebop, and $M=$ Music are sets with the following relationships: $J\subset M$ and $B\subset J$.

For the following exercises, the universal set is the set of single digit numbers, $U=\{0,1,2,3,4,5,6,7,8,9\}$. Find the complement of each subset of U.

Exercise 1.3.0.25

$A=\{6,7,8\}$

Exercise 1.3.0.26

$A=\{0,2,4,6,8\}$

Exercise 1.3.0.27

$A=\{\}$

Exercise 1.3.0.28

$A=\{0,1,4,6,8,9\}$

Exercise 1.3.0.29

$A=\{0,1,2,3,4,5,6,7,8,9\}$

Exercise 1.3.0.30

$A=\{0,1,3,4,5,6,7,9\}$

Exercise 1.3.0.31

$A=\{1,2,3,4,5,6,7,8,9\}$

Exercise 1.3.0.32

$A=\{0,3,6,9\}$

For the following exercises, the universal set is $U=$ \{Bashful, Doc, Dopey, Grumpy, Happy, Sleepy, Sneezy\}. Find the complement of each subset of U.

Exercise 1.3.0.33

$A=\{$ Happy, Bashful, Grumpy\}

Exercise 1.3.0.34

$A=\{$ Sleepy, Sneezy $\}$

Exercise 1.3.0.35

$A = \{\text{Doc}\}$

Exercise 1.3.0.36

$A = \{$ Doc, Dopey $\}$

Exercise 1.3.0.37

$A = \emptyset$

Exercise 1.3.0.38

$A = \{$ Doc, Grumpy, Happy, Sleepy, Bashful, Sneezy, Dopey\}

For the following exercises, the universal set is $U = \mathbb{N} = \{1, 2, 3, \ldots\}$. Find the complement of each subset of U.

Exercise 1.3.0.39

$A = \{1, 2, 3, 4, 5\}$

Exercise 1.3.0.40

$A = \{1, 3, 5, \ldots\}$

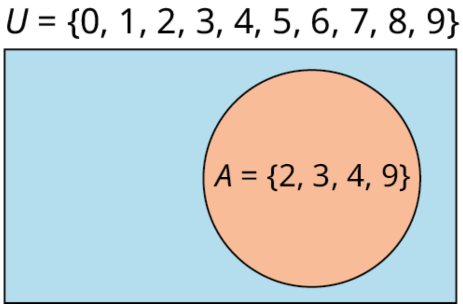
Exercise 1.3.0.41

$A = \{1\}$

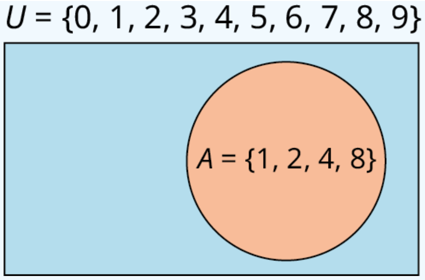
Exercise 1.3.0.42

$A = \{4, 5, 6, \ldots\}$

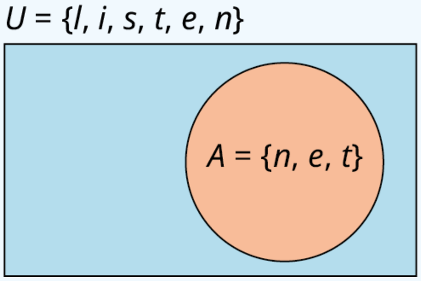
For the following exercises, use the Venn diagram to determine the members of the complement of set A, A'.



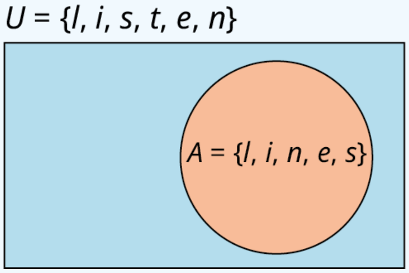
Exercise 1.3.0.44


U = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}
A = {1, 2, 4, 8}

Exercise 1.3.0.45


U = {l, i, s, t, e, n}
A = {n, e, t}

Exercise 1.3.0.46


U = {l, i, s, t, e, n}
A = {l, i, n, e, s}

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1.4: Set Operations with Two Sets

Figure 1.20 A large, multigenerational family contains an intersection and a union of sets. (credit: “Family Photo Shoot Bani Syakur” by Mainur Risyada/Flickr, CC BY 2.0)

Figure 1.4.1: A large, multigenerational family contains an intersection and a union of sets. (credit: “Family Photo Shoot Bani Syakur” by Mainur Risyada/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

- Determine the intersection of two sets.
- Determine the union of two sets.
- Determine the cardinality of the union of two sets.
- Apply the concepts of AND and OR to set operations.
- Draw conclusions from Venn diagrams with two sets.

The movie *Yours, Mine, and Ours* was originally released in 1968 and starred Lucille Ball and Henry Fonda. This movie, which is loosely based on a true story, is about the marriage of Helen, a widow with eight children, and Frank, a widower with ten children, who then have an additional child together. The movie is a comedy that plays on the interpersonal and organizational struggles of feeding, bathing, and clothing twenty people in one household.

If we consider the set of Helen's children and the set of Frank's children, then the child they had together is the intersection of these two sets, and the collection of all their children combined is the union of these two sets. In this section, we will explore the operations of union and intersection as it relates to two sets.

The Intersection of Two Sets

The members that the two sets share in common are included in the intersection of two sets. To be in the intersection of two sets, an element must be in both the first set and the second set. In this way, the intersection of two sets is a logical AND statement. Symbolically, A intersection B is written as: $A \cap B$. A intersection B is written in set builder notation as: $A \cap B = \{x | x \in A \text{ and } x \in B\}$.

Let us look at Helen's and Frank's children from the movie Yours, Mine, and Ours. Helen's children consist of the set $H = \{$ Colleen, Nick, Janette, Tommy, Jean, Phillip, Gerald, Theresa, Joseph\} and Frank's children are included in the set $F = \{$ Mike, Rusty, Greg, Rosemary, Loise, Susan, Veronica, Mary, Germaine, Joan, Joseph \}. H intersection F is the set of children they had together. $H \cap F = \{$ Joseph $\}$, because Joseph is in both set H and set F.

Example 1.23

Finding the Intersection of Set A and Set B

Set $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 3, 5, 7\}$. Find A intersection B.

Answer

The intersection of sets A and B include the elements that set A and B have in common: 3, 5, and 7. $A \cap B = \{3, 5, 7\}$.

Your Turn 1.23

1. Set $A = \{h, a, p, y\}$ and $B = \{s, a, d\}$. Find A intersection B.

Notice that if sets A and B are disjoint sets, then they do not share any elements in common, and A intersection B is the empty set, as shown in the Venn diagram below.

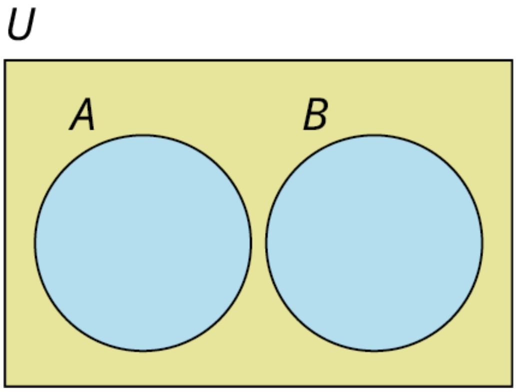


Figure 1.21

Example 1.24

Determining the Intersection of Disjoint Sets

Set $A = \{0, 2, 4, 6, 8\}$ and set $B = \{1, 3, 5, 7, 9\}$. Find $A \cap B$.

Answer

Because sets A and B are disjoint, they do not share any elements in common. So, the intersection of set A and set B is the empty set. $A \cap B = \emptyset$.

Your Turn 1.24

Set $A = \{$ red, yellow, blue $\}$ and set $B = \{$ orange, green, purple $\}$. Find $A \cap B$.

Notice that if set A is a subset of set B, then A intersection B is equal to set A, as shown in the Venn diagram below.

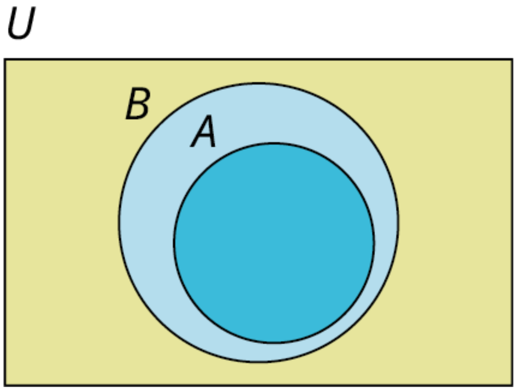


Figure 1.22

Example 1.25

Finding the Intersection of a Set and a Subset

Set $A = \{1, 3, 5, ...\}$ and set $B = \mathbb{N} = \{1, 2, 3, ...\}$ Find $A \cap B$.

Answer

Because set A is a subset of set B, A intersection B is equal to set A. $A \cap B = A = \{1, 3, 5, \ldots\}$, the set of odd natural numbers.

Your Turn 1.25

1. Set $A = \{a, b, c, \ldots, z\}$ and set $B = \{a, e, i, o, u\}$. Find $A \cap B$.

The Union of Two Sets

Like the union of two families in marriage, the union of two sets includes all the members of the first set and all the members of the second set. To be in the union of two sets, an element must be in the first set, the second set, or both. In this way, the union of two sets is a logical inclusive OR statement. Symbolically, A union B is written as: $A \cup B$. A union B is written in set builder notation as: $A \cup B = \{x | x \in A \text{ or } x \in B\}$.

Let us consider the sets of Helen's and Frank's children from the movie Yours, Mine, and Ours again. Helen's children is set $H = \{$ Colleen, Nick, Janette, Tommy, Jean, Phillip, Gerald, Theresa, Joseph\} and Frank's children is set $F = \{$ Mike, Rusty, Greg, Rosemary, Loise, Susan, Veronica, Mary, Germaine, Joan, Joseph $\}$. The union of these two sets is the collection of all nineteen of their children, $H \cup F = \{$ Colleen, Nick, Janette, Tommy, Jean, Phillip, Gerald, Theresa, Joseph, Mike, Rusty, Greg, Rosemary, Loise, Susan, Veronica, Mary, Germaine, Joan\}. Notice, Joseph is in both set H and set F, but he is only one child, so, he is only listed once in the union.

Example 1.26

Finding the Union of Sets A and B When A and B Overlap

Set $A = \{1, 3, 5, 7, 9\}$ and set $B = \{2, 3, 5, 7\}$. Find A union B.

Answer

A union B is the set formed by including all the unique elements in set A, set B, or both sets A and B: $A \cup B = \{1, 3, 5, 7, 9, 2\}$. The first five elements of the union are the five unique elements in set A. Even though 3, 5, and 7 are also members of set B, these elements are only listed one time. Lastly, set B includes the unique element 2, so 2 is also included as part of the union of sets A and B.

Your Turn 1.26

1. Set $A = \{h, a, p, y\}$ and set $B = \{s, a, d\}$. Find A union B.

When observing the union of sets A and B, notice that both set A and set B are subsets of A union B. Graphically, A union B can be represented in several different ways depending on the members that they have in common. If A and B are disjoint sets, then A union B would be represented with two disjoint circles within the universal set, as shown in the Venn diagram below.

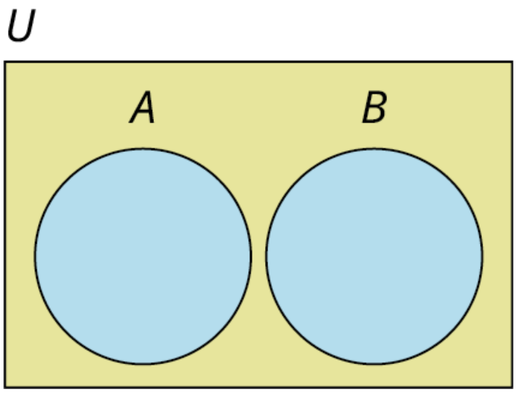


Figure 1.23 $A \cup B$

If sets A and B share some, but not all, members in common, then the Venn diagram is drawn as two separate circles that overlap.

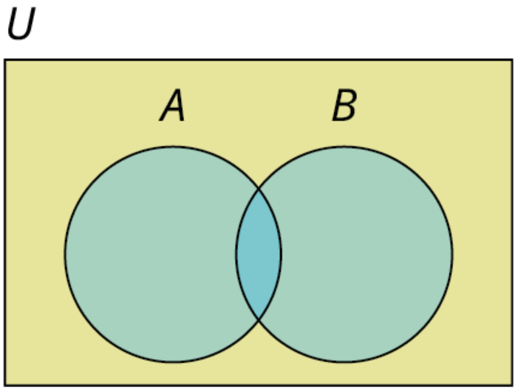


Figure 1.24

If every member of set A is also a member of set B, then A is a subset of set B, and A union B would be equal to set B. To draw the Venn diagram, the circle representing set A should be completely enclosed in the circle containing set B.

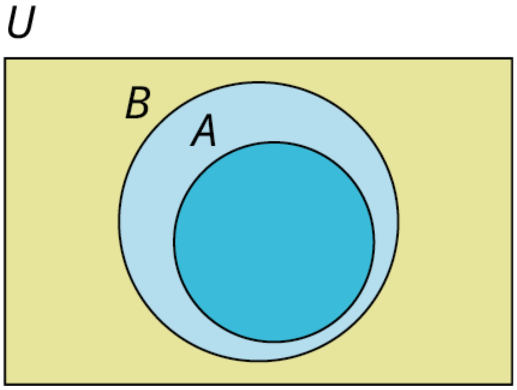


Figure 1.25

Example 1.27

Finding the Union of Sets A and B When A and B Are Disjoint

Set $A = \{0, 2, 4, 6, 8\}$ and set $B = \{1, 3, 5, 7, 9\}$. Find $A \cup B$.

Answer

Because sets A and B are disjoint, the union is simply the set containing all the elements in both set A and set B. $A \cup B = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

Your Turn 1.27

Set $A =$ { red, yellow, blue } and set $B =$ \{orange, green, purple }. Find $A \cup B$.

Example 1.28

Finding the Union of Sets A and B When One Set is a Subset of the Other

Set $A = \{1, 3, 5, ...\}$ and set $B = \mathbb{N} = \{1, 2, 3, ...\}$. Find $A \cup B$.

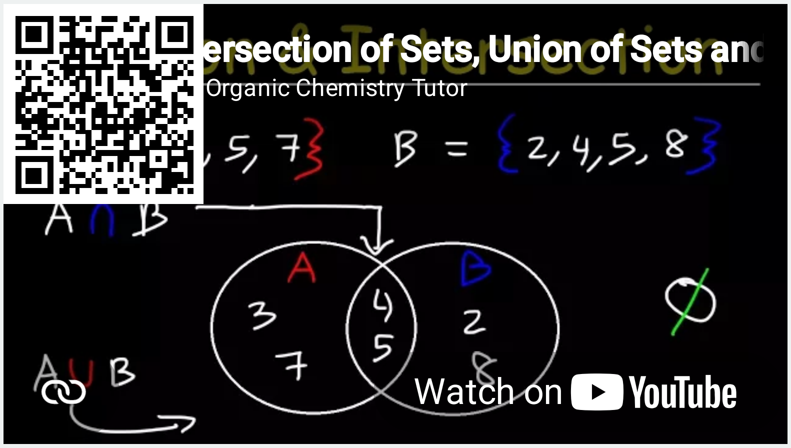
Answer

Because set A is a subset of set B, A union B is equal to set B. $A \cup B = \mathbb{N} = \{1, 2, 3, ...\} = B$.

Your Turn 1.28

1. Set $A = \{a, b, c, \ldots, z\}$ and set $B = \{a, e, i, o, u\}$. Find $A \cup B$.

Video

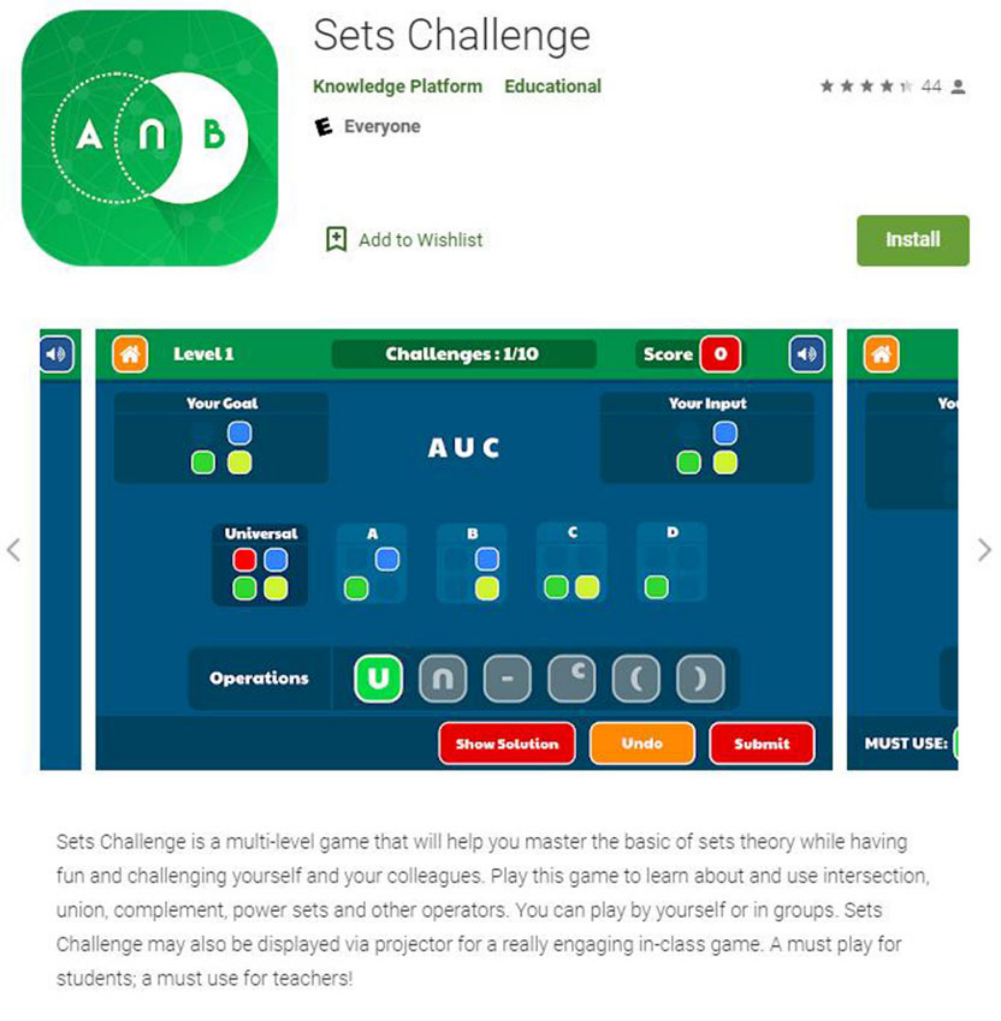

ersection of Sets, Union of Sets anc
Organic Chemistry Tutor
, 5, 7}
B = {2, 4, 5, 8}
A ∩ B
A
B
3
7
4
5
2
8
∅
A ∪ B
Watch on YouTube

The Basics of Intersection of Sets, Union of Sets and Venn Diagrams

Tech Check

Set Operation Practice

Sets Challenge is an application available on both Android and iPhone smartphones that allows you to practice and gain familiarity with the operations of set union, intersection, complement, and difference.



Sets Challenge

Knowledge Platform Educational

Everyone

44

Sets Challenge is a multi-level game that will help you master the basic of sets theory while having fun and challenging yourself and your colleagues. Play this game to learn about and use intersection, union, complement, power sets and other operators. You can play by yourself or in groups. Sets Challenge may also be displayed via projector for a really engaging in-class game. A must play for students; a must use for teachers!

A ∩ B
Add to Wishlist
Install
Level 1
Challenges : 1/10
Score 0
Your Goal
A U C
Your Input
Universal
A
B
C
D
Operations
U
∩
-
c
(
)
Show Solution
Undo
Submit
MUST USE:

Figure 1.26 Google Play Store image of Sets Challenge game. (credit: screenshot from Google Play)

The Sets Challenge application/game uses some notation that differs from the notation covered in the text.

- The complement of set A in this text is written symbolically as A', but the Sets Challenge game uses A^C to represent the complement operation.
- In the text we do not cover set difference between two sets A and B, represented in the game as $A - B$. In the game this operation removes from set A all the elements in $A \cap B$. For example, if set $A = \{a, b, c, d\}$ and set $B = \{b, d, f, h\}$ are subsets of the universal set $U = \{a, b, c, \ldots, z\}$, then $A - B = \{a, b, c, d\} - \{b, d\} = \{a, c\}$, and $B - A = \{b, d, f, h\} - \{b, d\} = \{f, h\}$. There is a project at the end of the chapter to research the set difference operation.

Determining the Cardinality of Two Sets

The cardinality of the union of two sets is the total number of elements in the set. Symbolically the cardinality of A union B is written, $n\,(A \cup B)$. If two sets A and B are disjoint, the cardinality of A union B is the sum of the cardinality of set A and the cardinality of set B. If the two sets intersect, then A intersection B is a subset of both set A and set B. This means that if we add the cardinality of set A and set B, we will have added the number of elements in A intersection B twice, so we must then subtract it once as shown in the formula that follows.

FORMULA

The **cardinality of A union B** is found by adding the number of elements in set A to the number of elements in set B, then subtracting the number of elements in the intersection of set A and set B. $n(A \cup B) = n(A) + n(B) - n(A \cap B)$ or $n(A \text{ or } B) = n(A) + n(B) - n(A \text{ and } B)$.

Checkpoint

If sets A and B are disjoint, then $n(A \cap B) = n(A \text{ and } B) = 0$ and the formula is still valid, but simplifies to $n(A \cup B) = n(A) + n(B)$.

Example 1.29

Determining the Cardinality of the Union of Two Sets

The number of elements in set A is 10, the number of elements in set B is 20, and the number of elements in A intersection B is 4. Find the number of elements in A union B.

Answer

Using the formula for determining the cardinality of the union of two sets, we can say $n(A \cup B) = n(A) + n(B) - n(A \cap B) = 10 + 20 - 4 = 26$.

Your Turn 1.29

1. If $n(A) = 23, n(B) = 17$, and $n(A \cap B) = 7$, then find $n(A \cup B)$.

Example 1.30

Determining the Cardinality of the Union of Two Disjoint Sets

If A and B are disjoint sets and the cardinality of set A is 37 and the cardinality of set B is 43, find the cardinality of A union B.

Answer

To find the cardinality of A union B, apply the formula, $n(A\cup B)=n(A)+n(B)-n(A\cap B)$. Because sets A and B are disjoint, $A\cap B$ is the empty set, therefore $n(A\cap B)=n(\emptyset)=0$ and $n(A\cup B)=37+43-0=80$.

Your Turn 1.30

1. If $A\cap B=\emptyset, n(A)=35$, and $n(B)=78$, then find $n(A\cup B)$.

Applying Concepts of “AND” and “OR” to Set Operations

To become a licensed driver, you must pass some form of written test and a road test, along with several other requirements depending on your age. To keep this example simple, let us focus on the road test and the written test. If you pass the written test but fail the road test, you will not receive your license. If you fail the written test, you will not be allowed to take the road test and you will not receive a license to drive. To receive a driver's license, you must pass the written test AND the road test. For an “AND” statement to be true, both conditions that make up the statement must be true. Similarly, the intersection of two sets A and B is the set of elements that are in both set A and set B. To be a member of A intersection B, an element must be in set A and also must be in set B. The intersection of two sets corresponds to a logical "AND" statement.

The union of two sets is a logical inclusive "OR" statement. Say you are at a birthday party and the host offers Leah, Lenny, Maya, and you some cake or ice cream for dessert. Leah asks for cake, Lenny accepts both cake and ice cream, Maya turns down both, and you choose only ice cream. Leah, Lenny, and you are all having dessert. The “OR” statement is true if at least one of the components is true. Maya is the only one who did not have cake or ice cream; therefore, she did not have dessert and the “OR” statement is false. To be in the union of two sets A and B, an element must be in set A or set B or both set A and set B.

Example 1.31

Applying the "AND" or "OR" Operation

$A=\{0,3,6,9,12\}$, $B=\{0,4,8,12,16\}$, and $C=\{1,2,3,5,8,13\}$.

Find the set consisting of elements in:

1. $A \text{ and } B$.
2. $A \text{ or } B$.
3. $A \text{ or } C$.
4. $(B \text{ and } C) \text{ or } A$.

Answer

1. $A \text{ and } B=A\cap B=\{0,12\}$, because only the elements 0 and 12 are members of both set A and set B.
2. $A \text{ or } B=A\cup B=\{0,3,4,6,8,9,12,16\}$, because the set A or B is the collection of all elements in set A or set B, or both.
3. $A \text{ or } C=A\cup C=\{0,1,2,3,5,6,8,9,12,13\}$, because the set A or C is the collection of all elements in set A or set C, or both.
4. $(B \text{ and } C) \text{ or } A=(B\cap C)\cup A$. Parentheses are evaluated first: $(B \text{ and } C)=B\cap C=\{8\}$, because the only member that both set B and set C share in common is 8. So, now we need to find $\{8\} \text{ or } \{0,3,6,9,12\}$, Because the word translates to the union operation, the problem becomes $\{8\}\cup\{0,3,6,9,12\}$, which is equal to $\{0,3,6,8,9,12\}$.

Your Turn 1.31

$A=\{h,a,p,y\}$, and $B=\{a,w,e,s,o,m\}$, and $C=\{m,a,t,h\}$.
Find the set consisting of elements in:
1. A or B.
2. A and C.
3. B or C.
4. $(A$ and $C)$ and B.

Example 1.32

Determine and Apply the Appropriate Set Operations to Solve the Problem

Don Woods is serving cake and ice cream at his Juneteenth celebration. The party has a total of 54 guests in attendance. Suppose 30 guests requested cake, 20 guests asked for ice cream, and 12 guests did not have either cake or ice cream.

1. How many guests had cake or ice cream?
2. How many guests had cake and ice cream?

Answer

1. The total number of people at the party is 54, and 12 people did not have cake or ice cream. Recall that the total number of elements in the universal set is always equal to the number of elements in a subset plus the number of elements in the complement of the set, $n(U) = n(A) + n(A')$. That means $54 = n\,(\text{cake or ice cream})\ +\ n\,(\text{not}\ (\text{cake or ice cream}))$, or equivalently,$n\,(\text{cake}\ \cup\ \text{ice cream}) =\ 54 - n\left((\text{cake}\ \cup\ \text{ice cream})'\right) = 54 - 12 = 42$. A total of 42 people at the party had cake or ice cream.
2. To determine the number of people who had both cake and ice cream, we need to find the intersection of the set of people who had cake and the set of people who had ice cream. From Question 1, the number of people who had cake or ice cream is 42. This is the union of the two sets. The formula for the union of two sets is $n(A \cup B) = n(A) + n(B) - n(A \cap B)$. Use the information given in the problem and substitute the known values into the formula to solve for the number of people in the intersection: $42 = 30 + 20 - n(A \cap B)$. Adding 30 and 20, the equation simplifies to $42 = 50 - n(\text{cake and ice cream})$. Which means $n(\text{cake and ice cream}) = 50 - 42 = 8$.

Your Turn 1.32

Ravi and Priya are serving soup and salad along with the main course at their wedding reception. The reception will have a total of 150 guests in attendance. A total of 92 soups and 85 salads were ordered, while 23 guests did not order any soup or salad.
1. How many guests had soup or salad or both?
2. How many guests had both soup and a salad?

Who Knew?

The Real Inventor of the Venn Diagram

John Venn, in his writings, references works by both John Boole and Augustus De Morgan, who referred to the circle diagrams commonly used to present logical relationships as Euler's circles. Leonhard Euler's works were published over 100 years prior to Venn's, and Euler may have been influenced by the works of Gottfried Leibniz.

So, why does John Venn get all the credit for these graphical depictions? Venn was the first to formalize the use of these diagrams in his book *Symbolic Logic*, published in 1881. Further, he made significant improvements in their design, including shading to highlight the region of interest. The mathematician C.L. Dodgson, also known as Lewis Carroll, built upon Venn's work by adding an enclosing universal set.

Invention is not necessarily coming up with an initial idea. It is about seeing the potential of an idea and applying it to a new situation.

References:

Margaret E. Baron. "A Note on the Historical Development of Logic Diagrams: Leibniz, Euler and Venn." *The Mathematical Gazette*, vol. 53, no. 384, 1969, pp. 113-125. *JSTOR*, www.jstor.org/stable/3614533. Accessed 15 July 2021.

Deborah Bennett. "Drawing Logical Conclusions." *Math Horizons*, vol. 22, no. 3, 2015, pp. 12-15. *JSTOR*, www.jstor.org/stable/10.4169/mathhorizons.22.3.12. Accessed 15 July 2021.

Drawing Conclusions from a Venn Diagram with Two Sets

All Venn diagrams will display the relationships between the sets, such as subset, intersecting, and/or disjoint. In addition to displaying the relationship between the two sets, there are two main additional details that Venn diagrams can include: the individual members of the sets or the cardinality of each disjoint subset of the universal set.

A Venn diagram with two subsets will partition the universal set into 3 or 4 sections depending on whether they are disjoint or intersecting sets. Recall that the complement of set A, written A', is the set of all elements in the universal set that are not in set A.

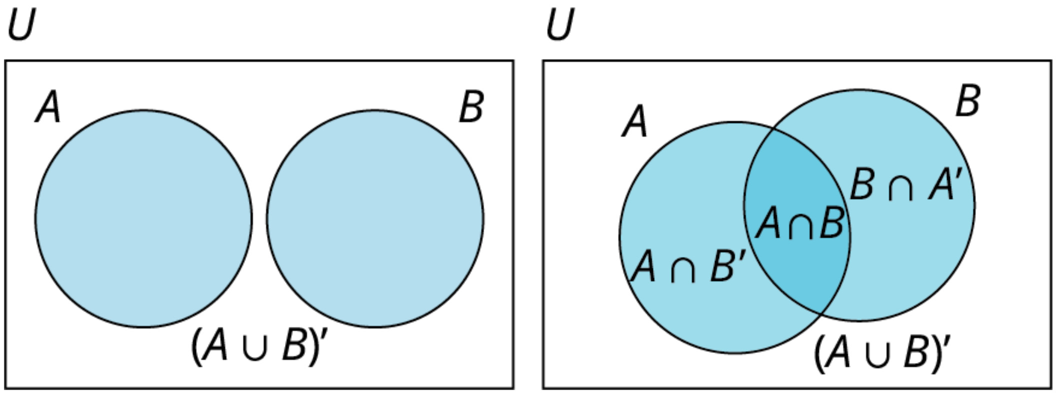


Figure 1.27 Side-by-side Venn diagrams with disjoint and intersecting sets, respectively.

Example 1.33

Using a Venn Diagram to Draw Conclusions about Set Membership

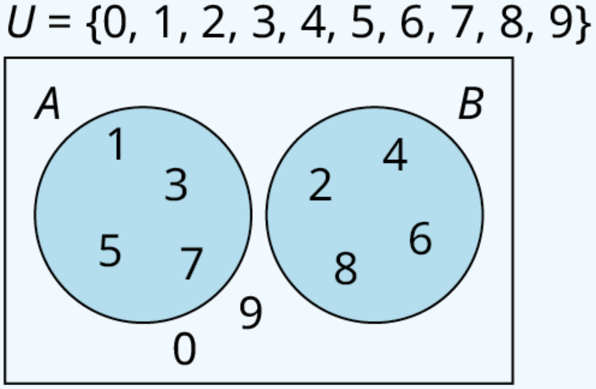


Figure 1.28

1. Find $A \cup B$.
2. Find $A \cap B$.
3. Find B'.
4. Find $n(B')$.

Answer

1. $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\}$, because A union B is the collection of all elements in set A or set B or both.
2. Because A and B are disjoint sets, there are no elements that are in both A and B. Therefore, A intersection B is the empty set, $A \cap B = \emptyset$.
3. The complement of set B is the set of all elements in the universal set that are not in set B: $B' = \{0, 1, 3, 5, 7, 9\}$.
4. The cardinality, or number of elements in set B', isn(B')=6.

Your Turn 1.33

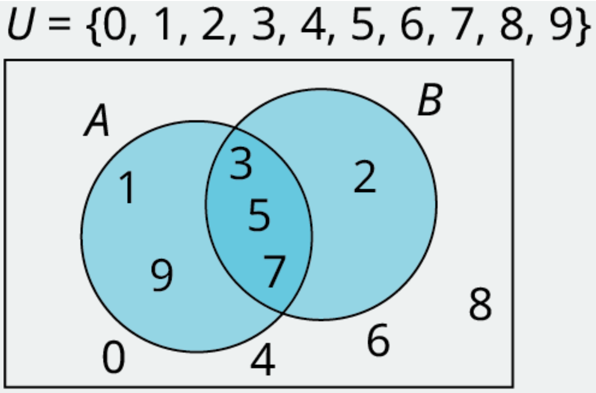


Figure 1.29 Venn diagram with two intersecting sets and members.
1. Find $A \cap B$.
2. Find $A \cup B$. 3. Find $A \cap B'$. 4. Find $n\left(A \cap B'\right)$.

Example 1.34

Using a Venn Diagram to Draw Conclusions about Set Cardinality

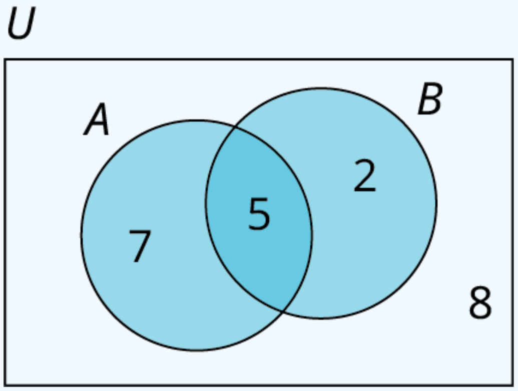


Figure 1.30 Venn diagram with two intersecting sets and number of elements in each section indicated.

1. Find $n(A \text{ or } B)$.
2. Find $n(A \text{ and } B)$.
3. Find $n(A)$.

Answer

1. The number of elements in A or B is the number of elements in A union B: $n(A \cup B) = n(\{2, 5, 7\}) = 14$.
2. The number of elements in A and B is the number of elements in A intersection B: $n(A \cap B) = 5$.
3. The number of elements in set A is the sum of all the numbers enclosed in the circle representing set A: $n(A) = n(\{7, 5\}) = 12$.

Your Turn 1.34

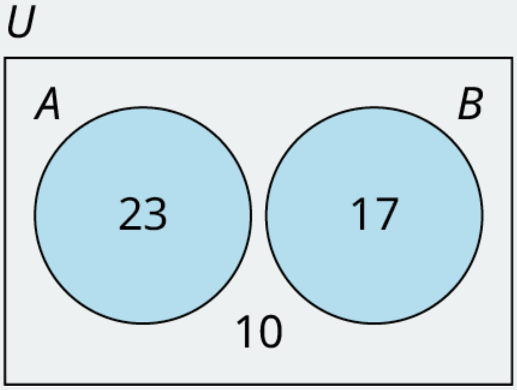


Figure 1.31 Venn diagram with two disjoint sets and number of elements in each section.
1. Find $n(A \text{ or } B)$.
2. Find $n(A \text{ and } B)$. 3. Find $n\left(A'\right)$.

Check Your Understanding

20. The ________ of two sets A and B is the set of all elements that they share in common.
21. The ________ of two sets A and B is the collection of all elements that are in set A or set B, or both set A and set B.
22. The union of two sets A and B is represented symbolically as ________ .
23. The intersection of two sets A and B is represented symbolically as ________ .
24. If set A is a subset of set B, then A intersection B is equal to set ________ .
25. If set A is a subset of set B, then A union B is equal to set ________ .
26. If set A and set B are disjoint sets, then A intersection B is the ________ set.
27. The cardinality of A union $B, n(A \cup B)$, is found using the formula: ________ .

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1.4.0: Exercises

For the following exercises, determine the union or intersection of the sets as indicated.

$$A=\{2,4,6,8,10,12\}, B=\{4,8,12,16,20\}, C=\{8,16,24,32,40\}, \text{ and } D=\{10,20,30,40,50\}$$

? Exercise 1.4.0.1

$B \cup C$

? Exercise 1.4.0.2

$A \cap D$

? Exercise 1.4.0.3

$D \cap C$

? Exercise 1.4.0.4

$A \cup D$

? Exercise 1.4.0.5

$A \cap (C \cup D)$

? Exercise 1.4.0.6

$B \cup (A \cap D)$

? Exercise 1.4.0.7

$D \cup (A \cap C)$

? Exercise 1.4.0.8

$C \cap (A \cup D)$

? Exercise 1.4.0.9

$B \cap (A \cap D)$

? Exercise 1.4.0.10

$B \cap (A \cap C)$

? Exercise 1.4.0.11

$B \cup (A \cup D)$

? Exercise 1.4.0.12

$B \cup (A \cup C)$

For the following exercises, use the sets provided to apply the “AND” or “OR” operation as indicated to find the resulting set.

$$U = \{a, b, c, \ldots, z\}, S = \{s, a, m, p, l, e\}, M = \{m, a, p\}, L = \{l, a, m, p\}, D = \{d, o, g\}, \text{ and } P = \{p, l, o, t\}$$

? Exercise 1.4.0.13

Find the set consisting of elements in S and P.

? Exercise 1.4.0.14

Find the set consisting of elements in M or D.

? Exercise 1.4.0.15

Find the set consisting of elements in P or M.

? Exercise 1.4.0.16

Find the set consisting of elements in M and D.

? Exercise 1.4.0.17

Find the set consisting of elements in L and M.

? Exercise 1.4.0.18

Find the set consisting of elements in L or M.

? Exercise 1.4.0.19

Find the set consisting of the elements in D or M or P.

? Exercise 1.4.0.20

Find the set consisting of the elements in S or M or P.

? Exercise 1.4.0.21

Find the set consisting of the elements in $(S$ or $D)$ and P.

? Exercise 1.4.0.22

Find the set consisting of the elements in S or $(D$ and $P)$.

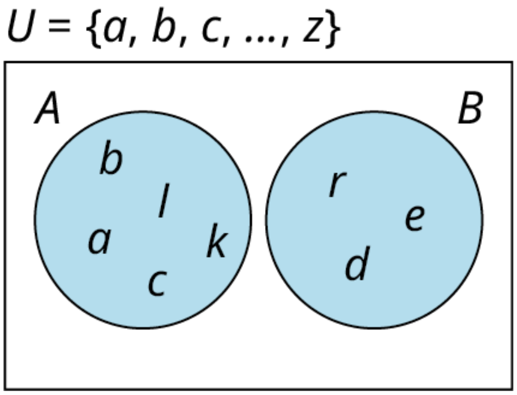
? Exercise 1.4.0.23

Find the set consisting of elements in U or $(P$ and $S)$.

? Exercise 1.4.0.24

Find the set consisting of elements in $(U \text{ or } P)$ and S.

For the following exercises, use the Venn diagram provided to answer the following questions about the sets.



? Exercise 1.4.0.25

Find $A \cup B$.

? Exercise 1.4.0.26

Find $A \cap B$.

? Exercise 1.4.0.27

Find $(A \cap B)'$.

? Exercise 1.4.0.28

Find $(A \cup B)'$.

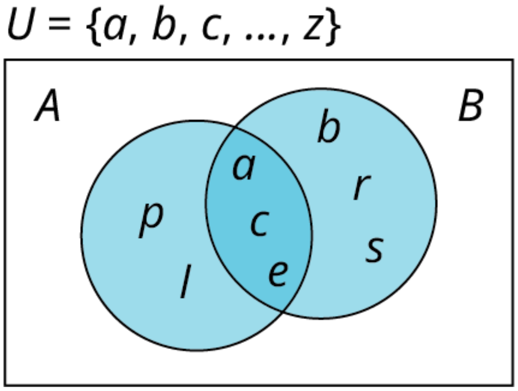
? Exercise 1.4.0.29

Find $A \cap B'$.

? Exercise 1.4.0.30

Find $B \cap A'$.

For the following exercises, use the Venn diagram provided to answer the following questions about the sets.



Exercise 1.4.0.31

Find $A \cap B$.

Exercise 1.4.0.32

Find $A \cup B$.

Exercise 1.4.0.33

Find $(A \cup B)'$.

Exercise 1.4.0.34

Find $(A \cap B)'$.

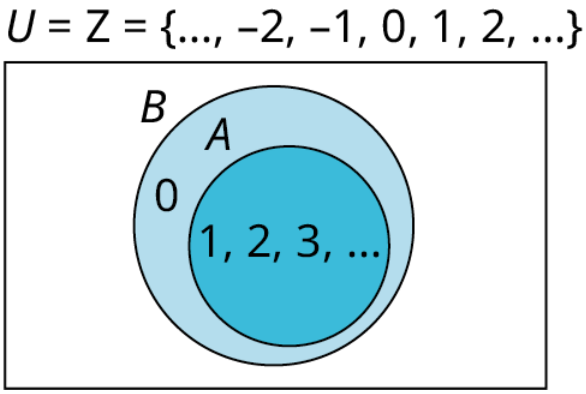
Exercise 1.4.0.35

Find $B \cap A'$.

Exercise 1.4.0.36

Find $A \cap B'$.

For the following exercises, use the Venn diagram provided to answer the following questions about the sets.



Exercise 1.4.0.37

Find $A \cup B$.

Exercise 1.4.0.38

Find $A \cap B$.

Exercise 1.4.0.39

Find $(A \cup B)'$.

Exercise 1.4.0.40

Find $(A \cap B)'$.

Exercise 1.4.0.41

Find $B \cap A'$.

Exercise 1.4.0.42

Find $A \cap B'$.

For the following exercises, determine the cardinality of the union of set A and set B.

Exercise 1.4.0.43

If set $A = \{$ red, white, blue $\}$ and set $B = \{$ green, white, red $\}$, find $n(A \cup B)$.

Exercise 1.4.0.44

If set $A = \{$ silver, gold, bronze $\}$ and set $B = \{$ silver, gold $\}$, find the number of elements in A or B.

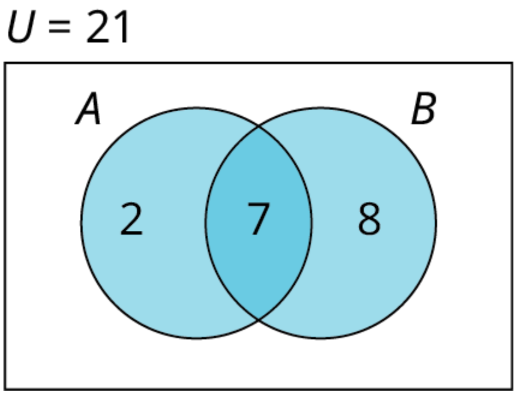
Exercise 1.4.0.45

If set $A = \{$ glass, plate, fork, knife $\}$ and set $B = \{$ bowl, spoon, cup $\}$, find the number of elements in A or B.

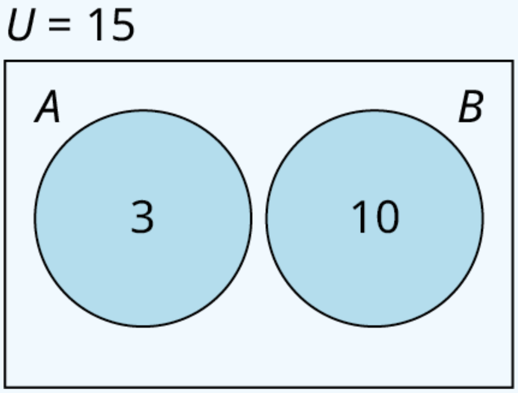
Exercise 1.4.0.46

If set $A = \{$ Algebra, Geometry, Biology, Physics, Chemistry, English $\}$ and Set $B = \{$ Algebra, English, History, Spanish, French, Music $\}$, find $n(A \cup B)$.

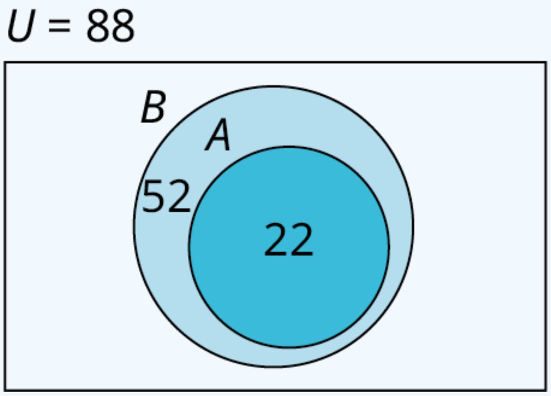
For the following exercises, use the Venn diagram to determine the cardinality of A union B.



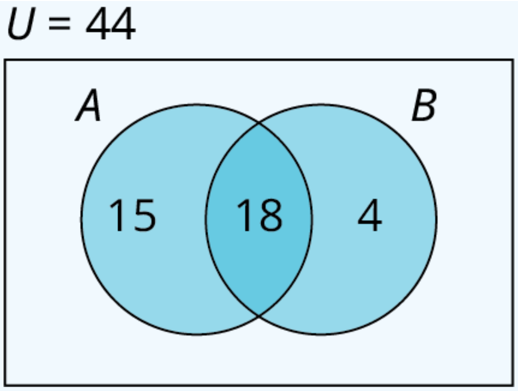
Exercise 1.4.0.48



Exercise 1.4.0.49



Exercise 1.4.0.50



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1.5: Set Operations with Three Sets

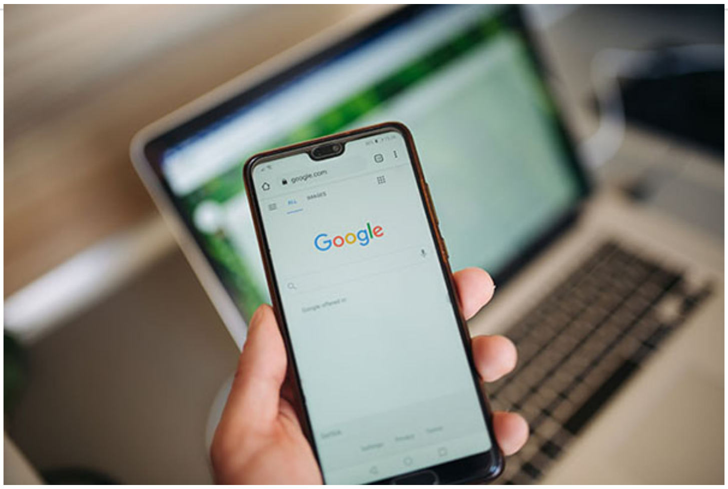


Figure 1.32: Companies like Google collect data on how you use their services, but the data requires analysis to really mean something. (credit: “Man holding smartphone and searches through google” by Nenad Stojkovic/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

- Interpret Venn diagrams with three sets.
- Create Venn diagrams with three sets.
- Apply set operations to three sets.
- Prove equality of sets using Venn diagrams.

Have you ever searched for something on the Internet and then soon after started seeing multiple advertisements for that item while browsing other web pages? Large corporations have built their business on data collection and analysis. As we start working with larger data sets, the analysis becomes more complex. In this section, we will extend our knowledge of set relationships by including a third set.

A Venn diagram with two intersecting sets breaks up the universal set into four regions; simply adding one additional set will increase the number of regions to eight, doubling the complexity of the problem.

Venn Diagrams with Three Sets

Below is a Venn diagram with two intersecting sets, which breaks the universal set up into four distinct regions.

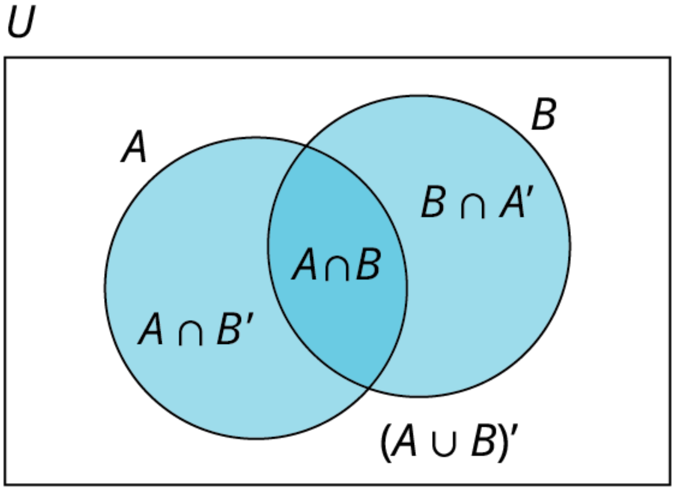


Figure 1.33:

Next, we see a Venn diagram with three intersecting sets, which breaks up the universal set into eight distinct regions.

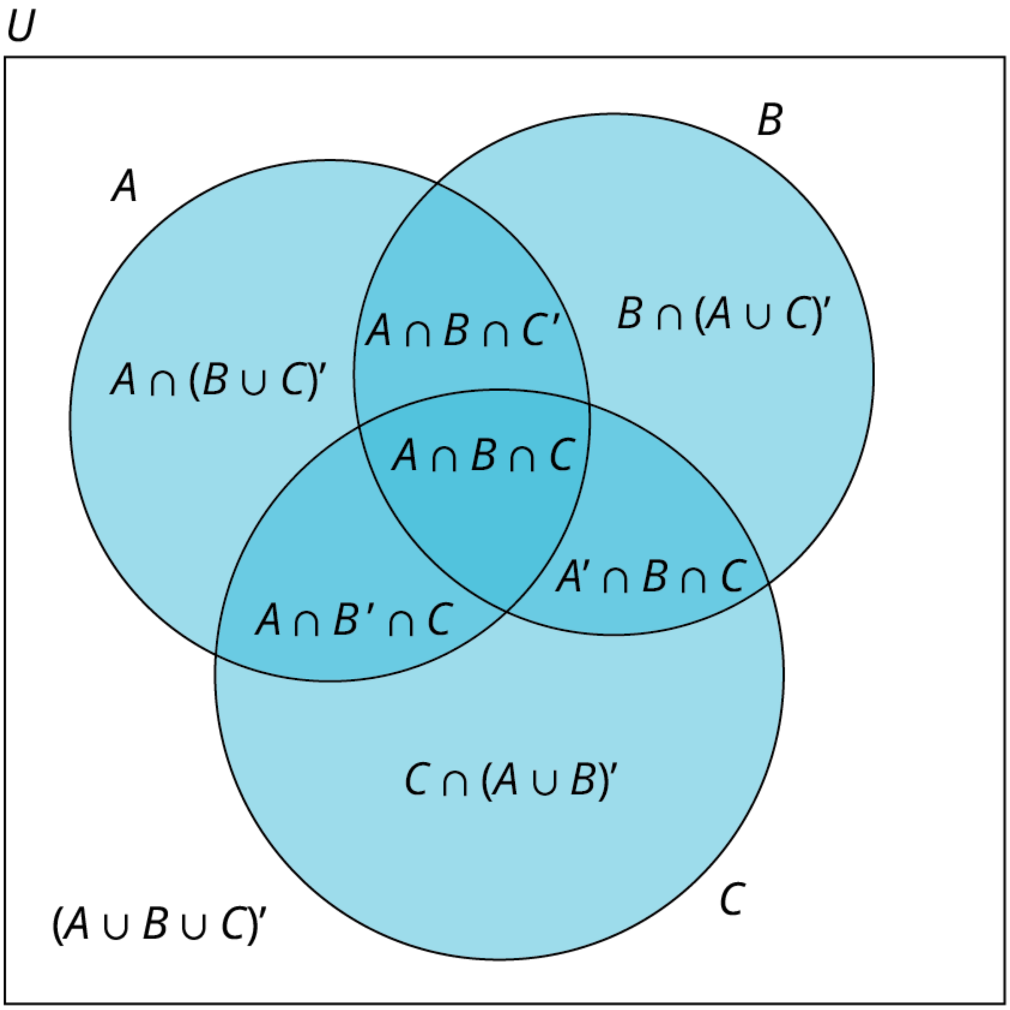
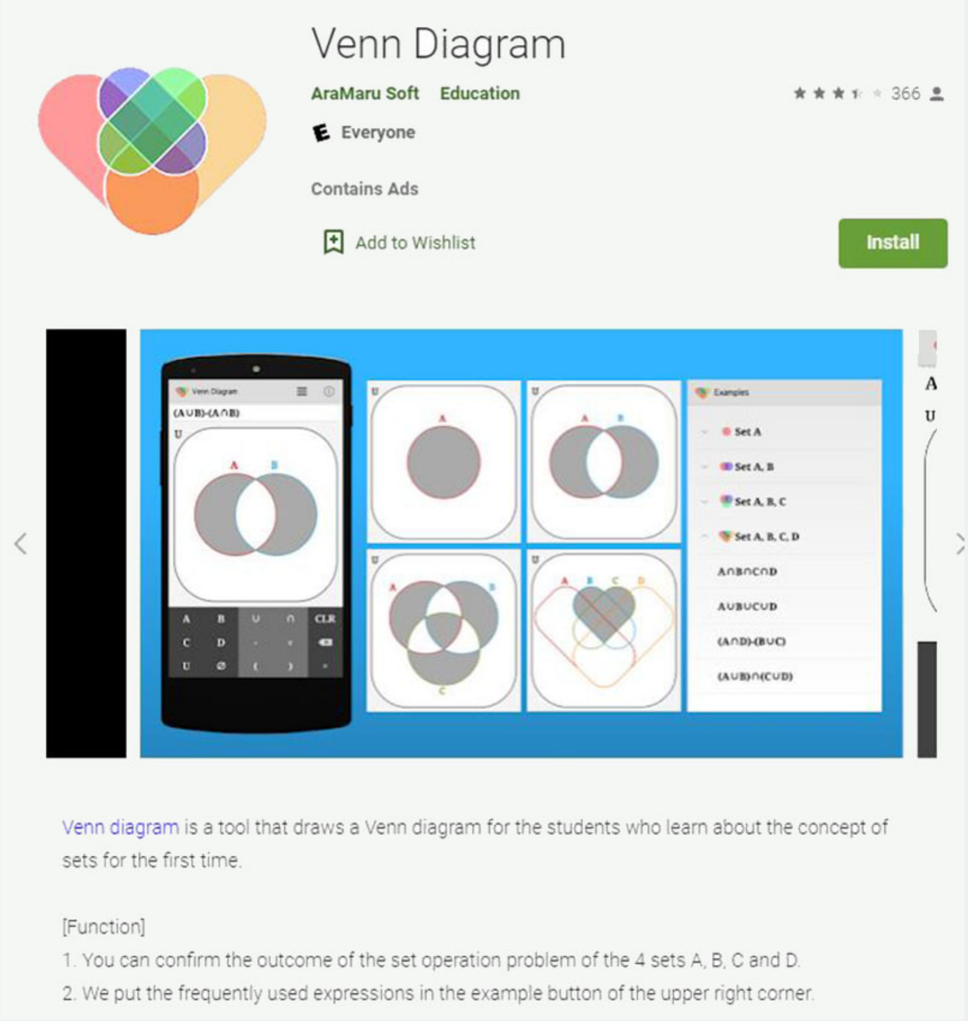

U
A
B
C
$A \cap (B \cup C)'$
$A \cap B \cap C'$
$B \cap (A \cup C)'$
$A \cap B \cap C$
$A' \cap B \cap C$
$A \cap B' \cap C$
$C \cap (A \cup B)'$
$(A \cup B \cup C)'$

Figure 1.34:

Tech Check: Shading Venn Diagrams

Venn Diagram is an Android application that allows you to visualize how the sets are related in a Venn diagram by entering expressions and displaying the resulting Venn diagram of the set shaded in gray.



Venn diagram is a tool that draws a Venn diagram for the students who learn about the concept of sets for the first time.

[Function]

1. You can confirm the outcome of the set operation problem of the 4 sets A, B, C and D.
2. We put the frequently used expressions in the example button of the upper right corner.

Figure 1.35: Google Play Store image of Venn Diagram app. (credit: screenshot from Google Play)

The Venn Diagram application uses some notation that differs from the notation covered in this text.

a. The complement of set A in this text is written symbolically as A', but the Venn Diagram app uses semanticsACAC to represent the complement operation.
b. The set difference operation, $-$, is available in the Venn Diagram app, although this operation is not covered in the text.

It is recommended that you explore this application to expand your knowledge of Venn diagrams prior to continuing with the next example.

In the next example, we will explore the three main blood factors, A, B and Rh. The following background information about blood types will help explain the relationships between the sets of blood factors. If an individual has blood factor A or B , those will be included in their blood type. The Rh factor is indicated with a + or a$-$. For example, if a person has all three blood factors, then their blood type would be AB^+. In the Venn diagram, they would be in the intersection of all three sets, $A \cap B \cap Rh^+$. If a person did not have any of these three blood factors, then their blood type would be O^-, and they would be in the set $(A \cup B \cup Rh^+)'$ which is the region outside all three circles.

✓ Example 1.35: Interpreting a Venn Diagram with Three Sets

Use the Venn diagram below, which shows the blood types of 100 people who donated blood at a local clinic, to answer the following questions.

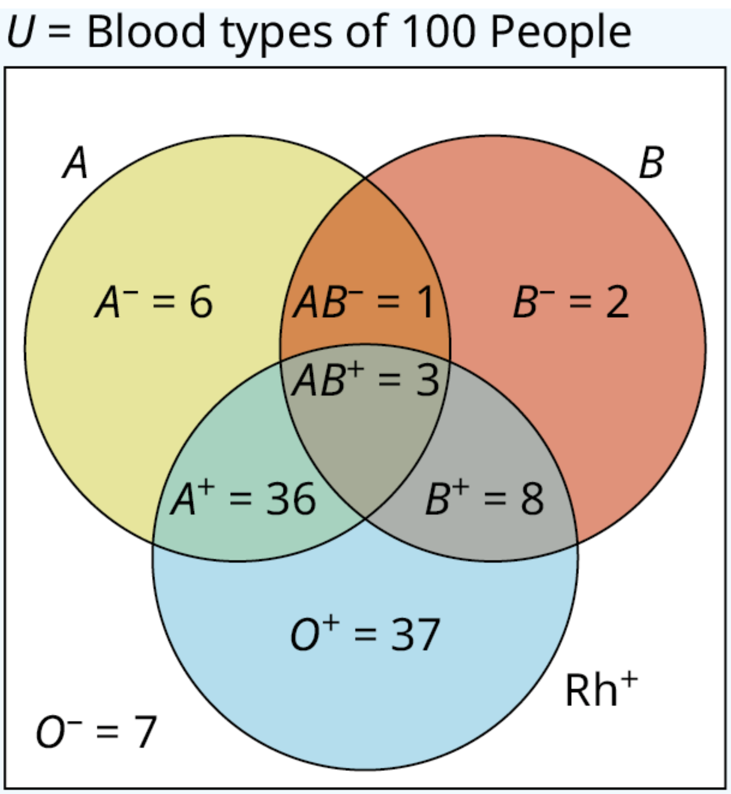


Figure 1.36

1. How many people with a type A blood factor donated blood?
2. Julio has blood type B^+. If he needs to have surgery that requires a blood transfusion, he can accept blood from anyone who does not have a type A blood factor. How many people donated blood that Julio can accept?
3. How many people who donated blood do not have the Rh^+ blood factor?
4. How many people had type A and type B blood?

Answer

1. The number of people who donated blood with a type A blood factor will include the sum of all the values included in the A circle. It will be the union of sets A^-, A^+, $AB^-$and AB^+. $n(A)=n\left(A^-\right)+n\left(A^+\right)+n\left(AB^-\right)+n\left(AB^+\right)=6+36+1+3=46$.

2. In part 1, it was determined that the number of donors with a type A blood factor is 46 . To determine the number of people who did not have a type A blood factor, use the following property, A' union is equal to U, which means $n(A)+n\left(A'\right)=n(U)$, and $n\left(A'\right)=n(U)-n(A)=100-46=54$. Thus, 54 people donated blood that Julio can accept.

3. This would be everyone outside the Rh^+circle, or everyone with a negative Rh factor, $n\left(Rh^-\right)=n\left(O^-\right)+n\left(A^-\right)+n\left(AB^-\right)+n\left(B^-\right)=7+6+1+2=16$.

4. To have both blood type A and blood type B , a person would need to be in the intersection of sets A and B. The two circles overlap in the regions labeled $AB^-$and AB^+. Add up the number of people in these two regions to get the total: $1+3=4$. This can be written symbolically as $n(A \text{ and } B)=n(A\cap B)=n\left(AB^-\right)+n\left(AB^+\right)=1+3=4$.

Your Turn 1.35

Use the same Venn diagram in the example above to answer the following questions.

How many people donated blood with a type B blood factor?

2. How many people who donated blood did not have a type B blood factor?

How many people who donated blood had a type B blood factor or were Rh$^+$?

Who Knew?: Blood Types

Most people know their main blood type of A, B, AB, or O and whether they are Rh^{+} or Rh^{-}, but did you know that the International Society of Blood Transfusion recognizes twenty-eight additional blood types that have important implications for organ transplants and successful pregnancy? For more information, check out this article:

Blood mystery solved: Two new blood types identified

Creating Venn Diagrams with Three Sets

In general, when creating Venn diagrams from data involving three subsets of a universal set, the strategy is to work from the inside out. Start with the intersection of the three sets, then address the regions that involve the intersection of two sets. Next, complete the regions that involve a single set, and finally address the region in the universal set that does not intersect with any of the three sets. This method can be extended to any number of sets. The key is to start with the region involving the most overlap, working your way from the center out.

✓ Example 1.36: Creating a Venn Diagram with Three Sets

A teacher surveyed her class of 43 students to find out how they prepared for their last test. She found that 24 students made flash cards, 14 studied their notes, and 27 completed the review assignment. Of the entire class of 43 students, 12 completed the review and made flash cards, nine completed the review and studied their notes, and seven made flash cards and studied their notes, while only five students completed all three of these tasks. The remaining students did not do any of these tasks. Create a Venn diagram with subsets labeled: “Notes,” “Flash Cards,” and “Review” to represent how the students prepared for the test.

Answer

Step 1: First, draw a Venn diagram with three intersecting circles to represent the three intersecting sets: Notes, Flash Cards, and Review. Label the universal set with the cardinality of the class.

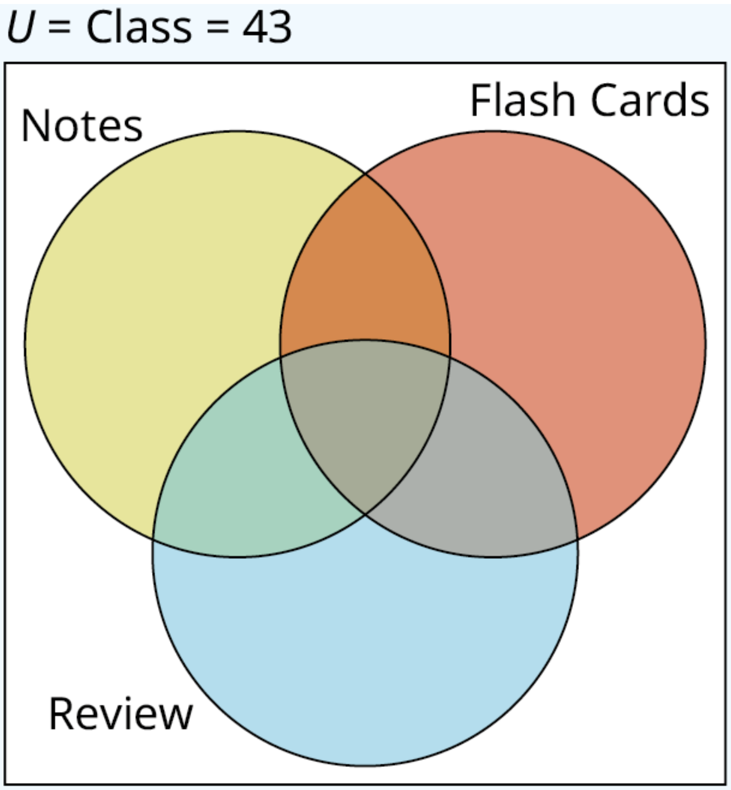

U = Class = 43
Notes
Flash Cards
Review

Figure 1.37:

Step 2: Next, in the region where all three sets intersect, enter the number of students who completed all three tasks.

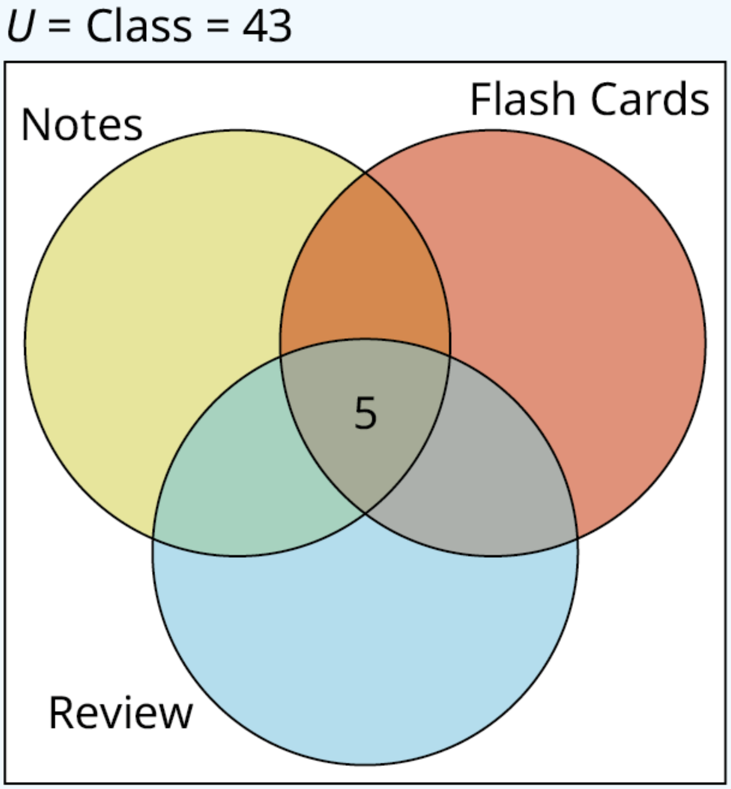

U = Class = 43
Notes
Flash Cards
5
Review

Figure 1.38:

Step 3: Next, calculate the value and label the three sections where just two sets overlap.

Review and flash card overlap

. A total of 12 students completed the review and made flash cards, but five of these twelve students did all three tasks, so we need to subtract: 12 − 5 = 7 12 − 5 = 7 . This is the value for the region where the flash card set intersects with the review set.

Review and notes overlap

. A total of 9 students completed the review and studied their notes, but again, five of these nine students completed all three tasks. So, we subtract: 9 − 5 = 4 9 − 5 = 4 . This is the value for the region where the review set intersects with the notes set.

Flash card and notes overlap

. A total of 7 students made flash cards and studied their notes; subtracting the five students that did all three tasks from this number leaves 2 students who only studied their notes and made flash cards. Add these values to the Venn diagram.

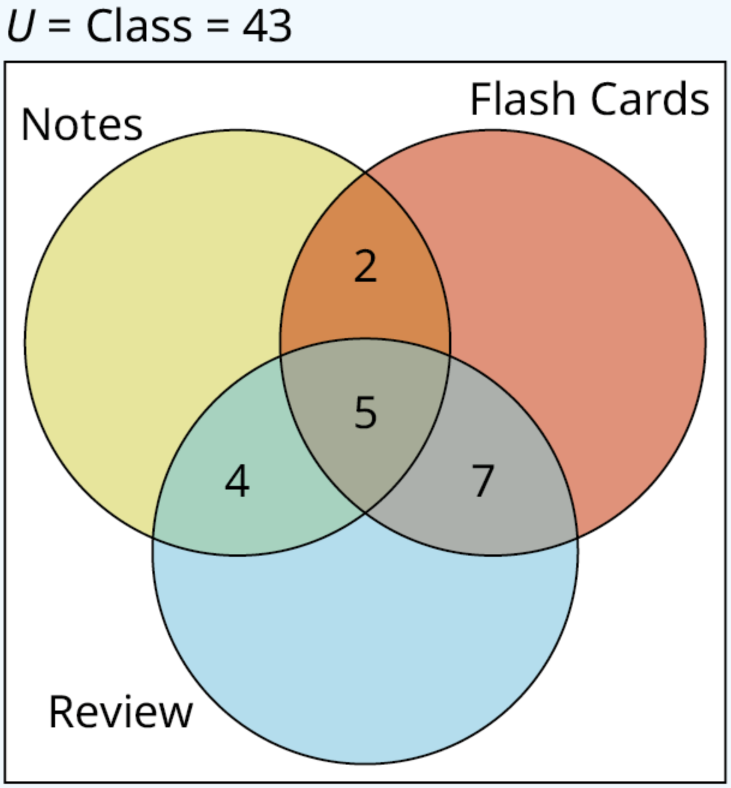
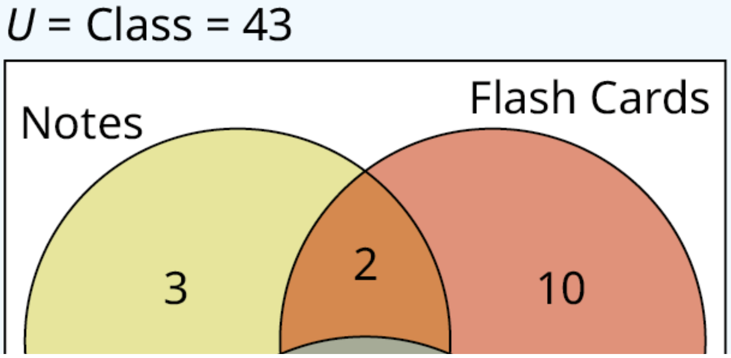


Figure 1.39:

Step 4: Now, repeat this process to find the number of students who only completed one of these three tasks.

A total of 24 students completed flash cards, but we have already accounted for 2 + 5 + 7 = 14 2 + 5 + 7 = 14 of these. Thus, 24 - 14 = 10 24 - 14 = 10 students who just made flash cards. A total of 14 students studied their notes, but we have already accounted for 4 + 5 + 2 = 11 4 + 5 + 2 = 11 of these. Thus, 14 - 11 = 3 14 - 11 = 3 students only studied their notes. A total of 27 students completed the review assignment, but we have already accounted for 4 + 5 + 7 = 16 4 + 5 + 7 = 16 of these, which means 27 - 16 = 11 27 - 16 = 11 students only completed the review assignment. Add these values to the Venn diagram.



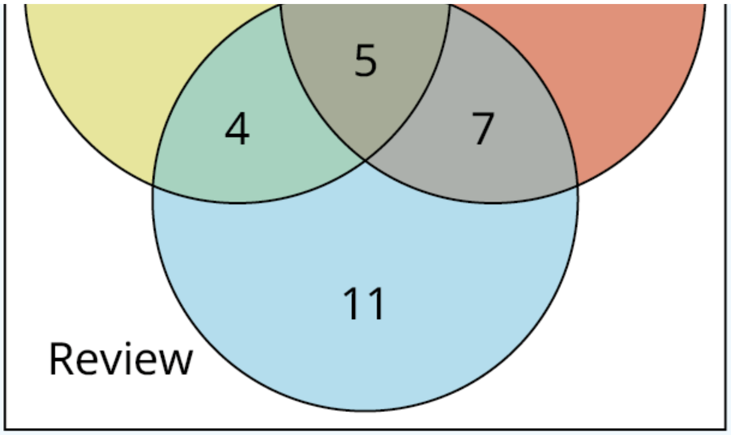


Figure 1.40:

Step 5: Finally, compute how many students did not do any of these three tasks. To do this, we add together each value that we have already calculated for the separate and intersecting sections of our three sets: $3+2+4+5+10+7+11=42$. Because there 43 students in the class, and $43-42=1$, this means only one student did not complete any of these tasks to prepare for the test. Record this value somewhere in the rectangle, but outside of all the circles, to complete the Venn diagram.

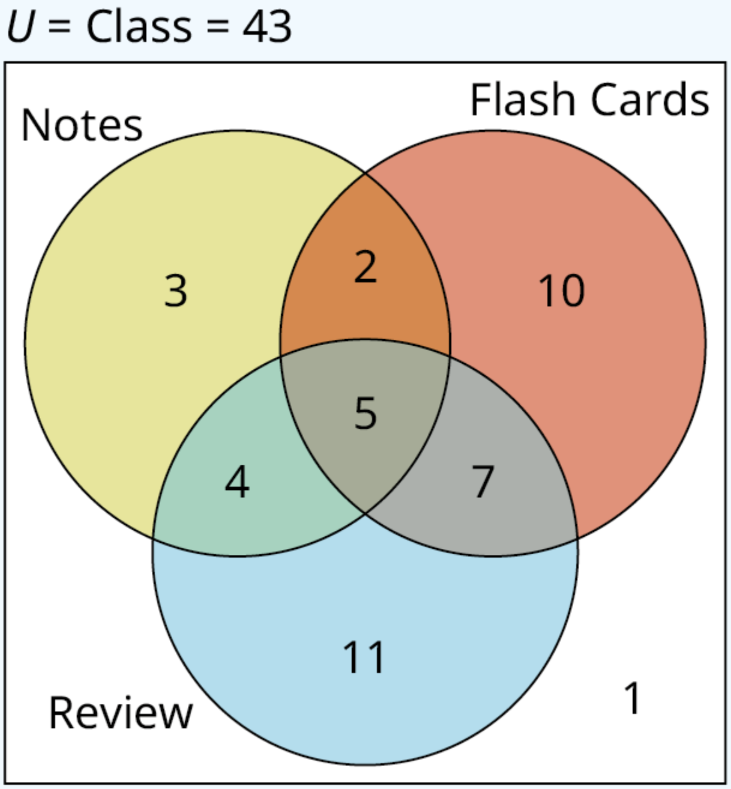


Figure 1.41

Your Turn 1.36

A group of 50 people attending a conference who preordered their lunch were able to select their choice of soup, salad, or sandwich. A total of 17 people selected soup, 29 people selected salad and 35 people selected a sandwich. Of these orders, 11 attendees selected soup and salad, 10 attendees selected soup and a sandwich, and 18 selected a salad and a sandwich, while eight people selected a soup, a salad, and a sandwich. Create a Venn diagram with subsets labeled “Soup,” “Salad,” and “Sandwich,” and label the cardinality of each section of the Venn diagram as indicated by the data.

Applying Set Operations to Three Sets

Set operations are applied between two sets at a time. Parentheses indicate which operation should be performed first. As with numbers, the inner most parentheses are applied first. Next, find the complement of any sets, then perform any union or intersections that remain.

✓ Example 1.37: Applying Set Operations to Three Sets

Perform the set operations as indicated on the following sets: $U = \{0,1,2,3,4,5,6,7,8,9,10,11,12\}$, $A = \{0,1,2,3,4,5,6\}$, $B = \{0,2,4,6,8,10,12\}$, and $C = \{0,3,6,9,12\}$.

1. Find $(A \cap B) \cap C$.
2. Find $A \cap (B \cup C)$.
3. Find $(A \cap B) \cup C'$.

Answer

1. Parentheses first, A intersection B equals $A \cap B = \{0,2,4,6\}$, the elements common to both A and B. $(A \cap B) \cap C = \{0,2,4,6\} \cap \{0,3,6,9,12\} = \{0,6\}$, because the only elements that are in both sets are 0 and 6.
2. Parentheses first, B union C equals $B \cup C = \{0,2,3,4,6,8,9,10,12\}$, the collection of all elements in set B or set C or both. $A \cap (B \cup C) = \{0,1,2,3,4,5,6\} \cap \{0,2,3,4,6,8,9,10,12\} = \{0,2,3,4,6\}$, because the intersection of these two sets is the set of elements that are common to both sets.
3. Parentheses first, A intersection B equals $A \cap B = \{0,2,4,6\}$. Next, find C'. The complement of set C is the set of elements in the universal set U that are not in set C. $C' = \{1,2,4,5,7,8,10,11\}$. Finally, find $(A \cap B) \cup C' = \{0,2,4,6\} \cup \{1,2,4,5,7,8,10,11\} = \{0,1,2,4,5,6,7,8,10,11\}$.

Your Turn 1.37

Using the same sets from Example 1.37, perform the set operations indicated.

Find $A \cap (B \cap C)$.

Find $(A \cap B) \cup (A \cap C)$.

Find $(A \cup C') \cap (B \cup C')$.

Notice that the answers to the Your Turn are the same as those in the Example. This is not a coincidence. The following equivalences hold true for sets:

- $A \cap (B \cap C) = (A \cap B) \cap C$ and $A \cup (B \cup C) = (A \cup B) \cup C$. These are the associative property for set intersection and set union.
- $A \cap B = B \cap A$ and $A \cup B = B \cup A$. These are the commutative property for set intersection and set union.
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. These are the distributive property for sets over union and intersection, respectively.

Proving Equality of Sets Using Venn Diagrams

To prove set equality using Venn diagrams, the strategy is to draw a Venn diagram to represent each side of the equality, then look at the resulting diagrams to see if the regions under consideration are identical.

Augustus De Morgan was an English mathematician known for his contributions to set theory and logic. De Morgan's law for set complement over union states that $(A \cup B)' = A' \cap B'$. In the next example, we will use Venn diagrams to prove De Morgan's law for set complement over union is true. But before we begin, let us confirm De Morgan's law works for a specific example. While showing something is true for one specific example is not a proof, it will provide us with some reason to believe that it may be true for all cases.

Let $U = \{1,2,3,4,5,6,7\}$, $A = \{2,3,4\}$, and $B = \{3,4,5,6\}$. We will use these sets in the equation $(A \cup B)' = A' \cap B'$. To begin, find the value of the set defined by each side of the equation.

Step 1: $A \cup B$ is the collection of all unique elements in set A or set B or both. $A \cup B = \{2,3,4,5,6\}$. The complement of A union B, $(A \cup B)'$, is the set of all elements in the universal set that are not in $A \cup B$. So, the left side the equation $(A \cup B)'$ is equal to the set $\{1,7\}$.

Step 2: The right side of the equation is $A' \cap B'$. A' is the set of all members of the universal set U that are not in set A. $A' = \{1,5,6,7\}$. Similarly, $B' = \{1,2,7\}$.

Step 3: Finally, $A' \cap B'$ is the set of all elements that are in both A' and B'. The numbers 1 and 7 are common to both sets, therefore, $A' \cap B' = \{1,7\}$. Because, $\{1,7\} = \{1,7\}$ we have demonstrated that De Morgan's law for set complement over union works for this particular example. The Venn diagram below depicts this relationship.

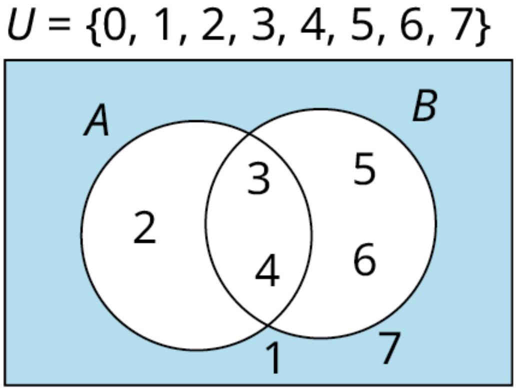


Figure 1.42:

✓ Example 1.38: Proving De Morgan's Law for Set Complement over Union Using a Venn Diagram

De Morgan's Law for the complement of the union of two sets A and B states that: $(A \cup B)' = A' \cap B'$. Use a Venn diagram to prove that De Morgan's Law is true.

Answer

Step 1: First, draw a Venn diagram representing the left side of the equality. The regions of interest are shaded to highlight the sets of interest. $A \cup B$ is shaded on the left, and $(A \cup B)'$ is shaded on the right.

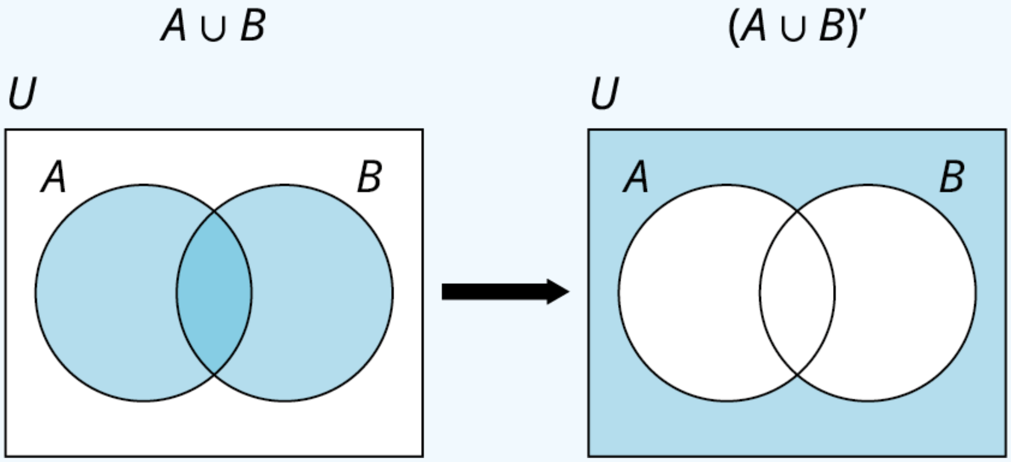
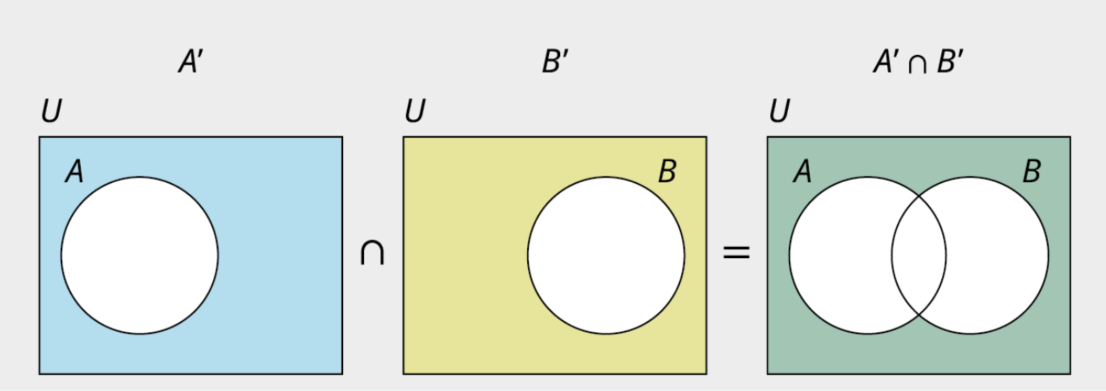


Figure 1.43:

Step 2: Next, draw a Venn diagram to represent the right side of the equation. A' is shaded and B' is shaded. Because A' and B' mix to form $A' \cap B'$ is also shaded.



Step 3: Verify the conclusion. Because the shaded region in the Venn diagram for $(A \cup B)'$ matches the shaded region in the Venn diagram for $A' \cap B'$, the two sides of the equation are equal, and the statement is true. This completes the proof that De Morgan's law is valid.

Your Turn 1.38

De Morgan's Law for the complement of the intersection of two sets A and B states that $(A \cap B)' = A' \cup B'$. Use a Venn diagram to prove that De Morgan's Law is true.

Check Your Understanding

When creating a Venn diagram with two or more subsets, you should begin with the region involving the most ___________ , then work your way from the center outward.

To construct a Venn diagram with three subsets, draw and label three circles that overlap in a common ___________ region inside the rectangle of the universal set to represent each of the three subsets.

In a Venn diagram with three sets, the area where all three sets, A, B, and C overlap is equal to the set ___________ .

When performing set operations with three or more sets, the order of operations is inner most ___________ first, then find the ___________ of any sets, and finally perform any union or intersection operations that remain.

To prove set equality using Venn diagrams, draw a Venn diagram to represent each side of the ___________ and then compare the diagrams to determine if they match or not. If they match, the statement is ___________, otherwise it is not.

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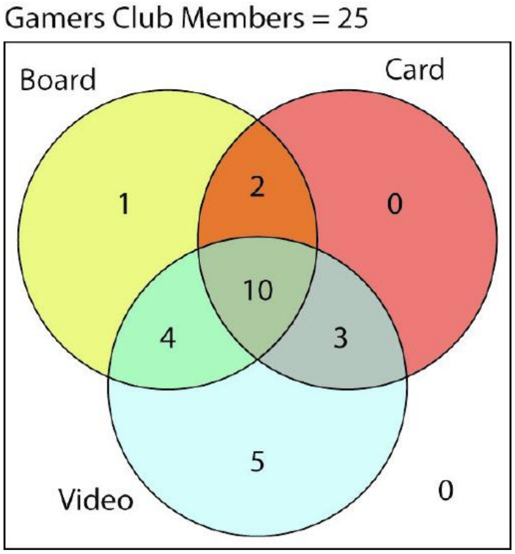
1.5.0: Exercises

A gamers club at Baily Middle School consisting of 25 members was surveyed to find out who played board games, card games, or video games. Use the results depicted in the Venn diagram below to answer the following exercises.



? Exercise 1.5.0.1

How many gamers club members play all three types of games: board games, card games, and video games?

? Exercise 1.5.0.2

How many gamers are in the set Board ∩ Video?

? Exercise 1.5.0.3

If Javier is in the region with a total of three members, what type of games does he play?

? Exercise 1.5.0.4

How many gamers play video games?

? Exercise 1.5.0.5

How many gamers are in the set Board ∪ Card?

? Exercise 1.5.0.6

How many members of the gamers club do not play video games?

? Exercise 1.5.0.7

How many members of this club only play board games?

Exercise 1.5.0.8

How many members of this club only play video games?

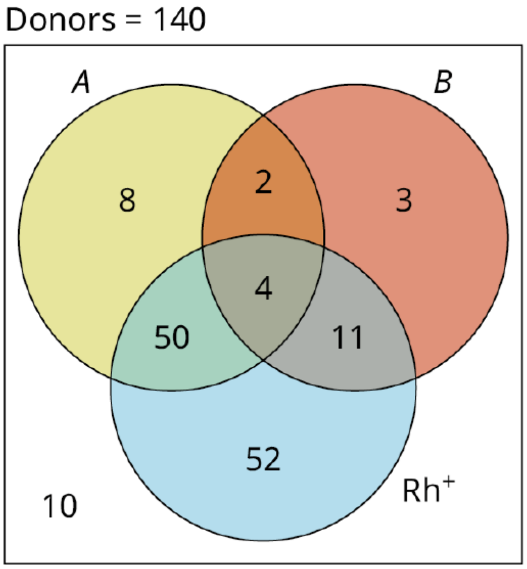
Exercise 1.5.0.9

How many members of the gamers club play video and card games?

Exercise 1.5.0.10

How many members of the gamers club are in the set Card'?

A blood drive at City Honors High School recently collected blood from 140 students, staff, and faculty. Use the results depicted in the Venn diagram below to answer the following exercises.



Exercise 1.5.0.11

Blood type $\mathrm{AB}^{+}$is the universal acceptor. Of the 140 people who donated at City Honors, how many had blood type AB^{+}?

Exercise 1.5.0.12

Blood type $\mathrm{O}^{-}$is the universal donor. Anyone needing a blood transfusion can receive this blood type. How many people who donated blood during this drive had O^{-}blood?

Exercise 1.5.0.13

How many people donated with a type A blood factor?

Exercise 1.5.0.14

How many people donated with a type A and type B blood factor (that is, they had type AB blood).

Exercise 1.5.0.15

How many donors were O^{+}?

Exercise 1.5.0.16

How many donors were not Rh^{+}?

Exercise 1.5.0.17

Opal has blood type A^{+}. If she needs to have surgery that requires a blood transfusion, she can accept blood from anyone who does not have a type B blood factor. How many people donated blood during this drive at City Honors that Opal can accept?

Exercise 1.5.0.18

Find $n\left(A \cap \mathrm{Rh}^{+}\right)$.

Exercise 1.5.0.19

Find $n\left(A \cup \mathrm{Rh}^{+}\right)$.

Exercise 1.5.0.20

Find $n\left(A \cap B \cap \mathrm{Rh}^{-}\right)$.

For the following exercises, create a three circle Venn diagram to represent the relationship between the described sets.

Exercise 1.5.0.21

The number of elements in the universal set, U, is $n(U)=48$. Sets A, B, and C are subsets of $U: n(A)=23, n(B)=25$, and $n(C)=17$. Also, $n(A \cap B)=15, n(B \cap C)=12, n(C \cap A)=11$, and $n(A \cap B \cap C)=8$.

Exercise 1.5.0.22

The number of elements in the universal set, U, is $n(U)=88$. Sets A, B, and C are subsets of $U: n(A)=31, n(B)=46: n(C)=33$. Also, $n(A \cap B)=24, n(B \cap C)=24, n(C \cap A)=26$, and $n(A \cap B \cap C)=22$.

Exercise 1.5.0.23

The number of elements in the universal set, U, is $n(U)=52$. Sets A, B, and C are subsets of $U: n(A)=23, n(B)=27$, and $n(C)=29$. Also, $n(A \cap B)=22, n(B \cap C)=21, n(C \cap A)=19$, and $n(A \cap B \cap C)=18$.

Exercise 1.5.0.24

The number of elements in the universal set, U, is $n(U)=144$. Sets A, B, and C are subsets of $U: n(A)=36, n(B)=64$, and $n(C)=81$. Also, $n(A \cap B)=26, n(B \cap C)=61, n(C \cap A)=29$, and $n(A \cap B \cap C)=25$.

Exercise 1.5.0.25

The universal set, U, has a cardinality of 36 . $n(A)=12, n(B)=12, n(C)=15, n(A \text{ and } B)=3, n(B \text{ and } C)=4, n(C \text{ and } A)=5$, $n(C \text{ and } A)=5$, and $n(A \text{ and } B \text{ and } C)=1$.

Exercise 1.5.0.26

The universal set, U, has a cardinality of $63. n(A)=29$, $n(B)=31, n(C)=41, n(A \text{ and } B)=12, n(B \text{ and } C)=16, n(C \text{ and } A)=18$, and $n(A \text{ and } B \text{ and } C)=5$.

? Exercise 1.5.0.27

The universal set, U, has a cardinality of 72 . $n(A)=32, n(B)=32, n(C)=44, n(A$ and $B)=18, n(B$ and $C)=22, n(C$ and $A)=26$, and $n(A$ and B and $C)=14$.

? Exercise 1.5.0.28

The universal set, U, has a cardinality of 81 . $n(A)=54, n(B)=41, n(C)=52, n(A$ and $B)=32, n(B$ and $C)=28, n(C$ and $A)=30$, and $n(A$ and B and $C)=21$.

? Exercise 1.5.0.29

The anime drawing club at Pratt Institute conducted a survey of its 42 members and found that 23 of them sketched with pastels, 28 used charcoal, and 17 used colored pencils. Of these, 10 club members used all three mediums, 18 used charcoal and pastels, 11 used colored pencils and charcoal, and 12 used colored pencils and pastels. The remaining club members did not use any of these three mediums.

? Exercise 1.5.0.30

A new SUV is selling with three optional packages: a sport package, a tow package, and an entertainment package. A dealership gathered the following data for all 31 of these vehicles sold during the month of July. A total of 18 SUVs included the entertainment package, 11 included the tow package, and 16 included the sport package. Of these, five SUVs included all three packages, seven were sold with both the tow package and sport package, 11 were sold with the entertainment and sport package, and eight were sold with the tow package and entertainment package. The remaining SUVs sold did not include any of these optional packages.

For the following exercises, perform the set operations as indicated on the following sets: $U=\{$ red, orange, yellow, green, blue, indigo, violet $\}, A=\{$ red, yellow, blue $\}, B=\{$ orange, green, violet $\}$, and $C=\{$ red, green, indigo $\}$.

? Exercise 1.5.0.31

Find $(A \cup B) \cap C$.

? Exercise 1.5.0.32

Find $(A \cap C) \cup B$.

? Exercise 1.5.0.33

Find $U \cap(B \cup C)$.

? Exercise 1.5.0.34

Find $(B \cap A) \cap U$.

? Exercise 1.5.0.35

Find $A \cap(B \cap C)^{\prime}$.

? Exercise 1.5.0.36

Find $A^{\prime} \cap(B \cup C)$.

For the following exercises, perform the set operations as indicated on the following sets: $U = \{20, 21, 22, \ldots, 29\}$, $A = \{21, 24, 27\}$, $B = \{20, 22, 24, 28\}$, and $C = \{21, 23, 25, 27\}$.

Exercise 1.5.0.37

Find A and B and C'.

Exercise 1.5.0.38

Find A' or B or C.

Exercise 1.5.0.39

Find $(A$ or $B)$ and C'.

Exercise 1.5.0.40

Find $(A$ or $B)$ or C'.

Exercise 1.5.0.41

Find $(A$ and $C)$ and B'.

Exercise 1.5.0.42

Find $(A \text{ or } B)'$ and C.

For the following exercises, use Venn diagrams to prove the following properties of sets:

Exercise 1.5.0.43

Commutative property for the union of two sets: $A \cup B = B \cup A$.

Exercise 1.5.0.44

Commutative property for the intersection of two sets: $A \cap B = B \cap A$.

Exercise 1.5.0.45

Associative property for the intersection of three sets: $(A \cap B) \cap C = A \cap (B \cap C)$.

Exercise 1.5.0.46

Associative property for the union of three sets: $A \cup (B \cup C) = (A \cup B) \cup C$.

Exercise 1.5.0.47

Distributive property for set intersection over set union: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Exercise 1.5.0.48

Distributive property for set union over set intersection: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

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SECTION OVERVIEW

1.6: Chapter Summary

1.6.0: Key Terms

1.6.1: Key Concepts

1.6.2: Videos

1.6.3: Formula Review

1.6.4: Projects

1.6.5: Chapter Review

1.6.6: Chapter Test

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1.6.0: Key Terms

1.1 Basic Set Concepts

- set
- elements
- well-defined set
- empty set
- roster method
- finite set
- infinite set
- natural numbers
- integer
- set-builder notation
- cardinality of a set
- countably infinite
- equal sets
- equivalent sets

1.2 Subsets

- subset
- proper subset
- equivalent subsets
- exponential notation

1.3 Understanding Venn Diagrams

- Venn diagram
- universal set
- disjoint set
- complement of a set

1.4 Set Operations with Two Sets

- intersection of two sets
- union of two sets

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1.6.1: Key Concepts

1.1 Basic Set Concepts

- Identify a set as being a well-defined collection of objects and differentiate between collections that are not well-defined and collections that are sets.
- Represent sets using both the roster or listing method and set builder notation which includes a description of the members of a set.
- In set theory, the following symbols are universally used:
  $\mathbb{N}$ - The set of natural numbers, which is the set of all positive counting numbers.

  $$\mathbb{N}=\{1,2,3,\ldots\}$$

  $\mathbb{Z}$ - The set of integers, which is the set of all the positive and negative counting numbers and the number zero.

  $$\mathbb{Z}=\{\ldots,-2,-1,0,1,2,\ldots\}$$

  $\mathbb{Q}$ - The set of rational numbers or fractions.

  $$\mathbb{Q}=\left\{\frac{p}{q} \mid p \text{ and } q \text{ are integers and } q \neq 0\right\}$$

- Distinguish between finite sets, infinite sets, and the empty set to determine the size or cardinality of a set.
- Distinguish between equal sets which have exactly the same members and equivalent sets that may have different members but must have the same cardinality or size.

1.2 Subsets

- Every member of a subset of a set is also a member of the set containing it. $A \subseteq B$
- A proper subset of a set does not contain all the members of the set containing it. There is a least one member of set B that is not a member of set A. $A \subset B$
- The number subsets of a finite set A with $n(A)$ members is equal to 2 raised to the $n(A)$ power.
- The empty set is a subset of every set and must be included when listing all the subsets of a set.
- Understand how to create and distinguish between equivalent subsets of finite and infinite sets that are not equal to the original set.

1.3 Understanding Venn Diagrams

- A Venn diagram is a graphical representation of the relationship between sets.
- In a Venn diagram, the universal set, U is the largest set under consideration and is drawn as a rectangle. All subsets of the universal set are drawn as circles within this rectangle.
- The complement of set A includes all the members of the universal set that are not in set A. A set and its complement are disjoint sets, they do not share any elements in common.
- To find the complement of set A remove all the elements of set A from the universal set U, the set that includes only the remaining elements is the complement of set A, A'.
- Determine the complement of a set using Venn diagrams, the roster method and set builder notation.

1.4 Set Operations with Two Sets

- The intersection of two sets, $A \cap B$ is the set of all elements that they have in common. Any member of A intersection B must be is both set A and set B.
- The union of two sets, $A \cup B$, is the collection of all members that are in either in set A, set B or both sets A and B combined.
- Two sets that share at least one element in common, so that they are not disjoint are represented in a Venn Diagram using two circles that overlap.
  - The region of the overlap is the set A intersection B, $A \cap B$.

    - The regions that include everything in the circle representing set A or the circle representing set B or their overlap is the set A union B, $A \cup B$.
- Apply knowledge of set union and intersection to determine cardinality and membership using Venn Diagrams, the roster method and set builder notation.

1.5 Set Operations with Three Sets

- A Venn diagram with two overlapping sets breaks the universal set up into four distinct regions. When a third overlapping set is added the Venn diagram is broken up into eight distinct regions.
- Analyze, interpret, and create Venn diagrams involving three overlapping sets.
    - Including the blood factors: A, B and Rh
    - To find unions and intersections.
    - To find cardinality of both unions and intersections.
- When performing set operations with three or more sets, the order of operations is inner most parentheses first, then fine the complement of any sets, then perform any union or intersection operations that remain.
- To prove set equality using Venn diagrams the strategy is to draw a Venn diagram to represent each side of the equality or equation, then look at the resulting diagrams to see if the regions under consideration are identical. If they regions are identical the equation represents a true statement, otherwise it is not true.

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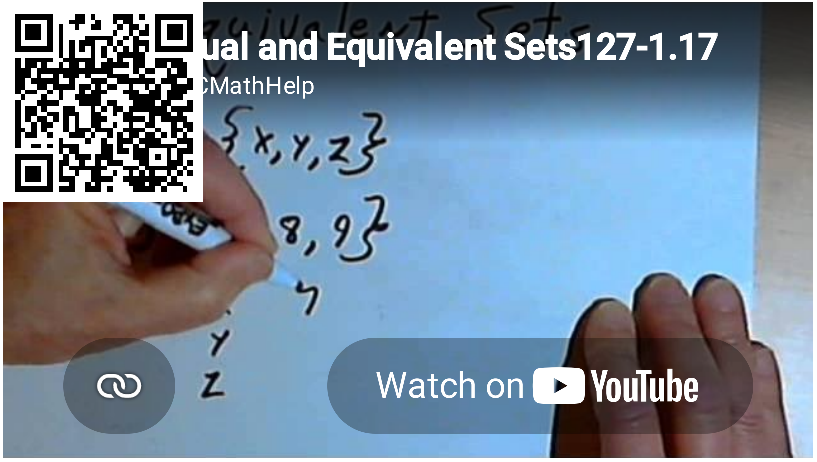
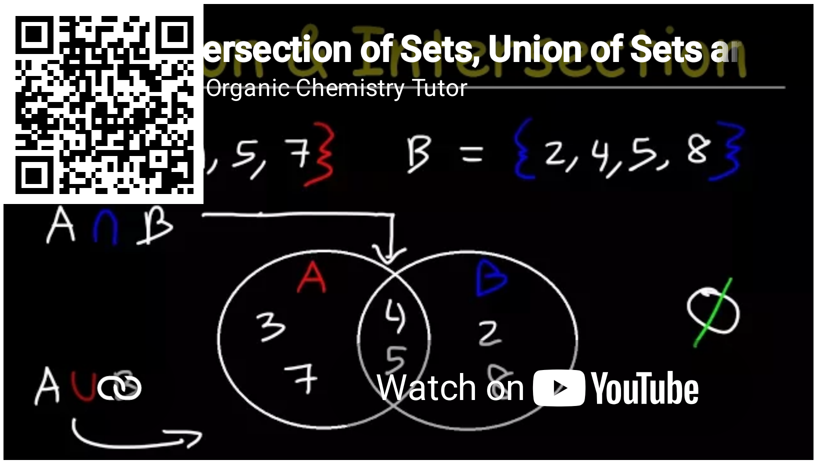
1.6.2: Videos

1.1 Basic Set Concepts


ual and Equivalent Sets127-1.17
CMathHelp
Watch on YouTube

- Equal and Equivalent Sets

1.4 Set Operations with Two Sets


ersection of Sets, Union of Sets ar
Organic Chemistry Tutor
Watch on YouTube

- The Basics of Intersection of Sets, Union of Sets and Venn Diagrams

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1.6.3: Formula Review

1.2 Subsets

The number of subsets of a finite set A is equal to 2 raised to the power of $n\left(A\right)$, where $n\left(A\right)$ is the number of elements in set A: Number of Subsets of Set $A = 2^{n(A)}$.

1.4 Set Operations with Two Sets

The cardinality of A union $B : n(A \cup B) = n(A) + n(B) - n(A \cap B)$

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1.6.4: Projects

Cardinality of Infinite Sets

In set theory, it has been shown that the set of irrational numbers has a cardinality greater than the set of natural numbers. That is, the set of irrational numbers is so large that it is uncountably infinite.

1. Perform a search with the phrase, "Who first proved that the real numbers are uncountable?"
    a. Who first proved that the real numbers are uncountable?
    b. What was the significance of this proof to the development of set theory and by extension other fields of mathematics?
2. Recent discoveries in the field of set theory include the solution to a 70-year-old problem previously thought to be unprovable. To learn more read this article:
    a. What does it mean for two infinite sets to have the same size?
    b. The real numbers are sometimes referred to as what?
    c. Summarize your understanding of the problem known as the "Continuum Hypothesis."
    d. Malliaris and Shelah's proof of this 70-year-old problem is opening up investigation in what two fields of mathematics?
3. Summarize your understanding of infinity.
    a. Define what it means to be infinite.
    b. Explain the difference between countable and uncountable sets.
    c. Research the difference between a discrete set and a continuous set, then summarize your findings.

Set Notation

In arithmetic, the operation of addition is represented by the plus sign, +, but multiplication is represented in multiple ways, including $\cdot, \times, *,$ and parentheses, such as 5(3). Several set operations also are written in different forms based on the preferences of the mathematician and often their publisher.

1. Search for "Set Complement" on the internet and list at least three ways to represent the complement of a set.
2. Both the Set Challenge and Venn Diagram smartphone apps highlighted in the Tech Check sections have an operation for set difference. List at least two ways to represent set difference and provide a verbal description of how to calculate the difference between two sets A and B.
3. When researching possible Venn diagram applications, the Greek letter delta, Δ appeared as a symbol for a set operator. List at least one other symbol used for this same operation.
4. Search for "List of possible set operations and their symbols." Find and select two symbols that were not presented in this chapter.

The Real Number System

The set of real numbers and their properties are studied in elementary school today, but how did the number system evolve? The idea of natural numbers or counting numbers surfaced prior to written words, as evidenced by tally marks in cave writing. Create a timeline for significant contributions to the real number system.

1. Use the following phrase to search online for information on the origins of the number zero: "History of the number zero." Then, record significant dates for the invention and common use of the number zero on your timeline.
2. Find out who is credited for discovering that the $\sqrt{2}$ is irrational and add this information to your timeline. Hint: Search for, "Who was the first to discover irrational numbers?"
3. Research Georg Cantor's contribution to the representation of real numbers as a continuum and add this to your timeline.
4. Research Ernst Zermelo's contribution to the real number system and add this to your timeline.

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1.6.5: Chapter Review

? Exercise 1.6.5.1

A ____________ is a well-defined collection of distinct objects.

? Exercise 1.6.5.2

A collection of well-defined objects without any members in it is called the____________

? Exercise 1.6.5.3

Write the set consisting of the last five letters of the English alphabet using the roster method.

? Exercise 1.6.5.4

Write the set consisting of the numbers 1 through 20 inclusive using the roster method and an ellipsis.

? Exercise 1.6.5.5

Write the set of all zebras that do not have stripes in symbolic form.

? Exercise 1.6.5.6

Write the set of negative integers using the roster method and an ellipsis.

? Exercise 1.6.5.7

Use set builder notation to write the set of all even integers.

? Exercise 1.6.5.8

Write the set of all letters in the word Mississippi and label it with a capital M.

? Exercise 1.6.5.9

Determine whether the following collection describes a well-defined set: "A group of these five types of apples: Granny Smith, Red Delicious, McIntosh, Fuji, and Jazz."

? Exercise 1.6.5.10

Determine whether the following collection describes a well-defined set: "A group of five large dogs."

? Exercise 1.6.5.11

Determine the cardinality of the set $A = \{$ Alabama, Alaska, Arkansas, Arizona $\}$.

? Exercise 1.6.5.12

Determine whether the following set is a finite set or an infinite set: $F = \{5, 10, 15, \ldots\}$.

? Exercise 1.6.5.13

Determine whether sets A and B are equal, equivalent, or neither: $A=\{a, b, c\}$ and $B=\{1,2,3,4\}$.

? Exercise 1.6.5.14

Determine if sets A and B are equal, equivalent, or neither: $A=\{a, b, c\}$ and $B=\{c, a, b\}$.

? Exercise 1.6.5.15

Determine if sets A and B are equal, equivalent, or neither: $A=\{a, b, c\}$ and $B=\{1,2,3\}$.

Subsets

? Exercise 1.6.5.16

If every member of set A is also a member of set B, then set A is a of set B.

? Exercise 1.6.5.17

Determine whether set A is a subset, proper subset, or neither a subset nor proper subset of set $B: A=\{s, o, n\}$ and B $=\{s, o, n, g\}$.

? Exercise 1.6.5.18

Determine whether set A is a subset, proper subset, or neither a subset nor proper subset of set $B: A=\{s, o, n\}$ and $B=\{s, o, l\}$.

? Exercise 1.6.5.19

Determine whether set A is a subset, proper subset, or neither a subset nor proper subset of set $B: A=\{s, o, n\}$ and B $=\{o, n, s\}$.

? Exercise 1.6.5.20

List all the subsets of the set $\{up$, down $\}$.

? Exercise 1.6.5.21

List all the subsets of the set $\{0\}$.

? Exercise 1.6.5.22

Calculate the total number of subsets of the set \{Scooby, Velma, Daphne, Shaggy, Fred\}.

? Exercise 1.6.5.23

Calculate the total number of subsets of the set \{top hat, thimble, iron, shoe, battleship, cannon\}.

? Exercise 1.6.5.24

Find a subset of the set $\{g, r, e, a, t\}$ that is equivalent, but not equal, to $\{t, e, a\}$.

? Exercise 1.6.5.25

Find a subset of the set $\{g, r, e, a, t\}$ that is equal to $\{t, e, a\}$.

? Exercise 1.6.5.26

Find two equivalent finite subsets of the set of natural numbers, $\mathbb{N} = \{1, 2, 3, \ldots\}$, with a cardinality of 4 .

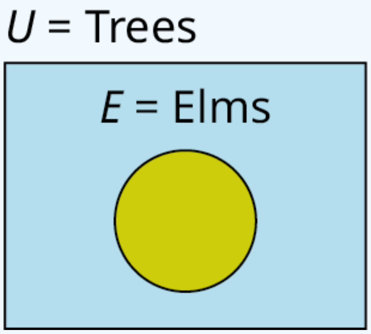
? Exercise 1.6.5.27

Find two equal finite subsets of the set of natural numbers, $\mathbb{N} = \{1, 2, 3, \ldots\}$, with a cardinality of 3 .

Understanding Venn Diagrams

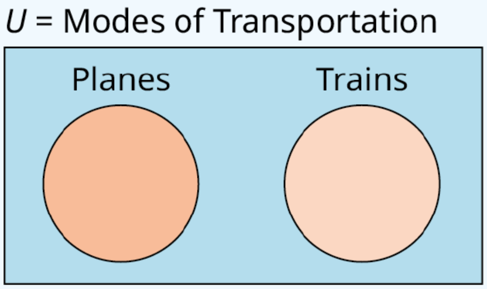
? Exercise 1.6.5.28

Use the Venn diagram below to describe the relationship between the sets, symbolically and in words:



? Exercise 1.6.5.29

Use the Venn diagram below to describe the relationship between the sets, symbolically and in words:



? Exercise 1.6.5.30

Draw a Venn diagram to represent the relationship between the described sets: Falcons $\subset$ Raptors.

? Exercise 1.6.5.31

Draw a Venn diagram to represent the relationship between the described sets: Natural numbers $\subset$ Integers $\subset$ Real numbers.

Exercise 1.6.5.32

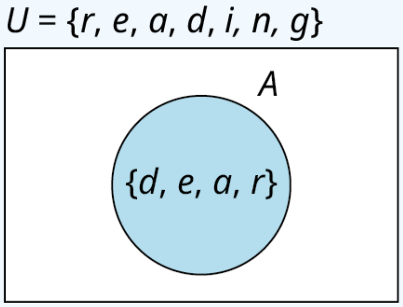
The universal set is the set $U=\{s, m, i, l, e\}$. Find the complement of the set $E=\{e, l, m\}$.

Exercise 1.6.5.33

The universal set is the set $U=\{1,2,3, \ldots\}$. Find the complement of the set $V=\{18,19,20, \ldots\}$.

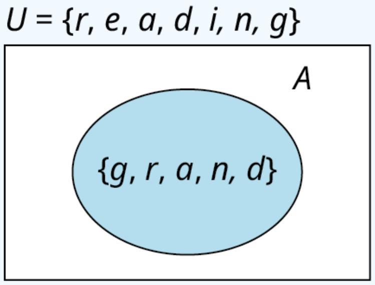
Exercise 1.6.5.34

Use the Venn diagram below to determine the members of the set A'.



Exercise 1.6.5.35

Use the Venn diagram below to determine the members of the set A'.



Set Operations with Two Sets

Determine the union and intersection of the sets indicated: $U=\{a, b, c, \ldots, z\}$, $S=\{s, c, r, a, b, l, e\}$, $B=\{b, r, a, c, e\}$, $C=\{c, r, a, b\}$, $R=\{r, i, s, k\}$, and $Q=\{q, u, i, z\}$.

Exercise 1.6.5.36

What is $S \cap R$?

Exercise 1.6.5.37

What is $S \cup B$? 38. Write the set containing the elements in sets B or Q.

Exercise 1.6.5.39

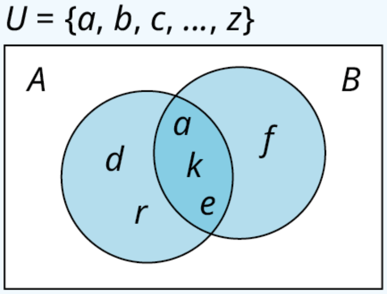
Write the set containing all the elements is both sets B and Q. 40. Find C intersection R.

? Exercise 1.6.5.41

Find C union R.

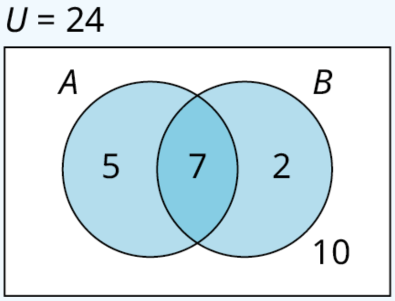
? Exercise 1.6.5.42

Find the cardinality of $C \cup R, n(C \cup R)$. 43. Find $n(S$ union $R)$. 44. Use the Venn diagram below to find $A \cap B$.


U = {a, b, c, ..., z}
A
B
d
r
a
k
e
f

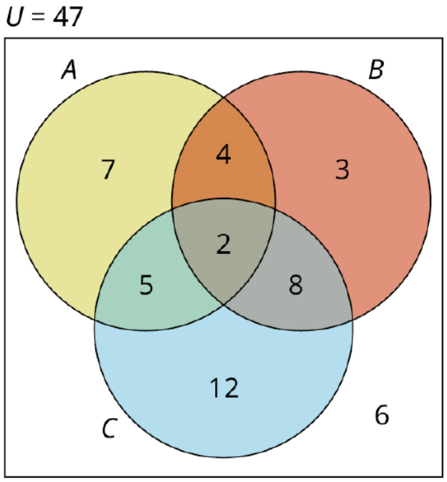
? Exercise 1.6.5.45

Use the Venn diagram below to find $n(A \cup B)$.


U = 24
A
B
5
7
2
10

Set Operations with Three Sets

Use the Venn diagram below to answer the following questions.


U = 47
A
B
7
4
3
2
5
8
12
C
6

? Exercise 1.6.5.46

Find $n(A \cup C)$.

Exercise 1.6.5.47

Find $n(B \cap C)$.

Exercise 1.6.5.48

A food truck owner surveyed a group of 50 customers about their preferences for hamburger condiments. After tallying the responses, the owner found that 24 customers preferred ketchup, 11 preferred mayonnaise, and 31 preferred mustard. Of these customers, eight preferred ketchup and mayonnaise, one preferred mayonnaise and mustard, and 13 preferred ketchup and mustard. No customer preferred all three. The remaining customers did not select any of these three condiments. Draw a Venn diagram to represent this data.

Exercise 1.6.5.49

Given $U = \{\mathrm{r, s, t, l, n, e, i, a}\}$, $R = \{\mathrm{r, e, s, t}\}$, $S = \{\mathrm{s, t, a, i, r}\}$, and $L = \{\mathrm{l, i, n, e, s}\}$, find $(S \cup R) \cap L'$.

Exercise 1.6.5.50

Use Venn diagrams to prove that if $A \subset B$, then $B' \subset A'$.

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1.6.6: Chapter Test

Chapter Test

1. Determine whether the following collection describes a well-defined set: "A group of small tomatoes."

Classify each of the following sets as either finite or infinite.

2. $\{1, 5, 9, \ldots\}$

3. $\{c \mid c$ is a cat $\}$

4. $\{1, 2, 3, \ldots, 1000\}$

5. $\{s, m, i, l, e\}$

6. $\{m \in \mathbb{N} \mid m = n^2$ where n is a natural number $\}$

Use the sets provided to answer the following questions: $U = \{31, 32, 33, \ldots, 50\}$, $A = \{35, 38, 41, 44, 47, 50\}$, $B = \{32, 36, 40, 44, 48\}$, and $C = \{31, 32, 41, 42, 48, 50\}$.

7. Find A or B.

8. Find B and C.

9. Determine if set A is equivalent to, equal to, or neither equal nor equivalent to set C. Justify your answer.

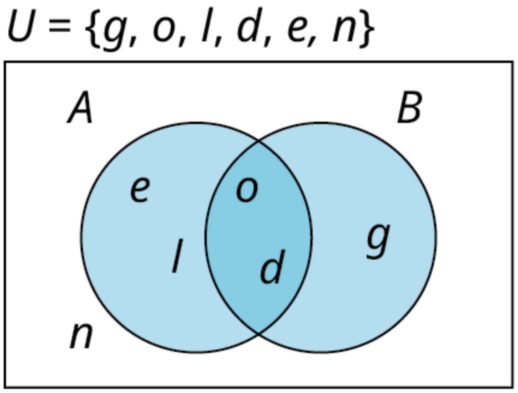
10. Find $n(A \cup C)$.

11. Find $A \cap (B \cap C)$.

12. Find $(A \cup B)' \cap C$.

13. Find $(A \cap B') \cup C$.

Use the Venn diagram below to answer the following questions.



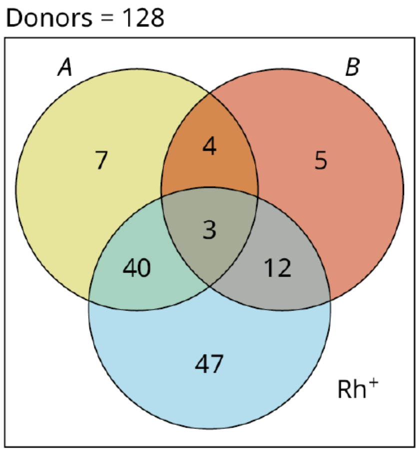
14. Find B'.

15. Find $A \cup B$.

16. Find $A \cap B'$.

17. Draw a Venn diagram to represent the relationship between the two sets: "All flowers are plants."

For the following questions, use the Venn diagram showing the blood types of all donors at a recent mobile blood drive.



Exercise 1.6.6.18

Find the number of donors who were O^-; that is, find $n\left((A \cup B \cup Rh^+)'\right)$.

Exercise 1.6.6.19

Find the number of donors who were A^+or B^+or AB^+.

Exercise 1.6.6.20

Use Venn diagrams to prove that if $A \subset B$, then $A \cap B = A$.

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CHAPTER OVERVIEW

2: Logic

2.0: Introduction
2.1: Statements and Quantifiers
  2.1.0: Exercises
2.2: Compound Statements
  2.2.0: New Page
2.3: Constructing Truth Tables
  2.3.0: Exercises
2.4: Truth Tables for the Conditional and Biconditional
  2.4.0: Exercises
2.5: Equivalent Statements
  2.5.0: Exercises
2.6: De Morgan’s Laws
  2.6.0: Exercises
2.7: Logical Arguments
  2.7.0: Exercises
2.8: Chapter Summary
  2.8.0: Key Terms
  2.8.1: Key Concepts
  2.8.2: Videos
  2.8.3: Projects
  2.8.4: Chapter Review
  2.8.5: Chapter Test

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2.0: Introduction

What is logic? Logic is the study of reasoning, and it has applications in many fields, including philosophy, law, psychology, digital electronics, and computer science.

In law, constructing a well-reasoned, logical argument is extremely important. The main goal of arguments made by lawyers is to convince a judge and jury that their arguments are valid and well-supported by the facts of the case, so the case should be ruled in their favor. Think about Thurgood Marshall arguing for desegregation in front of the U.S. Supreme Court during the *Brown v. Board of Education of Topeka* lawsuit in 1954, or Ruth Bader Ginsburg arguing for equality in social security benefits for both men and women under the law during the mid-1970s. Both these great minds were known for the preparation and thoroughness of their logical legal arguments, which resulted in victories that advanced the causes they fought for. Thurgood Marshall and Ruth Bader Ginsburg would later become well respected justices on the U.S. Supreme Court themselves.

Figure $2.0.1$: Logic is key to a well-reasoned argument, in both math and law. (credit: modification of work "Lady Justicia holding sword and scale bronze figurine with judge hammer on wooden table" by Jernej Furman/Flickr, CC BY 2.0)

In this chapter, we will explore how to construct well-reasoned logical arguments using varying structures. Your ability to form and comprehend logical arguments is a valuable tool in many areas of life, whether you're planning a dinner date, negotiating the purchase of a new car, or persuading your boss that you deserve a raise.

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2.1: Statements and Quantifiers

Figure 2.1.1: Construction of a logical argument, like that of a house, requires you to begin with the right parts. (credit: modification of work “Barn Raising” by Robert Stinnett/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

1. Identify logical statements.
2. Represent statements in symbolic form.
3. Negate statements in words.
4. Negate statements symbolically.
5. Translate negations between words and symbols.
6. Express statements with quantifiers of all, some, and none.
7. Negate statements containing quantifiers of all, some, and none.

Have you ever built a club house, tree house, or fort with your friends? If so, you and your friends likely started by gathering some tools and supplies to work with, such as hammers, saws, screwdrivers, wood, nails, and screws. Hopefully, at least one member of your group had some knowledge of how to use the tools correctly and helped to direct the construction project. After all, if your house isn't built on a strong foundation, it will be weak and could possibly fall apart during the next big storm. This same foundation is important in logic.

In this section, we will begin with the parts that make up all logical arguments. The building block of any logical argument is a logical statement, or simply a statement. A logical statement has the form of a complete sentence, and it must make a claim that can be identified as being true or false.

When making arguments, sometimes people make false claims. When evaluating the strength or validity of a logical argument, you must also consider the truth values, or the identification of true or false, of all the statements used to support the argument. While a false statement is still considered a logical statement, a strong logical argument starts with true statements.

Identifying Logical Statements

Figure $2.1.2$: Not all roses are red. (credit: “assorted pink yellow white red roses macro” by ProFlowers/Flickr, CC BY 2.0)

An example of logical statement with a false truth value is, “All roses are red.” It is a logical statement because it has the form of a complete sentence and makes a claim that can be determined to be either true or false. It is a false statement because not all roses are red: some roses are red, but there are also roses that are pink, yellow, and white. Requests, questions, or directives may be complete sentences, but they are not logical statements because they cannot be determined to be true or false. For example, suppose someone said to you, “Please, sit down over there.” This request does not make a claim and it cannot be identified as true or false; therefore, it is not a logical statement.

Exercise $2.1.1$: Identifying Logical Statements

Determine whether each of the following sentences represents a logical statement. If it is a logical statement, determine whether it is true or false.

1. Tiger Woods won the Master’s golf championship at least five times.
2. Please, sit down over there.
3. All cats dislike dogs.

Answer

1. This is a logical statement because it is a complete sentence that makes a claim that can be identified as being true or false. As of 2021, this statement is true: Tiger Woods won the Master’s in 1997, 2001, 2002, 2005 and 2019.
2. This is not a logical statement because, although it is a complete sentence, this request does not make a claim that can be identified as being either true or false.
3. This is a logical statement because it is a complete sentence that makes a claim that can be identified as being true or false. This statement is false because some cats do like some dogs.

Your Turn $2.1.1$

Determine whether each of the following sentences represents a logical statement. If it is a logical statement, determine whether it is true or false.

1. The Buffalo Bills defeated the New York Giants in Super Bowl XXV
2. Michael Jackson’s album *Thriller* was released in 1982.
3. Would you like some coffee or tea?

Representing Statements in Symbolic Form

When analyzing logical arguments that are made of multiple logical statements, symbolic form is used to reduce the amount of writing involved. Symbolic form also helps visualize the relationship between the statements in a more concise way in order to determine the strength or validity of an argument. Each logical statement is represented symbolically as a single lowercase letter, usually starting with the letter p.

To begin, you will practice how to write a single logical statement in symbolic form. This skill will become more useful as you work with compound statements in later sections.

? Exercise $2.1.2$: Representing Statements Using Symbolic Form

Write each of the following logical statements in symbolic form.

1. Barry Bonds holds the Major League Baseball record for total career home runs.
2. Some mammals live in the ocean.
3. Ruth Bader Ginsburg served on the U.S. Supreme Court from 1993 to 2020.

Answer

1. *p* : Barry Bonds holds the Major League Baseball record for total career home runs. The statement is labeled with a p. Once the statement is labeled, use p as a replacement for the full written statement to write and analyze the argument symbolically.
2. *q* : Some mammals live in the ocean. The letter q is used to distinguish this statement from the statement in question 1, but any lower-case letter may be used.
3. *r* : Ruth Bader Ginsburg served on the U.S. Supreme Court from 1993 to 2020. When multiple statements are present in later sections, you will want to be sure to use a different letter for each separate logical statement.

✓ Your Turn $2.1.2$

Write each of the following logical statements in symbolic form.

1. The movie *Gandhi* won the Academy Award for Best Picture in 1982.
2. Soccer is the most popular sport in the world.
3. All oranges are citrus fruits.

Who Knew?

Mathematics is not the only language to use symbols to represent thoughts or ideas. The Chinese and Japanese languages use symbols known as Hanzi and Kanji, respectively, to represent words and phrases. At one point, the American musician Prince famously changed his name to a symbol representing love.

BBC Prince Symbol Article

Negating Statements

Consider the false statement introduced earlier, “All roses are red.” If someone said to you, “All roses are red,” you might respond with, “Some roses are not red.” You could then strengthen your argument by providing additional statements, such as, “There are also white roses, yellow roses, and pink roses, to name a few.”

The negation of a logical statement has the opposite truth value of the original statement. If the original statement is false, its negation is true, and if the original statement is true, its negation is false. Most logical statements can be negated by simply adding or removing the word *not*. For example, consider the statement, “Emma Stone has green eyes.” The negation of this statement would be, “Emma Stone does not have green eyes.” The table below gives some other examples.

Table $2.1.1$

Logical Statement	Negation
Gordon Ramsey is a chef.	Gordon Ramsey is not a chef.

Logical Statement	Negation
Tony the Tiger does not have spots.	Tony the Tiger has spots.

The way you phrase your argument can impact its success. If someone presents you with a false statement, the ability to rebut that statement with its negation will provide you with the tools necessary to emphasize the correctness of your position.

Exercise $2.1.3$: Negating Logical Statements

Write the negation of each logical statement in words.

1. Michael Phelps was an Olympic swimmer.
2. Tom is a cat.
3. Jerry is not a mouse.

Answer

1. Add the word *not* to negate the affirmative statement: “Michael Phelps was not an Olympic swimmer.”
2. Add the word *not* to negate the affirmative statement: “Tom is not a cat.”
3. Remove the word *not* to negate the negative statement: “Jerry is a mouse.”

Your Turn $2.1.3$

Write the negation of each logical statement in words.

1. Ted Cruz was not born in Texas
2. Adele has a beautiful singing voice.
3. Leaves convert carbon dioxide to oxygen during the process of photosynthesis.

The symbol for negation, or not, in logic is the tilde, ~. So, not p is represented as $\sim p$. To negate a statement symbolically, remove or add a tilde. The negation of not (not p) is p. Symbolically, this equation is $\sim(\sim p) = p$.

Exercise $2.1.4$: Negating Logical Statements Symbolically

Write the negation of each logical statement symbolically.

1. p: Michael Phelps was an Olympic swimmer.
2. r: Tom is not a cat.
3. $\sim q$: Jerry is not a mouse.

Answer

1. To negate an affirmative logical statement symbolically, add a tilde: $\sim p$.
2. Because the symbol for this statement is r, its negation is $\sim r$.
3. The symbol for this statement is $\sim q$, so to negate it we simply remove the ~, because $\sim(\sim q) = q$. The answer is q.

Your Turn $2.1.4$

Write the negation of each logical statement symbolically.

1. p: Ted Cruz was not born in Texas
2. q: Adele has a beautiful voice.
3. r: Leaves convert carbon dioxide to oxygen during the process of photosynthesis.

Translating Negations Between Words and Symbols

In order to analyze logical arguments, it is important to be able to translate between the symbolic and written forms of logical statements.

? Exercise 2.1.5

Given the statements:

r: Elmo is a red Muppet.

p: Ketchup is not a vegetable.

1. Write the symbolic form of the statement, “Elmo is not a red Muppet.”
2. Translate the statement $\sim p$ into words.

Answer

1. “Elmo is not a red muppet” is the negation of “Elmo is a red muppet,” which is represented as *r*. Thus, we would write “Elmo is not a red muppet” symbolically as ~*r*.
2. Because $p$is the symbol representing the statement, “Ketchup is not a vegetable,” ~*p* is equivalent to the statement, “Ketchup is a vegetable.”

✓ Your Turn 2.1.5

Given the statements:
r: Woody and Buzz Lightyear are best friends.
p: Wonder Woman is not stronger than Captain Marvel.

1. Write the symbolic form of the statement, "Wonder Woman is stronger than Captain Marvel."
2. Translate the statement r into words

Expressing Statements with Quantifiers of All, Some, or None

A quantifier is a term that expresses a numerical relationship between two sets or categories. For example, all squares are also rectangles, but only some rectangles are squares, and no squares are circles. In this example, *all*, *some*, and *none* are quantifiers. In a logical argument, the logical statements made to support the argument are called premises, and the judgment made based on the premises is called the conclusion. Logical arguments that begin with specific premises and attempt to draw more general conclusions are called inductive arguments.

Consider, for example, a parent walking with their three-year-old child. The child sees a cardinal fly by and points it out. As they continue on their walk, the child notices a robin land on top of a tree and a duck flying across to land on a pond. The child recognizes that cardinals, robins, and ducks are all birds, then excitedly declares, "All birds fly!" The child has just made an inductive argument. They noticed that three different specific types of birds all fly, then synthesized this information to draw the more general conclusion that all birds can fly. In this case, the child's conclusion is false.

The specific premises of the child's argument can be paraphrased by the following statements:

- Premise: Cardinals are birds that fly.
- Premise: Robins are birds that fly.
- Premise: Ducks are birds that fly.

The general conclusion is: “All birds fly!”

All inductive arguments should include at least three specific premises to establish a pattern that supports the general conclusion. To counter the conclusion of an inductive argument, it is necessary to provide a counter example. The parent can tell the child about penguins or emus to explain why that conclusion is false.

On the other hand, it is usually impossible to prove that an inductive argument is true. So, inductive arguments are considered either strong or weak. Deciding whether an inductive argument is strong or weak is highly subjective and often determined by the background knowledge of the person making the judgment. Most hypotheses put forth by scientists using what is called the “scientific method” to conduct experiments are based on inductive reasoning.

In the following example, we will use quantifiers to express the conclusion of a few inductive arguments.

Exercise 2.1.6: Drawing General Conclusions to Inductive Arguments Using Quantifiers

For each series of premises, draw a logical conclusion to the argument that includes one of the following quantifiers: all, some, or none.

1. Squares and rectangles have four sides. A square is a parallelogram, and a rectangle is a parallelogram. What conclusion can be drawn from these premises?

2. $1+2=3, 6+7=13$, and $23+24=47$. Of these, 1 and 2,6 and 7 , and 23 and 24 are consecutive integers; 3,13 , and 47 are odd numbers. What conclusion can be drawn from these premises?

3. Sea urchins live in the ocean, and they do not breathe air. Sharks live in the ocean, and they do not breathe air. Eels live in the ocean, and they do not breath air. What conclusion can be drawn from these premises?

Answer

1. The conclusion you would likely come to here is “Some four-sided figures are parallelograms.” However, it would be incorrect to say that all four-sided figures are parallelograms because there are some four-sided figures, such as trapezoids, that are not parallelograms. This is a false conclusion.
2. From these premises, you may draw the conclusion “All sums of two consecutive counting numbers result in an odd number.” Most inductive arguments cannot be proven true, but several mathematical properties can be. If we let n n represent our first counting number, then $n + 1$ would be the next counting number and $n + (n + 1) = 2 n + 1$. Because $2n$ is divisible by two, it is an even number, and if you add one to any even number the result is always an odd number. Thus, the conclusion is true!
3. Based on the premises provided, with no additional knowledge about whales or dolphins, you might conclude “No creatures that live in the ocean breathe air.” Even though this conclusion is false, it still follows from the known premises and thus is a logical conclusion based on the evidence presented. Alternatively, you could conclude “Some creatures that live in the ocean do not breathe air.” The quantifier you choose to write your conclusion with may vary from another person’s based on how persuasive the argument is. There may be multiple acceptable answers.

✓ Your Turn 2.1.6

For each series of premises, draw a logical conclusion to the argument that includes one of the following quantifiers: all, some, or none.

1. $1+2=3$, $5+6=11$, and $14+15=29$. Of these, 1 and 2 are consecutive integers, 5 and 6 are consecutive integers, and 14 and 15 are consecutive integers. Also, their sums, 3, 11, and 29 are all prime numbers. Prime numbers are positive integers greater than one that are only divisible by one and the number itself. What conclusion can you draw from these premises?
2. A robin is a bird that lays blue eggs. A chicken is a bird that typically lays white and brown eggs. An ostrich is a bird that lays exceptionally large eggs. If a bird lays eggs, then they do not give live birth to their young. What conclusion can you draw from these premises?
3. All parallelograms have four sides. All rectangles are parallelograms. All squares are rectangles. What additional conclusion can you make about squares from these premises?

Checkpoint

It is not possible to prove definitively that an inductive argument is true or false in most cases.

Negating Statements Containing Quantifiers

Recall that the negation of a statement will have the opposite truth value of the original statement. There are four basic forms that logical statements with quantifiers take on.

- All **A** are B.
- Some A are B.
- No A are B.
- Some A are not B.

The negation of logical statements that use the quantifiers *all*, *some*, or *none* is a little more complicated than just adding or removing the word *not*.

For example, consider the logical statement, “All oranges are citrus fruits.” This statement expresses as a subset relationship. The set of oranges is a subset of the set of citrus fruit. This means that there are no oranges that are outside the set of citrus fruit. The negation of this statement would have to break the subset relationship. To do this, you could say, “At least one orange is not a citrus fruit.” Or, more concisely, “Some oranges are not citrus fruit.” It is tempting to say "No oranges are citrus fruit," but that would be incorrect. Such a statement would go beyond breaking the subset relationship, to stating that the two sets have nothing in common. The negation of "semanticsA is a subset of B" would be to state that "A is not a subset of B," as depicted by the Venn diagram in Figure $2.1.3$.

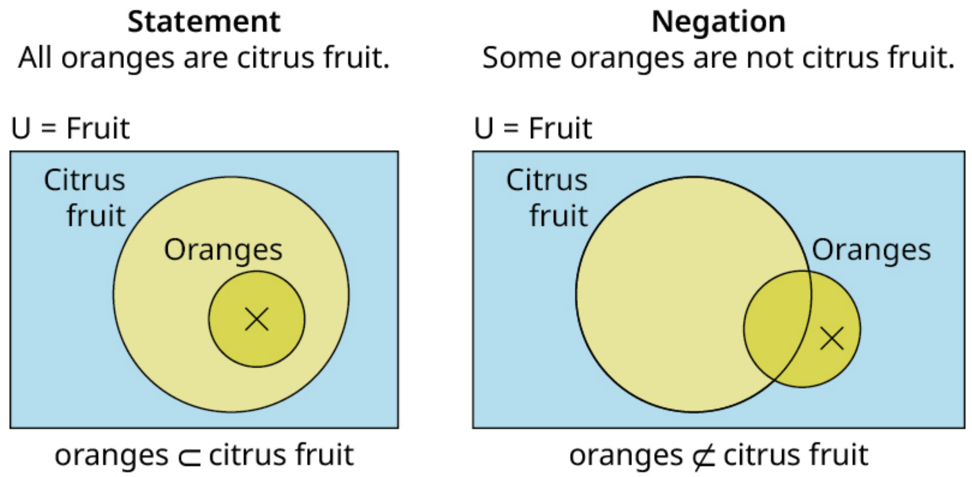
Statement
All oranges are citrus fruit.
Negation
Some oranges are not citrus fruit.
U = Fruit
Citrus fruit
Oranges
U = Fruit
Citrus fruit
Oranges
oranges ⊂ citrus fruit
oranges ⊄ citrus fruit

Figure $2.1.3$: The statement, “All oranges are citrus fruit,” is true, so its negation, “Some oranges are not citrus fruit,” is false.

Now, consider the statement, “No apples are oranges.” This statement indicates that the set of apples is disjointed from the set of oranges. The negation must state that the two are not disjoint sets, or that the two sets have a least one member in common. Their intersection is not empty. TThe negation of the statement, “A intersection B is the empty set,” is the statement that " A intersection B is not empty," as depicted in the Venn diagram in Figure $2.1.4$.

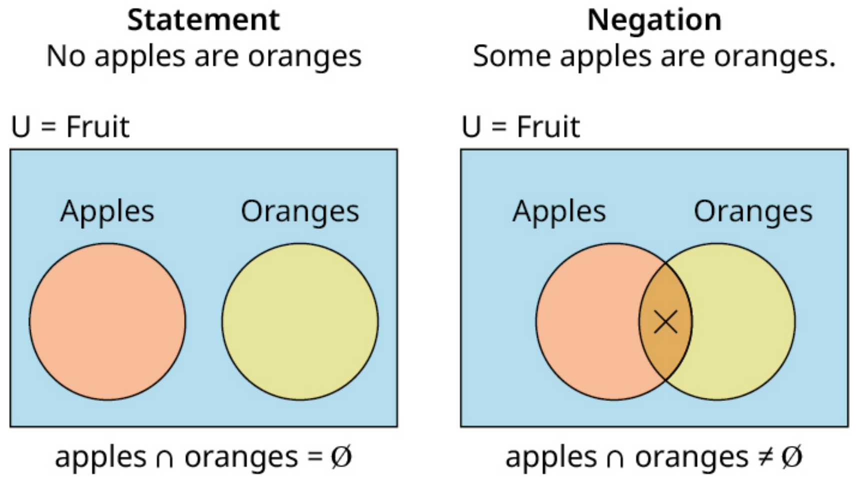
Statement
No apples are oranges
Negation
Some apples are oranges.
U = Fruit
Apples
Oranges
U = Fruit
Apples
Oranges
apples ∩ oranges = Ø
apples ∩ oranges ≠ Ø

Figure $2.1.4$:The negation of the true statement “No apples are oranges,” is the false statement, “Some apples are oranges.”

Table $2.1.2$ summarizes the four different forms of logical statements involving quantifiers and the forms of their associated negations, as well as the meanings of the relationships between the two categories or sets A and B.

Table $2.1.2$

Logical Statements with Quantifiers	Negation of Logical Statements w/Quantifiers

Logical Statements with Quantifiers	Negation of Logical Statements w/Quantifiers
Form: All A are B. Means: A is a subset of B, $A \subset B$. All zebras have stripes. (True)	Form: Some A are not B. Means: A is not a subset of B, $A \not\subset B$. Some zebras do not have stripes. (False)
Form: Some A are B. Means: A intersection B is not empty, $A \cap B \neq \emptyset$. Some fish are sharks. (True)	Form: No A are B. Means: A intersection B is empty, $A \cap B = \emptyset$. No fish are sharks. (False)
Form: No A are B. Means: A intersection B is empty, $A \cap B = \emptyset$. No trees are evergreens. (False)	Form: Some A are B. Means: A intersection B is not empty, $A \cap B \neq \emptyset$. Some trees are evergreens. (True)
Form: Some A are not B. Means: A is not a subset of B, $A \not\subset B$. Some horses are not mustangs. (True)	Form: All A are B. Means: A is a subset of B, $A \subset B$. All horses are mustangs. (False)

We covered sets in great detail in Chapter 1. To review, " A is a subset of B" means that every member of set A is also a member of set B. The intersection of two sets A and B is the set of all elements that they share in common. If A intersection B is the empty set, then sets A and B do not have any elements in common. The two sets do not overlap. They are disjoint.

Hearn Math
Logical Statements & Quantifiers

❓ Exercise 2.1.7: Negating Statements Containing Quantifiers *All*, *Some*, or *None*

Given the statements:

p: All leopards have spots.
r: Some apples are red.
s: No lemons are sweet.

Write each of the following symbolic statements in words.

1. $\sim p$
2. $\sim r$
3. $\sim s$

Answer

1. The statement "All leopards have spots" is p and has the form "All A are B." According to Table 2.1.2, the negation will have the form "Some A are not B." The negation of p is the statement, "Some leopards do not have spots."
2. The statement "Some apples are red" has the form "Some A are B." This indicates that the categories A and B overlap or intersect. According to Table 2.1.2, the negation will have the form, "No A are B," indicating that A and B do not intersect. This results in the opposite truth value of the original statement, so the negation of "Some apples are red" is the statement: "No apples are red."

3. Because s is the statement: "No lemons are sweet," s is asserting that the set of lemons does not intersect with the set of sweet things. The negation of $s, \sim s$, must make the opposite claim. It must indicate that the set of lemons intersects with the set of sweet things. This means at least one lemon must be sweet. The statement, "Some lemons are sweet" is $\sim s$. The negation of the statement, "No A are B," is the statement, "Some A are B," as indicated in Table 2.1.2.

✓ Your Turn 2.1.7

Given the statements:
$\sim p$: Some apples are not sweet.
r : No triangles are squares.
s : Some vegetables are green. Write each of the following symbolic statements in words.

1. p
2. $\sim r$
3. $\sim s$

Check Your Understanding

1. A __________ ___________ is a complete sentence that makes a claim that may be either true or false.
2. The ___________________ of a logical statement has the opposite truth value of the original statement.
3. If p represents the logical statement, “Marigolds are yellow flowers,” then ______ represents the statement, “Marigolds are not yellow flowers.”
4. The statement $(\ p)$ has the same truth value as the statement ________.
5. The logical statements used to support the conclusion of an argument are called _____________.
6. _________________________ arguments attempt to draw a general conclusion from specific premises.
7. All, some, and none are examples of ________________________, words that assign a numerical relationship between two or more groups.
8. The negation of the statement, “All giraffes are tall,” is __________________________________.

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2.1.0: Exercises

For the following exercises, determine whether the sentence represents a logical statement. If it is a logical statement, determine whether it is true or false.

? Exercise 2.1.0.1

A loan used to finance a house is called a mortgage.

? Exercise 2.1.0.2

All odd numbers are divisible by 2.

? Exercise 2.1.0.3

Please, bring me that notebook.

? Exercise 2.1.0.4

Robot, what's your function?

? Exercise 2.1.0.5

In English, a conjunction is a word that connects two phrases or parts of a sentence together.

? Exercise 2.1.0.6

$8-3=5$.

? Exercise 2.1.0.7

$7+3=11$.

? Exercise 2.1.0.8

What is 7 plus 3?

For the following exercises, write each statement in symbolic form.

? Exercise 2.1.0.9

Grammy award winning singer, Lady Gaga, was not born in Houston, Texas.

? Exercise 2.1.0.10

Bruno Mars performed during the Super Bowl halftime show twice.

? Exercise 2.1.0.11

Coco Chanel said, “The most courageous act is still to think for yourself. Aloud.”

? Exercise 2.1.0.12

Bruce Wayne is not Superman.

For the following exercises, write the negation of each statement in words.

? Exercise 2.1.0.13

Bozo is not a clown.

? Exercise 2.1.0.14

Ash is Pikachu’s trainer and friend.

? Exercise 2.1.0.15

Vanilla is the most popular flavor of ice cream.

? Exercise 2.1.0.16

Smaug is a fire breathing dragon.

? Exercise 2.1.0.17

Elephant and Piggy are not best friends.

? Exercise 2.1.0.18

Some dogs like cats.

? Exercise 2.1.0.19

Some donuts are not round.

? Exercise 2.1.0.20

All cars have wheels.

? Exercise 2.1.0.21

No circles are squares.

? Exercise 2.1.0.22

Nature’s first green is not gold.

? Exercise 2.1.0.23

The ancient Greek philosopher Plato said, “The greatest wealth is to live content with little.”

? Exercise $2.1.0.24$

All trees produce nuts.

For the following exercises, write the negation of each statement symbolically and in words.

? Exercise $2.1.0.25$

p: Their hair is red.

? Exercise $2.1.0.26$

q: My favorite superhero does not wear a cape.

? Exercise $2.1.0.27$

s: All wolves howl at the moon.

? Exercise $2.1.0.28$

t: Nobody messes with Texas.

? Exercise $2.1.0.29$

u: I do not love New York.

? Exercise $2.1.0.30$

v: Some cats are not tigers.

? Exercise $2.1.0.31$

q: No squares are not parallelograms.

? Exercise $2.1.0.32$

p: The President does not like broccoli.

For the following exercises, write each of the following symbolic statements in words.

? Exercise $2.1.0.33$

Given: p: Kermit is a green frog; translate p into words.

? Exercise $2.1.0.34$

Given: r: Mick Jagger is not the lead singer for The Rolling Stones; translate $r$into words.

? Exercise $2.1.0.35$

Given: q: All dogs go to heaven; translate q into words.

Exercise 2.1.0.36

Given: s: Some pizza does not come with pepperoni on it; translate s into words.

Exercise 2.1.0.37

Given: p: No pizza comes with pineapple on it; translate (p) into words.

Exercise 2.1.0.38

Given: r: Not all roses are red; translate (r) into words.

Exercise 2.1.0.39

Given: t: Thelonious Monk is not a famous jazz pianist; translate (t) into words.

Exercise 2.1.0.40

Given: v: Not all violets are blue; translate (v) into words.

For the following exercises, draw a logical conclusion from the premises that includes one of the following quantifiers: all, some, or none.

Exercise 2.1.0.41

The Ford Motor Company builds cars in Michigan. General Motors builds cars in Michigan. Chrysler builds cars in Michigan. What conclusion can be drawn from these premises?

Exercise 2.1.0.42

Michelangelo Buonarroti was a great Renaissance artist from Italy. Raphael Sanzio was a great Renaissance artist from Italy. Sandro Botticelli was a great Renaissance artist from Italy. What conclusion can you draw from these premises?

Exercise 2.1.0.43

Four is an even number and it is divisible by 2. Six is an even number and it is divisible by 2. Eight is an even number and it is divisible by 2. What conclusion can you draw from these premises?

Exercise 2.1.0.44

Three is an odd number and it is not divisible by 2. Seven is an odd number and it is not divisible by 2. Twenty-one is an odd number and it is not divisible by 2. What conclusion can you draw from these premises?

Exercise 2.1.0.45

The odd number 5 is not divisible by 3. The odd number 7 is not divisible by 3. The odd number 29 is not divisible by 3. What conclusion can you draw from these premises?

Exercise 2.1.0.46

Penguins are flightless birds. Emus are flightless birds. Ostriches are flightless birds. What conclusion can you draw from these premises?

? Exercise 2.1.0.47

Plants need water to survive. Animals need water to survive. Bacteria need water to survive. What conclusion can you draw from these premises?

? Exercise 2.1.0.48

A chocolate chip cookie is not sour. An oatmeal cookie is not sour. An Oreo cookie is not sour. What conclusion can you draw from these premises?

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2.2: Compound Statements

2.2.1:A person seeking their driver's license must pass two tests. A compound statement can be used to explain performance on both tests at once. (credit: modification of work “Drivers License -Teen driver” by State Farm/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

1. Translate compound statements into symbolic form.
2. Translate compound statements in symbolic form with parentheses into words.
3. Apply the dominance of connectives.

Suppose your friend is trying to get a license to drive. In most places, they will need to pass some form of written test proving their knowledge of the laws and rules for driving safely. After passing the written test, your friend must also pass a road test to demonstrate that they can perform the physical task of driving safely within the rules of the law.

Consider the statement: "My friend passed the written test, but they did not pass the road test." This is an example of a compound statement, a statement formed by using a connective to join two independent clauses or logical statements. The statement, “My friend passed the written test,” is an independent clause because it is a complete thought or idea that can stand on its own. The second independent clause in this compound statement is, “My friend did not pass the road test.” The word "but" is the connective used to join these two statements together, forming a compound statement. So, did your friend acquire their driving license?.

This section introduces common logical connectives and their symbols, and allows you to practice translating compound statements between words and symbols. It also explores the order of operations, or dominance of connectives, when a single compound statement uses multiple connectives.

Common Logical Connectives

Understanding the following logical connectives, along with their properties, symbols, and names, will be key to applying the topics presented in this chapter. The chapter will discuss each connective introduced here in more detail.

The joining of two logical statements with the word "and" or "but" forms a compound statement called a conjunction. In logic, for a conjunction to be true, all the independent logical statements that make it up must be true. The symbol for a conjunction is $\wedge$. Consider the compound statement, “Derek Jeter played professional baseball for the New York Yankees, and he was a shortstop.” If p represents the statement, “Derrick Jeter played professional baseball for the New York Yankees,” and if q represents the statement, “Derrick Jeter was a short stop,” then the conjunction will be written symbolically as $p \wedge q$.

The joining of two logical statements with the word “or” forms a compound statement called a disjunction. Unless otherwise specified, a disjunction is an inclusive *or* statement, which means the compound statement formed by joining two independent clauses with the word *or* will be true if a least one of the clauses is true. Consider the compound statement, "The office manager ordered cake for for an employee’s birthday or they ordered ice cream.” This is a disjunction because it combines the independent clause, “The office manager ordered cake for an employee’s birthday,” with the independent clause, “The office manager ordered ice cream,” using the connective, *or*. This disjunction is true if the office manager ordered only cake, only ice cream, or they ordered both cake and ice cream. Inclusive *or* means you can have one, or the other, or both!

Joining two logical statements with the word *implies*, or using the phrase “if *first statement*, then *second statement,”* is called a conditional or *implication*. The clause associated with the "if" statement is also called the hypothesis or *antecedent*, while the clause following the "then" statement or the word *implies* is called the **conclusion** or *consequent*. The conditional statement is like a one-way contract or promise. The only time the conditional statement is false, is if the hypothesis is true and the conclusion is false. Consider the following conditional statement, “If Pedro does his homework, then he can play video games.” The hypothesis/antecedent is the statement following the word *if*, which is “Pedro does/did his homework.” The conclusion/consequent is the statement following the word *then*, which is “Pedro can play his video games.”

Joining two logical statements with the connective phrase “if and only if” is called a biconditional. The connective phrase "if and only if" is represented by the symbol, $\leftrightarrow$. In the biconditional statement, $p \leftrightarrow q$, p is called the hypothesis or antecedent and q is called the conclusion or consequent. For a biconditional statement to be true, the truth values of p and q must match. They must both be true, or both be false.

The table below lists the most common connectives used in logic, along with their symbolic forms, and the common names used to describe each connective.

Table 2.2.1

Connective	Symbol	Name
and but	$\wedge$	conjunction
or	$\vee$	disjunction, inclusive or
not	$\sim$	negation
if ..., then implies	$\rightarrow$	conditional, implication
if and only if	$\leftrightarrow$	biconditional

Checkpoint

These connectives, along with their names, symbols, and properties, will be used throughout this chapter and should be memorized.

Exercise 2.2.1

Associate Connectives with Symbols and Names

For each of the following connectives, write its name and associated symbol.

1. or
2. implies
3. but

Answer

A compound statement formed with the connective word *or* is called a disjunction, and it is represented by the ∨ symbol.

A compound statement formed with the connective word *implies* or phrase “*if …, then”* is called a conditional statement or implication and is represented by the → symbol.

A compound statement formed with the connective words *but* or *and* is called a conjunction, and it is represented by the ∧ symbol.

Your Turn 2.2.1

For each connective write its name and associated symbol.

1. not

2. and
3. if and only if

Translating Compound Statements to Symbolic Form

To translate a compound statement into symbolic form, we take the following steps.

1. Identify and label all independent affirmative logical statements with a lower case letter, such as p, q, or r.
2. Identify and label any negative logical statements with a lowercase letter preceded by the negation symbol, such as $\sim p$, $\sim q$, or $\sim r$.
3. Replace the connective words with the symbols that represent them, such as $\wedge, \vee, \rightarrow, \text{or} \leftrightarrow$.

Consider the previous example of your friend trying to get their driver's license. Your friend passed the written test, but they did not pass the road test. Let p represent the statement, "My friend passed the written test." And, let $\sim q$ represent the statement, "My friend did not pass the road test." Because the connective *but* is logically equivalent to the word *and*, the symbol for *but* is the same as the symbol for *and*; replace *but* with the symbol $\wedge$. The compound statement is symbolically written as: $p \wedge \sim q$. My friend passed the written test, but they did not pass the road test.

Checkpoint

When translating English statements into symbolic form, do not assume that every connective "and" means you are dealing with a compound statement. The statement, "The stripes on the U.S. flag are red and white," is a simple statement. The word "white" doesn't express a complete thought, so it is not an independent clause and does not get its own symbol. This statement should be represented by a single letter, such as s*: The stripes on the U.S. flag are red and white.*

Exercise 2.2.2: Translating Compound Statements into Symbolic Form

Let p represent the statement, "It is a warm sunny day," and let q represent the statement, "the family will go to the beach." Write the symbolic form of each of the following compound statements.

1. If it is a warm sunny day, then the family will go to the beach.
2. The family will go to the beach, and it is a warm sunny day.
3. The family will not go to the beach if and only if it is not a warm sunny day.
4. The family not go to the beach, or it is a warm sunny day.

Answer

1. Replace "it is a warm sunny day" with p. Replace "the family will go to the beach." with q. Next. Next, because the connective is if ..., then place the conditional symbol, $\rightarrow$, between p and q. The compound statement is written symbolically as: $p \rightarrow q$.
2. Replace "The family will go to the beach" with q. Replace "it is a warm sunny day." with p. Next, because the connective is and, place the $\wedge$ symbol between q and p. The compound statement is written symbolically as: $q \wedge p$.
3. Replace "The family will not go to the beach. with $\sim q$. Replace "it is not a warm sunny day" with $\sim p$. Next, because the connective is or, if and only if, place the biconditional symbol, $\leftrightarrow$ between $\sim q$ and $\sim p$. The compound statement is written symbolically as: $\sim q \leftrightarrow \sim p$.
4. Replace "The family will not go to the beach" with $\sim q$. Replace "it is a warm sunny day" with p. Next, because the connective is or, place the $\vee$ symbol between $\sim q$ and p. The compound statement is written symbolically as: $\sim q \vee p$.

Your Turn 2.2.2

Let p represent the statement, "Last night it snowed," and let q represent the statement, "Today we will go skiing." Write the symbolic form of each of the following compound statements:

1. Today we will go skiing, but last night it did not snow.
2. Today we will go skiing if and only if it snowed last night.
3. Last night is snowed or today we will not go skiing.
4. If it snowed last night, then today we will go skiing.

Translating Compound Statements in Symbolic Form with Parentheses into Words

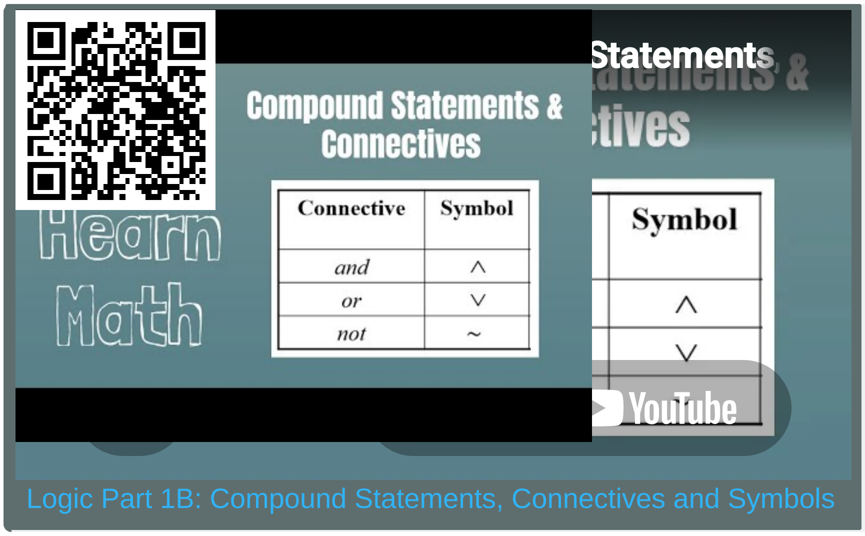
When parentheses are written in a logical argument, they group a compound statement together just like when calculating numerical or algebraic expressions. Any statement in parentheses should be treated as a single component of the expression. If multiple parentheses are present, work with the inner most parentheses first.

Consider your friend's struggles to get their license to drive. Let p represent the statement, "My friend passed the written test," let q represent the statement, "My friend passed the road test," and let r represent the statement, "My friend received a driver's license." The statement $(p \wedge q) \rightarrow r$ can be translated into words as follows: the statement $p \wedge q$ is grouped together to form the hypothesis of the conditional statement and r is the conclusion. The conditional statement has the form "if $p \wedge q$, then r." Therefore, the written form of this statement is: "If my friend passed the written test and they passed the road test, then my friend received a driver's license."

Sometimes a compound statement within parentheses may need to be negated as a group. To accomplish this, add the phrase, "it is not the case that" before the translation of the phrase in parentheses. For example, using p, q, and r of your friend obtaining a license, let's translate the statement $\sim (p \wedge q) \rightarrow \sim r$ into words.
In this case, the hypothesis of the conditional statement is $\sim (p \wedge q)$ and the conclusion is $\sim r$. To negate the hypothesis, add the phrase "it is not the case" before translating what is in parentheses. The translation of the hypothesis is the sentence, "It is not the case that my friend passed the written test and they passed the road test," and the translation of the conclusion is, "My friend did not receive a driver's license." So, a translation of the complete conditional statement, $\sim (p \wedge q) \rightarrow \sim r$ is: "If it is not the case that my friend passed the written test and the road test, then my friend did not receive a driver's license."

Checkpoint

It is acceptable to interchange proper names with pronouns and remove repeated phrases to make the written statement more readable, as long the meaning of the logical statement is not changed.



Connective	Symbol
and	$\wedge$
or	$\vee$
not	$\sim$

Logic Part 1B: Compound Statements, Connectives and Symbols

Exercise 2.2.3

Let p represent the statement, "My child finished their homework," let q represent the statement, "My child cleaned her room," let r represent the statement, "My child played video games," and let s represent the statement, "My child streamed a movie." Translate each of the following symbolic statements into words.

1. $\sim (p \wedge q)$
2. $(p \wedge q) \rightarrow (r \vee s)$
3. $\sim (r \vee s) \leftrightarrow \sim (p \wedge q)$

Answer

1. Replace with "It is not the case," and $\wedge$ with "and." One possible translation is: "It is not the case that my child finished their homework and cleaned their room."
2. The hypothesis of the conditional statement is, "My child finished their homework and cleaned their room." The conclusion of the conditional statement is, "My child played video games or streamed a movie." One possible translation of the entire statement is: "If my child finished their homework and cleaned their room, then they played video games or streamed a movie."
3. The hypothesis of the biconditional statement is $\sim(r \vee s)$ and is written in words as: "It is not the case that my child played video games or streamed a movie." The conclusion of the biconditional statement is $\sim(p \wedge q)$, which translates to: "It is not the case that my child finished their homework and cleaned their room." Because the biconditional, $\leftrightarrow$ translates to if and only if, one possible translation of the statement is: "It is not the case that my child played video games or streamed a movie if and only if it is not the case that my child finished their homework and cleaned their room."

✓ Your Turn 2.2.3

Let p represent the statement, "My roommates ordered pizza," let q represent the statement, "I ordered wings," and let r be the statement, "Our friends came over to watch the game." Translate the following statements into words.

1. $\sim r \rightarrow (p \vee q)$
2. $(p \wedge q) \rightarrow r$
3. $\sim(p \vee r)$

The Dominance of Connectives

The order of operations for working with algebraic and arithmetic expressions provides a set of rules that allow consistent results. For example, if you were presented with the problem $1 + 3 \times 2$, and you were not familiar with the order of operation, you might assume that you calculate the problem from left to right. If you did so, you would add 1 and 3 to get 4, and then multiply this answer by 2 to get 8, resulting in an incorrect answer. Try inputting this expression into a scientific calculator. If you do, the calculator should return a value of 7, not 8.

Use the Scientific Calculator found under the Resources tab to the far left

The order of operations for algebraic and arithmetic operations states that all multiplication must be applied prior to any addition. Parentheses are used to indicate which operation—addition or multiplication—should be done first. Adding parentheses can change and/or clarify the order. The parentheses in the expression $1 + (3 \times 2)$ indicate that 3 should be multiplied by 2 to get 6, and then 1 should be added to 6 to get 7: $1 + (3 \times 2) = 7$.

As with algebraic expressions, there is a set of rules that must be applied to compound logical statements in order to evaluate them with consistent results. This set of rules is called the dominance of connectives. When evaluating compound logical statements, connectives are evaluated from least dominant to most dominant as follows:

1. Parentheses are the least dominant connective. So, any expression inside parentheses must be evaluated first. Add as many parentheses as needed to any statement to specify the order to evaluate each connective.
2. Next, we evaluate negations.
3. Then, we evaluate conjunctions and disjunctions from left to right, because they have equal dominance.
4. After evaluating all conjunctions and disjunctions, we evaluate conditionals.
5. Lastly, we evaluate the most dominant connective, the biconditional. If a statement includes multiple connectives of equal dominance, then we will evaluate them from left to right.

See Figure $2.2.2$ for a visual breakdown of the dominance of connectives.

Dominance	Connective	Symbol	Evaluate
Least Dominant	Parentheses	( )	First
↓	Negation	~	↓
↓	Disjunction/Conjunction	∨, ∧	Left to right or add parentheses to specify order because or/and have equal dominance.
↓	Conditional	→	↓
Most Dominant	Biconditional	↔	Last

Figure 2.2.2

Let’s revisit your friend’s struggles to get their driver’s license. Let p represent the statement, “My friend passed the written test,” let q represent the statement, “My friend passed the road test,” and let r represent the statement, “My friend received a driver’s license.” Let's use the dominance of connectives to determine how the compound statement $p \wedge {\sim} q \to r$ should be evaluated.

Step 1: There are no parentheses, which is least dominant of all connectives, so we can skip over that.

Step 2: Because negation is the next least dominant, we should evaluate ${\sim} q$ first. We could add parentheses to the statement to make it clearer which connecting needs to be evaluated first: $p \wedge {\sim} q \to r$ is equivalent to $p \wedge ({\sim} q) \to r$.

Step 3: Next, we evaluate the conjunction, $\wedge$. $p \wedge ({\sim} q) \to r$ is equivalent to $(p \wedge ({\sim} q)) \to r$.

Step 4: Finally, we evaluate the conditional, $\to$, as this is the most dominant connective in this compound statement.

Who Knew?

When using spreadsheet applications, like Microsoft Excel or Google Sheets, have you noticed that core functions, such as sum, average, or rate, have the same name and syntax for use? This is not a coincidence. Various standards organizations have developed requirements that software developers must implement to meet the conditions set by governments and large customers worldwide. The OpenDocument Format OASIS Standard enables transferring data between different office productivity applications and was approved by the International Standards Organization (ISO) and IEC on May 6, 2006.

Prior to adopting these standards, a government entity, business, or individual could lose access to their own data simply because it was saved in a format no longer supported by a proprietary software product—even data they were required by law to preserve, or data they simply wished to maintain access to.

Just as rules for applying the dominance of connectives help maintain understanding and consistency when writing and interpreting compound logical statements and arguments, the standards adopted for database software maintain global integrity and accessibility across platforms and user bases.

? Exercise 2.2.4

For each of the following compound logical statements, add parentheses to indicate the order to evaluate the statement. Recall that parentheses are evaluated innermost first.

1. $p \wedge \sim q \vee r$
2. $q \to \sim p \wedge r$
3. $\sim (p \vee q) \leftrightarrow \sim p \wedge \sim q$

Answer

1. Because negation is the least dominant connective, we evaluate it first: $p \wedge (\sim q) \vee r$. Because conjunction and disjunction have the same dominance, we evaluate them left to right. So, we evaluate the conjunction next, as indicated by the additional set of parentheses: $(p \wedge (\sim q)) \vee r$. The only remaining connective is the disjunction, so it is evaluated

last, as indicated by the third set of parentheses. The complete solution is: $((p \wedge (\sim q)) \vee r)$.

2. Negation has the lowest dominance, so it is evaluated first: $q \rightarrow (\sim p) \wedge r$. The remaining connectives are the conditional and the conjunction. Because conjunction has a lower precedence than the conditional, it is evaluated next, as indicated by the second set of parentheses: $q \rightarrow ((\sim p) \wedge r)$. The last step is to evaluate the conditional, as indicated by the third set of parentheses: $(q \rightarrow ((\sim p) \wedge r))$.
3. This statement is known as De Morgan's Law for the negation of a disjunction. It is always true. Section 2.6 of this chapter will explore De Morgan's Laws in more detail.
   - First, we evaluate the negations on the right side of the biconditional prior to the conjunction.
   - Then, we evaluate the disjunction on the left side of the biconditional, followed by the negation of the disjunction on the left side.
   - Lastly, after completely evaluating each side of the biconditional, we evaluate the biconditional. It does not matter which side you begin with.

   The final solution is: $(\sim (p \vee q)) \leftrightarrow ((\sim p) \wedge (\sim q))$.

✓ Example 2.2.4

For each of the following compound logical statements, add parentheses to indicate the order in which to evaluate the statement. Recall that parentheses are evaluated innermost first.

1. $p \vee q \wedge \sim r$
2. $\sim p \rightarrow q \vee r$
3. $\sim p \vee \sim q \leftrightarrow \sim (p \wedge q)$

WORK IT OUT: Logic Terms and Properties – Matching Game

Materials: For every group of four students, include 30 cards (game, trading, or playing cards), 30 individual clear plastic gaming card sleeves, and 30 card-size pieces of paper. Prepare a list of 60 questions and answers ahead of time related to definitions and problems in Statements and Quantifiers and Compound Statements. Provide each group of four students with 20 questions and their associated answers. Instruct each group to select 15 of the 20 questions, and then, for each problem selected, create one card with the question and one card with the answer. Instruct the groups to then place each problem and answer in a separate card sleeve. Once they create 15 problem cards and 15 answer cards, have students shuffle the set of cards.

To play the game, groups should exchange their card decks to ensure no team begins playing with the deck that they created. Split each four-person group into teams of two students. After shuffling the cards, one team lays cards face down on their desk in a five-by-six grid. The other team will go first.

Each player will have a turn to try matching the question with the correct answer by flipping two cards to the face up position. If a team makes a match, they get to flip another two cards; if they do not get a match, they flip the cards face down and it is the other team’s turn to flip over two cards. The game continues in this manner until teams match all question cards with their corresponding answer cards. The team with the most set of matching cards wins.

In the first module of this chapter, we evaluated the truth value of negations. In the following modules, we will discuss how to evaluate conjunctions, disjunctions, conditionals, and biconditionals, and then evaluate compound logical statements using truth tables and our knowledge of the dominance of connectives.

Check Your Understanding

1. A ___________ ___________ is a logical statement formed by combining two or more statements with connecting words, such as and, or, but, not, and if …, then.
2. A ______ is a word or symbol used to join two or more logical statements together to form a compound statement.
3. The most dominant connective is the ______.
4. ______ are used to specify which logical connective should be evaluated first when evaluating a compound statement.
5. Both ______ and ______ have equal dominance and are evaluated from left to right when no parentheses are present in a compound logical statement.

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2.2.0: New Page

For the following exercises, translate each compound statement into symbolic form. Given p: “Layla has two weeks for vacation,” q: “Marcus is Layla’s friend,” r: “Layla will travel to Paris, France,” and s: “Layla and Marcus will travel together to Niagara Falls, Ontario.”

? Exercise 2.2.0.1

If Layla has two weeks for vacation, then she will travel to Paris, France.

? Exercise 2.2.0.2

Layla and Marcus will travel together to Niagara Falls, Ontario or Layla will travel to Paris, France.

? Exercise 2.2.0.3

If Marcus is not Layla’s friend, then they will not travel to Niagara Falls, Ontario together.”

? Exercise 2.2.0.4

Layla and Marcus will travel to Niagara Falls, Ontario together if and only if Layla and Marcus are friends.

? Exercise 2.2.0.5

If Layla does not have two weeks for vacation and Marcus is Layla’s friend, then Marcus and Layla will travel together to Niagara Falls, Ontario.

? Exercise 2.2.0.6

If Layla has two weeks for vacation and Marcus is not her friend, then she will travel to Paris, France.

For the following exercises, translate each compound statement into symbolic form. Given p: “Tom is a cat,” q: “Jerry is a mouse,” r: “Spike is a dog,” s: “Tom chases Jerry,” and t: “Spike catches Tom.”

? Exercise 2.2.0.7

Jerry is a mouse and Tom is a cat.

? Exercise 2.2.0.8

If Tom chases Jerry, then Spike will catch Tom.

? Exercise 2.2.0.9

If Spike does not catch Tom, then Tom did not chase Jerry.

? Exercise 2.2.0.10

Tom is a cat and Spike is a dog, or Jerry is not a Mouse.

Exercise 2.2.0.11

It is not the case that Tom is not a cat and Jerry is not a mouse.

Exercise 2.2.0.12

Spike is not a dog and Jerry is a mouse if and only if Tom chases Jerry, but Spike does not catch Tom.

For the following exercises, translate the symbolic form of each compound statement into words. Given p: “Tracy Chapman plays guitar,” q: “Joan Jett plays guitar,” r: “All rock bands include guitarists,” and s: “Elton John plays the piano.”

Exercise 2.2.0.13

$p \vee r$

Exercise 2.2.0.14

$\sim s \rightarrow \sim q$

Exercise 2.2.0.15

$(p \wedge q) \leftrightarrow r$

Exercise 2.2.0.16

$\sim r \wedge (q \vee s)$

Exercise 2.2.0.17

$\sim (p \wedge \sim q)$

Exercise 2.2.0.18

$(q \rightarrow \sim r) \leftrightarrow (\sim p \vee \sim r)$

For the following exercises, translate the symbolic form of each compound statement into words. Given p: “The median is the middle number,” q: “The mode is the most frequent number,” r: “The mean is the average number,” s: “The median, mean, and mode are equal,” and t: “The data set is symmetric.”

Exercise 2.2.0.19

$t \rightarrow s$

Exercise 2.2.0.20

$p \wedge (q \wedge r)$

Exercise 2.2.0.21

$\sim t \rightarrow \sim s$

Exercise 2.2.0.22

$(r \wedge p) \leftrightarrow q$

Exercise 2.2.0.23

$(t \rightarrow \sim q) \vee (r \rightarrow s)$

Exercise 2.2.0.24

$\sim (q \vee r) \rightarrow t$

For the following exercises, apply the proper dominance of connectives by adding parentheses to indicate the order in which the statement must be evaluated.

Exercise 2.2.0.25

$p \rightarrow q \vee r$

Exercise 2.2.0.26

$p \wedge q \leftrightarrow \sim r$

Exercise 2.2.0.27

$p \vee r \vee \sim q$

Exercise 2.2.0.28

$p \wedge \sim q \wedge r$

Exercise 2.2.0.29

$p \wedge r \vee s \wedge t$

Exercise 2.2.0.30

$q \rightarrow \sim r \leftrightarrow \sim p \vee \sim r$

Exercise 2.2.0.31

$p \rightarrow r \vee s \leftrightarrow \sim t$

Exercise 2.2.0.32

$\sim (t \wedge s) \vee (p \rightarrow q) \wedge \sim r$

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2.3: Constructing Truth Tables

Figure 2.3.1: Just like solving a puzzle, a computer programmer must consider all potential solutions in order to account for each possible outcome. (credit: modification of work "Deadline Xmas 2010" by Allan Henderson/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

1. Interpret and apply negations, conjunctions, and disjunctions.
2. Construct a truth table using negations, conjunctions, and disjunctions.
3. Construct a truth table for a compound statement and interpret its validity.

Are you familiar with the Choose Your Own Adventure book series written by Edward Packard? These gamebooks allow the reader to become one of the characters and make decisions that affect what happens next, resulting in different sequences of events in the story and endings based on the choices made. Writing a computer program is a little like what it must be like to write one of these books. The programmer must consider all the possible inputs that a user can put into the program and decide what will happen in each case, then write their program to account for each of these possible outcomes.

A truth table is a graphical tool used to analyze all the possible truth values of the component logical statements to determine the validity of a statement or argument along with all its possible outcomes. The rows of the table correspond to each possible outcome for the given logical statement identified at the top of each column. A single logical statement p has two possible truth values, true or false. In truth tables, a capital T will represent true values, and a capital F will represent false values.

In this section, you will use the knowledge built in Statements and Quantifiers and Compound Statements to analyze arguments and determine their truth value and validity. A logical argument is valid if its conclusion follows from its premises, regardless of whether those premises are true or false. You will then explore the truth tables for negation, conjunction, and disjunction, and use these truth tables to analyze compound logical statements containing these connectives.

Interpret and Apply Negations, Conjunctions, and Disjunctions

The negation of a statement will have the opposite truth value of the original statement. When p is true, $\sim p$ is false, and when p is false, $\sim p$ is true.

Exercise 2.3.1

For each logical statement, determine the truth value of its negation.

1. p: $3+5=8$.
2. q: All horses are mustangs.
3. $\sim r$: New Delhi is not the capital of India.

Answer

p is true because 3 + 5 does equal 8; therefore, the negation of p, ~ p : 3 + 5 ≠ 8 ~ p : 3 + 5 ≠ 8 , is false.

q is false because there are other types of horses besides mustangs, such as Clydesdales or Arabians; therefore, the negation of *q*, ~ *q* , is true.

~ *r* is false because New Delhi is the capital of India; therefore, the negation of ~*r*, *r* , is true.

✓ Your Turn ✓ 2.3.1

For each logical statement, determine the truth value of its negation.

1. p: $3 \times 5 = 14$.
2. q: Some houses are built with bricks
3. r: Abuja is the capital of Nigeria.

A conjunction is a logical *and* statement. For a conjunction to be true, both statements that make up the conjunction must be true. If at least one of the statements is false, the *and* statement is false.

1. $p \wedge q$
2. $\sim q \wedge r$
3. $\sim p \wedge q$

Answer

1. p is true, and q is false. Because one statement is true, and the other statement is false, this makes the complete conjunction false.
2. q is false, so $\sim q$ is true, and r is true. Therefore, both statements are true, making the complete conjunction true.
3. $\sim p$ is false, and q is false. Because both statements are false, the complete conjunction is false.

✓ Your Turn ✓ 2.3.2

Given p : Yellow is a primary color, q : Blue is a primary color, and r : Green is a primary color, determine the truth value of each conjunction.

1. $p \wedge q$
2. $q \wedge r$
3. $\sim r \wedge p$.

Checkpoint

The only time a conjunction is true is if both statements that make up the conjunction are true.

A disjunction is a logical inclusive *or* statement, which means that a disjunction is true if one or both statements that form it are true. The only way a logical inclusive *or* statement is false is if both statements that form the disjunction are false.

1. $p \vee q$
2. $\sim q \vee r$
3. $\sim p \vee q$

Answer

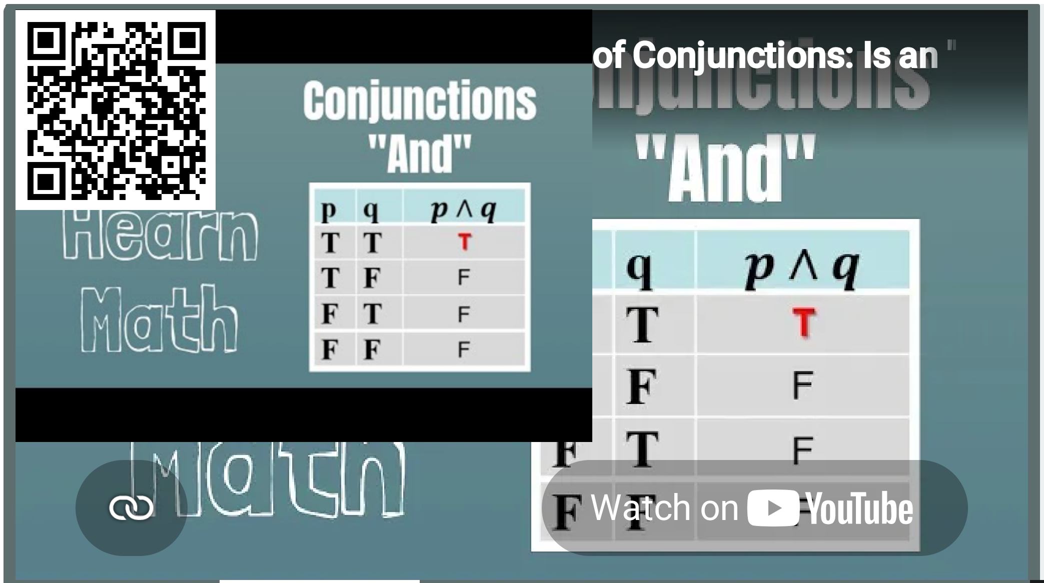
1. p **is true, and** q **is false. One statement is true, and one statement is false, which makes the complete disjunction true.**
2. q **is false, so** $\sim q$ **is true, and** r **is true. Therefore, one statement is true, and the other statement is true, which makes the complete disjunction true.**
3. $\sim p$ **is false, and** q **is false. When all of the component statements are false, the disjunction is false.**

✓ Your Turn ✓ 2.3.3

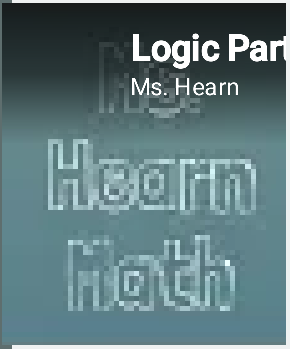
Given p : Yellow is a primary color, q : Blue is a primary color, and r : Green is a primary color, determine the truth value of each disjunction.

1. $p \vee q$
2. $\sim p \vee r$
3. $q \vee r$

In the next example, you will apply the dominance of connectives to find the truth values of compound statements containing negations, conjunctions, and disjunctions and use a table to record the results. When constructing a truth table to analyze an argument where you can determine the truth value of each component statement, the strategy is to create a table with two rows. The first row contains the symbols representing the components that make up the compound statement. The second row contains the truth values of each component below its symbol. Include as many columns as necessary to represent the value of each compound statement. The last column includes the complete compound statement with its truth value below it.


of Conjunctions: Is an
Conjunctions "And"
Hearn Math
p q p ∧ q
T T T
T F F
F T F
F F F
Watch on YouTube

Logic Part 2: Tr njunctions: Is an "AND" statement true or false?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Logic Part 3: Truth Values of Disjunctions: Is an "OR" statement true or false?

1. $\sim p \wedge q \vee r$
2. $\sim p \vee q \wedge r$
3. $\sim (p \wedge r) \vee q$

Answer

1. **Step 1:** The statement " $\sim p \wedge q \vee r$ " contains three basic logical statements, p, q, and r, and three connectives, $\sim, \wedge, \vee$. When we place parentheses in the statement to indicate the dominance of connectives, the statement becomes $(((\sim p) \wedge q) \vee r)$.
Step 2: After we have applied the dominance of connectives, we create a two row table that includes a column for each basic statement that makes up the compound statement, and an additional column for the contents of each parentheses. Because we have three sets of parentheses, we include a column for $\sim p$, the innermost parentheses, a column for $(\sim p) \wedge q$, the next set of parentheses, and $((\sim p) \wedge q) \vee r$ in the last column for the third parentheses.
Step 3: Once the table is created, we determine the truth value of each statement starting from left to right. The truth values of p, q, and r are true, false, and true, respectively, so we place T, F, and T in the second row of the table. Because p is true, $\sim p$ is false.
Step 4: Next, evaluate $\sim p \wedge q$ from the table: $\sim p$ is false, and q is also false, so $\sim p \wedge q$ is false, because a conjunction is only true if both of the statements that make it are true. Place an F in the table below its heading.
Step 5: Finally, using the table, we understand that $(\sim p \wedge q)$ is false and r is true, so the complete statement $(\sim p \wedge q) \vee r$ is false or true, which is true (because a disjunction is true whenever at least one of the statements that make it is true). Place a T in the last column of the table. The complete statement $\sim p \wedge q \vee r$ is true.

2. **Step 1**: Applying the dominance of connectives to the original compound statement $\sim p \vee q \wedge r$, we get $(((\sim p) \vee q) \wedge r)$.
Step 2: The table needs columns for $p, q, r, \sim p, (\sim p) \vee q$, and $((\sim p) \vee q) \wedge r$.
Step 3:The truth values of p, q, r, and $\sim p$ are the same as in Question 1.
Step 4: Next, $\sim p \vee q$ is false or false, which is false, so we place an F below this statement in the table. This is the only time that a disjunction is false.
Step 5: Finally, $\sim p \vee q$ and r are the conjunction of the statements, $\sim p \vee q$ and r, and so the expressions evaluate to false and true, which is false. Recall that the only time an "and" statement is true is when both statements that form it are also true. The complete statement $\sim p \vee q \wedge r$ is false.

3. **Step 1**: Applying the dominance of connectives to the original statement, we have: $((\sim (p \wedge r)) \vee q)$.
Step 2: So, the table needs the following columns: $p, q, r, p \wedge r, \sim (p \wedge r)$, and $\sim (p \wedge r) \vee q$.
Step 3: The truth values of p, q, and r are the same as in Questions 1 and 2.
Step 4: From the table it can be seen that $p \wedge r$ is true and true, which is true. So the negation of p and r is false, because the negation of a statement always has the opposite truth value of the original statement.
Step 5: Finally, $\sim (p \wedge r) \vee q$ is the disjunction of $\sim (p \wedge r)$ with q, and so we have false or false, which makes the complete statement false.

T	F	T	T	F	F

✓ Your Turn ✓ 2.3.4

Given p : Yellow is a primary color, q : Blue is a primary color, and r : Green is a primary color, determine the truth value of each compound statement, by constructing a truth table.

1. $\sim q \wedge p \vee r$
2. $p \vee q \wedge \sim r$
3. $\sim (p \wedge r) \wedge q$

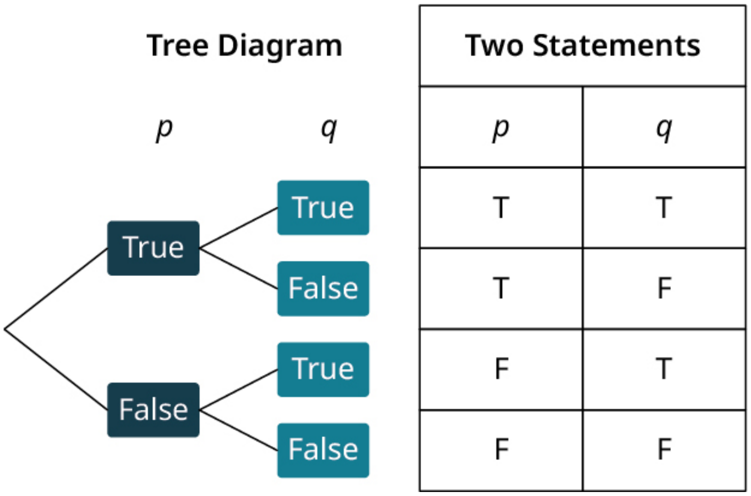
Construct Truth Tables to Analyze All Possible Outcomes

Recall from Statements and Questions that the negation of a statement will always have the opposite truth value of the original statement; if a statement p is false, then its negation $\sim p$ is true, and if a statement p is true, then its negation $\sim p$ is false. To create a truth table for the negation of statement p, add a column with a heading of $\sim p$, and for each row, input the truth value with the opposite value of the value listed in the column for p, as depicted in the table below.

Negation	
p	**~ p**
T	F
F	T

Conjunctions and disjunctions are compound statements formed when two logical statements combine with the connectives “and” and “or” respectively. How does that change the number of possible outcomes and thus determine the number of rows in our truth table? The multiplication principle, also known as the *fundamental counting principle*, states that the number of ways you can select an item from a group of n items and another item from a group with m items is equal to the product of m and n. Because each proposition p and q has two possible outcomes, true or false, the multiplication principle states that there will be $2 \times 2 = 4$ possible outcomes: {TT, TF, FT, FF}.

The tree diagram and table in Figure $2.3.2$ demonstrate the four possible outcomes for two propositions p and q.



Two Statements	
p	q
T	T
T	F
F	T
F	F

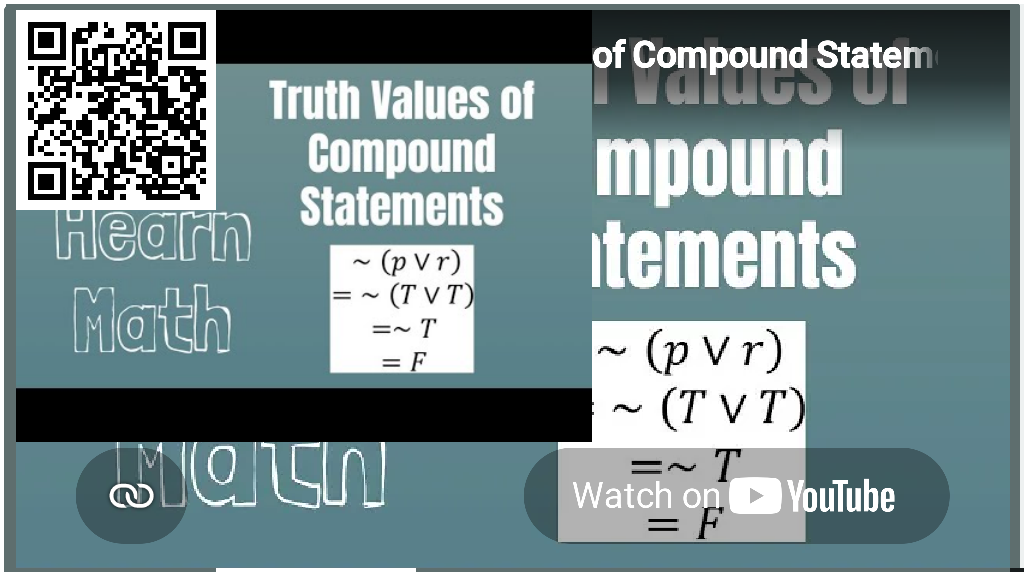
Figure $2.3.2$:

A conjunction is a logical *and* statement. For a conjunction to be true, both the statements that make up the conjunction must be true. If at least one of the statements is false, the *and* statement is false.

A disjunction is a logical inclusive *or* statement. Which means that a disjunction is true if one or both statements that make it are true. The only way a logical inclusive *or* statement is false is if both statements that make up the disjunction are false.

Conjunction (AND)				Disjunction (OR)		
p	q	$p \wedge q$		p	q	$p \vee q$
T	T	T		T	T	T
T	F	F		T	F	T

Conjunction (AND)				Disjunction (OR)		
p	q	$p \wedge q$		p	q	$p \vee q$
F	T	F		F	T	T
F	F	F		F	F	F


Truth Values of Compound Statements
$\sim(p \vee r)$
$= \sim(T \vee T)$
$=\sim T$
$= F$
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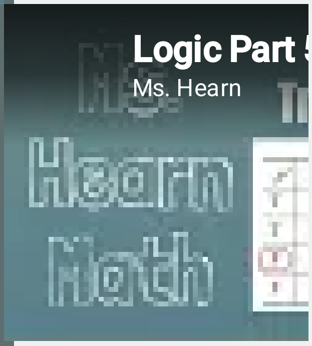
Logic Part 4: Tr ompound Statements with "and", "or", and "not"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Logic Part 5: What are truth tables? How do you set them up?

Exercise 2.3.5: Constructing Truth Tables to Analyze Compound Statements

Construct a truth table to analyze all possible outcomes for each of the following arguments.

1. $p \vee \sim q$
2. $\sim(p \wedge q)$
3. $(p \vee \sim q) \wedge r$

Answer

1. Step 1: Because there are two basic statements, p and q, and each of these has two possible outcomes, we will have $2(2) = 4$ rows in our table to represent all possible outcomes: TT, TF, FT, and FF. The columns will include $p, q, \sim q$, and $p \vee \sim q$.
Step 2: Every value in column $\sim q$ will have the opposite truth value of the corresponding value in column q : F, T, F, and T.
Step 3: To complete the last column, evaluate each element in column p with the corresponding element in column $\sim q$ using the connective or.

p	q	$\sim q$	$p \vee \sim q.$

p	q	$\sim q$	$p \vee \sim q.$
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

2. Step 1: The columns will include $p, q, p \wedge q$, and $\sim (p \wedge q)$. Because there are two basic statements, p and q, the table will have four rows to account for all possible outcomes.
Step 2: The $p \wedge q$ column will be completed by evaluating the corresponding elements in columns p and q respectively with the and connective.
Step 3: The final column, $\sim (p \wedge q)$, will be the negation of the $p \wedge q$ column.

p	q	$p \wedge q$	$\sim (p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

3. Step 1: This statement has three basic statements, p, q, and r. Because each basic statement has two possible truth values, true or false, the multiplication principal indicates there are $2(2)(2) = 8$ possible outcomes. So eight rows of outcomes are needed in the truth table to account for each possibility. Half of the eight possibilities must be true for the first statement, and half must be false.
Step 2: So, the first column for statement p, will have four T's followed by four F's. In the second column for statement q, when p is true, half the outcomes for q must be true and the other half must be false, and the same pattern will repeat for when p is false. So, column q will have TT, FF, FF, FF.

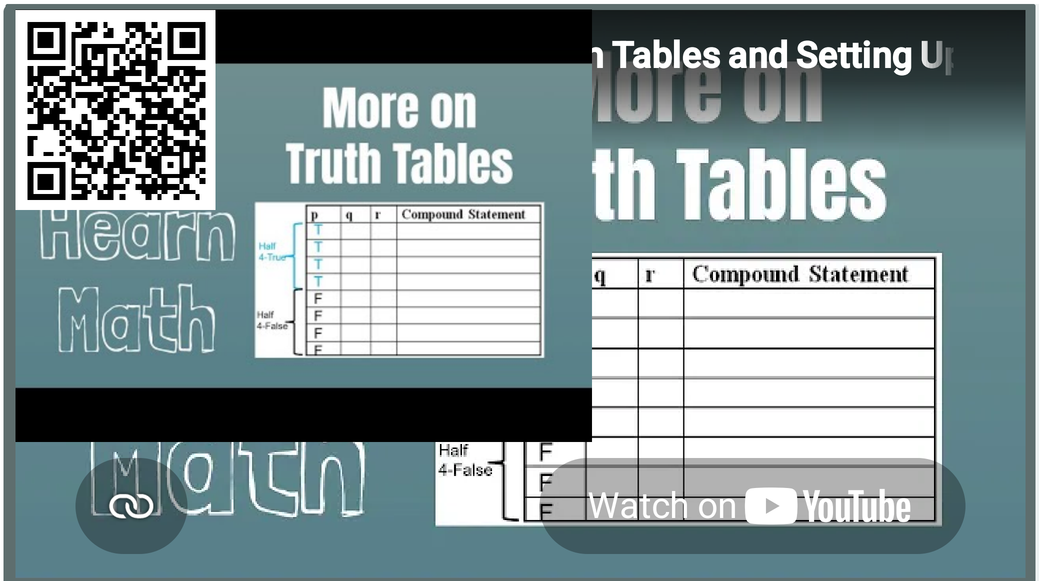
Step 3: The column for the third statement, r, must alternate between T and F. Once, the three basic propositions are listed, you will need a column for $\sim, p \vee \sim q$, and $(p \vee \sim q) \wedge r$.
Step 4: The column for the negation of $q, \sim q$, will have the opposite truth value of each value in column q.
Step 5: Next, fill in the truth values for the column containing the statement $p \vee \sim q$. The or statement is true if at least one of p or $\sim q$ is true, otherwise it is false.
Step 6: Finally, fill in the column containing the conjunction $(p \vee \sim q) \wedge r$. To evaluate this statement, combine column $p \vee \sim q$ and column r with the and connective. Recall, that only time "and" is true is when both values are true, otherwise the statement is false. The complete truth table is:

p	q	r	$\sim q$	$p \vee \sim q$	$(p \vee \sim q) \wedge r$
T	T	T	F	T	T
T	T	F	F	T	F
T	F	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	F
F	T	F	F	F	F
F	F	T	T	T	T
F	F	F	T	T	F

✓ Your Turn ✓ $2.3.5$

Construct a truth table to analyze all possible outcomes for each of the following arguments.

1. $p \wedge \sim q$
2. $\sim (p \vee q)$
3. $(p \wedge \sim q) \vee r$



Logic Part 6: More on Truth Tables and Setting Up Rows and Column Headings

Determine the Validity of a Truth Table for a Compound Statement

A logical statement is valid if it is always true regardless of the truth values of its component parts. To test the validity of a compound statement, construct a truth table to analyze all possible outcomes. If the last column, representing the complete statement, contains only true values, the statement is valid.

? Exercise $2.3.6$: Determining the Validity of Compound Statements

Construct a truth table to determine the validity of each of the following statements.

1. $\sim p \wedge q$
2. $\sim (p \wedge \sim p)$

Answer

1. Step 1: Because there are two statements, p and q, and each of these has two possible outcomes, there will be $2(2) = 4$ rows in our table to represent all possible outcomes: TT, TF, FT, and FF.
Step 2: The columns, will include $p, q, \sim p$ and $\sim p \wedge q$. Every value in column $\sim p$ will have the opposite truth value of the corresponding value in column p.
Step 3: To complete the last column, evaluate each element in column $\sim p$ with the corresponding element in column q using the connective and. The last column contains at least one false, therefore the statement $\sim p \wedge q$ is not valid.

p	q	$\sim p$	$\boldsymbol{\sim p \wedge q}$
T	T	F	F
T	F	F	F
F	T	T	T
F	F	T	F

2. **Step 1**: Because the statement $\sim (p\wedge \sim p)$ only contains one basic proposition, the truth table will only contain two rows. Statement p may be either true or false.
Step 2: The columns will include $p, \sim p, p\wedge \sim p$, and $\sim (p\wedge \sim p)$. Evaluate column $p\wedge \sim p$ with the and connective, because the symbol $\wedge$ represents a conjunction or logical and statement. True and false is false, and false and true is also false.
Step 3: The final column is the negation of each entry in the third column, both of which are false, so the negation of false is true. Because all the truth values in the final column are true, the statement $\sim (p\wedge \sim p)$ is valid.

p	$\sim p$	$p\wedge \sim p$	$\sim (p\wedge \sim p)$
T	F	F	T
F	T	F	T

Your Turn2.3.6

Construct a truth table to determine the validity of each of the following statements.
1. $p\vee \sim p$
2. $\sim p\vee \sim q$

Check Your Understanding

1. A logical argument is _____ if its conclusion follows from its premises.
2. A logical statement is valid if it is always _____.
3. A _____ _____ is a tool used to analyze all the possible outcomes for a logical statement.
4. The truth table for the conjunction, $p \wedge q$, has _____ rows of truth values.
5. The truth table for the negation of a logical statement, p, has _____ rows of truth values.

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2.3.0: Exercises

For the following exercises, find the truth value of each statement.

? Exercise 2.3.0.1

$p: 7 \times 3 = 21$. What is the truth value of $\sim p$?

? Exercise 2.3.0.2

q : The sun revolves around the Earth. What is the truth value of $\sim q$?

? Exercise 2.3.0.3

$\sim r$: The acceleration of gravity is $9.81 \mathrm{~m}/\mathrm{sec}^2$. What is the truth value of r ?

? Exercise 2.3.0.4

s : Dan Brown is not the author of the book, The Davinci Code. What is the truth value of $\sim(\sim s)$?

? Exercise 2.3.0.5

t : Broccoli is a vegetable. What is the truth value of $\sim(\sim t)$?

For the following exercises, given $p: 1+2=3, q$: Five is an even number, and r : Seven is a prime number, find the truth value of each of the following statements.

? Exercise 2.3.0.6

$\sim q$

? Exercise 2.3.0.7

$p \wedge q$

? Exercise 2.3.0.8

$p \vee q$

? Exercise 2.3.0.9

$\sim p \vee \sim q$

? Exercise 2.3.0.10

$p \wedge \sim q$

? Exercise 2.3.0.11

$p \wedge r$

Exercise 2.3.0.12

$q \wedge r$

Exercise 2.3.0.13

$q \wedge \sim r$

Exercise 2.3.0.14

$q \vee \sim r$

Exercise 2.3.0.15

$\sim (p \wedge r)$

Exercise 2.3.0.16

$p \vee q \wedge r$

Exercise 2.3.0.17

$\sim p \vee (q \wedge r)$

Exercise 2.3.0.18

$\sim (q \wedge r) \vee \sim p$

Exercise 2.3.0.19

$q \vee r \vee p$

Exercise 2.3.0.20

$\sim q \wedge r \wedge p$

For the following exercises, complete the truth table to determine the truth value of the proposition in the last column.

Exercise 2.3.0.21

p	q	r	$\sim p$	$\sim p \vee q$	$(\sim p \vee q) \wedge r$
T	T	T			

Exercise 2.3.0.22

p	q	r	$\sim p$	$\sim p \vee q$	$(\sim p \vee q) \wedge r$
F	T	F			

Exercise 2.3.0.23

p	q	r	$\sim p$	$\sim r$	$\sim p\wedge q(\sim p\wedge q)\vee\sim r$
F	F	F			

Exercise 2.3.0.24

p	q	r	$\sim p$	$\sim r$	$\sim p\vee q$	$(\sim p\vee q)\vee\sim r$
F	F	F				

For the following exercises, given p : All triangles have three sides, q : Some rectangles are not square, and r : A pentagon has eight sides, determine the truth value of each compound statement by constructing a truth table.

Exercise 2.3.0.25

$\sim r\wedge q\wedge p$

Exercise 2.3.0.26

$\sim(q\wedge p)\vee r$

Exercise 2.3.0.27

$\sim p\vee q\wedge r$

Exercise 2.3.0.28

$\sim p\vee\sim q\vee r$ For the following exercises, construct a truth table to analyze all the possible outcomes for the following arguments.

Exercise 2.3.0.29

$\sim q\wedge q$

Exercise 2.3.0.30

$\sim p\vee\sim q$

Exercise 2.3.0.31

$\sim p\wedge\sim q$

Exercise 2.3.0.32

$p\wedge q\vee r$ For the following exercises, construct a truth table to determine the validity of each statement.

Exercise 2.3.0.33

$\sim q\vee q$

? Exercise $2.3.0.34$

$p \wedge \sim q$

? Exercise $2.3.0.35$

$p \wedge q \vee \sim p$

? Exercise $2.3.0.36$

$(p \wedge q) \vee (\sim p \wedge \sim q)$

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2.4: Truth Tables for the Conditional and Biconditional

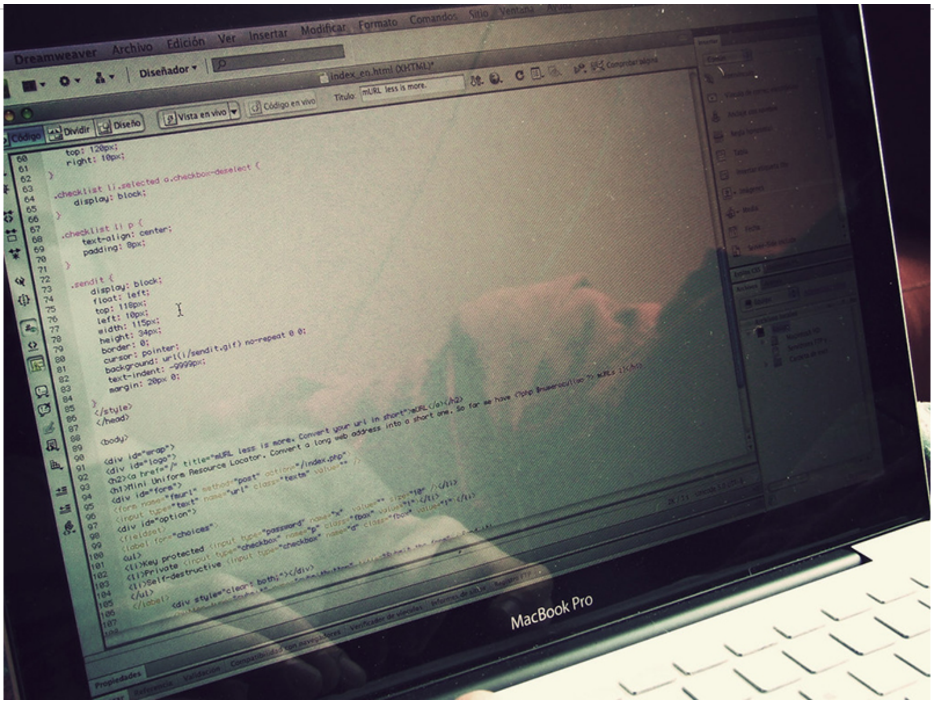


Figure $2.4.1$: If-then statements use logic to execute directions. (credit: “Coding” by Carlos Varela/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

1. Use and apply the conditional to construct a truth table.
2. Use and apply the biconditional to construct a truth table.
3. Use truth tables to determine the validity of conditional and biconditional statements.

Computer languages use if-then or if-then-else statements as decision statements:

- *If* the hypothesis is true, *then* do something.
- Or, *if* the hypothesis is true, *then* do something; else do something else.

For example, the following representation of computer code creates an if-then-else decision statement:

Check value of variable i.

If $i < 1$, then print "Hello, World!" else print "Goodbye".

In this imaginary program, the if-then statement evaluates and acts on the value of the variable i. For instance, if $i = 0$, the program would consider the statement $i < 1$ as true and “Hello, World!” would appear on the computer screen. If instead, $i = 3$, the program would consider the statement $i < 1$ as false (because 3 is greater than 1), and print “Goodbye” on the screen.

In this section, we will apply similar reasoning without the use of computer programs.

People in Mathematics

The Countess of Lovelace, Ada Lovelace, is credited with writing the first computer program. She wrote an algorithm to work with Charles Babbage’s Analytical Engine that could compute the Bernoulli numbers in 1843. In doing so, she became the first person to write a program for a machine that would produce more than just a simple calculation. The computer programming language ADA is named after her.

Reference: Posamentier, Alfred and Spreitzer Christian, “Chapter 34 Ada Lovelace: English (1815-1852)” pp. 272-278, *Math Makers: The Lives and Works of 50 Famous Mathematicians*, Prometheus Books, 2019.

Use and Apply the Conditional to Construct a Truth Table

A conditional is a logical statement of the form if p, then q. The conditional statement in logic is a promise or contract. The only time the conditional, $p \rightarrow q$, is false is when the contract or promise is broken.

For example, consider the following scenario. A child's parent says, "If you do your homework, then you can play your video games." The child really wants to play their video games, so they get started right away, finish within an hour, and then show their parent the completed homework. The parent thanks the child for doing a great job on their homework and allows them to play video games. Both the parent and child are happy. The contract was satisfied; true implies true is true.

Now, suppose the child does not start their homework right away, and then struggles to complete it. They eventually finish and show it to their parent. The parent again thanks the child for completing their homework, but then informs the child that it is too late in the evening to play video games, and that they must begin to get ready for bed. Now, the child is really upset. They held up their part of the contract, but they did not receive the promised reward. The contract was broken; true implies false is false.

So, what happens if the child does not do their homework? In this case, the hypothesis is false. No contract has been entered, therefore, no contract can be broken. If the conclusion is false, the child does not get to play video games and might not be happy, but this outcome is expected because the child did not complete their end of the bargain. They did not complete their homework. False implies false is true. The last option is not as intuitive. If the parent lets the child play video games, even if they did not do their homework, neither parent nor child are going to be upset. False implies true is true.

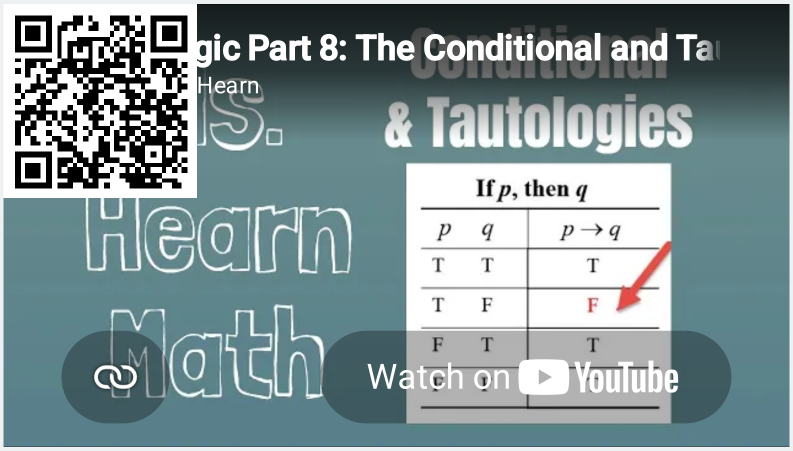
The truth table for the conditional statement below summarizes these results.

p	q	$p \rightarrow q$
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

Checkpoint

Notice that the only time the conditional statement, $p \rightarrow q$, is false is when the hypothesis, p, is true and the conclusion, q, is false.

Video



Logic Part 8: The Conditional and Tautologies

Exercise 2.4.1: Constructing Truth Tables for Conditional Statements

Assume both of the following statements are true: p: My sibling washed the dishes, and q: My parents paid them \$5.00. Create a truth table to determine the truth value of each of the following conditional statements.

1. $p \rightarrow q$

2. $p \to {\sim}q$
3. ${\sim}p \to q$

Answer

Because p is true and q is true, the statement p → q is, “If my sibling washed the dishes, then my parents paid them $5.00.” My sibling did wash the dishes, since p p is true, and the parents did pay the sibling $5.00, so the contract was entered and completed. The conditional statement is true, as indicated by the truth table representing this case:
T → T = T.

p	q	$p \to q$
T	T	**T**

p → ~ q p

2. p → ~ q p → ~ q translates to the statement, “If my sibling washed the dishes, then my parents did not pay them $5.00.” p p is true, but ~ q ~ q is false. The sibling completed their end of the contract, but they did not get paid. The contract was broken by the parents. The conditional statement is false, as indicated by the truth table representing this case:

T → F = F.

p	q	${\sim}q$	$p \to {\sim}q$
T	T	F	**F**

3. ~ p → q translates to the statement, “If my sibling did not wash the dishes, then my parents paid them $5.00.” ~ p is false, but q is true. The sibling did not do the dishes. No contract was entered, so it could not be broken. The parents decided to pay them $5.00 anyway. The conditional statement is true, as indicated by the truth table representing this case: F → T = T.

p	q	${\sim}p$	${\sim}p \to q$
T	T	F	**T**

✓ Your Turn 2.4.1

Assume p is true and q is false. p: Kevin vacuumed the living room, and q: Kevin's parents did not let him borrow the car. Create a truth table to determine the truth value of each of the following conditional statements.

1. $p \to q$
2. $p \to {\sim}q$
3. ${\sim}p \to q$

? Exercise 2.4.2: **Determining Validity of Conditional Statements**

Construct a truth table to analyze all possible outcomes for each of the following statements then determine whether they are valid.

1. $p \wedge q \to {\sim}q$
2. $p \to {\sim}p \vee q$

Answer

1. Applying the dominance of connectives, the statement $p \wedge q \to {\sim}q$ is equivalent to $(p \wedge q) \to ({\sim}q)$. So, the columns of the truth table will include p, q, $p \wedge q$, ${\sim}q$, and $p \wedge q \to {\sim}q$. Because there are only two basic propositions, p and q, the table will have 2 (2) = 4 2 (2) = 4 rows of truth values to account for all the possible outcomes. The statement is not valid because the last column is not all true.

p	q	$p \wedge q$	$\sim q$	$p \wedge q \to \sim q$
T	T	T	F	**F**
T	F	F	T	**T**
F	T	F	F	**T**
F	F	F	T	**T**

Applying the dominance of connectives, the statement $p \to \sim p \vee q$ is equivalent to $(p) \to ((\sim p) \vee q)$. So, the columns of the truth table will include p , q , ~ p, ~ p ∨ q, and $p \to (\sim p \vee q)$. Because there are only two basic propositions, p and q , the table will have 2 (2) = 4 2 (2) = 4 rows of truth values to account for all the possible outcomes. The statement is not valid because the last column is not all true.

p	q	$\sim p$	$\sim p \vee q$	$p \to (\sim p \vee q)$
T	T	F	T	**T**
T	F	F	F	**F**
F	T	T	T	**T**
F	F	T	T	**T**

✓ Your Turn 2.4.2

Construct a truth table to analyze all possible outcomes for each of the following statements, then determine whether they are valid.

1. $q \to \sim p \vee q$
2. $\sim p \to q \wedge p$

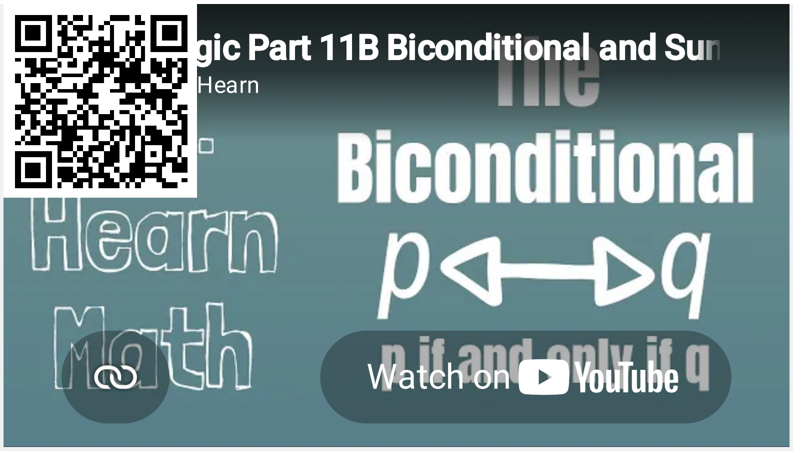
Use and Apply the Biconditional to Construct a Truth Table

The biconditional, $p \leftrightarrow q$, is a two way contract; it is equivalent to the statement $(p \to q) \wedge (q \to p)$. A biconditional statement, $p \leftrightarrow q$, is true whenever the truth value of the hypothesis matches the truth value of the conclusion, otherwise it is false.

The truth table for the biconditional is summarized below.

p	q	$p \leftrightarrow q$
T	T	**T**
T	F	**F**
F	T	**F**
F	F	**T**

Video



Logic Part 11B: Biconditional and Summary of Truth Value Rules in Logic

? Exercise 2.4.3: Constructing Truth Tables for Biconditional Statements

Assume both of the following statements are true: p : The plumber fixed the leak, and q : The homeowner paid the plumber $\$150.00$. Create a truth table to determine the truth value of each of the following biconditional statements.

1. $p \leftrightarrow q$
2. $p \leftrightarrow \sim q$
3. $\sim p \leftrightarrow \sim q$

Answer

1. Because p p is true and q q is true, the statement p ↔ q p ↔ q is “The plumber fixed the leak if and only if the homeowner paid them $150.00.” Because both p p and q q are true, the leak was fixed and the plumber was paid, meaning both parties satisfied their end of the bargain. The biconditional statement is true, as indicated by the truth table representing this case: T ↔ T = T.

p	q	$p \leftrightarrow q$
T	T	**T**

2. p → ~ q p → ~ q translates to the statement, “The plumber fixed the leak if and only if the homeowner did not pay them $150.” If the plumber fixed the leak and the homeowner did not pay them, the homeowner will have broken their end of the contract. The biconditional statement is false, as indicated by the truth table representing this case:
T ↔ F = F.

p	q	$\sim q$	$p \leftrightarrow \sim q$
T	T	F	**F**

3. ~ p ↔ ~ q ~ p ↔ ~ q translates to the statement, “The plumber did not fix the leak if and only if the homeowner did not pay them $150.” In this case, neither party—the plumber nor the homeowner—entered into the contract. The leak was not repaired, and the plumber was not paid. No agreement was broken. The biconditional statement is true, as indicated by the truth table representing this case: F ↔ F = T.

p	q	$\sim p$	$\sim q$	$\sim p \leftrightarrow \sim q$
T	T	F	F	**T**

✓ Your Turn 2.4.3

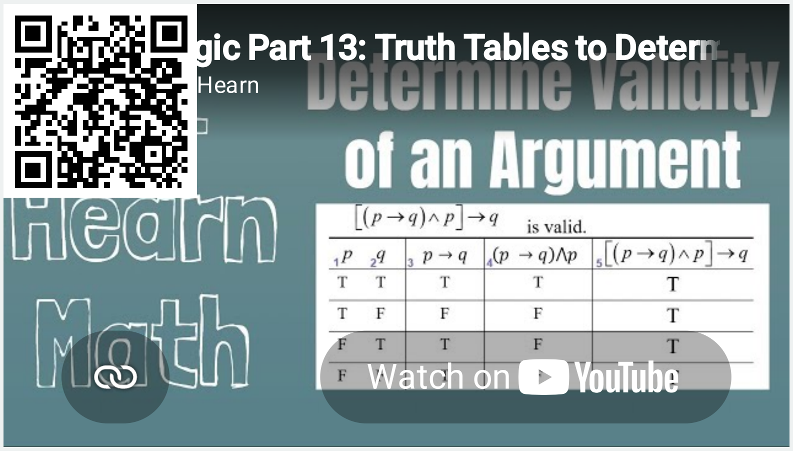
Assume p is true and q is false: p : The contractor fixed the broken window, and q : The homeowner paid the contractor $\$200$. Create a truth table to determine the truth value of each of the following biconditional statements.

1. $p \leftrightarrow q$
2. $p \leftrightarrow \sim q$
3. $\sim p \leftrightarrow q$

Checkpoint

The biconditional, $p \leftrightarrow q$, is true whenever the truth values of p and q match, otherwise it is false.

Video


gic Part 13: Truth Tables to Detern
Hearn
Determine Validity
of an Argument
Hearn
Math
$[(p \to q) \wedge p] \to q$ is valid.
Watch on YouTube

Logic Part 13: Truth Tables to Determine if Argument is Valid or Invalid

? Exercise 2.4.1: Determining Validity of Biconditional Statements

Construct a truth table to analyze all possible outcomes for each of the following statements, then determine whether they are valid.

1. $p \wedge q \leftrightarrow p \wedge \sim q$
2. $p \vee q \leftrightarrow \sim p \vee q$
3. $p \rightarrow q \leftrightarrow \sim q \rightarrow \sim p$
4. $p \wedge q \rightarrow \sim r \leftrightarrow p \wedge q \wedge r$

Answer

1. Applying the dominance of connectives, the statement $p \wedge q \leftrightarrow p \wedge \sim q$ is equivalent to $(p \wedge q) \leftrightarrow (p \wedge (\sim q))$. So, the columns of the truth table will include $p, q, p \wedge q, \sim q, p \wedge \sim q$ and $(p \wedge q) \leftrightarrow (p \wedge \sim q)$. Because there are only two basic propositions, p and q, the table will have $2(2) = 4$ rows of truth values to account for all the possible outcomes. The statement is not valid because the last column is not all true.

p	q	$p \wedge q$	$\sim q$	$p \wedge \sim q$	$(p \wedge q) \leftrightarrow (p \wedge \sim q)$
T	T	T	F	F	**F**
T	F	F	T	T	**F**
F	T	F	F	F	**T**
F	F	F	T	F	**T**

2. Applying the dominance of connectives, the statement $p \vee q \leftrightarrow \sim p \vee q$ is equivalent to $(p \vee q) \leftrightarrow ((\sim p) \vee q)$. So, the columns of the truth table will include $p, q, p \vee q, \sim p, \sim p \vee q$, and $(p \vee q) \leftrightarrow (\sim p \vee q)$. Because there are only two basic propositions, p and q, the table will have $2(2) = 4$ rows of truth values to account for all the possible outcomes. The statement is not valid because the last column is not all true.

p	q	$p \vee q$	$\sim p$	$\sim p \vee q$	$(p \vee q) \leftrightarrow (\sim p \vee q)$
T	T	T	F	T	**T**
T	F	T	F	F	**F**
F	T	T	T	T	**T**
F	F	F	T	T	**F**

3. Applying the dominance of connectives, the statement $p \rightarrow q \leftrightarrow \sim q \rightarrow \sim p$ is equivalent to $(p \rightarrow q) \leftrightarrow ((\sim q) \rightarrow (\sim p))$. So, the columns of the truth table will include $p, q, p \rightarrow q, \sim q, \sim p, \sim q \rightarrow \sim p$, and $(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$. Because there are only two basic propositions, p and q the table will have $2(2) = 4$ rows of truth values to account for all the possible outcomes. The statement is valid because the last column is all true.

p	q	$p \rightarrow q$	$\sim q$	$\sim p$	$\sim q \rightarrow \sim p$	$(p \rightarrow \mathbf{q}) \leftrightarrow (\sim q \rightarrow \sim p)$
T	T	T	F	F	T	**T**
T	F	F	T	F	F	**T**
F	T	T	F	T	T	**T**
F	F	T	T	T	T	**T**

4. Applying the dominance of connectives, the statement $p \wedge q \rightarrow \sim r \leftrightarrow p \wedge q \wedge r$ is equivalent to $((p \wedge q) \rightarrow (\sim r)) \leftrightarrow ((p \wedge q) \wedge r)$. So, the columns of the truth table will include $p, q, r, \sim r, p \wedge q, (p \wedge q) \wedge \sim r, (p \wedge q) \wedge r$, and $((p \wedge q) \rightarrow (\sim r)) \leftrightarrow ((p \wedge q) \wedge r)$. Because there are three basic propositions, p, q, and r, the table will have $2(2)(2) = 8$ rows of truth values to account for all the possible outcomes. The statement is not valid because the last column is not all true.

p	q	r	$\sim r$	$p \wedge q$	$(p \wedge q) \rightarrow \sim r$	$(p \wedge q) \wedge r$	$(p \wedge q \rightarrow \sim r) \leftrightarrow (p \wedge q$
T	T	T	F	T	F	T	**T**
T	T	F	T	T	T	F	**T**
T	F	T	F	F	T	F	**F**
T	F	F	T	F	T	F	**F**
F	T	T	F	F	T	F	**F**
F	T	F	T	F	T	F	**F**
F	F	T	F	F	T	F	**F**
F	F	F	T	F	T	F	**F**

✓ Your Turn 2.4.1

Construct a truth table to analyze all possible outcomes for each of the following statements, then determine whether they are valid.

1. $\sim (p \wedge q) \leftrightarrow (\sim p \vee \sim q)$

2. $\sim p \leftrightarrow q \wedge p$

3. $p \rightarrow q \leftrightarrow \sim p \vee q$

4. $p \wedge q \rightarrow r \leftrightarrow \sim p \vee \sim q \vee r$

Check Your Understanding

1. 1. In logic, a conditional statement can be thought of as a _____________.
2. If the hypothesis, p, of a conditional statement is true, the _____, q, must also be true for the conditional statement \\($p \to q$ to be true.
3. If the _______________ of a conditional statement is false, the conditional statement is true.
4. The symbolic form of the _______________________ statement is $p \leftrightarrow q$
5. The _______________________ statement is equivalent to the statement $(p \to q) \wedge (q \to p)$.
6. p if and only if q is _____________ whenever the truth value of p matches the truth value of q, otherwise it is false.

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2.4.0: Exercises

For the following exercises, complete the truth table to determine the truth value of the proposition in the last column.

Exercise 2.4.0.1

p	q	p	$p \to q$
T	T		

Exercise 2.4.0.2

p	q	q	$p \to q$
T	T		

Exercise 2.4.0.3

p	q	p	$p \leftrightarrow q$
F	T		

Exercise 2.4.0.4

p	q	q	$p \leftrightarrow q$
F	T		

Exercise 2.4.0.5

p	q	r	p	$p \wedge q$	$(p \wedge q) \to r$
F	T	F			

Exercise 2.4.0.6

p	q	r	p	r	$p \wedge q$	$(p \wedge q) \to r$
F	F	F				

Exercise 2.4.0.7

p	q	r	p	r	$p \vee q$	$(p \vee q) \leftrightarrow r$
F	F	F				

Exercise 2.4.0.8

p	q	r	p	r	$p \wedge q$	$(p \wedge q) \leftrightarrow r$
T	F	F				

Exercise 2.4.0.9

p	q	r	p	r	$p \vee q$	$p \to r$	$(p \vee r) \leftrightarrow (p \to r)$
F	F	F					

Exercise 2.4.0.10

p	q	r	p	r	$p \wedge q$	$p \to r$	$(p \wedge q) \leftrightarrow (p \to r)$
T	T	T					

For the following exercises, assume these statements are true: p: Faheem is a software engineer, q: Ann is a project manager, r: Giacomo works with Faheem, and s: The software application was completed on time. Translate each of the following statements to symbols, then construct a truth table to determine its truth value.

Exercise 2.4.0.11

If Giacomo works with Faheem, then Faheem is not a software engineer.

Exercise 2.4.0.12

If the software application was not completed on time, then Ann is not a project manager.

Exercise 2.4.0.13

The software application was completed on time if and only if Giacomo worked with Faheem.

Exercise 2.4.0.14

Ann is not a project manager if and only if Faheem is a software engineer.

Exercise 2.4.0.15

If the software application was completed on time, then Ann is a project manager, but Faheem is not a software engineer.

Exercise 2.4.0.16

If Giacomo works with Faheem and Ann is a project manager, then the software application was completed on time.

Exercise 2.4.0.17

The software application was not completed on time if and only if Faheem is a software engineer or Giacomo did not work with Faheem.

Exercise 2.4.0.18

Faheem is a software engineer or Ann is not a project manager if and only if Giacomo did not work with Faheem and the software application was completed on time.

? Exercise 2.4.0.19

Ann is a project manager implies Faheem is a software engineer if and only if the software application was completed on time implies Giacomo worked with Faheem.

? Exercise 2.4.0.20

If Giacomo did not work with Faheem implies that the software application was not completed on time, then Ann was not the project manager.

For the following exercises, construct a truth table to analyze all the possible outcomes and determine the validity of each argument.

? Exercise 2.4.0.21

$p \vee \sim q \rightarrow q$

? Exercise 2.4.0.22

$\sim q \rightarrow p \wedge \sim q$

? Exercise 2.4.0.23

$(p \rightarrow q) \leftrightarrow q$

? Exercise 2.4.0.24

$(p \rightarrow q) \leftrightarrow p$

? Exercise 2.4.0.25

$\sim (p \vee q) \leftrightarrow (\sim p \wedge \sim q)$

? Exercise 2.4.0.26

$(p \rightarrow q) \wedge p \rightarrow q$

? Exercise 2.4.0.27

$p \rightarrow q \rightarrow r$

? Exercise 2.4.0.28

$(p \rightarrow q) \wedge (q \rightarrow r) \leftrightarrow (p \rightarrow r)$

? Exercise 2.4.0.29

$p \vee (q \wedge r) \leftrightarrow (p \vee q) \wedge (p \vee r)$

? Exercise 2.4.0.30

$p \vee (q \vee r) \leftrightarrow (p \vee q) \vee r$

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2.5: Equivalent Statements

Figure $2.5.1$: How your logical argument is stated affects the response, just like how you speak when holding a conversation can affect how your words are received. (credit: modification of work by Goelshivi/Flickr, Public Domain Mark 1.0)

Learning Objectives

After completing this section, you should be able to:

1. Determine whether two statements are logically equivalent using a truth table.
2. Compose the converse, inverse, and contrapositive of a conditional statement

Have you ever had a conversation with or sent a note to someone, only to have them misunderstand what you intended to convey? The way you choose to express your ideas can be as, or even more, important than what you are saying. If your goal is to convince someone that what you are saying is correct, you will not want to alienate them by choosing your words poorly.

Logical arguments can be stated in many different ways that still ultimately result in the same valid conclusion. Part of the art of constructing a persuasive argument is knowing how to arrange the facts and conclusion to elicit the desired response from the intended audience.

In this section, you will learn how to determine whether two statements are **logically equivalent** using truth tables, and then you will apply this knowledge to compose logically equivalent forms of the conditional statement. Developing this skill will provide the additional skills and knowledge needed to construct well-reasoned, persuasive arguments that can be customized to address specific audiences.

Checkpoint

An alternate way to think about logical equivalence is that the truth values have to match. That is, whenever p is true, q is also true, and whenever p is false, q is also false.

Determine Logical Equivalence

Two statements, p and q, are logically equivalent when $p \leftrightarrow q$ is a valid argument, or when the last column of the truth table consists of only true values. When a logical statement is always true, it is known as a tautology. To determine whether two statements p and

q are logically equivalent, construct a truth table for $p \leftrightarrow q$ and determine whether it valid. If the last column is all true, the argument is a tautology, it is valid, and p is logically equivalent to q; otherwise, p is not logically equivalent to q.

Exercise $2.5.1$: Determining Logical Equivalence with a Truth Table

Create a truth table to determine whether the following compound statements are logically equivalent.

1. $p \rightarrow q; \sim p \rightarrow \sim q$
2. $p \rightarrow q; \sim p \vee q$

Answer

1. Construct a truth table for the biconditional formed by using the first statement as the hypothesis and the second statement as the conclusion, $(p \rightarrow q) \leftrightarrow (\sim p \rightarrow \sim q)$.

p	q	$p \rightarrow q$	$\sim p$	$\sim q$	$\sim p \rightarrow \sim q$	$(p \rightarrow q) \leftrightarrow (\sim p \rightarrow \sim q)$
T	T	T	F	F	T	**T**
T	F	F	F	T	T	**F**
F	T	T	T	F	F	**F**
F	F	T	T	T	T	**T**

Because the last column it not all true, the biconditional is not valid and the statement $p \rightarrow q$ is not logically equivalent to the statement $\sim p \rightarrow \sim q$.

2. Construct a truth table for the biconditional formed by using the first statement as the hypothesis and the second statement as the conclusion,
$(p \rightarrow q) \leftrightarrow (\sim p \vee q)$.

p	q	$p \rightarrow q$	$\sim p$	$\sim p \vee q$	$(p \rightarrow q) \leftrightarrow (\sim p \vee q)$
T	T	T	F	T	**T**
T	F	F	F	F	**T**
F	T	T	T	T	**T**
F	F	T	T	T	**T**

Because the last column is true for every entry, the biconditional is valid and the statement $p \rightarrow q$ is logically equivalent to the statement $\sim p \vee q$. Symbolically, $p \rightarrow q \equiv \sim p \vee q$.

Your Turn $2.5.1$

Create a truth table to determine whether the following compound statements are logically equivalent.

1. $p \rightarrow q; q \rightarrow \sim p$
2. $p \rightarrow q; p \vee \sim q$

Compose the Converse, Inverse, and Contrapositive of a Conditional Statement

The converse, inverse, and contrapositive are variations of the conditional statement, $p \rightarrow q$.

- The converse is if q then p, and it is formed by interchanging the hypothesis and the conclusion. The converse is logically equivalent to the inverse.
- The inverse is if $\sim p$ then $\sim q$, and it is formed by negating both the hypothesis and the conclusion. The inverse is logically equivalent to the converse.

- The contrapositive is if $\sim q$ then $\sim p$, and it is formed by interchanging and negating both the hypothesis and the conclusion. The contrapositive is logically equivalent to the conditional.

The table below shows how these variations are presented symbolically.

				Conditional	Contrapositive	Converse	Inverse
p	q	$\sim p$	$\sim q$	$p \to q$	$\sim q \to \sim p$	$q \to p$	$\sim p \to \sim q$
T	T	F	F	T	T	T	T
T	F	F	T	F	F	T	T
F	T	T	F	T	T	F	F
F	F	T	T	T	T	T	T

Exercise 2.5.2: Writing the Converse, Inverse, and Contrapositive of a Conditional Statement

Use the statements, p: Harry is a wizard and q: Hermione is a witch, to write the following statements:

1. Write the conditional statement, $p \to q$, in words.
2. Write the converse statement, $q \to p$, in words.
3. Write the inverse statement, $\sim p \to \sim q$, in words.
4. Write the contrapositive statement, $\sim q \to \sim p$, in words.

Answer

1. The conditional statement takes the form, “if p p , then q q ,” so the conditional statement is: “If Harry is a wizard, then Hermione is a witch.” Remember the *if* … *then* … words are the connectives that form the conditional statement.
2. The converse swaps or interchanges the hypothesis, p p , with the conclusion, q q . It has the form, “if q q , then p p .” So, the converse is: "If Hermione is a witch, then Harry is a wizard."
3. To construct the inverse of a statement, negate both the hypothesis and the conclusion. The inverse has the form, “if ~ p ~ p , then ~ q ~ q ,” so the inverse is: "If Harry is not a wizard, then Hermione is not a witch."
4. The contrapositive is formed by negating and interchanging both the hypothesis and conclusion. It has the form, “if ~ q ~ q , then ~ p ~ p ," so the contrapositive statement is: "If Hermione is not a witch, then Harry is not a wizard."

✓ Your Turn 2.5.2

Use the statements, p : Elvis Presley wore capes and q : Some superheroes wear capes, to write the following statements:

1. Write the conditional statement, $p \to q$, in words.
2. Write the converse statement, $q \to p$, in words.
3. Write the inverse statement, $\sim p \to \sim q$, in words.
4. Write the contrapositive statement, $\sim q \to \sim p$, in words.

Exercise 2.5.3

Use the conditional statement, “If all dogs bark, then Lassie likes to bark,” to identify the following.

1. Write the hypothesis of the conditional statement and label it with a p.
2. Write the conclusion of the conditional statement and label it with a q.
3. Identify the following statement as the converse, inverse, or contrapositive: “If Lassie likes to bark, then all dogs bark.”
4. Identify the following statement as the converse, inverse, or contrapositive: “If Lassie does not like to bark, then some dogs do not bark.”
5. Which statement is logically equivalent to the conditional statement?

Answer

1. The hypothesis is the phrase following the *if*. The answer is p: All dogs bark. Notice, the word *if* is not included as part of the hypothesis.
2. The conclusion of a conditional statement is the phrase following the *then*. The word *then* is not included when stating the conclusion. The answer is: q: Lassie likes to bark.
3. “Lassie likes to bark” is q and “All dogs bark” is p. So, “If Lassie likes to bark, then all dogs bark,” has the form “if q, then p,” which is the form of the converse. “Lassie does not like to bark” is $\sim q$ and “Some dogs do not bark” is $\sim p$. The statement, “If Lassie does not like to bark, then some dogs do not bark,” has the form “if $\sim q$, then $\sim p$,” which is the form of the contrapositive.
4. The contrapositive $\sim q \rightarrow \sim p$ is logically equivalent to the conditional statement $p \rightarrow q$.

✓ Your Turn 2.5.3

Use the conditional statement, “If Dora is an explorer, then Boots is a monkey,” to identify the following:

1. Write the hypothesis of the conditional statement and label it with a p�.

2. Write the conclusion of the conditional statement and label it with a q�.

3. Identify the following statement as the converse, inverse, or contrapositive: “If Dora is not an explorer, then Boots is not a monkey.”

4. Identify the following statement as the converse, inverse, or contrapositive: “If Boots is a monkey, then Dora is an explorer.”

5. Which statement is logically equivalent to the inverse?

? Exercise 2.5.4

Assume the conditional statement, $p \rightarrow q$: “If Chadwick Boseman was an actor, then Chadwick Boseman did not star in the movie *Black Panther*” is false, and use it to answer the following questions.

1. Write the converse of the statement in words and determine its truth value.
2. Write the inverse of the statement in words and determine its truth value.
3. Write the contrapositive of the statement in words and determine its truth value.

Answer

The only way the conditional statement can be false is if the hypothesis, p: Chadwick Boseman was an actor, is true and the conclusion, q: Chadwick Boseman did not star in the movie *Black Panther*, is false. The converse is $q \rightarrow p$, and it is written in words as: “If Chadwick Boseman did not star in the movie *Black Panther*, then Chadwick Boseman was an actor.” This statement is true, because false → true is true.

The inverse has the form “ $\sim p \rightarrow \sim q$.” The written form is: “If Chadwick Boseman was not an actor, then Chadwick Boseman starred in the movie *Black Panther*.” Because p is true, and q is false, $\sim p$ is false, and $\sim q$ is true. This means the inverse is false → true, which is true. Alternatively, from Question 1, the converse is true, and because the inverse is logically equivalent to the converse it must also be true.

The contrapositive is logically equivalent to the conditional. Because the conditional is false, the contrapositive is also false. This can be confirmed by looking at the truth values of the contrapositive statement. The contrapositive has the form “ $\sim q \rightarrow \sim p$.” Because q is false and p is true, $\sim$ q is true and $\sim p$ is false. Therefore, $\sim q \rightarrow \sim p$ is true → false, which is false. The written form of the contrapositive is “If Chadwick Boseman starred in the movie *Black Panther*, then Chadwick Boseman was not an actor.”

✓ Your Turn 2.5.4

Assume the conditional statement $p \rightarrow q$: “If my friend lives in San Francisco, then my friend does not live in California” is false, and use it to answer the following questions.

1. Write the converse of the statement in words and determine its truth value.

2. Write the inverse of the statement in words and determine its truth value.
3. Write the contrapositive of the statement in words and determine its truth value.

Check Your Understanding

1. Two statements p and q are logically equivalent to each other if the biconditional statement, $p \leftrightarrow q$ is ________________.
2. The _____ statement has the form, “p then q.”
3. The contrapositive is _____________ ___________ to the conditional statement, and has the form, "if q, then p."
4. The _________________ of the conditional statement has the form, "if p, then q."
5. The _________________ of the conditional statement is logically equivalent to the _______________ and has the form, "if q then p."

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2.5.0: Exercises

For the following exercises, determine whether the pair of compound statements are logically equivalent by constructing a truth table.

Exercise $2.5.0.1$

Converse: $q \rightarrow p$ and inverse: $\sim p \rightarrow\sim q$

Exercise $2.5.0.2$

Conditional: $p \rightarrow q$ and contrapositive: $\sim q \rightarrow\sim p$

Exercise $2.5.0.3$

Inverse: $\sim p \rightarrow\sim q$ and contrapositive: $\sim q \rightarrow\sim p$

Exercise $2.5.0.4$

Conditional: $p \rightarrow q$ and converse: $q \rightarrow p$

Exercise $2.5.0.5$

$\sim p \rightarrow q$ and $p\vee \sim q$

Exercise $2.5.0.6$

$\sim p \rightarrow q$ and $p \vee q$

Exercise $2.5.0.7$

$\sim (p \wedge q)$ and $\sim p\wedge \sim q$

Exercise $2.5.0.8$

$\sim (p \wedge q)$ and $\sim p\vee \sim q$

Exercise $2.5.0.9$

$p \wedge (q \vee r)$ and $(p \wedge q) \vee (p \wedge r)$

Exercise $2.5.0.10$

$p \wedge (q \vee r)$ and $(p \wedge q) \vee r$

For the following exercises, answer the following:

a. Write the conditional statement $p \rightarrow q$ in words.
b. Write the converse statement $q \rightarrow p$ in words.
c. Write the inverse statement $\sim p \rightarrow\sim q$ in words.
d. Write the contrapositive statement $\sim q \rightarrow\sim p$ in words.

Exercise 2.5.0.11

p : Six is afraid of Seven and q : Seven ate Nine.

Exercise 2.5.0.12

p : Hope is eternal and q : Despair is temporary.

Exercise 2.5.0.13

p : Tom Brady is a quarterback and q : Tom Brady does not play soccer.

Exercise 2.5.0.14

p : Shakira does not sing opera and q : Shakira sings popular music.

Exercise 2.5.0.15

p :The shape does not have three sides and q : The shape is not a triangle.

Exercise 2.5.0.16

p : All birds can fly and q : Emus can fly.

Exercise 2.5.0.17

p : Penguins cannot fly and q : Some birds can fly.

Exercise 2.5.0.18

p : Some superheroes do not wear capes and q : Spiderman is a superhero.

Exercise 2.5.0.19

p : No Pokémon are little ponies and q : Bulbasaur is a Pokémon.

Exercise 2.5.0.20

p : Roses are red, and violets are blue and q : Sugar is sweet, and you are sweet too.

For the following exercises, use the conditional statement: "If Clark Kent is Superman, then Lois Lane is not a reporter," to answer the following questions.

Exercise 2.5.0.21

Write the hypothesis of the conditional statement, label it with a p, and determine its truth value.

Exercise 2.5.0.22

Write the conclusion of the conditional statement, label it with a q, and determine its truth value.

Exercise 2.5.0.23

Identify the following statement as the converse, inverse, or contrapositive, and determine its truth value: "If Clark Kent is not Superman, then Lois Lane is a reporter."

Exercise 2.5.0.24

Identify the following statement as the converse, inverse, or contrapositive, and determine its truth value: "If Lois Lane is a reporter, then Clark Kent is not Superman."

Exercise 2.5.0.25

Which form of the conditional is logically equivalent to the converse?

For the following exercises, use the conditional statement: "If The Masked Singer is not a music competition, then Donnie Wahlberg was a member of New Kids on the Block," to answer the following questions.

Exercise 2.5.0.26

Write the hypothesis of the conditional statement, label it with a p, and determine its truth value.

Exercise 2.5.0.27

Write the conclusion of the conditional statement, label it with a q, and determine its truth value.

Exercise 2.5.0.28

Identify the following statement as the converse, inverse, or contrapositive, and determine its truth value: "If Donnie Wahlberg was a member of New Kids on the Block, then The Masked Singer is not a music competition."

Exercise 2.5.0.29

Identify the following statement as the converse, inverse, or contrapositive, and determine its truth value: "If The Masked Singer is a music competition, then Donnie Wahlberg was not a member of New Kids on the Block."

Exercise 2.5.0.30

Which form of the conditional is logically equivalent to the contrapositive, $\sim q \rightarrow \sim p$?

For the following exercises, use the conditional statement: "If all whales are mammals, then no fish are whales," to answer the following questions.

Exercise 2.5.0.31

Write the hypothesis of the conditional statement, label it with a p, and determine its truth value.

Exercise 2.5.0.32

Write the conclusion of the conditional statement, label it with a q, and determine its truth value.

Exercise 2.5.0.33

Identify the following statement as the converse, inverse, or contrapositive, and determine its truth value: "If some fish are whales, then some whales are not mammals."

? Exercise $2.5.0.34$

Write the inverse in words and determine its truth value.

? Exercise $2.5.0.35$

Write the converse in words and determine its truth value.

For the following exercises, use the conditional statement: "If some parallelograms are rectangles, then some circles are not symmetrical," to answer the following questions.

? Exercise $2.5.0.36$

Write the hypothesis of the conditional statement, label it with a p, and determine its truth value.

? Exercise $2.5.0.37$

Write the conclusion of the conditional statement, label it with a q, and determine its truth value.

? Exercise $2.5.0.38$

Write the converse in words and determine its truth value.

? Exercise $2.5.0.39$

Write the contrapositive in words and determine its truth value.

? Exercise $2.5.0.40$

Write the inverse in words and determine its truth value.

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2.6: De Morgan’s Laws

Figure 2.6.1: De Morgan’s Laws were key to the rise of logical mathematical expression and helped serve as a bridge for the invention of the computer. (credit: modification of work “Golden Gate Bridge (San Francisco Bay, California, USA)” by James St. John/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

1. Use De Morgan’s Laws to negate conjunctions and disjunctions.
2. Construct the negation of a conditional statement.
3. Use truth tables to evaluate De Morgan’s Laws.

The contributions to logic made by Augustus De Morgan and George Boole during the 19th century acted as a bridge to the development of computers, which may be the greatest invention of the 20th century. **Boolean** logic is the basis for computer science and digital electronics, and without it the technological revolution of the late 20th and early 21st centuries—including the creation of computer chips, microprocessors, and the Internet—would not have been possible. Every modern computer language uses Boolean logic statements, which are translated into commands understood by the underlying electronic circuits enabling computers to operate. But how did this logic get its name?

People in Mathematics: George Boole

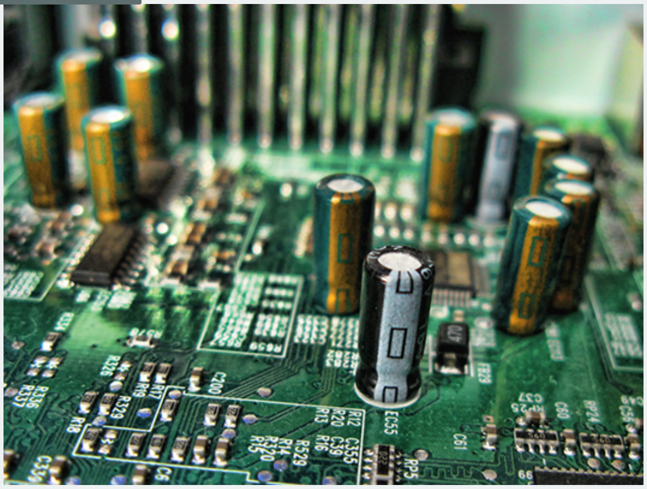

Figure 2.6.2: Boole’s algebra of logic was foundational in the design of digital computer circuits. (credit: “Circuit Board” by Squeezyboy/Flickr, CC BY 2.0)

George Boole was born in Lincolnshire, England in 1815. He was the son of a cobbler who provided him some initial education, but Boole was mostly self-taught. He began teaching at 16 years of age, and opened his own school at the age of 20. In 1849, at the age of 34, he was appointed Professor of Mathematics at Queens College in Cork, Ireland. In 1853, he published the paper, *An Investigation of the Laws of Thought, on Which Are Founded the Mathematical Theories of Logic and Probabilities*, which is the treatise that the field of Boolean algebra and digital circuitry was built on.

Reference: Posamentier, Alfred and Spreitzer Christian, “Chapter 35 George Boole: English (1815-1864)” pp. 279-283, *Math Makers: The Lives and Works of 50 Famous Mathematicians*, Prometheus Books, 2019.

Negation of Conjunctions and Disjunctions

In Chapter 1, Example 1.37 used a Venn diagram to prove De Morgan’s Law for set complement over union. Because the complement of a set is analogous to negation and union is analogous to an *or* statement, there are equivalent versions of De Morgan’s Laws for logic.

FORMULA

De Morgan’s Law for negation of a conjunction: $\sim(p \wedge q) \equiv {\sim}p \vee {\sim}q$

De Morgan’s Law for the negation of a disjunction: $\sim(p \vee q) \equiv {\sim}p \wedge {\sim}q$

Negation of a conditional: $\sim(p \to q) \equiv p \wedge {\sim}q$

Writing conditional as a disjunction: $p \to q \equiv {\sim}p \vee q$

Checkpoint

Recall that the symbol for logical equivalence is: $\equiv$.

De Morgan’s Laws allow us to write the negation of conjunctions and disjunctions without using the phrase, “It is not the case that …” to indicate the parentheses. Avoiding this phrase often results in a written or verbal statement that is clearer or easier to understand.

Exercise 2.6.1: Applying De Morgan’s Law for Negation of Conjunctions and Disjunctions

Write the negation of each statement in words without using the phrase, “It is not the case that.”

1. Kristin is a biomedical engineer and Thomas is a chemical engineer.
2. A person had cake or they had ice cream.

Answer

1. Kristin is a biomedical engineer and Thomas is a chemical engineer has the form " $p \wedge q$," where p is the statement, "Kristin is a biomedical engineer," and q is the statement, "Thomas is a chemical engineer." By De Morgan's Law, the negation of a conjunction, $\sim(p \wedge q)$, is logically equivalent to $\sim p \vee \sim q$. $\sim p$ is "Kristen is not a biomedical engineer," and $\sim q$ is "Thomas is not a chemical engineer." By De Morgan's Law, the solution has the form " $\sim p \vee \sim q$," so the answer is: "Kristin is not a biomedical engineer or Tom is not a chemical engineer."
2. A person had cake or they had ice cream has the form " $p \vee q$," where p is the statement, "A person had cake," and q is the statement, "A person had ice cream." By De Morgan's Law for the negations of a disjunction, $\sim(p \vee q) \equiv \sim p \wedge \sim q$. The solution is the statement: "A person did not have cake and they did not have ice cream."

Your Turn 2.6.1

Write the negation of each statement in words without using the phrase, *it is not the case that*.

1. Jackie played softball or she ran track.

2. Brandon studied for his certification exam, and he passed his exam.

is not a chemical engineer.”

Negation of a Conditional Statement

The negation of any statement has the opposite truth values of the original statement. The **negation of a conditional**, ${\sim}(p \to q)$, is the conjunction of p and not q, $p \wedge {\sim}q$. Consider the truth table below for the negation of the conditional.

p	q	$p \to q$	${\sim}(p \to q)$
T	T	T	F
T	F	F	T
F	T	T	F
F	F	T	F

The only time the negation of the conditional statement is true is when p is true, and q is false. This means that $\sim(p \to q)$ is logically equivalent to $p \wedge {\sim} q$, as the following truth table demonstrates.

p	q	$p \to q$	$\sim(p \to q)$	$\sim q$	$p \wedge {\sim} q$	$\sim(p \to q) \leftrightarrow (p \wedge {\sim} q)$
T	T	T	F	F	F	**T**
T	F	F	T	T	T	**T**
F	T	T	F	F	F	**T**
F	F	T	F	T	F	**T**

? Exercise 2.6.2: Constructing the Negation of a Conditional Statement

Write the negation of each conditional statement.

1. If Adele won a Grammy, then she is a singer.
2. If Henrik Lundqvist played professional hockey, then he did not win the Stanley Cup.

Answer

1. The negation of the conditional statement, $p \to q$, is the statement, $p \wedge \sim q$. The hypothesis of the conditional statement is p : "Adele won a Grammy," and conclusion of the conditional statement is q : "Adele is a singer." The negation of the conclusion, $\sim q$, is the statement: "She is not a singer." Therefore, the answer is $p \wedge \sim q$: "Adele won a Grammy, and she is not a singer."

2. The hypothesis is p : "Henrik Lundqvist played professional hockey," and the conclusion of the conditional statement is q : "He did not win the Stanley Cup." The negation of q is the statement: "He won the Stanley Cup." The negation of the conditional statement is equal to $p \wedge \sim q$: "Henrick Lundqvist played professional hockey, and he won the Stanley Cup."

✓ Your Turn 2.6.2

Write the negation of each conditional statement.

1. If Edna Mode makes a new superhero costume, then it will not include a cape.

2. If I had pancakes for breakfast, then I used maple syrup.

? Exercise 2.6.3: Constructing the Negation of a Conditional Statement with Quantifiers

Write the negation of each conditional statement.

1. If all cats purr, then my partner’s cat purrs.
2. If a penguin is a bird, then some birds do not fly.

Answer

1. The negation of the conditional statement $p \to q$ is the statement $p \wedge \sim q$. The hypothesis of the conditional statement is p : "All cats purr," and the conclusion of the conditional statement is q : "My partner's cat purrs." The negation of the conclusion, $\sim q$, is the statement: "My partner's cat does not purr." Therefore, the answer is $p \wedge \sim q$: "All cats purr, but my partner's cat does not purr."

2. The hypothesis is p : "A penguin is a bird," and the conclusion of the conditional statement is q : "Some birds do not fly." The negation of q is the statement: "All birds fly." Therefore, the negation of the conditional statement is equal to $p \wedge \sim q$: "A penguin is a bird, and all birds fly."

Your Turn 2.6.3

Write the negation of each conditional statement.

1. If some people like ice cream, then ice cream makers will make a profit.
2. If Raquel cannot play video games, then nobody will play video games.

Many of the properties that hold true for number systems and sets also hold true for logical statements. The following table summarizes some of the most useful properties for analyzing and constructing logical arguments. These properties can be verified using a truth table.

Property	Conjunction (AND)	Disjunction (OR)
Commutative	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
Associative	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
Distributive	$p \wedge (q \wedge r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
De Morgan's	$\sim(p \wedge q) \equiv \sim p \vee \sim q$	$\sim(p \vee q) \equiv \sim p \wedge \sim q$
Conditional	$\sim(p \rightarrow q) \equiv p \wedge \sim q$	$p \rightarrow q \equiv \sim p \vee q$

Exercise 2.6.4

Negating a Conditional Statement with a Conjunction or Disjunction

Write the negation of each conditional statement applying De Morgan's Law.

1. If mom needs to buy chips, then Mike had friends over and Bob was hungry.
2. If Juan had pizza or Chris had wings, then dad watched the game.

Answer

1. The conditional has the form "If p then q or r," where p is "Mom needs to buy chips," q is "Mike had friends over," and r is "Bob was hungry." The negation of $p \rightarrow (q \wedge r)$ is $p \wedge \sim (q \wedge r)$. Applying De Morgan's Law to the statement $\sim (q \wedge r)$ the result is $\sim q \vee \sim r$, so our conditional statement becomes $p \wedge (\sim q \vee \sim r)$. By the distributive property for conjunction over disjunction, this statement is equivalent to $(p \wedge \sim q) \vee (p \wedge \sim r)$. Translating the statement $(p \wedge \sim q) \vee (p \wedge \sim r)$ into words, the solution is: "Mom needs to buy chips and Mike did not have friends over, or Mom needs to buy chips and Bob was not hungry."

2. The conditional has the form "If p or q, then r," where p is "Juan had pizza," q is "Chris had wings," and r is "Dad watched the game." The negation of $(p \vee q) \rightarrow r$ is $(p \vee q) \wedge \sim r$. By the distributive property for disjunction over conjuction, the statement is equivalent to $(p \vee \sim r) \wedge (q \vee \sim r)$. Translating the statement $(p \vee \sim r) \wedge (q \vee \sim r)$ into words, the solution is: "Juan had pizza or dad did not watch the game, and Chris had wings or dad did not watch the game."

Your Turn 2.6.4

Write the negation of each conditional statement applying De Morgan's Law.

1. If Eric needs to replace the light bulb, then Marcus left the light on all night or Dan broke the bulb.
2. If Trenton went to school and Regina went to work, then Merika cleaned the house.

Evaluating De Morgan's Laws with Truth Tables

In Chapter 1, you learned that you could prove the validity of De Morgan's Laws using Venn diagrams. Truth tables can also be used to prove that two statements are logically equivalent. If two statements are logically equivalent, you can use the form of the statement that is clearer or more persuasive when constructing a logical argument.

The next example will prove the validity of one of De Morgan's Laws using a truth table. The same procedure can be applied to any two logical statement that you believe are equivalent. If the last column of the truth table is a tautology, then the two statements are logically equivalent.

? Exercise $2.6.5$: Verifying De Morgan's Law for Negation of a Conjunction

Construct a truth table to verify De Morgan's Law for the negation of a conjunction, $\sim(p \wedge q) \ \equiv \sim p \vee \sim q$, is valid.

Answer

Step 1: To verify any logical equivalence, you must first replace the logical equivalence symbol, $\equiv$, with the biconditional symbol, $\leftrightarrow$. The statement $\sim(p \wedge q) \equiv \sim p \vee \sim q$ becomes $\sim(p \wedge q) \leftrightarrow \sim p \vee \sim q$

Step 2: Next, you create a truth table for the statement. Because we have two basic statements, p, and q, the truth table will have four rows to account for all the possible outcomes. The columns will be $p, q, \sim p, \sim q, p \wedge q, \sim(p \wedge q), \sim p \vee \sim q$, and the biconditional statement is $\sim(p \wedge q) \leftrightarrow \sim p \vee \sim q$.

p	q	$p \wedge q$	$\sim(p \wedge q)$	$\sim p$	$\sim q$	$\sim p \vee \sim q$	$\sim(p \wedge q) \leftrightarrow (\sim p \vee \sim q)$
T	T	T	F	F	F	F	**T**
T	F	F	T	F	T	T	**T**
F	T	F	T	T	F	T	**T**
F	F	F	T	T	T	T	**T**

Step 3: Finally, verify that the statement is valid by confirming it is a tautology. In this instance, the last column is all true. Therefore, the statement is valid and De Morgan's Law for the negation of a conjunction is verified.

✓ Your Turn $2.6.5$

1. Construct a truth table to verify De Morgan's Law for the negation of a disjunction, $\sim(p \vee q) \equiv \sim p \wedge \sim q$, is valid.

Check Your Understanding

1. De Morgan's Law for the negation of a conjunction states that \($(p \wedge q)$is logically equivalent to ____________________.

2. De Morgan's Law for the negation of a disjunction states that $(p \vee q)$ is logically equivalent to ___________________.

3. The negation of the conditional statement, $(p \rightarrow q)$, is logically equivalent to ______________.

$(\ (p \rightarrow q)) \equiv p \rightarrow q$, which means the conditional statement is logically equivalent to $(p \wedge \ q)$. Apply _________________________ to the statement $(p \wedge \ q)$to show that the conditional statement $p \rightarrow q \equiv \ p \vee q$.

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2.6.0: Exercises

For the following exercises, use De Morgan's Laws to write each statement without parentheses.

? Exercise $2.6.0.1$

$\sim(\sim p \vee q)$

? Exercise $2.6.0.2$

$\sim(\sim p \wedge \sim q)$

? Exercise $2.6.0.3$

$\sim(p \wedge \sim q)$

? Exercise $2.6.0.4$

$\sim(\sim p \vee \sim q)$

For the following exercises, use De Morgan's Laws to write the negation of each statement in words without using the phrase, "It is not the case that, ..."

? Exercise $2.6.0.5$

Sergei plays right wing and Patrick plays goalie.

? Exercise $2.6.0.6$

Mario is a carpenter, or he is a plumber.

? Exercise $2.6.0.7$

Luigi is a plumber, or he is not a video game character.

? Exercise $2.6.0.8$

Ralph Macchio was the original Karate Kid, and karate is not for defense only.

? Exercise $2.6.0.9$

Some people like broccoli, but my siblings did not like broccoli.

? Exercise $2.6.0.10$

Some people do not like chocolate or all people like pizza.

For the following exercises, write each statement as a conjunction or disjunction in symbolic form by applying the property for the negation of a conditional.

? Exercise $2.6.0.11$

$\sim(p \rightarrow q)$

Exercise 2.6.0.12

$\sim(p \rightarrow \sim q)$

Exercise 2.6.0.13

$\sim(\sim p \rightarrow q)$

Exercise 2.6.0.14

$\sim(\sim p \rightarrow \sim q)$

Exercise 2.6.0.15

$\sim(p \wedge q \rightarrow \sim r)$

Exercise 2.6.0.16

$\sim(p \rightarrow q \vee r)$

Exercise 2.6.0.17

$\sim(p \vee q \rightarrow \sim r)$

Exercise 2.6.0.18

$(p \rightarrow q \wedge r)$

For the following exercises, write the negation of each conditional in words by applying the property for the negation of a conditional.

Exercise 2.6.0.19

If a student scores an 85 on the final exam, then they will receive an A in the class.

Exercise 2.6.0.20

If a person does not pass their road test, then they will not receive their driver's license.

Exercise 2.6.0.21

If a student does not do their homework, then they will not play video games.

Exercise 2.6.0.22

If a commuter misses the bus, then they will not go to work today.

Exercise 2.6.0.23

If a racecar driver gets pulled over for speeding, then they will not make it to the track on time for the race.

Exercise 2.6.0.24

If Rene Descartes was a philosopher, then he was not a mathematician.

Exercise 2.6.0.25

If George Boole invented Boolean algebra and Thomas Edison invented the light bulb, then Pacman is not the best video game ever.

Exercise 2.6.0.26

If Jonas Salk created the polio vaccine, then his child received the vaccine or his child had polio.

Exercise 2.6.0.27

If Billie Holiday sang the blues or Cindy Lauper sang about true colors, then John Lennon was not a Beatle.

Exercise 2.6.0.28

If Percy Jackson is the lightning thief and Artemis Fowl is a detective, then Artemis Fowl will catch Percy Jackson.

Exercise 2.6.0.29

If all rock stars are men, then Pat Benatar is not a rock star.

Exercise 2.6.0.30

If Lady Gaga is a rock star, then some rock stars are women.

Exercise 2.6.0.31

If yellow combined with blue makes green, then all colors are beautiful.

Exercise 2.6.0.32

If leopards have spots and zebras have stripes, then some animals are not monotone in color. For the following exercises, construct a truth table to verify that the logical property is valid.

Exercise 2.6.0.33

$p \rightarrow q \equiv \sim p \vee q$

Exercise 2.6.0.34

$p \rightarrow \sim q \equiv \sim p \vee \sim q$

Exercise 2.6.0.35

$\sim p \rightarrow q \equiv p \vee q$

Exercise $2.6.0.36$

$\sim (p \to \sim q \vee \sim r) \equiv p \wedge q \wedge r$

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2.7: Logical Arguments

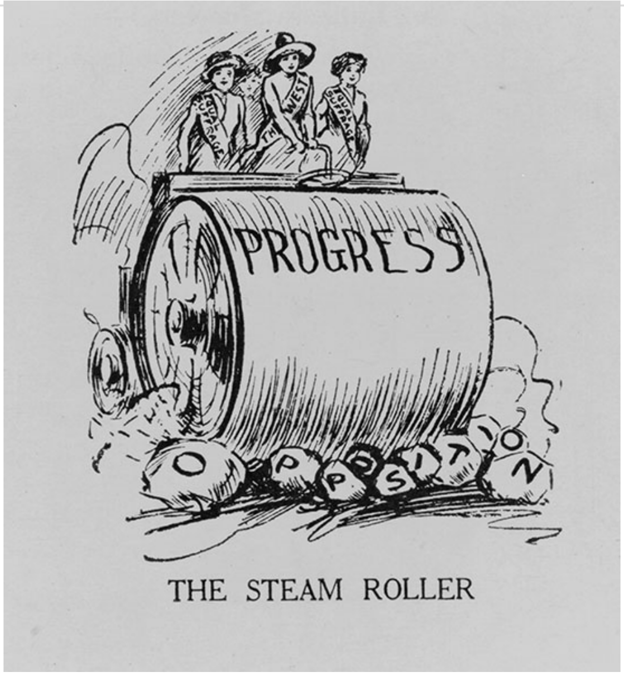


Figure 2.7.1: Not all logical arguments are valid, and the ongoing fight for equal rights proves that much progress has yet to be made. (credit: "The Steam Roller" by Library of Congress Prints and Photographs Division, public domain) ;

Learning Objectives

After completing this section, you should be able to:

1. Apply the law of detachment to determine the conclusion of a pair of statements.
2. Apply the law of denying the consequent to determine the conclusion for pairs of statements.
3. Apply the chain rule to determine valid conclusions for pairs of true statements.

The previous sections of this chapter provide the foundational skills for constructing and analyzing logical arguments. All logical arguments include a set of premises that support a claim or conclusion; but not all logical arguments are valid and sound. A logical argument is valid if its conclusion follows from the premises, and it is sound if it is valid and all of its premises are true. A false or deceptive argument is called a fallacy. Many types of fallacies are so common that they have been named.

Who Knew?

In 1936, Dale Carnegie published his first book, titled *How to Win Friends and Influence People*. It was marketed as training materials for the improvement of public speaking and negotiation skills, and the methods it presented are still used today. Carnegie famously said, “When dealing with people, remember you are not dealing with creatures of logic, but creatures of emotion.”

People who put forth fallacious logical arguments often take advantage of our susceptibility to emotional appeals, to try to convince us that what they are saying is true. The study of logic helps us combat this weakness through recognition and learning to focus on the facts and structure of the argument.

This section focuses on the two main forms that logical arguments can take. While inductive arguments attempt to draw a more general conclusion from a pattern of specific premises, deductive arguments attempt to draw specific conclusions from at least one or more general premises. Deductive arguments can be proven to be valid using Venn diagrams or truth tables.

Inductive arguments generally cannot be proven to be true. They are judged as being strong or weak, but, like any opinion, whether you believe an argument is strong or weak often depends on your knowledge of the topic being discussed along with the evidence being provided in the premises. *Hasty generalization* is the name given to any fallacy that presents a weak inductive argument.

Checkpoint

Be careful! Premises may be true or false. If a premise is false, the claims made by the argument should be questioned.

Law of Detachment

The law of detachment is a valid form of a conditional argument that asserts that if both the conditional, $p \rightarrow q$, and the hypothesis, p, are true, then the conclusion q must also be true. The law of detachment is also called affirming the hypothesis (or antecedent) and modus ponens. Symbolically, it has the form $((p \rightarrow q) \wedge p) \rightarrow q$.

Law of Detachment	
Premise:	$p \rightarrow q$
Premise:	p
Conclusion:	$\therefore q$

Checkpoint

The $\therefore$ is read as the word, "therefore."

Looking at the truth table for the conditional statement, the only time the conditional is true is when the hypothesis p is also true. The only place this happens is in the first row, where q is also true, confirming that the law of detachment is a valid argument.

p	q	$p \rightarrow q$
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

Another way to verify that the law of detachment is a valid argument is to construct a truth table for the argument $((p \rightarrow q) \wedge p) \rightarrow q$ and verify that it is a tautology.

p	q	$p \rightarrow q$	$(p \rightarrow q) \wedge p$	$((p \rightarrow q) \wedge p) \rightarrow q$
T	T	T	T	**T**
T	F	F	F	**T**
F	T	T	F	**T**
F	F	T	F	**T**

Venn diagrams may also be used to verify deductive arguments, which include conditional premises. Consider the statement $p \rightarrow q$:Then verify if the conclusion is also represented by the Venn diagram of the premises. If it is, the argument is valid. If it is not, the argument is not valid. The set of guitarists is drawn as a subset of the set of musicians to represent the premise.

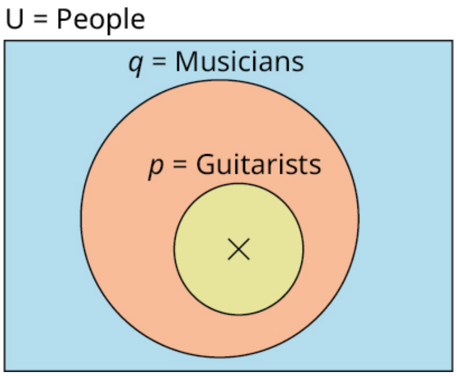


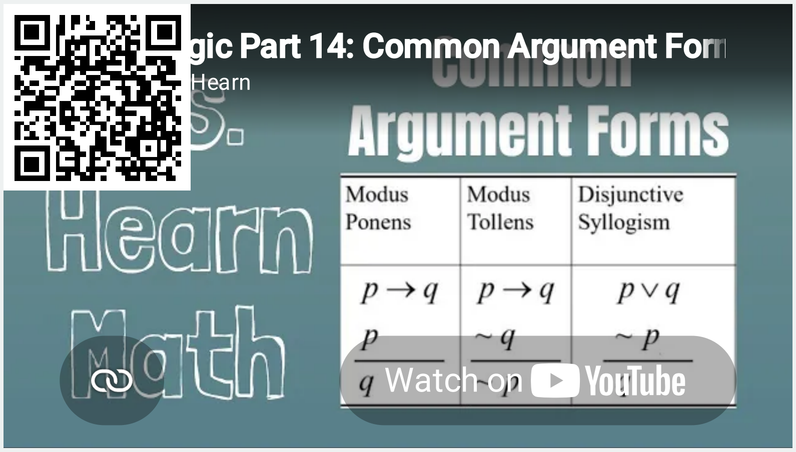
Figure $2.7.2$

Now, examine the Venn diagram to verify if the conclusion is included in the picture. The conclusion is q. Because the $\times$ is in the set p, and p is a subset of q, $\times$ is also in q; therefore, the law of detachment is a valid argument.

Checkpoint

Remember that an argument can be valid without being true. For the argument to be proven true, it must be both valid and sound. An argument is sound if all its premises are true.

Video



Modus Ponens	Modus Tollens	Disjunctive Syllogism
$p \rightarrow q$ p q	$p \rightarrow q$ $\sim q$ $\sim p$	$p \vee q$ $\sim p$

[Logic Part 14: Common Argument Forms like Modus Ponens and Tollens]

Exercise $2.7.1$: Applying the Law of Detachment to Determine a Valid Conclusion

Each pair of statements represents the premises in a logical argument. Based on these premises, apply the law of detachment to determine a valid conclusion.

1. If Leonardo da Vinci was an artist, then he painted the *Mona Lisa*. Leonardo da Vinci was an artist.
2. If Michael Jordan played for the Chicago Bulls, then Michael Jordan was not a soccer player. Michael Jordan played for the Chicago Bulls.
3. If all fish have gills, then clown fish have gills. All fish have gills.

Answer

1. The premises are $p \rightarrow q$: If Leonardo da Vinci was an artist, then he painted the Mona Lisa, and p : Leonardo da Vinci was an artist. This argument has the form of the law of detachment, so, the conclusion is q : Leonardo da Vinci painted the Mona Lisa.

2. The premises follow the form of the law of detachment, so a valid conclusion would be q. The premises are $p \rightarrow q$: If Michael Jordan played for the Chicago Bulls, then Michael Jordan was not a soccer player, and p : Michael Jordan played for the Chicago Bulls. The conclusion that follows from the premises is q : Michael Jordan was not a soccer player.

3. The premises are $p \rightarrow q$: If all fish have gills, then clown fish have gills, and p : All fish have gills. This argument has the form of the law of detachment, so the conclusion is q : clown fish have gills.

Your Turn 2.7.1

Each pair of statements represents the premises in a logical argument. Based on these premises, apply the law of detachment to determine a valid conclusion.

1. If my classmate likes history, then some people like history. My classmate likes history.
2. If you do not like to read, then some people do not like reading. You do not like to read.
3. If the polygon has five sides, then it is not an octagon. The polygon has five sides.

Another form of a valid conditional argument is called the law of denying the consequent, or modus tollens. Recall, that the conditional statement, $p \rightarrow q$, is logically equivalent to the contrapositive, $\sim q \rightarrow \sim p$. So, if the conditional statement is true, then the contrapositive statement is also true. By the law of detachment, if $\sim q$ is also true, then it follows that $\sim p$ must also be true. Symbolically, it has the form $((p \rightarrow q) \wedge \sim q) \rightarrow \sim p$.

Law of Denying the Consequent	
Premise:	$p \rightarrow q$
Premise:	$\sim q$
Conclusion:	$\therefore \sim p$

Checkpoint

The conditional statement can also be described as, "If antecedent, then consequent."This is where the law of denying the consequent gets its name.

To verify if the law of denying the consequent is a valid argument, construct a truth table for the argument, $((p \rightarrow q) \wedge \sim q) \rightarrow \sim p$, and verify that it is a tautology.

p	q	$\sim p$	$\sim q$	$p \rightarrow q$	$(p \rightarrow q) \wedge \sim q$	$((p \rightarrow q) \wedge \sim q) \rightarrow \sim p$
T	T	F	F	T	F	**T**
T	F	F	T	F	F	**T**
F	T	T	F	T	F	**T**
F	F	T	T	T	T	**T**

To verify an argument of this form using a Venn diagram, again consider the premise: $p \rightarrow q$

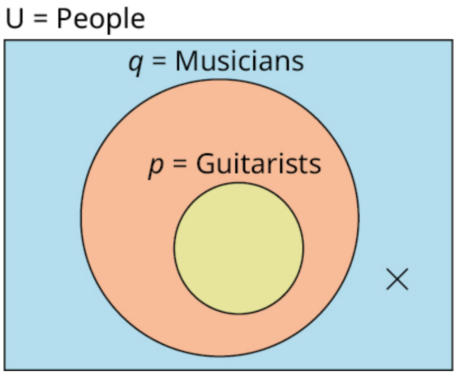


Figure 2.7.3

Because the × is also outside the set of guitarists, the statement $\sim p$ follows from the premises and the argument is valid.

? Exercise 2.7.2: Applying the Law of Denying the Consequent to Determine a Valid Conclusion

Each pair of statements represents the premises in a logical argument. Based on these premises, apply the law of denying the consequent to determine a valid conclusion.

1. If Leonardo da Vinci was an artist, then he painted the *Mona Lisa*. Leonardo da Vinci did not paint the *Mona Lisa*.
2. If Michael Jordan played for the Chicago Bulls, then Michael Jordan was not a soccer player. Michael Jordan was a soccer player.
3. If all fish have gills, then clown fish have gills. Clown fish do not have gills.

Answer

1. The premises are $p \to q$: If Leonardo da Vinci was an artist, then he painted the Mona Lisa, and $\sim q$: Leonardo da Vinci did not paint the Mona Lisa. This argument has the form of the law of denying the consequent, so the conclusion is $\sim p$: Leonardo da Vinci was not an artist.
2. The premises follow the form of the law of denying the consequent, so a valid conclusion would be $\sim p$. The premises are: $p \to q$: If Michael Jordan played for the Chicago Bulls, then Michael Jordan was not a soccer player, and $\sim q$: Michael Jordan was a soccer player. The conclusion that follows from the premises is $\sim p$: Michael Jordan did not play for the Chicago Bulls.
3. The premises are $p \to q$: If all fish have gills, then clown fish have gills, and $\sim q$: Clown fish do not have gills. This argument has the form of the law denying the consequent, so the conclusion is $\sim p$: Some fish do not have gills.

✓ Your Turn 2.7.2

Each pair of statements represents the premises in a logical argument. Based on these premises, apply the law of denying the consequent to determine a valid conclusion.

1. If my classmate likes history, then some people like history. Nobody likes history.

2. If Homer does not like to read, then some people do not like reading. All people like reading.

3. If the polygon has five sides, then it is not an octagon. The polygon is an octagon.

Chain Rule for Conditional Arguments

The chain rule for conditional arguments is another form of a valid conditional argument. It is also called hypothetical syllogism or the transitivity of implication. Recall that the conditional statement $p \to q$ can also be read as p implies q. This is where the name *transitivity of implication* comes from. The transitive property for numbers states that, if $3 < 4$ and $4 < 5$, then it follows that $3 < 5$. The chain rule extends this property to conditional statements. If the premises of the argument consist of two conditional statements, with the form "$p \to q$" and "$q \to r$," then it follows that $p \to r$. Symbolically, it has the form $((p \to q) \wedge (q \to r)) \to (p \to r)$.

Chain Rule for Conditional Arguments	
Premise:	$p \to q$
Premise:	$q \to r$
Conclusion:	$\therefore p \to r$

To verify the chain rule for conditional arguments, construct a truth table for the argument, $((p \to q) \wedge (q \to r)) \to (p \to r)$, and verify that it is a tautology.

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (p \to r)$	$p \to r$	$((p \to q) \wedge (q \to r))$ -
T	T	T	T	T	T	T	**T**
T	T	F	T	F	F	F	**T**
T	F	T	F	T	F	T	**T**
T	F	F	F	T	F	F	**T**
F	T	T	T	T	T	T	**T**
F	T	F	T	F	F	T	**T**
F	F	T	T	T	T	T	**T**

p	q	r	$p \rightarrow q$	$q \rightarrow r$	$(p \rightarrow q) \wedge (p \rightarrow r)$	$p \rightarrow r$	$((p \rightarrow q) \wedge (q \rightarrow r))$ -
F	F	F	T	T	T	T	**T**

To verify an argument of this form using a Venn diagram, again consider the premise $p \rightarrow q$:

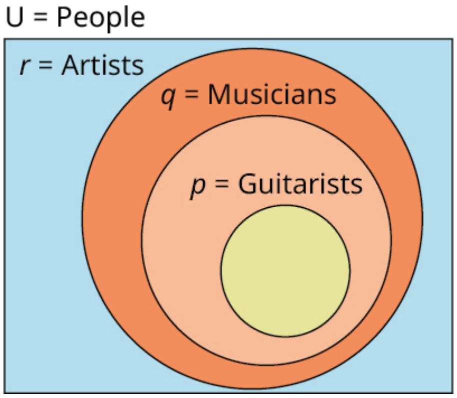


Figure 2.7.4

Exercise 2.7.3: Applying the Chain Rule for Conditional Arguments to Determine a Valid and Sound Conclusion

Each pair of statements represents true premises in a logical argument. Based on these premises, apply the chain rule for conditional arguments to determine a valid and sound conclusion.

1. If my roommate goes to work, then my roommate will get paid. If my roommate gets paid, then my roommate will pay their bills.
2. If robins can fly, then some birds can fly. If some birds can fly, then we will watch birds fly.
3. If Irma is a teacher, then Irma has a college degree. If Irma has a college degree, then Irma graduated from college.

Answer

1. The premises are $p \rightarrow q$: "If my roommate goes to work, then they will get paid," and $q \rightarrow r$: "If my roommate gets paid, then my roommate will pay their bills." This argument has the form of the chain rule for conditional arguments, so the valid conclusion will have the form " $p \rightarrow r$." Because all the premises are true, the valid and sound conclusion of this argument is: "If my roommate goes to work, then my roommate will pay their bills."

2. The premises are $p \rightarrow q$: "If robins can fly, then some birds can fly," and $q \rightarrow r$: "If some birds can fly, then we will watch them fly." This argument has the form of the chain rule for conditional arguments, so, the valid conclusion will have the form " $p \rightarrow r$." Because all the premises are true, the valid and sound conclusion of this argument is: "If robins can fly, then we will watch birds fly."

3. The premises are $p \rightarrow q$: (see line 1 of solution 1 and 2 above) "If Irma is a teacher, then Irma has a college degree," and $q \rightarrow r$: "If Irma has a college degree, then Irma graduated from college." This argument has the form of the chain rule for conditional arguments, so the valid conclusion will have the form " $p \rightarrow r$." Because all the premises are true, the valid and sound conclusion of this argument is: "If Irma is a teacher, then Irma graduated from college."

✓ Your Turn 2.7.3

Each pair of statements represent true premises in a logical argument. Based on these premises, apply the chain rule for conditional arguments to determine a valid and sound conclusion.

1. If my roommate does not go to work, then my roommate will not get paid. If my roommate does not get paid, then they will not be able to pay their bills.

2. If penguins cannot fly, then some birds cannot fly. If some birds cannot fly, then we will watch the news.

3. If Marcy goes to the movies, then Marcy will buy popcorn. If Marcy buys popcorn, then she will buy water.

Check Your Understanding

1. A ___________________ is a logical statement used as a fact to support the conclusion of an argument.
2. A logical argument is ________________ if its conclusion follows from the premises.

3. A logical argument that attempts to draw a more general conclusion from a pattern of specific premises is called an ________________________ argument.
4. A ________________________ argument draws specific conclusions from more general premises.
5. Not all arguments are true. A false or deceptive argument is called a ____________________.
6. If an argument is valid and all of its premises are true, then it is considered _________________.

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2.7.0: Exercises

For the following exercises, analyze the argument and identify the form of the argument as the law of detachment, the law of denying the consequent, the chain rule for conditional arguments, or none of these.

? Exercise $2.7.0.1$

If Apple Inc. releases a new iPhone, then customers will buy it. Customers did not buy a new iPhone. Therefore, Apple Inc. did not release a new iPhone.

? Exercise $2.7.0.2$

In the animated movie Toy Story, if Paul Newman turned down the role of voicing Woody, then Tom Hanks was chosen for the role. Tom Hanks was chosen as the voice for Woody, therefore, Paul Newman turned down the role of voicing Woody in Toy Story.

? Exercise $2.7.0.3$

$p \to q$ and $q \to r. \therefore p \to r.$

? Exercise $2.7.0.4$

$p \to q$ and $p. \therefore q.$

? Exercise $2.7.0.5$

$p \to q$ and $\sim q. \therefore \sim p.$

? Exercise $2.7.0.6$

If all people are created equal, then all people are the same with respect to the law. If all people are the same with respect to the law, then justice is blind. Therefore, if all people are created equal, then justice is blind.

? Exercise $2.7.0.7$

If I mow the lawn, then my caregiver will pay me twenty dollars. I mowed the lawn. Therefore, my caregiver paid me twenty dollars.

? Exercise $2.7.0.8$

If Robin Williams was a comedian, then some comedians are funny. No comedians are funny. Therefore, Robin Williams was not a comedian.

For the following exercises, each pair of statements represents the premises in a logical argument. Based on these premises, apply the law of detachment to determine and write a valid conclusion.

? Exercise $2.7.0.9$

$p \to \sim q$ and $p.$

? Exercise 2.7.0.10

$\sim p \to q$ and $\sim p$.

? Exercise 2.7.0.11

If Richard Harris played Dumbledore, then Daniel Radcliffe played Harry Potter. Richard Harris played Dumbledore.

? Exercise 2.7.0.12

If Emma Watson is an actor, then Emma Watson starred as Belle in the movie Beauty and the Beast. Emma Watson is an actor.

? Exercise 2.7.0.13

If some Granny Smiths are available, then we will make an apple pie. Some Granny Smiths are available.

? Exercise 2.7.0.14

If Peter Rabbit lost his coat, then all rabbits must avoid Mr. McGregor's garden. Peter Rabbit lost his coat. For the following exercises, each pair of statements represents the premises in a logical argument. Based on these premises, apply the law of denying the consequent to determine and write a valid conclusion.

? Exercise 2.7.0.15

If Greg and Ralph are friends, then Greg will not play a prank on Ralph. Greg played a prank on Ralph.

? Exercise 2.7.0.16

If Drogon is not a dragon, then Daenerys ruled Westeros. Daenerys did not rule Westeros.

? Exercise 2.7.0.17

$p \to \sim q$ and q.

? Exercise 2.7.0.18

$\sim p \to q$ and $\sim q$.

? Exercise 2.7.0.19

If all dragons breathe fire, then rainwings are not dragons. Rainwings are dragons.

? Exercise 2.7.0.20

If some pirates have parrots as pets, then some parrots do not like crackers. All parrots like crackers.

For the following exercises, each pair of statements represent true premises in a logical argument. Based on these premises, apply the chain rule for conditional arguments to determine a valid and sound conclusion.

? Exercise 2.7.0.21

$\sim p \to q$ and $q \to \sim r$.

? Exercise 2.7.0.22

$r \rightarrow\sim q$ and $\sim q \rightarrow\sim p$.

? Exercise 2.7.0.23

$q \rightarrow r$ and $p \rightarrow q$.

? Exercise 2.7.0.24

$\sim r \rightarrow p$ and $q \rightarrow\sim r$.

? Exercise 2.7.0.25

If Mr. Spock is a science officer, then Montgomery Scott is an engineer. If Montgomery Scott is an engineer, then James T. Kirk is the captain.

? Exercise 2.7.0.26

If Prince Charles is a character from Star Wars, then Luke Skywalker is not a Jedi. If Luke Skywalker is not a Jedi, then Darth Vader is not his father.

For the following exercises, each pair of statements represent true premises in a logical argument. Based on these premises, state a valid conclusion based on the form of the argument.

? Exercise 2.7.0.27

If my siblings drink milk out of the carton, then they will leave the carton on the counter. My siblings did not leave the carton on the counter.

? Exercise 2.7.0.28

If my friend likes to bowl, then my partner does not like to play softball. My friend likes to bowl.

? Exercise 2.7.0.29

If mathematics is fun, then students will study algebra. If students study algebra, then they will score a 100 on their final exam.

? Exercise 2.7.0.30

If all fleas bite and our dog has fleas, then our dog will scratch a lot. Our dog will not scratch a scratch a lot.

? Exercise 2.7.0.31

If the toddler is not tall, then they will use a stepladder to reach the cookie jar. If the toddler will use a stepladder to reach the cookie jar, then they will drop the jar. If they drop the cookie jar, then they will not eat any cookies.

? Exercise 2.7.0.32

If you do not like to dance, then you will not go to the club. You went to the club.

For the following exercises, use a truth table or construct a Venn diagram to prove whether the following arguments are valid.

? Exercise 2.7.0.33

Denying the hypothesis: $p \to q$ and $\sim p$. Therefore, $\sim q$.

? Exercise 2.7.0.34

Affirming the consequent: $p \to q$ and q. Therefore, p.

? Exercise 2.7.0.35

$\sim p \vee q$ and p. Therefore, q.

? Exercise 2.7.0.36

$p \wedge q \to r$ and $\sim r$. Therefore, $\sim p \vee \sim q$.

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SECTION OVERVIEW

2.8: Chapter Summary

2.8.0: Key Terms

2.8.1: Key Concepts

2.8.2: Videos

2.8.3: Projects

2.8.4: Chapter Review

2.8.5: Chapter Test

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2.8.0: Key Terms

Key Terms

2.1 Statements and Quantifiers

- logic
- logical statement
- truth values
- symbolic form
- negation of a logical statement
- quantifier
- premises
- conclusion
- inductive logical arguments

2.2 Compound Statements

- compound statement
- connective
- conjunction
- disjunction
- conditional
- hypothesis
- conclusion
- biconditional
- dominance of connectives

2.3 Constructing Truth Tables

- Truth table
- Multiplication principle
- Valid

2.5 Equivalent Statements

- logically equivalent
- tautology
- inverse
- converse
- contrapositive

2.6 De Morgan’s Laws

- Boolean logic
- negation of a conditional

2.7 Logical Arguments

- sound
- fallacy
- deductive arguments
- law of detachment
- law of denying the consequent
- chain rule for conditional arguments

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2.8.1: Key Concepts

Key Concepts

2.1 Statements and Quantifiers

- Logical statements have the form of a complete sentence and make claims that can be identified as true or false.
- Logical statements are represented symbolically using a lowercase letter.
- The negation of a logical statement has the opposite truth value of the original statement.
- Be able to
  - Determine whether a sentence represents a logical statement.
  - Write and translate logical statements between words and symbols.
  - Negate logical statements, including logical statements containing quantifiers of *all, some, and none*.

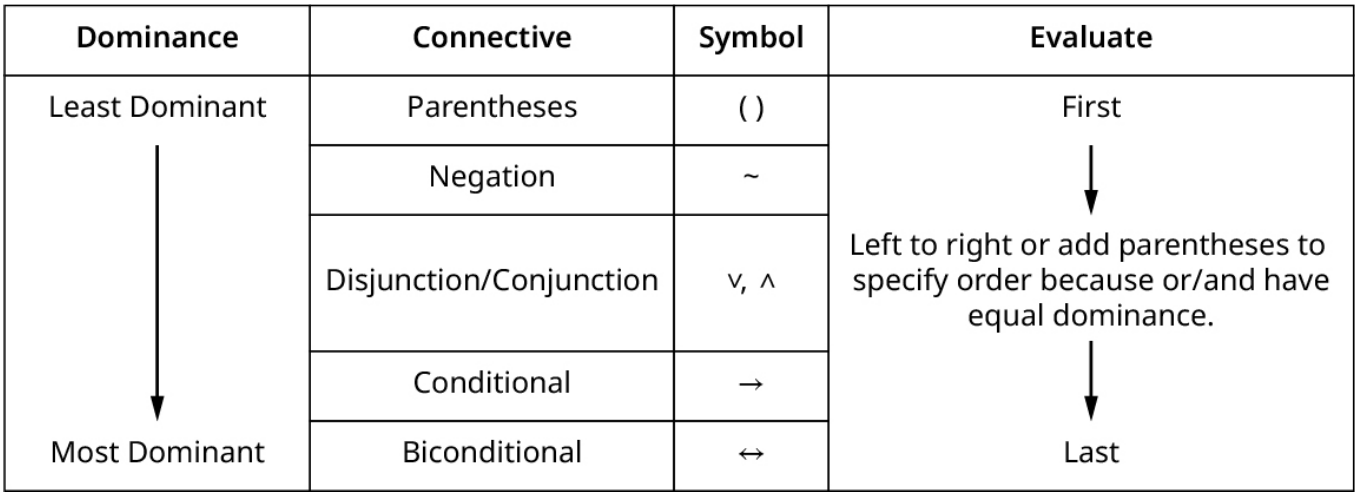
2.2 Compound Statements

- Logical connectives are used to form compound logical statements by using words such as *and, or, and if …, then*.
- A conjunction is a compound logical statement formed by combining two statements with the words “and” or “but.” If the two independent clauses are represented by p and q, respectively, then the conjunction is written symbolically as $p \wedge q$. For the conjunction to be true, both p and q must be true.
- A disjunction joins two logical statements with the *or* connective. In, logic *or* is inclusive. For an *or* statement to be true at least one statement must be true, but both may also be true.
- A conditional statement has the form if p, then q, where p and q are logical statements. The only time the conditional statement is false is when p is true, and q is false.
- The biconditional statement is formed using the connective if and only for the biconditional statement to be true, the true values of p and q, must match. If p is true then q must be true, if p is false, then q must be false.
- Translate compound statements between words and symbolic form.

Connective	Symbol	Name
and but	$\wedge$	conjunction
or	$\vee$	disjunction, inclusive or
not	~	negation
if ..., then implies	$\rightarrow$	conditional, implication
if and only if	$\leftrightarrow$	biconditional

- The dominance of connectives explains the order in which compound logical statements containing multiple connectives should be interpreted.
- The dominance of connectives should be applied in the following order

- Parentheses
- Negations
- Disjunctions/Conjunctions, left to right
- Conditionals
- Biconditionals



Dominance	Connective	Symbol	Evaluate
Least Dominant	Parentheses	( )	First
↓	Negation	~	↓
	Disjunction/Conjunction	∨, ∧	Left to right or add parentheses to specify order because or/and have equal dominance.
↓	Conditional	→	↓
Most Dominant	Biconditional	↔	Last

Figure 2.18

2.3 Constructing Truth Tables

- Determine the true values of logical statements involving negations, conjunctions, and disjunctions.
  - The negation of a logical statement has the opposite true value of the original statement.
  - A conjunction is true when both p and q are true, otherwise it is false.
  - A disjunction is false when both p and q are false, otherwise it is true.
- Know how to construct a truth table involving negations, conjunctions, and disjunctions and apply the dominance of connectives to determine the truth value of a compound logical statement containing, negations, conjunctions, and disjunctions.

Negation			Conjunction (AND)				Disjunction (OR)		
p	$\sim p$		p	q	$p \wedge q$		p	q	$p \vee q$
T	F		T	T	T		T	T	T
F	T		T	F	F		T	F	T
			F	T	F		F	T	T
			F	F	F		F	F	F

- A logical statement is valid if it is always true. Know how to construct a truth table for a compound statement and use it to determine the validity of compound statements involving negations, conjunctions, and disjunctions.

2.4 Truth Tables for the Conditional and Biconditional

- The conditional statement, if p then q, is like a contract. The only time it is false is when the contract has been broken. That is, when p is true, and q is false.

Conditional		
p	q	$p \rightarrow q$
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

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2.8.2: Videos

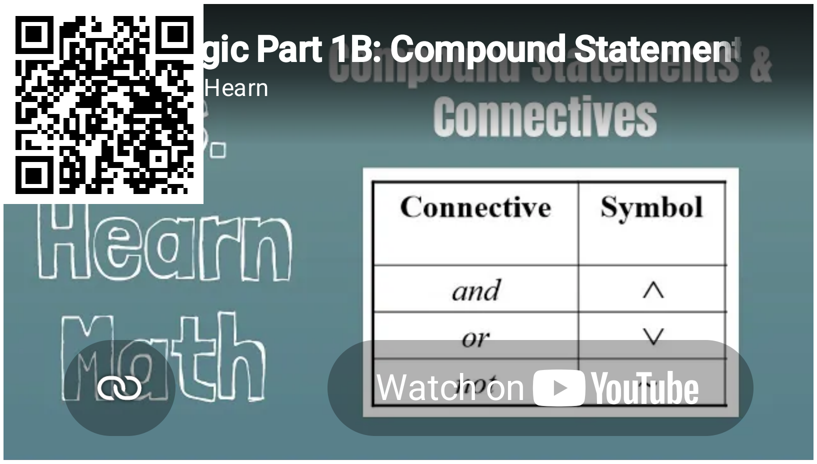
Videos

2.1 Statements and Quantifiers

gic Part 1A: Logic Statements and
Hearn
Logical Statements & Quantifiers
Hearn Math
Watch on YouTube

- Logic Part 1A: Logic Statements and Quantifiers

2.2 Compound Statements


gic Part 1B: Compound Statement
Hearn
Compound Statements & Connectives
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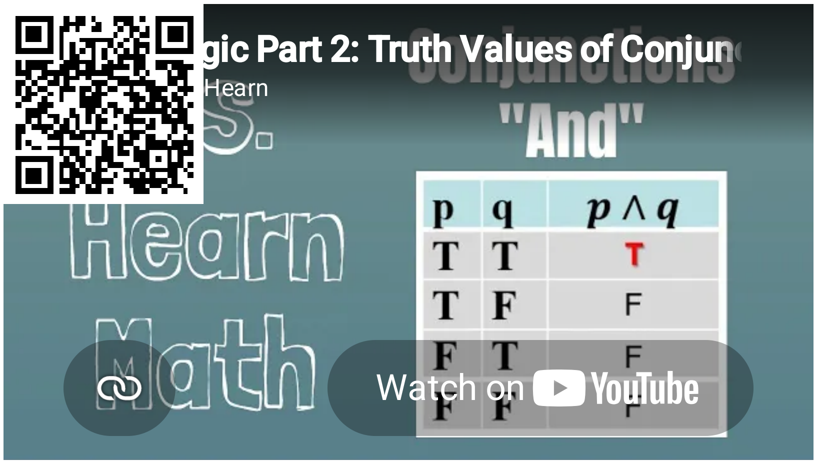
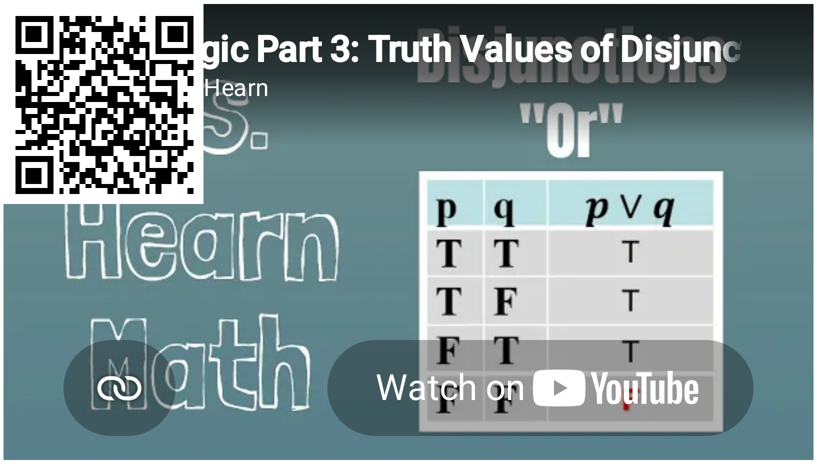
Symbol
and
∧
or
∨
Hearn Math
Watch on YouTube

- Logic Part 1B: Compound Statements, Connectives and Symbols

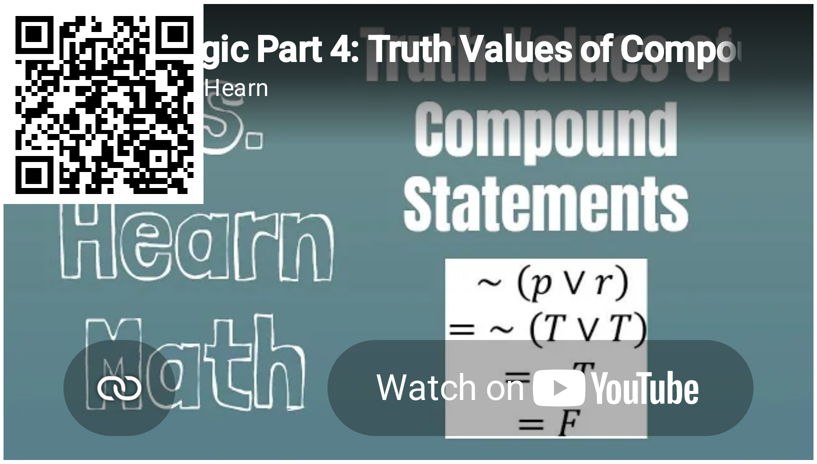
2.3 Constructing Truth Tables


gic Part 2: Truth Values of Conjun
Hearn
Conjunctions
"And"
p q p ∧ q
T T T
T F F
F T F
F F F
Hearn Math
Watch on YouTube

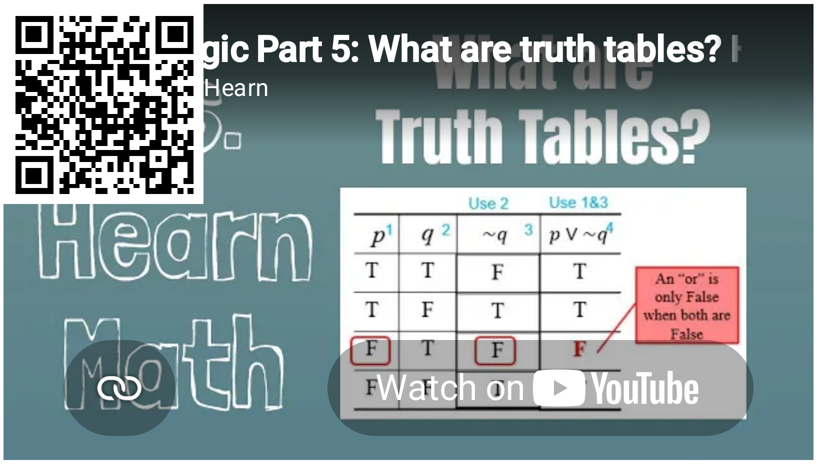
- Logic Part 2: Truth Values of Conjunctions: Is an "AND" statement true or false?


gic Part 3: Truth Values of Disjunc
Hearn
Disjunctions
"Or"
p q p ∨ q
T T T
T F T
F T T
Hearn Math
Watch on YouTube

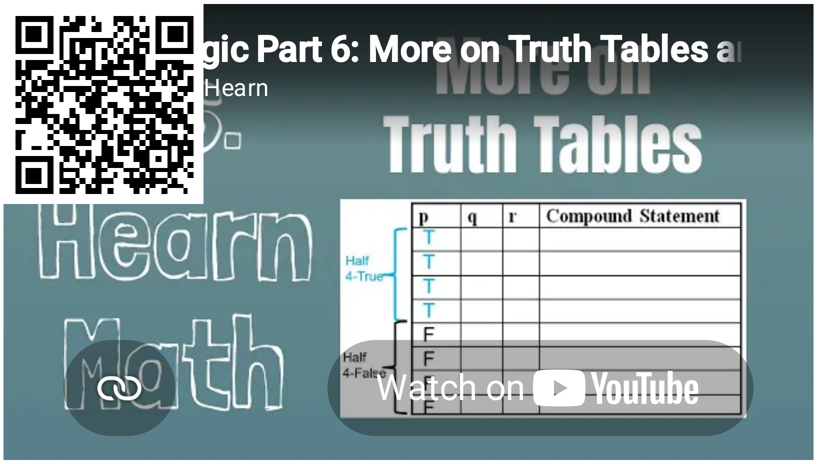
- Logic Part 3: Truth Values of Disjunctions: Is an "OR" statement true or false?


gic Part 4: Truth Values of Compo
Hearn
Truth Values of Compound Statements
~ (p ∨ r)
= ~ (T ∨ T)
= F
Hearn Math
Watch on YouTube

- Logic Part 4: Truth Values of Compound Statements with "and", "or", and "not"

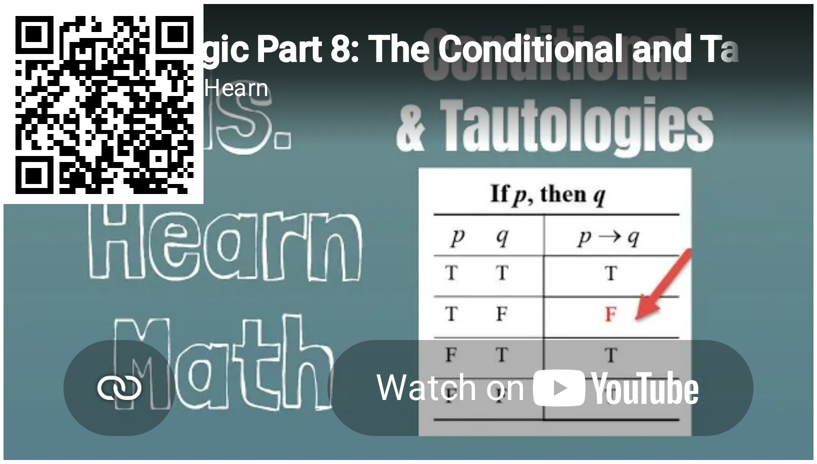

gic Part 5: What are truth tables?
Hearn
What are Truth Tables?
Use 2
Use 1&3
p 1
q 2
~q 3
p ∨ ~q 4
T T F T
T F T T
F T F F
An "or" is only False when both are False
Hearn Math
Watch on YouTube

- Logic Part 5: What are truth tables? How do you set them up?

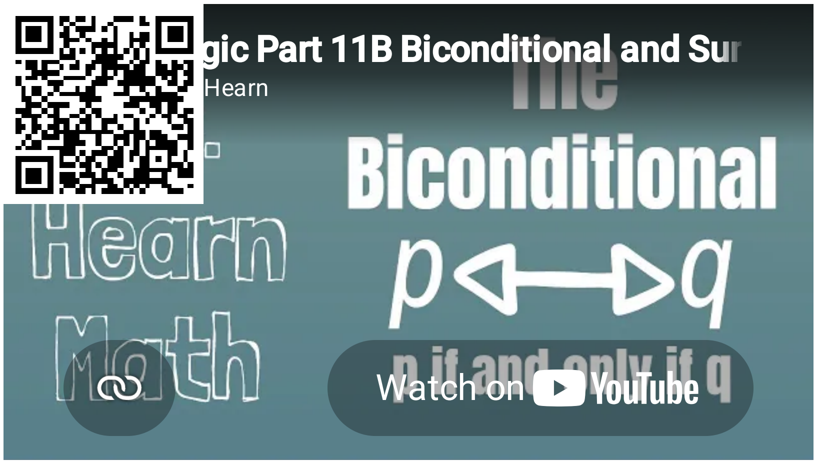

gic Part 6: More on Truth Tables a
Hearn
More on
Truth Tables
p
q
r
Compound Statement
Half 4-True
Half 4-False
Hearn
Math
Watch on YouTube

- Logic Part 6: More on Truth Tables and Setting Up Rows and Column Headings

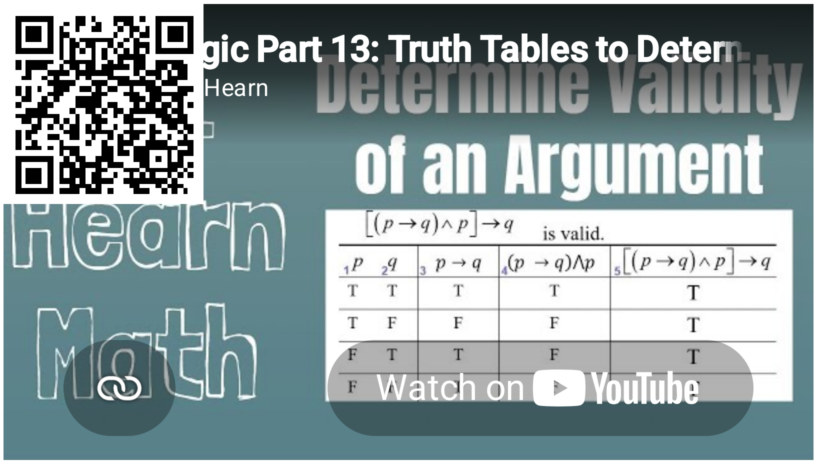
2.4 Truth Tables for the Conditional and Biconditional


gic Part 8: The Conditional and Ta
Hearn
Conditional
& Tautologies
If p, then q
p q p → q
T T T
T F F
F T T
Hearn
Math
Watch on YouTube

- Logic Part 8: The Conditional and Tautologies

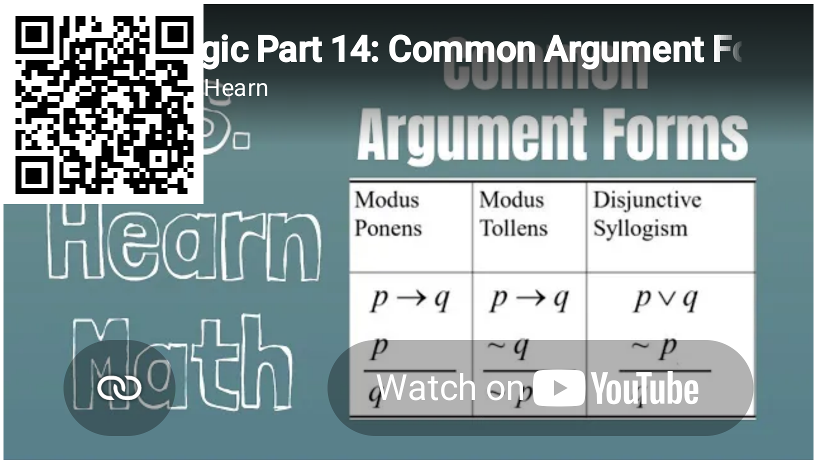

gic Part 11B Biconditional and Su
Hearn
The
Biconditional
p ⟷ q
p if and only if q
Hearn
Math
Watch on YouTube

- Logic Part 11B Biconditional and Summary of Truth Value Rules in Logic


gic Part 13: Truth Tables to Deter
Hearn
Determine Validity
of an Argument
[(p→q)∧p]→q is valid.
Hearn
Math
Watch on YouTube

- Logic Part 13: Truth Tables to Determine if Argument is Valid or Invalid

2.7 Logical Arguments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Modus Ponens
Modus Tollens
Disjunctive Syllogism
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Watch on YouTube

- Logic Part 14: Common Argument Forms like Modus Ponens and Tollens

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2.8.3: Projects

Projects

Logic Gates

Logic gates are the basis for all digital circuits.

1. Research and document the following terms: logic gate, OR gate, AND gate, and NOT gate.
2. Construct a diagram of a NAND gate, NOR gate, and a XOR gate by using at least two of the following gates: AND, OR, and NOT.
3. Digital electronics use a 1 for true or on, and a 0 for false or off. Create a truth table documenting all possible cases using 0s and 1s for the NAND gate, NOR gate and XOR gate.
4. Use a truth table to explain how XOR is related to the biconditional statement.

Logical Fallacies

Fallacies are false or deceptive logical arguments.

1. Research and document the structure of five of the following named fallacies: hasty generalization, limited choice, false cause, appeal to popularity, appeal to emotion, appeal to authority, personal attack, gamblers' ruin, slippery slope, and circular reasoning.
2. Create a presentation highlighting one of the five fallacies researched in the previous question. The presentation must include an introductory slide with the title of the fallacy and the form or structure of the argument. The second slide must include an example of this fallacy as used in a commercial, a political cartoon or a current event or new article. The third slide must include an explanation of why the example on slide to is a representative example of the fallacy. The last slide must include citations for any materials used. No textbooks should be used as reference.

Careers in Logic

Lawyers, mathematicians, and computer programmers are a few of the careers that require knowledge of logic.

1. What career are you interested in? Research how knowledge of logic applies to your chosen field of study. Then, write a cover letter for a position in your field you'd like to apply to. In the cover letter, include how your knowledge of logic qualifies you for the position you are applying for. If you do not think logic is important for your given career choice, find a position where logic is an essential element of the position and complete the project by pretending you are writing a cover letter for that job.

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2.8.4: Chapter Review

Statements and Quantifiers

Fill in the blanks to complete the following sentences.

? Exercise 2.8.4.1

The _______________ of a logical statement has the opposite truth value of the original statement.

? Exercise 2.8.4.2

________________ are logical statements presented as the facts used to support the conclusion of a logical argument.

Determine whether each of the following sentences represents a logical statement, also called a proposition. If it is a logical statement, determine whether it is true or false.

? Exercise 2.8.4.3

Where is the restroom?

? Exercise 2.8.4.4

No even numbers are odd numbers.

? Exercise 2.8.4.5

4 + 3 = 8.

Write the negation of each following statement symbolically and in words.

? Exercise 2.8.4.6

p: Pink Floyd's album *The Wall* is not a rock opera.

? Exercise 2.8.4.7

q: Some dogs are Labrador retrievers.

? Exercise 2.8.4.8

r: Some universities are not expensive.

Draw a logical conclusion to the following arguments, and include in both one of the following quantifiers: all, some, or none.

? Exercise 2.8.4.9

Spaghetti noodles are made with wheat, ramen noodles are made with wheat, and lo mein noodles are made with wheat.

? Exercise 2.8.4.10

A Porsche Boxster does not have four doors, a Volkswagen Beetle does not have four doors, and a Mazda Miata does not have four doors.

Compound Statements

Fill in the blanks to complete the following sentences.

Exercise 2.8.4.11

____________________ are words or symbols used to join two or more logical statement together to from a compound statement.

Exercise 2.8.4.12

____________________ and ____________________ have equal dominance and are evaluated from left to right when no parentheses are present in a compound logical statement.

Translate each compound statement below into symbolic form. Given: p: “Tweety Bird is a bird,” q: “Bugs is a bunny,” r: “Bugs says, ‘What’s up, Doc?’,” s: “Sylvester is a cat,” and t: “Sylvester chases Tweety Bird.”

Exercise 2.8.4.13

If Tweety Bird is a bird, then Sylvester will not chase him.

Exercise 2.8.4.14

Tweety Bird is a bird and Sylvester chases him if and only if Bugs says, “What’s up Doc?”

Translate the symbolic form of each compound logical statement below into words.

Given: p: “Tweety Bird is a bird,” q: “Bugs is a bunny,” r: “Bugs says, ‘What’s up, Doc?’,” s: “Sylvester is a cat,” and t: “Sylvester chases Tweety Bird.”

Exercise 2.8.4.15

$\sim q \vee p \rightarrow \sim s$

Exercise 2.8.4.16

$\sim (p \wedge \sim s) \leftrightarrow q \rightarrow r$

For each of the following compound logical statements, apply the proper dominance of connectives by adding parentheses to indicate the order to evaluate the statement.

Exercise 2.8.4.17

$p \vee q \wedge r \rightarrow \sim s \wedge t$

Exercise 2.8.4.18

$\sim p \rightarrow q \vee r \leftrightarrow p \wedge s \rightarrow \sim t$

Constructing Truth Tables

Fill in the blanks to complete the sentences.

Exercise 2.8.4.19

A _______________ is true if at least one of its component statements is true.

Exercise 2.8.4.20

For a _____________________ to be true, all of its component statements must be true.

Given the statements, p: “No fish are mammals,” q: “All lions are cats,” and r: “Some birds do not lay eggs,” construct a truth table to determine the truth value of each compound statement below.

Exercise 2.8.4.21

$p\wedge \sim r$

Exercise 2.8.4.22

$\sim (p\vee \sim r)$

Exercise 2.8.4.23

$\sim p\vee q\wedge \sim (\sim r)$

Construct a truth table to analyze all the possible outcomes of the following statements, and determine whether the statements are valid.

Exercise 2.8.4.24

$\sim p\vee q\wedge p$

Exercise 2.8.4.25

$\sim p\vee q\vee \sim q$

Truth Tables for the Conditional and Biconditional

Fill in the blanks to complete the following sentences.

Exercise 2.8.4.26

If the , p, of a conditional statement is true, then the conclusion, q, must also be true for the conditional statement $p\rightarrow q$ to be true.

Exercise 2.8.4.27

The biconditional statement $p\leftrightarrow q$ is whenever the truth value of p matches the truth value of q, otherwise it is

Complete the truth tables below to determine the truth value of the proposition in the last column.

Exercise 2.8.4.28

p	q	r	$p\vee q$	$\sim (p\vee q)$	r	$\sim (p\vee q)\rightarrow \sim r$
F	F	T				

? Exercise 2.8.4.29

p	q	q	$p \to q$	$\sim (p \to q)$	$p \wedge \sim q$	$\sim (p \to q) \leftrightarrow (p \wedge \sim q)$
T	F					

Assume the following statements are true. p: “Poof is a baby fairy,” q: “Timmy Turner has fairly odd parents,” r: “Cosmo and Wanda will grant Timmy’s wishes,” and t: “Timmy Turner is 10 years old.” Translate each of the following statements into symbolic form, then determine its truth value.

? Exercise 2.8.4.30

If Timmy Turner is 10 years old and Poof is not a baby fairy, then Timmy Turner has fairly odd parents.

? Exercise 2.8.4.31

Cosmos and Wanda will not grant Timmy’s wishes if and only if Timmy Turner is 10 years old or he does not have fairly odd parents.

? Exercise 2.8.4.32

Construct a truth table to analyze all the possible outcomes and determine the validity of the following argument.

$p \vee q \; \leftrightarrow \;\; q \to \;\; p$

Equivalent Statements

Fill in the blanks to complete the sentences below. 33. The _________________ is logically equivalent to the inverse $\sim p \to \sim q$ $p \to q$.

? Exercise 2.8.4.34

The _________________ is logically equivalent to the conditional $p \to q p \to q$.

Use the conditional statement, $p \to q p \to q$: “If Novak makes the basket, then Novak’s team will win the game," to answer the following questions. 35. Write the conclusion of the conditional statement in words and label it appropriately.

? Exercise 2.8.4.36

Write the hypothesis of the conditional statement in words and label it appropriately.

? Exercise 2.8.4.37

Identify the following statement as the converse, inverse, or contrapositive: “If Novak does not make the basket, then his team will not win the game.”

? Exercise 2.8.4.38

Identify the following statement as the converse, inverse, or contrapositive: “If Novak’s team wins the game, then he made the basket.”

De Morgan’s Laws

Fill in the blanks to complete the sentences.

Exercise 2.8.4.39

De Morgan’s Law for the negation of a disjunction states that \($(p \vee q) \equiv$______________.

Exercise 2.8.4.40

De Morgan’s Law for the negation of a conjunctions states that ______________\($\equiv \ \ p \ \vee \ q$.

Exercise 2.8.4.41

Apply De Morgan’s Law to write the statement without parentheses: \($(\ p \wedge q)$.

Exercise 2.8.4.42

Apply the property for the negation of a conditional to write the statement as a conjunction or disjunction: \($(\ p \wedge q \rightarrow \ r)$.

Exercise 2.8.4.43

Write the negation of the conditional statement in words: If Thomas Edison invented the phonograph, then albums are made of vinyl, or the transistor radio was the first portable music device.

Exercise 2.8.4.44

Construct a truth table to verify that the logical property is valid: $(\ p \rightarrow \ q) \equiv \ p \wedge q$.

Logical Arguments

Fill in the blanks to complete the sentences below.

Exercise 2.8.4.45

The ______ ___ ___________ is a valid logical argument with premises, $p \rightarrow q$ and p, used to support the conclusion, q.

Exercise 2.8.4.46

The chain rule for conditional arguments states that the _____________________ property applies to conditional arguments, so that: \($(p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$.

Assume each pair of statements represents true premises in a logical argument. Based on these premises, state a valid conclusion that is consistent with the form of the argument.

Exercise 2.8.4.47

If the Tampa Bay Buccaneers did not win Super Bowl LV, then Tom Brady was not their quarterback. Tom Brady was the Tampa Bay Buccaneers quarterback.

Exercise 2.8.4.48

If $\ q$, then p and if r, then $\ q$.

? Exercise 2.8.4.49

If Kamala Harris is the vice president of the United States, then Kamala Harris is the president of the U.S. Senate. Kamala Harris is the vice president of the United States.

? Exercise 2.8.4.50

Construct a truth table or Venn diagram to prove whether the following argument is valid. If the argument is valid, determine whether it is sound.

If all frogs are brown, then Kermit is not a frog. Kermit is a frog. Therefore, some frogs are not brown.

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2.8.5: Chapter Test

Chapter Test

Determine whether each of the following sentences represent a logical statement. If it is a logical statement, determine whether it is true or false.

1. $1+2-3=0$

2. Please, sit down over there.

3. All mammals lay eggs.

Write the negation of each statement below.

4. Some monkeys do not have tails.

5. $p \wedge \sim q$

6. If the plumber does not remove the clog, then the homeowner will not pay the plumber.

Given: p : Frodo is a hobbit, q : Gandalf is a wizard, r : Frodo and Samwise will take the ring to Mordor, and s : Gollum will help Frodo get into Mordor.
Translate the symbolic form of each compound logical statement into words.

7. $p \vee q \leftrightarrow r$

8. $\sim(\sim r \vee \sim s)$

Translate the written form of each compound logical statement into symbolic form.

9. Frodo and Samwise will take the ring to Mordor or Gandalf is not a wizard and Frodo is a hobbit.

10. If Gollum will not help Frodo get into Mordor, then Gandalf is not a wizard and Frodo is not a hobbit.

For each of the following compound logical statements, apply the proper dominance of connectives by adding parentheses to indicate the order in which the statement must be evaluated

11. $\sim p \rightarrow q \leftrightarrow r$

12. $\sim(p \wedge q) \leftrightarrow \sim p \vee \sim q$

13. Complete the truth table to determine the truth value of the proposition in the last column.

p	q	$p \wedge q$	$\sim q$	$p \vee \sim q$	$(p \wedge q) \leftrightarrow (p \vee \sim q)$
F	T				

Given the true statements p : "A right triangle has one 90-degree angle," q : "The triangle is a right triangle," r : " $a^2+b^2=c^2,''$ and s : "The longest side of a triangle is c implies $a+b$ must be $>c$." Write each of the following compound statements in symbolic form, then construct a truth table to determine the truth value of the compound statement.

14. If a triangle is a right triangle, then it does not have one 90-degree angle or $a^2+b^2=c^2$.

15. The triangle is a right triangle, or a right triangle does not have a 90-degree angle, if and only if it is not the case that the longest side of a triangle is c implies $a+b$ must be $>c$.

Use the conditional statement, $p \rightarrow q$: "If Phil Mickelson is 50 years old, then Phil Mickelson won the Player's Championship," to answer the following questions.

16. Write the converse statement in words.

17. If the conditional statement is true, and the hypothesis is true, what is a valid conclusion to the argument?

18. If the conditional statement is true, and the conclusion is false, what is a valid conclusion to the argument?

19. Construct a truth table to analyze all the possible outcomes and determine the validity of the following argument.

$$\sim p \vee q \leftrightarrow q \rightarrow p$$

20. Construct a truth table or Venn diagram to prove whether the following argument is valid. If the argument is valid, determine whether it is sound.
If John Mayer played MTV unplugged, then some guitars are acoustic. John Mayer played MTV unplugged. Therefore, some guitars are acoustic.

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CHAPTER OVERVIEW

3: Real Number Systems and Number Theory

- 3.0: Introduction
- 3.1: Prime and Composite Numbers
  - 3.1.0: Exercises
- 3.2: The Integers
  - 3.2.0: Exercises
- 3.3: Order of Operations
  - 3.3.0: Exercises
- 3.4: Rational Numbers
  - 3.4.0: Exercises
- 3.5: Irrational Numbers
  - 3.5.0: Exercises
- 3.6: Real Numbers
  - 3.6.0: Exercises
- 3.7: Clock Arithmetic
  - 3.7.0: Exercises
- 3.8: Exponents
  - 3.8.0: Exercises
- 3.9: Scientific Notation
  - 3.9.0: Exercises
- 3.10: Arithmetic Sequences
  - 3.10.0: Exercises
- 3.11: Geometric Sequences
  - 3.11.0: Exercises
- 3.12: Chapter Summary
  - 3.12.0: Key Terms
  - 3.12.1: Key Concepts
  - 3.12.2: Videos
  - 3.12.3: Formula Review
  - 3.12.4: Projects
  - 3.12.5: Chapter Review
  - 3.12.6: Chapter Test

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3.0: Introduction

Encryption is used to secure online banking, for secure online shopping, and for browsing privately using VPNs (Virtual Private Networks). We need encryption (using prime numbers) for a secure exchange of information. For a prime number to be useful for encryption, though, it has to be large. Encryption uses a composite number that is the product of two very large primes. In order to break the encryption, one must determine the two primes that were used to form the composite number. If the two prime numbers used are sufficiently large, even the fastest computer cannot determine those two prime numbers in a reasonable amount of time. It would take a computer 300 trillion years to crack the current encryption standard.

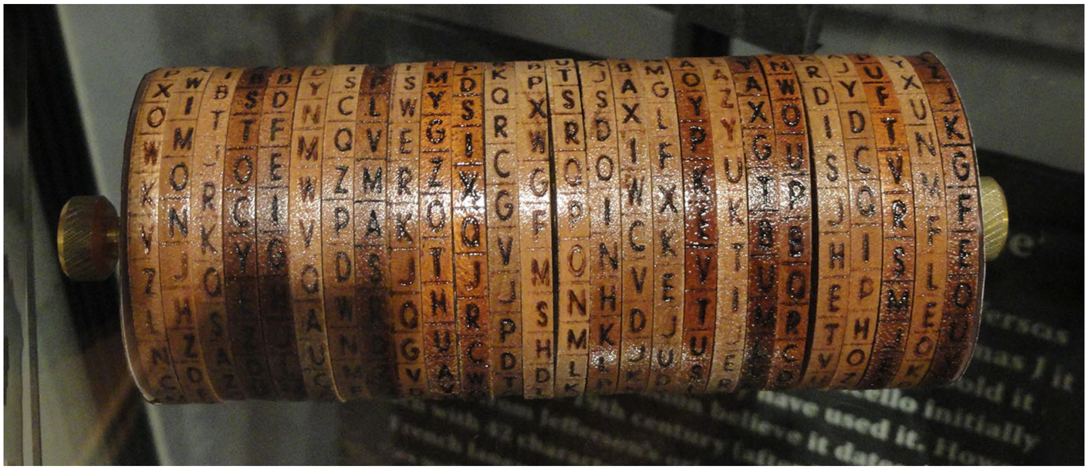

Figure 3.1 Encryption of computers and messages use very large prime numbers. (credit: modification of work "Jefferson cylinder cipher (replica)" by Daderot/Wikimedia Commons, Public Domain)

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3.1: Prime and Composite Numbers

Learning Objectives

After completing this section, you should be able to:

- Apply divisibility rules.
- Define and identify numbers that are prime or composite.
- Find the prime factorization of composite numbers.
- Find the greatest common divisor.
- Use the greatest common divisor to solve application problems.
- Find the least common multiple.
- Use the least common multiple to solve application problems.

Encryption, which is needed for the secure exchange of information (i.e., online banking or shopping) is based on prime numbers. Encryption uses a composite number that is the product of two very large prime numbers. To break the encryption, the two primes that were used to form the composite number need to be determined. If the two prime numbers used are sufficiently large, even the fastest computer cannot determine those two prime numbers in a reasonable amount of time. It would take a computer 300 trillion years to crack the current encryption standard.

Figure $3.1.1$: Computers are protected using encryption based on prime numbers. (credit: “Data Security” by Blogtrepreneur/Flickr, CC BY 2.0)

Applying Divisibility Rules

Before we begin our investigation of divisibility, we need to know some facts about important sets of numbers:

- The counting numbers are referred to as the natural numbers. This set of numbers, $\{1, 2, 3, 4, \ldots\}$, is denoted with the symbol $\mathbb{N}$.
- Another important set of numbers is the integers. The integers are the natural numbers, along with 0 , and the negatives of the natural numbers. This set is often written as $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$. We denote the integers with the symbol $\mathbb{Z}$.
- Notice that $\mathbb{N}$ is a proper subset of $\mathbb{Z}$, or, $\mathbb{N} \subset \mathbb{Z}$. All the ideas of this section apply to the natural numbers, while only some apply to all the integers.

Divisibility is when the integer n is divisible by m, if n can be written as m times another integer. Equivalently, there is no remainder when n is divided by m. There are many occasions when separating items into equal groups comes into play to ensure an equal distribution of whole items. For example, Francis, a preschool art teacher, has 15 students in one class. Francis has 225 sheets of construction paper and wants to provide each student with an equal number of pieces. To know if he will use all the construction paper, Francis is really asking if 225 can be evenly divided into 15 groups.

✓ Example 3.1.1: Determining if a Number Divides Another Number

Determine if 36 is divisible by 4.

Answer

We could divide 36 by 4 and see if there is a remainder, or we could see if we can write 36 as 4 times another integer. If we divide 36 by 4, we see $36 \div 4 = 9$ with no remainder. We see that 36 is divisible by 4. We can write 36 as 4 times another integer, $36 = 4 \times 9$. By the definition of divisibility, 36 is divisible by 4.

Your Turn 3.1.1

Determine if 54 is divisible by 9.

You can quickly check if a number is divisible by 2, 3, 4, 5, 6, 9, 10, and 12. Each has an easy-to-identify feature, or rule, that indicates the divisibility by those numbers, as shown in the following table.

Divisor	Rule
2	Last digit is even
3	Add the digits of the number together. If that sum is divisible by 3, then so is the original number
4	Look at only the last two digits. If this number is divisible by 4, so is the original number
5	Look at only the last digit. If it is a 5 or a 0, then the original number is divisible by 5
6	If the number passes the rule for divisibility by 2 and for 3, then the number is divisible by 6
9	Add the digits of the number together. If that number is divisible by 9, then so is the original number
10	Look at only the last digit. If it is a 0, then the original number is divisible by 10
12	If the number passes the rule for 3 and 4, the number is divisible by 12

✓ Example 3.1.2: Using Divisibility Rules

Using divisibility rules, determine if 245 is divisible by 5.

Answer

Since the last digit is a 5, the number 245 is divisible by 5 because the rule states if the last digit of the number is a 5 or a 0, then the original number is divisible by 5.

Your Turn 3.1.12

1. Using divisibility rules, determine if 45,730 is divisible by 5.

✓ Example $3.1.3$: Using Divisibility Rules

Using divisibility rules, determine if 25,983 is divisible by 9.

Answer

The divisibility rule for 9 is when the digits of the number are added, the sum is divisible by 9. So, we calculate the sum of the digits.

$2+5+9+8+3=27$. Since 27 is divisible by 9, so is the original number 25,983.

Your Turn $3.1.3$

Using divisibility rules, determine if 342,887 is divisible by 9.

✓ Example $3.1.4$: Using Divisibility Rules

Can 298 coins be stacked into 6 stacks with an equal number of coins in each stack?

Answer

In order for the coins to be in equal-sized stacks, 298 would need to be divisible by 6. The divisibility rule for 6 is that the number passes the divisibility rules for both 2 and 3. Since the last digit is even, 298 is divisible by 2. To determine if 298 is divisible by 3, we first add the digits of the number: $2+9+8=19$. Since 19 is not divisible by 3, neither is 298. Because 298 is not divisible by 3, it is also not divisible by 6, which means they cannot be put into 6 equal stacks of coins.

Your Turn $3.1.4$

1.Can 43,568 pieces of mail be separated into 6 bins with the same number of pieces of mail per bin?

✓ Example $3.1.5$: Using Divisibility Rules

Using divisibility rules, determine if 4,259 is divisible by 10.

Answer

The divisibility rule for 10 is that the last digit of the number is 0. Since the last digit of 4,259 is not 0, then 4,259 is not divisible by 10.

Your Turn $3.1.5$

Using divisibility rules, determine if 87,762 is divisible by 10.

✓ Example $3.1.6$: Using Divisibility Rules

Using divisibility rules, determine if 936,276 is divisible by 4.

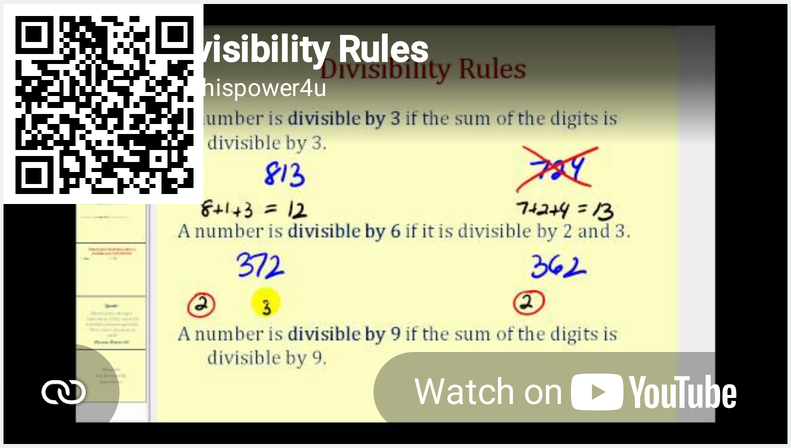
Answer

The divisibility rule for 4 is to check the last two digits of the number. If the number formed by the last two digits of the original number is divisible by 4, then so is the original number. The last two digits make the number 76 and 76 is divisible by 4, since $76=4\times 19$. Since 76 is divisible by 4, so is 936,276.

Your Turn 3.1.6

Using divisibility rules, determine if 43,568 is divisible by 4.

Video



umber is divisible by 3 if the sum of the digits is divisible by 3.

813

8+1+3 = 12

~~724~~

7+2+4 = 13

A number is divisible by 6 if it is divisible by 2 and 3.

372

362

A number is divisible by 9 if the sum of the digits is divisible by 9.

Divisibility Rules

Prime and Composite Numbers

Sometimes, a natural number has only two unique divisors, 1 and itself. For instance, 7 and 19 are prime. In other words, there is no way to divide a prime number into groups with an equal number of things, unless there is only one group, or those groups have one item per group. Other natural numbers have more than two unique divisors, such as 4, or 26. These numbers are called composite. The number 1 is special; it is neither prime nor composite.

To determine if a number is prime or composite, you have to determine if the number has any divisors other than 1 and itself. The divisibility rules are useful here, and can quickly show you if a number has a divisor on that list.

However, if none of those divide the number, you still have to check all other possible prime divisors. What are the prime numbers that are possibly divisors of the number you are checking? You need only check the prime numbers up to the square root of the number in question. For instance, if you want to know if 2,117 is prime, you need to determine if any primes up to the square root of 2,117 (which is 46.0 when rounded to one decimal place) divide 2,117. If any of those primes are divisors of the number in question, then the number is composite. If none of those primes work, then the number is itself prime.

We can check divisibility with whatever tool we wish. Divisibility rules are quick for some prime divisors (2 and 5 come to mind) but aren't quick for other values (like 11). In place of divisibility rules, we could just use a calculator. If the prime number divides the number evenly (that is, there is no decimal or fractional part), then the number is divisible by that prime. Table 3.1 is a quick list of the prime numbers up to 50. There are 15 prime numbers less than 50.

Table 3.1 Prime Numbers Less than 50

2	3	5	7	11
13	17	19	23	29
31	37	41	43	47

Example 3.1.7: Determining If a Number Is Prime or Composite

Determine if 2,117 is prime or composite.

Answer

The square root of 2,117 is 46.0 (rounded to one decimal place). So, we need to check if 2,117 is divisible by any prime up to 46.

Step 1: First we'll use the rules of divisibility we learned earlier:

- We can tell 2,117 is not divisible by 2, as the last digit isn't even.
- 2,117 is not divisible by 5 (the last digit isn't 0 or 5).
- Add the digits of 2,117 to get 11, which is not divisible by 3. So, 2,117 is also not divisible by 3.

Step 2: Now we repeat the process for all the primes up to 46.

Using a calculator, we find that 2,117 divided by the prime numbers 7, 11, 13, 17, 19, and 23 results in a remainder, a decimal part. So, we know that 2,117 is not divisible by these prime numbers. (You should check these results yourself.)

Moving on, we check the next prime: 29. Using the calculator to divide 2,117 by 29 results in 73. Since there is no decimal part, 2,117 is divisible by 29.

This means that 2,117 is not a prime number, but rather, a composite number. Writing 2,117 as the product of 29 and another natural number, $2{,}117 = 29 \times 73$.

Your Turn 3.1.7

Determine if 1,429 is prime or composite.

✓ Example 3.1.8: Determining if a Number Is Prime or Composite

Determine if 423 is prime or composite.

Answer

The square root of 423 is 20.57 (rounded to two decimal places). So, we need to check if 423 is divisible by any prime up to 20.

Step 1: Check 2. We can tell 423 is not divisible by 2, as the last digit isn't even.

Step 2: Check 5. It is not divisible by 5 (the last digit isn't 0 or 5).

Step 3: Check 3. To check if 423 is divisible by 3, we use the divisibility rule for 3. When we take the sum of the digits of 423, the result is 9. Since 9 is divisible by 3, so is 423.

Since 423 is divisible by 3, then 423 is a composite number. Writing 423 as the product of 3 and another natural number, $423 = 3 \times 141$.

Your Turn 3.1.8

Determine if 859 is prime or composite.

✓ Example 3.1.9: Determining if a Number Is Prime or Composite

Determine if 1,034 is prime or composite

Answer

A quick inspection of 1,034 shows it is divisible by 2 since the last digit is even, and so 1,034 is a composite number.

Your Turn 3.1.9

Determine if 5,067,322 is prime or composite.

✓ Example 3.1.10: Determining if a Number Is Prime or Composite

Determine if 2,917 is prime or composite.

Answer

The square root of 2,917 is 50.01 (rounded to two decimal places). So, we need to check if 2,917 is divisible by any prime up to 50.

Step 1: Check 2. We can tell 2,917 is not divisible by 2, as the last digit isn't even.

Step 2: Check 5. It is it divisible by 5 (the last digit isn't 0 or 5).

Step 3: Check 3. Using the divisibility rule for 3, we take the sum of the digits of 2,917, which is 19. Since 19 is not divisible by 3, neither is 2,917.

Step 4: Check the rest of the primes up to 50 using a calculator. When 2,917 is divided by every prime number up to 50, the result has a decimal part.

Since no prime up to 50 divides 2,917, it is a prime number.

Your Turn 3.1.10

Determine if 1,477 is prime.

Who Knew?: ILLEGAL PRIMES

Large primes are a hot commodity. Using two very large primes (some have more than 22 million digits!) is necessary for secure encryption. Anyone who has a new prime that is large enough can use that prime to create a new encryption. Of course, whoever discovers a large prime could sell it to a security company. These primes are so useful for encryption, it is necessary to protect that intellectual property. In fact, at least one prime number was declared illegal.

Video

L #7: This Number is Illegal
dover Productions
illegal
Watch on YouTube

Illegal Prime Number

People in Mathematics: Sophie Germain

Sophie GERMAIN (portrait de), à l'âge de 14 ans.

Figure $3.1.2$: Sophie Germain (credit: “Sophie Germain at 14 years,” Illustration from histoire du socialisme, approx. 1880, Wikimedia Commons, public domain)

Born into a wealthy French family in 1776, Sophie Germain discovered and fell in love with mathematics by browsing her father’s books. Clandestine study, hard and tenacious work, and a mathematical mindset did not lead to college, however, as she was not allowed to attend. She did manage, through friends, to obtain problem sets and submit brilliant solutions under the name Monsieur LaBlanc. One of her great interests was number theory, which is the study of properties of integers. One of her theorems, titled “Sophie Germain’s Theorem,” partially solved one of the great mathematical mysteries, Fermat’s Last Theorem. From this she discovered what are now known as Sophie Germain Primes. A Sophie Germain Prime is a prime number that can be written in the form $2p+1$2p+1, where p is a prime number. For instance, 23 is prime: $2(23)+1=47$, which is prime, so 47 is a Sophie Germain Prime. It should be noted that $2p+1$2p+1, where p is a prime number, may or may not be prime (check for $p=7$!).

Finding the Prime Factorization of Composite Numbers

Before we can start with prime factorization, let’s remind ourselves what it means to factor a number. We factor a number by identifying two (or more) numbers that, when multiplied, result in the original number. For instance, 3 and 24, when multiplied, give 72. So, 72 can be factored into 3×24. Notice that we could have factored the 72 differently, say as $72 = 6 \times 12$, or $72 = 2 \times 36$, or even as $72 = 3 \times 4 \times 6$.

Finding the prime factorization of a composite number means writing the number as the product of all of its prime factors. For example, $80 = 2 \times 2 \times 2 \times 2 \times 5$. Notice that all the numbers being multiplied on the right-hand side are prime numbers. Sometimes prime numbers repeat themselves in the factorization. When prime factors do repeat, we may write the prime factorization using exponents, as in $80 = 2^4 \times 5$. In that equation, the 2 is raised to the 4th power. The 4 is the exponent, and the 2 is the **base**. More generally, in the exponent notation semanticsabab, the number represented by a is the base, and the number represented by b is the exponent.

One has to wonder if finding the prime factorization could result in different factorizations. The **Fundamental Theorem of Arithmetic** tells us that there is only one prime factorization for a given natural number.

Fundamental Theorem of Arithmetic

Every natural number, other than 1, can be expressed in exactly one way, apart from the arrangement, as a product of primes.

The process of finding the prime factorization of a number is iterative, which means we do a step, then repeat it until we cannot do the step any longer. The step we use is to identify one prime factor of the number, then write the number as the prime factor times another factor. We repeat this step on the other, newly found, factor. This step is repeated until no more primes can be factored from the remaining factor. This is easier to see and explain with an example.

✓ Example $3.1.11$: Finding the Prime Factorization

Find the prime factorization of 140.

Answer

Step 1: Identify a prime number that divides 140. Since 140 is even (the last digit is even), 2 divides 140. We then factor the 2 out of the 140, giving us $140 = 2 \times 70$.

Step 2: With the other factor, 70, find a prime factor of 70. Since 70 is also even, 2 divides 70. We factor the 2 out of the 70 and the factorization is now $140 = 2 \times 2 \times 35 = 2^2 \times 35$.

Notice that we expressed the two factors of 2 as 2^2.

Step 3: Look to the remaining factor, 35. The last digit of 35 is 5, so 5 is a factor of 35. We factor the 5 out of the 35. The factorization is now $140 = 2^2 \times 5 \times 7$.

Step 4: Look to the remaining factor, 7. Since 7 is prime, the process is complete.

The prime factorization of 140 is $2^2 \times 5 \times 7$.

Your Turn $3.1.11$

Find the prime factorization of 90.

Factor Trees

A useful tool for helping with prime factorization is a **factor tree**. To create a factor tree for the natural number n (where n is not 1), perform the following steps:

Step 1: If n is prime, you're done. If n is composite, continue to the next step.

Step 2: Identify two divisors of n, call them a and b.

Step 3: Write the number n down, and draw two branches extending down (or to the right) of the number n.

Step 4: Label the end of one branch semanticsaFigure 3.4.

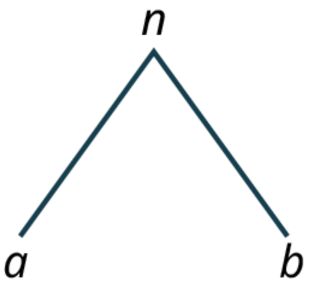


Figure $3.1.3$

Step 5: If a and b are prime, the tree is complete. When a number at this step is a prime number, we refer to it as a leaf of the tree diagram.

Step 6: If either a or b are composite, repeat Steps 2 through 4 for a and b.

Step 7: The process stops when the leaves are all prime.

Step 8: The prime factorization is then the product of all the leaves.

This is best seen in an example.

✓ Example 3.1.12: Finding the Prime Factorization

Find the prime factorization of 66.

Answer

Since 66 is even, 2 is a factor.

Step 1: Factor out the 2. The factorization is $66 = 2 \times 33$Figure 3.5.

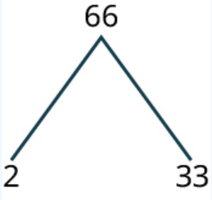


Figure ✓ 3.1.4

Since 2 is a factor, that branch is done, and 2 is a leaf.

Step 2: The 33, though, is divisible by 3, and is the product of 3 and 11. We attach that to the factor tree (Figure 3.6).

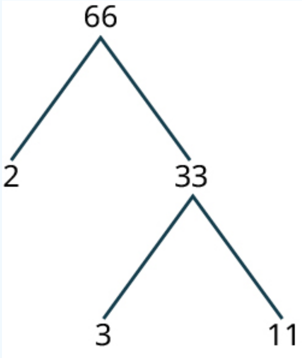


Figure ✓ 3.1.5

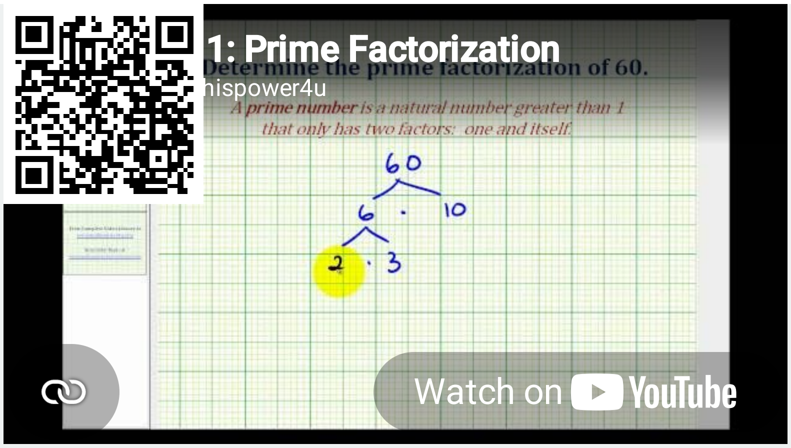
Since the 2, 3 and 11 are all prime, the factor tree is done.

The prime factorization of 66 is the product of the leaves, so $66 = 2 \times 3 \times 11$. The factorization is complete.

Your Turn 3.1.12

Find the prime factorization of 85.

Video


1: Prime Factorization
Determine the prime factorization of 60.
hispower4u
A prime number is a natural number greater than 1
that only has two factors: one and itself.
60
6 · 10
2 · 3
Watch on YouTube

Using a Factor Tree to Find the Prime Factorization

✓ Example $3.1.13$: Finding the Prime Factorization

Find the prime factorization of 135.

Answer

The number 135 is divisible by 3, and so 3 is a factor of 135.

Step 1: Factor out the 3. The factorization is $135 = 3 \times 45$Figure ✓ $3.1.6$),

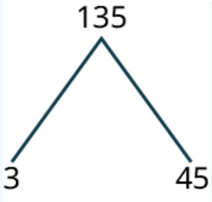


Figure ✓ 3.1.6

45 is also divisible by 3.

Step 2: Factor out a 3 from 45. The other factor is 15. The factor tree is shown in Figure ✓ 3.1.7.

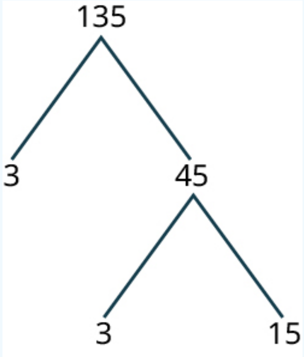


Figure ✓ \(\PageIndex{7}\)

15 is also divisible by 3.

Step 3: The factors of 15 are 3 and 5. The factor tree is shown in Figure ✓ 3.1.8.

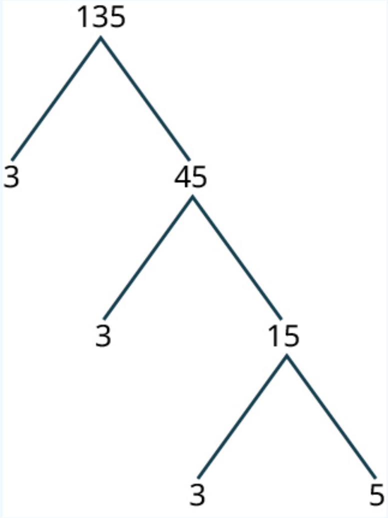


Figure ✓ 3.1.8

All the leaves are prime, so the process is complete. The prime factorization of 135 is $3^3 \times 5$.

Your Turn 3.1.13

Find the prime factorization of 280.

Example 3.1.14: Identifying Prime Factors

How many different prime factors does 10,241 have?

Answer

To know how many different prime factors 10,241 has, we need the prime factorization of 10,241.

Step 1: Use divisibility rules, to see that the number 10,241 is not divisible by 2 or by 3 (the sum of the digits is 8), or by 5. However, it is divisible by 7.

Step 2: After factoring the 7, the factorization is $10,241 = 7 \times 1463$; 1,463 is also divisible by 7.

Step 3: After factoring the 7, the factorization is $10,241 = 7 \times 7 \times 209 = 7^2 \times 209$. The number 209 is not divisible by 7.

Step 4: Check the next prime number: 11; 11 does divide 1,463.

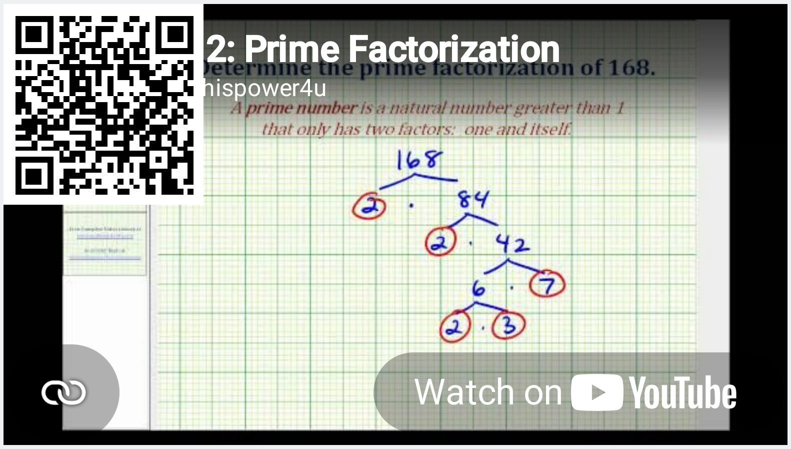
Step 5: After factoring the 11, the factorization is $10,241 = 7^2 \times 11 \times 19$.

Since 19 is prime, the prime factorization of 10,241 is complete. We see that 10,241 has three different prime factors: 7, 11, and 19.

Your Turn 3.1.14

Find the number of different prime factors of 180.

Video


2: Prime Factorization
hispower4u
Determine the prime factorization of 168.
A prime number is a natural number greater than 1 that only has two factors: one and itself.
168
2 · 84
2 · 42
6 · 7
2 · 3
Watch on YouTube

Finding the Prime Factorization of 168

Tech Check: Using Wolfram Alpha to Find Prime Factorizations

The Wolfram Alpha website is a powerful resource available for free to use. It is designed using AI so that it understands natural language requests. For instance, typing the question “What is the prime factorization of 543,390?” gets a rather quick answer of $2 \times 3 \times 5 \times 59 \times 307$. So, if you want to find the prime factorization of a number, you can simply ask Wolfram Alpha to find the prime factorization of the number.

Finding the Greatest Common Divisor

Two numbers often have more than one divisor in common (all pairs of natural numbers have the common divisor 1). When listing the common divisors, it’s often the case that the largest is of interest. This divisor is called the greatest common divisor and is denoted **GCD**. It is also sometimes referred to as the greatest common factor (GCF).

For instance, 6 is the greatest common divisor of 12 and 18. We can see this by listing all the divisors of each number and, by inspection, select the largest value that shows up in each list.

The divisors of 12	1, 2, 3, 4, 6, 12
The divisors of 18	1, 2, 3, 6, 9

It is easy to see that 6 is the largest value that appears in both lists.

✓ Example $3.1.15$: Finding the Greatest Common Divisor Using Lists

Find the greatest common divisor of 1,400 and 250 by listing all divisors of each number.

Answer

We create a list of all the divisors of 1,400 and of 250, and choose the largest one.

The divisors of 1,400 are

1, 2, 4, 5, 7, 8, 10, 14, 20, 25, 28, 35, 40, 50, 56, 70, 100, 140, 175, 200, 280, 350, 700, 1,400.

The divisors of 250 are

1, 2, 5, 10, 25, 50, 125, 250.

The largest value that appears on both lists is 50, so the greatest common divisor of 1,400 and 250 is 50.

Your Turn $3.1.15$

Find the greatest common divisor of 270 and 99 by listing all divisors of each number.

Listing all the divisors of the numbers in the set will always work, but for some relatively small numbers, the set of all divisors can become pretty big, and finding them all can be a chore. Another approach to finding the greatest common divisor is to use the prime factorization of the numbers. To do so, use the following steps:

Step 1: Find the prime factorization of the numbers.

Step 2: Identify the prime factors that appear in every number’s prime factorization. These are called the common prime factors.

Step 3: Identify the smallest exponent of each prime factor identified in Step 2 in the prime factorizations.

Step 4: Multiply the prime factors identified in Step 2 raised to the powers identified in Step 3. The result is the greatest common divisor.

✓ Example $3.1.16$: Finding the Greatest Common Divisor Using Prime Factorization

Find the greatest common divisor of 1,400 and 250 by using their prime factorizations.

Answer

Step 1: Find the prime factorizations of the numbers.

The prime factorization of 1,400 is $2^3 \times 5^2 \times 7$.

The prime factorization of 250 is $250 = 2 \times 5^3$.

Step 2: Identify the prime factors that appear in every number’s prime factorization.

The common prime factors are 2 and 5.

Step 3: Identify the smallest exponent of each prime identified in Step 2 in the prime factorizations.

The exponent of common prime factor 2 in the prime factorization of 1,400 is 3, and in the prime factorization of 250 is 1. The smallest of those exponents is 1.

The exponent of the common prime factor 5 in the prime factorization of 1,400 is 2 and is in the prime factorization of 250 is 3. The smallest of these exponents is 2.

Step 4: Multiply the prime factors identified in Step 2 raised to the powers identified in Step 3.

This gives $2^1 \times 5^2 = 50$. The greatest common divisor of 1,400 and 250 is $2 \times 5^2 = 50$.

Notice that the answer matches the one we found in Example 3.15.

Your Turn 3.1.16

Using prime factorization, determine the greatest common divisor of 36 and 128.

✓ Example 3.1.17: Finding the Greatest Common Divisor Using Prime Factorization

Find the greatest common divisor of 600 and 784 by using their prime factorizations.

Answer

Step 1: Find the prime factorizations of the numbers.

The prime factorization of 600 is $2^3 \times 3 \times 5^2$.

The prime factorization of 784 is $2^4 \times 7^2$.

Step 2: Identify the prime factors that appear in every number's prime factorization.

There is only one common prime factor, 2.

Step 3: Identify the smallest exponent of each prime from identified in Step 2 in the prime factorizations.

The exponent of 2 in the prime factorization of 600 is 3. The exponent of 2 in the prime factorization of 784 is 4. So, the smallest exponent of 2 is 3.

Step 4: Multiply the prime factors identified in Step 2 raised to the powers identified in Step 3.

This gives $2^3 = 8$. The greatest common divisor of 600 and 784 is 8.

Your Turn 3.1.17

Using prime factorization, determine the greatest common divisor of 120 and 200.

People in Mathematics: Srinivasa Ramanujan

Figure 3.1.9: Srinivasa Ramanujan (credit: Srinivasa Ramanujan, photo by Konrad Jacobs/Oberwolfach Photo Collection/public domain)

Ramanujan was born in southern India in 1887, during British rule. He was a self-taught mathematician, who, while in high school, began working through a two-volume text of mathematical theorems and results. His work included examination of the distribution of primes. He eventually came to the attention of British mathematician, G.H. Hardy. During one visit, Hardy mentioned to Ramanujan that his taxicab number was 1,729, remarking that 1,729 appeared to be a rather dull number. To which Ramanujan responded, “It is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways.”

Tech Check: Using Desmos to Find the GCD

To find the GCD of a set of numbers in Desmos, type gcd(first_number,second_number…) and Desmos will display the GCD of the numbers, as the numbers are typed!

Video

eat Common Factor Using Desmos
hispower4u
Common Factor Using Desmos
gcd( )
Watch on YouTube

Using Desmos to Find the GCD

Applications of the Greatest Common Divisor

The greatest common divisor has uses that are related to other mathematics (reducing fractions) but also in everyday applications. We'll look at two such applications, which have very similar underlying structures. In each case, something must divide the groups or measurements equally.

✓ Example $3.1.18$: Calculating Floor Tile Size

Suppose you have a rectangular room that is 570 cm wide and 450 cm long. You wish to cover the floor of the room with square tiles. What's the largest size square tile that can be used to cover this floor?

Answer

Using squares means that the length and width of the tiles are equal. To ensure we are using full tiles, the side length of the square tiles must divide the length of the room and the width of the room. Since we want the largest square tiles, we need the GCD of the width and length of the room or the GCD of 570 and 450.

Step 1: Find the prime factorizations of the numbers.

The prime factorization of 570 is $2 \times 3 \times 5 \times 19$.

The prime factorization of 450 is $2 \times 3^2 \times 5^2$.

Step 2: Identify the prime factors that appear in every number's prime factorization.

The common prime factors are 2, 3, and 5.

Step 3: Identify the smallest exponent of each prime identified in Step 2 in the prime factorizations.

The exponents of 2 in the prime factorizations of 570 and 450 are 1 and 1. So the smallest exponent for 2 is 1.

The exponents of 3 in the prime factorizations of 570 and 450 are 1 and 2, so the smallest exponent for 3 is 1.

The exponents of 5 in the prime factorizations of 570 and 450 are 1 and 2, so the smallest exponent for 5 is 1.

Step 4: Multiply the prime factors identified in Step 2 raised to the powers identified in Step 3.

This gives $2^1 \times 3^1 \times 5^1 = 30$. The GCD of 450 and 570 is 30, so the largest size square tile that can be used to cover the floor is 30 cm by 30 cm.

Your Turn $3.1.18$

You are designing a brick patio made of square bricks 5 cm thick, but you need to determine the width and length of those bricks. The patio will be 400 cm by 540 cm. What are the largest size square bricks that can be used so that you do not need to cut any bricks?

✓ Example $3.1.19$: Organizing Books Per Shelf

Suppose you want to organize books onto shelves, and you want the shelves to hold the same number of books. Each shelf will only contain one genre of book. You have 24 sci-fi, 42 fantasy, and 30 horror books. How many books can go on each shelf?

Answer

Since we want shelves that hold an equal number of books, and a shelf can only hold one genre of book, we need a number that will equally divide 24, 42, and 30. So, we need the GCD of the number of books of each genre or the GCD of 24, 42, and 30.

Step 1: Find the prime factorizations of the numbers.

The prime factorization of 24 is $2^3 \times 3$.

The prime factorization of 42 is $2 \times 3 \times 7$.

The prime factorization of 30 is $2 \times 3 \times 5$.

Step 2: Identify the prime factors that appear in every number’s prime factorization.

The common prime factors are 2 and 3.

Step 3: Identify the smallest exponent of each prime identified in Step 2 in the prime factorizations.

The smallest exponent of 2 and 3 in the factorizations is 1.

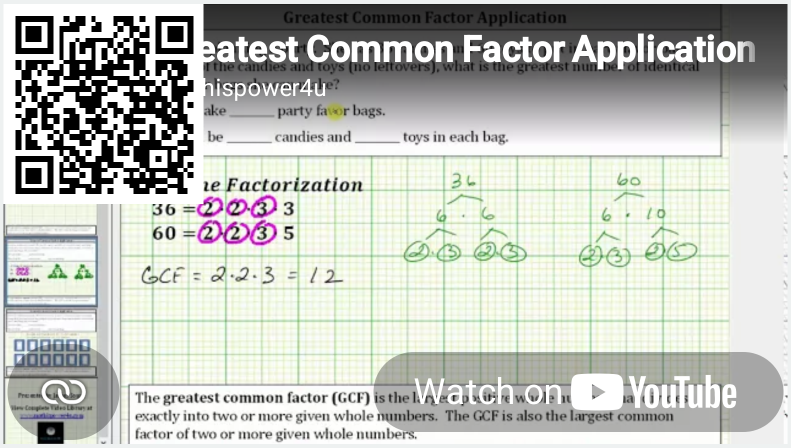
Step 4: Multiply the prime factors identified in Step 2 raised to the powers identified in Step 3.

This gives $2^1 \times 3^1 = 6$. The GCD of 24, 42, and 30 is 6, so the largest number of books that can go on a shelf is 6.

Your Turn 3.1.19

There are three gym classes. The number of students in the classes is 21, 35, and 28. What is the largest team size that can be formed if teams from every class must have the same number of students?

Video


Greatest Common Factor Application
eatest Common Factor Application
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Applying the GCD

Finding the Least Common Multiple

The flip side to a divisor of a number is a multiple of a number. For example, 5 is a divisor of 45 and so 45 is a multiple of 5. More generally, if the number a divides the number b, then b is a multiple of a.

This drives the idea of the least common multiple of a set of numbers. A common multiple of a set of numbers is a multiple of each of those numbers. For instance, 45 is a common multiple of 9 and 5, because 45 is a multiple of 9 (9 divides 45) and 45 is also a multiple of 5 (5 divides 45). The least common multiple (LCM) of a set of number is the smallest positive common multiple of that set of numbers.

There are (at least) three ways to find the LCM of a set of numbers, and we will explore two of them. One way is to create a list of multiples of each number in the set, and then identify the smallest multiple that appears in those lists.

Example 3.1.20: Finding the Least Common Multiple Using Lists

Find the LCM of 24 and 90 by listing multiples and choosing the smallest common multiple.

Answer

Create a list of the multiples of each number.

Step 1: The first 15 multiples for 24:

24, 48, 72, 96, 120, 144, 168, 192, 216, 240, 264, 288, 312, 336, 360.

Step 2: The first 15 multiples for 90:

90, 180, 270, 360, 450, 540, 630, 720, 810, 900, 990, 1,080, 1,170, 1,260, 1,350.

There is one multiple common to these lists, which is 360. So, 360 is the LCM of 24 and 90.

Your Turn $3.1.20$

Use lists to find the LCM of 12 and 15.

The second method we can use is to find the prime factorizations of the number in the set to build the LCM of the numbers based on the prime divisors of the numbers. The LCM can be built from the prime factorization of the numbers in the set in a similar way as when finding the greatest common divisor. Here are the steps for using the prime factorization process for finding the LCM.

Step 1: Find the prime factorization of each number.

Step 2: Identify each prime that is present in any of the prime factorizations.

Step 3: Identify the largest exponent of each prime identified in Step 2 in the prime factorizations.

Step 4:. Multiply the prime factors identified in Step 2 raised to the powers identified in Step 3.

✓ Example $3.1.21$: Finding the Least Common Multiple Using Prime Factorization

Use the prime factorizations of 24 and 90 to identify their LCM.

Answer

Step 1: Find the prime factorization of each number.

$24 = 2^3 \times 3$

$90 = 2 \times 3^2 \times 5$

Step 2: Identify each prime that is present in any of the prime factorizations.

The prime numbers present in the prime factorizations are 2, 3, and 5.

Step 3: Identify the largest exponent of each prime identified in Step 2 in the prime factorizations.

Prime	2	3	5
Largest exponent	3	2	1

Step 4: Compute the LCM by multiplying the prime factors identified in Step 2 raised to the powers identified in Step 3.

The LCM for 24 and 90 is $2 \times 2 \times 2 \times 3 \times 3 \times 5 = 2^3 \times 3^2 \times 5^1 = 360$.

Your Turn $3.1.21$

Use prime factorization to find the LCM of 20 and 28.

✓ Example $3.1.22$: Finding the Least Common Multiple Using Prime Factorization

Use the prime factorizations of 36, 66, and 250 to identify the LCM.

Answer

Step 1: Find the prime factorization of each number.

$36 = 2^2 \times 3^2$

$66 = 2 \times 3 \times 11$

$250 = 2 \times 5^3$

Step 2: Identify each prime that is present in any of the prime factorizations.

The prime numbers present in the prime factorizations are 2, 3, 5, and 11.

Step 3: Identify the largest exponent of each prime identified in Step 2 in the prime factorizations.

Prime	2	3	5	11
Largest exponent	2	2	3	1

Step 4: Compute the LCM by multiplying the prime factors identified in Step 2 raised to the powers identified in Step 3.

The LCM for 36, 66, and 250 is $2 \times 2 \times 3 \times 3 \times 5 \times 5 \times 5 \times 11 = 2^2 \times 3^2 \times 5^3 \times 11^1 = 49{,}500$.

Your Turn $3.1.22$

Use the prime factorization method to find the LCM of 150, 240, and 462.

Using lists for three or more numbers, particularly larger numbers, could take quite a bit of time. Frequently, as in this example, the prime factorization process is much quicker. In practice, you can use either listing or prime factorization to find the LCM.

✓ Example $3.1.23$: Finding the Least Common Multiple Using Both Methods

Find the LCM of 20, 36, and 45 using lists and prime factorization.

Answer

Step 1: Use listing. List the multiples:

20	20, 40, 60, 80, 100, 120, 140, 160, 180, 200, 220
36	36, 72, 108, 144, 180, 216, 252, 288, 324, 360
45	45, 90, 135, 180, 225, 270, 315, 360, 405, 450, 495, 540

The smallest value that appears on all the lists is 180, so 180 is the LCM of 20, 36, and 45.

Step 2: Find the prime factorization of each number.

$20 = 2^2 \times 5$

$36 = 2^2 \times 3^2$

$45 = 3^2 \times 5$

Step 3: Identify each prime that is present in any of the prime factorizations.

The prime numbers present in the prime factorizations are 2, 3, 5.

Step 4: Identify the largest exponent of each prime identified in Step 3 in the prime factorizations.

Prime	2	3	5
Largest exponent	2	2	1

Step 5: Compute the LCM by multiplying the prime factors identified in Step 3 raised to the powers identified in Step 4.

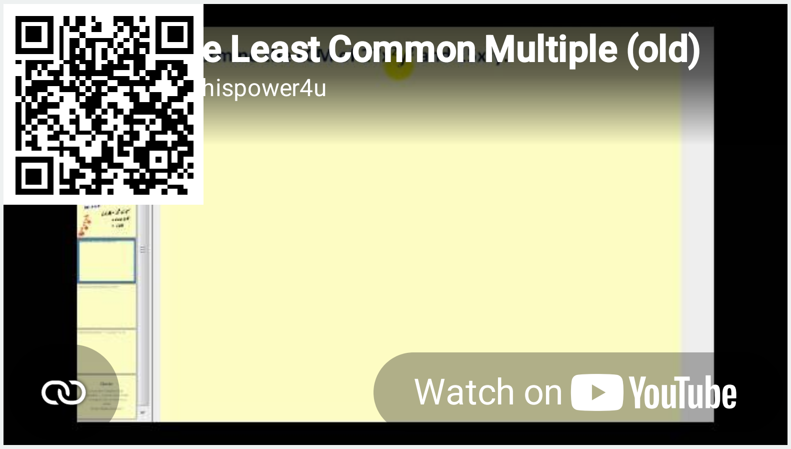
The LCM for 20, 36, and 45 is $2^2 \times 3^2 \times 5^1 = 180$.

Both listing and prime factorization produced the same result: the LCM is 180.

Your Turn 3.1.23

Find the LCM of 18, 24, and 40 using lists and prime factorization.

Video

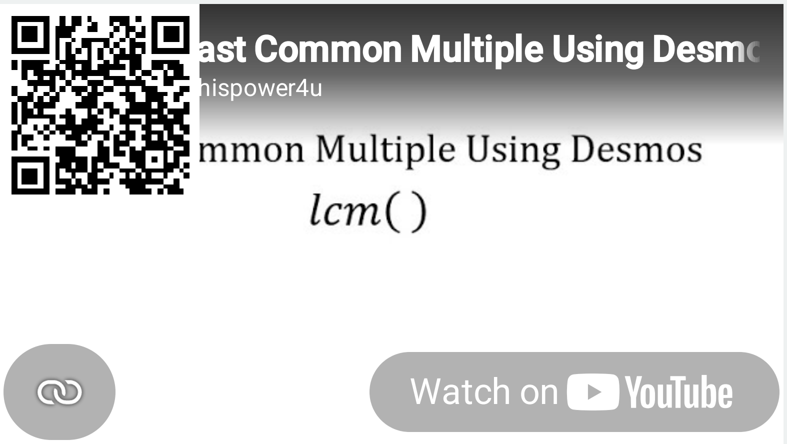

e Least Common Multiple (old)
hispower4u
Watch on YouTube

Finding the LCM

Tech Check: Using Desmos to Find the LCM

To find the LCM of a set of numbers in Desmos, type lcm(first_number,second_number…) and Desmos will display the LCM of the numbers, as the numbers are typed!

Video


ast Common Multiple Using Desmc
hispower4u
mmon Multiple Using Desmos
lcm( )
Watch on YouTube

Using Desmos to find the LCM

Applications of the Least Common Multiple

Some applications of LCM involve events that repeat at fixed intervals, such as visits to a location. Other applications involve getting things to be of equal magnitude when using parts that are different sizes (see Example 3.25, for instance). In each case, we may be looking to determine when two processes “line up.”

Example 3.1.24: Determining Scheduling Overlap Using the Least Common Multiple

Two students, João, and Amelia, meet one day at an assisted living facility where they volunteer. João volunteers every 6 days. Amelia volunteers every 10 days. How many days will it be until they are both volunteering on the same day again?

Answer

If we list the days that each student will volunteer, it becomes clear that we could solve this problem using the LCM of 6 and 10.

João will be at the assisted living facility 6, 12, 18, 24, 30, 36, 42, and 48 days later.

Amelia will be at the assisted living facility 10, 20, 30, 40, 50, and 60 days later.

The smallest number appearing on both lists is 30. João and Amelia will once more be volunteering together 30 days later.

Your Turn 3.1.24

The sun, Venus, and Jupiter all line up on a given day. Venus orbits the sun once every 255 days. Jupiter orbits the sun every 4,330 days (we'll ignore the decimal values of days for each orbit). How many days will it be until they line up again?

Example 3.1.25: Determining the Minimum Height Using the LCM

A team-building exercise has teams build a house of cards as high as possible. However, the cards for different teams are of different sizes. Team 1 uses 10 cm × 10 cm cards, while Team 2 uses 8 cm × 8 cm cards. What is the minimum height when the two teams will be tied? Ignore the width of the cards.

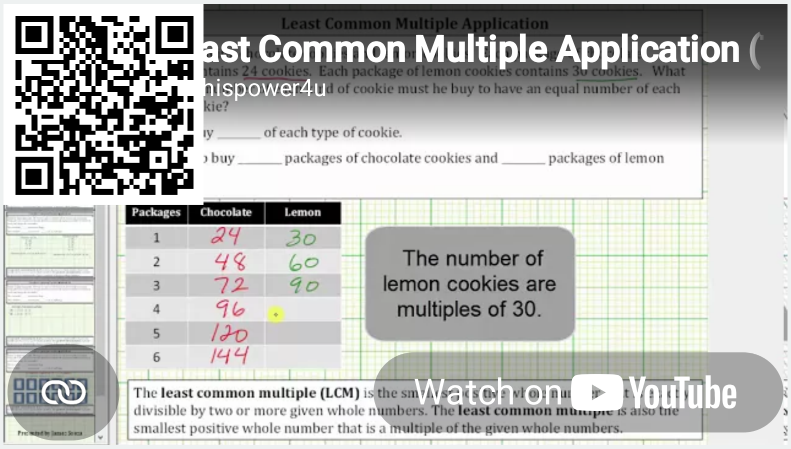
Answer

This is an example where we want to put together objects with different sizes. We want to know the minimum height when they are tied, or when the houses of cards line up the first time. The heights of the towers built using the 10 cm × 10 cm cards will be 10, 20, 30, 40, 50, and 60 cm tall. When the 8 cm × 8 cm cards are used, the tower will be 8, 16, 24, 32, 40, 48, and 56 cm tall. The smallest number appearing on both lists is 40. The first time they are tied is when the two towers are 40 cm tall.

Your Turn 3.1.25

In an Internet giveaway, every 130th person who submits a survey receives $250, and every 900th person receives a free cell phone. How many submissions must be received for the first person to receive both prizes?

Video


Least Common Multiple Application
ast Common Multiple Application (
hispower4u
The number of lemon cookies are multiples of 30.
Watch on YouTube

Packages	Chocolate	Lemon
1	24	30
2	48	60
3	72	90
4	96	
5	120	
6	144	

Application of LCM

WORK IT OUT: Prime Number Life Cycles—Cicadas

Cicadas are known to have life cycles of 13 or 17 years, which are prime numbers. Why would a prime number life cycle be an evolutionary advantage? To figure this out, we have to explore how common multiples work with prime numbers.

Make a conjecture regarding the LCM of a prime number and another number. Test this conjecture with a few examples of your own making.

Check Your Understanding

1. Identify which of the following numbers are prime and which are composite.

$$31, 56, 213, 48, 701$$

2. Find the prime factorization of 4,570 .

3. Find the greatest common divisor of 410 and 144.

4. Find the least common multiple of 45 and 70 .

5. You want to fill gift bags for children in the after-school program where you volunteer. You have 30 crayons, 20 sticker sheets, and 70 bite-sized candies. If each gift bag must contain the same number of crayons, sticker sheets, and bite-sized candies, what is the maximum number of bags that can be filled?

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3.1.0: Exercises

For the following exercises, use divisibility rules to determine if each of the following is divisible by 2.

? Exercise 3.1.0.1

24

? Exercise 3.1.0.2

37

? Exercise 3.1.0.3

1,345,321

For the following exercises, use divisibility rules to determine if each of the following is divisible by 3.

? Exercise 3.1.0.4

48

? Exercise 3.1.0.5

210

? Exercise 3.1.0.6

5,345,324

For the following exercises, use divisibility rules to determine if each of the following is divisible by 5 .

? Exercise 3.1.0.7

130

? Exercise 3.1.0.8

237

? Exercise 3.1.0.9

1,345,321

For the following exercises, use divisibility rules to determine if each of the following is divisible by 9 .

? Exercise 3.1.0.10

48

? Exercise 3.1.0.11

210

? Exercise $3.1.0.12$

5,345,325

For the following exercises, use divisibility rules to determine if each of the following is divisible by 12.

? Exercise $3.1.0.13$

48

? Exercise $3.1.0.14$

210

? Exercise $3.1.0.15$

5,355,324

? Exercise $3.1.0.16$

Determine which of the following numbers are prime: $\{3, 27, 77, 131, 457\}$

? Exercise $3.1.0.17$

Determine which of the following numbers are prime: $\{31, 97, 188, 389\}$

For the following exercises, find the prime factorization of the given number.

? Exercise $3.1.0.18$

12

? Exercise $3.1.0.19$

53

? Exercise $3.1.0.20$

72

? Exercise $3.1.0.21$

345

? Exercise $3.1.0.22$

938

? Exercise $3.1.0.23$

36,068

? Exercise 3.1.0.24

8,211,679

For the following exercises, find the greatest common divisor of the given set of numbers.

? Exercise 3.1.0.25

$\{45, 245\}$

? Exercise 3.1.0.26

$\{11, 24\}$

? Exercise 3.1.0.27

$\{56, 44\}$

? Exercise 3.1.0.28

$\{150, 600\}$

? Exercise 3.1.0.29

$\{1, 746, 28, 324\}$

? Exercise 3.1.0.30

$\{30, 40, 75\}$

? Exercise 3.1.0.31

$\{19, 45, 70\}$

? Exercise 3.1.0.32

$\{293, 7, 298, 19, 229\}$

? Exercise 3.1.0.33

$\{3, 927, 473, 82, 709, 1, 210, 121\}$

? Exercise 3.1.0.34

Make a list of the common divisors of 12 and 18 . What is the GCD of 12 and 18 ? Which of the other common divisors of 12 and 18 divide the GCD?

? Exercise 3.1.0.35

Make a list of the common divisors of 20 and 84 . What is the GCD? Which of the other common divisors of 20 and 84 also divide the GCD?

Exercise $3.1.0.36$

Make a list of the common divisors of 120 and 88 . What is the GCD? Which of the other common divisors of 120 and 88 also divide the GCD?

Exercise $3.1.0.37$

Based on the answers to 34,35 , and 36 , make a conjecture about the GCD of two numbers, and the other common divisors of those numbers.

Exercise $3.1.0.38$

Rebecca wants to cut two lengths of board into equal length pieces, with no leftover piece. The two boards are 230 cm long and 370 cm long. What is the longest length that Rebecca can cut from these boards so that all the cut boards are the same length?

Exercise $3.1.0.39$

Yasmin is playing with her younger brother, Cameron. They are grouping Skittles by color. They have 14 green, 10 yellow, and 8 purple Skittles. Each group must have the same number of green, the same number of yellow, and the same number of purple Skittles. What's the maximum number of piles that Sophia can build with Cameron?

Exercise $3.1.0.40$

Gathii is creating a tile backsplash for his kitchen. He wants to use square tiles to cover a $330 \text{ cm} \times 12 \text{ cm}$ area. What is the largest size square tile he can use to create this backsplash?

Exercise $3.1.0.41$

Deiji is designing a contest where teams will be given the same number of toothpicks, 5 oz. paper cups, and 2 cm length pieces of string. She has 420 pieces of string, 300 paper cups, and 1,610 toothpicks. What is the maximum number of teams she can have so that every team gets an equal number of pieces of string, paper cups, and toothpicks?

For the following exercises, find the least common multiple of the given set of numbers.

Exercise $3.1.0.42$

$\{30, 40\}$

Exercise $3.1.0.43$

$\{11, 24\}$

Exercise $3.1.0.44$

$\{14, 45\}$

Exercise $3.1.0.45$

$\{200, 450\}$

? Exercise 3.1.0.46

$\{38,077,9,088,687\}$

? Exercise 3.1.0.47

$\{36,42,70\}$

? Exercise 3.1.0.48

$\{7,13,36\}$

? Exercise 3.1.0.49

$\{4,450,864,339,889,157,339\}$

? Exercise 3.1.0.50

Benjamin and Mia both work at the Grease Fire diner, a local eatery. Benjamin has every 4th day off, and Mia has every 6th day off. How many days pass until they have another day off together?

? Exercise 3.1.0.51

A lunar month is 30 days (rounding off). A new lunar month begins on a Saturday. How many days is it until a lunar month begins on a Saturday again?

? Exercise 3.1.0.52

Isabella is creating a collage for a project and wants a horizontal cut in the collage. The cut will be made by using purple strips of cloth that are 28 mm long, and yellow strips of paper that are 36 mm long. What is the minimum length of the cut can she make using strips with those lengths?

? Exercise 3.1.0.53

Asteroids are objects that orbit the sun. The smallest distance that an asteroid gets to the sun during its orbit is called the perihelion. Asteroids also have orbital periods, or the time it takes to go around the sun exactly one time. The asteroid Ceres has an orbital period (number of days to circle the sun) of 1,680 days. The asteroid Hygiea has an orbital period of 2,031 days. Suppose they are at their perihelion on the same day. How many days will it be before Ceres and Hygiea are at their perihelion on the same day again?

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3.2: The Integers

Learning Objectives

After completing this section, you should be able to:

- Define and identify numbers that are integers.
- Graph integers on a number line.
- Compare integers.
- Compute the absolute value of an integer.
- Add and subtract integers.
- Multiply and divide integers.

Positive net wealth is when the total value of a person's assets, such as their home, their 401(k), their car, and savings account balance, exceed that of their debts, such as car loans, mortgages, or credit card debt. However, when the total value of debt exceeds the total value of assets, then the person has negative net wealth. Expressing the negative net wealth as a negative number allows people to work with the positive net values and negative net values with the same mathematical processes, and in the same applications. This section introduces the integers and operations with integers.

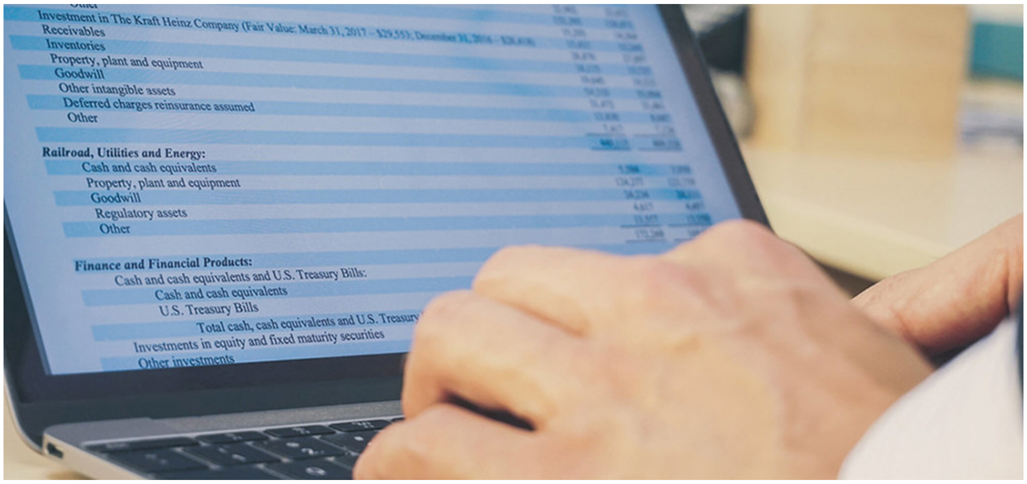


Figure $3.2.1$: A ledger comparing assets to debts, resulting in net wealth. (credit: modification of work "Reviewing Financial Statements" by Mary Cullen/Flickr, CC BY 2.0)

Defining and Identifying Integers

Extending the counting numbers to include negative numbers and zero forms the integers. Any other number that cannot be written as $\{\ldots-3,-2,-1,0,1,2,3,\ldots\}$ is not an integer.

Example $3.2.1$: Identifying Integers

Which of the following are integers and which are not?

-3	Is an integer, as it is the negative of a counting number
$\sqrt{24}$	This is not written as an integer. Entering the square root of 24 in a calculator, such as desmos, the result is 4.899 (rounded off). Since this is not an integer, then $\sqrt{24}$ is not an integer.
36/4	Since 36 divided by 4 is 9, and 9 is an integer, then 36/4 is an integer.
45	Is an integer, as it is a counting number
63.9	Is not an integer, because it is not a counting number and not the negative of a counting number.

2/7	Dividing 2 by 7 results in a number less than 1, but greater than 0, so is between two consecutive integers. So, 2/7 is not an integer.
−16.0	Is an integer, since the decimal part is 0

Your Turn $3.2.1$

Which of the following are integers?

- -214
- $38/11$
- $\sqrt{90}$
- $\sqrt{121}$
- $420/35$

Graphing Integers on a Number Line

Integers are often imagined as steps along a path. You start at 0, and going to the left is going backward, or in the negative direction, while going to the right is going forward, or in the positive direction. A number line (Figure 3.12) helps envision the integers. This also means that an integer gives magnitude (size) and direction (positive is to the right, negative is to the left). Graphing an integer on the number line means placing a solid dot at the integer on the number line.

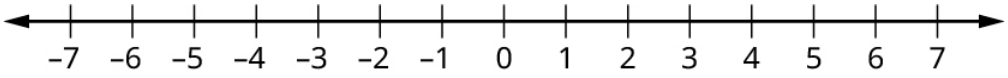
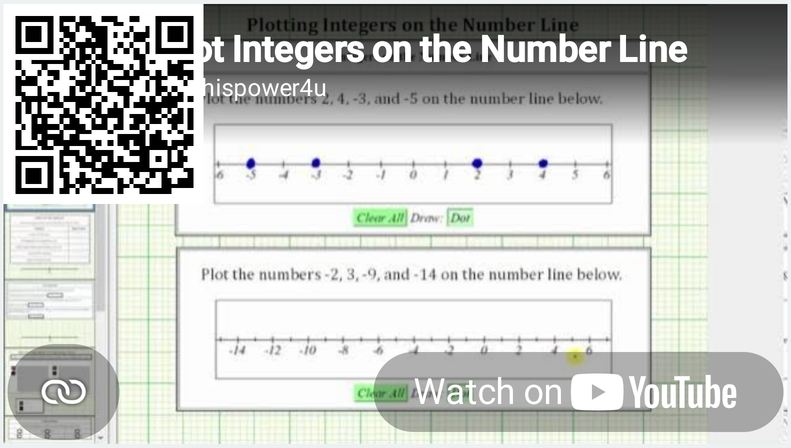


Figure $3.2.2$:

Video


Plotting Integers on the Number Line

Plot the numbers 2, 4, -3, and -5 on the number line below.

Plot the numbers -2, 3, -9, and -14 on the number line below.

Graphing Integers on the Number Line

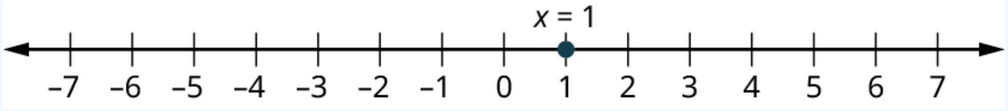
Example $3.2.2$: Graphing Numbers on the Number Line

Graph the following on the number line:

1. 1
2. −4
3. 3

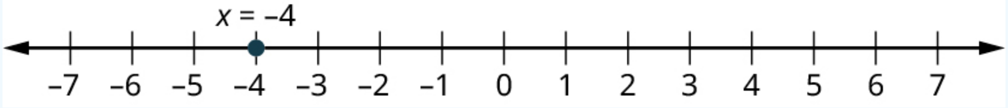
Answer

1.



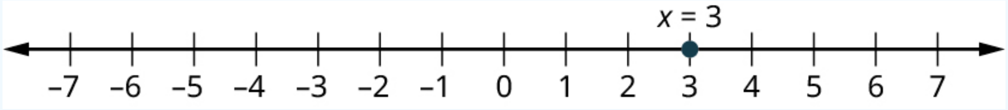
x = 1
−7 −6 −5 −4 −3 −2 −1 0 1 2 3 4 5 6 7

Figure ✓ 3.2.3



x = −4
−7 −6 −5 −4 −3 −2 −1 0 1 2 3 4 5 6 7

Figure ✓ 3.2.4:



x = 3
−7 −6 −5 −4 −3 −2 −1 0 1 2 3 4 5 6 7

Figure ✓ 3.2.5

Your Turn 3.2.2

Graph the following numbers on the number line:

1. -10
2. 4
3. 0

Comparing Integers

When determining if one quantity or size is larger than another, we know it means there is more of whatever is being discussed. In terms of positive integers, we can envision that larger integers are further to the right on the number line. This idea applies to negative integers also. This means that $\boldsymbol{a}$ is greater than $\boldsymbol{b}$ when a is to the right of b on the number line. We write $a > b$. When a is greater than b, we can also say that $\boldsymbol{b}$ is less than $\boldsymbol{a}$. On the number line, b would be to the left of a. We write $b < a$.

We need to recognize that a >b Figure 3.16. On this number line, a is to the right of b, so $a > b$. But this means b is to the left of a, so $b < a$.

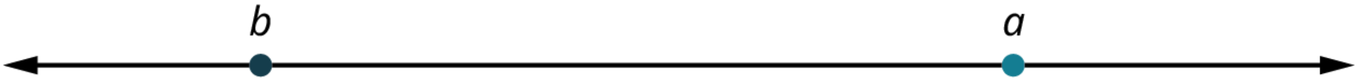
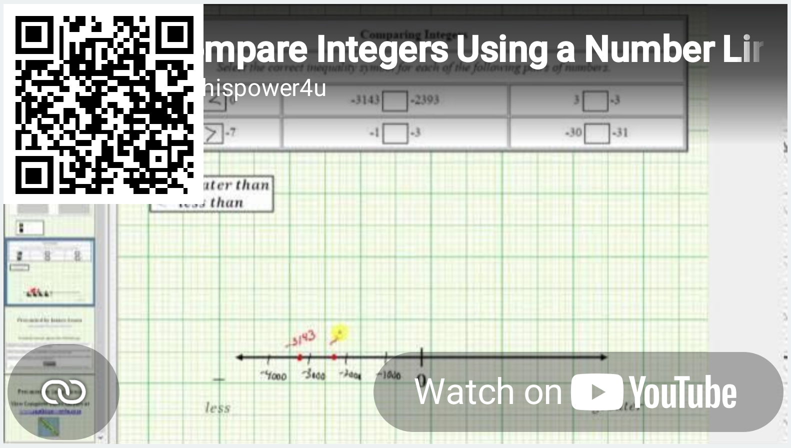


Figure 3.2.6:

Video



mpare Integers Using a Number Lir
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Watch on YouTube

Comparing Integers Using the Number Line

✓ Example 3.2.3: Comparing Integers Using a Number Line

Determine which of −6 and 4 is larger using a number line, and express that using both the greater than and the less than notations.

Answer

To illustrate this, we use a number line (Figure 3.17).

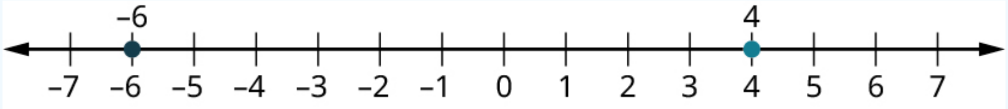


Figure ✓ 3.2.7:

Since −6 is to the left of 4, then −6 is less than 4. We can write this as $-6 < 4$. Another way of expressing this is that 4 is greater than −6. So we can also write $4 > -6$ $4 > -6$.

Your Turn 3.2.3

Determine which of −38 and 27 is larger using a number line, and express that using both the greater than and the less than notations.

✓ Example 3.2.4: Comparing Negative Integers

Determine which of −6 and −2 is larger, and express that using both the greater than and the less than notations.

Answer

To illustrate this, we use a number line (Figure 3.18).

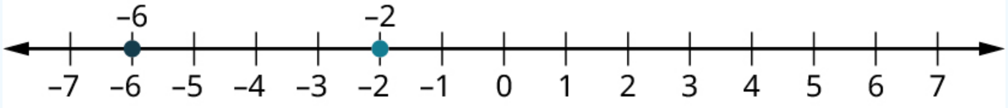


Figure ✓ 3.2.8:

Since −6 is to the left of −2, then −6 is less than −2. We can write this as $-6 < -2$.

Another way of expressing this is that −2 is greater than −6. So we can also write $-2 > -6$.

Checkpoint

Warning: People often ignore the negative signs, and think than since 6 is greater than 2, −6 is greater than −2. To avoid that error, remember that the greater number is to the right on the number line.

Your Turn 3.2.4

Determine which of −63 and −213 is larger, and express that using both the greater than and the less than notations.

✓ Example 3.2.5: Comparing Integers by Quantity

Determine which of 27 and 410 is larger, and express that using both the greater than and the less than notations.

Answer

When thinking about quantity, 410 is more than 27. So, 410 is greater than 27 and 27 is less than 410. We can write this as $410 > 27$ or as $27 < 410$.

Your Turn $3.2.5$

Determine which of 101 and 98 is larger, and express that using both the greater than and the less than notations.

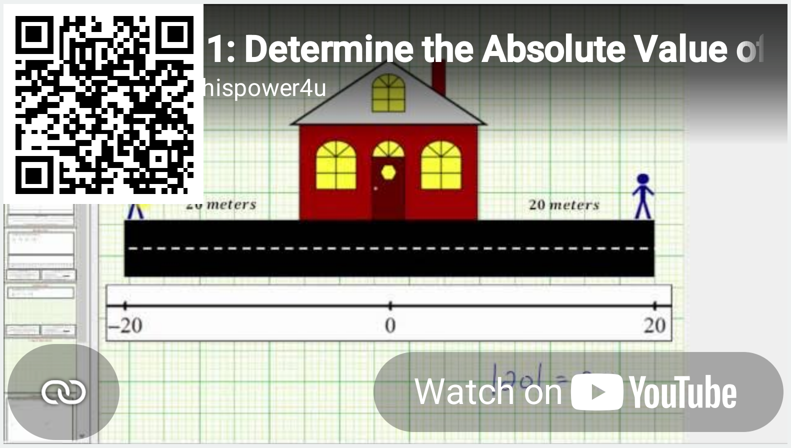
The Absolute Value of an Integer

When talking about graphing integers on the number line, one interpretation suggests it is like walking along a path. Negative is going to the left of 0, and positive going to the right. If you take 30 steps to the right, you are 30 steps away from 0. On the other hand, when you take 30 steps to the left, you are still 30 steps away from 0. So, in a way, even though one is negative and the other positive, these two numbers, 30 and −30, are equal since both are 30 steps away from 0. The absolute value of an integer n is the distance from n to 0, regardless of the direction. The notation for absolute value of the integer n is $|n|$.

If we think of an integer as both direction and magnitude (size), absolute value is the magnitude part.

Calculating the absolute value of an integer is very straightforward. If the integer is positive, then the absolute value of the integer is just the integer itself. If the integer is negative, then to compute the absolute value of the integer, simply remove the negative sign. Keeping in mind the number line as a path, when you've gone 10 steps to the left of 0, you have still taken 10 steps, and the direction does not matter.

Video



Evaluating the Absolute Value of an Integer

Example $3.2.6$: Calculating the Absolute Value of a Positive Integer

Calculate |19|.

Answer

Since the number inside the absolute value symbol is positive, the absolute value is just the number itself. So |19| = 19.

Your Turn $3.2.6$

Calculate |38|.

Example $3.2.7$: Calculating the Absolute Value of a Negative Integer

Calculate |−435|.

Answer

Since the number inside the absolute value is negative, the absolute value removes the negative sign. So $|-435| = 435$.

Your Turn 3.2.7

Calculate $|-81|$.

Adding and Subtracting Integers

You may recall having approached adding and subtracting integers using the number line from earlier in your academic life. Adding a positive integer results in moving to the right on the number line. Adding a negative integer results in moving to the left. Subtracting a positive integer results in a move to the left on the number line. But subtracting a negative integer results in a move to the right.

This leads to a few adding and subtracting rules, such as:

Rule 1: Subtracting a negative is the same as adding a positive.

Rule 2: Adding two negative integers always results in a negative integer.

Rule 3: Adding two positive integers always results in a positive integer.

Rule 4: The sign when adding integers with opposite signs is the same as the integer with the larger absolute value.

These rules are good to keep in the back of your mind, as they can serve as a quick error check when you use a calculator.

✓ Example 3.2.8: Adding Integers

Use your calculator to calculate $4 + (-7)$. Explain how the answer agrees with what was expected.

Answer

Using a calculator, we find that $4 + (-7) = -3$. Since we are adding integers with opposite signs, the sign of the answer matches the sign of the integer with the larger absolute value which $|-7|=7$.

Your Turn 3.2.8

Use your calculator to calculate $(-18) + 11$. Explain how the answer agrees with what was expected.

✓ Example 3.2.9: Subtracting Positive Integers

Use your calculator to calculate $18 - 9$. Explain how the answer agrees with what was expected.

Answer

Using a calculator, we find that $18 - 9 = 9$. Since 18 was larger than 9, we expected the difference to be positive.

Your Turn 3.2.9

Use your calculator to calculate $38 - 100$. Explain how the answer agrees with what was expected.

✓ Example 3.2.10: Subtracting with Negative Integers

Use your calculator to calculate $27 - (-13)$. Explain how the answer agrees with what was expected.

Answer

Using a calculator, we find that 27 – (−13) = 40. Since we're subtracting a negative number, it is the same as adding a positive, so this is the same as 27 + 13 = 40.

Your Turn 3.2.10

Use your calculator to calculate 45 − (−26). Explain how the answer agrees with what was expected.

Example 3.2.11: Adding Integers with Opposite Signs

Use your calculator to calculate (−13) + 90. Explain how the answer agrees with what was expected.

Answer

Using a calculator, we find that (−13) + 90 = 77. Since we are adding integers with opposite signs, the sign of the answer matches the sign of the integer with the larger absolute value, which is positive since 90 is positive.

Your Turn 3.2.11

Use your calculator to calculate 19 + (−36). Explain how the answer agrees with what was expected.

One use of negative numbers is determining net worth, which is all the weath someone owns less all that someone owes. Sometimes net worth is positive (which is good), and sometimes net worth is negative (which can be stressful).

Example 3.2.12: Calculating Net Worth

Jennifer is owed $50 from her friend Janice, but owes her friend Pat $87. What is Jennifer's net worth?

Answer

Net worth is the amount that one is owed minus the amount one owes. Jennifer is owed $50 but owes $87. So, her net worth is $50 – $87 = −$37. The negative indicates that Jennifer owes more than she is owed.

Your Turn 3.2.12

Christian is owed $180 from his friend Chanel, but owes his friend Jeff $91. What is Christian's net worth?

Multiplying and Dividing Integers

Similar to addition and subtraction, the signs of the integers impact the results when multiplying and dividing integers. The rules are fairly straightforward, but again rely on the direction on the number line. There are only two rules.

Rule 1: When multiplying or dividing two integers with the same sign, the result is positive.

Rule 2: When multiplying or dividing two integers with opposite signs, the result is negative.

Just as before, these rules can serve as a quick error check when using a calculator.

Example 3.2.13: Multiplying Positive Integers

Use your calculator to calculate 4 × 8. Explain how the answer agrees with what was expected.

Answer

Entering 4 × 8 into your calculator, the result is 32. This agrees with our expectation. The numbers have the same signs, so the result is positive.

Your Turn 3.2.13

Use your calculator to calculate 81 × 26. Explain how the answer agrees with what was expected.

✓ Example 3.2.14: Multiplying Integers with Different Signs

Use your calculator to calculate 9 × (−10). Explain how the answer agrees with what was expected.

Answer

Entering 9 × (−10) into your calculator, the result is −90. This agrees with our expectation. The numbers have opposite signs, so the result is negative.

Your Turn 3.2.14

Use your calculator to calculate (−18) × 13. Explain how the answer agrees with what was expected.

✓ Example 3.2.15: Dividing Integers with Different Signs

Use your calculator to calculate 400/(−25). Explain how the answer agrees with what was expected.

Answer

Entering 400/(−25) into your calculator, the result is −16. This agrees with our expectation. The numbers have opposite signs, so the result is negative.

Your Turn 3.2.15

Use your calculator to calculate (−116)/4. Explain how the answer agrees with what was expected.

✓ Example 3.2.16: Dividing Negative Integers

Use your calculator to calculate −750/(−3). Explain how the answer agrees with what was expected.

Answer

Entering −750/(−3) into your calculator, the result is 250. This agrees with our expectation. The numbers have the same signs, so the result is positive.

Your Turn 3.2.16

Use your calculator to calculate (−77)/(−11). Explain how the answer agrees with what was expected.

At the end of a season, a team may wish to buy their coach an end-of season gift. It makes sense to share the cost equally among the members. To do so, the team would need to find the average (or mean) cost per member. The average (or mean) of a set of numbers is the sum of the numbers divided by the number values that are being averaged.

✓ Example 3.2.17: Finding the Average of a Set of Numbers

The daily low temperatures in Barrie, Ontario, for the week of February 14, 2021, were −20°, −12°, −15°, −23°, −17°, −13°, and −19° degrees Celsius. What was the average daily temperature for the week of February 14, 2021, in Barrie?

Answer

Step 1: To find the average daily temperature, we first need to add the temperatures.

$(-20)+(-12)+(-15)+(-23)+(-17)+(-13)+(-19)=-119$

Step 2: That sum will then be divided by 7 since we are averaging over seven days, giving −119/7 = −17. So, the average daily temperature in Barrie, Ontario the week of February 14, 2021, was −17° Celsius.

Your Turn 3.2.17

Banks and credit cards often base their interest on the average daily balance of an account, which is the average of the balance from each day of the period. The account balance of Jada's checking account on each day of the week of December 13, 2020, was $1,250, $673, −$1,500, $1,000, $785, $785, and $710. What was Jada's average daily balance for the week of December 13, 2020? Assume Jada pays no fees for a negative balance.

Check Your Understanding

1. Identify which of the following numbers are integers:

$-4, 15.2, \sqrt{2}, \frac{3}{20}, 430$

2. Graph the following integers on the number line: $4, -2, 7$.

3. Place these integers in increasing order: $4, -2, -7, 10, -13$.

4. Calculate $|-7|$.

5. Calculate 4 - (-9).

6. Calculate $(-3) \times (-12)$.

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3.2.0: Exercises

Exercise 3.2.0.1

Identify all the integers in the following list: $4, -17, 8, 0.5, \sqrt{7}, \frac{1}{9}, -300$.

Exercise 3.2.0.2

Identify all integers in the following list: $-9.2, 13, -1, \sqrt{47}, \frac{\sqrt{3}}{2}, 567, -300$.

For the following exercises, plot the integers on the same number line.

Exercise 3.2.0.3

$4, -2, 10, 0$

Exercise 3.2.0.4

$-6, -3, 10, 1, 4$

Exercise 3.2.0.5

$-3, -10, 7, 2$

Exercise 3.2.0.6

$2, 4, 8, -2, -5$

For the following exercises, determine if the comparison is true or false.

Exercise 3.2.0.7

$-3 < -10$

Exercise 3.2.0.8

$-10 > -3$

Exercise 3.2.0.9

$7 > -6$

Exercise 3.2.0.10

$-6 < 7$

Exercise 3.2.0.11

$18 < 20$

? Exercise 3.2.0.12

$20 < 18$

? Exercise 3.2.0.13

What are two numbers with an absolute value of 76 ?

? Exercise 3.2.0.14

What are two numbers with an absolute value 87 ?

? Exercise 3.2.0.15

Determine |-67|.

? Exercise 3.2.0.16

Determine |98|.

? Exercise 3.2.0.17

Determine |61|.

? Exercise 3.2.0.18

Determine |-903|.

? Exercise 3.2.0.19

What are two numbers that are 23 away from 0 ?

? Exercise 3.2.0.20

What are two numbers 13 away from 0 ?

For the following exercises, complete the indicated calculation.

? Exercise 3.2.0.21

$47 + 200$

? Exercise 3.2.0.22

$67 + (-86)$

? Exercise 3.2.0.23

$(-86) + 104$

? Exercise 3.2.0.24

$13-(-54)$

? Exercise 3.2.0.25

$(-45)-(-26)$

? Exercise 3.2.0.26

$(-13)+(-102)$

? Exercise 3.2.0.27

38×12

? Exercise 3.2.0.28

$(-12)\times(-10)$

? Exercise 3.2.0.29

$14\times(-5)$

? Exercise 3.2.0.30

$421\times(-13)$

? Exercise 3.2.0.31

$(-412)\times 504$

? Exercise 3.2.0.32

$(-8,919)\times(-3,401)$

? Exercise 3.2.0.33

$(-130)\div(-65)$

? Exercise 3.2.0.34

$450\div 9$

? Exercise 3.2.0.35

$2000\div(-40)$

? Exercise 3.2.0.36

$(-910)\div 14$

Exercise 3.2.0.37

The daily low temperatures, in degrees centigrade, in Fargo for the week of January 17, 2021, were -9, $-17, -18, -14, -17, -19$, and -11 . What was the average low temperature in Fargo that week?

Exercise 3.2.0.38

Riley collects checks for a fundraiser supporting the homeless in town. Through Venmo, they collect the following amounts: $\$20, \$20, \$50, \$75, \$250, \$10, \$15, \$65, \$30, \$15$. What was the average donation that Riley collected?

Exercise 3.2.0.39

Heath has $\$495$ in an account. They will collect two paychecks this week, one for $\$150$ and the other for $\$250$. Heath also will pay three bills, one for $\$50$, one for $\$110$, and one for $\$300$. After all those transactions, how much will Heath have in their account?

Exercise 3.2.0.40

Five diners decide to split the check evenly. The total bill comes to $\$475$. How much does each diner owe?

Exercise 3.2.0.41

There are many people who are single-issue voters, which means that they will vote for (or against!) a candidate based on one issue and one issue only. Suppose a politician wants to earn votes based on single issues: Issue 1, Issue 2, and Issue 3. By publicly supporting issue 1, the politician gains 127 voter but loses 154. By publicly supporting issue 2, the politician gains 350 voters but loses 83 . By publicly denouncing Issue 3, the politician gains 306 voters but loses 158. By publicly taking those stances, what is the politician's net gain or loss in number of voters?

Exercise 3.2.0.9

$7>-6$

Exercise 3.2.0.10

$-6<7$

Exercise 3.2.0.11

$18<20$

Exercise 3.2.0.12

$20<18$

Exercise 3.2.0.13

What are two numbers with an absolute value of 76 ?

Exercise 3.2.0.14

What are two numbers with an absolute value 87 ?

? Exercise 3.2.0.15

Determine |-67|.

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? Exercise 3.2.0.17

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? Exercise 3.2.0.18

Determine |-903|.

? Exercise 3.2.0.19

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What are two numbers 13 away from 0 ?

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$47+200$

? Exercise 3.2.0.22

$67+(-86)$

? Exercise 3.2.0.23

$(-86)+104$

? Exercise 3.2.0.24

$13-(-54)$

? Exercise 3.2.0.25

$(-45)-(-26)$

? Exercise 3.2.0.26

$(-13)+(-102)$

Exercise 3.2.0.27

38×12

Exercise 3.2.0.28

$(-12) \times (-10)$

Exercise 3.2.0.29

$14 \times (-5)$

Exercise 3.2.0.30

$421 \times (-13)$

Exercise 3.2.0.31

$(-412) \times 504$

Exercise 3.2.0.32

$(-8,919) \times (-3,401)$

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$(-130) \div (-65)$

Exercise 3.2.0.34

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Exercise 3.2.0.36

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3.3: Order of Operations

Learning Objectives

After completing this module, you should be able to:

- Simplify expressions using order of operations.
- Simplify expressions using order of operations involving grouping symbols.

Calculates else sure someone be rules expect explicit we what that needs to need make, we to that them what be calculate to calculated first.

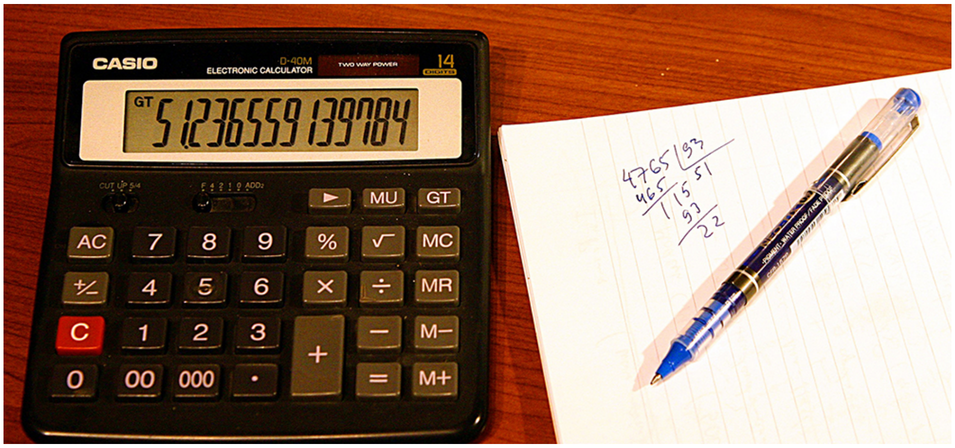


Figure 3.19: Calculators may automatically apply order of operations to calculations. (credit: “Precision” by Leonid Mamchenkov/Flickr, CC BY 2.0)

You probably read that sentence and couldn't make heads or tails of it. Seems like it might concern calculations, but maybe it concerns needs? You may even be attempting to unscramble the sentence as you read it, placing words in the order you might expect them to appear in. The reason that the sentence makes no sense is that the words don't follow the order you expect them to follow. Unscrambled, the sentence was intended to be “To be sure that someone else calculates when we expect them to calculate, we need rules that make explicit what needs to be calculated first.”

Similarly, when working with math expressions and equations, if we don't follow the rules for order of operations, arithmetic expressions make no sense. Just a simple expression would be problematic if we didn't have some rules to tell us what to calculate first. For instance, $4 \times 2^2 + 3 + 5^2$ can be calculated in many ways. You could get 5,184. Or, you could get 80. Or, 96. The issue is that without following a set of rules for calculation, the same expression will give various results. In case you are curious, using the appropriate order of operations, we find $4 \times 2^2 + 3 + 5^2 = 44$.

Simplify Expressions Using Order of Operations

The order in which mathematical operations is performed is a convention that makes it easier for anyone to correctly calculate. They follow the acronym EMDAS:

E	Exponents
M/D	Multiplication and division
A/S	Addition and Subtraction

So, what does EMDAS tell us to do? In an equation, moving left to right, we begin by calculating all the exponents first. Once the exponents have been calculated, we again move left to right, calculating the multiplications and divisions, one at a time. Multiplication and division hold the same position in the ordering, so when you encounter one or the other at this step, do it. Once the multiplications and divisions have been calculated, we again move left to right, calculating the additions and subtractions, one at a time. Additions and subtractions hold the same position in the ordering, so when you encounter one or the other at this step, do it. (You may have previously learned the order of operations as PEMDAS, with parentheses first; we will add that aspect later on.) We'll explore this as we work an example.

✓ Example $3.3.1$: Using Two Order of Operations

Calculate $21-4\times 13$.

Answer

There are no exponents in this expression, so the next operations to check are multiplication and division.

Step 1: Moving left to right, the first multiplication encountered is 4 multiplied by 13. We perform that operation first.

$21-\mathbf{4\times 13}$
$=21-\mathbf{52}$

Step 2: The only operation remaining is the subtraction.

$21-52=-31.$

So, $21-4\times 13=-31$.

Your Turn 3.3.1

1. Calculate $43+18\times 15$.

✓ Example $3.3.2$: Using Two Order of Operations

Calculate 4×8^3.

Answer

Step 1: Moving left to right, we see there is an exponent. We calculate the exponent first.

$4\times\mathbf{8^3}=4\times(\mathbf{8\times 8\times 8})=4\times\mathbf{512}$

Step 2: The only operation remaining is the multiplication.

$4\times 512=2{,}048$

So, $4\times 8^3=2{,}048$.

Your Turn 3.3.2

1. Calculate $60^2/50$.

✓ Example $3.3.3$: Using Three Order of Operations

Calculate $2+3^2\times 4$.

Answer

Step 1: To calculate this, move left to right, and compute all the exponents first. The only exponent we see is the squaring of the 3, so that is calculated first.

$2+3^2\times 4=2+(\mathbf{3\times 3})\times 4=2+\mathbf{9}\times 4$

Step 2: Since the exponents are all calculated, now calculate all the multiplications and divisions moving left to right. The only multiplication or division present is 9 times 4.

$2+9\times 4=2+\mathbf{36}$

Step 3: Moving left to right, perform the additions and subtractions. There is only one such operation, 2 plus 36.

$2+36=38$

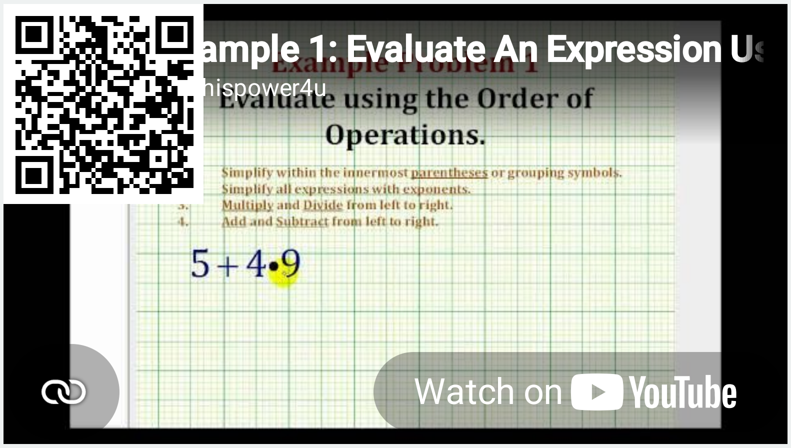
So, $2+3^2\times 4=38$.

Your Turn 3.3.3

1. Calculate $3\times 6^3-18$.

Even if the expression being calculated gets more complicated, we perform the operations in the order: EMDAS.

Video


Example 1: Evaluate An Expression Us
mathispower4u
Example Problem 1
Evaluate using the Order of Operations.
Simplify within the innermost parentheses or grouping symbols.
Simplify all expressions with exponents.
3. Multiply and Divide from left to right.
4. Add and Subtract from left to right.
5+4•9
Watch on YouTube

Order of Operations 1

Example 3.3.4: Using Eight Order of Operations

Correctly apply the order of operations to compute the following:

$4-25\times 6/10\times 3^2+7\times 2^3$.

Answer

Step 1: To do so, calculate the exponents first, moving left to right. There are two occurrences of exponents in the expression, 3 squared and 2 cubed.

$4-25\times 6/10\times \mathbf{3^2}+7\times \mathbf{2^3}=4-25\times 6/10\times \mathbf{9}+7\times \mathbf{8}$

Step 2: Now that the exponents are calculated, perform the multiplication and division, moving left to right. The first is the product of 25 and 6.

$4-\mathbf{25\times 6}/10\times 9+7\times 8$
$=4-\mathbf{150}/10\times 9+7\times 8$

Step 3: Next is the 150 divided by 10.

$4-\mathbf{150/10}\times 9+7\times 8$
$=4-\mathbf{15}\times 9+7\times 8$

Step 4: Next is 15 multiplied by 9.

$4-\mathbf{15\times 9}+7\times 8$
$=4-\mathbf{135}+7\times 8$

Step 5: Finally, multiply the 7 and 8.

$4-135+\mathbf{7\times 8}$
$=4-135+\mathbf{56}$

As all the multiplications and divisions have been calculated, the additions and subtractions are performed, moving left to right.

$\mathbf{4-135}+56$
$=\mathbf{-131}+56$
$=\mathbf{-131+56}$
$=\mathbf{-75}$

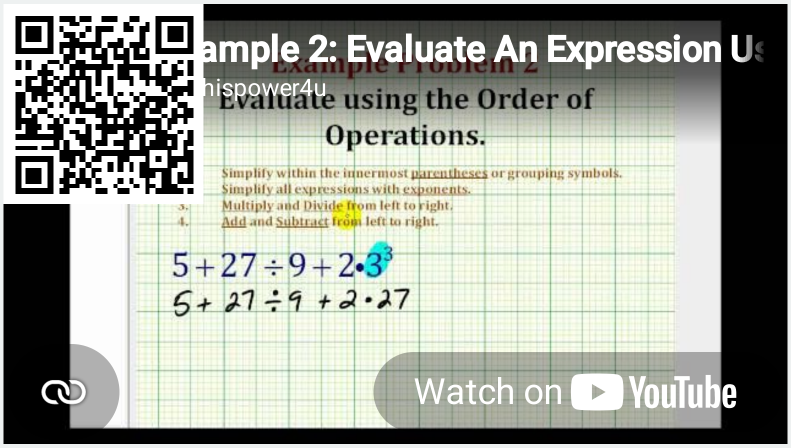
The computed value is −75.

Your Turn $3.3.4$

Correctly apply the order of operations to compute the following:

$$13+3^4\times 8/6\times 5^2-3\times 4$$

Video


ample 2: Evaluate An Expression Us
hispower4u
Example Problem 2
Evaluate using the Order of Operations.
Simplify within the innermost parentheses or grouping symbols.
Simplify all expressions with exponents.
3. Multiply and Divide from left to right.
4. Add and Subtract from left to right.
5+27÷9+2·3³
5+27÷9+2·27
Watch on YouTube

Order of Operations 2

Example $3.3.5$: Using Six Order of Operations

Correctly apply the rules for the order of operations to accurately compute the following:

$10-3\times 5^3/15+56/4$.

Answer

Step 1: Calculate exponents first, moving left to right:

$10-3\times \mathbf{5^3}/15+56/4$
$=10-3\times \mathbf{125}/15+56/4$

Step 2: Multiply and divide, moving left to right:

$10-\mathbf{3\times 125}/15+56/4$
$=10-\mathbf{375/15}+56/4$
$=10-25+\mathbf{56/4}$
$=10-25+\mathbf{14}$

Step 3: Add and subtract, moving left to right:

$\mathbf{10-25}+14$
$=\mathbf{-15+14}$
$=-1$

Your Turn 3.3.5

Correctly apply the rules for the order of operations to accurately compute the following:

$$12/(-4)+8\times 9/12\times 2^3-24\times 25/10$$

Example 3.3.6: Using Order of Operations

Correctly apply the rules for the order of operations to accurately compute the following: $(-8)/2\times 3-9\times 2^4/12+9\times(-4)^2/2^3$.

Answer

Step 1: Calculate the exponents first, moving left to right:

$$(-8)/2\times 3-9\times \mathbf{2^4}/12+9\times(-4)^2/2^3$$
$$=(-8)/2\times 3-9\times 16/12+9\times \mathbf{12^2}/2^3$$
$$=(-8)/2\times 3-9\times 16/12+9\times 144/\mathbf{2^3}$$
$$=(-8)/2\times 3-9\times 16/12+9\times 144/8$$

Step 2: Multiply and divide, moving left to right:

$$=\mathbf{(-8)/2}\times 3-9\times 16/12+9\times 144/8$$
$$=\mathbf{(-4)\times 3}-9\times 16/12+9\times 144/8$$
$$=(-12)-\mathbf{9\times 16}/12+9\times 144/8$$
$$=(-12)-\mathbf{144/12}+9\times 144/8$$
$$=(-12)-12+\mathbf{9\times 144}/8$$
$$=(-12)-12+\mathbf{1{,}296/8}$$
$$=(-12)-12+162$$

Step 3: Add and subtract, moving left to right:

$$=\mathbf{(-12)-12}+162=\mathbf{(-24)+162}=138$$

Your Turn 3.3.6

Correctly apply the rules for the order of operations to accurately compute the following:

$$3\times 4^3\times 7+24/6\times 7^2-9/3\times 8$$

Using the Order of Operations Involving Grouping Symbols

We have examined how to use the order of operations, denoted by EMDAS, to correctly calculate expressions. However, there may be expressions where a multiplication should happen before an exponent, or a subtraction before a division. To indicate an operation should be performed out of order, the operation is placed inside parentheses. When parentheses are present, the operations inside the parentheses are performed first. Adding the parentheses to our list, we now have PEMDAS, as shown below.

P	Parentheses	
E	Exponents	
M/D	Multiplication and division	(division is just the multiplication by the reciprocal)
A/S	Addition and subtraction	(subtraction is just the addition of the negative)

As said previously, parentheses indicate that some operation or operations will be performed outside the standard order of operation rules. For instance, perhaps you want to multiply 4 and 7 before squaring. To indicate that the multiplication happens before the exponent, the multiplication is placed inside parentheses: $(4\times 7)^2$.

This means operations inside the parentheses take precedence, or happen before other operations. Now, the first step in calculating arithmetic expressions using the order of operations is to perform operations inside parentheses first. Inside the parentheses, you follow the order of operation rules EMDAS.

✓ Example $3.3.7$: Prioritizing Parentheses in the Order of Operations

Correctly apply the rules for the order of operations to accurately compute the following:

$(10-3)\times 5^3$.

Answer

Step 1: Perform all calculations within the parentheses before all other operations.

$(\mathbf{10-3})\times 5^3=\mathbf{7}\times 5^3$

Step 2: Since all parentheses have been cleared, move left to right, and compute all the exponents next.

$7\times \mathbf{5^3}=7\times \mathbf{125}$

Step 3: Perform all multiplications and divisions moving left to right.

$\mathbf{7}\times\mathbf{125}=\mathbf{875}$

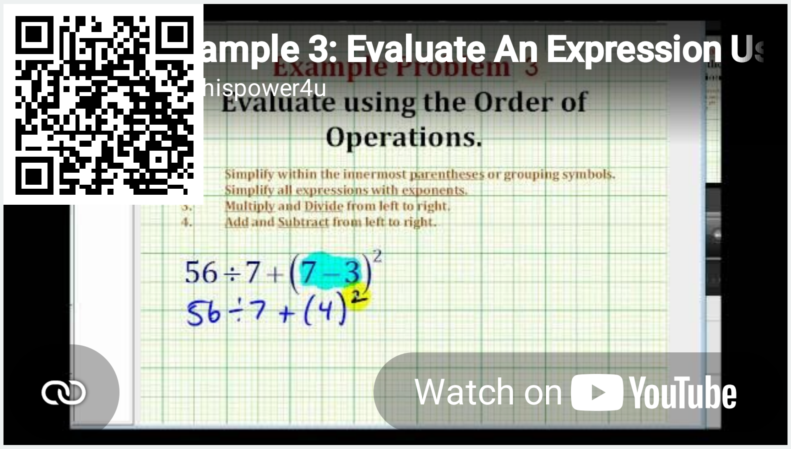
Your Turn $3.3.7$

Correctly apply the rules for the order of operations to accurately compute the following:

$8-(25-2^2)/7$.

Be aware that there can be more than one set of parentheses, and parentheses within parentheses. When one set of parentheses is inside another set, do the innermost set first, and then work outward.

Video



Example Problem 3

Evaluate using the Order of Operations.

Simplify within the innermost parentheses or grouping symbols.
Simplify all expressions with exponents.
3. Multiply and Divide from left to right.
4. Add and Subtract from left to right.

$56\div 7+(7-3)^2$
$56\div 7+(4)^2$

Order of Operations 3

✓ Example $3.3.8$: Working Innermost Parentheses in the Order of Operations

Correctly apply the rules for order of operations to accurately compute the following:

$4+2\times(3^2-(2+5)^2\times 4)/(3+8)$.

Answer

Step 1: Perform all calculations within the parentheses before other operations. Evaluate the innermost parentheses first. We can work separate parentheses expressions at the same time. The innermost set of parentheses has the 2 + 5 inside. The

3 + 8 is in a separate set of parentheses, so that addition can occur at the same time as the 2 + 5.

$$4+2\times(3^2-(\mathbf{2+5})^2\times 4)/(\mathbf{3+8})$$
$$=4+2\times(3^2-(\mathbf{7})^2\times 4)/(\mathbf{11})$$

Step 2: Now that those parentheses have been handled, move on to the next set of parentheses. Applying the order of operation rules inside that set of parentheses, the exponent is evaluated first, then the multiplication, and then the addition.

$$4+2\times(\mathbf{3^2-7^2\times 4})/11$$
$$=4+2\times(\mathbf{3^2-49\times 4})/11$$
$$=4+2\times(\mathbf{-187})/11$$

Step 3: Since all parentheses have been cleared, apply the EMDAS rules to finish the calculation.

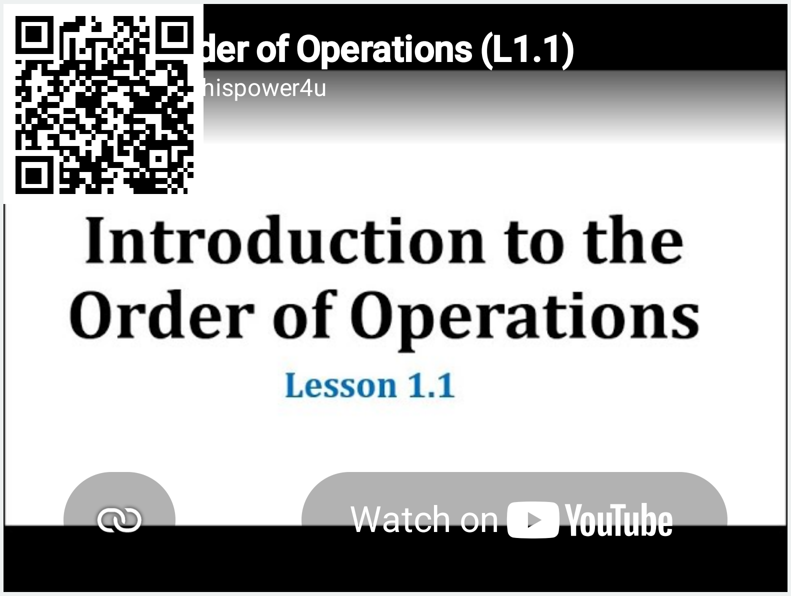
$$4+\mathbf{2\times(-187)}/11=4-\mathbf{374/11}=\mathbf{4-34}=-30$$

Your Turn 3.3.81

Correctly apply the rules for the order of operations to accurately compute the following:

$$(8-6)^2\times 100-((48/6-3)^2-4\times 7)$$

Video



Order of Operations 4

Check Your Understanding

1. Which operation has highest precedence?
2. Which is performed first, exponents or addition?
3. Calculate $2\times 3^2-5\times 8$.
4. What is used to indicate operations that should be performed out of order?
5. Calculate $(4-3)^2+27\times 8^2\div 6^2$.

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3.3.0: Exercises

? Exercise 3.3.0.1

Which operations have the lowest precedence in order of operations?

? Exercise 3.3.0.2

If many operations have the same precedence in an expression, in what order should the operations be performed?

? Exercise 3.3.0.3

Which operations have the same precedence in order of operations?

? Exercise 3.3.0.4

After all operations in parentheses have been performed, which operations should be performed next? For the following exercises, perform the indicated calculation.

? Exercise 3.3.0.5

$4-5\times 6$

? Exercise 3.3.0.6

$34-10\times 6$

? Exercise 3.3.0.7

$18+8\times 13$

? Exercise 3.3.0.8

$40+12\times 17$

? Exercise 3.3.0.9

$50\div 2+8$

? Exercise 3.3.0.10

$72\div 6+18$

? Exercise 3.3.0.11

$13+4\times 3^2$

? Exercise 3.3.0.12

$45-6\times 7^3$

? Exercise 3.3.0.13

$6^2 \times 5 - 13 \times 9^2$

? Exercise 3.3.0.14

$14^2 \times 8 - 5 \times 2^4$

? Exercise 3.3.0.15

$450 \div 3^2 - 56 \div 2^2$

? Exercise 3.3.0.16

$1,000 \div 5^3 - 7 \times 8^4$

? Exercise 3.3.0.17

$38 \times 6 - 4 + 5 \times 18 \div 10$

? Exercise 3.3.0.18

$15 \times 7 + 23 - 6 \times 40 \div 24$

? Exercise 3.3.0.19

$600 \div 12 \times 5 + 40 - 6^2$

? Exercise 3.3.0.20

$2 \times 12^3 \div 8 \times 3 - 4^5$

? Exercise 3.3.0.21

$10 \times 6^3 \div 2 \times 5 + 3^2 - 240 \div 8 \times 9$

? Exercise 3.3.0.22

$45 \div 15 \times 6 + 7^2 - 4 \times 15 \times 18 \div 12 \times 3$

? Exercise 3.3.0.23

$(4 + 3) \times 2$

? Exercise 3.3.0.24

$6 \times (12 + 8)$

? Exercise 3.3.0.25

$(14 - 25) \times 2$

? Exercise 3.3.0.26

$(86-61) \div 5$

? Exercise 3.3.0.27

$(3+2)^3$

? Exercise 3.3.0.28

$(17-12)^4$

? Exercise 3.3.0.29

$(45-60)^2$

? Exercise 3.3.0.30

$(90-101)^3$

? Exercise 3.3.0.31

$(3 \times 4)^3 \div 8+5$

? Exercise 3.3.0.32

$(6-9)^2 \times 5-4$

? Exercise 3.3.0.33

$4-(6+3) \times 5+11$

? Exercise 3.3.0.34

$12+(13-6) \times 8-130$

? Exercise 3.3.0.35

$15 \times (4-1)^2+5 \times (17+2 \times 3)+4$

? Exercise 3.3.0.36

$18 \times (12-6)^4-8 \times (30-14 \times 16)-90$

? Exercise 3.3.0.37

$4 \times (5+2 \times (6+8))+10 \times 3^2$

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3.4: Rational Numbers

Learning Objectives

1. Define and identify numbers that are rational.
2. Simplify rational numbers and express in lowest terms.
3. Add and subtract rational numbers.
4. Convert between improper fractions and mixed numbers.
5. Convert rational numbers between decimal and fraction form.
6. Multiply and divide rational numbers.
7. Apply the order of operations to rational numbers to simplify expressions.
8. Apply density property of rational numbers.
9. Solve problems involving rational numbers.
10. Use fractions to convert between units.
11. Define and apply percent.
12. Solve problems using percent.

We are often presented with percentages or fractions to explain how much of a population has a certain feature. For example, the 6-year graduation rate of college students at public institutions is 57.6%, or 72/125. That fraction may be unsettling. But without the context, the percentage is hard to judge. So how does that compare to private institutions? There, the 6-year graduation rate is 65.4%, or 327/500. Comparing the percentages is straightforward, but the fractions are harder to interpret due to different denominators. For more context, historical data could be found. One study reported that the 6-year graduation rate in 1995 was 56.4%. Comparing that historical number to the recent 6-year graduation rate at public institutions of 57.6% shows that there hasn't been much change in that rate.

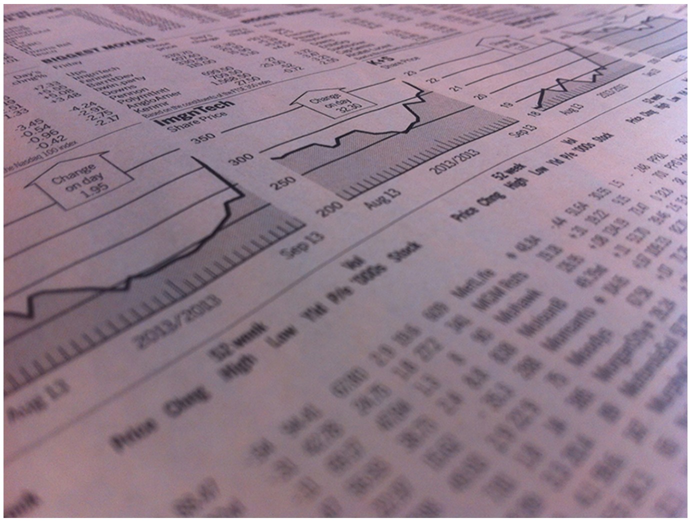


Figure 3.20: Stock gains and losses are often represented as percentages.(credit: "stock market quotes in newspaper" by Andreas Poike/Flickr, CC BY 2.0)

Defining and Identifying Numbers That Are Rational

A rational number (called rational since it is a ratio) is just a fraction where the numerator is an integer and the denominator is a non-zero integer. As simple as that is, they can be represented in many ways. It should be noted here that any integer is a rational number. An integer, n, written as a fraction of two integers is $\frac{n}{1}$. In its most basic representation, a rational number is an integer divided by a non-zero integer, such as $\frac{3}{12}$

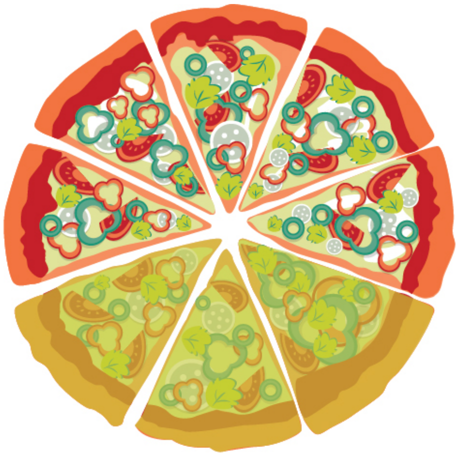


Figure $3.4.1$: Pizza cut in 8 slices, with 3 slices highlighted

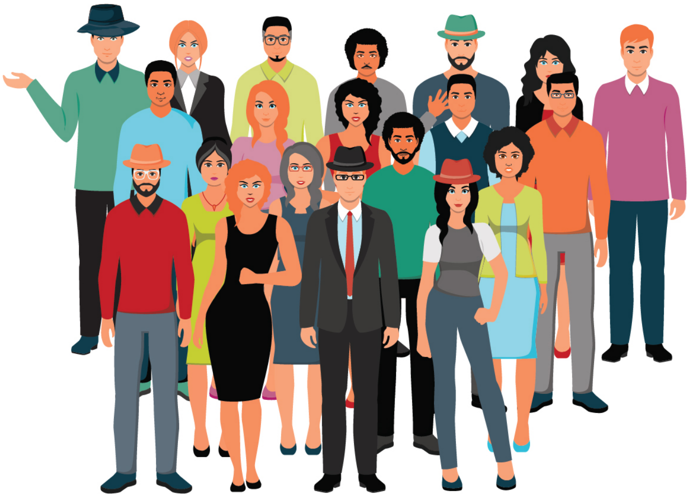


Figure $3.4.2$: Group of 20 people, with 5 people wearing hats

Another representation of rational numbers is as a mixed number, such as $2\frac{5}{8}$.

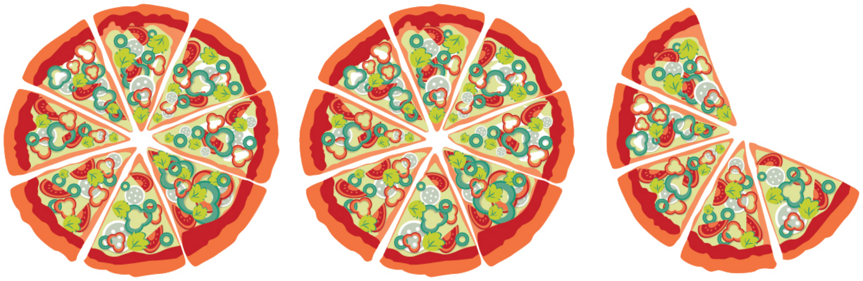


Figure $3.4.3$: Two whole pizzas and one partial pizza

Rational numbers may also be expressed in decimal form; for instance, as 1.34. When 1.34 is written, the decimal part, 0.34, represents the fraction $\frac{34}{100}$, and the number 1.34 is equal to $1\frac{34}{100}$. However, not all decimal representations are rational numbers.

A number written in decimal form where there is a last decimal digit (after a given decimal digit, all following decimal digits are 0) is a **terminating decimal**, as in 1.34 above. Alternately, any decimal numeral that, after a finite number of decimal digits, has digits equal to 0 for all digits following the last non-zero digit.

All numbers that can be expressed as a terminating decimal are rational. This comes from what the decimal represents. The decimal part is the fraction of the decimal part divided by the appropriate power of 10. That power of 10 is the number of decimal digits present, as for 0.34, with two decimal digits, being equal to $\frac{34}{100}$.

Another form that is a rational number is a decimal that repeats a pattern, such as 67.1313… When a rational number is expressed in decimal form and the decimal is a repeated pattern, we use special notation to designate the part that repeats. For example, if we have the repeating decimal 4.3636…, we write this as $4.\overline{36}$. The bar over the 36 indicates that the 36 repeats forever.

If the decimal representation of a number does not terminate or form a repeating decimal, that number is not a rational number.

One class of numbers that is not rational is the **square roots** of integers or rational numbers that are not **perfect squares**, such as $\sqrt{10}$ and $\sqrt{\frac{25}{6}}$. More generally, the number b is the square root of the number a if $a =$. The notation for this is $b = \sqrt{a}$, where the symbol $\sqrt{\ }$ is the square root sign. An integer perfect square is any integer that can be written as the square of another integer. A rational perfect square is any rational number that can be written as a fraction of two integers that are perfect squares.

Sometimes you may be able to identify a perfect square from memory. Another process that may be used is to factor the number into the product of an integer with itself. Or a calculator (such as Desmos) may be used to find the square root of the number. If the calculator yields an integer, the original number was a perfect square.

Tech Check: Using Desmos to Find the Square Root of a Number

When Desmos is used, there is a tab at the bottom of the screen that opens the keyboard for Desmos. The keyboard is shown below. On the keyboard (Figure $3.4.4$) is the square root symbol $(\sqrt{\ })$. To find the square root of a number, click the square root key, and then type the number. Desmos will automatically display the value of the square root as you enter the number.

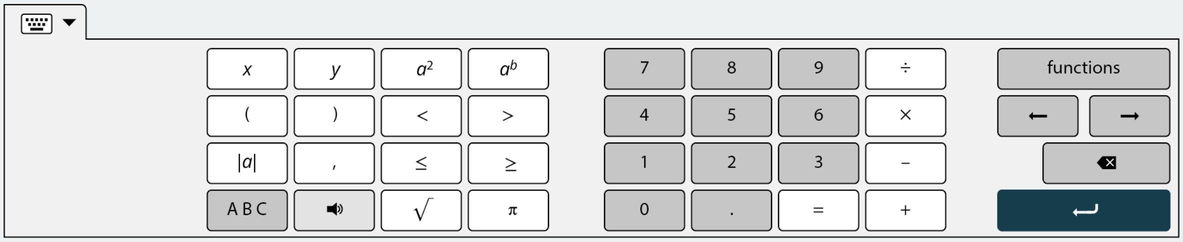


Figure $3.4.4$: Desmos keyboard with square root key

Example $3.4.1$: Identifying Perfect Squares

Which of the following are perfect squares?

1. 45
2. 144

Answer

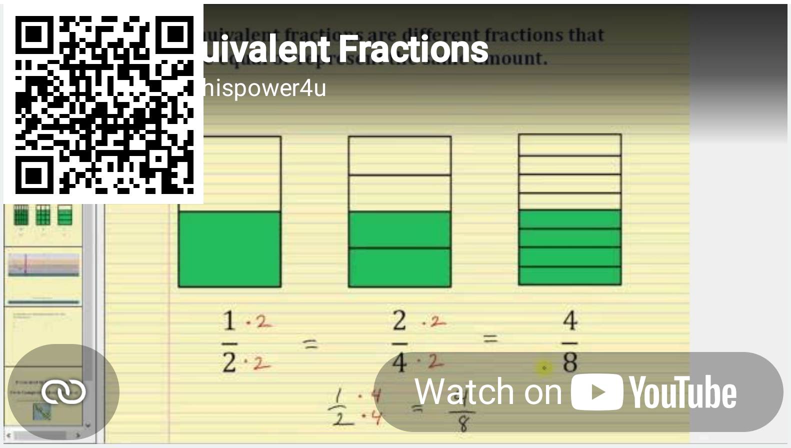
1. We could attempt to find the perfect square by factoring. Writing all the factor pairs of 45 results in $1 \times 45, 3 \times 15$, and 5×9. None of the pairs is a square, so 45 is not a perfect square. Using a calculator to find the square root of 45, we obtain 6.708 (rounded to three decimal places). Since this was not an integer, the original number was not a perfect square.
2. We could attempt to find the perfect square by factoring. Writing all the factor pairs of 144 results in $1 \times 144, 2 \times 72, 3 \times 48, 6 \times 24, 8 \times 18$, and 12×12. Since the last pair is an integer multiplied by itself, 144 is a perfect square. Alternately, using Desmos to find the square root of 144, we obtain 12. Since the square root of 144 is an integer, 144 is a perfect square.

Your Turn $3.4.1$

Determine if the following are perfect squares:

1. 45
2. 144

Video


uivalent Fractions
hispower4u
$\frac{1 \cdot 2}{2 \cdot 2} = \frac{2 \cdot 2}{4 \cdot 2} = \frac{4}{8}$
$\frac{1 \cdot 4}{2 \cdot 4} = \frac{4}{8}$
Watch on YouTube

Introduction to Fractions

✓ Example ✓ 3.4.2 ✓ : Identifying Rational Numbers

Determine which of the following are rational numbers:

1. $\sqrt{73}$
2. 4.556
3. $3\frac{1}{5}$
4. $\frac{41}{17}$
5. $5.\overline{64}$

Answer

1. Since 73 is not a perfect square, its square root is not a rational number. This can also be seen when a calculator is used. Entering $\sqrt{73}$ into a calculator results in 8.544003745317 (and then more decimal values after that). There is no repeated pattern, so this is not a rational number.
2. Since 4.556 is a decimal that terminates, this is a rational number.
3. $3\frac{1}{5}$ is a mixed number, so it is a rational number.
4. $\frac{41}{17}$ is an integer divided by an integer, so it is a rational number.
5. $5.646464\ldots$ is a decimal that repeats a pattern, so it is a rational number.

Your Turn 3.4.2

Determine which of the following are rational numbers:

$\sqrt{13}$

$-13.\overline{21}$

$\frac{-48}{-16}$

$-4\frac{18}{19}$

14.1131

Simplifying Rational Numbers and Expressing in Lowest Terms

A rational number is one way to express the division of two integers. As such, there may be multiple ways to express the same value with different rational numbers. For instance, $\frac{4}{5}$ and $\frac{12}{15}$ are the same value. If we enter them into a calculator, they both equal 0.8. Another way to understand this is to consider what it looks like in a figure when two fractions are equal.

In Figure 3.25, we see that $\frac{3}{5}$ of the rectangle and $\frac{9}{15}$ of the rectangle are equal areas.

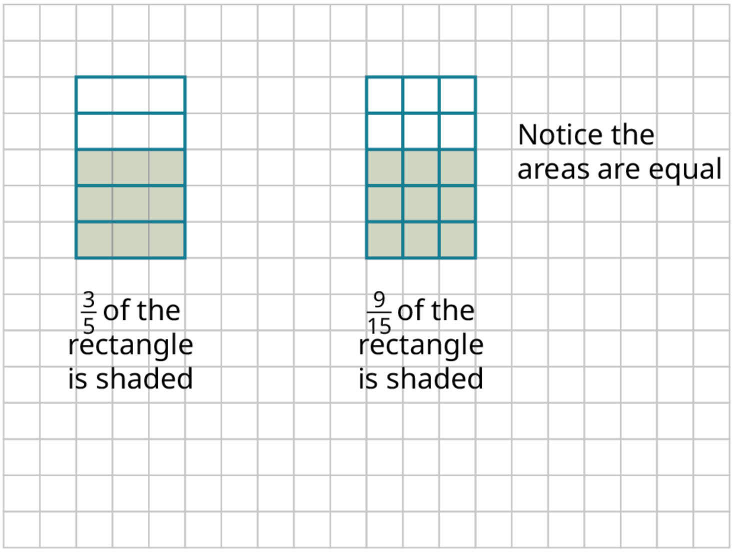

Notice the areas are equal
$\frac{3}{5}$ of the rectangle is shaded
$\frac{9}{15}$ of the rectangle is shaded

Figure $3.4.5$: Two Rectangles with Equal Areas

They are the same proportion of the area of the rectangle. The left rectangle has 5 pieces, three of which are shaded. The right rectangle has 15 pieces, 9 of which are shaded. Each of the pieces of the left rectangle was divided equally into three pieces. This was a multiplication. The numerator describing the left rectangle was 3 but it becomes 3×3, or 9, as each piece was divided into three. Similarly, the denominator describing the left rectangle was 5, but became 5×3, or 15, as each piece was divided into 3. The fractions $\frac{3}{5}$ and $\frac{9}{15}$ are **equivalent** because they represent the same portion (often loosely referred to as equal).

This understanding of equivalent fractions is very useful for conceptualization, but it isn't practical, in general, for determining when two fractions are equivalent. Generally, to determine if the two fractions $\frac{a}{b}$ and $\frac{c}{d}$ are equivalent, we check to see that $a \times d = b \times c$. If those two products are equal, then the fractions are equal also.

✓ Example ✓ $3.4.3$: Determining If Two Fractions Are Equivalent

Determine if $\frac{12}{30}$ and $\frac{14}{35}$ are equivalent fractions.

Answer

Applying the definition, $a = 12, b = 30, c =$ and $d =$. So $a \times d = 12 \times 35 =$. Also, $b \times c = 30 \times 14 =$. Since these values are equal, the fractions are equivalent.

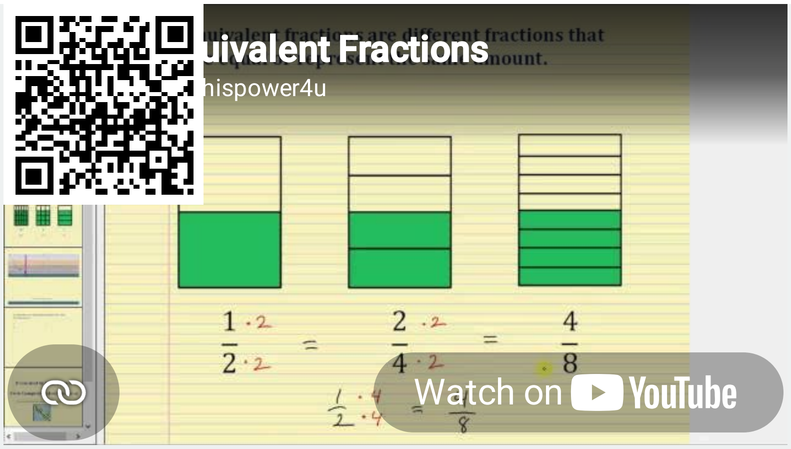
Your Turn $3.4.3$

Determine if $\frac{8}{14}$ and $\frac{12}{26}$ are equivalent fractions.

That $a \times d = b \times c$ indicates the fractions $\frac{a}{b}$ and $\frac{c}{d}$ are equivalent is due to some algebra. One property of natural numbers, integers, and rational numbers (also irrational numbers) is that for any three numbers a, b, and c with $c \neq 0$, if $a = b$, then $a/c = b/c$. In other words, when two numbers are equal, then dividing both numbers by the same non-zero number, the two newly obtained numbers are also equal. We can apply that to $a \times d$ and $b \times c$, to show that $\frac{a}{b}$ and $\frac{c}{d}$ are equivalent if $a \times d = b \times c$.

If $a \times d = b \times c$, and $c \neq 0, d \neq 0$, we can divide both sides by and obtain the following: $\frac{a \times d}{c} =$. We can divide out the c on the right-hand side of the equation, resulting in $\frac{a \times d}{c} =$. Similarly, we can divide both sides of the equation by d and obtain the following: $\frac{a \times d}{c \times d} =$. We can divide out d the on the left-hand side of the equation, resulting in $\frac{a}{c} =$. So, the rational numbers $\frac{a}{c}$ and $\frac{b}{d}$ are equivalent when $a \times d = b \times c$.

Video


uivalent Fractions
hispower4u
Watch on YouTube
$\frac{1\cdot 2}{2\cdot 2} = \frac{2\cdot 2}{4\cdot 2} = \frac{4}{8}$

Equivalent Fractions

Recall that a **common divisor** or **common factor** of a set of integers is one that divides all the numbers of the set of numbers being considered. In a fraction, when the numerator and denominator have a common divisor, that common divisor can be **divided out**. This is often called **canceling the common factors** or, more colloquially, as **canceling**.

To show this, consider the fraction $\frac{36}{63}$. The numerator and denominator have the common factor 3. We can rewrite the fraction as $\frac{36}{63} = \frac{12\times 3}{21\times 3}$. The common divisor 3 is then divided out, or canceled, and we can write the fraction as $\frac{12\times \cancel{3}}{21\times \cancel{3}} = \frac{12}{21}$. The 3s have been crossed out to indicate they have been divided out. The process of dividing out two factors is also referred to as **reducing the fraction**.

If the numerator and denominator have no common positive divisors other than 1, then the rational number is in **lowest terms**.

The process of dividing out common divisors of the numerator and denominator of a fraction is called **reducing the fraction**.

One way to reduce a fraction to lowest terms is to determine the GCD of the numerator and denominator and divide out the GCD. Another way is to divide out common divisors until the numerator and denominator have no more common factors.

✓ Example ✓ 3.4.4: Reducing Fractions to Lowest Terms

Express the following rational numbers in lowest terms:

1. $\frac{36}{48}$
2. $\frac{100}{250}$
3. $\frac{51}{136}$

Answer

1. One process to reduce $\frac{36}{48}$ to lowest terms is to identify the GCD of 36 and 48 and divide out the GCD. The GCD of 36 and 48 is 12.

   Step 1: We can then rewrite the numerator and denominator by factoring 12 from both.

   $\frac{36}{48} = \frac{12\times 3}{12\times 4}$

   Step 2: We can now divide out the 12s from the numerator and denominator.

   $\frac{36}{48} = \frac{\cancel{12}\times 3}{\cancel{12}\times 4} = \frac{3}{4}$

   So, when $\frac{36}{48}$ is reduced to lowest terms, the result is $\frac{3}{4}$.

   Alternately, you could identify a common factor, divide out that common factor, and repeat the process until the remaining fraction is in lowest terms.

   Step 1: You may notice that 4 is a common factor of 36 and 48.

   Step 2: Divide out the 4, as in $\frac{36}{48} = \frac{4\times 9}{4\times 12} = \frac{\cancel{4}\times 9}{\cancel{4}\times 12} = \frac{9}{12}$.

Step 3: Examining the 9 and 12, you identify 3 as a common factor and divide out the 3, as in $\frac{9}{12} = \frac{\cancel{3}\times 3}{\cancel{3}\times 4} = \frac{3}{4}$. The 3 and 4 have no common positive factors other than 1, so it is in lowest terms.

So, when $\frac{36}{48}$ is reduced to lowest terms, the result is $\frac{3}{4}$.

2. **Step 1:** To reduce $\frac{100}{250}$ to lowest terms, identify the GCD of 100 and 250. This GCD is 50.

   Step 2: We can then rewrite the numerator and denominator by factoring 50 from both.

   $\frac{100}{250} = \frac{50\times 2}{50\times 5}$.

   Step 3: We can now divide out the 50s from the numerator and denominator.

   $\frac{100}{250} = \frac{\cancel{50}\times 2}{\cancel{50}\times 5} = \frac{2}{5}$

   So, when $\frac{100}{250}$ is reduced to lowest terms, the result is $\frac{2}{5}$.

3. **Step 1:** To reduce $\frac{51}{136}$ to lowest terms, identify the GCD of 51 and 136. This GCD is 17.

   Step 2: We can then rewrite the numerator and denominator by factoring 17 from both.

   $\frac{51}{136} = \frac{17\times 3}{17\times 8}$

   Step 3: We can now divide out the 17s from the numerator and denominator.

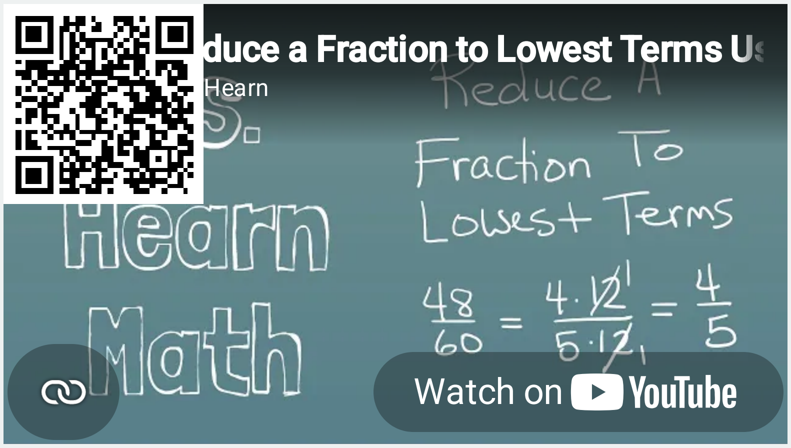
   $\frac{51}{136} = \frac{\cancel{17}\times 3}{\cancel{17}\times 8} = \frac{3}{8}$

   So, when $\frac{51}{136}$ is reduced to lowest terms, the result is $\frac{3}{8}$.

Your Turn 3.4.4

Express $\frac{252}{840}$ in lowest terms.

Video


duce a Fraction to Lowest Terms Us
Hearn
Reduce A Fraction To Lowest Terms
48/60 = 4·12/5·12 = 4/5
Hearn Math
Watch on YouTube

Reducing Fractions to Lowest Terms

Tech Check: Using Desmos to Find Lowest Terms

Desmos is a free online calculator. Desmos supports reducing fractions to lowest terms. When a fraction is entered, Desmos immediately calculates the decimal representation of the fraction. However, to the left of the fraction, there is a button that, when clicked, shows the fraction in reduced form.

Video

w to simplify fractions using DESM
y Chang
Watch on YouTube

Using Desmos to Reduce a Fraction

Adding and Subtracting Rational Numbers

Adding or subtracting rational numbers can be done with a calculator, which often returns a decimal representation, or by finding a common denominator for the rational numbers being added or subtracted.

Tech Check: Using Desmos to Add Rational Numbers in Fractional Form

To create a fraction in Desmos, enter the numerator, then use the division key (/) on your keyboard, and then enter the denominator. The fraction is then entered. Then click the right arrow key to exit the denominator of the fraction. Next, enter the arithmetic operation (+ or –). Then enter the next fraction. The answer is displayed dynamically (calculates as you enter). To change the Desmos result from decimal form to fractional form, use the fraction button (Figure 3.26) on the left of the line that contains the calculation:

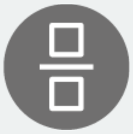


Figure $3.4.6$: Fraction button on the Desmos keyboard

Example $3.4.5$: Adding Rational Numbers Using Desmos

Calculate $\frac{23}{42}+\frac{9}{56}$ using Desmos.

Answer

Enter $\frac{23}{42}+\frac{9}{56}$ in Desmos. The result is displayed as 0.708333333333 (which is $0.708\overline{3}$). Clicking the fraction button to the left on the calculation line yields $\frac{17}{24}$.

Your Turn $3.4.5$

Calculate $\frac{124}{297}+\frac{3}{125}$.

Performing addition and subtraction without a calculator may be more involved. When the two rational numbers have a common denominator, then adding or subtracting the two numbers is straightforward. Add or subtract the numerators, and then place that value in the numerator and the common denominator in the denominator. Symbolically, we write this as Math input error

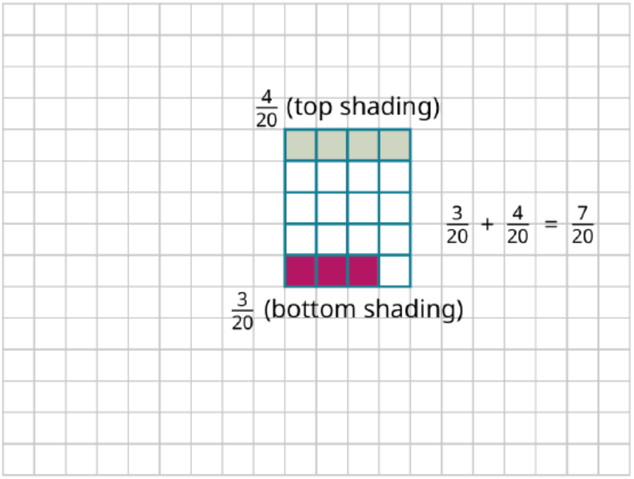

$\frac{4}{20}$ (top shading)
$\frac{3}{20} + \frac{4}{20} = \frac{7}{20}$
$\frac{3}{20}$ (bottom shading)

Figure 3.4.7: Partially Shaded Rectangle

It is customary to then write the result in lowest terms.

FORMULA

If c is a non-zero integer, then $\frac{a}{c} \pm \frac{b}{c} =$ mfrac.

Example 3.4.6: Adding Rational Numbers with the Same Denominator

Calculate $\frac{13}{28} + \frac{7}{28}$.

Answer

Since the rational numbers have the same denominator, we perform the addition of the numerators, $13 + 7$, and then place the result in the numerator and the common denominator, 28, in the denominator. $\frac{13}{28} + \frac{7}{28} = \frac{13+7}{28} = \frac{20}{28}$

Once we have that result, reduce to lowest terms, which gives $\frac{20}{28} = \frac{4\times 5}{4\times 7} = \frac{\cancel{4}\times 5}{\cancel{4}\times 7} = \frac{5}{7}$.

Your Turn 3.4.6

Calculate $\frac{38}{73} + \frac{7}{73}$.

Example \(\PageIndex{7}\): Subtracting Rational Numbers with the Same Denominator

Calculate $\frac{45}{136} - \frac{17}{136}$.

Answer

Since the rational numbers have the same denominator, we perform the subtraction of the numerators, $45 - 17$, and then place the result in the numerator and the common denominator, 136, in the denominator. $\frac{45}{136} - \frac{17}{136} - \frac{45-17}{136} = \frac{28}{136}$

Once we have that result, reduce to lowest terms, this gives $\frac{28}{136} = \frac{4\times 7}{4\times 34} = \frac{\cancel{4}\times 7}{\cancel{4}\times 34} = \frac{7}{34}$.

Your Turn 3.4.7

1. Calculate $\frac{21}{40} - \frac{8}{40}$.

When the rational numbers do not have common denominators, then we have to transform the rational numbers so that they do have common denominators. The common denominator that reduces work later in the problem is the LCM of the numerator and denominator. When adding or subtracting the rational numbers $\frac{a}{b}$ and $\frac{c}{d}$, we perform the following steps.

Step 1: Find $LCM(b, d)$.

Step 2: Calculate $n = \frac{LCM(b,d)}{b}$ and $m = \frac{LCM(b,d)}{d}$.

Step 3: Multiply the numerator and denominator of $\frac{a}{b}$ by n, yielding $\frac{a \times n}{b \times n}$.

Step 4: Multiply the numerator and denominator of $\frac{c}{d}$ by m, yielding $\frac{c \times m}{d \times m}$.

Step 5: Add or subtract the rational numbers from Steps 3 and 4, since they now have the common denominators.

You should be aware that the common denominator is $LCM(b,d)$. For the first denominator, we have $b \times n = b \times \frac{LCM(b,d)}{b} = LCM(b,d)$, since we multiply and divide $LCM(b,d)$ by the same number. For the same reason, $d \times m = d \times \frac{LCM(b,d)}{b} = LCM(b,d)$.

Example $3.4.8$: Adding Rational Numbers with Unequal Denominators

Calculate $\frac{11}{18} + \frac{2}{15}$.

Answer

The denominators of the fractions are 18 and 15, so we label $b =$ and $d =$.

Step 1: Find LCM(18,15). This is 90.

Step 2: Calculate n and m. $n = \frac{90}{18} = 5$ and $m = \frac{90}{15} = 6$.

Step 3: Multiplying the numerator and denominator of $\frac{11}{18}$ by $n = 5$ yields $\frac{11 \times 5}{18 \times 5} = \frac{55}{90}$.

Step 4: Multiply the numerator and denominator of $\frac{2}{15}$ by $m = 6$ yields $\frac{2 \times 6}{15 \times 6} = \frac{12}{90}$.

Step 5: Now we add the values from Steps 3 and 4: $\frac{55}{90} + \frac{12}{90} = \frac{67}{90}$.

This is in lowest terms, so we have found that $\frac{11}{18} + \frac{2}{15} = \frac{67}{90}$.

Your Turn $3.4.8$

Calculate $\frac{4}{9} + \frac{7}{12}$.

Example $3.4.9$: Subtracting Rational Numbers with Unequal Denominators

Calculate $\frac{14}{25} - \frac{9}{70}$.

Answer

The denominators of the fractions are 25 and 70, so we label $b =$ and $d =$.

Step 1: Find LCM(25,70). This is 350.

Step 2: Calculate n and m: $n = \frac{350}{25} =$ and $m = \frac{350}{70} = 5$.

Step 3: Multiplying the numerator and denominator of $\frac{14}{25}$ by $n =$ yields $\frac{14 \times 14}{25 \times 14} = \frac{196}{350}$.

Step 4: Multiplying the numerator and denominator of $\frac{9}{70}$ by $m = 5$ yields $\frac{9 \times 5}{70 \times 5} = \frac{45}{350}$.

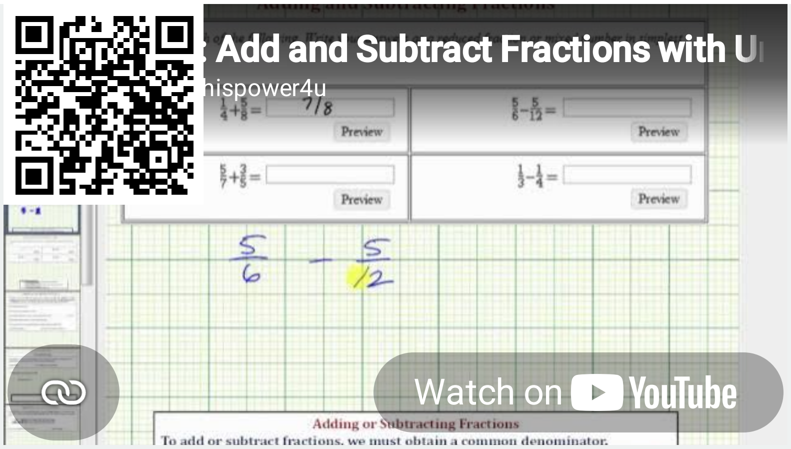
Step 5: Now we subtract the value from Step 4 from the value in Step 3: $\frac{196}{350} - \frac{45}{350} = \frac{151}{350}$.

This is in lowest terms, so we have found that $\frac{14}{25} - \frac{9}{70} = \frac{151}{350}$.

Your Turn $3.4.9$

Calculate $\frac{10}{99} - \frac{17}{300}$.

Video


Add and Subtract Fractions with U
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Adding or Subtracting Fractions
To add or subtract fractions, we must obtain a common denominator.

Adding and Subtracting Fractions with Different Denominators

Converting Between Improper Fractions and Mixed Numbers

One way to visualize a fraction is as parts of a whole, as in $\frac{5}{12}$ of a pizza. But when the numerator is larger than the denominator, as in $\frac{23}{12}$, then the idea of parts of a whole seems not to make sense. Such a fraction is an **improper fraction.** That kind of fraction could be written as an integer plus a fraction, which is a **mixed number**. The fraction $\frac{23}{12}$ rewritten as a mixed number would be $1\frac{11}{12}$. Arithmetically, $1\frac{11}{12}$ is equivalent to $1+\frac{11}{12}$, which is read as “one and 11 twelfths.”

Improper fractions can be rewritten as mixed numbers using division and remainders. To find the mixed number representation of an improper fraction, divide the numerator by the denominator. The quotient is the integer part, and the remainder becomes the numerator of the remaining fraction.

✓ Example $3.4.10$: Rewriting an Improper Fraction as a Mixed Number

Rewrite $\frac{48}{13}$ as a mixed number.

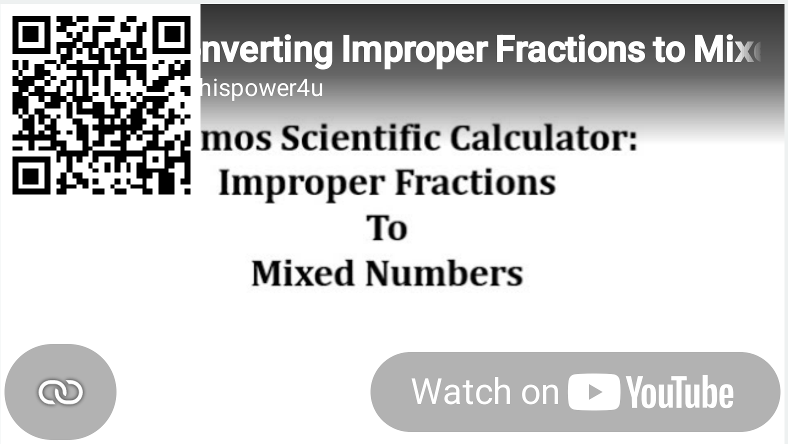
Answer

When 48 is divided by 13, the result is 3 with a remainder of 9. So, we can rewrite $\frac{48}{13}$ as $3\frac{9}{13}$.

Your Turn $3.4.10$

Rewrite $\frac{95}{26}$ as a mixed number.

Video


nverting Improper Fractions to Mix
hispower4u
Watch on YouTube

mos Scientific Calculator:
Improper Fractions
To
Mixed Numbers

Converting an Improper Fraction to a Mixed Number Using Desmos

Similarly, we can convert a mixed number into an improper fraction. To do so, first convert the whole number part to a fraction by writing the whole number as itself divided by 1, and then add the two fractions.

Alternately, we can multiply the whole number part and the denominator of the fractional part. Next, add that product to the numerator. Finally, express the number as that product divided by the denominator.

✓ Example $3.4.11$: Rewriting a Mixed Number as an Improper Fraction

Rewrite $5\frac{4}{9}$ as an improper fraction.

Answer

Step 1: Multiply the integer part, 5, by the denominator, 9, which gives $5 \times 9 = 45$.

Step 2: Add that product to the numerator, which gives $45 + 4 = 49$.

Step 3: Write the number as the sum, 49, divided by the denominator, 9, which gives $\frac{49}{9}$.

Your Turn $3.4.11$

Rewrite $9\frac{5}{14}$ as an improper fraction.

Tech Check: Using Desmos to Rewrite a Mixed Number as an Improper Fraction

Desmos can be used to convert from a mixed number to an improper fraction. To do so, we use the idea that a mixed number, such as $5\frac{6}{11}$, is another way to represent $5 + \frac{6}{11}$. If $5 + \frac{6}{11}$ is entered in Desmos, the result is the decimal form of the number. However, clicking the fraction button to the left will convert the decimal to an improper fraction, $\frac{61}{11}$. As an added bonus, Desmos will automatically reduce the fraction to lowest terms.

Converting Rational Numbers Between Decimal and Fraction Forms

Understanding what decimals represent is needed before addressing conversions between the fractional form of a number and its **decimal form**, or writing a number in **decimal notation**. The decimal number 4.557 is equal to $4\frac{557}{1{,}000}$. The decimal portion, .557, is 557 divided by 1,000. To write any decimal portion of a number expressed as a terminating decimal, divide the decimal number by 10 raised to the power equal to the number of decimal digits. Since there were three decimal digits in 4.557, we divided 557 by $10^3 = 1000$.

Decimal representations may be very long. It is convenient to **round off** the decimal form of the number to a certain number of decimal digits. To round off the decimal form of a number to n (decimal) digits, examine the $(n + 1)$st decimal digit. If that digit is 0, 1, 2, 3, or 4, the number is rounded off by writing the number to the nth decimal digit and no further. If the $(n + 1)$st decimal digit is 5, 6, 7, 8, or 9, the number is rounded off by writing the number to the nth digit, then replacing the nth digit by one more than the nth digit.

✓ Example $3.4.12$: Rounding Off a Number in Decimal Form to Three Digits

Round 5.67849 to three decimal digits.

Answer

The third decimal digit is 8. The digit following the 8 is 4. When the digit is 4, we write the number only to the third digit. So, 5.67849 rounded off to three decimal places is 5.678.

Your Turn $3.4.12$

Round 5.1082 to three decimal places.

Rounding Off a Number in Decimal Form to Four Digits

Round 45.11475 to four decimal digits.

Answer

The fourth decimal digit is 7. The digit following the 7 is 5. When the digit is 5, we write the number only to the fourth decimal digit, 45.1147. We then replace the fourth decimal digit by one more than the fourth digit, which yields 45.1148. So, 45.11475 rounded off to four decimal places is 45.1148.

Your Turn $3.4.13$

Round 18.6298 to two decimal places.

To convert a rational number in fraction form to decimal form, use your calculator to perform the division.

✓ Example $3.4.14$: Converting a Rational Number in Fraction Form into Decimal Form

Convert $\frac{47}{25}$ into decimal form.

Answer

Using a calculator to divide 47 by 25, the result is 1.88.

Your Turn $3.4.14$

Convert $\frac{48}{30}$ into decimal form.

Converting a terminating decimal to the fractional form may be done in the following way:

Step 1: Count the number of digits in the decimal part of the number, labeled n.

Step 2: Raise 10 to the nth power.

Step 3: Rewrite the number without the decimal.

Step 4: The fractional form is the number from Step 3 divided by the result from Step 2.

This process works due to what decimals represent and how we work with mixed numbers. For example, we could convert the number 7.4536 to fractional from. The decimal part of the number, the .4536 part of 7.4536, has four digits. By the definition of decimal notation, the decimal portion represents $\frac{4{,}536}{10^4} = \frac{4{,}536}{10{,}000}$. The decimal number 7.4536 is equal to the improper fraction $7\frac{4{,}536}{10{,}000}$. Adding those to fractions yields $\frac{74{,}536}{10{,}000}$.

✓ Example $3.4.15$: Converting from Decimal Form to Fraction Form with Terminating Decimals

Convert 3.2117 to fraction form.

Answer

Step 1: There are four digits after the decimal point, so $n = 4$.

Step 2: Raise 10 to the fourth power, $10^4 = 10{,}000$.

Step 3: When we remove the decimal point, we have 32,117.

Step 4: The fraction has as its numerator the result from Step 3 and as its denominator the result of Step 2, which is the fraction $\frac{32{,}117}{10{,}000}$.

Your Turn 3.4.15

Convert 17.03347 to fraction form.

The process is different when converting from the decimal form of a rational number into fraction form when the decimal form is a repeating decimal. This process is not covered in this text.

Multiplying and Dividing Rational Numbers

Multiplying rational numbers is less complicated than adding or subtracting rational numbers, as there is no need to find common denominators. To multiply rational numbers, multiply the numerators, then multiply the denominators, and write the numerator product divided by the denominator product. Symbolically, $\frac{a}{b} \times \frac{c}{d} = \frac{a\times c}{b\times d}$. As always, rational numbers should be reduced to lowest terms.

FORMULA

If b and d are non-zero integers, then $\frac{a}{b} \times \frac{c}{d} = \frac{a\times c}{b\times d}$.

Example 3.4.16: Multiplying Rational Numbers

Calculate $\frac{12}{25} \times \frac{10}{21}$.

Answer

Multiply the numerators and place that in the numerator, and then multiply the denominators and place that in the denominator.

$\frac{12}{25} \times \frac{10}{21} = \frac{12\times 10}{25\times 21} = \frac{120}{525}$

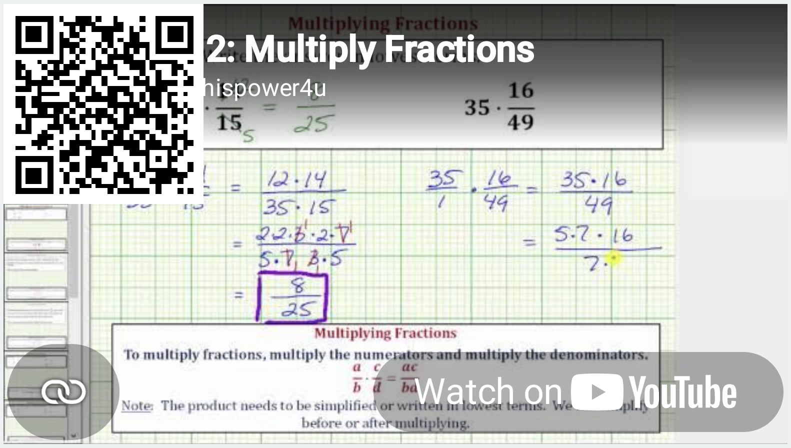
This is not in lowest terms, so this needs to be reduced. The GCD of 120 and 525 is 15.

$\frac{120}{525} = \frac{15\times 8}{15\times 35} = \frac{8}{35}$

Your Turn 3.4.16

Calculate $\frac{45}{88} \times \frac{28}{75}$.

Video



Multiplying Fractions

As with multiplication, division of rational numbers can be done using a calculator.

✓ Example 3.4.17: Dividing Decimals with a Calculator

Calculate 3.45 ÷ 2.341 using a calculator. Round to three decimal places if necessary.

Answer

Using a calculator, we obtain 1.473729175565997. Rounding to three decimal places we have 1.474.

Your Turn 3.4.17

Calculate $45.63 \div 17.13$ using a calculator. Round to three decimal places, if necessary.

Before discussing division of fractions without a calculator, we should look at the reciprocal of a number. The **reciprocal** of a number is 1 divided by the number. For a fraction, the reciprocal is the fraction formed by switching the numerator and denominator. For the fraction $\frac{a}{b}$, the reciprocal is $\frac{b}{a}$. An important feature for a number and its reciprocal is that their product is 1.

When dividing two fractions by hand, find the reciprocal of the **divisor** (the number that is being divided into the other number). Next, replace the divisor by its reciprocal and change the division into multiplication. Then, perform the multiplication. Symbolically, $\frac{b}{a} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} =$. As before, reduce to lowest terms.

FORMULA

If b, c and d are non-zero integers, then $\frac{b}{a} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} =$.

✓ Example 3.4.18: Dividing Rational Numbers

1. Calculate $\frac{4}{21} \div \frac{6}{35}$.
2. Calculate $\frac{1}{8} \div \frac{5}{28}$.

Answer

1. **Step 1:** Find the reciprocal of the number being divided by $\frac{6}{35}$. The reciprocal of that is $\frac{35}{6}$.

   Step 2: Multiply the first fraction by that reciprocal.

   $\frac{4}{21} \div \frac{6}{35} = \frac{4}{21} \times \frac{35}{6} = \frac{140}{126}$

   The answer, $\frac{140}{126}$ is not in lowest terms. The GCD of 140 and 126 is 14. Factoring and canceling gives $\frac{140}{126} = \frac{14 \times 10}{14 \times 9} = \frac{10}{9}$.

2. **Step 1:** Find the reciprocal of the number being divided by, which is $\frac{5}{28}$. The reciprocal of that is $\frac{28}{5}$.

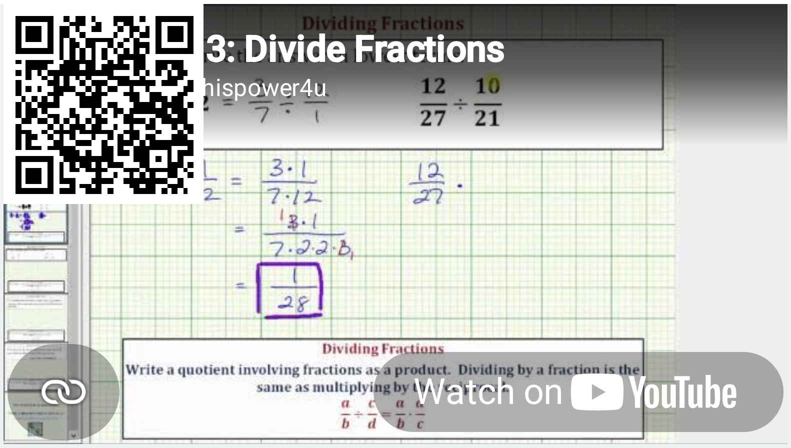
   Step 2: Multiply the first fraction by that reciprocal: $\frac{1}{8} \div \frac{5}{28} = \frac{1}{8} \times \frac{28}{5} = \frac{28}{40}$

   The answer, $\frac{28}{40}$, is not in lowest reduced form. The GCD of 28 and 40 is 4. Factoring and canceling gives $\frac{28}{40} = \frac{4 \times 7}{4 \times 10} = \frac{7}{10}$.

Your Turn 3.4.18

1. Calculate $\frac{46}{175} \div \frac{69}{285}$.
2. Calculate $\frac{3}{40} \div \frac{42}{55}$.

Video



Dividing Fractions

Applying the Order of Operations to Simplify Expressions

The order of operations for rational numbers is the same as for integers, as discussed in Order of Operations. The order of operations makes it easier for anyone to correctly calculate and represent. The order follows the well-known acronym PEMDAS:

P	Parentheses
E	Exponents
M/D	Multiplication and division
A/S	Addition and subtraction

The first step in calculating using the order of operations is to perform operations inside the parentheses. Moving down the list, next perform all exponent operations moving from left to right. Next (left to right once more), perform all multiplications and divisions. Finally, perform the additions and subtractions.

✓ Example $3.4.19$: Applying the Order of Operations with Rational Numbers

Correctly apply the rules for the order of operations to accurately compute $\left(\frac{5}{7}-\frac{2}{7}\right)\times 2^3$.

Answer

Step 1: To calculate this, perform all calculations within the parentheses before other operations.

$\left(\frac{5}{7}-\frac{2}{7}\right)\times 2^3=\left(\frac{3}{7}\right)\times 2^3$

Step 2: Since all parentheses have been cleared, we move left to right, and compute all the exponents next.

$\left(\frac{3}{7}\right)\times 2^3=\left(\frac{3}{7}\right)\times 8$

Step 3: Now, perform all multiplications and divisions, moving left to right.

$\left(\frac{3}{7}\right)\times 8=\frac{24}{7}$

Your Turn $3.4.19$

Correctly apply the rules for the order of operations to accurately compute $\left(\frac{3}{16}+\frac{7}{16}\right)^2+\frac{1}{5}\div\frac{3}{10}$

Example 3.4.20: Applying the Order of Operations with Rational Numbers

Correctly apply the rules for the order of operations to accurately compute $4+\frac{2}{3} \div\left(\left(\frac{5}{9}\right)^2-\left(\frac{2}{3}+5\right)\right)^2$.

Answer

To calculate this, perform all calculations within the parentheses before other operations. Evaluate the innermost parentheses first. We can work separate parentheses expressions at the same time.

Step 1: The innermost parentheses contain $\frac{2}{3}+5$. Calculate that first, dividing after finding the common denominator.

$$\begin{aligned}
&4+\frac{2}{3} \div\left(\left(\frac{5}{9}\right)^2-\left(\mathbf{\frac{2}{3}+5}\right)\right)^2 \\
&=4+\frac{2}{3} \div\left(\left(\frac{5}{9}\right)^2-\left(\mathbf{\frac{2}{3}+\frac{5}{1}}\right)\right)^2 \\
&=4+\frac{2}{3} \div\left(\left(\frac{5}{9}\right)^2-\left(\mathbf{\frac{2}{3}+\frac{15}{3}}\right)\right)^2 \\
&=4+\frac{2}{3} \div\left(\left(\frac{5}{9}\right)^2-\left(\mathbf{\frac{17}{3}}\right)\right)^2
\end{aligned}$$

Step 2: Calculate the exponent in the parentheses, $\left(\frac{5}{9}\right)^2$.

$$\begin{aligned}
&4+\frac{2}{3} \div\left(\left(\mathbf{\frac{5}{9}}\right)^{\mathbf{2}}-\left(\frac{17}{3}\right)\right)^2 \\
&=4+\frac{2}{3} \div\left(\left(\mathbf{\frac{25}{81}}\right)-\left(\frac{17}{3}\right)\right)^2
\end{aligned}$$

Step 3: Subtract inside the parentheses is done, using a common denominator.

$$\begin{aligned}
&4+\frac{2}{3} \div\left(\left(\mathbf{\frac{25}{81}}\right)\mathbf{-}\left(\mathbf{\frac{17}{3}}\right)\right)^2 \\
&4+\frac{2}{3} \div\left(\left(\mathbf{\frac{25}{81}}\right)\mathbf{-}\left(\mathbf{\frac{17 \times 27}{3 \times 27}}\right)\right)^2 \\
&4+\frac{2}{3} \div\left(\left(\mathbf{\frac{25}{81}}\right)\mathbf{-}\left(\mathbf{\frac{459}{81}}\right)\right)^2 \\
&4+\frac{2}{3} \div\left(\left(\mathbf{\frac{-434}{81}}\right)\right)^2
\end{aligned}$$

Step 4: At this point, evaluate the exponent and divide.

$$\begin{aligned}
&4+\frac{2}{3} \div\left(\left(\mathbf{\frac{-434}{81}}\right)\right)^2 \\
&4+\frac{2}{3} \div\left(\mathbf{\frac{188,356}{6,561}}\right) \\
&=4+\mathbf{\frac{2}{3}} \times\left(\mathbf{\frac{6,561}{188,356}}\right) \\
&=\mathbf{4}+\mathbf{\frac{2,187}{94,178}}
\end{aligned}$$

Step 5: Add.

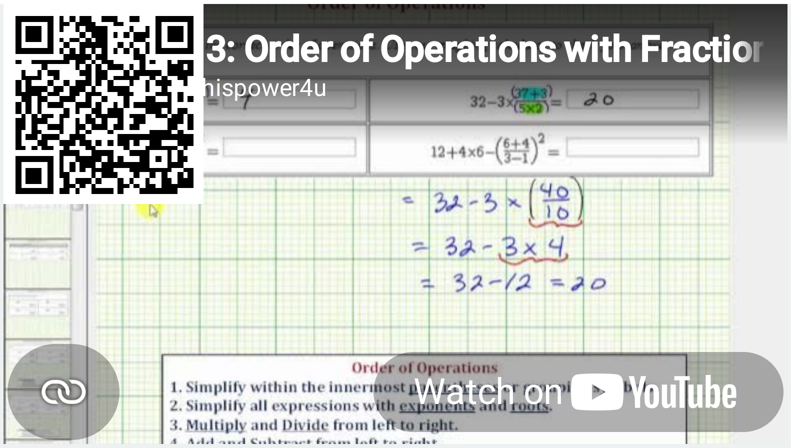
$$\begin{aligned}
&\mathbf{4}+\mathbf{\frac{2,187}{94,178}} \\
&=\frac{378,899}{94,178}
\end{aligned}$$

Had this been done on a calculator, the decimal form of the answer would be 4.0232 (rounded to four decimal places).

Your Turn 3.4.20

Correctly apply the rules for the order of operations to accurately compute $\left(\frac{3}{5}+2\right) \times\left(\frac{4}{5}-\frac{1}{2}\right)^2 \div \frac{11}{15}$.

Video



Order of Operations Using Fractions

Applying the Density Property of Rational Numbers

Between any two rational numbers, there is another rational number. This is called the **density property** of the rational numbers.

Finding a rational number between any two rational numbers is very straightforward.

Step 1: Add the two rational numbers.

Step 2: Divide that result by 2.

The result is always a rational number. This follows what we know about rational numbers. If two fractions are added, then the result is a fraction. Also, when a fraction is divided by a fraction (and 2 is a fraction), then we get another fraction. This two-step process will give a rational number, provided the first two numbers were rational.

✓ Example 3.4.21: Applying the Density Property of Rational Numbers

Demonstrate the density property of rational numbers by finding a rational number between $\frac{4}{11}$ and $\frac{7}{12}$.

Answer

To find a rational number between $\frac{4}{11}$ and $\frac{7}{12}$:

Step 1: Add the fractions.

$$\frac{4}{11}+\frac{7}{12}=\frac{4\times 12}{11\times 12}+\frac{7\times 11}{12\times 11}=\frac{48}{132}+\frac{77}{132}=\frac{125}{132}$$

Step 2: Divide the result by 2. Recall that to divide by 2, you multiply by the reciprocal of 2. The reciprocal of 2 is $\frac{1}{2}$, as seen below.

$$\frac{125}{132}\div 2=\frac{125}{132}\times\frac{1}{2}=\frac{125}{264}$$

So, one rational number between $\frac{4}{11}$ and $\frac{7}{12}$ is $\frac{125}{264}$.

We could check that the number we found is between the other two by finding the decimal representation of the numbers. Using a calculator, the decimal representations of the rational numbers are $0.363636\ldots$, $0.473484848\ldots$, and $0.5833333\ldots$. Here it is clear that $\frac{125}{264}$ is between $\frac{4}{11}$ and $\frac{7}{12}$.

Your Turn 3.4.21

Demonstrate the density property of rational numbers by finding a rational number between $\frac{27}{13}$ and $\frac{21}{10}$.

Solving Problems Involving Rational Numbers

Rational numbers are used in many situations, sometimes to express a portion of a whole, other times as an expression of a ratio between two quantities. For the sciences, converting between units is done using rational numbers, as when converting between gallons and cubic inches. In chemistry, mixing a solution with a given concentration of a chemical per unit volume can be solved with rational numbers. In demographics, rational numbers are used to describe the distribution of the population. In dietetics, rational numbers are used to express the appropriate amount of a given ingredient to include in a recipe. As discussed, the application of rational numbers crosses many disciplines.

Example $3.4.22$: Mixing Soil for Vegetables

James is mixing soil for a raised garden, in which he plans to grow a variety of vegetables. For the soil to be suitable, he determines that $\frac{2}{5}$ of the soil can be topsoil, but $\frac{2}{5}$ needs to be peat moss and $\frac{1}{5}$ has to be compost. To fill the raised garden bed with 60 cubic feet of soil, how much of each component does James need to use?

Answer

In this example, we know the proportion of each component to mix, and we know the total amount of the mix we need. In this kind of situation, we need to determine the appropriate amount of each component to include in the mixture. For each component of the mixture, multiply 60 cubic feet, which is the total volume of the mixture we want, by the fraction required of the component.

Step 1: The required fraction of topsoil is $\frac{2}{5}$, so James needs $60 \times \frac{2}{5}$ cubic feet of topsoil. Performing the multiplication, James needs $60 \times \frac{2}{5} = \frac{120}{5} = 24$ (found by treating the fraction as division, and 120 divided by 5 is 24) cubic feet of topsoil.

Step 2: The required fraction of peat moss is also $\frac{2}{5}$, so he also needs $60 \times \frac{2}{5}$ cubic feet, or $60 \times \frac{2}{5} = \frac{120}{5} = 24$ cubic feet of peat moss.

Step 3: The required fraction of compost is $\frac{1}{5}$. For the compost, he needs $60 \times \frac{1}{5} = \frac{60}{5} = 12$ cubic feet.

Your Turn $3.4.22$

Ashley wants to study for 10 hours over the weekend. She plans to spend half the time studying math, a quarter of the time studying history, an eighth of the time studying writing, and the remaining eighth of the time studying physics. How much time will Ashley spend on each of those subjects?

Example$3.4.23$: Determining the Number of Specialty Pizzas

At Bella's Pizza, one-third of the pizzas that are ordered are one of their specialty varieties. If there are 273 pizzas ordered, how many were specialty pizzas?

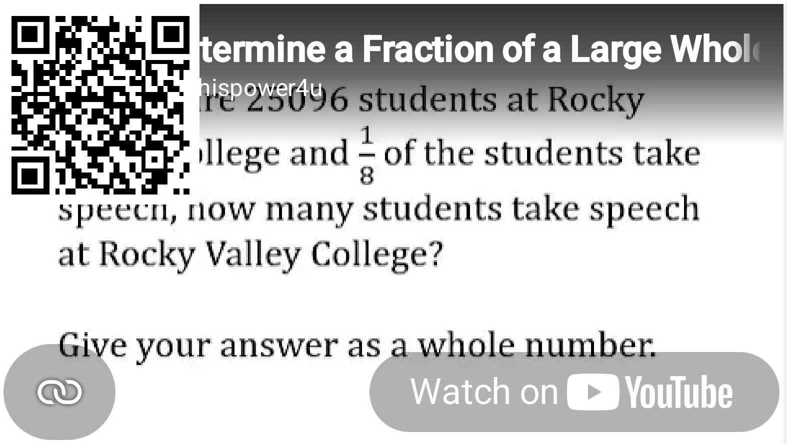
Answer

One-third of the whole are specialty pizzas, so we need one-third of 273, which gives $\frac{1}{3} \times 273 = \frac{273}{3} = 91$, found by dividing 273 by 3. So, 91 of the pizzas that were ordered were specialty pizzas.

Your Turn $3.4.23$

Danny, a nutritionist, is designing a diet for her client, Callum. Danny determines that Callum's diet should be 30% protein. If Callum consumes 2,400 calories per day, how many calories of protein should Danny tell Callum to consume?

Video



re 25096 students at Rocky llege and $\frac{1}{8}$ of the students take speech, how many students take speech at Rocky Valley College?

Give your answer as a whole number.

Finding a Fraction of a Total

Using Fractions to Convert Between Units

A common application of fractions is called **unit conversion**, or **converting units**, which is the process of changing from the units used in making a measurement to different units of measurement.

For instance, 1 inch is (approximately) equal to 2.54 cm. To convert between units, the two equivalent values are made into a fraction. To convert from the first type of unit to the second type, the fraction has the second unit as the numerator, and the first unit as the denominator.

From the inches and centimeters example, to change from inches to centimeters, we use the fraction $\frac{2.54\text{ cm}}{1\text{ in}}$. If, on the other hand, we wanted to convert from centimeters to inches, we'd use the fraction $\frac{1\text{ in}}{2.54\text{ cm}}$. This fraction is multiplied by the number of units of the type you are converting *from*, which means the units of the denominator are the same as the units being multiplied.

✓ Example 3.4.24: Converting Liters to Gallons

It is known that 1 liter (L) is 0.264172 gallons (gal). Use this to convert 14 liters into gallons.

Answer

We know that 1 liter = 0.264172 gal. Since we are converting from liters, when we create the fraction we use, make sure the liter part of the equivalence is in the denominator. So, to convert the 14 liters to gallons, we multiply 14 by $\frac{1\text{ gal}}{0.264172\text{ gal}/1\text{ liter}}$. Notice the gallon part is in the numerator since we're converting *to* gallons, and the liter part is in the denominator since we are converting *from* liters. Performing this and rounding to three decimal places, we find that 14 liters is $14\text{ liter}\times\frac{0.264172\text{ gal}}{1\text{ liter}}=3.69841\text{ gal}$.

Your Turn 3.4.24

One mile is equal to 1.60934 km. Convert 200 miles to kilometers. Round off the answer to three decimal places.

✓ Example 3.4.25: Converting Centimeters to Inches

It is known that 1 inch is 2.54 centimeters. Use this to convert 100 centimeters into inches.

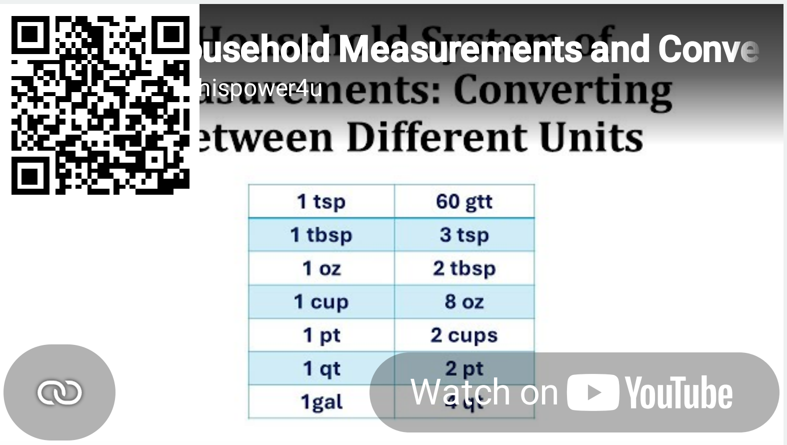
Answer

We know that 1 inch = 2.54 cm. Since we are converting from centimeters, when we create the fraction we use, make sure the centimeter part of the equivalence is in the denominator, $\frac{1\text{ in}}{2.54\text{ cm}}$. To convert the 100 cm to inches, multiply 100 by $\frac{1\text{ in}}{2.54\text{ cm}}$. Notice the inch part is in the numerator since we're converting *to* inches, and the centimeter part is in the denominator since we are converting *from* centimeters. Performing this and rounding to three decimal places, we obtain $100\text{ cm}\times\frac{1\text{ in}}{2.54\text{ cm}}=39.370\text{ in}$. This means 100 cm equals 39.370 in.

Your Turn 3.4.25

It is known that 4 quarts equals 3.785 liters. If you have 25 quarts, how many liters do you have? Round off to three decimal places.

Video


usehold Measurements and Conve
hispower4u
Household System of Measurements: Converting Between Different Units
Watch on YouTube

1 tsp	60 gtt
1 tbsp	3 tsp
1 oz	2 tbsp
1 cup	8 oz
1 pt	2 cups
1 qt	2 pt
1gal	4 qt

Converting Units

Defining and Applying Percent

A **percent** is a specific rational number and is literally per 100. n percent, denoted n%, is the fraction $\frac{n}{100}$.

Example 3.4.26: Rewriting a Percentage as a Fraction

Rewrite the following as fractions:

1. 31%
2. 93%

Answer

1. Using the definition and $n =$, 31% in fraction form is $\frac{31}{100}$.
2. Using the definition and $n =$, 93% in fraction form is $\frac{93}{100}$.

Your Turn 3.4.26

Rewrite the following as fractions:

1. 4%
2. 50%

Example 3.4.27: Rewriting a Percentage as a Decimal

Rewrite the following percentages in decimal form:

1. 54%
2. 83%

Answer

1. Using the definition and $n =$, 54% in fraction form is $\frac{54}{100}$. Dividing a number by 100 moves the decimal two places to the left; 54% in decimal form is then 0.54.
2. Using the definition and $n =$, 83% in fraction form is $\frac{83}{100}$. Dividing a number by 100 moves the decimal two places to the left; 83% in decimal form is then 0.83.

Your Turn 3.4.27

Rewrite the following percentages in decimal form:

1. 14%
2. 7%

You should notice that you can simply move the decimal two places to the left without using the fractional definition of percent.

Percent is used to indicate a fraction of a total. If we want to find 30% of 90, we would perform a multiplication, with 30% written in either decimal form or fractional form. The 90 is the **total**, 30 is the **percentage**, and 27 (which is 0.30×90) is the **percentage of the total**.

FORMULA

$n\%$ of x items is $\frac{n}{100} \times x$. The x is referred to as the **total**, the n is referred to as the **percent** or **percentage**, and the value obtained from $\frac{n}{100} \times x$ is the **part** of the total and is also referred to as the **percentage of the total**.

Example 3.4.28: Finding a Percentage of a Total

1. Determine 40% of 300.
2. Determine 64% of 190.

Answer

1. The total is 300, and the percentage is 40. Using the decimal form of 40% and multiplying we obtain $0.40 \times 300 = 120$.
2. The total is 190, and the percentage is 64. Using the decimal form of 64% and multiplying we obtain $0.64 \times 190 = 121.6$.

Your Turn 3.4.28

1. Determine 25% of 1,200.

2. Determine 53% of 1,588.

In the previous situation, we knew the total and we found the percentage of the total. It may be that we know the percentage of the total, and we know the percent, but we don't know the total. To find the total if we know the percentage the percentage of the total, use the following formula.

FORMULA

If we know that $n\%$ of the total is x, then the total is $\frac{100 \times x}{n}$.

Example 3.4.29: Finding the Total When the Percentage and Percentage of the Total Are Known

1. What is the total if 28% of the total is 140?
2. What is the total if 6% of the total is 91?

Answer

1. 28 is the percentage, so $n =$. 28% of the total is 140, so $x =$. Using those we find that the total was $\frac{100\times140}{28} = 500$.
2. 6 is the percentage, so $n = 6$. 6% of the total is 91, so $x =$. Using those we find that the total was $\frac{100\times91}{6} = 1{,}516.6$.

Your Turn 3.4.29

What is the total if 25% of the total is 30?

What is the total if 45% of the total is 360?

The percentage can be found if the total and the percentage of the total is known. If you know the total, and the percentage of the total, first divide the part by the total. Move the decimal two places to the right and append the symbol %. The percentage may be found using the following formula.

FORMULA

The percentage, n, of b that is a is $\frac{a}{b} \times 100\%$.

✓ Example $3.4.30$: Finding the Percentage When the Total and Percentage of the Total Are Known

Find the percentage in the following:

1. Total is 300, percentage of the total is 60.
2. Total is 440, percentage of the total is 176.

Answer

1. The total is 300; the percentage of the total is 60. Calculating yields 0.2. Moving the decimal two places to the right gives 20. Appending the percentage to this number results in 20%. So, 60 is 20% of 300.
2. The total is 440; the percentage of the total is 176. Calculating yields 0.4. Moving the decimal two places to the right gives 40. Appending the percentage to this number results in 40%. So, 176 is 40% of 440.

Your Turn $3.4.30$

Find the percentage in the following:

Total is 1,000, percentage of the total is 70.

Total is 500, percentage of the total is 425.

Solve Problems Using Percent

In the media, in research, and in casual conversation percentages are used frequently to express proportions. Understanding how to use percent is vital to consuming media and understanding numbers. Solving problems using percentages comes down to identifying which of the three components of a percentage you are given, the total, the percentage, or the percentage of the total. If you have two of those components, you can find the third using the methods outlined previously.

✓ Example $3.4.31$: Percentage of Students Who Are Sleep Deprived

A study revealed that 70% of students suffer from sleep deprivation, defined to be sleeping less than 8 hours per night. If the survey had 400 participants, how many of those participants had less than 8 hours of sleep per night?

Answer

The percentage of interest is 70%. The total number of students is 400. With that, we can find how many were in the percentage of the total, or, how many were sleep deprived. Applying the formula from above, the number who were sleep deprived was $0.70 \times 400 = 280$; 280 students on the study were sleep deprived.

Your Turn $3.4.31$

Riley has a daily calorie intake of 2,200 calories and wants to take in 20% of their calories as protein. How many calories of protein should be in their daily diet?

✓ Example $3.4.32$: Amazon Prime Subscribers

There are 126 million users who are U.S. Amazon Prime subscribers. If there are 328.2 million residents in the United States, what percentage of U.S. residents are Amazon Prime subscribers?

Answer

We are asked to find the percentage. To do so, we divide the percentage of the total, which is 126 million, by the total, which is 328.2 million. Performing this division and rounding to three decimal places yields $\frac{126}{328.2} = 0.384$. The decimal is moved to the right by two places, and a % sign is appended to the end. Doing this shows us that 38.4% of U.S. residents are Amazon Prime subscribers.

Your Turn $3.4.32$

A small town has 450 registered voters. In the primaries, 54 voted. What percentage of registered voters in that town voted in the primaries?

✓ Example $3.4.33$: Finding the Percentage When the Total and Percentage of the Total Are Known

Evander plays on the basketball team at their university and 73% of the athletes at their university receive some sort of scholarship for attending. If they know 219 of the student-athletes receive some sort of scholarship, how many student-athletes are at the university?

Answer

We need to find the total number of student-athletes at Evander’s university.

Step 1: Identify what we know. We know the percentage of students who receive some sort of scholarship, 73%. We also know the number of athletes that form the part of the whole, or 219 student-athletes.

Step 2: To find the total number of student-athletes, use $\frac{100 \times x}{n}$, with $x =$ and $n =$. Calculating with those values yields $\frac{100 \times 219}{73} = 300$.

So, there are 300 total student-athletes at Evander’s university

Your Turn $3.4.33$

A store declares a deep discount of 40% for an item, which they say will save $30. What was the original price of the item?

Check Your Understanding

Identify which of the following are rational numbers. $-41, \sqrt{13}, \frac{4}{3}, 2.75, 0.2\overline{13}$

1. Express $\frac{18}{30}$ in lowest terms.
2. Calculate $\frac{3}{8} + \frac{5}{12}$ and express in lowest terms.
3. Convert 0.34 into fraction form.
4. Convert $\frac{47}{12}$ into a mixed number.
5. Calculate $\frac{2}{9} \times \frac{21}{22}$ and express in lowest terms.
6. Calculate $\frac{2}{5} \div \frac{3}{10} + \frac{1}{6}$.
7. Identify a rational number between $\frac{7}{8}$ and $\frac{20}{21}$.
8. Rewrite $3\frac{2}{7}$ as an improper fraction.
9. Lina decides to save $\frac{1}{8}$ of her take-home pay every paycheck. Her most recent paycheck was for $\$882$. How much will she save from that paycheck?
10. Determine 38% of 600 .
11. A microchip factory has decided to increase its workforce by 10%. If it currently has 70 employees, how many new employees will the factory hire?

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3.4.0: Exercises

For the following exercises, identify which of the following are rational numbers.

? Exercise 3.4.0.1

4.598

? Exercise 3.4.0.2

$\sqrt{144}$

? Exercise 3.4.0.3

$\sqrt{131}$

For the following exercises, reduce the fraction to lowest terms

? Exercise 3.4.0.4

$\frac{8}{10}$

? Exercise 3.4.0.5

$\frac{30}{105}$

? Exercise 3.4.0.6

$\frac{36}{539}$

? Exercise 3.4.0.7

$\frac{231}{490}$

? Exercise 3.4.0.8

$\frac{750}{17,875}$

For the following exercises, do the indicated conversion. If it is a repeating decimal, use the correct notation.

? Exercise 3.4.0.9

Convert $\frac{25}{6}$ to a mixed number.

? Exercise 3.4.0.10

Convert $\frac{240}{53}$ to a mixed number.

? Exercise 3.4.0.11

Convert $2\frac{3}{8}$ to an improper fraction.

? Exercise 3.4.0.12

Convert $15\frac{7}{30}$ to an improper fraction.

? Exercise 3.4.0.13

Convert $\frac{4}{9}$ to decimal form.

? Exercise 3.4.0.14

Convert $\frac{13}{20}$ to decimal form.

? Exercise 3.4.0.15

Convert $\frac{27}{625}$ to decimal form.

? Exercise 3.4.0.16

Convert $\frac{11}{14}$ to decimal form.

? Exercise 3.4.0.17

Convert 0.23 to fraction form and reduce to lowest terms.

? Exercise 3.4.0.18

Convert 3.8874 to fraction form and reduce to lowest terms.

For the following exercises, perform the indicated operations. Reduce to lowest terms.

? Exercise 3.4.0.19

$\frac{3}{5}+\frac{3}{10}$

? Exercise 3.4.0.20

$\frac{3}{14}+\frac{8}{21}$

? Exercise 3.4.0.21

$\frac{13}{36}-\frac{14}{99}$

? Exercise 3.4.0.22

$\frac{13}{24}-\frac{4}{117}$

? Exercise 3.4.0.23

$\frac{3}{7}\times\frac{21}{48}$

Exercise 3.4.0.24

$\frac{48}{143} \times \frac{77}{120}$

Exercise 3.4.0.25

$\frac{14}{27} \div \frac{7}{12}$

Exercise 3.4.0.26

$\frac{44}{75} \div \frac{484}{285}$

Exercise 3.4.0.27

$\left(\frac{3}{5} + \frac{2}{7}\right) \times \frac{10}{21}$

Exercise 3.4.0.28

$\frac{3}{8} \times \left(\frac{13}{12} - \frac{35}{36}\right)$

Exercise 3.4.0.29

$\left(\frac{3}{7} + \frac{5}{16}\right)^2 - \frac{5}{12}$

Exercise 3.4.0.30

$\frac{3}{8} \times \left(\frac{4}{9} - \frac{1}{8}\right)^2$

Exercise 3.4.0.31

$\left(\frac{2}{5} \times \left(\frac{7}{8} - \frac{2}{3}\right)\right)^2 \div \left(\frac{4}{9} + \frac{5}{6}\right) + \frac{7}{12}$

Exercise 3.4.0.32

$\left(\frac{1}{5} \div \left(\frac{3}{10} + \frac{11}{15}\right)\right) \times \left(\frac{2}{21} + \frac{5}{9}\right) - \left(\frac{8}{15} \div \frac{4}{33}\right)^2$

Exercise 3.4.0.33

Find a rational number between $\frac{8}{17}$ and $\frac{15}{28}$

Exercise 3.4.0.34

Find a rational number between $\frac{3}{50}$ and $\frac{13}{98}$.

Exercise 3.4.0.35

Find two rational numbers between $\frac{3}{10}$ and $\frac{19}{45}$.

Exercise 3.4.0.36

Find three rational numbers between $\frac{5}{12}$ and $\frac{175}{308}$.

Exercise 3.4.0.37

Convert 24% to fraction form and reduce completely.

Exercise 3.4.0.38

Convert 95\% to fraction form and reduce completely.

Exercise 3.4.0.39

Convert 0.23 to a percentage.

Exercise 3.4.0.40

Convert 1.22 to a percentage.

Exercise 3.4.0.41

Determine 30\% of 250.

Exercise 3.4.0.42

Determine 75% of 600.

Exercise 3.4.0.43

If 25% of a group is 41 members, how many members total are in the group?

Exercise 3.4.0.44

If 80% of the total is 60 , how much is in the total?

Exercise 3.4.0.45

13 is what percent of 20 ?

Exercise 3.4.0.46

80 is what percent of 320 ?

Exercise 3.4.0.47

Professor Donalson's history of film class has 60 students. Of those students, $\frac{2}{5}$ say their favorite movie genre is comedy. How many of the students in Professor Donalson's class name comedy as their favorite movie genre?

Exercise 3.4.0.48

Naia's dormitory floor has 80 residents. Of those, $\frac{3}{8}$ play Fortnight for at least 15 hours per week. How many students on Naia's floor play Fortnight at least 15 hours per week?

Exercise 3.4.0.49

In Tara's town there are 24,000 people. Of those, $\frac{13}{100}$ are food insecure. How many people in Tara's town are food insecure?

Exercise 3.4.0.50

Roughly $\frac{4}{5}$ of air is nitrogen. If an enclosure holds 2,000 liters of air, how many liters of nitrogen should be expected in the enclosure?

Exercise 3.4.0.51

To make the dressing for coleslaw, Maddie needs to mix it with $\frac{3}{5}$ mayonnaise and $\frac{2}{5}$ apple cider vinegar. If Maddie wants to have 8 cups of dressing, how many cups of mayonnaise and how many cups of apple cider vinegar does Maddie need?

Exercise 3.4.0.52

Malika is figuring out their schedule. They wish to spend $\frac{4}{15}$ of their time sleeping, $\frac{1}{3}$ of their time studying and going to class, $\frac{1}{5}$ of their time at work, and $\frac{2}{15}$ of their time doing other activities, such as entertainment or exercising. There are 168 hours in a week. How many hours in a week will Malika spend:

a. Sleeping?
b. Studying and going to class?
c. Not sleeping?

Exercise 3.4.0.53

Roughly 20.9% of air is oxygen. How much oxygen is there in 200 liters of air?

Exercise 3.4.0.54

65% of college students graduate within 6 years of beginning college. A first-year cohort at a college contains 400 students. How many are expected to graduate within 6 years?

Exercise 3.4.0.55

A 20% discount is offered on a new laptop. How much is the discount if the new laptop originally cost $\$700$?

Exercise 3.4.0.56

Leya helped at a neighborhood sale and was paid 5% of the proceeds. If Leya is paid $\$171.25$, what were the total proceeds from the neighborhood sale?

Exercise 3.4.0.57

Unit Conversion. 1 kilogram (kg) is equal to 2.20462 pounds. Convert 13 kg to pounds. Round to three decimal places, if necessary.

Exercise 3.4.0.58

Unit Conversion. 1 kilogram (kg) is equal to 2.20462 pounds. Convert 200 pounds to kilograms. Round to three decimal places, if necessary.

? Exercise $3.4.0.59$

Unit Conversion. There are 12 inches in a foot, 3 feet in a yard, and 1,760 yards in a mile. Convert 10 miles to inches. To do so, first convert miles to yards. Next, convert the yards to feet. Last, convert the feet to inches.

? Exercise $3.4.0.60$

Unit Conversion. There are 1,000 meters (m) in a kilometer (km), and 100 centimeters (cm) in a meter. Convert 4 km to centimeters.

? Exercise $3.4.0.61$

Markup. In this exercise, we introduce the concept of markup. The markup on an item is the difference between how much a store sells an item for and how much the store paid for the item. Suppose Wegmans (a northeastern U.S. grocery chain) buys cereal at $\$1.50$ per box and sells the cereal for $\$2.29$.

a. Determine the markup in dollars.
b. The markup is what percent of the original cost? Round the percentage to one decimal place.

? Exercise $3.4.0.62$

In this exercise, we explore what happens when an item is marked up by a percentage, and then marked down using the same percentage.
Wegmans purchases an item for $\$5$ per unit. The markup on the item is 25%.

a. Calculate the markup on the item, in dollars.
b. What is the price for which Wegmans sells the item? This is the price Wegmans paid, plus the markup.
c. Suppose Wegmans then offers a 25\% discount on the sale price of the item (found in part b). In dollars, how much is the discount?
d. Determine the price of the item after the discount (this is the sales price of the item minus the discount). Round to two decimal places.
e. Is the new price after the markup and discount equal to the price Wegmans paid for the item? Explain.

? Exercise $3.4.0.63$

Repeated Discounts. In this exercise, we explore applying more than one discount to an item. Suppose a store cuts the price on an item by 50%, and then offers a coupon for 25% off any sale item. We will find the price of the item after applying the sale price and the coupon discount.

a. The original price was $\$150$. After the 50% discount, what is the price of the item?
b. The coupon is applied to the discount price. The coupon is for 25%. Find 25% of the sale price (found in part a).
c. Find the price after applying the coupon (this is the value from part a minus the value from part b).
d. The total amount saved on the item is the original price after all the discounts. Determine the total amount saved by subtracting the final price paid (part c) from the original price of the item.
e. Determine the effective discount percentage, which is the total amount saved divided by the original price of the item.
f. Was the effective discount percentage equal to 75%, which would be the 50% plus the 25% ? Explain.

Converting Repeating Decimals to a Fraction

It was mentioned in the section that repeating decimals are rational numbers. To convert a repeating decimal to a rational number, perform the following steps:

- Step 1: Label the original number S.
- Step 2: Count the number of digits, n, in the repeating part of the number.
- Step 3: Multiply S by 10^n, and label this as $10^n \times S$.
- Step 4: Determine $10^n - 1$.

- Step 5: Calculate $10^n \times S - S$. If done correctly, the repeating part of the number will cancel out.
- Step 6: If the result from Step 5 has decimal digits, count the number of decimal digits in the number from Step 5. Label this m.
- Step 7: Remove the decimal from the result of Step 5.
- Step 8: Add m zeros to the end of the number from Step 4.
- Step 9: Divide the result from Step 7 by the result from Step 8. This is the fraction form of the repeating decimal.

? Exercise $3.4.0.64$

Convert $0.\overline{7}$ to fraction form.

? Exercise $3.4.0.65$

Convert $0.\overline{45}$ to fraction form.

? Exercise $3.4.0.66$

Convert $3.1\overline{5}$ to fraction form.

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3.5: Irrational Numbers

Learning Objectives

1. Define and identify numbers that are irrational.
2. Simplify irrational numbers and express in lowest terms.
3. Add and subtract irrational numbers.
4. Multiply and divide irrational numbers.
5. Rationalize fractions with irrational denominators.

The Pythagoreans were a philosophical sect in ancient Greece. Their philosophy included reincarnation and purifying the mind through the study and contemplation of mathematics and science. One of their principles was the cosmos is ruled by order, specifically mathematics and music. They even held mystic beliefs about specific numbers and figures. For example, the number 1 was associated with the mind and essence. Four represented justice, as it is the first product of two even numbers. Most famously, though, is the association with the Pythagorean Theorem, which states that in a right triangle, the sum of the squares of the shorter sides of the triangle (the legs) equals the square of the longer side (the hypotenuse). Even the ancient Egyptians used this relationship, as triangles with side measures 3, 4, and 5 were often used in surveying following the flooding of the Nile.

Figure

There is a story of a Pythagorean, Hippasus, discovering that not all numbers could be expressed as fractions. In other words, not all numbers were rational numbers. The story ends with Hippasus, who shared this, or in some versions discovered it, put to death by drowning for sharing this fact, that not all quantities could be expressed as the ratio of two natural numbers.

As colorful as that story may be, it is likely false, as there are no contemporary sources to corroborate it. But it does seem to mark the discovery that not all quantities or measures were fractions of numbers. And so, irrational numbers were discovered.

Video

thagoras – the Mystic Philosopher
mon's Cave
PRESENTS:
PHYLOSOPHER AND MYSTIC
PYTHAGORAS
Watch on YouTube

The Philosophy of the Pythagoreans

Defining and Identifying Numbers That Are Irrational

We defined rational numbers in the last section as numbers that could be expressed as a fraction of two integers. **Irrational numbers** are numbers that cannot be expressed as a fraction of two integers. Recall that rational numbers could be identified as those whose decimal representations either terminated (ended) or had a repeating pattern at some point. So irrational numbers must be those whose decimal representations do not terminate or become a repeating pattern.

One collection of irrational numbers is **square roots** of numbers that aren't **perfect squares**. x is the square root of the number a, denoted $\sqrt{a}$, if x^2 =ax2=a. The number a is the perfect square of the integer n if a =n2. The rational number semanticsab is a perfect square if both a and b are perfect squares.

One method of determining if an integer is a perfect square is to examine its prime factorization. If, in that factorization, all the prime factors are raised to even powers, the integer is a perfect square. Another method is to attempt to factor the integer into an integer squared. It is possible that you recognize the number as a perfect square (such as 4 or 9). Or, if you have a calculator at hand, use the calculator to determine if the square root of the integer is an integer.

✓ Example $3.5.1$: Identifying Perfect Squares

Determine which of the following are perfect squares.

1. 45
2. 81
3. $\frac{9}{28}$
4. $\frac{144}{400}$

Answer

1. The prime factorization of 45 is $45 = 3^2 \times 5$. Since the 5 is not raised to an even power, 45 is not a perfect square.
2. The prime factorization of 81 is 3^4. All the prime factors are raised to even powers, so 81 is a perfect square.
3. We must determine if both the numerator and denominator of $\frac{9}{28}$ are perfect squares for the rational number to be a perfect square. The numerator is 9, and as mentioned above, 9 is a perfect square (it is 3 squared). Now we check the prime factorization of the denominator, 28, which is $28 = 2^2 \times 7$. Since 7 is not raised to an even power, 28 is not a perfect square. Since the denominator is not a perfect square, $\frac{9}{28}$ is not a perfect square.
4. We must determine if both the numerator and denominator of $\frac{144}{400}$ are perfect squares for the rational number to be a perfect square. The numerator is 144. The prime factorization of 144 is $144 = 2^4 \times 3^2$. Since all the prime factors of 144 are raised to even powers, 144 is a perfect square. Now we check the prime factorization of the denominator, 400, which is $400 = 2^4 \times 5^2$. Since all the prime factors of 400 are raised to even powers, 400 is a perfect square. Since the numerator and denominator of $\frac{144}{400}$ are perfect squares, $\frac{144}{400}$ is a perfect square.

Your Turn $3.5.1$

Determine which of the following are perfect squares.

1. 36
2. 27
3. $\frac{9}{49}$
4. $\frac{12}{221}$

Tech Check: Using Desmos to Determine if a Number Is a Perfect Square

Desmos may be used to determine if a number is a perfect square by using its square root function. When Desmos is opened, there is a tab in the lower left-hand corner of the Desmos screen. This tab opens the Desmos keypad, shown in Figure $3.5.2$.

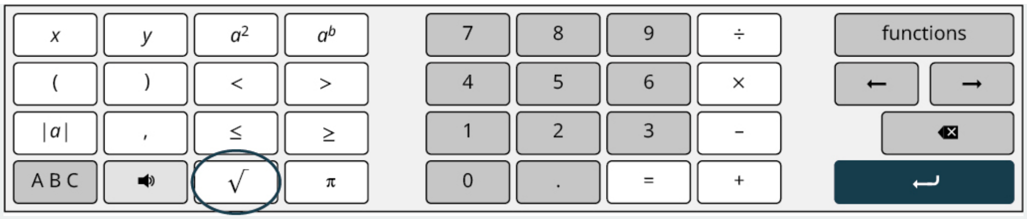
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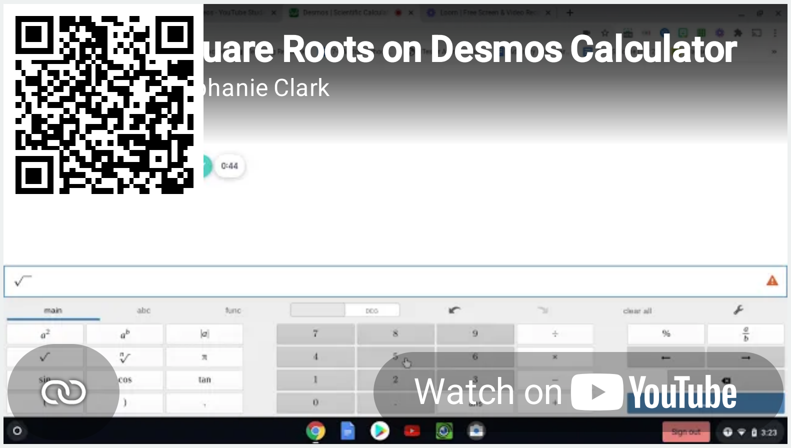
Figure $3.5.2$: Desmos keyboard with square root key circles

There you find the key for the square root, which is circled in Figure $3.5.2$. To find the square root of a number, click the square root key, which begins a calculation, and then enter the value for which you want a square root. If the result is an integer, then the number is a perfect square.

Video


uare Roots on Desmos Calculator
hanie Clark
Watch on YouTube

Using Desmos to Find the Square Root of a Number

Another collection of irrational numbers is based on the special number, **pi**, denoted by the Greek letter π

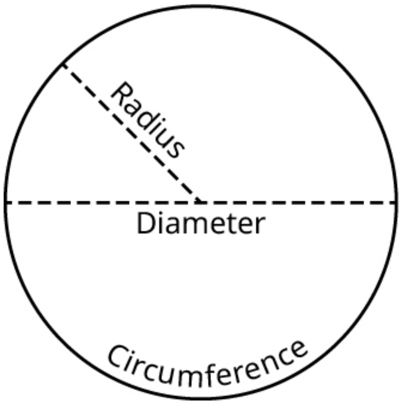

Radius
Diameter
Circumference

Figure $3.5.3$: Circle with radius, diameter, and circumference labeled

Any multiple or power of π is an irrational number.

Any number expressed as a rational number times an irrational number is an irrational number also. When an irrational number takes that form, we call the rational number the **rational part**, and the irrational number the **irrational part**. It should be noted that a rational number plus, minus, multiplied by, or divided by any irrational number is an irrational number.

Example 3.5.2: Identifying Irrational Numbers

Identify which of the following numbers are irrational.

1. $\sqrt{35}$
2. $0.\overline{15}$
3. $\sqrt{121}$
4. semantics4π4π

Answer

1. 35 can be factored as 5×7, showing that 35 is not the square of an integer or a rational number. This mean its square root is an irrational number.
2. Since $0.\overline{15}$ is a decimal with a repeating pattern, it is rational, so it is not an irrational number.
3. $121 = 11^2$. Since 121 is the square of an integer, its square root is a rational number.
4. Since semantics4π4π is a multiple of pi, it is irrational. In this case, the rational part of the number is 4, while the irrational part is semanticsππ.

Your Turn 3.5.2

Identify which of the following numbers are irrational.

1. $\sqrt{225}$
2. $3\sqrt{5}$
3. $\sqrt{80}$
4. $20 - 3\pi$

Who Knew?: Euler-Mascheroni Constant

Determining if a number is rational or irrational is not trivial. There are numbers that defied such classification for quite a long time. One such is the Euler-Mascheroni constant. The Euler-Mascheroni constant is used in mathematics, and is primarily associated with the natural logarithm, which is a mathematical function. The constant has been around since around 1790. However, it was unknown if this constant was rational or irrational until 2013, at which point it was proven to be irrational.

Simplifying Square Roots and Expressing Them in Lowest Terms

To **simplify a square root** means that we rewrite the square root as a rational number times the square root of a number that has no perfect square factors. The act of changing a square root into such a form is simplifying the square root.

The number inside the square root symbol is referred to as the **radicand**. So in the expression $\sqrt{a}$ the number a is referred to as the radicand.

Before discussing how to simplify a square root, we need to introduce a rule about square roots. The square root of a product of numbers equals the product of the square roots of those number. Written symbolically, $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$.

FORMULA

For any two numbers a and b, $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$.

Using this formula, we can factor an integer inside a square root into a perfect square times another integer. Then the square root can be applied to the perfect square, leaving an integer times the square root of another integer. If the number remaining under the square root has no perfect square factors, then we've simplified the irrational number into lowest terms. To simplify the irrational number into lowest terms when n is an integer:

Step 1: Determine the largest perfect square factor of n, which we denote semanticsa2.

Step 2: Factor n into semanticsa2×b.

Step 3: Apply √a2×b=a2×b.

Step 4: Write $\sqrt{n}$ in its simplified form, semanticsab.

When a square root has been simplified in this manner, a is referred to as the rational part of the number, and $\sqrt{b}$ is referred to as the irrational part.

✓ Example 3.5.3: Simplifying a Square Root

Simplify the irrational number $\sqrt{180}$ and express in lowest terms. Identify the rational and irrational parts.

Answer

Begin by finding the largest perfect square that is a factor of 180. We can do this by writing out the factor pairs of 180:

$$1 \times 180 \quad 2 \times 90 \quad 3 \times 60 \quad 4 \times 45 \quad 5 \times 36 \quad 6 \times 30 \quad 9 \times 20 \quad 10 \times 18 \quad 12 \times 15$$

Looking at the list of factors, the perfect squares are 4, 9, and 36. The largest is 36, so we factor the into $36 \times 5 = 6^2 \times 5$. In the formula, $a = 6$ and $b = 5$. Apply $\sqrt{}$a2×b=a2×ba2×b=a2×b.

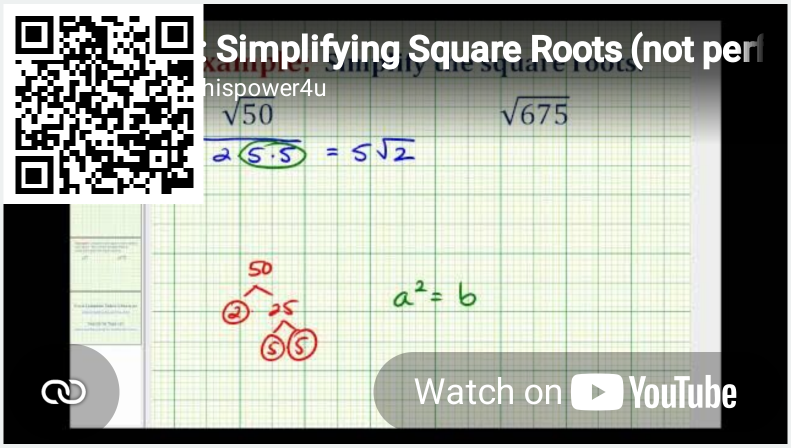
$\sqrt{6^2 \times 5} = \sqrt{6^2} \times \sqrt{5}$

The simplified form of $\sqrt{180}$ is $6\sqrt{5}$. In this example, the 6 is the rational part, and the $\sqrt{5}$ is the irrational part.

Your Turn 3.5.3

Simplify the irrational number $\sqrt{550}$ and express in lowest terms. Identify the rational and irrational parts.

Video


Simplifying Square Roots (not perf
hispower4u
xample: Simplify the square roots
√50
√675
2(5·5) = 5√2
50
2
25
5 5
a² = b
Watch on YouTube

Simplifying Square Roots

✓ Example 3.5.4: Simplifying a Square Root

Simplify the irrational number $\sqrt{330}$ and express in lowest terms. Identify the rational and irrational parts.

Answer

Begin by finding the largest perfect square that is a factor of 330. We can do this by writing out the factor pairs of 330:

$$1 \times 330 \quad 2 \times 165 \quad 3 \times 110 \quad 5 \times 66 \quad 6 \times 55 \quad 10 \times 33 \quad 11 \times 30 \quad 15 \times 22$$

Looking at the list of factors, there are no perfect squares other than 1, which means $\sqrt{330}$ is already expressed in lowest terms. In this case, 1 is the rational part, and $\sqrt{330}$ is the irrational part. **Though we could write this as $1\sqrt{330}$, but the product of 1 and any other number is just the number.**

Your Turn 3.5.4

Simplify the irrational number $\sqrt{733}$ and express in lowest terms. Identify the rational and irrational parts.

Example 3.5.51: Simplifying a Square Root

Simplify the irrational number $\sqrt{2{,}548}$ and express in lowest terms. Identify the rational and irrational parts.

Answer

Begin by finding the largest perfect square that is a factor of 2,548. We can do this by writing out the factor pairs of 2,548:

$$1 \times 2548 \quad 2 \times 1274 \quad 4 \times 637 \quad 7 \times 364 \quad 13 \times 196 \quad 14 \times 182 \quad 26 \times 98 \quad 28 \times 91 \quad 49 \times 52$$

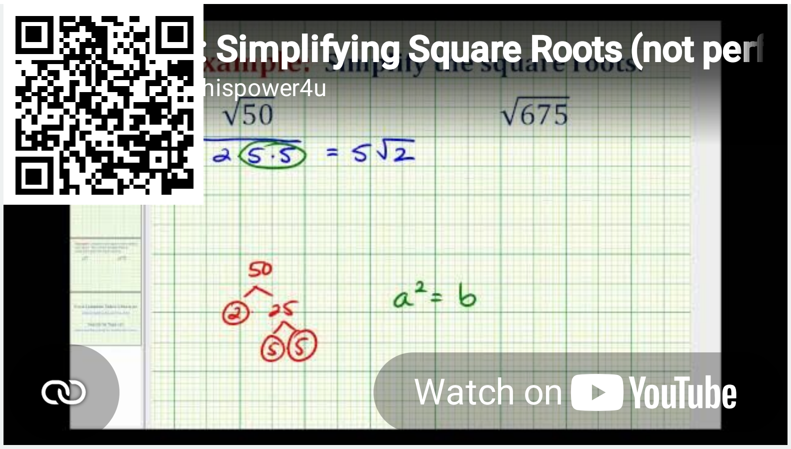
Looking at the list of factors, the perfect squares are 4, 49, and 196. The largest is 196, so we factor the 2,548 into $196 \times 13 = 14^2 \times 13$. In the formula, a =14a=14 and $b = 5$. Apply $\sqrt{}$a2×b=a2×ba2×b=a2×b.

$\sqrt{14^2 \times 13} = \sqrt{14^2} \times \sqrt{13}$

The simplified form of $\sqrt{2{,}548}$ is $14\sqrt{13}$. In this example, 14 is the rational part, and $\sqrt{13}$ is the irrational part.

Your Turn 3.5.5

Simplify the irrational number $\sqrt{1,815}$.



Simplifying Square Roots

Adding and Subtracting Irrational Numbers

Just like any other number we've worked with, irrational numbers can be added or subtracted. When working with a calculator, enter the operation and a decimal representation will be given. However, there are times when two irrational numbers may be added or subtracted without the calculator. This can happen only when the irrational parts of the irrational numbers are the same.

To add or subtract two irrational numbers that have the same irrational part, add or subtract the rational parts of the numbers, and then multiply that by the common irrational part.

FORMULA

Let our first irrational number be semantics\(a \times x\), where \(a\) is the rational and x the irrational parts.

Let the other irrational number be semantics\(b×x\), where b is the rational and x the irrational parts.

Then \a $\times x \pm$ \b $\times x = (a \pm b) \times x$.

✓ Example 3.5.6: Subtracting Irrational Numbers with Similar Irrational Parts

If possible, subtract the following irrational numbers without using a calculator. If this is not possible, state why.

$$3\sqrt{7}-8\sqrt{7}$$

Answer

Since these two irrational numbers have the same irrational part, $\sqrt{7}$, we can subtract without using a calculator. The rational part of the first number is 3. The rational part of the second number is 8. Using the formula yields $3\sqrt{7}-8\sqrt{7}=(3-8)\times\sqrt{7}=-5\sqrt{7}$.

Your Turn 3.5.6

If possible, subtract the following irrational numbers without using a calculator. If this is not possible, state why. $41\sqrt{15}-23\sqrt{15}$

✓ Example 3.5.1: Adding Irrational Numbers with Similar Irrational Parts

If possible, add the following irrational numbers without using a calculator. If this is not possible, state why.

\[35π+17π3 \nonumber\]

Answer

Since these two irrational numbers have the same irrational part, semanticsπ, the addition can be performed without using a calculator. The rational part of the first number is 35. The rational part of the second number is 17. Using the formula yields 35π+17π=(35+17)×π=52π.

Your Turn 3.5.7

If possible, add the following irrational numbers without using a calculator. If this is not possible, state why.$4.1\pi+3.2\pi$

✓ Example 3.5.8: Subtracting Irrational Numbers with Different Irrational Parts

If possible, subtract the following irrational numbers without using a calculator. If this is not possible, state why.

$$19\sqrt{3}-5.6\sqrt{7}$$

193−5.67

Answer

The two numbers being subtracted do not have the same irrational part, so the operation cannot be performed.

Your Turn 3.5.8

If possible, subtract the following irrational numbers without using a calculator. If this is not possible, state why. $2.1\sqrt{45}-3.7\sqrt{5}$

Multiplying and Dividing Irrational Numbers

Just like any other number that we've worked with, irrational numbers can be multiplied or divided. When working with a calculator, enter the operation and a decimal representation will be given. Sometimes, though, you may want to retain the form of the irrational number as a rational part times an irrational part. The process is similar to adding and subtracting irrational numbers when they are in this form. We do not need the irrational parts to match. Even though they need not match, they do need to be similar, such as both irrational parts are square roots, or both irrational parts are multiples of pi. Also, if the irrational parts are square roots, we may need to reduce the resulting square root to lowest terms.

When multiplying two square roots, use the following formula. It is the same formula presented during the discussion of simplifying square roots.

FORMULA

For any two positive numbers a and b, $\sqrt{a} \times \sqrt{b} = \sqrt{a \times b}$.

When dividing two square roots, use the following formula.

FORMULA

For any two positive numbers a and b, with b not equal to 0, $\sqrt{a} \div \sqrt{b} = \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$.

To multiply or divide irrational numbers with similar irrational parts, do the following:

Step 1: Multiply or divide the rational parts.

Step 2: If necessary, reduce the result of Step 1 to lowest terms. This becomes the rational part of the answer.

Step 3: Multiply or divide the irrational parts.

Step 4: If necessary, reduce the result from Step 3 to lowest terms. This becomes the irrational part of the answer.

Step 5: The result is the product of the rational and irrational parts.

✓ Example $3.5.9$: Dividing Irrational Numbers with Similar Irrational Parts

Perform the following operations without a calculator. Simplify if possible.

1. $3\sqrt{15} \div \left(8\sqrt{3}\right)$
2. $14.7\sqrt{135} \div \left(3\sqrt{5}\right)$.

Answer

1. In this division problem, $3\sqrt{15} \div \left(8\sqrt{3}\right)$, notice that the irrational parts of these numbers are similar. They are both square roots, so follow the steps given above.

   Step 1: Divide the rational parts. $3 \div 8 = \frac{3}{8}$

   Step 2: If necessary, reduce the result of Step 1 to lowest terms. The 3 and 8 have no common factors, so $\frac{3}{8}$ is already in lowest terms.

   Step 3: Divide the irrational parts. $\sqrt{15} \div \sqrt{3} = \frac{\sqrt{15}}{\sqrt{3}} = \sqrt{\frac{15}{3}}$

   Step 4: If necessary, reduce the result from Step 3 to lowest terms. The radicand can be reduced, which yields $\sqrt{5}$.

   Step 5: The result is the product of the rational and irrational parts, which is $\frac{3}{8}\sqrt{5}$.

2. In this division problem, $14.7\sqrt{135} \div \left(3\sqrt{5}\right)$, notice that the irrational parts of these numbers are similar. They are both square roots, so follow the steps given above.

   Step 1: Divide the rational parts. $14.7 \div 3 = 4.9$

   Step 2: If necessary, reduce the result of Step 1 to lowest terms. This rational number is expressed as a decimal so will not be reduced.

   Step 3: Divide the irrational parts. $\sqrt{135} \div \sqrt{5} = \frac{\sqrt{135}}{\sqrt{5}} = \sqrt{\frac{135}{5}}$

   Step 4: If necessary, reduce the result from Step 3 to lowest terms. The radicand can be reduced, which yields $\sqrt{\frac{135}{5}} = \sqrt{27} = \sqrt{9 \times 3} = 3\sqrt{3}$.

   Step 5: The result is the product of the rational and irrational parts, which is $4.9 \times 3\sqrt{3} = 14.7\sqrt{3}$.

Your Turn 3.5.9

Perform the following operations without a calculator. Simplify if possible.

1. $84\sqrt{132} \div (14\sqrt{11})$
2. $57\sqrt{792} \div (25\sqrt{2})$

Example 3.5.10: Multiplying Irrational Numbers with Similar Irrational Parts

Perform the following operations without a calculator. Simplify if possible.

1. $\left(19\sqrt{3}\right) \times \left(5.6\sqrt{12}\right)$
2. 13π×8π

Answer

1. In this multiplication problem, $\left(19\sqrt{3}\right) \times \left(5.6\sqrt{12}\right)$, notice that the irrational parts of these numbers are similar. They are both square roots. Follow the process above.

   Step 1: Multiply the rational parts. $19 \times 5.6 = 106.4$

   Step 2: If necessary, reduce the result of Step 1 to lowest terms. This rational number is expressed as a decimal and will not be reduced.

   Step 3: Multiply the irrational parts. $\sqrt{3} \times \sqrt{12} = \sqrt{3 \times 12} = \sqrt{36}$

   Step 4: If necessary, reduce the result from Step 3 to lowest terms. The radicand is 36, which is the square of 6. The irrational part reduces to $\sqrt{36} = 6$.

   Step 5: The result is the product of the rational and irrational parts, which is $106.4 \times 6 = 638.4$.

   Notice that sometimes multiplying or dividing irrational numbers can result in a rational number.

2. In this multiplication problem, semantics13π×8π13π×8π, notice that the irrational parts of these numbers are the same, semanticsππ. Follow the process above.

   Step 1: Multiply the rational parts. $13 \times 8 = 104$

   Step 2: If necessary, reduce the result of Step 1 to lowest terms. That result is an integer.

   Step 3: Multiply the irrational parts. π×π=π2π×π=π2

   Step 4: If necessary, reduce the result from Step 3 to lowest terms. This cannot be reduced.

   Step 5: The result is the product of the rational and irrational parts, which is semantics104π2104π2.

Your Turn 3.5.10

Perform the following operations without a calculator. Simplify if possible.

1. $(1.2\sqrt{21}) \times (4.5\sqrt{14})$
2. $38\pi \div (2\pi)$

Rationalizing Fractions with Irrational Denominators

Fractions often represent that some amount is being equally divided into some number of parts. But to conceptualize a fraction in that manner, the denominator needs to be an integer. An irrational number in the denominator interferes with that interpretation of a fraction. Fractions that have denominators that are just the square root of an integer can be altered into fractions with integer denominators using a process called **rationalizing the denominator**. The process relies on the following property of square roots: $\sqrt{a} \times \sqrt{a} = a$ and the following property of fractions: $\frac{a}{b}$ =acbcab=acbc for any non-zero number c.

Using these two properties, when a fraction has a square root in the denominator, we can eliminate that square root. Multiply the numerator and denominator by that square root from the denominator, $\frac{a}{\sqrt{b}}$ =abb×bab=abb×b. Then apply $\sqrt{a} \times \sqrt{a} = a$ to the denominator, yielding $\frac{a\sqrt{b}}{\sqrt{b}\times\sqrt{b}}$ =abbabb×b=abb. Notice that there is no longer a square root in the denominator, which allows for interpreting the fraction as dividing a whole into equal parts.

Example 3.5.11: Rationalizing the Denominator

Rationalize the denominator of the following:

1. $\frac{5}{\sqrt{7}}$
2. $\frac{3\sqrt{6}}{2\sqrt{10}}$

Answer

1. The square root in the denominator is $\sqrt{7}$. In order to rationalize the denominator of $\frac{5}{\sqrt{7}}$, we need to multiply the numerator and denominator by $\sqrt{7}$ and simplify.
   $\frac{5}{\sqrt{7}}=\frac{5\sqrt{7}}{\sqrt{7}\times\sqrt{7}}=\frac{5\sqrt{7}}{7}$
   The square root is in simplified form, so the final answer is $\frac{5\sqrt{7}}{7}$.
2. The square root in the denominator is $\sqrt{10}$.
   Step 1: In order to rationalize the denominator of $\frac{3\sqrt{6}}{2\sqrt{10}}$, we need to multiply the numerator and denominator by $\sqrt{10}$ and simplify.
   $\frac{3\sqrt{6}}{2\sqrt{10}}=\frac{3\sqrt{6}\times\sqrt{10}}{2\sqrt{10}\times\sqrt{10}}=\frac{3\sqrt{60}}{2\times10}=\frac{3\sqrt{60}}{20}$

   Step 2: The 60 under the square root can be factored into the following factor pairs:

   $1\times60\quad 2\times30\quad 3\times20\quad 4\times15\quad 5\times12\quad 6\times10$

   Step 3: The largest square factor of 60 is 4, so we simplify the $\sqrt{60}$ in the numerator into $2\sqrt{15}$. We also cancel any common factors.

   $\frac{3\sqrt{60}}{20}=\frac{3\times2\sqrt{15}}{20}=\frac{6\sqrt{15}}{20}=\frac{3\sqrt{15}}{10}$

   This is completely simplified.

Your Turn3.5.11

Rationalize the denominator of the following:

1. $\frac{24}{\sqrt{15}}$
2. $\frac{11\sqrt{14}}{6\sqrt{21}}$

There are occasions when the denominator is irrational but is the sum of two numbers where one or both involve square roots. For instance, $\frac{5}{4+\sqrt{3}}$. The process used earlier required that the denominator was the square root of a number and would not work here. However, this type of denominator can be rationalized. In order to rationalize such a denominator, we will multiply the numerator and denominator of the fraction by the **conjugate** of the denominator. The conjugate of $a+b$ is $a–b$. We say that $a+b$ and $a–b$ are **conjugate numbers**.

So, the conjugate of $-3+\sqrt{10}$ is just $-3-\sqrt{10}$. But why is this of interest? The reason is because it leads to the **difference of squares** formula, which is used to factor the difference of two squares. Or, for our purposes, in reverse it allows us to eliminate a square root.

FORMULA

For any two numbers, a and b, $a^2-b^2=(a-b)(a+b)$.

Looking at that formula, you should see that the two factors on the right-hand side of the equals sign are conjugates of one another. So, for our purposes, we're interested in $(a-b)(a+b)=a^2-b^2$. This tells us that when we multiply $a+b$ by its conjugate, we get a squared minus b squared, or a^2–b2a2−b2. But how is this useful? Let's return to the fraction above, $\frac{5}{4+\sqrt{3}}$. The denominator is $4+\sqrt{3}$. Its conjugate is $4-\sqrt{3}$. According to the formula, and letting a $=4$ and $b=\sqrt{3}$, we see that

$\left(4+\sqrt{3}\right)\left(4-\sqrt{3}\right)=4^2-\left(\sqrt{3}\right)^2$. But $\left(\sqrt{3}\right)^2$ is just 3. That means the product is $16-3$ or 13. This no longer has a square root. We use this to rationalize the denominator.

We will also need the **distributive property** of numbers.

FORMULA

For any three numbers a, b, and c, $a\times(b\pm c)=a\times b\pm a\times c$. This is called the distributive property.

Example 3.5.12: Rationalizing the Denominator Using Conjugates

Rationalize the denominator of $\frac{4}{6+\sqrt{10}}$.

Answer

Step 1: We recognize that the denominator is the sum of two numbers where one or both involve square roots. This means the conjugate can be used to remove the square root from the denominator.

Step 2: To do so, we multiply the numerator and the denominator each by the conjugate of the denominator. Since the denominator is $6+\sqrt{10}$, the conjugate we will use is $6-\sqrt{10}$.

Step 3: The conjugate is multiplied by the numerator and the denominator.

$$\frac{4}{6+\sqrt{10}}\times\frac{6-\sqrt{10}}{6-\sqrt{10}}$$

Step 4: Remembering how a number times its conjugate works, this becomes

$$\frac{4}{6+\sqrt{10}}\times\frac{6-\sqrt{10}}{6-\sqrt{10}}=\frac{4\times(6-\sqrt{10})}{6^2-\left(\sqrt{10}\right)^2}.$$

Step 5: In the numerator, we apply the distributive property. Using it yields

$$\frac{4\times\left(6-\sqrt{10}\right)}{6^2-\left(\sqrt{10}\right)^2}=\frac{24-4\sqrt{10}}{36-10}=\frac{24-4\sqrt{10}}{26}.$$

Step 6: Notice that the denominator no longer contains a square root. It has been rationalized. If desired, this can then be written as a rational number minus an irrational number, by recalling that $\frac{a-b}{c}=\frac{a}{c}$–bca−bc=ac−bc.

Applying that to the answer, we have $\frac{4}{6+\sqrt{10}}=\frac{24-4\sqrt{10}}{26}=\frac{24}{26}-\frac{4\sqrt{10}}{26}$.

Step 7: With a bit of cancellation, this reduces to $\frac{4}{6+\sqrt{10}}=\frac{24}{26}-\frac{4\sqrt{10}}{26}=\frac{12}{13}-\frac{2\sqrt{10}}{13}$.

Your Turn 3.5.12

Rationalize the denominator of $\frac{15}{5-\sqrt{13}}$.

Check Your Understanding

1. Simplify the following square root: $\sqrt{500}$.
2. Perform the following operation: $3\sqrt{7}-10\sqrt{7}$.
3. Perform the following operation: $8\sqrt{10}\times3\sqrt{2}$.
4. Rationalize the denominator of the following: $\frac{4}{\sqrt{7}}$.

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3.5.0: Exercises

? Exercise 3.5.0.1

Identify which of the following numbers are irrational: $\sqrt{441}, 4.33, \sqrt{70}, 5+9\pi$

? Exercise 3.5.0.2

Identify which of the following numbers are irrational: $\frac{13}{\sqrt{46}}, 4+13\pi, \sqrt{144}, \frac{5}{9}$

For the following exercises, simplify the square root by expressing it in lowest terms.

? Exercise 3.5.0.3

$\sqrt{12}$

? Exercise 3.5.0.4

$\sqrt{75}$

? Exercise 3.5.0.5

$\sqrt{605}$

? Exercise 3.5.0.6

$\sqrt{45}$

? Exercise 3.5.0.7

$\sqrt{112}$

? Exercise 3.5.0.8

$\sqrt{396}$

? Exercise 3.5.0.9

$\sqrt{2,940}$

? Exercise 3.5.0.10

$\sqrt{2,400}$

? Exercise 3.5.0.11

$\sqrt{3,240}$

? Exercise 3.5.0.12

$\sqrt{5,472}$

For the following exercises, perform the arithmetic operations without a calculator, if possible. If it is not possible, state why.

? Exercise 3.5.0.13

$4\sqrt{3}+2\sqrt{3}$

? Exercise 3.5.0.14

$8\sqrt{5}+3\sqrt{5}$

? Exercise 3.5.0.15

$9\sqrt{7}-15\sqrt{7}$

? Exercise 3.5.0.16

$\sqrt{13}-15\sqrt{13}$

? Exercise 3.5.0.17

$8\pi-13\sqrt{2}$

? Exercise 3.5.0.18

$7\sqrt{5}+6\sqrt{14}$

? Exercise 3.5.0.19

$7.2\pi+8.6\pi$

? Exercise 3.5.0.20

$14.5\pi-5.8\pi$

? Exercise 3.5.0.21

$19.8\sqrt{12}-6.1\sqrt{3}$

? Exercise 3.5.0.22

$7.3\sqrt{45}-6.8\sqrt{20}$

? Exercise 3.5.0.23

$(4\sqrt{15})\times(3\sqrt{10})$

? Exercise 3.5.0.24

$(7\sqrt{33})\times(8\sqrt{66})$

? Exercise 3.5.0.25

$(4.5\sqrt{154})\div(3\sqrt{77})$

? Exercise 3.5.0.26

$(70\sqrt{30}) \div (14\sqrt{6})$

For the following exercises, rationalize the denominators of the fractions, and then simplify.

? Exercise 3.5.0.27

$\frac{3}{\sqrt{2}}$

? Exercise 3.5.0.28

$\frac{8}{\sqrt{6}}$

? Exercise 3.5.0.29

$\frac{5}{\sqrt{13}}$

? Exercise 3.5.0.30

$\frac{9}{\sqrt{35}}$

? Exercise 3.5.0.31

$\frac{12}{\sqrt{40}}$

? Exercise 3.5.0.32

$\frac{20}{\sqrt{150}}$

? Exercise 3.5.0.33

Determine the conjugate of $5+\sqrt{6}$.

? Exercise 3.5.0.34

Determine the conjugate of $10-\sqrt{13}$.

? Exercise 3.5.0.35

Determine the conjugate of $4+3\sqrt{5}$.

? Exercise 3.5.0.36

Determine the conjugate of $5\sqrt{15}+8\sqrt{13}$.

? Exercise 3.5.0.37

Find the product of $3+2\sqrt{7}$ and its conjugate.

? Exercise $3.5.0.38$

Find the product of $2\sqrt{3}-5$ and its conjugate.

For the following exercises, rationalize the denominator of the fraction, and then simplify the fraction.

? Exercise $3.5.0.39$

$\frac{4}{1+\sqrt{3}}$

? Exercise $3.5.0.40$

$\frac{6}{5+\sqrt{7}}$

? Exercise $3.5.0.41$

$\frac{-4}{5-\sqrt{10}}$

? Exercise $3.5.0.42$

$\frac{10}{\sqrt{6}+7}$

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3.6: Real Numbers

Learning Objectives

After completing this section, you should be able to:

1. Define and identify numbers that are real numbers.
2. Identify subsets of the real numbers.
3. Recognize properties of real numbers.

Have you ever been impressed by the speed at which someone can do math in their head? Most of us at one time or another have witnessed a person speed through mental math, an impressive feat that often bests calculators. One such person is Neelkantha Bhanu Prakash. As of September 20, 2020, he is considered the world’s fastest human calculator. He currently holds four world records. How does someone do that, though? Have they memorized lots of arithmetic facts? Are they simply brilliant?

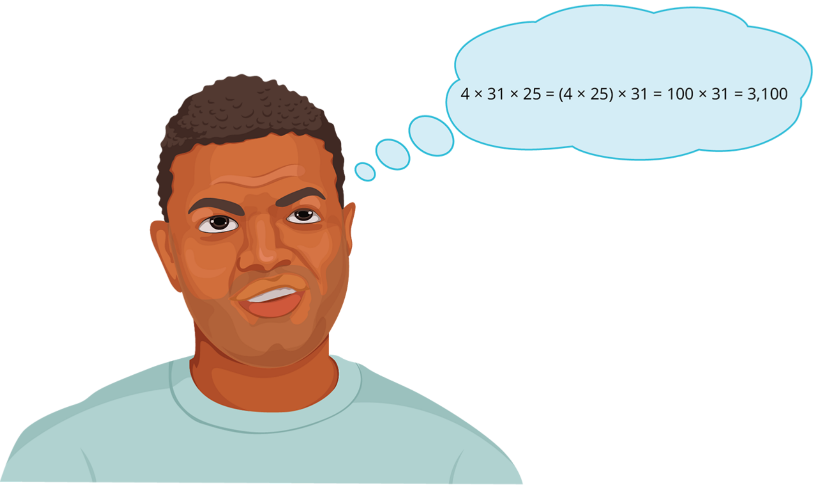
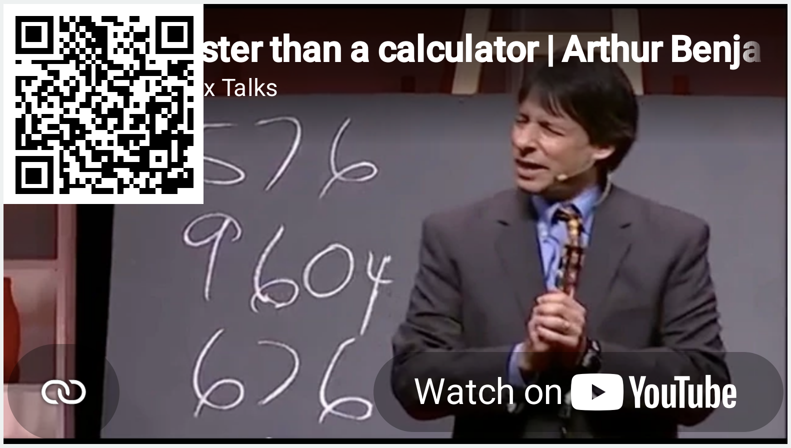

$4 \times 31 \times 25 = (4 \times 25) \times 31 = 100 \times 31 = 3{,}100$

Figure 3.6.1: Quick mental math involves using the known properties of real numbers.

The answer isn't simple so much as it is about knowledge. Real numbers behave in some very regular ways, following rules that can be learned. In this section, those rules are explored.

Watch the video of Arthur Benjamin’s TED Talk to learn about another mathematician with remarkable mental abilities.

Note



Arthur Benjamin TED Talk, Faster Than a Calculator

Defining and Identifying Real Numbers

Real numbers are the rational and irrational numbers combined. The real numbers represent the collection of all physical distances that exist, along with 0 and the negatives of those physical distances. For example, if you take a measure of three units, and divide that distance into eight (8) equal lengths, the distance you have formed is semantics38 (Figure $3.6.2$).

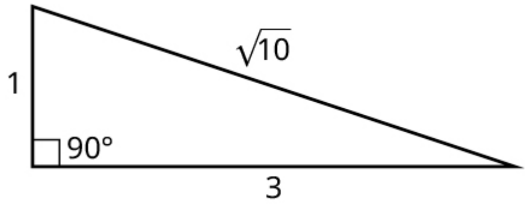


Figure $3.6.2$: Right triangle

Of course, if we name something the real numbers, there must be numbers that aren't real. Otherwise, they'd just be called the numbers. One such not real number, one that cannot be a length, is $\sqrt{-1}$. It is part of a collection of numbers called the complex numbers, it is denoted with the letter i. As an extension, the square root of any negative number is not a real number, but instead a complex number.

To determine if a number is real, check to see if there are any negatives under a square root or any semanticsi'si's. If there are any present, the number is not real.

✓ Example 3.96: Identifying Real Numbers

Determine if each of the following are real numbers:

1. $\frac{4\sqrt{3}}{7}$
2. 13.3381
3. $17\sqrt{-8}$

Answer

1. $\frac{4\sqrt{3}}{7}$ is a real number, as there are no negatives under the square roots, nor is there any factor of i.
2. 13.3381 is a rational number, and so it is a real number.
3. $17\sqrt{-8}$ is not a real number, as there is a negative number under the square root.

Your Turn 3.96

Determine if each of the following are real numbers: 1. $\frac{8}{15}$ 2. $4+17i$ 3. $-17\sqrt{46}$

Identifying Subsets of Real Numbers

The real numbers were built out of pieces, including integers, rational numbers, and irrational numbers. As such, the real numbers have named subsets, as shown in the table below.

Set Name	Set Symbol	Set Description
Natural Numbers	$\mathbb{N}$	The counting numbers
Whole Numbers		The counting numbers and 0
Integers	$\mathbb{Z}$	The natural numbers, their negatives, and 0
Rational Numbers	$\mathbb{Q}$	Fractions of integers
Irrational Numbers	$\mathbb{P}$	Numbers that cannot be written as a fraction of integers
Real Numbers	$\mathbb{R}$	The union of the rational and irrational numbers, all possible physical lengths, and their negatives

When we categorize numbers using these sets, we use the smallest set that they belong to. For instance, -7 is an integer, and a rational number, and a real number. The smallest set to which -7 belongs is integer, so we'd say it belongs to the integers.

We can also represent the relationships between the different sets of real numbers using set notation. All natural numbers are integers, but there are integers that are not natural numbers, so $\mathbb{N} \subset \mathbb{Z}$. Similarly, every integer is a rational number, but there are rational numbers that are not integers, so $\mathbb{Z} \subset \mathbb{Q}$. The same is true of the rational numbers and the real numbers, so $\mathbb{Q} \subset \mathbb{R}$.

There is no agreed-upon symbol for the irrational numbers. If we represent the irrationals as the set \$A\$, we should note that the following are true: $\mathbb{Q} \cup A = \mathbb{R}$ and $\mathbb{Q} \cap A = \emptyset$. Recall that this means the irrationals are the complement of the rational numbers in the universal set of real numbers.

✓ Example 3.97: Categorizing Numbers

Identify all subsets of the real numbers to which the following real numbers belong:

1. 14
2. -14.223
3. $\sqrt{17}$

Answer

1. 14 is a natural number, integer, and rational number.
2. -14.223 is a rational number.
3. $\sqrt{17}$ is an irrational number.

Your Turn 3.97

Identify all subsets of the real numbers to which the following real numbers belong:

1. $14\sqrt{3}$
2. -147 3. $\frac{37}{150}$

✓ Example 3.98: Categorizing Numbers within a Venn Diagram

Place the following numbers correctly in the Venn diagram (Figure $3.6.3$).

$$-4\sqrt{2} \quad -10 \quad \frac{37}{150} \quad 41 \quad \frac{1}{20} \quad 4\pi$$

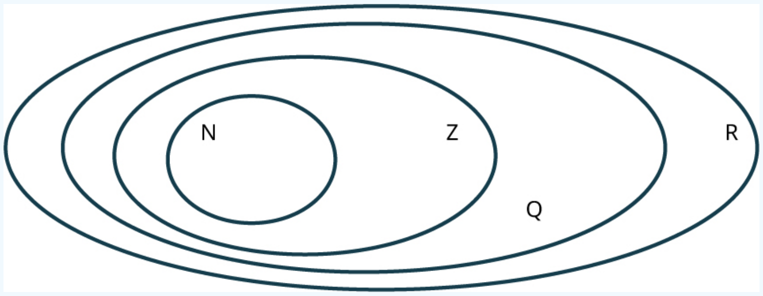


Figure $3.6.3$

Answer

Since $-4\sqrt{2}$ is irrational, it belongs in the real numbers, but outside the rational numbers (Figure $3.6.4$).

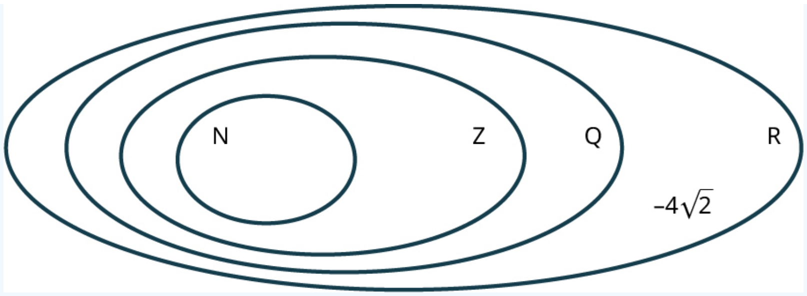

N
Z
Q
R
$-4\sqrt{2}$

Figure 3.6.4

Since −10 is an integer, it belongs in the integers but outside the natural numbers (Figure 3.6.5).

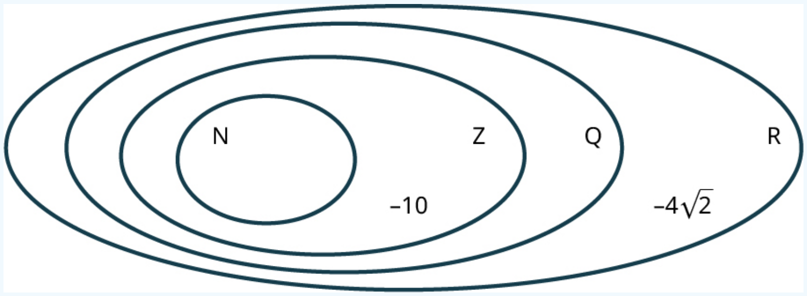

N
Z
Q
R
−10
$-4\sqrt{2}$

Figure 3.6.5

Since \(\frac{37}{150}\) is a rational number, it belongs in the rational numbers but not in the integers (Figure 3.6.6).

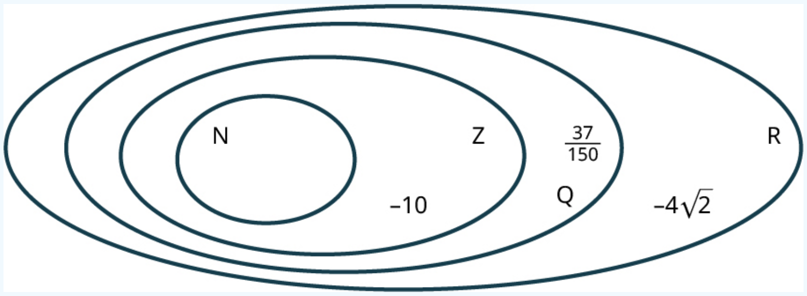

N
Z
$\frac{37}{150}$
R
−10
Q
$-4\sqrt{2}$

Figure 3.6.6

Since 41 is a natural number, it belongs in the natural numbers circle (Figure 3.6.7).

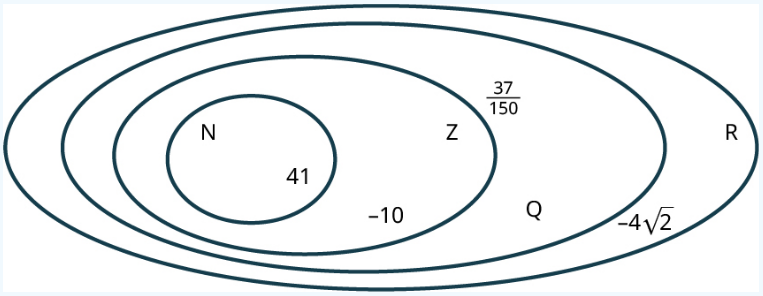

$\frac{37}{150}$
N
Z
R
41
−10
Q
$-4\sqrt{2}$

Figure 3.6.7

Since semantics\(\frac{1}{20}\) is a rational number, it belongs in the rational numbers but not in the integers (Figure 3.6.8).

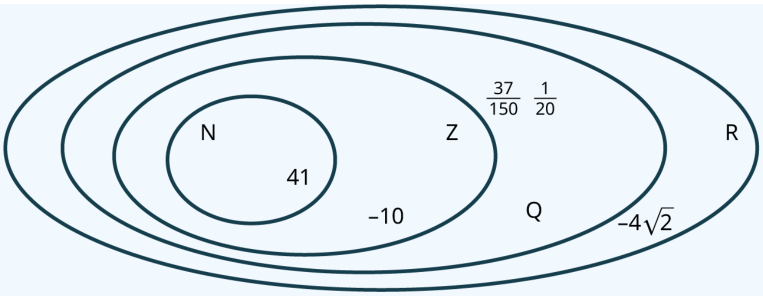

37/150
1/20
N
Z
R
41
−10
Q
−4√2

Figure 3.6.8

Since 4π is irrational, it belongs in the real numbers, but outside the rational numbers (Figure 3.6.9).

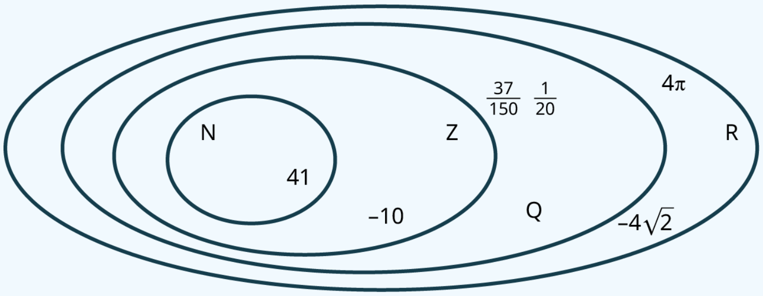

37/150
1/20
4π
N
Z
R
41
−10
Q
−4√2

Figure 3.6.9

Exercise 3.6.1

Place the following numbers correctly into the Venn diagram.

$$4 \quad 13.863 \quad 15 \quad 871 \quad 5\sqrt{7} \quad -3\pi$$

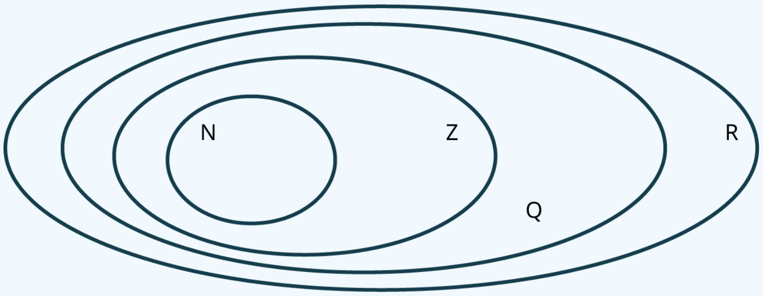
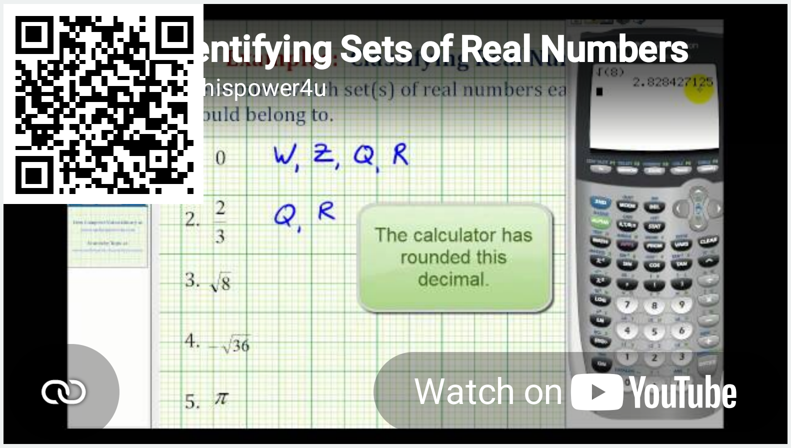


Figure 3.6.10

Note


ntifying Sets of Real Numbers
hispower4u
set(s) of real numbers ea
ould belong to.
0
W, Z, Q, R
2. 2/3
Q, R
The calculator has rounded this decimal.
3. √8
4. −√36
5. π
2.828427125
Watch on YouTube

Identifying Sets of Real Numbers

Recognizing Properties of Real Numbers

The real numbers behave in very regular ways. These behaviors are called the properties of the real numbers. Knowing these properties helps when evaluating formulas, working with equations, or performing algebra. Being familiar with these properties is helpful in all settings where numbers are used and manipulated. For example, when multiplying $4 \times 13 \times 25$, you could multiply the 4 and 25 first. If you know that product is 100, it makes the multiplication easier.

The table below is a partial list of properties of real numbers.

Property	Example	In Words
Distributive property $a \times (b+c) = a \times b + a \times c$	$5 \times (3+4) = 5 \times 3 + 5 \times 4$	Multiplication distributes across addition
Commutative property of addition $a + b = b + a$	$3 + 7 = 7 + 3$	Numbers can be added in any order
Commutative property of multiplication $a \times b = b \times a$	$10 \times 4 = 4 \times 10$	Numbers can be multiplied in any order
Associative property of addition $a + (b + c) = (a + b) + c$	$4 + (3 + 8) = (4 + 3) + 8$	Doesn't matter which pair of numbers is added first
Associative property of multiplication $a \times (b \times c) = (a \times b) \times c$	$2 \times (5 \times 7) = (2 \times 5) \times 7$	Doesn't matter which pair of numbers is multiplied first
Additive identity property $a + 0 = a$	$17 + 0 = 17$	Any number plus 0 is the number
Multiplicative identity property $a \times 1 = a$	$21 \times 1 = 21$	Any number times one is the number
Additive inverse property $a + (-a) = 0$	$14 + (-14) = 0$	Every number plus its negative is 0
Multiplicative inverse property $a \times \left(\frac{1}{a}\right)$ =1a×(1a)=1, provided $a \neq 0$	$3 \times \left(\frac{1}{3}\right) = 1$	Every non-zero number times its reciprocal is 1

The names of the properties are suggestive. The **commutative properties**, for example, suggest commuting, or moving. **Associative properties** suggest which items are associated with others, or if order matters in the computation. The **distributive property** addresses how a number is distributed across parentheses.

✓ Example 3.99: Identifying Properties of Real Numbers

In each of the following, identify which property of the real numbers is being applied.

1. $4 + (8 + 13) = (4 + 8) + 13$
2. $34 \times \left(\frac{1}{34}\right) = 1$
3. $14 + 27 = 27 + 14$

Answer

1. Here, the pair of numbers that is added first is switched. This is the associative property of addition.
2. Here, a number is multiplied by its reciprocal, resulting in 1. This is the multiplicative inverse property.
3. Here, the order in which numbers are added is switched. This is the commutative property of addition.

Your Turn 3.99

In each of the following, identify which property of the real numbers is being applied. 1. $5 \times (6+19) = 5 \times 6 + 5 \times 19$ 2. $41.7 + (-41.7) = 0$

Using these properties to perform arithmetic quickly relies on spotting easy numbers to work with. Look for numbers that add to a multiple of 10, or multiply to a multiple of 10 or 100.

✓ Example 3.100: Using Properties of Real Numbers in Calculations

Use properties of the real numbers and mental math to calculate the following:

1. $2 \times 13 \times 50$
2. $13 + 84 + 27$
3. $9 \times 16 \times 11$

Answer

1. Notice that $2 \times 50 = 100$, so that becomes the multiplication to do first. Use the commutative property of multiplication to change the order of the numbers being multiplied.

   $$2 \times 13 \times 50 = 2 \times 50 \times 13 = 100 \times 13 = 1,300$$

2. Notice that $13 + 27 = 40$, so that becomes the addition to do first. Use the commutative property of addition to change the order in which the numbers are added.

   $$13 + 84 + 27 = 13 + 27 + 84 = 40 + 84 = 124$$

3. Notice that $9 \times 11 = 99$. Using that, the problem can be changed to 99×16. That, however, doesn't look easy at all. But 99 $= (100 - 1)$. Using the distributive property, we rewrite and expand this as

   $$99 \times 16 = (100 - 1) \times 16 = 100 \times 16 - 1 \times 16 = 1,600 - 16.$$

   The last step is subtraction, so the final answer is 1,584 . So, multiplying by 99 is the same as multiplying by 100, and then subtracting the other number once.

Your Turn 3.100

Use properties of real numbers and mental math to calculate the following:

Check Your Understanding

Which of the following are real numbers: $4 + i, \sqrt{77}, -19, 38.902$?

Indicate which of the sets are subsets of the others: $\mathbb{Q}, \mathbb{R}, \mathbb{N}, \mathbb{Z}$.

Which property is demonstrated here: $3 \times 6 + 3 \times 15 = 3 \times (6 + 15)$?

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3.6.0: Exercises

For the following exercises, identify each number as a natural number, an integer, a rational number, or a real number.

? Exercise $3.6.0.1$

$\frac{1}{3}$

? Exercise $3.6.0.2$

16

? Exercise $3.6.0.3$

$\sqrt{19}$

? Exercise $3.6.0.4$

$-\frac{3}{7}$

? Exercise $3.6.0.5$

π

? Exercise $3.6.0.6$

-47

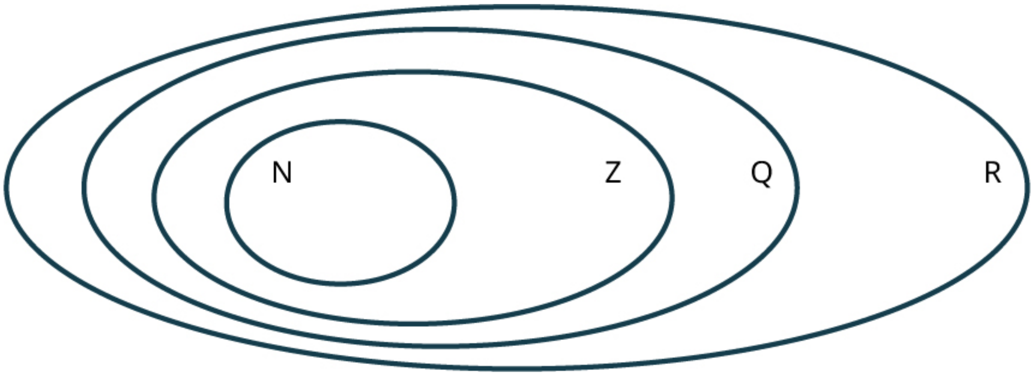
? Exercise $3.6.0.7$

13

? Exercise $3.6.0.8$

$\frac{31}{7}$

For the following exercises, correctly place the numbers in the Venn diagram.



For the following exercises, identify the property of real numbers that is being illustrated.

Exercise 3.6.0.11

$14+38.9=38.9+14$

Exercise 3.6.0.12

$37.12+98=98+37.12$

Exercise 3.6.0.13

$38\times 16=16\times 38$

Exercise 3.6.0.14

$6.3\times\sqrt{14}=\sqrt{14}\times 6.3$

Exercise 3.6.0.15

$4+(8+\sqrt{7})=(4+8)+\sqrt{7}$

Exercise 3.6.0.16

$13\times\left(\frac{6}{11}\times 14\right)=\left(13\times\frac{6}{11}\right)\times 14$

Exercise 3.6.0.17

$(5.6\times 8.7)\times 6=5.6\times(8.7\times 6)$

Exercise 3.6.0.18

$(3.8+\sqrt{5})+13=3.8+(\sqrt{5}+13)$

Exercise 3.6.0.19

$4\times(5+\sqrt{3})=4\times 5+4\times\sqrt{3}$

Exercise 3.6.0.20

$\sqrt{74}+(-\sqrt{74})=0$

Exercise 3.6.0.22

$17\times\left(\frac{1}{17}\right)=1$

Exercise 3.6.0.23

$4\sqrt{35}+0=4\sqrt{35}$

? Exercise 3.6.0.24

$\sqrt{17} \times (5 + 3.6) = \sqrt{17} \times 5 + \sqrt{17} \times 3.6$

? Exercise 3.6.0.26

$\frac{3}{\sqrt{7}} \times 1 = \frac{3}{\sqrt{7}}$

? Exercise 3.6.0.27

$3\pi + 7\sqrt{21} = 7\sqrt{21} + 3\pi$

? Exercise 3.6.0.28

$14 \times (3\sqrt{2} + 11.6) = 14 \times 3\sqrt{2} + 14 \times 11.6$

For the following exercises, use properties of real numbers and mental math to calculate the expression.

? Exercise 3.6.0.29

$43 + 62 + 17$

? Exercise 3.6.0.30

$106 + 75 + 94$

? Exercise 3.6.0.31

$5 \times 13 \times 4$

? Exercise 3.6.0.32

$4 \times 72 \times 5$

? Exercise 3.6.0.33

$46 + 77 + 23 + 24 + 103$

? Exercise 3.6.0.34

$23 + 98 + 75 + 12 + 77$

? Exercise 3.6.0.35

$4 \times 13 \times 25$

? Exercise 3.6.0.36

$50 \times 23 \times 2$

? Exercise $3.6.0.37$

13×99

? Exercise $3.6.0.38$

15×39

? Exercise $3.6.0.39$

43×101

? Exercise $3.6.0.40$

16×999

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3.7: Clock Arithmetic

Learning Objectives

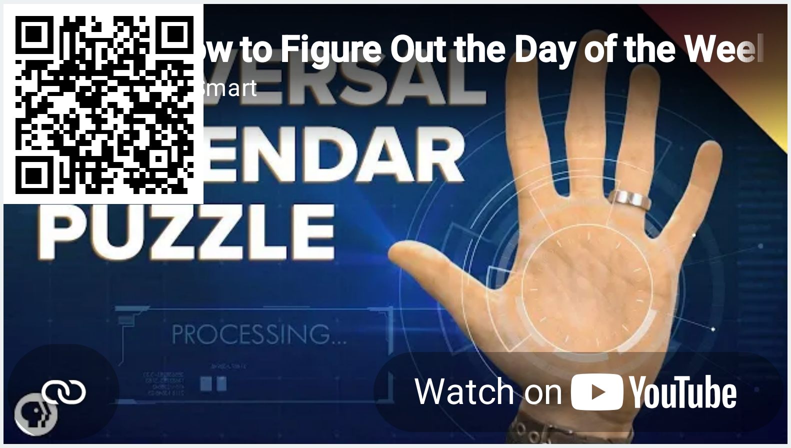
After completing this section, you should be able to:

- Add, subtract, and multiply using clock arithmetic.
- Apply clock arithmetic to calculate real-world applications.

Online shopping requires you to enter your credit card number, which is then sent electronically to the vendor. Using an ATM involves sliding your bank card into a reader, which then reads, sends, and verifies your card. Swiping or tapping for a purchase in a brick–and-mortar store is how your card sends its information to the machine, which is then communicated to the store’s computer and your credit card company. This information is read, recorded, and transferred many times. Each instance provides one more opportunity for error to creep into the process, a misrecorded digit, transposed digits, or missing digits. Fortunately, these card numbers have a built-in error checking system that relies on modular arithmetic, which is often referred to as clock arithmetic. In this section, we explore clock, or modular, arithmetic.

Figure $3.7.1$: If a credit card number is entered incorrectly, error checking algorithms will often catch the mistake. (credit: modification of work “Senior couple at home checking finance on credit card from above” by Nenad Stojkovic/Flickr, CC BY 2.0)

Video



Determining the Day of the Week for Any Date in History

Adding, Subtracting, and Multiplying Using Clock Arithmetic

When we do arithmetic, numbers can become larger and larger. But when we work with time, specifically with clocks, the numbers cycle back on themselves. It will never be 49 o’clock. Once 12 o’clock is reached, we go back to 1 and repeat the numbers. If it's

11 AM and someone says, “See you in four hours,” you know that 11 AM plus 4 hours is 3 PM, not 15 AM (ignoring military time for now). Math worked on the clock, where numbers restart after passing 12, is called clock arithmetic.

Clock arithmetic hinges on the number 12. Each cycle of 12 hours returns to the original time (Figure 3.42). Imagine going around the clock one full time. Twelve hours pass, but the time is the same. So, if it is 3:00, 14 hours later and two hours later both read the same on the clock, 5:00. Adding 14 hours and adding 2 hours are identical. As is adding 26 hours. And adding 38 hours.

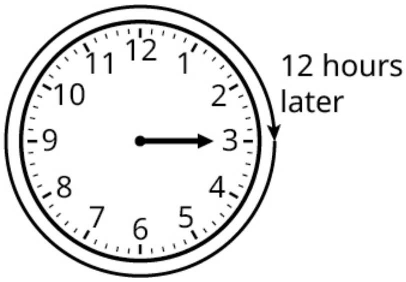


Figure $3.7.2$: Clock showing 3:00 with arrow going around the clock one full time, or 12 hours

What do 2, 14, 26, and 38 have in common in relation to 12? When they are divided by 12, they each have a remainder of 2. That's the key. When you add a number of hours to a specific time on the clock, first divide the number of hours being added by 12 and determine the remainder. Add that remainder to the time on the clock to know what time it will be.

A good visualization is to wrap a number line around the clock, with the 0 at the starting time. Then each time 12 on the number line passes, the number line passes the starting spot on the clock. This is referred to as modulo 12 arithmetic. Even though the process says to divide the number being added by 12, first perform the addition; the result will be the same if you add the numbers first, and then divide by 12 and determine the remainder.

In general terms, let n be a positive integer. Then n modulo 12, written (n mod 12), is the remainder when n is divided by 12. If that remainder is x, we would write $n = x$ (mod 12).

Checkpoint

Caution: 12 mod 12 is 0. So, if a mod 12 problem ends at 0, that would be 12 on the clock.

✓ Example ✓ 3.7.1: Determining the Value of a Number modulo 12

Find the value of the following numbers modulo 12:

1. 34
2. 539
3. 156

Answer

To determine the value of a number modulo 12, divide the number by 12 and record the remainder.

1. To find the value 34 modulo 12:

   Step 1: Determine the remainder when 34 is divided by 12 using long division. The largest multiple of 12 that is less than or equal to 34 is 24, which is the product of 12 and 2.

$$\begin{array}{r} 2 \\ 12\overline{)34} \\ \underline{24} \end{array}$$

Step 2: Performing the subtraction yields 10.

$$\begin{array}{r} 2 \\ 12\overline{)34} \\ \underline{24} \\ 10 \end{array}$$

Since that subtraction resulted in a number less than 12, that is the remainder, 10. The value of 34 modulo 12 is 10, or 34 = 10 (mod 12).

1. To find the value 539 modulo 12:

   Step 1: Determine the remainder when 539 is divided by 12 using long division. We first look to the first two digits of 539, 53. The largest multiple of 12 that is less than or equal to 53 is 48, which is the product of 12 and 4.

$$\begin{array}{r} 4 \\ 12\overline{)539} \\ \underline{48} \end{array}$$

Step 2: Performing the subtraction results in 5.

$$\begin{array}{r} 4 \\ 12\overline{)539} \\ \underline{48} \\ 5 \end{array}$$

Step 3: Now, the 9 is brought down.

$$\begin{array}{r} 4 \\ 12\overline{)539} \\ \underline{48} \\ 59 \end{array}$$

Step 4: The largest multiple of 12 that is less than or equal to 59 is once more 48 itself, which is 12×4.

$$\begin{array}{r} 44 \\ 12\overline{)539} \\ \underline{48} \\ 59 \\ \underline{48} \end{array}$$

Step 5: Finishing the process, the 48 is subtracted from the 59, yielding 11.

$$\begin{array}{r} 44 \\ 12\overline{)539} \\ \underline{48} \\ 59 \\ \underline{48} \\ 11 \end{array}$$

We've used all the digits of 539, and the last subtraction resulted in a number less than 12, so that number, 11, is the remainder. The value of 539 modulo 12 is 11, or, 539 = 11 (mod 12).

1. To find the value 156 modulo 12:

   Step 1: Determine the remainder when 156 is divided by 12 using long division. We first look to the first two digits of 156, 15. The largest multiple of 12 that is less than or equal to 15 is 12 itself, which is the product of 12 and 1.

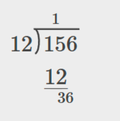
$$\begin{array}{r} 1 \\ 12\overline{)156} \\ \underline{12} \end{array}$$

Step 2: Performing the subtraction results in 3.

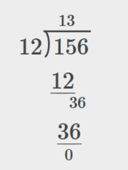
$$\begin{array}{r} 1 \\ 12\overline{)156} \\ \underline{12} \\ 3 \end{array}$$

Step 3: Now, the 6 is brought down.

Step 4: The largest multiple of 12 that is less than or equal to 36 is 36 itself, which is 12×3.


1
12)156
12
36

Step 5: Finishing the process, the 36 is subtracted from the 36, yielding 0.


13
12)156
12
36
36
0

We've used all the digits of 156, and the last subtraction resulted in a number less than 12, so that number, 0, is the remainder. The value of 156 modulo 12 is 0, or, 156 = 0 (mod 12).

We should note here that, had we been speaking of time, the 0 would be interpreted as 12:00.

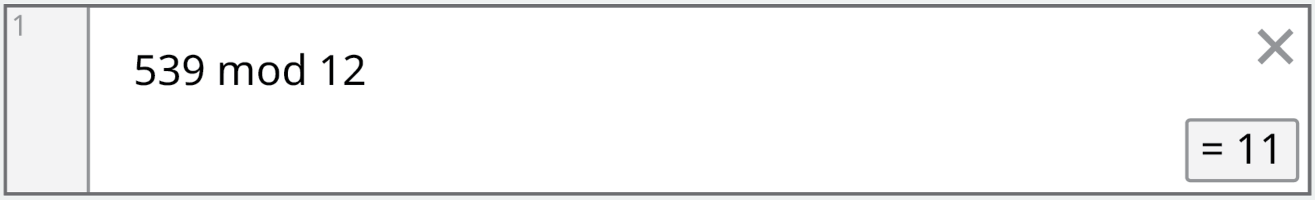
Your Turn 3.7.1

Find the value of the following numbers modulo 12.

1. 93
2. 387

Tech Check: Using Desmos to Determine the Value of a Number module 12

Desmos may be used to determine the value of a number modulo 12. It is flexible enough to find the value of a number modulo of any other integer you want. To determine the value of semanticsnFigure 3.43.



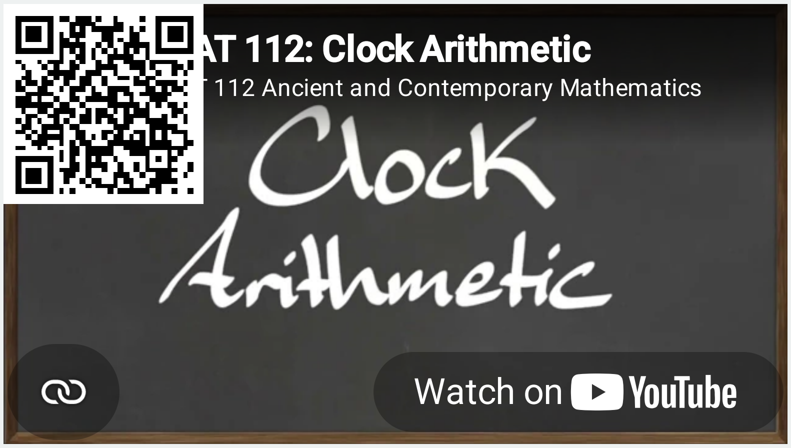
539 mod 12

= 11

Figure 3.43: Display of 539 modulo 12

Clock arithmetic is modulo 12 arithmetic but applied to time. As time is divided into 12 hours that repeat a cycle, we use modulo 12 for clock arithmetic.

Video


AT 112: Clock Arithmetic
112 Ancient and Contemporary Mathematics
Clock Arithmetic
Watch on YouTube

Clock Arithmetic

✓ Example ✓ 3.7.2: Adding with Clock Arithmetic

If it's 3:00, what time will it be in 89 hours?

Answer

To find that future time, we may determine the value of 89 (mod 12), either by long division or by using a calculator, such as Desmos. Then add the result to 3:00. Entering **mod(89,12)** in Desmos results in 5. Adding 5 hours, which was 89 (mod12), to 3:00 results in 8:00.

Your Turn 3.7.2

If it is 9:00 now, what time will it be in 43 hours?

Subtracting time on the clock works in much the same way as addition. Find the value of the number of hours being subtracted modulo 12, then subtract that from the original time.

✓ Example ✓ 3.7.3: Subtracting with Clock Arithmetic

If it is 4:00 now, what time was it 67 hours ago?

Answer

To find that past time, we may determine the value of 67 (mod 12), either by long division or by using a calculator, such as Desmos. Then subtract the result to 4:00. Entering **mod(67,12)** in Desmos results in 7. Subtracting 7 hours from 4:00 results in −3:00. We know, though, that time is not represented with negative times. This value, −3:00, indicates three hours before 12:00, which is 9:00. So, 67 hours before 4:00 was 9:00. We see this in the Figure 3.44.

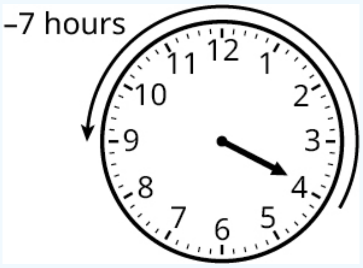


Figure 3.44: Clock showing 7 hours subtracted from 4:00

Your Turn 3.7.3

If it is 7:00 now, what time was it 34 hours ago?

Recall that clock arithmetic was referred to as modulo 12 arithmetic. Multiplying in modulo 12 also relies on the remainder when dividing by 12. To multiply modulo 12 is just to multiply the two numbers, and then determine the remainder when divided by 12.

✓ Example ✓ 3.7.4: Multiplying modulo 12

What is the product of 11 and 45 modulo 12?

Answer

We begin by multiplying 11 and 45, which is 495. Next, we find 495 modulo 12, either by dividing the result by 12 to determine the remainder, or by using a calculator. Entering **mod(495,12)** in Desmos yields 3. Had long division been used, the remainder would be 3. So $11 \times 45 = 3$ modulo 12.

Your Turn 3.7.4

What is the product of 4 and 19 modulo 12?

Calculating Real-World Applications with Clock Arithmetic

✓ Example ✓ 3.7.5: Applying Clock Arithmetic

Suppose it is 3:00, and you decide to check your email every 5 hours. What time will it be when you check your email the ninth time?

Answer

> If you check your email every 5 hours nine times, that ninth check will occur 45 hours after 3:00, which is an addition of 45 hours to 3:00. So, we find 45 modulo 12, which is 9. Nine hours after 3:00 is 12:00. It will be 12:00 when you check your email the ninth time.

Your Turn 3.7.5

You have agreed to text your friend every 3 hours while driving across the country. You began your trip at 8 AM. What time will it be when you text your friend the 15th time?

Clock arithmetic processes can be applied to days of the week. Every 7 days the day of the week repeats, much like every 12 hours the time on the clock repeats. The only difference will be that we work with remainders after dividing by 7. In technical terms, this is referred to as modulo 7. More generally, let n be a positive integer. Then n modulo 7, written n mod 7, is the remainder when n is divided by 7. If that value is x, we may write $n = x$ (mod 7).

✓ Example ✓ 3.7.6: Applying Clock Arithmetic to Days of the Week

Your family has a cat, and no one wants to empty the litter box. However, it has to be done daily. The six of you agree to take turns, so everyone has to empty the litter box every 6 days. You empty the box on a Thursday. What day will you empty the box for the 10th time?

Answer

> The first time you emptied the litter box was on a Thursday. So,the 10th time you empty the litter box will be 9 times later (you've already had your first turn, so 9 turns left!). This will happen 54 (9 times 6) days later. Finding the value of 54 modulo 7, using division to determine the remainder or using a calculator to find the value of 54 modulo 7 gives the answer 5. Five days after a Thursday is Tuesday.

Your Turn 3.7.6

Your family shares the cooking duties in the home. You've agreed to prepare the meal every 5 days. The last time you prepared dinner was a Tuesday. What day of the week will it be after you've prepared the meals 20 more times?

Check Your Understanding

For the following exercises, use clock arithmetic to perform the following:

1. $7 + 19$
2. $8 - 31$
3. 5×37
4. Calene calls her mother every fourth day. She calls on a Monday. What day of the week will it be on Calene's eighth time calling after that?

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3.7.0: Exercises

? Exercise $3.7.0.1$

Explain what modulo 12 means.

? Exercise $3.7.0.2$

Explain what modulo 7 means.

? Exercise $3.7.0.3$

What is 75 modulo 12 ?

? Exercise $3.7.0.4$

What is 139 modulo 12 ?

? Exercise $3.7.0.5$

What is 38 modulo 7 ?

? Exercise $3.7.0.6$

What is 83 modulo 7 ?

For the following exercises, use clock arithmetic (mod 12), to perform the indicated calculation.

? Exercise $3.7.0.7$

$7+13$

? Exercise $3.7.0.8$

$8+19$

? Exercise $3.7.0.9$

$4+27$

? Exercise $3.7.0.10$

$3+100$

? Exercise $3.7.0.11$

$9-15$

? Exercise $3.7.0.12$

$6-27$

? Exercise 3.7.0.13

4×18

? Exercise 3.7.0.14

7×29

? Exercise 3.7.0.15

11×38

? Exercise 3.7.0.16

6×23

? Exercise 3.7.0.17

It is $8:00$. What time will it be in 70 hours?

? Exercise 3.7.0.18

It is a Thursday. What day of the week will it be in 100 days?

? Exercise 3.7.0.19

It is Monday. What day of the week will it be in 58 days?

? Exercise 3.7.0.20

It is $3:00$. What time of the day will it be in 150 hours?

? Exercise 3.7.0.21

It is $6:00$. What time was it 34 hours ago?

? Exercise 3.7.0.22

It is $2:00$. What time was it 100 hours ago?

? Exercise 3.7.0.23

A trucker passes through Kokomo, Indiana, once every 9 days. They come through Kokomo on a Wednesday. What day of the week will the driver pass through Kokomo after 8 more visits?

? Exercise 3.7.0.24

Jason checks his email every 5 hours. He checks it at 6 PM one day. What time of the day will it be when he checks his mail the 50th time after that 6 PM check?

? Exercise $3.7.0.25$

Mickey gets a new prescription of a drug that she needs to take every day. The prescription is for 250 days. She takes the first pill of the new bottle on a Friday. What day of the week will her prescription run out?

? Exercise $3.7.0.26$

Zainab visits the nursing home every 5 days. She visits on a Sunday. What day of the week will it be when she visits it for the 7th time after that?

? Exercise $3.7.0.27$

Micaela has to check in with her boss every 14 hours. If she checks in at $3:00$, what time will it be when she checks in the 10th time after that?

? Exercise $3.7.0.28$

Tracy has an alarm set for every 4 hours. It goes off at 3:00. What time will it be when the alarm goes off the 20th time after that?

? Exercise $3.7.0.29$

Dejan must check his blood sugar every 5 hours. He checks his blood sugar at $4:00$. What time will it be when Dejan checks their his sugar the 40th time after that?

? Exercise $3.7.0.30$

Latanjana is in the hospital, where her blood pressure is checked every 3 hours. If her blood pressure is checked at 5:00, what time will it be when her blood pressure is checked the 13th time after that?

Months come in twelves, just as hours do. This means that months can be calculated using modulo 12, just like hours. For the following exercises, calculate what month it will be for each exercise.

? Exercise $3.7.0.31$

Micaela works for a sprinkler maintenance company and runs a routine check on the Harris's sprinkler system every third month. If Micaela checks the system in an April, what month will it be when Micaela returns to the Harris's for the 11th time after that?

? Exercise $3.7.0.32$

Dana runs a half marathon every 5 months. She runs one in a May. What month will it be when she runs her 8th marathon after that?

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3.8: Exponents

Learning Objectives

Apply the rules of exponents to simplifying expressions.

Sometimes, we look for shorthand when writing or expressing something that simply takes too long. The use of LOL and tl;dr. This shorthand only works if everyone reading the shorthand knows what it stands for. Using exponents is a similar instance. Writing out a long string of a number times itself over and over takes too much time, and eventually one would forget how many of the value has been written or read. For example,

$$8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8$$

There has to be a shorter and more efficient way to write 8 times itself $1, 2, 3 \ldots \ldots \text{hmmmm}, 19$ times.

And that’s the role that exponents play in mathematics. They are shorthand for multiplying a number by itself a number of times. Without it, calculations would become a mess and we’d have to write a lot more.

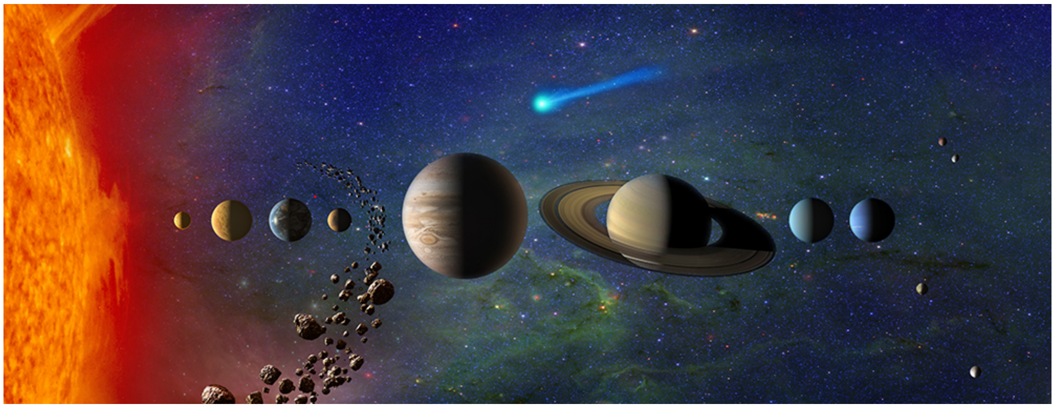


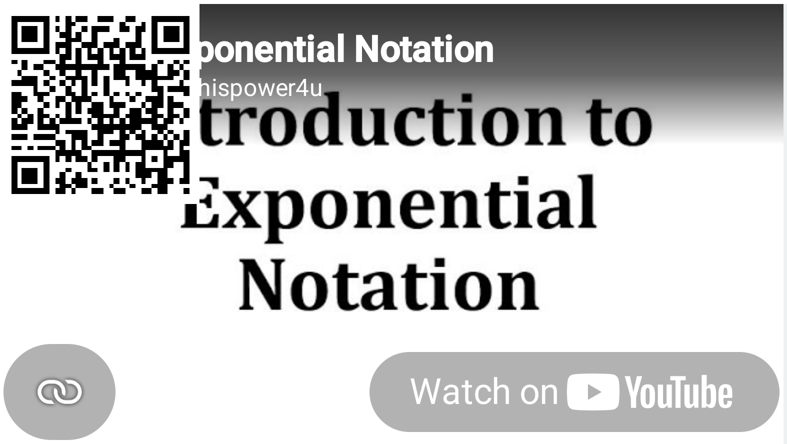
Figure 3.45: Astronomical distances are written using exponents. (credit: “Our Solar System (Artist's Concept)” by NASA/Jet Propulsion Laboratory-Caltech/Public Domain)

Applying the Rules of Exponents to Simplify Expressions

Squaring a number is multiplying it by itself, and has that name because it is the area of a square with that side length. Cubing a number is finding the volume of a cube with that length of sides. That’s why we refer to 5^2 as five squared, or 10^3 as ten cubed. Exponents represent that multiplication.

Let’s remind ourselves of the terminology associated with exponents and what exponents represent. Suppose you want to multiply a number, let’s label that number a, by itself some number of times. Let’s label the number of times b. We denote that as semanticsabab. We say a raised to the bth power. When we write or see 7^5, we call the 7 the base and we call 5 the exponent. What it represents is 7 multiplied by itself 5 times. This means exponents are used as a shorthand for repeated multiplications, where we write $7^5 = 7 \times 7 \times 7 \times 7 \times 7$. We would write 7^5 and say seven to the fifth power.

Video



Exponential Notation

The definitions of base and exponent make it possible to understand the exponent rules.

Product Rule for Exponents

The first rule we examine is the product rule, $a^n a^m = a^{n+}$m. This rule means that when we multiply a base raised to a power times the same base to another power, the result is the base raised to the sum of the powers. To demonstrate, consider $9^3 \times 9^5$. If we apply the product rule to that we get $9^3 \times 9^5 = 9^{3+5} = 9^8$. This can be tested by looking at the multiplications that are represented. The 9^3 is 9 times itself 3 times, while 9^5 is 9 times itself 5 times. Substituting those into $9^3 \times 9^5$ we see $9^3 \times 9^5 = (9 \times 9 \times 9) \times (9 \times 9 \times 9 \times 9 \times 9) = 9^8$, which is what the formula told us would happen.

The first rule we examine is the product rule, $a^n a^m = a^{n+m}$. This rule means that when we multiply a base raised to a power times the same base to another power, the result is the base raised to the sum of the powers. To demonstrate, consider $9^3 \times 9^5$. If we apply the product rule to that we get $9^3 \times 9^5 = 9^{3+5} = 9^8$. This can be tested by looking at the multiplications that are represented. The 9^3 is 9 times itself 3 times, while 9^5 is 9 times itself 5 times. Substituting those into $9^3 \times 9^5$ we see $9^3 \times 9^5 = (9 \times 9 \times 9) \times (9 \times 9 \times 9 \times 9 \times 9) = 9^8$, which is what the formula told us would happen.

Checkpoint

Caution: The product rule only applies when the bases are the same. If the bases are different, we do not apply this rule.

FORMULA

If a number, a, raised to a power, n, is then multiplied by a raised to another power, m, the result is $a^n a^m = a^{n+}$manam=an+m.

✓ Example ✓ 3.8.1: Using the Product Rule for Exponents

If possible, use the product rule to simplify the following:

1. $21^9 \times 21^{15}$
2. $5^9 \times 8^4$

Answer

1. We can apply the product rule to simplify the expression because the bases are the same and we are multiplying.

$$21^9 \times 21^{15} = 21^{(9+15)} = 21^{24}$$

2. Since the bases are not the same (one is 5, the other 8), this cannot be simplified using the product rule for exponents.

Your Turn 3.8.1

If possible, use the product rule to simplify the following: 1. $12^{13} \times 12^{8}$ 2. $3^{6} \times 4^{10}$

These rules can be applied to unknowns too.

Example 3.8.2: Using the Product Rule for Exponents of Unknowns

Use the product rule to simplify
$a^4 \times a^{10}$.

Answer

The bases are the same, and we are multiplying, so we apply the multiplication rule to simplify the expression.

$$a^4 \times a^{10} = a^{(4+10)} = a^{14}$$

msup

Your Turn 3.8.2

Use the product rule to simplify $b^6 \times b^3$.

Quotient Rule for Exponents

The next rule we examine is the quotient, or division, rule.

FORMULA

When a number, a, raised to a power, n, is divided by a raised to another power, m, then the result is

$$\frac{a^n}{a^m} = a^{(n-m)}$$

This rule means that when we divide a base raised to a power by the same base to another power, the result is the base raised to the difference of the powers. To demonstrate, consider $\frac{14^{13}}{14^{6}}$. If we apply the quotient rule to that, we get $\frac{14^{13}}{14^{6}} = 14^{13-6} = 14^{7}$. This can be tested by looking at the division that is represented. Remember, 14^{13} is 14 multiplied to itself 13 times, while 14^{6} is 14 multiplied to itself 6 times. Substituting those into $\frac{14^{13}}{14^{6}}$ gives the following:

$$\frac{4^{13}}{4^{6}} = \frac{4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4}{4 \times 4 \times 4 \times 4 \times 4 \times 4}$$

We see here that there are a LOT of fours to be divided out.

$$\begin{aligned} &= \frac{4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4}{4 \times 4 \times 4 \times 4 \times 4 \times 4} \\ &= \frac{4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4}{1} \\ &= 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \end{aligned}$$

What remains is 4 to the 7th power,

$$4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 = 4^7.$$

All of the work above confirmed what the formula told us would be the result.

Checkpoint

Caution: The quotient rule only applies when the bases are the same. If the bases are different, we do not apply this rule.

Example 3.8.3: Using the Quotient Rule for Exponents

Use the quotient rule to simplify $\frac{5^{19}}{5^{11}}$.

Answer

We can apply the quotient rule to simplify the expression since the bases are the same and we are dividing.

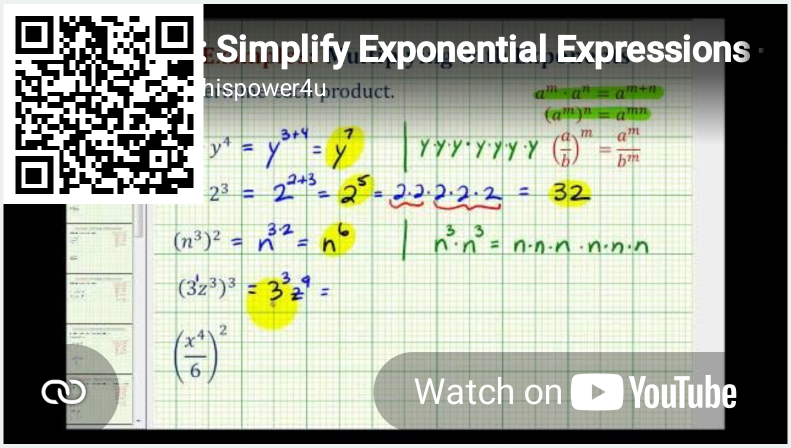
$$\frac{5^{19}}{5^{11}} = 5^{(19-11)} = 5^{8}$$

msup

Your Turn 3.8.3

Use the quotient rule to simplify $\frac{b^6}{b^4}$.

Video



Product and Quotient Rule for Exponents

A natural consequence of the quotient rule is what it means to raise a non-zero number to the zeroth power. Let's look at the simplification when the exponents are equal.

$$\frac{3^6}{3^6} = 3^{(6-6)} = 3^0$$

We know that a number divided by itself is 1, so $\frac{3^6}{3^6} = 1$. From that is must be that $\frac{3^6}{3^6} = 3^0 = 1$. This provides the rule for a number raised to the power 0: $a \neq 0$.

FORMULA

If you have a non-zero number a, then $a^o = 1$.

Distributive Rule for Exponents

The next rule we look to is a distributive rule for exponents.

FORMULA

If you have a product, $(a \times b)$, and raise it to an exponent, n, then $(a \times b)^n = a^n \times b^n$.

This means that when we have two numbers multiplied together, and that is raised to a power, it is the same as raising each of the numbers to the same power first, then multiplying. For example, $(3 \times 7)^4 = 3^4 \times 7^4$. This can be explained using the definition of exponents and multiplying all the factors.

$$(3\times 7)^4=(3\times 7)\times(3\times 7)\times(3\times 7)\times(3\times 7)$$

We may change the order in which numbers are multiplied. This is the commutative property of the real numbers. This can be written as $3\times 3\times 3\times 3\times 7\times 7\times 7\times 7$. Using exponents, that shortens to $3^4\times 7^4$.

This also works in the other direction, $a^n\times b^n=(a\times b)^n$. Read this way, if we have one base raised to an exponent, and another base raised to the same exponent, we can multiply the bases and raise that product to the shared exponent. For instance, $7^8\times 11^8$ $=(7\times 11)^8=77^8$

Checkpoint

Caution: The exponent distributive rule, $a^n\times b^n=(a\times b)^n$, only works if the exponents are the same.

✓ Example $3.8.43$: Using the Distributive Rule for Exponents

Use the exponent distributive rule to expand $(6\times 13)^7$.

Answer

Applying the distributive rule to the product, we get

$$(6\times 13)^7=6^7\times 13^7.$$

Your Turn $3.8.4$

Use the exponent distributive rule to expand $(2\times 19)^{14}$.

✓ Example $3.8.5$: Using the Distributive Rule for Exponents

Use the exponent distributive rule to expand $(c\times d)^{10}$.

Answer

Applying the distributive rule to the product, we get

$$(c\times d)^{10}=c^{10}\times d^{10}$$

Your Turn $3.8.5$

Use the exponent distributive rule to expand $(a\times b)^6$.

This distribution also works for quotients. A fraction raised to an exponent equals the numerator raised to the exponent divided by the denominator raised to the exponent. For example,

$$\left(\frac{3}{5}\right)^7=\frac{3^7}{5^7}.$$

Demonstrating this is similar to the previous rule.

FORMULA

When you have a fraction, $\frac{a}{b}$, raised to an exponent, n, then

$$\left(\frac{a}{b}\right)^n=\frac{a^n}{b^n}.$$

Example 3.8.6: Using the Distributive Rule for Exponents with Fractions

Use the exponent distributive rule to expand the following:

1. $\left(\frac{4}{9}\right)^6$
2. $\left(\frac{3}{b}\right)^{11}$

Answer

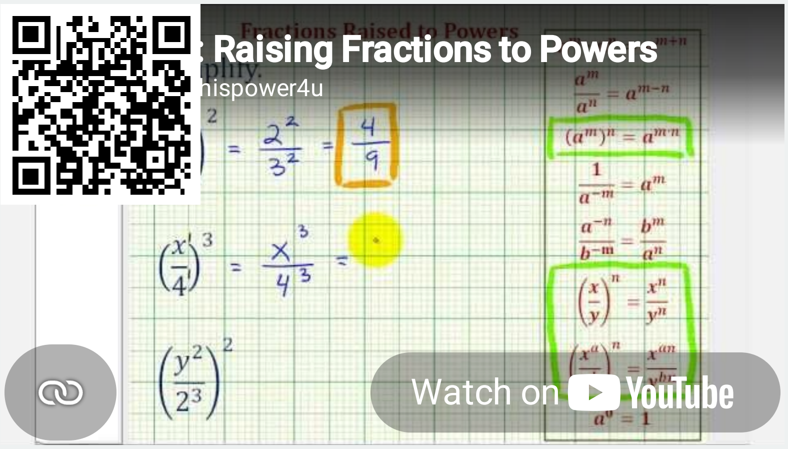
1. Applying the distributive rule to the quotient, we get $\left(\frac{4}{9}\right)^6 = \frac{4^6}{9^6}$.
2. Applying the distributive rule to the quotient, we get $\left(\frac{3}{b}\right)^{11} =$ mfrac311b11(3b)11=311b11.

Your Turn 3.8.6

Use the exponent distributive rule to expand the following:

- $\left(\frac{14}{5}\right)^9$
- $\left(\frac{a}{18}\right)^5$

Video



Fraction Raised to a Power

Power Rule

In the previous two sets of rules, we've seen exponents applied to products and quotients. Now we look to exponents applied to other exponents. For example, $\left(3^6\right)^4 = 3^{(6\times4)} = 3^{24}$. This can be explained by examining what the outer exponent does. We raise 3^6 to the fourth power, so we multiply 3^6 by itself 4 times, $\left(3^6\right)^4 = 3^6 \times 3^6 \times 3^6 \times 3^6$. Now if we apply the product rule for exponents, this becomes $3^{(6+6+6+6)} = 3^{24}$.

FORMULA

If you raise a non-zero base, say a, to an exponent n, and raise that to another exponent, m, you get the base raised to the product of the exponents, which is

$$\left(a^n\right)^m = a^{(n\times m)}.$$

Example 3.8.7: Raising an Exponent to an Exponent

Expand the following:

1. $\left(6^7\right)^3$
2. $\left(b^{12}\right)^4$

Answer

1. Using the power rule of exponents, $\left(6^7\right)^3 = 6^{(7\times3)} = 6^{21}$.
2. Using the power rule of exponents, msupb12)4=b(12×4)=b48(b12)4=b(12×4)=b48.

Your Turn 3.8.7

Expand the following:

1. $\left(11^4\right)^{12}$
2. $\left(a^7\right)^6$

Negative Exponent Rule

Up until now, we've only looked at positive exponents. The last exponent rule we look at is what negative exponents represent. Recall the quotient rule: $\frac{a^n}{a^m} = a^{(n+m)}$. What would happen if the exponent in the denominator was larger than that in the numerator? For example, $\frac{4^5}{4^5}$. If we apply the quotient rule, we obtain $\frac{4^5}{4^7} = 4^{5-7} = 4^{-2}$. We need to make sense of that negative exponent. To do so, we can expand the quotient and see what happens:

$$\frac{4^5}{4^7} = \frac{4\times4\times4\times4\times4}{4\times4\times4\times4\times4\times4\times4}.$$

When we divide out common factors, only two factors of 4 are left in the denominator, as we see here: $\frac{1}{4\times4}$. Using exponent notation, this is $\frac{1}{4^2}$. Since 4^{-2} and $\frac{1}{4^2}$ represent the same number, $\frac{4^5}{4^7}$, they are equal. This demonstrates how negative exponents are defined.

FORMULA

$$a^{-n} = \frac{1}{a^n}$$

provided that $a =\neq 0$.

Similarly,

$$\frac{1}{a^{-n}} = a^n.$$

Example 3.8.8: Eliminating Negative Exponents

Convert the following to expressions with no negative exponent:

1. $3^4 \times 5^{-8}$
2. $a^{-9} \times b^5$
3. $\frac{7}{c^{-2}}$

Answer

1. Using the negative exponent rule on the 5^{-8} and multiplying, $3^4 \times 5^{-8} = 3^4 \times \frac{1}{5^8} = \frac{3^4}{5^8}$.
2. Using the negative exponent rule on the a^{-9} and multiplying, $a^{-9} \times b^5 = \frac{1}{a^9} \times b^5 = \frac{b^5}{a^9}$.
3. Begin by rewriting the expression as $\frac{7}{c^{-2}} = \frac{7}{1} \times \frac{1}{c^{-2}}$. Apply the negative exponent rule to $\frac{1}{c^{-2}}$ in the expression, which becomes $\frac{7}{1} \times \frac{1}{c^{-2}} = 7 \times c^2$, which has no negative exponents.

Your Turn 3.8.8

Convert the following to expressions with no negative exponent:

1. $12^{-3} \times 7^5$
2. $c^{-7} \times 5^3$

Example 3.8.9: Eliminating Denominators by Using Negative Exponents

Use negative exponents to rewrite the following expressions with no denominator:

1. $\frac{7^3}{13^9}$
2. $\frac{c^4}{d^8}$

Answer

1. Rewrite the expression $\frac{7^3}{13^9}$ as $\frac{7^3}{1} \times \frac{1}{13^9}$. Then use the definition of negative exponents to rewrite the $\frac{1}{13^9}$ as 13^{-9}. Last, multiply, yielding $\frac{7^3}{1} \times \frac{1}{13^9} = 7^3 \times 13^{-9}$.
2. Rewrite the expression $\frac{c^4}{d^8}$ as $\frac{c^4}{1} \times \frac{1}{d^8}$. Then use the definition of negative exponents to rewrite the $\frac{1}{d^8}$ as d^{-8}. Last, multiply, yielding $\frac{c^4}{1} \times \frac{1}{d^8} = c^4 \times d^{-8}$.

Your Turn 3.8.9

Use negative exponents to rewrite the following expressions with no denominator:

1. $\frac{6^3}{13^8}$
2. $\frac{c^5}{2^9}$

The table below shows a summary of the exponent rules from this section.

Rule	Example	In Words
Product Rule $a^n a^m = a^{n+}$manam=an+m	$8^2 \times 8^5 = 8^7$	A base raised to a power, times the same based raised to another power, is the base raised to the sum of the powers.
Quotient Rule $\frac{a^n}{a^m} = a^{(n-m)}$	$\frac{11^{14}}{11^{12}} = 11^{12}$	A base raised to a power, divided by the same based raised to another power, is the base raised to the difference of the powers.
Zero Power Rule a^0 =1a0=1 provided that $a \neq 1$	$412^0 = 1$	Any non-zero number raised to the zeroth power equals 1.
Distributive Rule, Multiplication $(a \times b)^n = a^n \times b^n$	$(14 \times 31)^9 = 14^9 \times 31^9$	Exponents distribute across multiplication.
Distributive Rule, Division $\left(\frac{a}{b}\right)^n =$ mfrac anbn(ab)n=anbn	$\left(\frac{62}{91}\right)^8 = \frac{62^8}{91^8}$	Exponents distribute across division.
Power Rule $(a^n)^m = a^{(n \times m)}$	$(5^9)^{15} = 5^{135}$	A base raised to a power, raised to another power, is the base raised to the first power times the second power.
Negative Exponent Rule a^{-n} =1ana−n=1an provided that $a \neq 0$	$6^{-8} = \frac{1}{6^8}$ $\frac{1}{12^7} = 12^{-7}$	A base raised to a negative exponent is 1 divided by the base raised to the positive power, and vice versa.

These rules often occur in tandem with each other, but it requires that you carefully apply the rules.

Example 3.8.10: Simplifying Expressions Using Exponent Rules

Simplify the following:

1. $\left(\frac{4^2 \times 7}{9^3}\right)^5$
2. $\left(\frac{5a^4}{b^9}\right)^6$

Answer

1. **Step 1:** To simplify this, we start by distributing the power 5 across the quotient:

$$\left(\frac{4^2 \times 7}{9^3}\right)^5 = \frac{\left(4^2 \times 7\right)^5}{\left(9^3\right)^5}$$

   Step 2: We distribute the power 5 in the numerator across that multiplication:

$$\left(\frac{4^2 \times 7}{9^3}\right)^5 = \frac{\left(4^2 \times 7\right)^5}{\left(9^3\right)^5} = \frac{\left(4^2\right)^5 \times 7^5}{\left(9^3\right)^5}$$

   Step 3: We apply the power rule where indicated:

$$\left(\frac{4^2 \times 7}{9^3}\right)^5 = \frac{\left(4^2 \times 7\right)^5}{\left(9^3\right)^5} = \frac{\left(4^2\right)^5 \times 7^5}{\left(9^3\right)^5} = \frac{4^{(2\times5)} \times 7^5}{9^{(3\times5)}} = \frac{4^{10} \times 7^5}{9^{15}}$$

2. **Step 1:** To simplify this, we start by distributing the power 6 across the quotient:

$$\left(\frac{5a^4}{b^9}\right)^9 = \frac{\left(5 \times a^4\right)^6}{\left(b^9\right)^6}$$

   Step 2: We distribute the power 5 in the numerator across that multiplication:

$$\frac{\left(5 \times a^4\right)^6}{\left(b^9\right)^6} = \frac{(5)^6 \times \left(a^4\right)^6}{\left(b^9\right)^6}$$

   Step 3: We apply the power rule where indicated:

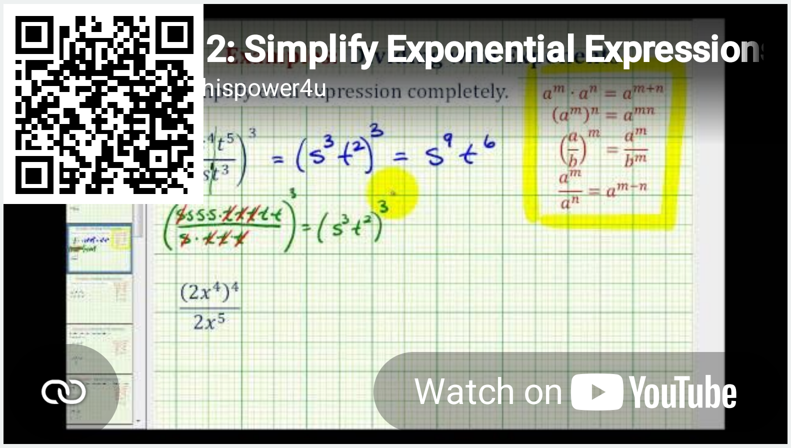
$$\frac{(5)^6 \times \left(a^4\right)^6}{\left(b^9\right)^6} = \frac{5^6 a^{24}}{b^{54}}$$

Your Turn 3.8.10

Simplify the following:

1. $\left(\frac{7^9}{10^5 \times 6^3}\right)^8$
2. $\left(\frac{4}{a^9 b^6}\right)^2$

Video



Simplifying Expressions with Exponents

Check Your Understanding

1. Simplify $a^3 \times a^5$.
2. Simplify $\frac{5^4}{5^8}$.
3. Simplify $(6b)^9$.
4. Simplify $\left(\frac{c}{7}\right)^3$.
5. Simplify $\left(\frac{3a^2}{4b^5}\right)^6$.

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3.8.0: Exercises

For the following exercises, simplify the expression.

? Exercise 3.8.0.1

$4^5 \times 4^2$

? Exercise 3.8.0.2

$3^3 \times 3^6$

? Exercise 3.8.0.3

$a^2 \times a^7$

? Exercise 3.8.0.4

$b^7 \times b^{12}$

? Exercise 3.8.0.5

$\frac{4^6}{4^2}$

? Exercise 3.8.0.6

$\frac{15^{14}}{15^{11}}$

? Exercise 3.8.0.7

$\frac{c^9}{c^4}$

? Exercise 3.8.0.8

$\frac{a^{14}}{a^5}$

? Exercise 3.8.0.9

$\frac{11^3}{11^7}$

? Exercise 3.8.0.10

$\frac{7^{14}}{7^{23}}$

? Exercise 3.8.0.11

$\frac{b^6}{b^{19}}$

? Exercise 3.8.0.12

$\frac{d^8}{d^{17}}$

? Exercise 3.8.0.13

$(4 \times 3)^4$

? Exercise 3.8.0.14

$(5 \times 8)^7$

? Exercise 3.8.0.15

$(3c)^6$

? Exercise 3.8.0.16

$(n \times m)^9$

? Exercise 3.8.0.17

$\left(\frac{7}{2}\right)^8$

? Exercise 3.8.0.18

$\left(\frac{a}{6}\right)^{11}$

? Exercise 3.8.0.19

$\left(\frac{4}{c}\right)^9$

? Exercise 3.8.0.20

$\left(6^4\right)^8$

? Exercise 3.8.0.21

$\left(12^5\right)^3$

? Exercise 3.8.0.22

$\left(x^6\right)^2$

? Exercise 3.8.0.23

$\left(b^6\right)^{11}$

? Exercise 3.8.0.24

$\left(\frac{3b}{5}\right)^4$

? Exercise 3.8.0.25

$\left(\frac{a}{2b}\right)^5$

? Exercise 3.8.0.26

$\left(\frac{3a^4}{7}\right)^3$

? Exercise 3.8.0.27

$\left(\frac{6x^7}{11}\right)^8$

? Exercise 3.8.0.28

$\left(\frac{2x^4}{b^7}\right)^3$

? Exercise 3.8.0.29

$\left(\frac{7r^9}{a^4}\right)^{12}$

For the following exercises, rewrite the expression without a denominator.

? Exercise 3.8.0.30

$\frac{3}{a^4}$

? Exercise 3.8.0.31

$\frac{5}{b^6}$

? Exercise 3.8.0.32

$\frac{4}{b^2c^3}$

? Exercise 3.8.0.33

$\frac{9}{x^4y^5}$

For the following exercises, rewrite the expression without negative exponents.

? Exercise 3.8.0.34

$12^4 \times 5^{-3}$

? Exercise 3.8.0.35

$3b^{-12}$

? Exercise 3.8.0.36

$\frac{15}{a^{-10}}$

? Exercise 3.8.0.37

$\frac{6x^{-5}}{y^{-7}}$

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3.9: Scientific Notation

Learning Objectives

1. Write numbers in standard or scientific notation.
2. Convert numbers between standard and scientific notation.
3. Add and subtract numbers in scientific notation.
4. Multiply and divide numbers in scientific notation.
5. Use scientific notation in computing real-world applications.

The amount of information available on the Internet is simply incomprehensible. One estimate for the amount of data that will be on the Internet by 2025 is 175 Zettabytes. A single zettabyte is one billion trillion. Written out, it is 1,000,000,000,000,000,000,000. One estimate is that we're producing 2.5 quintillion bytes of data per day. A quintillion is a trillion trillion, or, written out, 1,000,000,000,000,000,000. To determine how many days it takes to increase the amount of information that is on the Internet by 1 zettabyte, divide these two numbers, a zettabyte being 1,000,000,000,000,000,000,000, and 2.5 quintillion, being 2,500,000,000,000,000,000, shows it takes 400 days to generate 1 zettabyte of information. But doing that calculation is awkward with a calculator. Keeping track of the zeros can be tedious, and a mistake can easily be made.

$$\frac{5\text{ mol C}}{1} \times \frac{6.02 \times 10^{23}\text{ atoms C}}{1\text{ mol C}} = 3.01 \times 10^{24}\text{ atoms C}$$

Figure 3.9.1: Calculations in the sciences often involve numbers in scientific notation form.

On the other end of the scale, a human red blood cell has a diameter of 7.8 micrometers. One micrometer is one millionth of a meter. Written out, 7.8 micrometers is 0.0000078 meters. Smaller still is the diameter of a virus, which is about 100 nanometers in diameter, where a nanometer is a billionth of a meter. Written out, 100 nanometers is 0.0000001 meters. To compare that to engineered items, a single transistor in a computer chip can be 14 nanometers in size (0.000000014 meters). Smaller yet is the diameter of an atom, at between 0.1 and 0.5 nanometers.

Sometimes we have numbers that are incredibly big, and so have an incredibly large number of digits, or sometimes numbers are incredibly small, where they have a large number of digits after the decimal. But using those representations of the names of the sizes makes comparing and computing with these numbers problematic. That's where scientific notation comes in.

Writing Numbers in Standard or Scientific Notation Form

When we say that a number is in scientific notation, we are specifying the form in which that number is written. That form begins with an integer with an absolute value between 1 and 9, then perhaps followed the decimal point and then some more digits. This is then multiplied by 10 raised to some power. When the number only has one non-zero digit, the scientific notation form is the digit multiplied by 10 raised to an exponent. When the number has more than one non-zero digit, the scientific notation form is a single digit, followed by a decimal, which is then followed by the remaining digits, which is then multiplied by 10 to a power.

The following numbers are written in scientific notation:

- 1.45×10^3
- -8.345×10^{-4}
- 3×10^2
- 3.14159×10^0

The following numbers are not written in scientific notation:

- 1.45 because it isn't multiplied by 10 raised to a power
- -50.053×10^7 because the absolute value of −50.053 is not at least 1 and less than 10

- 41.7×10^9 because 41.7 is not at least 1 and less than 10
- 0.036×10^{-3} because 0.036 is not at least one and less than 10

✓ Example $3.9.1$: Identifying Numbers in Scientific Notation

Which of the following numbers are in scientific notation? If the number is not in scientific notation, explain why it is not.

1. -9.67×10^{20}
2. 145×10^{-8}
3. 1.45

Answer

1. The number -9.67×10^{20} is in scientific notation because the absolute value of −9.67 is at least 1 and less than 10.
2. The number 145×10^{-8} is not in scientific notation because 145 is not at least 1 and less than 10.
3. The number 1.45 is not in scientific notation form. Even though it is at least 1 but less than 10, it is not multiplied by 10 raised to a power.

Your Turn $3.9.1$

Which of the following numbers are in scientific notation? If the number is not in scientific notation, explain why it is not.

1. 42.67×10^{13}
2. 7.113×10^{-2}
3. -80.91

Some numbers are so large or so small that it is impractical to write them out fully. Avogadro's number is important in chemistry. It represents the number of units in 1 mole of any substance. The substance many be electrons, atoms, molecules, or something else. Written out, the number is: 602,214,076,000,000,000,000,000. Another example of a number that is impractical to write out fully is the length of a light wave. The wavelength of the color blue is about 0.000000450 to 0.000000495 meters. Such numbers are awkward to work with, and so scientific notation is often used. We need to discuss how to convert numbers into scientific notation, and also out of scientific notation.

Recall that multiplying a number by 10 adds a 0 to the end of the number or moves the decimal one place to the right, as in $43 \times 10 = 430$ or $3.89 \times 10 = 38.9$. And if you multiply by 100, it adds two zeros to the end of the number or moves the decimal two places to the right, and so on. For example, $38 \times 100{,}000 = 3{,}800{,}000$ and $32.998 \times 10{,}000 = 329{,}980$. Multiplying a number by 1 followed by some number of zeros just adds that many zeros to the end of the number or moves the decimal place that many places to the right. Numbers written as 1 followed by some zeros are just powers of 10, as in $10^1 = 10$, $10^2 = 100$, $10^3 = 1{,}000$, etc. Generally, msup10n=10...0⏟n10n=10...0⏟n.

We can use this to write very large numbers. For instance, Avogadro's number is 602,214,076,000,000,000,000,000, which can be written as $6.02214076 \times 10^{23}$. The multiplication moves the decimal 23 places to the right.

Similarly, when we divide by 10, we move the decimal one place to the left, as in $\frac{46.7}{10} = 4.67$. If we divide by 100, we move the decimal two places to the left, as in $\frac{3.456}{100} = 0.03456$. In general, when you divide a number by a 1 followed by n zeros, you move the decimal n places to the left, as in $\frac{8{,}244.902}{1{,}000{,}000} = 0.008244902$. This denominator could be written as 10^6. If we use that in the expression and allow for negative exponents, rewrite the number as $\frac{8{,}244.902}{1{,}000{,}000} = \frac{8{,}244.902}{10^6} = 8{,}244.902 \times 10^{-6}$. With this, we can write division by a 1 followed by n zeros as multiplication by 10 raised to $-n$.

Using that information, we can demonstrate how to convert from a number in standard form into scientific notation form.

Case 1: The number is a single-digit integer.

In this case, the scientific notation form of the number is semanticsdigit×101digit×101.

Case 2: The absolute value of the number is less than 1.

Follow the process below.

- **Step 1:** Count the number of zeros between the decimal and the first non-zero digit. Label this n.

- **Step 2:** Starting with the first non-zero digit of the number, write the digits. If the number was negative, include the negative sign.
- **Step 3:** If there is more than one digit, place the decimal after the first digit from Step 2.
- **Step 4:** Multiply the number from Step 3 by msup10n+110n+1.

Case 3: The absolute value of the number is 10 or larger.

Follow the process below.

- **Step 1:** Count the number of digits that are to the left of the decimal point. Label this n.
- **Step 2:** Write the digits of the number without the decimal place, if one was present. If the number was negative, include the negative sign.
- **Step 3:** If there is more than one digit, place the decimal point after the first digit.
- **Step 4:** Multiply the number from Step 3 by msup10n−110n−1.

✓ Example3.9.2: Writing a Number in Scientific Notation

Write the following numbers in scientific notation form:

1. 428.9
2. −0.00000981
3. 8

Answer

1. Since the absolute value of 428.9 is 10 or larger, so we use the process from Case 3, above.

   Step 1: There are three digits to the left of the decimal point, so $n = 3$.

   Step 2: Write the digits of the number without the decimal place, which is 4289.

   Step 3: Since there is more than one digit, place the decimal point after the first digit. We now have 4.289.

   Step 4: Since $n = 3$, we multiply 4.289 by 10 raised to the second power, 4.289×10^2.

   The scientific notation form of 428.9 is 4.289×10^2.

2. Since the absolute value of −0.00000981 is less than 1, we use the process from Case 2.

   Step 1: The number of zeros between the decimal and the first non-zero digit is 5, so $n = 5$.

   Step 2: We write the non-zero digits, including the negative sign, yielding −981.

   Step 3: The decimal gets placed to the right of the first digit, resulting in −9.81.

   Step 4: Since $n = 5$, we multiply −9.81 by 10 raised to the fourth power, -9.81×10^{-6}.

   The scientific notation form of −0.00000981 is 9.81×10^{-6}.

3. Since 8 is a single-digit integer, apply Case 1. The scientific notation form of 8 is 8×10^1.

Your Turn 3.9.2

Write the following numbers in scientific notation form:

1. -38300
2. 0.0045
3. 1

When we write numbers in scientific notation form, we can manipulate the representation of the number by moving the decimal around, and making an appropriate change to the exponent of the 10. For instance, let's look at 145.8141×10^8. If we wanted to move the decimal one place to the left, we'd have to increase the power of 10, as shown here: $145.8141 \times 10^8 = 14.58141 \times 10^9$. Since we moved the decimal one to the left, we balance that with moving the exponent up by one. Similarly, if we move the decimal one place to the right, we have to balance that by moving the exponent one to the left, or subtracting one from the exponent, as shown here: $145.8141 \times 10^8 = 1458.141 \times 10^7$. Generally, for a number in the form semanticsnumber×10nnumber×10n:

- If you move the decimal to the left by k digits, you increase the exponent by k.
- If you move the decimal to the right by k digits, you decrease the exponent by k digits.

✓ Example 3.9.3: Increasing the Exponent

Change 456.142×10^5 by moving the decimal two places to the left.

Answer

> Since we are moving the decimal to the left by two places, we increase the exponent of 10 by 2, so that the exponent is now 7. This gives us $456.142 \times 10^5 = 4.56142 \times 10^7$.

Your Turn 3.9.3

Change 46.113×10^8 by moving the decimal four places to the left.

✓ Example 3.9.4: Decreasing the Exponent

Change 12.3×10^2 by moving the decimal five places to the right.

Answer

> Since we are moving the decimal to the right by five places, we decrease the exponent of 10 by 5, so that exponent is now −3. This give us $12.3 \times 10^2 = 1230000.0 \times 10^{-3}$.

Your Turn 3.9.4

Change 149.11×10^{-4} by moving the decimal two places to the right

Converting Numbers from Scientific Notation to Standard Form

In the previous section, converting a number from standard form to scientific notation was explored. Now, we explore converting from scientific notation back into standard form. Doing so involves moving the decimal according to the power of the 10. The decimal is moved a number of steps equal to the exponent of the 10. As demonstrated previously, when the exponent of the 10 is negative, the decimal is moved to the left and when the exponent of the 10 is positive, the decimal is moved to the right.

✓ Example 3.9.5: Converting from Scientific Notation to Standard Form

Convert the following into standard form:

1. 2.78×10^9
2. 9.04×10^{-8}

Answer

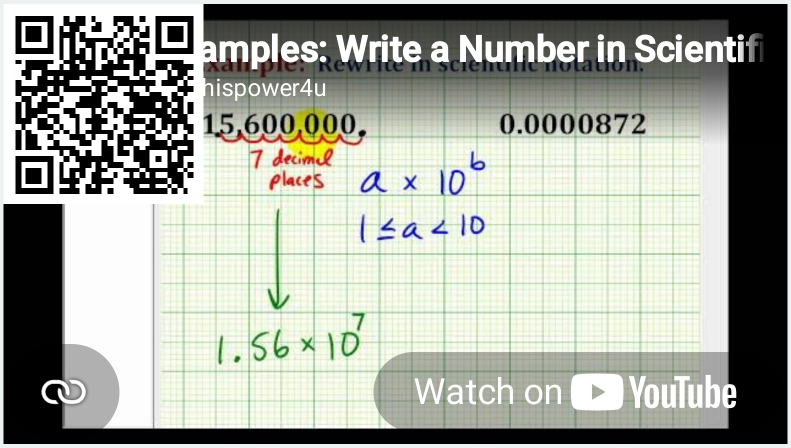
> 1. Since the exponent is positive, the decimal moves nine places to the right, so 2.78×10^9 is $2{,}780{,}000{,}000$.
> 2. Since the exponent is negative, the decimal moves eight places to the left, so 9.04×10^{-8} is 0.0000000904.

Your Turn 3.9.5

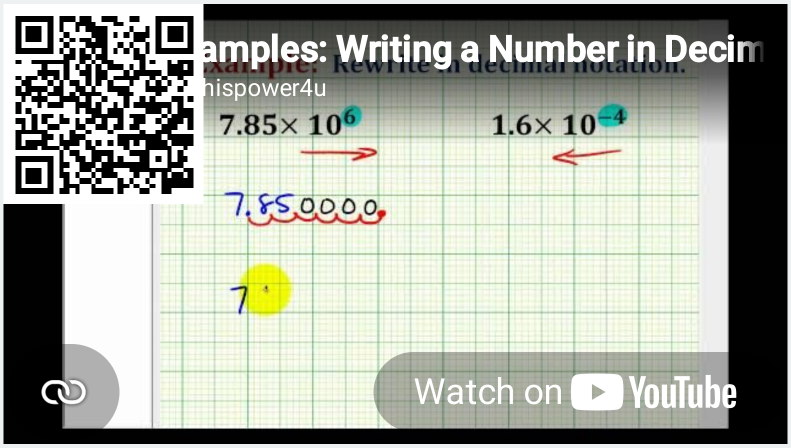
Convert the following into standard form:

1. 1.02×10^6
2. 4.09×10^{-5}

Video


amples: Write a Number in Scientifi
hispower4u
15,600,000.
0.0000872
7 decimal places
a × 10^6
1 ≤ a < 10
1.56 × 10^7
Watch on YouTube

Converting from Standard Form to Scientific Notation Form

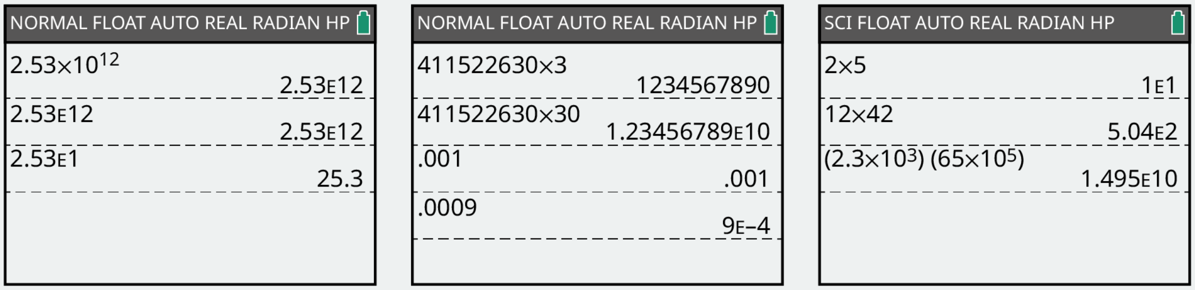

amples: Writing a Number in Decim
hispower4u
7.85× 10^6
1.6× 10^-4
7.850000
7
Watch on YouTube

Converting from Scientific Notation Form to Standard Form

Tech Check: Scientific Notation on a Calculator

Most scientific and graphing calculators come with the ability to directly convert from standard form to scientific notation. On the TI-83, it is accessed through the MODE menus. For a commonly used, free phone scientific calculator, the calculator can be forced to work in scientific notation mode through its settings.

Some calculators, such as the Desmos online calculator, display scientific notation as a number times 10 to a power as you've seen in this section. However, some calculators indicate scientific notation by replacing the semantics×10nFigure 3.47 shows what you may see on a TI-84.



```
NORMAL FLOAT AUTO REAL RADIAN HP
2.53×10^12
                       2.53E12
2.53E12
                       2.53E12
2.53E1
                          25.3
```

```
NORMAL FLOAT AUTO REAL RADIAN HP
411522630×3
                    1234567890
411522630×30
               1.23456789E10
.001
                          .001
.0009
                          9E-4
```

```
SCI FLOAT AUTO REAL RADIAN HP
2×5
                           1E1
12×42
                        5.04E2
(2.3×10^3) (65×10^5)
                      1.495E10
```

Accessing E

Using normal mode

Using sci mode

Figure 3.9.2: Calculator screens

Adding and Subtracting Numbers in Scientific Notation

To add or subtract numbers in scientific notation, the numbers first need to have the same exponent for the 10s. It is possible to add the following since the powers of 10 match: $4.5 \times 10^4 + 3.15 \times 10^4 = 7.65 \times 10^4$

Notice that the number parts were added, but the exponent part remained the same. This is due to the distributive property of the real numbers. The 10^4 is factored from the two terms, as shown: $4.5 \times 10^4 + 3.15 \times 10^4 = (4.5 + 3.15) \times 10^4 = 7.65 \times 10^4$

Numbers in scientific notation can be added or subtracted directly using a calculator. Simply enter the values in scientific form and set your calculator to display scientific notation.

Example 3.9.6: Adding and Subtracting Numbers in Scientific Notation with the Same Powers of 10

Calculate the following:

1. $3.8 \times 10^{-3} + 1.006 \times 10^{-3}$
2. $9.61 \times 10^8 - 3.85 \times 10^8$

Answer

1. Since the powers of 10 match, we use the distributive property of real numbers to factor 10^{-3} from the numbers. We then add the number parts separately to get 4.806. $3.8 \times 10^{-3} + 1.006 \times 10^{-3} = (3.8 + 1.006) \times 10^{-3} = 4.806 \times 10^{-3}$
2. Since the powers of 10 match, we use the distributive property of real numbers to factor 10^8 from the numbers. We then subtract the number parts separately to get 5.76.

Your Turn 3.9.6

Calculate the following:

- $7.57 \times 10^{13} + 2.031 \times 10^{13}$
- $3.03 \times 10^{-6} - 1.5 \times 10^{-6} 7.57 \times 10^{13} + 2.031 \times 10^{13}$

Adding and subtracting in scientific notation is straightforward when the exponents are the same. There are two issues that can arise. The first issue is what to do if after adding or subtracting the result is not in scientific notation.

Example 3.9.7: Correcting an Answer to Scientific Notation After Adding or Subtracting

Calculate the following:

1. $7.03 \times 10^{13} + 8.5 \times 10^{13}$
2. $4.3 \times 10^{21} - 4.613 \times 10^{21}$

Answer

1. Since the powers of 10 match, we add the number parts and multiply that by 10^{13}:
   $7.03 \times 10^{13} + 8.5 \times 10^{13} = (7.03 + 8.5) \times 10^{13} = 15.53 \times 10^{13}$.

   However, 15.53×10^{13} is not in scientific notation because the absolute value of 15.53 is more than 10. To put this number in scientific notation, the decimal needs to move one to the left. To balance that move, the power of 10 must be increased by 1. So, the answer in scientific notation is 1.553×10^{14}.
2. Since the powers of 10 match, we add the number parts: $4.3 \times 10^{21} - 4.613 \times 10^{21} = (4.3 - 4.613) \times 10^{21} = -0.313 \times 10^{21}$

Your Turn 3.9.7

Calculate the following:

1. \(5.08 \times {10^3} + 6.9 \times {10^3}\(
2. \(8.968 \times {10^{ - 38}} - 8.761 \times {10^{ - 38}}\(

The second issue that might be encountered when adding or subtracting is that the powers of 10 do not match. In that case, one of the numbers must be changed so that the powers of 10 match. It is easiest to make the smaller power of 10 larger to match the other power of 10.

For example, to perform the following, $4.5\times10^5+3.9\times10^3$, we'd change the 3.9×10^3 so that the power of 10 is 5. To do so, we need to increase the power of 10 and move the decimal in the number part two places to the left. That would alter 3.9×10^3 into 0.039×10^5. We would use 0.039×10^5 in the addition problem, so that the exponents match, allowing the addition to occur. $4.5\times10^5+3.9\times10^3=4.5\times10^5+0.039\times10^5=(4.5+0.039)\times10^5=4.539\times10^5$

The steps to take when the exponents of the 10s are not equal are:

Step 1: Increase the smaller exponent to equal the larger exponent. Label the amount increased as n.

Step 2: For the number with the smaller power of 10, move the decimal point of the number part to the left n places.

Step 3: Perform the addition or subtraction.

Step 4: If the result is not in scientific notation, adjust the number to be in scientific notation.

✓ Example 3.9.18: Adding Numbers in Scientific Notation with Different Powers of 10

Calculate the following:

$$6.1\times10^4+4.8\times10^5$$

Answer

Step 1: The lower exponent is 4. To make this equal to the larger exponent, we increased it by 1.

Step 2: Since the smaller exponent was increased by 1, move the decimal one to the left, so the addition become $6.1\times10^4+4.8\times10^5=0.61\times10^5+4.8\times10^5$.

Step 3: Now add the numbers, $0.61\times10^5+4.8\times10^5=5.41\times10^5$

Step 4: The result is in scientific notation, so no additional adjustment is necessary.

$6.1\times10^4+4.8\times10^5=5.41\times10^5$

Your Turn 3.9.8

Calculate the following:

$$1.14\times10^{-43}+2.56\times10^{-46} \tag{3.9.1}$$

✓ Example 3.9.9: Subtracting Numbers in Scientific Notation with Different Powers of 10

Calculate the following:

$$7.9\times10^{-15}-6.8\times10^{-13} \tag{3.9.2}$$

Answer

Step 1: The lower exponent is −15 and the larger is −13. To make −15 equal to the larger exponent, we increased it by 2.

Step 2: Since the smaller exponent increased by 2, move the decimal two to the left. The subtraction changes to $7.9\times10^{-15}-6.8\times10^{-13}=0.079\times10^{-13}-6.8\times10^{-13}$.

Step 3: Subtract the numbers, $0.079\times10^{-13}-6.8\times10^{-13}=-6.721\times10^{-13}$.

Step 4: The result is in scientific notation, so no additional adjustment is necessary.
$7.9\times10^{-15}-6.8\times10^{-13}=-6.721\times10^{-13}$

Your Turn 3.9.9

Calculate the following:

$$9.15 \times 10^{28} - 7.23 \times 10^{26} \tag{3.9.3}$$

Multiplying and Dividing Numbers in Scientific Notation

Multiplying and dividing numbers in scientific notation is somewhat easier than adding or subtracting, because the exponents of the 10s do not have to match. However, it is much more likely that the result will not be in scientific notation, and so that will have to be adjusted at the end. Generally, we multiply or divide the number parts of the two values, and then apply exponent rules to the 10 raised to the powers.

To multiply two numbers in scientific notation:

Step 1: Multiply the number parts.

Step 2: Add the exponents of the 10s.

Step 3: The result is the answer from Step 1 times 10 raised to the answer from Step 2.

Step 4: If the number is not in scientific notation, adjust it appropriately.

Example 3.9.10: Multiplying Numbers in Scientific Notation

Calculate the following:

1. $(4.3 \times 10^3) \times (1.8 \times 10^7)$
2. $(5 \times 10^{-13}) \times (7.3 \times 10^6)$

Answer

1. **Step 1:** Multiply the number parts to get $4.3 \times 1.8 = 7.74$.

   Step 2: Add the exponents of the 10s to get $3 + 7 = 10$.

   Step 3: The result is then 7.74×10^{10}.

   Step 4: This number is already in scientific notation, so no additional adjustment is necessary, $(4.3 \times 10^3) \times (1.8 \times 10^7) = 7.74 \times 10^{10}$.

2. **Step 1:** Multiply the number parts to get $5 \times 7.3 = 36.5$.

   Step 2: Add the exponents of the 10s to get $-13 + 6 = -7$.

   Step 3: The result then is 36.5×10^{-7}.

   Step 4: Since the number is not in scientific notation, it must be adjusted. To put 36.5×10^{-7} into scientific notation, the decimal moves one to the left, so the exponent would be increased by 1, giving 3.65×10^{-6}.

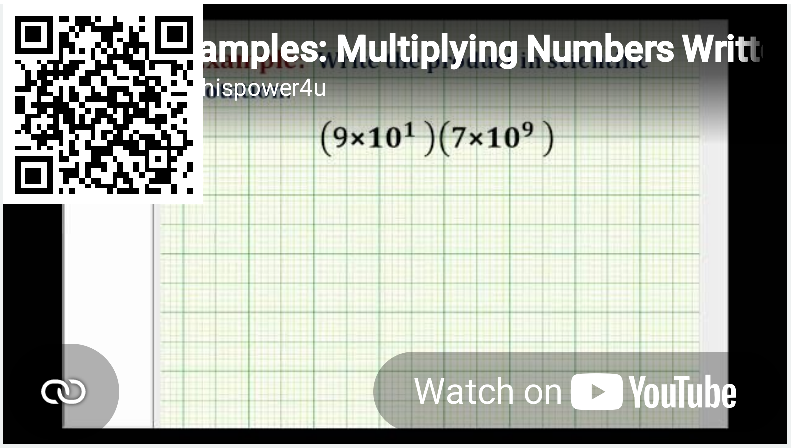
   $(5 \times 10^{-13}) \times (7.3 \times 10^6) = 3.65 \times 10^{-6}$

Your Turn 3.9.10

Calculate the following:

1. $(2.29 \times 10^3) \times (3 \times 10^4)$
2. $(6.91 \times 10^{-3}) \times (9.1 \times 10^5)$

Video


amples: Multiplying Numbers Writt
hispower4u
$(9\times10^1)(7\times10^9)$
Watch on YouTube

Multiplying Numbers in Scientific Notation

Dividing Numbers in Scientific Notation

To divide two numbers that are in scientific notation:

Step 1: Divide the number parts.

Step 2: Subtract the exponent of the denominator from the exponent of the numerator.

Step 3: The answer is the result from Step 1 times 10 raised to the result from Step 2.

Step 4: If the number is not in scientific notation, adjust it appropriately.

Example 3.9.11: Dividing Numbers in Scientific Notation

Calculate the following:

1. $(8.4\times10^{31})\ /\ (2.1\times10^{7})$
2. $(4.14\times10^{-13})\ /\ (8.28\times10^{9})$

Answer

1. **Step 2:** Subtract the exponent of the denominator from the exponent of the numerator to get $37-7=24$.

   Step 3: The result is then 4×10^{24}.

   Step 4: This number is already in scientific notation, so no adjustment is necessary. $(8.4\times10^{31})\ /\ (2.1\times10^{7})=4\times10^{24}$
2. **Step 2:** Subtract the exponent of the denominator from the exponent of the numerator to get $-13-9=-22$.

   Step 3: The result then is 0.5×10^{-22}.

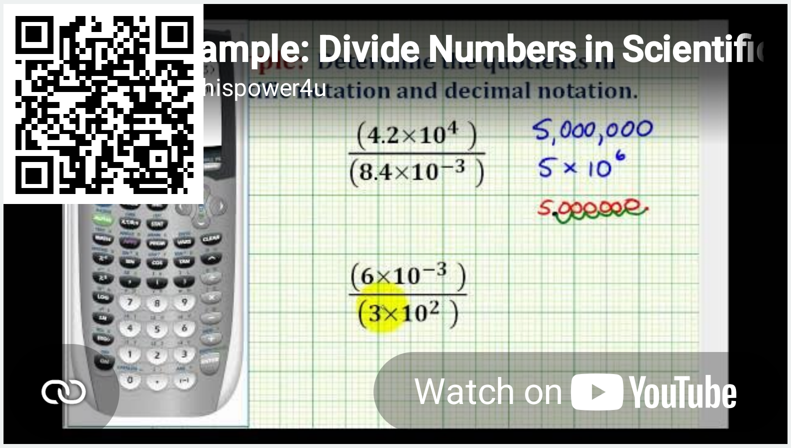
   Step 4: Since this number is not in scientific notation, it must be adjusted. To put 0.5×10^{-22} into scientific notation, the decimal needs to move one to the right, so the exponent is decreased by 1, giving 5×10^{-23}. $(4.14\times10^{-13})\ /\ (8.28\times10^{9})=5\times10^{-23}$

Your Turn 3.9.11

Calculate the following:

1. $\left(3.6\times10^{-2}\right)/\left(1.5\times10^{3}\right)$
2. $\left(1.8\times10^{4}\right)/\left(4.8\times10^{3}\right)$

Video



Dividing Numbers in Scientific Notation

Using Scientific Notation in Computing Real-World Applications

As noted at the start of this section, scientific notation is useful when the standard representation of a number is awkward or impractical, which occurs when the numbers being used are extremely large or extremely small. For example, Venus is 67,667,000 miles from the sun. In scientific notation, this is 6.7667×10^7. Planetary and galaxy distances is one set of numbers that is easier to express using scientific notation.

✓ Example 3.9.12: Calculating Distances

How much farther from the sun is Earth compared to Venus if Venus is 6.7667×10^7 miles from the sun and Earth is 9.1692×10^7 miles from the sun?

Answer

To determine how much farther Earth is compared to Venus, we'd subtract the distances. $9.1692 \times 10^7 - 6.7667 \times 10^7 = 2.4025 \times 10^7$.

So, Earth is 2.4025×10^7 miles farther from the sun than Venus.

Your Turn 3.9.21

Earlier we saw that a single transistor in a computer chip 0.000000014 meters, or 1.4×10^{-8} m, in size, and that the diameter of an atom could be 0.2 nanometers, or 2×10^{-10} m in size. How much larger is the transistor than the atom?

✓ Example 3.9.13: Calculating Probability

The probability of winning the Mega Millions lottery is published as 3.304693×10^{-9}. The probability of being hit by lightning is approximated to be 2×10^{-6}. How many times more likely are you to be hit by lightning than win the Mega Millions?

Answer

To find out how many times more likely you are to be hit by lightning, divide the probability of being hit by lightning by the probability of winning the Mega Millions.

Step 1: Divide the number parts to get = 0.6052 (rounded to the fourth digit).

Step 2: Subtract the exponent of the denominator from the exponent of the numerator to get $-6-(-9)=3$.

Step 3: The result then is 0.6052×10^3.

Step 4: Since this number is not in scientific notation, it must be adjusted. To put 0.6052×10^3 into scientific notation, the decimal needs to move one place to the right, so the exponent is decreased by 1, giving 6.052×10^2.

You are 6.052×10^2, or 605.2, times more likely to be hit by lightning than you are to win the Mega Millions.

Your Turn 3.9.31

Mercury is about 3.114×10^7 miles from the sun. Neptune is about 2.781×10^9 miles from the sun. How many times further is Neptune from the sun than Mercury?

Example 3.9.14: Calculating Time and Length

Sometimes it is entertaining to determine the time it takes for something to happen. Fingernails grow about 8.032×10^{-11} km per minute. How many kilometers long would fingernails be after 6×10^4 minutes?

Answer

To find the length of the fingernails after the specified time, we multiply their rate of growth and the time they've grown. $(8.032 \times 10^{-11}) \times (6 \times 10^4) = 48.192 \times 10^{-7} = 4.8192 \times 10^{-6}$

So, after 6×10^4 minutes, the fingernails would be 4.8192×10^{-6} km long. To put this in perspective, 1×10^{-6} km is a millimeter, and 6×10^4 minutes is about 4.16 days. So, after about 4.16 days, fingernails have grown about 4.8 millimeters.

Your Turn 3.9.14

There are approximately 1×10^{12} grains of sand in a cubic meter. If the number of grains of sand on the Australian coastline is roughly 7.5×10^{21} grains, roughly how many cubic meters of sand is there on the Australian coastline? of sand on the Australian coastline is roughly 7.5×10^{21} grains, roughly how many cubic meters of sand is there on the Australian coastline?

Example 3.9.15: Calculating Data Generated

As mentioned in the opening to this section, it is estimated that we're producing 2.5 quintillion bytes of data per day. A good estimate is that there are 7.674 billion people on the planet. Convert both of those numbers to scientific notation, and then determine how much data is being generated per person each day.

Answer

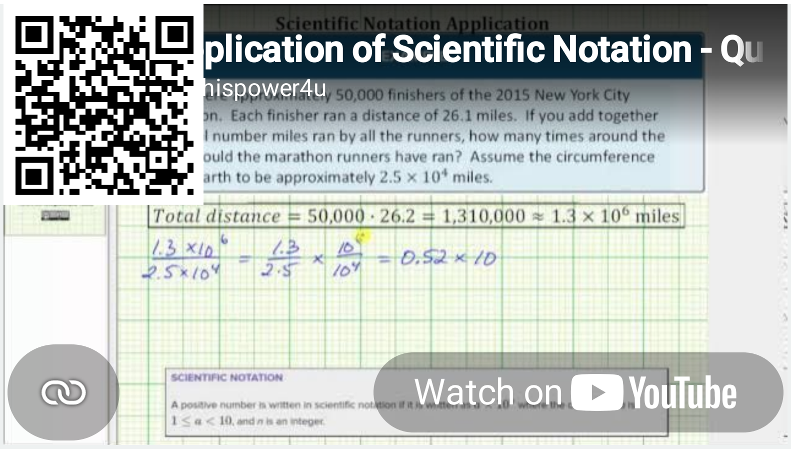
Written in standard form, 2.5 quintillion is 2,500,000,000,000,000,000. Changing that to scientific notation, move the decimal 18 places, so 2.5 quintillion bytes = 2.5×10^{18} bytes. Writing 7.647 billion in scientific notation would be 7.647×10^9 because a billion is 1,000,000,000 = 10^9. So, to find out how much data is being produced daily per person, we would divide these two numbers. $\frac{2.5 \times 10^{18}}{7.647 \times 10^9} = 0.327 \times 10^9 = 3.27 \times 10^8$

In standard form, that's 327,000,000 bytes per person, so 327 million bytes of data daily are being produced per person.

Your Turn 3.9.15

Humans collectively exhale approximately 6.4235×10^{12} pounds of carbon dioxide per year. There are approximately 7.647×10^9 humans currently living on Earth. How many pounds of carbon dioxide does a single human, on average, exhale per year?

Video


Scientific Notation Application
plication of Scientific Notation - Qu
hispower4u
Total distance = 50,000 · 26.2 = 1,310,000 ≈ 1.3 × 10⁶ miles
SCIENTIFIC NOTATION
Watch on YouTube

Application of Scientific Notation

What Numbers Could Be Considered “Too Big” or “Too Small”?

One wonders when the numbers we represent become too large or small for consideration. Perhaps the following examples put limits on what is meaningful. The number of particles in the known universe has been estimated at 4×10^{80} particles. The smallest distance that has been measured is $1 \times 10^{-}$18m1×10−18m, though the theoretical smallest measurable value is $1 \times 10^{-}$35m1×10−35m. The distance across the universe is $4.4 \times 10^{26}\ \mathrm{m}$. Considering what those numbers represent, the extreme largest and extreme smallest, they might be numbers that constrain what we should reasonably be expected to deal with.

Check Your Understanding

1. Write 0.00456 in scientific notation.
2. Write 5.67×10^{8} in standard form.
3. Calculate $4.5 \times 10^{3} + 9.8 \times 10^{2}$.
4. Calculate $2.5 \times 10^{5} - 9.8 \times 10^{6}$.
5. Calculate $(7.4 \times 10^{4}) \times (4.8 \times 10^{3})$.
6. Calculate \(\frac\).
7. The distance from Earth to the moon is 1.514×10^{10} inches. The thickness of a dollar bill is 4.3×10^{-3} inches. How many dollar bills must be stacked so the pile reaches the moon?

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3.9.0: Exercises

For the following exercises, convert numbers to scientific notation.

? Exercise $3.9.0.1$

0.0134

? Exercise $3.9.0.2$

0.0000761

? Exercise $3.9.0.3$

$3,400$

? Exercise $3.9.0.4$

$8,980,000$

For the following exercises, convert numbers to standard form.

? Exercise $3.9.0.5$

9.01×10^5

? Exercise $3.9.0.6$

3.78×10^7

? Exercise $3.9.0.7$

4.32×10^{-3}

? Exercise $3.9.0.8$

5.781×10^{-5}

For the following exercises, the numbers are not in scientific notation. Convert them to scientific notation.

? Exercise $3.9.0.9$

37.65×10^4

? Exercise $3.9.0.10$

0.0034×10^6

? Exercise $3.9.0.11$

0.0834×10^{-7}

Exercise 3.9.0.12

14.56×10^{-3}

For the following exercises, make the conversions required.

Exercise 3.9.0.13

The distance from the sun to the star Polaris is about 3,056,000,000,000,000 km. Express that distance in scientific notation.

Exercise 3.9.0.14

The distance from us to the next-closest galaxy is about 662,000,000,000,000,000 km. Express that distance in scientific notation.

Exercise 3.9.0.15

The mass of a grain of sand is about 6.66×10^{-4} g. Convert that mass to standard form.

Exercise 3.9.0.16

The diameter of a cell is about 2×10^{-6} m. Convert that diameter to standard form.

Exercise 3.9.0.17

The equatorial circumference of Earth is approximately 4.007×10^{4} km. Convert that circumference to standard form.

Exercise 3.9.0.18

The straight-line distance from Buffalo, NY, to Buenos Aires, Argentina, is approximately 8.86×10^{6} m. Convert that distance to standard form.

Exercise 3.9.0.19

The mass of a proton is approximately 1.67×10^{-27} kg. Convert that mass to standard form.

Exercise 3.9.0.20

The diameter of a housefly egg is approximately 1.2×10^{-3} m. Convert that diameter to standard form.

Exercise 3.9.0.21

The tallest building in the world, the Burj Khalifa in Dubai, stands at 829.8 m tall. Convert that height to scientific notation.

Exercise 3.9.0.22

Using the rings of the shell, the age of an Icelandic clam is 507 years. Express that age in scientific notation.

Calculate the following:

Exercise 3.9.0.23

$1.3 \times 10^{2} + 3.8 \times 10^{2}$

? Exercise 3.9.0.24

$7.8 \times 10^{12} + 1.1 \times 10^{12}$

? Exercise 3.9.0.25

$3.36 \times 10^{4} + 2.71 \times 10^{4}$

? Exercise 3.9.0.26

$4.58 \times 10^{9} + 1.93 \times 10^{9}$

? Exercise 3.9.0.27

$8.1 \times 10^{-17} + 1.6 \times 10^{-17}$

? Exercise 3.9.0.28

$4.506 \times 10^{-3} + 3.908 \times 10^{-3}$

? Exercise 3.9.0.29

$8.602 \times 10^{-25} + 1.096 \times 10^{-25}$

? Exercise 3.9.0.30

$2.0557 \times 10^{-6} + 1.001 \times 10^{-6}$

? Exercise 3.9.0.31

$5.2 \times 10^{4} - 4.1 \times 10^{4}$

? Exercise 3.9.0.32

$9.48 \times 10^{15} - 6.78 \times 10^{15}$

? Exercise 3.9.0.33

$7.81 \times 10^{-7} - 4.62 \times 10^{-7}$

? Exercise 3.9.0.34

$4.53 \times 10^{-3} - 2.79 \times 10^{-3}$

? Exercise 3.9.0.35

$4.6 \times 10^{-5} + 9.1 \times 10^{-5}$

? Exercise 3.9.0.36

$6.7 \times 10^{8} + 5.7 \times 10^{8}$

? Exercise 3.9.0.37

$4.13 \times 10^{-4} + 7.93 \times 10^{-4}$

? Exercise 3.9.0.38

$5.671 \times 10^{-8} + 9.073 \times 10^{-8}$

? Exercise 3.9.0.39

$4.513 \times 10^{6} + 7.856 \times 10^{4}$

? Exercise 3.9.0.40

$7.135 \times 10^{8} + 5.143 \times 10^{6}$

? Exercise 3.9.0.41

$3.17 \times 10^{-3} + 5.92 \times 10^{-5}$

? Exercise 3.9.0.42

$4.503 \times 10^{-6} + 3.119 \times 10^{-4}$

? Exercise 3.9.0.43

$(4.5 \times 10^{4}) \times (1.2 \times 10^{6})$

? Exercise 3.9.0.44

$(3.45 \times 10^{7}) \times (2.81 \times 10^{-3})$

? Exercise 3.9.0.45

$(3.1 \times 10^{8}) \times (2.7 \times 10^{-5})$

? Exercise 3.9.0.46

$(6.32 \times 10^{-4}) \times (1.31 \times 10^{-5})$

? Exercise 3.9.0.47

$(3.91 \times 10^{6}) \times (8.13 \times 10^{2})$

? Exercise 3.9.0.48

$(7.12 \times 10^{-11}) \times (6.61 \times 10^{-5})$

? Exercise 3.9.0.49

$(3.45 \times 10^{4}) \div (1.5 \times 10^{6})$

? Exercise $3.9.0.50$

$(1.4 \times 10^3) \div (5.6 \times 10^{-5})$

For the following exercises, apply your understanding of scientific notation to real-world applications.

? Exercise $3.9.0.51$

When stretched out, a strand of human DNA is, on average, 2.066×10^2 cm. One centimeter, or 1 cm, is 1×10^{-5} km. Determine the average length of a strand of human DNA in kilometers.

? Exercise $3.9.0.52$

One approximation of the average number of cells in the human body is 3×10^{13} cells (30 trillion!!!). If the DNA of each cell were stretched out and laid end to end, what would be the total length of the DNA in km? Use your answer from Exercise 51 for the length, in kilometers, of DNA.

? Exercise $3.9.0.53$

The equatorial circumference of Earth is approximately 4.007×10^4 km. Use the answer from Exercise 52 to determine the number of times that the stretched out human DNA would encircle Earth.

? Exercise $3.9.0.54$

The average stride length of a 1.651 m tall woman is 6.6×10^{-1} meters. If such a person could walk from Buffalo, NY, to Buenos Aires, Argentina, in a straight line, how many steps would that person need to take? See Exercise 18 for the distance from Buffalo, NY, to Buenos Aires, Argentina.

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3.10: Arithmetic Sequences

Learning Objectives

1. Identify arithmetic sequences.
2. Find a given term in an arithmetic sequence.
3. Find the nth term of an arithmetic sequence.
4. Find the sum of a finite arithmetic sequence.
5. Use arithmetic sequences to solve real- world applications

As we saw in the previous section, we are adding about 2.5 quintillion bytes of data per day to the Internet. If there are 550 quintillion bytes of data today, then there will be 552.5 quintillion bytes tomorrow, and 555 quintillion bytes in 2 days. This is an example of an arithmetic sequence. There are many situations where this concept of fixed increases comes into play, such as raises or table arrangements.

Identifying Arithmetic Sequences

A sequence of numbers is just that, a list of numbers in order. It can be a short list, such as the number of points earned on each assignment in a class, such as {10, 10, 8, 9, 10, 6, 10}. Or it can be a longer list, even infinitely long, such as the list of prime numbers. For example, here's a sequence of numbers, specifically, the squares of the first 12 natural numbers.

{1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144}

Each value in the sequence is called a term. Terms in the list are often referred to by their location in the sequence, as in the n

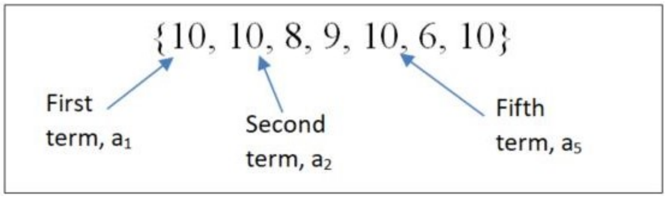


Figure 3.10.1: Sequence showing first, second, and fifth terms

The notation we use with sequences is a letter, which represents a term in the sequence, and a subscript, which indicates what place the term is in the sequence. For the sequence {1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144}, we will use the letter a as a value in the sequence, and so semanticsa5a5 would be the term in the sequence at the fifth position. That number is 25, so we can write a_5 =25a5=25.

In this section, we focus on a special kind of sequence, one referred to as an arithmetic sequence. Arithmetic sequences have terms that increase by a fixed number or decrease by a fixed number, called the constant difference (denoted by d), provided that value is not 0. This means the next term is always the previous term plus or minus a specified, constant value. Another way to say this is that the difference between any consecutive terms of the sequence is always the same value.

To see a constant difference, look at the following sequence: {7, 15, 23, 31, 39, 47, 55, 63, 71, 79, 87}. Figure 3.49 illustrates that each term of the sequence is the previous term plus 8. Eight is the constant difference here.

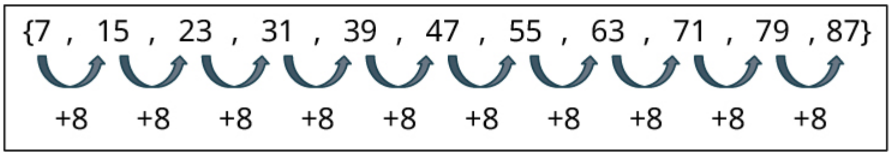


Figure 3.10.2: Sequence of numbers with 8 added to each term

Example 3.10.1 : Identifying Arithmetic Sequences

Determine if the following sequences are arithmetic sequences. Explain your reasoning.

1. $\{4, 7, 10, 13, 16, 19, 22, 25, \ldots\}$
2. $\{20, 40, 80, 160, 320, 640\}$
3. $\{7, 1, -5, -11, -17, -23, -29, -34, -40\}$

Answer

1. In the sequence $\{4, 7, 10, 13, 16, 19, 22, 25, \ldots\}$, every term is the previous term plus 3. The ellipsis indicates that the pattern continues, which means keep adding 3 to the previous term to get the new term. Therefore, this is an infinite arithmetic sequence.
2. In the sequence $\{20, 40, 80, 160, 320, 640\}$, terms increase by various amounts, for instance from term 1 to term 2, the sequence increases by 20, but from term 2 to term 3 the sequence increases by 40. So, this is not an arithmetic sequence.
3. In the sequence $\{7, 1, -5, -11, -17, -23, -29, -34, -40\}$, every term is the previous term minus 6, so this is an arithmetic sequence.

Your Turn 3.10.2

Determine if the following sequences are arithmetic sequences. Explain your reasoning.

$\{7.6, 5.4, 3.2, 1.0, -1.2, -3.4, -5.6, -7.8, -10.0\}$

$\{14, 16, 18, 22, 28, 40, 32, 0\}$

$\{14, 20, 26, 32, 38, 44, 50, 56, 62, 68, 74, 80, \ldots\}$

Arithmetic sequences can be expressed with a formula. When we know the first term of an arithmetic sequence, which we label semanticsa1a1, and we know the constant difference, which is denoted d, we can find any other term of the arithmetic sequence. The formula for the semanticsithith term of an arithmetic sequence is $a_i = a_1 + d \times (i - 1)$.

FORMULA

If we have an arithmetic sequence with first term semanticsa1a1 and constant difference d, then the semanticsithith term of the arithmetic sequence is $a_i = a_1 + d \times (i - 1)$.

Let's examine the formula with this arithmetic sequence: $\{4, 7, 10, 13, 16, 19, 22, 25, \ldots\}$. In this sequence a_1 =4a1=4 and $d = 3$. The table below shows the values calculated.

i, Place in Sequence	semanticsaiai, semanticsithith Term	Value of Term	Term Written as $a_1 + 3 \times (i - 1)$
1	semanticsa1a1	4	$4 + 3 \times 0$
2	semanticsa2a2	7	$4 + 3 \times 1$
3	semanticsa3a3	10	$4 + 3 \times 2$
4	semanticsa4a4	13	$4 + 3 \times 3$
5	semanticsa5a5	16	$4 + 3 \times 4$
i	semanticsaiai		$4 + 3 \times (i - 1)$

We can see how the semanticsithith term can be directly calculated. In this sequence, the formula is $a_1 + 3 \times (i - 1)$ where the first term, semanticsa1a1, is 4 and the constant difference d is 3. We can then determine the semantics47th47th term of this sequence: msuba47=4+3×(47−1)=4+3×46=4+138=142a47=4+3×(47−1)=4+3×46=4+138=142.

Example 3.10.3: Calculating a Term in an Arithmetic Sequence

Identify semanticsa1a1 and d for the following arithmetic sequence. Use this information to determine the semantics60th60th term.

$$\{18, 31, 44, 57, 70, 83, \ldots\}$$

Answer

Inspecting the sequence shows that a_1 =18a1=18 and d =13d=13. We use those values in the formula, with i =60i=60.

msub

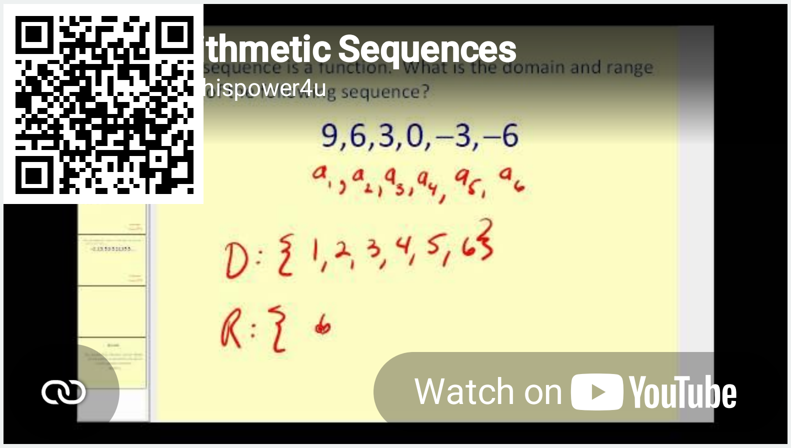
a60=a1+d×(i−1)=18+13×(60−1)=18+13×59=18+767=785a60=a1+d×(i−1)=18+13×(60−1)=18+13×59=18+767=785

Your Turn 3.10.3

Identify a_1 and d for the following arithmetic sequence. Use this information to determine the 86th term.

$\{4.5, 8.1, 11.7, 15.3, 18.9, 22.5, 26.1, \ldots\}$

Video



Arithmetic Sequences

If we know two terms of the sequence, it is possible to determine the general form of an arithmetic sequence, $a_i = a_1 + d \times (i - 1)$.

FORMULA

If we have the ith term of an arithmetic sequence, semanticsaiai, and the jth term of the sequence, semanticsajaj, then the constant difference is $d =$ mfracaij−id=aj−aij−i and the first term of the sequence is $a_1 = a_i - d(i - 1)$.

Example

3.10.4: Determining First Term and Constant Difference Using Two Terms

A sequence is known to be arithmetic. Two of its terms are a_7 =56a7=56 and msuba19=104a19=104. Use that information to find the constant difference, the first term, and then the semantics50th50th term of the sequence.

Answer

To find the constant difference, use $d =$ mfracaij−id=aj−aij−i. The location of the terms is given by the subscript of the two a terms, $i = 7$ and j =19j=19. So, the constant difference can be calculated as such:

$$d = \frac{104 - 56}{19 - 7} = \frac{48}{12} = 4$$

The constant difference of 4 is then used to find semanticsa1a1.

$$a_1 = a_i - d(i - 1) = a_7 - 4(7 - 1) = 56 - 4 \times 6 =$$

32a1=ai−d(i−1)=a7−4(7−1)=56−4×6=32.

So $d = 4$ and a_1 =32a1=32.

With this information, the semantics50th50th term can be found.

msub

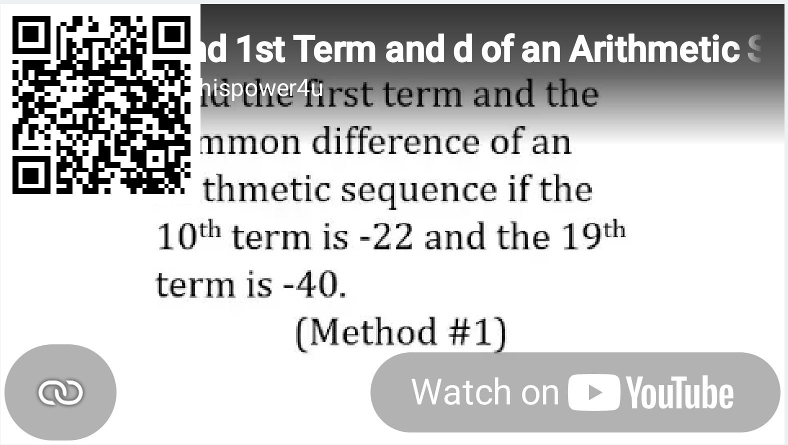
a50=a1+d×(i−1)=32+4×(50−1)=32+4×49=32+196=228a50=a1+d×(i−1)=32+4×(50−1)=32+4×49=32+196=228.

The semantics50th50th term is msuba50=228a50=228.

Your Turn 3.10.4

A sequence is known to be arithmetic. Two of the terms are $a_{14} = 41$ and $a_{38} = 161$. Use that information to find the constant difference and the first term. Then determine the 151st term of the sequence.

Video


d 1st Term and d of an Arithmetic S
hispower4u
d the first term and the
mmon difference of an
thmetic sequence if the
10th term is -22 and the 19th
term is -40.
(Method #1)
Watch on YouTube

Finding the First Term and Constant Difference for an Arithmetic Sequence

Finding the Sum of a Finite Arithmetic Sequence

Sometimes we want to determine the sum of the numbers of a finite arithmetic sequence. The formula for this is fairly straightforward.

FORMULA

The sum of the first n terms of a finite arithmetic sequence, written semanticssnsn, with first and last term semanticsa1a1 and semanticsanan, respectively, is $s_n = n($mfracan2)sn=n(a1+an2).

Example 3.10.5 : Finding the Sum of a Finite Arithmetic Sequence

What is the sum of the first 60 terms of an arithmetic sequence with $a_1 = 4.5$ and $d = 2.5$?

Answer

The formula requires the first and last terms of the sequence. The first term is given, $a_1 = 4.5$. The semantics60th60th term is needed. Using the formula $a_1 = a_i + d(i - 1)$ provides the value for the semantics60th60th term.

msub

a60=4.5+2.5(60−1)=4.5+2.5×59=4.5+147.5=152a60=4.5+2.5(60−1)=4.5+2.5×59=4.5+147.5=152.

Applying the formula $s_n = n($mfracan2)sn=n(a1+an2) provides the sum of the first 60 terms.

msub

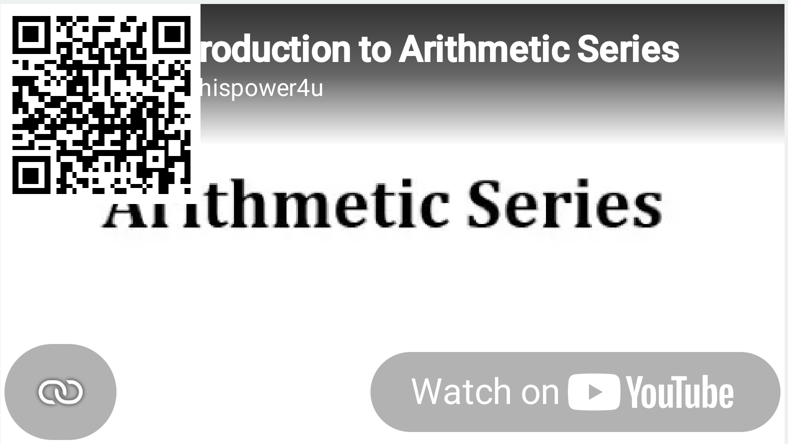
s60=60(4.5+1522)=60×156.52=4,695s60=60(4.5+1522)=60×156.52=4,695.

The sum of the first 60 terms is 4,695.

Your Turn 3.10.5

What is the sum of the first 101 terms of an arithmetic sequence with $a_1 = 13$ and $d = 2.25$?

Video



Finding the Sum of a Finite Arithmetic Sequence

Using Arithmetic Sequences to Solve Real-World Applications

Applications of arithmetic sequences occur any time some quantity increases by a fixed amount at each step. For instance, suppose someone practices chess each week and increases the amount of time they study each week. The first week the person practices for 3 hours, and vows to practice 30 more minutes each week. Since the amount of time practicing increases by a fixed number each week, this would qualify as an arithmetic sequence.

Example 3.10.6: Applying an Arithmetic Sequence

Jordan has just watched *The Queen's Gambit* and decided to hone their skills in chess. To really improve at the game, Jordan decides to practice for 3 hours the first week, and increase their time spent practicing by 30 minutes each week. How many hours will Jordan practice chess in week 20?

Answer

Jordan's practice scheme is an arithmetic sequence, as it increases by a fixed amount each week. The first week there are 3 hours of practice. This means a_1 =3a1=3. Jordan increases the time spent practicing by 30 minutes, or half an hour, each week. This means $d = 0.5$. Using those values, and that we want to know the amount of time Jordan will study in week 20, we determine the time in week 20 using $a_i = a_1 + d \times (i - 1)$.

$$a_{20} = 3 + 0.5 \times (20 - 1) = 3 + 0.5 \times 19 = 3 + 9.5 = 12.5$$

So, Jordan will practice 12.5 hours in week 20.

Your Turn 3.10.6

Christina decides to save money for after graduation. Christina starts by setting aside $10. Each week, Christina increases the amount she saves by $5. How much money will Christina save in week 52?

Example 3.10.7: Finding the Sum of a Finite Arithmetic Sequence

Let's check back in on Jordan. Recall, Jordan had just watched *The Queen's Gambit* and decided to hone their skills, practicing for 3 hours the first week, and increasing the time spent practicing by 30 minutes each week. How many hours total will Jordan have practiced chess after 30 weeks of practice?

Answer

To calculate the total amount of time that Jordan practiced, we need to use $s_n = n($mfrac an2)sn=n(a1+an2). The formula requires the first and last terms of the sequence. Since Jordan practiced 3 hours in the first week, the first term is a_1 =3a1=3. Because we want the total practice time after 30 weeks, we need the semantics30th30th term. Because the constant difference is $d = 0.5$, the semantics30th30th term is msuba30=3+0.5(30−1)=3+0.5×29=3+14.5=17.5a30=3+0.5(30−1)=3+0.5×29=3+14.5=17.5.

Applying the formula $s_n = n($mfrac an2)sn=n(a1+an2) provides the sum of the first 30 terms.

msub

s30=30(3+17.52)=60×20.52=615s30=30(3+17.52)=60×20.52=615.

This means that Jordan practiced a total of 615 hours after 30 weeks.

Your Turn 3.10.7

In a theater, the first row has 24 seats. Each row after that has 2 more seats. How many total seats are there if there are 40 rows of seat in the theater?

Who Knew?: The Fibonacci Sequence

Not all sequences are arithmetic. One special sequence is the Fibonacci sequence, which is the sequence that has as its first two terms 1 and 1. Every term thereafter is the sum of the previous two terms. The first nine terms of the Fibonacci sequence are 1, 1, 2, 3, 5, 8, 13, 21, and 34.

This sequence is found in nature, architecture, and even music! In nature, the Fibonacci sequence describes the spirals of sunflower seeds, certain galaxy spirals, and flower petals. In music, the band Tool used the Fibonacci sequence in the song “Lateralus.” The Fibonacci sequence even relates to architecture, as it is closely related to the golden ratio.

Video

w Tool Used Math to Create "Latera
phonic
MATHEMATICS OF
LATERALUS
0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, 10946, 17711, 28657, 46368, 75025, 121393, 196418, 317811
Watch on YouTube

Fibonacci Sequence and “Lateralus”

Check Your Understanding

1. Is the following an arithmetic sequence? Explain.

{3, 6, 9, 15, 25, 39, 90}

2. What is the 7th term of the following sequence?

{1, 5, 7, 100, 4, -17, 8, 100, 19, 7.6, 345}

3. In an arithmetic sequence, the first term is 10 and the constant difference is 4.5. What is the 135th term?

4. If the eighth term of an arithmetic sequence is 35 and the 40th term is 131, what is the constant difference and the first term of the sequence?

5. What is the sum of the first 100 terms of the arithmetic sequence with first term 4 and constant difference 7?

6. A new marketing firm began with 30 people in its survey group. The firm adds 4 people per day. How many people will be in their survey group after 100 days?

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3.10.0: Exercises

For the following exercises, determine if the sequence is an arithmetic sequence.

Exercise $3.10.0.1$

$\{3, 7, 11, 15, 25, 100, \ldots\}$

Exercise $3.10.0.2$

$\{27, 24, 21, 18, 15, 12, 9, \ldots\}$

Exercise $3.10.0.3$

$\{6, -1, -8, -15, -23, -31, -39, \ldots\}$

Exercise $3.10.0.4$

$\{-5, 4, 13, 22, 31, 40, 49, 58, 67, \ldots\}$

Exercise $3.10.0.5$

$\{14, 19, 24, 29, 34, 50, 60\}$

Exercise $3.10.0.6$

$\{3.9, 2.3, 0.7, -0.9, -2.5, -4.1, -5.7, \ldots\}$

Exercise $3.10.0.7$

$\{4, -8, 12, -16, 20, -24, 28, -32, \ldots\}$

Exercise $3.10.0.8$

$\{1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots\}$

For the following exercises, the sequences given are arithmetic sequences. Determine the constant difference for each sequence. Verify that each term is the previous term plus the constant difference.

Exercise $3.10.0.9$

$\{18, 68, 118, 168, 218, 268, \ldots\}$

Exercise $3.10.0.10$

$\{13, 35, 57, 79, 101, 123, 145, 167, \ldots\}$

Exercise $3.10.0.11$

$\{14, 11, 8, 5, 2, -1, -4, \ldots\}$

Exercise 3.10.0.12

$\{4.5, 1.9, -0.7, -3.3, -5.9, \ldots\}$

Exercise 3.10.0.13

$\{-27, -13, 1, 15, 29, 43, 57, 71, \ldots\}$

Exercise 3.10.0.14

$\{3.8, 10.6, 17.4, 24.2, 31, 37.8, 44.6, \ldots\}$

For the following exercises, the first term and the constant difference of an arithmetic sequence is given. Using that information, determine the indicated term of the sequence.

Exercise 3.10.0.15

$a_1 = 12$, $d = 11$, find a_{20}.

Exercise 3.10.0.16

$b_1 = 5$, $d = 8$, find b_{38}.

Exercise 3.10.0.17

$c_1 = 48$, $d = -7$, find c_{50}.

Exercise 3.10.0.18

$a_1 = 110$, $d = -16$, find a_{27}.

Exercise 3.10.0.19

$t_1 = 15.3$, $d = 4.2$, find t_{17}.

Exercise 3.10.0.20

$b_1 = 23.8$, $d = 11.7$, find b_{120}.

Exercise 3.10.0.21

$b_1 = 27.45$, $d = -3.67$, find b_{40}.

Exercise 3.10.0.22

$a_1 = 67.4$, $d = -12.3$, find a_{200}.

For the following exercises, two terms of an arithmetic sequence are given. Using that information, identify the first term and the constant difference.

Exercise 3.10.0.23

$a_5 = 27$, $a_{15} = 77$

? Exercise 3.10.0.24

$b_{10} = 47$, $b_{25} = 137$

? Exercise 3.10.0.25

$a_9 = 38$, $a_{45} = 189.2$

? Exercise 3.10.0.26

$a_6 = 43$, $a_{41} = -377$

? Exercise 3.10.0.27

$a_4 = -12.3$, $a_{54} = -106.5$

? Exercise 3.10.0.28

$a_{12} = 45.9$, $a_{60} = -563.7$

For the following exercises, the first term and the constant difference is given for an arithmetic sequence. Use that information to find the sum of the first n terms of the sequence, s_n.

? Exercise 3.10.0.29

$a_1 = 15$, $d = 7$, calculate s_{10}

? Exercise 3.10.0.30

$a_1 = 2$, $d = 13$, calculate s_{20}.

? Exercise 3.10.0.31

$a_1 = 105$, $d = 0.3$, calculate s_{15}.

? Exercise 3.10.0.32

$a_1 = 56.2$, $d = 1.1$, calculate a_{35}.

? Exercise 3.10.0.33

$a_1 = 450$, $d = -20$, calculate s_{20}.

? Exercise 3.10.0.34

$a_1 = 1400$, $d = -35$, calculate s_{40}.

For the following exercises, apply your knowledge of arithmetic sequences to these real-world scenarios.

? Exercise 3.10.0.35

A collection is taken up to support a family in need. The initial amount in the collection is $135. Everyone places $20 in the collection. When the 35th person puts their $20 in the collection, how much is present in the collection?

Exercise $3.10.0.36$

There are 50 songs on a playlist. Every minute, 3 more songs are added to the playlist. How many songs are on the playlist after 40 minutes have passed?

Exercise $3.10.0.37$

One genre on Netflix has 1,000 shows. Every week, 20 shows are added to that genre. After 15 weeks, how many shows are in that genre?

Exercise $3.10.0.38$

A new local band has 10 people come to their first show. News of the band spreads afterwards. Each week, 4 more people attend their show than the previous week. After

50 weeks, how many people are at their show?

Exercise $3.10.0.39$

The Jester Comic book store is going out of business and is taking in no new inventory. Its inventory is currently 13,563 titles. Each day after, they sell or give away 250 titles. After 15 days, how many titles are left?

Exercise $3.10.0.40$

Jasmyn has decided to train for a marathon. In week one, Jasmyn runs 5 miles. Each week, Jasmyn increased the running distance by 2 miles. How many miles will Jasmyn run in week 13 of the training schedule?

Exercise $3.10.0.41$

A 42-gallon bathtub sits with 14 gallons in it. The faucet is turned on and is now being filled at the rate of 2.2 gallons per minute, but is draining slowly, at 1.8 gallons per minute. After 20 minutes, how many gallons are in the tub?

Exercise $3.10.0.42$

A trained diver is 250 feet deep. The diver is nearly out of air and needs to surface. However, the diver can only comfortably ascend 30 feet per minute. How deep is the diver after ascending for 5 minutes?

Exercise $3.10.0.43$

Jaclyn, an investor, begins a start-up to revitalize homes in South Bend, Indiana. She begins with $10,000, making her investor 1. Each investor that joins will invest $500 more than the previous investor. How much does the 50th investor invest in the project? With that 50th investor, what is the total amount invested in the project?

Exercise $3.10.0.44$

Jasmyn has decided to train for a marathon. In week one, Jasmyn runs 5 miles. Each week, Jasmyn increased the running distance by 2 miles. After training for 14 weeks, how many total miles will Jasmyn have run?

Exercise $3.10.0.45$

The base of a pyramidal structure has 144 blocks. Each level above has 5 fewer blocks than the previous level. How many total blocks are there if the pyramidal structure has 25 levels?

? Exercise 3.10.0.46

As part of a deal, a friend tells you they will give you $10 on day 1, $20 on day 2, $30 on day 3, for all 30 days of a month. At the end of that month, what is the total amount your friend has given you?

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3.11: Geometric Sequences

Learning Objectives

1. Identify geometric sequences.
2. Find a given term in a geometric sequence.
3. Find the nth term of a geometric sequence.
4. Find the sum of a finite geometric sequence.
5. Use geometric sequences to solve real-world applications.

One of the concerns when investing is the doubling time, which is length of time it takes for the value of the investment to be twice, or double, that of its starting value. A shorter doubling times means the investment gets bigger, sooner. For example, if you invest $\$200$ in an account with an 8 -year doubling time, then in 8 years the value of the account will be double the starting amount, or $2 \times \$200 = \$400$. After another 8 years (for a total of 16 years) the investment would be twice its value after the first 8 years, or $2 \times (2 \times \$400) = 2 \times (\$400) = \$800$. Every 8 years, the investment would double again, so after the third 8 -year period, the investment would be worth $2 \times 2 \times (2 \times \$400) = \$1,600$. This process exhibits exponential growth, an application of geometric sequences, which is explored in this section.

Figure 3.11.1: Savings grows in a geometric sequence. (credit: modification of "A big part of financial freedom is having your heart and mind free from worry about the what-ifs of life. – Suze Orman" by Morgan/Flickr, CC BY 2.0)

Identifying Geometric Sequences

We know what a sequence is, but what makes a sequence a geometric sequence? In an arithmetic sequence, each term is the previous term plus the constant difference. So, you add a (possibly negative) number at each step. In a geometric sequence, though, each term is the previous term multiplied by the same specified value, called the common ratio. In the sequence $\{3, 6, 12, 24, 48, 96, 192, 384, 728, 1456\}$

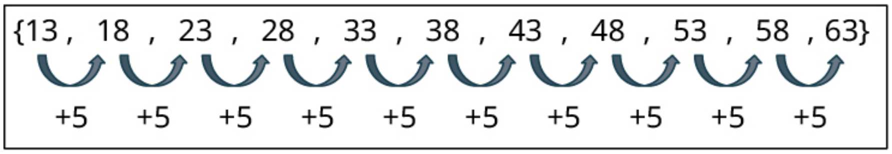


Figure 3.11.2: Arithmetic sequence

Each term in this arithmetic sequence is the previous term plus 5.

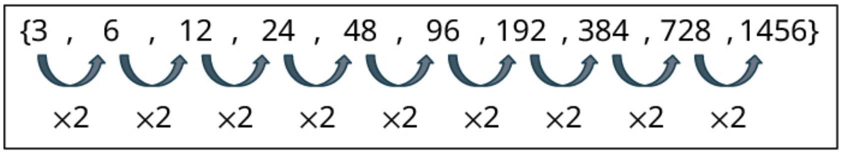


Figure 3.11.3: Geometric sequence

Each term in this geometric sequence is the previous term times 2.

In the sequence $\{3, 6, 12, 24, 48, 96, 192, 384, 728, 1456\}$, the numbers get big fairly quickly, and stay positive. However, that's not always the case with geometric sequences. Depending on the value of the common ratio, the terms could increase each time (like in the one shown in Figure 3.51), or the terms can get smaller each time, or the terms can alternate between positive and negative values. It all depends on the value of the common ratio, r.Math input error

Consider this geometric sequence:

$$\{5, 15, 45, 135, 405, 2025 ...\}$$

Each term is the previous term times 5, which means the common ratio is 5. This common ratio is larger than 1, and so the terms increase each time. Now, look at this geometric sequence:

$$\{2, -6, 18, -54, 162, -486, 1458 ...\}$$

Each term is the previous term times −3, and the sign of the terms alternate from positive to negative. Then, there's this geometric sequence:

$$\left\{9, 3, 1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27} ...\right\}$$

Each term is the previous term times $\frac{1}{3}$, and the terms decrease each time. What we should take away from these three examples is if the common ratio is a positive number larger than 1, then the sequence increases. If the common ratio is a negative number, then the sign of the terms alternates between positive and negative. If the common ratio is between 0 and 1, then the terms decrease.

Two special cases of geometric sequences are when the constant ratio is 1 and when the common ratio is 0. When the constant ratio is 1, every term of the sequence is the same, as in $\{3, 3, 3, 3, 3, 3, 3, 3, 3\}$. This is referred to as a constant sequence. When the constant ratio is 0, the first term can be any number, but every term after the first term is 0, as in $\{-43.2, 0, 0, 0, 0, 0, 0, 0\}$.

✓ Example ✓ 3.11.1: Identifying Geometric Sequences

For each sequence, determine if the sequence is a geometric sequence. If so, identify the common ratio.

1. $\{5, 20, 80, 320, 1{,}280, 5{,}120, 20{,}480, ...\}$
2. $\{-3, 6, -12, 24, 11, 33\}$
3. $\{4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...\}$

Answer

1. In the sequence $\{5, 20, 80, 320, 1{,}280, 5{,}120, 20{,}480, ...\}$, the jump from 5 to 20 is a multiplication by 4, as is the next jump to 80, and the next to 320. Each term is 4 times the previous term. Since each term is 4 times the previous, this is a geometric sequence. The common ratio is 4.
2. In the sequence $\{-3, 6, -12, 24, 11, 33\}$, notice that 6 is −3 times −2. The jump from 6 to −12 is another multiplication by negative. So, if this is a geometric sequence, each term should be the previous term times −2. But the change from 24 to 11 is not a multiplication by −2, This means the sequence is not a geometric sequence.
3. In the sequence $\{4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...\}$, the change from 4 to 2 is a multiplication by $\frac{1}{2}$, as is the next jump, from 2 to 1, as is the next from 1 to $\frac{1}{2}$. Each term is $\frac{1}{2}$ times the previous term. Since each term is $\frac{1}{2}$ times the previous, this is a geometric sequence. The common ratio is $\frac{1}{2}$.

Your Turn 3.11.1

For each sequence, determine if the sequence is a geometric sequence. If so, identify the common ratio.

1. $\{-1, -5, -25, -125, -625, -3125, , ...\}$
2. $\{-3, 6, -12, 24, 11, 33\}$
3. $\{-500, 50, -5, \frac{1}{2}, -\frac{1}{2}0, ...\}$

As with arithmetic sequences, the first term of a geometric sequence is labeled semanticsa1a1. The number that is multiplied by each term is called the common ratio and is denoted r. So, if the first term is known, a_1, and the common ratio is known, r, then the semanticsnthnth term, semanticsanan, can be calculated with the formula a_n =a1rn−1an=a1rn−1.

FORMULA

The nth term of the geometric sequence, semanticsanan, with first term semanticsa1a1 and common ratio r, is a_n =a1rn−1an=a1rn−1.

Return to the sequence $\{3, 6, 12, 24, 48, 96, 192, 384, 728, \ldots\}$. We observe that the first term is 3, so a_1 =3a1=3. We also found that the common ratio is 2, so $r = 2$. The table below shows how any term can be calculated using just semanticsa1a1 and r.

i, Place in Sequence	a_i,ithai,ith,Term	Value of Term	Term Written as a1×ri−1a1×ri−1
1	semanticsa1a1	3	3×2^0
2	semanticsa2a2	6	3×2^1
3	semanticsa3a3	12	3×2^2
4	semanticsa4a4	24	3×2^3
5	semanticsa5a5	48	3×2^4
i	semanticsaiai		3×2i−13×2i−1

✓ Example $3.11.2$: Determining the Value of a Specific Term in a Geometric Sequence

In the following geometric sequences, determine the indicated term of the geometric sequence with a given first term and common ratio.

1. Determine the semantics9th term of the geometric sequence with semanticsa1×6a1×6 and $r = 3$.
2. Determine the semantics11th term of the geometric sequence with a_1 =2a1=2 and $r = -5$.

Answer

1. Using a_n =a1rn−1an=a1rn−1 with a_1 =6a1=6, $r = 3$, and $n = 9$, we calculate

   $$a_9 =$$

   a1r9−1=6×(3)9−1=6×(3)8=6×6561=39366a9=a1r9−1=6×(3)9−1=6×(3)8=6×6561=39366

   .

   The semantics9th9th term of the geometric sequence with a_1 =6a1=6 and $r = 3$ is a_9 =39366a9=39366.
2. Using a_n =a1rn−1an=a1rn−1 with a_1 =2a1=2, $r = -5$, and $n =$, we calculate

   msub

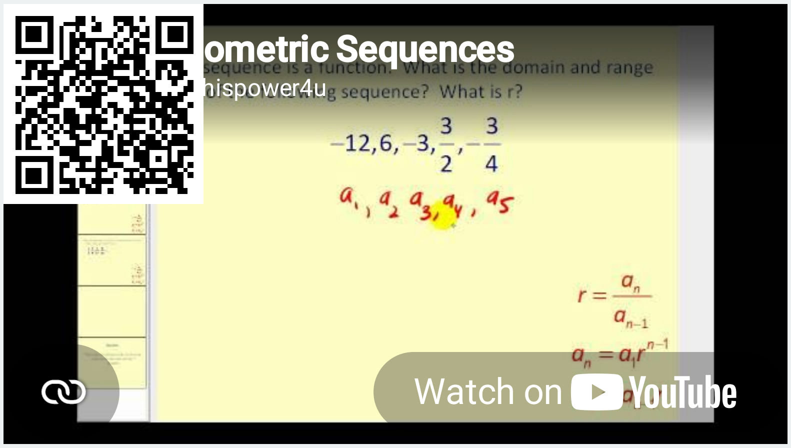
   a11=a1r11−1=2×(−5)11−1=2×(−5)10=2×9,765,625=19,531,250a11=a1r11−1=2×(−5)11−1=2×(−5)10=2×9,765,625=19,531,250

Your Turn $3.11.2$

In the following geometric sequences, determine the indicated term of the geometric sequence with a given first term and common ratio.

1. Determine the 12th
2. Determine the 5th

Video


ometric Sequences
hispower4u
sequence is a function. What is the domain and range
g sequence? What is r?
$-12, 6, -3, \frac{3}{2}, -\frac{3}{4}$
a_1, a_2, a_3, a_4, a_5
$r = \frac{a_n}{a_{n-1}}$
$a_n = a_1 r^{n-1}$
Watch on YouTube

Geometric Sequences

Finding the Sum of a Finite Geometric Sequence

As with arithmetic sequences, it is possible to add the terms of the geometric sequence. Like arithmetic sequences, the formula for the finite sum of the terms of a geometric sequence has a straightforward formula.

FORMULA

The sum of the first n terms of a finite geometric sequence, written semanticssnsn, with first term semanticsa1a1 and common ratio r, is $s_n = a_1\left(\frac{1-r^{n-1}}{1-r}\right)$ provided that $r \neq 1$.

Example 3.11.3: Calculating the Sum of a Finite Geometric Sequence

1. What is the sum of the first 13 terms of the geometric sequence with first term $a_1 =$5 and common ratio $r = 3$?
2. What is the sum of the first 7 terms of the geometric sequence with first term $a_1 =$16 and common ratio $r =$18?

Answer

1. The sum of the first 13 terms of this geometric sequence is 1,328,600.
2. Using $a_1 =$16, $r =$18, and $n = 7$, we find that the sum is:

$$s_7 = a_1\left(\frac{1-r^{n-1}}{1-r}\right) = 16\times\left(\frac{1-\left(\frac{1}{8}\right)^{7-1}}{1-\left(\frac{1}{8}\right)}\right) = 16\times\left(\frac{1-\left(\frac{1}{8}\right)^{6}}{\frac{7}{8}}\right) = 16\times\left(\frac{1-\frac{1}{262,144}}{\frac{7}{8}}\right) = 16\times\left(\frac{\frac{262,143}{262,144}}{\frac{7}{8}}\right) \\ = 16\times\left(\frac{262,143}{229,376}\right) = \frac{262,143}{14,336} = 18.2856 \tag{3.11.1}$$

The sum of the first 7 terms of this geometric sequence is 18.2856.

Your Turn 3.11.3

1. What is the sum of the first 10 terms of the geometric sequence with first term $a_1 = 7''style = ''font - size : 1.1rem;$
2. What is the sum of the first 6 terms of the geometric sequence with first term $a_1 = 27$

Using Geometric Sequences to Solve Real-World Applications

Geometric sequences have a multitude of applications, one of which is compound interest. Compound interest is something that happens to money deposited into an account, be it savings or an individual retirement account, or IRA. The interest on the account is calculated and added to the account at regular intervals. This means the interest that was earned later gains its own interest. This allows the money to grow faster. If that interest is added every month, we say it is compounded monthly. If the interest is added daily, then we say it is compounded daily. The amount of money that is deposited into the account is called the principal and is denoted P. The account earns money on that principal. The amount it earns is a percentage of the money in the account. The interest rate, expressed as a decimal, is denoted r.

FORMULA

If you deposit P dollars in an account that earns interest compounded yearly, then the amount in the account, A, after t years is calculated with the formula: $A = P(1+r)^t$. This is a geometric sequence, with constant ratio $(1+r)$ and first term a_1 =Pa1=P.

Example 3.11.4: Calculating Interest Compounded Yearly

Daryl deposits $1,000 in an account earning 4% interest compounded yearly. How much money is in the account after 25 years?

Answer

Using $A = P(1+r)^t$ with P =1000, r $= 0.04$, and t =25, we find that $A = P(1+r)^t = 1{,}000 \times (1+0.04)^{25} = 1{,}000 \times (1.04)^{25} = 1{,}000 \times 2.66583633 = 2{,}665.85$. After 25 years, there is $\$2{,}665.84$ in the account.

Your Turn 3.11.4

Sophia deposited $4,000 in an account that earns 5.5% interest compounded yearly. After 20 years, Sophia withdrew all the money in the account to pay for her child's college. How much money was in the account when Sophia withdrew the money?

Another application of geometric sequences is exponential growth. This arises in biology quite frequently, especially in relation to bacterial cultures, but also with other organism population models. In bacterial cultures, the time it takes the population to double is often recorded. This time to double is the same, regardless of how big the population gets. So, if the population doubles after 3 hours, it doubles again after another 3 hours, and again after another 3 hours, and so on. Put into geometric sequence language, it has a common ratio of 2.

Example 3.11.5: Doubling a Bacterial Culture

When *Escherichia coli* (*E. coli*) is in a broth culture at 37°C, the population of *E. coli* doubles in number with 30 organisms, how many *E. coli* bacteria are present in the culture after 16 hours?

Answer

Since the population is doubling every 20 minutes, this is a geometric sequence situation with common ratio $r = 2$. The culture begins with 30 organisms, so a_1 =30a1=30. The time,16 hours, is 48 twenty-minute periods, so we're looking for the 48th term in the sequence. Using these values in the geometric sequence formula gives

msuba48=a1rn−1=30×248−1=30×247=30×(1.40737×1014)=4.22212×1015a48=a1rn−1=30×248−1=30×247=30×(1.40737×1014)=4.22212×1015.

So, after 16 hours, the culture contains 4.22212×10^{15} *E. coli* organisms. That's more than 4,000 trillion bacteria.

Your Turn 3.11.5

When *Streptococcus lactis* (*S. lactis*) is in a milk culture at 37°C, the population of *S. lactis* doubles in number every 30 minutes. If the culture began with 15 organisms, how many *S. lactis* bacteria are present in the culture after 20 hours?

Example 3.11.6: Applying the Sum of a Finite Geometric Sequence

A player places one grain of rice on the first square of a chess board. On the second square, the player places 2 grains of rice. On the third square, the player places 4 grains of rice. On each successive square of the board, the player doubles the number of grains of rice placed on the chess board. When the player places the last rice on the 64th square, how many total grains of rice have been placed on the board?

Answer

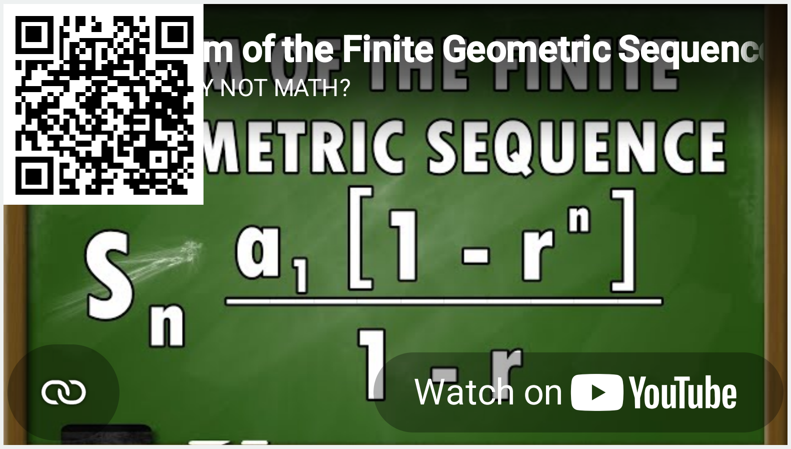
Since the number of grains of rice is doubled at each step, this is a geometric sequence with first term a_1 =1a1=1 and common ratio $r = 2$. Rice is placed on 64 total squares, so we want the sum of the first 64 terms. Using this information and the formula, the total number of grains of rice on the board will be: msubs64=a1(1−rn−11−r)=1×(1−264−11−2)=(1−263−1)=−(1−263)=−(−9.2233720369×1018)=9.2233720369×1018s64=a1(1−rn−11−r)=1×(1−264−11−2)=(1−263−1)=−(1−263)=−(−9.2233720369×1018)=9.2233720369×1018

That's a 20-digit number!

Your Turn 3.11.6

You have a square 1 meter on each side. You begin by coloring one half of the square blue. Then you color half the remaining area blue. Then you color half the remaining area blue once more. At each step, you color half the remaining area. What is the total area you have colored blue after performing this process 15 times?

Video


m of the Finite Geometric Sequenc
Y NOT MATH?
METRIC SEQUENCE
$S_n = \frac{a_1[1 - r^n]}{1 - r}$
Watch on YouTube

Sum of a Finite Geometric Sequence

Check Your Understanding

1. Is the following a geometric sequence? Explain. {3, 6, 12, 24, 48, 96, 192}
2. Find the common ratio of the geometric sequence {3, −30, 300, −3,000, …}.
3. In a geometric sequence, the first term is 10 and the common ratio is 1.5. What is the 15th term?
4. What is the sum of the first 100 terms of the geometric sequence with first term 4 and common ratio 0.3?
5. $15,000 is deposited in an account the yields 4.2% interest compounded annually. How much is in the account after 17 years?

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3.11.0: Exercises

For the following exercises, determine if the sequence is a geometric sequence.

Exercise 3.11.0.1

$\{3, 7, 11, 15, 25, 100, \ldots\}$

Exercise 3.11.0.2

$\{2, 4, 8, 16, 32, \ldots\}$

Exercise 3.11.0.3

$\{9, 0.9, 0.09, 0.009, 0.00009, \ldots\}$

Exercise 3.11.0.4

$\{262{,}144, 65{,}536, 16{,}384, 4{,}096, 1{,}024, \ldots\}$

Exercise 3.11.0.5

$\{14, 19, 24, 29, 34, 50, 60\}$

Exercise 3.11.0.6

$\{3.9, 2.3, 0.7, -0.9, -2.5, -4.1, -5.7, ..\}$

Exercise 3.11.0.7

$\{4, -8, 16, -32, 64, -128, 256, \ldots\}$

Exercise 3.11.0.8

$\{8, -4, 2, -1, 0.5, -0.25, 0.125, -0.0625, \ldots\}$

For the following exercises, the sequences given are geometric sequences. Determine the common ratio for each. Verify that each term is the previous term times the common ratio.

Exercise 3.11.0.9

$\{3, 6, 12, 24, 48, 96, \ldots\}$

Exercise 3.11.0.10

$\{8, 24, 72, 216, 648, 1944, \ldots\}$

Exercise 3.11.0.11

$\{15, 3, 0.6, 0.12, 0.024, 0.0048, 0.00096, \ldots\}$

Exercise 3.11.0.12

$\{52, 26, 13, 6.5, 3.25, 1.625, 0.8125, 0.40265 \ldots\}$

Exercise 3.11.0.13

$\{18, -18, 18, -18, 18-18, \ldots\}$

Exercise 3.11.0.14

$\{48, -12, 3-0.75, 0.1875, -0.046875, \ldots\}$

For the following exercises, the first term and the common ratio of a geometric sequence is given. Using that information, determine the indicated term of the sequence.

Exercise 3.11.0.15

$a_1 = 5, r = 3$

Exercise 3.11.0.16

$b_1 = 7$

Exercise 3.11.0.17

$c_1 = 11, r = 4$

Exercise 3.11.0.18

$a_1 = 2, r = 7$

Exercise 3.11.0.19

$t_1 = 100, r = \frac{1}{5}$

Exercise 3.11.0.20

$b_1 = 56, r = 0.25$

Exercise 3.11.0.21

$b_1 = 13, r = -2$

Exercise 3.11.0.22

$a_1 = 11, r = -3$

Exercise 3.11.0.23

$a_1 = 12, r = -\frac{1}{3}$

Exercise 3.11.0.24

$a_1 = 100$, $r = -10$

For the following exercises, the first term and the common ratio is given for a geometric sequence. Use that information to find the sum of the first n

Exercise 3.11.0.25

$a_1 = 3$

Exercise 3.11.0.26

$a_1 = 5$, $r = 3$

Exercise 3.11.0.27

$a_1 = 4$, $r = 5$

Exercise 3.11.0.28

$a_1 = 48$, $r = 2$

Exercise 3.11.0.29

$a_1 = 450$, $r = 0.5$

Exercise 3.11.0.30

$a_1 = 300$, $r = 0.25$

Exercise 3.11.0.31

$a_1 = 3$, $r = -2$

Exercise 3.11.0.32

$a_1 = 5$, $r = -4$

For the following exercises, apply your understanding of geometric sequences to real-world applications.

Exercise 3.11.0.33

Lactobacilius acidophilus (*L. acidophilus*) is a bacterium that grows in milk. In optimal conditions, its population doubles every 26 minutes. If a culture starts with 20 *L. acidophilus* bacteria, how many bacteria will there be after 390 minutes? Hint: This means the 26-minute time period has occurred 15 times.

Exercise 3.11.0.34

Bacillus megaterium (*B. megaterium*) is a bacterium that grows in sucrose salts. In optimal conditions, its population doubles every 25 minutes. If a culture starts with 30 *B. megaterium* bacteria, how many bacteria will there be after 1,000 minutes? Hint: This means the 25-minute time period has occurred 40 times.

? Exercise 3.11.0.35

Alex and Jill deposit $4,000 in an account bearing 5% interest compounded yearly. If they do not deposit any more money in that account, how much will it be worth in 30 years?

? Exercise 3.11.0.36

Kerry and Megan deposit $6,000 dollars in and account bearing 4% compounded yearly. If they do not deposit any more money in that account, how much will be in the account after 40 years?

? Exercise 3.11.0.37

You decide to color a square that measures 1 m on each side in a very particular manner. You first cut the square in half vertically. You color one side of the square with purple. On the side of the square that was not colored, you draw a line dividing that region horizontally exactly in half. You color the lower half blue. Now, you cut the remaining quarter of the square precisely in half with a vertical line. You color the left side red. You repeat this process 12 times. After you color that 12th piece, what is the total area you have colored?

? Exercise 3.11.0.38

Consider the geometric sequence with first term 0.9 and common ratio of 0.1. What is the sum of the first 5 terms?

? Exercise 3.11.0.39

Repeat Exercise 38, for the sum of the first 10 terms.

For the following questions, recall that the formula for interest compounded yearly is $A = P(1+r)^t$

? Exercise 3.11.0.40

Returning to Kerry and Megan (Exercise 36), what would their account be worth if their account was compounded monthly?

? Exercise 3.11.0.41

Returning to Alex and Jill (Exercise 35), what would their account be worth if their account was compounded monthly?

? Exercise 3.11.0.42

Imagine your family tree. You have two parents. Your parents have two parents: your grandparents. And so on. How many great-great-great-great-grandparents do you have? Hint: This would be six generations back.

? Exercise 3.11.0.43

Imagine your family tree. You have two parents. Your parents have two parents: your grandparents. And so on. How many great (20 times) grandparents do you have? Hint: This would be 22 generations back.

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SECTION OVERVIEW

3.12: Chapter Summary

3.12.0: Key Terms

3.12.1: Key Concepts

3.12.2: Videos

3.12.3: Formula Review

3.12.4: Projects

3.12.5: Chapter Review

3.12.6: Chapter Test

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3.12.0: Key Terms

Key Terms

3.1 Prime and Composite Numbers

- natural numbers
- factor of a number
- multiple of a number
- prime number
- composite number
- prime factorization
- greatest common divisor (GCD)
- least common multiple (LCM)

3.2 The Integers

- integer
- absolute value
- average of a set of numbers

3.3 Order of Operations

- order of operations
- PEMDAS

3.4 Rational Numbers

- density property of rational numbers
- improper fraction
- lowest terms
- mixed number
- rational number
- repeating decimal
- terminating decimal

3.5 Irrational Numbers

- conjugate numbers
- difference of squares
- irrational numbers
- lowest terms
- rationalize the denominator

3.6 Real Numbers

- complex number
- imaginary number
- real number

3.7 Clock Arithmetic

- clock arithmetic
- modulo 7
- modulo 12

3.8 Exponents

- base
- exponent

3.9 Scientific Notation

- scientific notation
- standard notation

3.10 Arithmetic Sequences

- sequence
- term of a sequence
- arithmetic sequence
- first term
- constant difference

3.11 Geometric Sequences

- geometric sequence
- common ratio

3.11 Geometric Sequences

- common ratio
- geometric sequence

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3.12.1: Key Concepts

Key Concepts

3.1 Prime and Composite Numbers

- The natural numbers can be categorized as 1, prime numbers, and composite numbers.
- Prime numbers have as their only factors 1 and themselves.
- Composite numbers have at least three distinct factors.
- Composite numbers can be written in their prime factorization form, which is found by repeatedly factoring prime factors from the number.
- The greatest common divisor (GCD) of a set of numbers is the largest integer that divides all of the numbers in the set. The prime factorizations of the numbers can be used to identify the greatest common divisor.
- The least common multiple (LCM) of a set of numbers is the smallest integer that is divisible by all of the numbers in the set. The prime factorizations of the numbers can be used to identify the least common multiple.
- There are various ways that the GCD and LCM are applied.

3.2 The Integers

- A set of numbers that can be built from the natural numbers are the integers, which consist of the natural numbers, zero (0), and the negatives of the natural numbers.
- Integers are often graphed on a number line, which helps display the relative positions and values of those numbers.
- The number line can be used to visualize when one integer is larger than or smaller than another integer.
- Arithmetic operations with integers are similar to the operations with natural numbers, except that the sign (positive or negative) of the numbers will determine the sign (positive or negative) of the result.

3.3 Order of Operations

- Establishing shared rules on which arithmetic operations are calculated first is necessary. Without them, different people may find different values for the same expression.
- The highest precedence is with expressions in parentheses. This allows parts of an expression to be calculated in an order different than the basic order of operations.
- The lowest precedence is addition and subtraction, as they are the basis for all other calculations.
- Multiplication and division have precedence over addition and subtraction, as they are representations of repeated addition or subtraction.
- Exponents have precedence over multiplication and division, as they represent repeated multiplication and division.

3.4 Rational Numbers

- Rational numbers are fractions of integers, and can always be written as an integer divided by an integer.
- The numerator and denominator of a fraction may have common factors. In such cases, the fraction can be reduced by canceling common factors. When the numerator and denominator of a fraction have no common factors, the fraction is said to be reduced.
- An improper fraction is one with a numerator larger than the denominator. Such a fraction can be rewritten as an integer plus a proper fraction. This is called a mixed number.
- Using division and remainder, an improper fraction may be written as a mixed number.
- A mixed number can be converted to an improper fraction by reversing the process for changing an improper fraction to a mixed number.
- The arithmetic operations or addition, subtraction, multiplication and division can all be performed on rational numbers.
- Addition and subtraction of rational numbers can be performed after a common denominator has been identified, and the fractions have been converted to forms having the common denominator.
- Multiplication and division of rational numbers can be performed without regard to common denominators.
- Between any two rational numbers, there is always another rational number. This is the density property of the rational numbers.

3.5 Irrational Numbers

- Irrational numbers are numbers that cannot be written as an integer divided by another integer. One example is pi, denoted π. Another collection of irrational numbers are natural numbers that are not perfect squares.

- Some irrational numbers can be written as a rational part multiplied by an irrational part. If two irrational numbers have the same irrational parts, they can be added or subtracted.
- When irrational numbers are similar, on can multiply and divide the numbers without a calculator.
- Since $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$, and $\sqrt{a} \div \sqrt{b} = \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$, products and quotients of square roots can be determined.
- Because $\sqrt{a^2} =$ and $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$, it is possible to simplify square root expressions so the radicand contains no perfect square factors.
- When a fraction has an irrational number as its denominator, it is possible to convert the denominator into a rational number using its conjugate. Doing so involves multiplying the numerator and denominator by the conjugate of the denominator, and then applying the difference of squares formula.
- With a single square root term
- Using conjugate numbers for two term denominators

3.6 Real Numbers

- Real numbers is the collection of all rational and irrational numbers. Conceptually, it is the collection of all values that can be represented on a number line, or, as a length along with sign.
- The subsets of the real numbers include the natural numbers, integers, rational numbers and irrational numbers. The natural numbers are a subset of the integers, which is a subset of the rational numbers. The rational and irrational numbers are disjoint sets.
- The real numbers, due to order of operation rules and that performing arithmetic operations on real number always results in a real number, have arithmetic properties that apply in all cases. There include the distributive property, the commutative property, and the associative property. Also, every real number has an additive inverse and, except for zero (0), have a multiplicative inverse.

3.7 Clock Arithmetic

- Clock arithmetic uses the idea that after 12 o'clock comes 1 o'clock. For clock arithmetic, this means that every time 12 is passed in an arithmetic process, the next number is 1, not 13.
- To determine the clock result of an arithmetic operation, divide the final result by 12 and keep the remainder. If the remainder is 0, then the time is 12 o'clock.
- Clock arithmetic is technically called modulo 12 arithmetic. To perform modulo 12 arithmetic, calculate the expression, then divide the result by 12. The modulo 12 result is the remainder.
- Days, in our system, pass in groups of seven. To calculate in day arithmetic, modulo 7 is used. To perform modulo 7 arithmetic, calculate the expression, then divide the result by 7. The modulo 7 result is the remainder.

3.8 Exponents

- Exponents are used to express multiplying a number by itself a number of times. The number being multiplied by itself is the base. The number of times it is multiplied by itself is the exponent, which is often referred to as the power.
- Understanding that exponents represent repeated multiplication of a base makes it possible to establish some rules for combining exponential expressions, using the product rule, the quotient rule, and the power rule. Additionally, it allows us to formulate distributive rules for exponents.
- Any non-zero number raised to the 0th power is 1. This makes the definition of the 0th power consistent with the division rule for exponents.
- For consistency, negative exponents represent the reciprocal of the base raised to the power, so that $a^{-n} = \frac{1}{a^n}$, provided that $a \neq 0$.

3.9 Scientific Notation

- Some numbers are so large or so small that writing the number out is clumsy and make it difficult to determine the true size of the number. Scientific notation makes the number more readable and make the relative size of the number immediately apparent.
- A number written in scientific notation is a number at least 1 and smaller than 10 multiplied by 10 raised to an exponent. Converting between scientific notation and standard notation involves correctly applying multiplication and division by powers of 10, which in practice equates to understanding how moving the decimal point of a number impacts the exponent of 10.
- Adding and subtracting numbers in base 10 requires the exponent of 10 in each number be the same. Once the numbers are converted to have the same exponent with the ten, then the numbers are added or subtracted as indicated, with the power of 10

remaining the same. If the result is not in scientific notation (for instance, the number has exceeded 10), then then number must be converted into scientific notation.
- Multiplying and dividing numbers in scientific notation is done by multiplying or dividing the number parts, then multiplying or dividing the 10 raised to the power parts, then multiplying those two results. If the new number is not in scientific notation, then the result must be converted into scientific notation.

3.10 Arithmetic Sequences

- A sequence is a list of numbers. Any individual number in that list, or sequence, is a term of the sequence. A specific term of a sequence is denoted by the sequence symbol with a subscript indicating where the term in the sequence is.
- A special form of a sequence is an arithmetic sequence. Each arithmetic sequence is determined by its first term and its constant difference. Any term in an arithmetic sequence is determined by adding the constant difference to the preceding term.
- If the first term and the constant difference of an arithmetic sequence are known, then any term of the sequence can be found directly.
- Because arithmetic sequences follow such a strict pattern, the sum of the first n terms of an arithmetic sequence can be determined with the formula $s_n = n\left(\frac{a_1+a_n}{2}\right)$.

3.11 Geometric Sequences

- A special form of a sequence is a geometric sequence. Each geometric sequence is determined by its first term and its constant ratio. Any term in a geometric sequence is determined by multiplying the constant ratio to the preceding term.
- If the first term and the constant ratio of a geometric sequence are known, then any term of the sequence can be found directly.
- Because geometric sequences follow such a strict pattern, the sum of the first n terms of a geometric sequence can be determined with the formula $s_n = a_1\left(\frac{1-r^{n-1}}{1-r}\right)$.
- Finding the sum of a finite geometric sequence
- Applying arithmetic sequences

3.11 Geometric Sequences

- Geometric sequence.
- Finding an arbitrary term in a geometric sequence.
- Constant ratio.
- Finding the sum of a finite geometric sequence.
- Applying arithmetic sequences.

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3.12.2: Videos

Videos

- Divisibility Rules
- Illegal Prime Number
- Using a Factor Tree to Find the Prime Factorization
- Finding the Prime Factorization of 168
- Using Desmos to find the GCD
- Applying the GCD
- Finding the LCM
- Using Desmos to find the LCM
- Application of LCM

- Graphing Integers on the Number Line
- Comparing Integers Using the Number Line
- Evaluating the Absolute Value of an Integer

- Order of Operations 1
- Order of Operations 2
- Order of Operations 3
- Order of Operations 4

- Introduction to Fractions
- Equivalent Fractions
- Reducing Fractions to Lowest Terms
- Using Desmos to Reduce a Fraction
- Adding and Subtracting Fractions with Different Denominators
- Converting an Improper Fraction to a Mixed Number Using Desmos
- Multiplying Fractions
- Dividing Fractions
- Order of Operations Using Fractions
- Finding a Fraction of a Total
- Converting Units

- The Philosophy of the Pythagoreans
- Using Desmos to Find the Square Root of a Number
- Simplifying Square Roots
- Rationalizing the Denominator

- Properties of the Real Numbers 1
- Properties of the Real Numbers 2
- Properties of the Real Numbers 3

- Arthur Benjamin TED talk, Faster than a Calculator
- Identifying Sets of Real Numbers
- Properties of the Real Numbers #1
- Properties of the Real Numbers #2
- Properties of the Real Numbers #3

- Determining the Day of the Week for Any Date in History
- Clock Arithmetic

- Exponential Notation
- Product and Quotient Rule for Exponents
- Fraction Raised to a Power
- Simplifying Expressions with Exponents

- Converting from Standard Form to Scientific Notation Form
- Converting from Scientific Notation Form to Standard Form
- Multiplying Numbers in Scientific Notation
- Dividing Numbers in Scientific Notation
- Application of Scientific Notation

- Arithmetic Sequences
- Finding the First Term and Constant Difference for an Arithmetic Sequence
- Finding the Sum of a Finite Arithmetic Sequence
- Fibonacci Sequence and “Lateralus”

- Geometric Sequences
- Sum of a Finite Geometric Sequence

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3.12.3: Formula Review

Formula Review

3.4 Rational Numbers

- $\frac{a}{c} \pm \frac{b}{c} = \text{mfrac}$
- $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$
- $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d} \frac{a}{b} \div \frac{c}{d} = \frac{a}{d} \times \frac{d}{c} = \frac{a \times d}{b \times c}$

3.5 Irrational Numbers

- $$\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$$
- $$a \times x \pm b \times x = (a \pm b)$$
- $$\sqrt{a} \div \sqrt{b} = \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$
- $$a^2 - b^2 = (a - b)(a + b)$$

3.6 Real Numbers

- $a \times (b + c) = a \times b + a \times c$
- $a + b = b + a$
- $a \times b = b \times a$
- $a + (b + c) = (a + b) + c$
- $a \times (b \times c) = (a \times b) \times c$
- $a + 0 = a$
- $a \times 1 = a$
- $a + (-a) = 0$
- $a \times \left(\frac{1}{a}\right) =$

3.8 Exponents

- $$a^n a^m = a^{n+}$$
- $$\frac{a^n}{a^m} = a^{(n-m)}$$
- $a^0 =$, provided that $a \neq 0$
- $$(a \times b)^n = a^n \times b^n$$
- $$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$
- $$(a^n)^m = a^{(n \times m)}$$
- $a^{-n} =$, provided that $a \neq 0$

3.10 Arithmetic Sequences

- $a_i = a_1 + d \times (i - 1)$
- $d = \text{mfrac}$
- $a_1 = a_i - d(i - 1)$
- $s_n = n\left(\text{mfrac}\right)$

3.11 Geometric Sequences

- $a_n = a_1 r^n \text{msup}$
- $s_n = a_1 \left(\frac{1 - r^{n-1}}{1 - r}\right)$

3.11 Geometric Sequences

$a_n = a_1 r^n$msup

$s_n = a_1\left(\frac{1-r^{n-1}}{1-r}\right)$

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3.12.4: Projects

Projects

Encryption began at least as far back as the Roman Empire. During the reign of Caesar, a particular cypher was used, fittingly named the Caesar Cypher. This encryption process granted the Romans a great tactical advantage. Even if a message was intercepted, it would not make sense to the person intercepting the message.

Find four instances when encryption was used and cracked over the course of history.

The Golden Ratio in Art and Architecture

The golden ratio has been used in art and architecture as far back as ancient Greece (possibly further). It also appears in South America (Incan architecture). Find five instances of the use of the golden ratio in art or architecture and describe its use in each of those instances.

Your Budget

Budgeting either is, or will shortly be, an important aspect of your life. Managing money well reduces stress in your life, and provides space for planning for future expenses, such as vacations or home improvements.

Imagine your life 10 years from now. Estimate your monthly income. Identify expenses you will encounter monthly (mortgage or rent, car payment, insurance, entertainment, etc.). Decide on an amount you plan to save monthly (this is treated as an expense). Create a spreadsheet with those values. Record your monthly net income (your income minus your expenses). Determine how much money you will have saved over the course of 5 years (ignore interest). Write a reflection on your anticipated financial health.

Estimating Pi

The value of pi is the ratio of the circumference of a circle to the diameter of the circle. It is also equal to the ratio of the area of the circle to the square of the radius of the square.

Research three ways to physically estimate pi.

Estimate pi using all three processes you found.

Present your process and solutions in class.

Design Your Own Shift Cypher

A cypher is a message written in such a way as to mask its contents. Changing a message into its cypher form is called encryption. Decryption or deciphering is the process of changing a cyphertext message back into the original (legible) message. One process of encryption is to scramble the letters, symbols, and punctuation of a message according to a mathematical rule. One rule that could be used for such a cypher is addition in a chosen modulus. In this project, you will create such a cypher, encrypt a message, and then decrypt the message.

Step 1: Choose the letters, symbols, and punctuation marks you want to allow in your messages. This should include at least the uppercase letters and a space character. This is your character set.

Step 2: Count the number of characters you will use. Label this number n.

Step 3: Pair each character of your character set an integer from 0 to $(n-1)$. Do not assign more than one character to an integer.

Step 4: Choose an integer between 1 and $(n-1)$. This will be the number used to create the cypher. Label this number s.

Step 5: Write a message using your character set.

Step 6: Replace every character in your message by the integer with which it was paired in Step 3.

Step 7: For every number, x, from Step 7, perform the addition $x+s$ (mod n).

Step 8: Replace every number found in Step 7 with the character with which it was paired in Step 3. This is your cyphertext.

To decrypt your cyphertext, reverse the steps above.

Step 1: Replace the cyphertext characters with the paired values.

Step 2: For each value x, perform the *subtraction* $x–s$ (mod n).

Step 3: Replace the numbers from Step 2 with their paired characters from the character set.

The message is then deciphered.

Design Your Own Cypher Using Multiplication

A cypher is a message written in such a way as to mask its contents. Changing a message into its cypher form is called encryption. Decryption or deciphering is the process of changing a cyphertext message back into the original (legible) message. One process of encryption is to scramble the letters, symbols, and punctuation of a message according to a mathematical rule. One rule that could be used for such a cypher is multiplication in a chosen modulus. In this project, you will create such a cypher, encrypt a message, then decrypt the message.

Step 1: Choose the letters, symbols, and punctuation marks you want to allow in your messages. This should include at least the uppercase letters and a space character. This is your character set.

Step 2: Count the number of characters you will use. Label this number n.

Step 3: Pair each character of your character set an integer from 0 to $(n-1)$. Do not assign more than one character to an integer.

Step 4: Choose an integer, labeled s, between 1 and $(n-1)$ so that $\text{GCF}\,(n,s)=1$. This will be the number used to create the cypher.

Step 5: Write a message using your character set.

Step 6: Replace every character in your message by the integer with which it was paired in Step 3.

Step 7: For every number, x, from Step 6, perform the multiplication $x-s$ (mod n).

Step 8: Replace every number found in Step 7 with the character with which it was paired in Step 3. This is your cyphertext.

Before beginning to decrypt in this cypher, you need to know the **multiplicative inverse** of the value you chose as *s*.

Step 1: The multiplicative inverse of s is the number that, when multiplied by s in your modulus, equals 1. To find this, you will have to multiply s and every number between 2 and (n–1) until the product is 1 (mod n). Once this number is found, the message can be decrypted. Call this number r.

Step 2: To decrypt your cyphertext, replace the cyphertext characters with the paired values.

Step 3: For each of the value, x, perform the multiplication $x \times r (\bmod n)$.

Step 4: Replace the numbers from Step 3 with their paired characters from the character set.

The message is then deciphered.

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3.12.5: Chapter Review

Prime and Composite Numbers

? Exercise 3.12.5.1

Identify which of the following numbers are prime, composite, or neither: 201, 34, 17, 1, 37.

? Exercise 3.12.5.2

Find the prime factorization of 500 .

? Exercise 3.12.5.3

Find the greatest common divisor of 80 and 340 .

? Exercise 3.12.5.4

Find the greatest common divisor of 30,40 , and 70 .

? Exercise 3.12.5.5

Find the least common multiple of 45 and 60.

? Exercise 3.12.5.6

Bella and JJ volunteer at the zoo. Bella volunteers every 8 days, while JJ volunteers every 14 days. How many days pass between days they volunteer together?

The Integers

? Exercise 3.12.5.7

Identify all the integers in the following list: $-4, \frac{2}{7}, 15, 97.5, 0, \sqrt{122}$.

? Exercise 3.12.5.8

Plot the following on the same number line: $1, 5, -2, 0$.

? Exercise 3.12.5.9

What two numbers have absolute value of 18 ?

? Exercise 3.12.5.10

Calculate $14-(-12)$.

? Exercise 3.12.5.11

Calculate $(-19) \times 4$.

? Exercise $3.12.5.12$

Calculate $(-240) \div (-12)$.

? Exercise $3.12.5.13$

Six students rent a house together. The total monthly rent (including heat and electricity) is $\$3,120$. If they all pay an equal amount, how much does each student pay?

Order of Operations

? Exercise $3.12.5.14$

Calculate $5 \times 3 - (-12)$.

? Exercise $3.12.5.15$

Calculate $4^2 \times 6 + 2^3$.

? Exercise $3.12.5.16$

$(4-6) \times 2$

? Exercise $3.12.5.17$

$(8-1)^3 - 3 \times 9$

? Exercise $3.12.5.18$

$120 - (51 + 12) \div 3^2$

Rational Numbers

? Exercise $3.12.5.19$

Reduce $\frac{90}{153}$ to lowest terms.

? Exercise $3.12.5.20$

Convert $\frac{430}{25}$ to a mixed number and reduce to lowest terms.

? Exercise $3.12.5.21$

Convert $\frac{9}{5}$ to decimal form.

? Exercise $3.12.5.22$

Convert $\frac{5}{11}$ to decimal form.

? Exercise $3.12.5.23$

Calculate $\frac{4}{15} + \frac{9}{10}$ and reduce to lowest terms.

Exercise 3.12.5.24

Compute $\frac{3}{10} \div \frac{12}{25}$ and reduce to lowest terms.

Exercise 3.12.5.25

Determine 30% of 400 .

Exercise 3.12.5.26

18 is what percent of 40 ?

Exercise 3.12.5.27

In Professor Finnegan's Science Fiction course, there are 60 students. Of those, 15% say they've read A Hitchhiker's Guide to the Galaxy. How many of the students have read that book?

Irrational Numbers

Exercise 3.12.5.28

Simplify the square root by expressing it in lowest terms: $\sqrt{275}$.

Exercise 3.12.5.29

Calculate $4\sqrt{13}-15\sqrt{13}$ without a calculator. If not possible, explain why.

Exercise 3.12.5.30

Calculate $(2.3\sqrt{5}) \times (4.2\sqrt{3})$ without a calculator. If not possible, explain why.

Exercise 3.12.5.31

Rationalize the denominator of $\frac{6}{\sqrt{22}}$, and simplify the fraction.

Exercise 3.12.5.32

Find the conjugate of $4\sqrt{7}+3$ and find the product of $4\sqrt{7}+3$ and its conjugate.

Exercise 3.12.5.33

Rationalize the denominator of $\frac{3}{8+\sqrt{34}}$ and simplify the fraction.

Real Numbers

Exercise 3.12.5.34

Identify the numbers of the following list as a natural number, an integer, a rational number, or a real number: $\sqrt{37}, -0.43, 18, -43, 12\pi$.

Exercise 3.12.5.35

Identify the property of real numbers illustrated here: $14+19=19+14$.

Exercise 3.12.5.36

Identify the property of real numbers illustrated here: $87.4+0=87.4$.

Exercise 3.12.5.37

Identify the property of real numbers illustrated here: $2\times(\sqrt{31}-12)=2\sqrt{31}-2\times 12$.

Exercise 3.12.5.38

Identify the property of real numbers illustrated here: $98\times 41=41\times 98$.

Exercise 3.12.5.39

Use mental math to calculate 21×99.

Clock Arithmetic

Exercise 3.12.5.40

Determine 74 modulo 9.

Exercise 3.12.5.41

Use clock arithmetic to calculate $13+25$.

Exercise 3.12.5.42

Use clock arithmetic to calculate 4×8.

Exercise 3.12.5.43

It is Wednesday. What day of the week will it be in 44 days?

Exercise 3.12.5.44

It is $4:00$. What time will it be in 100 hours?

Exercise 3.12.5.45

Security guards with Acuriguard submit a report on campus activity every 4 days. If they make a report on a Monday, what day of the week will it be after 10 more reports?

Exponents

Exercise 3.12.5.46

Use exponent rules to simplify $17^8\times 17^3$.

Exercise $3.12.5.47$

Use exponent rules to simplify $\frac{15^{9}}{15^{-5}}$.

Exercise $3.12.5.48$

Use exponent rules to simplify $(3k)^{6}$.

Exercise $3.12.5.49$

Use exponent rules to simplify $\left(\frac{y}{4}\right)^{11}$.

Exercise $3.12.5.50$

Use exponent rules to simplify $\left(51^{3}\right)^{6}$.

Exercise $3.12.5.51$

Use exponent rules to simplify $\left(\frac{3x^{4}}{5}\right)^{7}$.

Exercise $3.12.5.52$

Rewrite $\frac{5^{2}}{x^{4}}$ without a denominator.

Exercise $3.12.5.53$

Rewrite $16z^{-9}$ without negative exponents.

Scientific Notation

Exercise $3.12.5.54$

Convert 0.0000452 to scientific notation.

Exercise $3.12.5.55$

Convert 3.01×10^{5} to standard notation.

Exercise $3.12.5.56$

Convert 42.9×10^{-9} to scientific notation.

Exercise $3.12.5.57$

Calculate $4.51 \times 10^{5} - 9.11 \times 10^{4}$.

Exercise $3.12.5.58$

Calculate $\left(9.15 \times 10^{3}\right) \div \left(3 \times 10^{8}\right)$.

? Exercise 3.12.5.59

The Sextans Dwarf Spheroidal Galaxy has diameter 8,400 light years (ly). Express this in Scientific notation.

? Exercise 3.12.5.60

The Reticulum II Galaxy has diameter 3.78×10^2 light years (ly), while the Andromeda Galaxy has a diameter of 2×10^5. How many times bigger is the Andromeda Galaxy compared to the Reticulum II Galaxy?

Arithmetic Sequences

? Exercise 3.12.5.61

Determine the common difference of the following sequence: $\{19, 13, 7, 1, -5, ...\}$.

? Exercise 3.12.5.62

Find the 25th term, a_{25}, of the arithmetic sequence with $a_1 = 13$ and $d = 1.7$.

? Exercise 3.12.5.63

Find the first term and the common difference of the arithmetic sequence with 8 th term $b_8 = 50$ and 15 th term $b_{15} = 106$.

? Exercise 3.12.5.64

Find the sum of the first 30 terms, s_{30}, for the arithmetic sequence with first term $a_1 = -4$ and common difference $d = 3.5$.

? Exercise 3.12.5.65

Jem makes a stack of 5 pennies. Each day, Jem adds three pennies to the stack. How many pennies are in the stack after 10 days?

Geometric Sequences

? Exercise 3.12.5.66

Determine the common ratio of the following geometric sequence: $\{6, 18, 54, 162, ...\}$.

? Exercise 3.12.5.67

Find the 6th term, c_6, of the geometric sequence with $c_1 = 400$ and $r = 0.25$.

? Exercise 3.12.5.68

Find the sum of the first 12 terms, s_{12}, for the geometric sequence with first term $a_1 = -4$ and common ratio $r = -1.25$.

? Exercise 3.12.5.69

Carolann and Tyler deposit $\$8,500$ in an account bearing 5.5% interest compounded yearly. If they do not deposit any more money in that account, how much will be in the account after 15 years?

Exercise $3.12.5.70$

The total number of ebooks sold in 2013 was 242 million ($a_1 = 242$). Each year, the number of ebooks sold has declined by $3\% (r = 0.97)$. How many ebooks were sold between 2013 and 2022?

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3.12.6: Chapter Test

? Exercise $3.12.6.1$

Find the prime factorization of 300.

? Exercise $3.12.6.2$

Tiles will be used to cover an area that is $650 \text{ cm} \times 1,200 \text{ cm}$. What is the largest size square tile that can be used so that all the tiles used are full tiles?

? Exercise $3.12.6.3$

Calculate the following: $3 \times \left(\frac{4}{3} \times (14-8)+3\right)-4 \times \left(8-2^2\right)$.

? Exercise $3.12.6.4$

What does PEMDAS stand for?

? Exercise $3.12.6.5$

David's tax return is for $\$1,560$. He decides to spend 20% of that return. How much does David spend?

? Exercise $3.12.6.6$

Convert $\frac{305}{26}$ into a mixed number.

? Exercise $3.12.6.7$

Convert $0.\overline{659}$ to a fraction of integers.

? Exercise $3.12.6.8$

Simplify the following square root: $\sqrt{6237}$.

? Exercise $3.12.6.9$

What is the conjugate of $4-\sqrt{10}$?

? Exercise $3.12.6.10$

Rationalize the denominator of $\frac{2}{10+\sqrt{13}}$.

? Exercise $3.12.6.11$

What two sets of numbers comprise the real numbers?

? Exercise $3.12.6.12$

Which property of the real numbers is shown: $4(a+7)=4a+4\times 7$?

? Exercise 3.12.6.13

Georita believes it will take 30 hours for her and her family to drive to their vacation. If they leave at $2:00$, what time should they arrive (ignore AM/PM)?

? Exercise 3.12.6.14

Suppose a bacterium in the gut has a generation time (time to divide) of 16 hours. If it first divides at 4:00, what time will it be when they divide the 80th time afterward?

? Exercise 3.12.6.15

Simplify $\left(\frac{9x}{a^2}\right)^5$.

? Exercise 3.12.6.16

The moon is 3.844×10^8 m from Earth. A dollar bill has a thickness of 1.0922×10^{-4} m. If dollar bills could be stacked perfectly, how many would it take to reach the moon?

? Exercise 3.12.6.17

A single long table can seat 8 people, 3 on each side and 1 on each end. If a second table is added to the first, end to end, then 14 people can sit at the table, 6 per side and 1 at each end. Adding another table adds another 6 people. How many people can sit at a table made by placing 10 of these tables end to end?

? Exercise 3.12.6.18

The first row of a theater seats 25 people. Each following row seats 2 more people. If there are 80 rows in the theater, how many people, total, can sit in the theater?

? Exercise 3.12.6.19

Alice deposits $\$2,500$ in a bond yielding 6% interest compounded annually. How much is the bond worth in 20 years?

? Exercise 3.12.6.20

What is the 15th term of a geometric sequence with first term 5 and common ratio 3 ?

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CHAPTER OVERVIEW

4: Number Representation and Calculation

- 4.0: Introduction
- 4.0: Hindu-Arabic Positional System
  - 4.0.0: Exercises
- 4.1: Early Numeration Systems
  - 4.1.0: Exercises
- 4.2: Converting with Base Systems
  - 4.2.0: Exercises
- 4.3: Addition and Subtraction in Base Systems
  - 4.3.0: Exercises
- 4.4: Multiplication and Division in Base Systems
  - 4.4.0: Exercises
- 4.5: Chapter Summary
  - 4.5.0: Key Terms
  - 4.5.1: Key Concepts
  - 4.5.2: Videos
  - 4.5.3: Projects
  - 4.5.4: Chapter Review
  - 4.5.5: Chapter Test

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4.0: Introduction

Right now, almost all cultures use the familiar Hindu-Arabic numbering system, which uses the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 along with place values based on powers of ten. This is a relatively recent development. The system didn’t develop until the 6th or 7th century C.E. and took some time to spread across the world, which means other cultures at other times had to develop their own methods of counting and recording quantity. Being different cultures and different times means there were significant differences in counting systems. Cultures needed to count and measure time for agriculture and for religious observations. It was needed for trade. Some languages had words only for one, two, and many. Other cultures developed more complex ways to represent quantity, with the Oksapmin people of New Guinea using an astonishing 27 words for their system.

Figure $4.0.1$ Different cultures developed different ways to record quantity. (credit: modification of work "Tally sticks from the Swiss Alps" by Sandstein, Swiss Alpine Museum permanent collection/Wikimedia Commons, CC BY 3.0)

Representing these quantities in a recorded form likely began with a simple marking system, where one scratch on a stick or bone represented one of whatever was being counted. We still see this today with tally marks. These systems use repeated symbols to represent more than one. We also have systems where different symbols represent different quantities but still use some repetition, such as in Roman numerals.

Other systems were devised that rely on place values, like the Hindu-Arabic system in use today. Place value systems needed a zero, though, and weren't immediately recognized and took time to develop. And within these positional systems there is variation. Some systems counted in twenties, others in tens, and some in a mix (adding another reason to visit Hawaii). Even now, though we all use and think using tens, computers are designed to work in groups of two, which requires a different perspective on numbering.

In this chapter, we explore different numbering systems and grouping systems, eventually discussing base 2, the language of computers.

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4.0: Hindu-Arabic Positional System

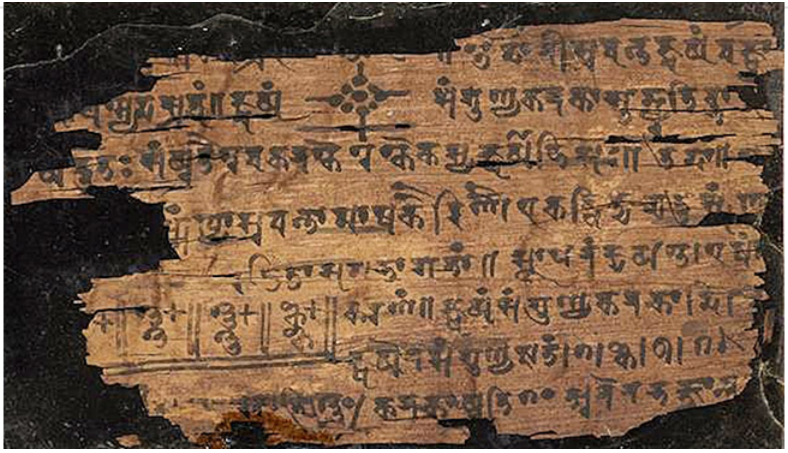


Figure 4.0.1: This manuscript is an early example of Hindu numerals. (credit: modification of work “Bakshali manuscript”, Bodleian Libraries/ University of Oxford, public domain)

Learning Objectives

After completing this section, you should be able to:

- Evaluate an exponential expression.
- Convert a Hindu-Arabic numeral to expanded form.
- Convert a number in expanded form to a Hindu-Arabic numeral.

The modern system of counting and computing isn’t necessarily natural. That different symbols are used to indicate different quantities or amounts is a relatively new invention. Simple marking by scratches or dots, one for each item being counted, was the norm long into human history. The modern system doesn’t use repeated symbols to indicate more than one of a thing. It uses the place of a digit in a numeral to determine what that digit represents. A numeral is a symbol used to represent a number. A number is an abstract idea that represents quantity or amount.

Being clear about the difference between numeral and number is important. Just like a person can be called by various names, such as brother, father, husband, uncle, they are all representing the same person, John Smith. The person John Smith is the number, and the names brother, father, husband, and uncle are the numerals.

Who Knew?: Hindu-Arabic Numerals

The numerals we currently use are referred to as Hindu-Arabic numerals, although they have changed as time has passed. Early forms of the numerals for 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 began in India, and passed through Persia to the Middle East. Place value was also employed in the early systems of India. Once this system was in north Africa and the Middle East, it spread to Europe, eventually replacing Roman numerals. Over time, the original symbols transformed into our modern ones. Read this article for another perspective on how the symbols began (based on the moon!).

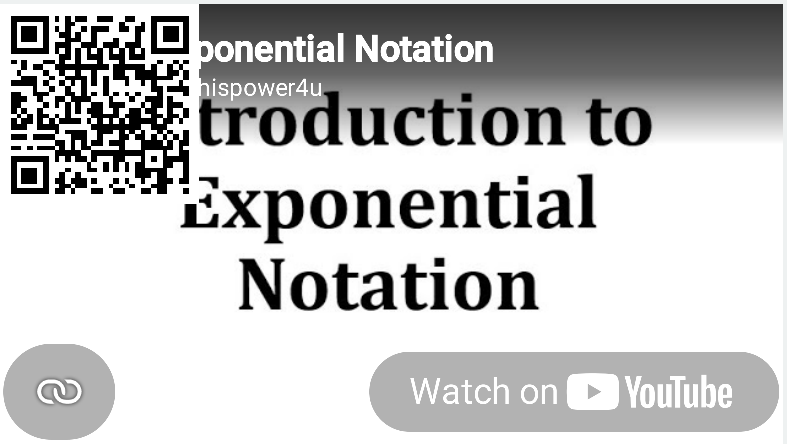
The system we use for counting and computing uses place values based on powers of 10. In this section, we review exponents and our positional system.

Evaluating Exponential Expressions

Most modern numerical systems depend on place values, where the quantity represented depends not only on the digit, but also on where the digit is in the number. The place value is a power of some specific number, which means most numbering systems are actually exponential expressions. An exponential expression is any mathematical expression that includes exponents. So, evaluating such an expression means performing the calculation. In this chapter, we will be using exponents that are positive integer values. Before we do so, let’s remind ourselves about exponents and what they represent. Suppose you want to multiply a number. Let’s label that number a, by itself some number of times. Let’s label the number of times b. We denote that as semanticsabab. We say a, or the **base**, raised to the bth power, or the **exponent**. For example, if we are multiplying 13 by itself eight times, we write 13^8 and say 13 to the eighth power.

When computing exponential expressions, we should be careful to remember the order of operations. Using the order of operation rules, calculations inside the parentheses are done first, then exponents are calculated, then multiplication and division calculations are performed, and then addition and subtraction.

Video



Exponential Notation

Example 4.0.1: Evaluating an Exponential Expression

Evaluate the following exponential expressions.

1. $4 \times 5^2 + 2 \times 6^3$
2. $6 \times 8^2 + 3 \times 8^1 + 4 \times 8^0$
3. $3 \times 10^2 + 0 \times 10^1 + 6 \times 10^0$

Answer

1. To evaluate, or calculate, this expression, we use order of operations, which means the exponents are done first, then multiplications, and then additions.

$$4 \times 5^2 + 2 \times 6^3 = 4 \times 5 \times 5 + 2 \times 6 \times 6 \times 6 = 4 \times 25 + 2 \times 216 = 100 + 432 = 532$$

2. To evaluate the expression, we use the order of operations, which means the exponents are done first, then the multiplications, then the additions. Remember that any base raised to the exponent 0 is 1.

$$6 \times 8^2 + 3 \times 8^1 + 4 \times 8^0 = 6 \times 8 \times 8 + 3 \times 8 + 4 \times 1 = 6 \times 64 + 3 \times 8 + 4 \times 1 = 384 + 24 + 4 = 412$$

3. To evaluate the expression, we use the order of operations, which means the exponents are done first, then the multiplications, and then the additions. Remember that any base raised to the exponent 0 is 1.

$$3 \times 10^2 + 0 \times 10^1 + 6 \times 10^0 = 3 \times 100 + 0 \times 10 + 6 \times 1 = 300 + 0 + 6 = 306$$

Your Turn 4.0.1

Evaluate the following exponential expressions.

$3 \times 2^5 + 5 \times 8^2$

$5 \times 7^3 + 2 \times 7^2 + 5 \times 7^1 + 3 \times 7^0$

$1 \times 10^4 + 7 \times 10^3 + 4 \times 10^2 + 8 \times 10^1 + 8 \times 10^0$

Converting Hindu-Arabic Numerals to Expanded Form

When you see the number 738, and you speak the number out loud, what do you say? You probably said “seven hundred thirty-eight” while wondering what point could possibly be made by asking this. What you didn’t say was “seven, and three, and eight.” A pre-K student might say that. Which should make you wonder, why?

The reason is that you’ve been taught place values, or the positions of digits in a number that determine the values of those digits. You know that in a three-digit number, the first digit is hundreds, the second digit is tens, and the last digit is ones. These place values rely on powers of 10, which makes this system a base 10 system.

This sense of place value is what makes our system of numbers so useful. You’ve also been taught the Hindu-Arabic numeration system. This system, which uses the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9, and also employs place value based on powers of 10, is in use today.

Writing a number using these place values is writing them in expanded form. For a number with n digits, the expanded form is the first digit times 10 raised to one less than n, plus each following digit times 10 raised to one less than the previous power of 10. For example, the number 738 would be written as $7 \times 10^2 + 3 \times 10^1 + 8 \times 10^0$.

What about a four-digit number, like 5,825? Out loud, we’d say five thousand, seven hundred twenty-five. In expanded form, it would be $5 \times 10^3 + 8 \times 10^2 + 2 \times 10^1 + 5 \times 10^0$. Notice that the largest exponent is one less than the number of digits, and that the exponents go down by one as we move through the number.

People in Mathematics: Aryabhata of Kusumapura and Brahmagupta

The Hindu-Arabic numeral system developed in India, and Aryabhata of Kusumapura is credited with the place value notation in the 5th century. However, the system wasn’t as complete as it could be, until. roughly a century later, when Brahmagupta introduced the symbol for 0. The 0 is necessary to indicate that a given place value has been skipped, as in 4,098. In 4,098, the 10^2 power is skipped. Without such a symbol, 4,098 and 498 look similar. The value of both the place value notation and the introduction of the symbol 0 cannot be overstated, for math and the sciences.

✓ Example $4.0.2$: Writing a Number in Expanded Form

Write the following in expanded form.

1. 563
2. 4,821
3. 903,786

Answer

1. **Step 1:** Since there are three digits in 563, n is 3. So, this is the first digit times 10 raised to the power of 2, so we start with 5×10^2.

   Step 2: Then we add the next digit, 6, multiplied by 10 to a power one less than the previous, at which point we have $5 \times 10^2 + 6 \times 10^1$.

   Step 3: Finally, the last digit is multiplied by 10 to the zeroth power and added to the previous. This results in $5 \times 10^2 + 6 \times 10^1 + 3 \times 10^0$.

2. **Step 1:** Since there are four digits in 4,821, n is 4. We multiply the first digit, 4, by 10 raised to the power of 3, which is 4×10^3.

   Step 2: Then we add the next digit, 8, multiplied by 10 to a power one less than the previous, at which point we have $4 \times 10^3 + 8 \times 10^2$.

   Step 3: We continue to the next digit, lowering the exponent of 10 by one. Now we have $4 \times 10^3 + 8 \times 10^2 + 2 \times 10^1$.

   Step 4: Finally, the last digit is multiplied by 10 to the zeroth power and added to the previous. This results in $4 \times 10^3 + 8 \times 10^2 + 2 \times 10^1 + 1 \times 10^0$.

3. Since there are six digits in 903,786, n is 6. So, we begin the process with 9 times 10 raised to the 5th power and continue through the numbers, reducing the exponent of 10 by one each time. This results in $9 \times 10^5 + 0 \times 10^4 + 3 \times 10^3 + 7 \times 10^2 + 8 \times 10^1 + 6 \times 10^0$.

Your Turn 4.0.2

Write the following in expanded form. 1. 924 2. 1,279 3. $4,130,045$

Converting Numbers in Expanded Form to Hindu-Arabic Numerals

Converting from expanded form back into a Hindu-Arabic numeral is the reverse process of expanding a number, and is equivalent to evaluating the exponential expression.

✓ Example 4.0.3: Converting a Number from Expanded Form to a Hindu-Arabic Numeral

Convert the following into Hindu-Arabic numerals.

1. $3 \times 10^2 + 4 \times 10^1 + 8 \times 10^0$
2. $5 \times 10^3 + 0 \times 10^2 + 9 \times 10^1 + 9 \times 10^0$
3. $6 \times 10^6 + 2 \times 10^5 + 0 \times 10^4 + 9 \times 10^3 + 1 \times 10^2 + 1 \times 10^1 + 7 \times 10^0$

Answer

1. Evaluating the expression results in: $3 \times 10^2 + 4 \times 10^1 + 8 \times 10^0 = 3 \times 100 + 4 \times 10 + 8 \times 1 = 300 + 40 + 8 = 348$
2. Evaluating the expression results in: $5 \times 10^3 + 0 \times 10^2 + 9 \times 10^1 + 9 \times 10^0 = 5 \times 1000 + 0 \times 100 + 9 \times 10 + 9 \times 1 = 5000 + 0 + 90 + 9 = 5099$
3. Evaluating the expression results in: $$\begin{aligned} &6 \times 10^6 + 2 \times 10^5 + 0 \times 10^4 + 9 \times 10^3 + 1 \times 10^2 + 1 \times 10^1 + 7 \times 10^0 \\ &= 6 \times 1{,}000{,}000 + 2 \times 100{,}000 + 0 \times 10{,}000 + 9 \times 1{,}000 + 1 \times 100 + 1 \times 10 + 7 \times 1 \\ &= 6{,}000{,}000 + 200{,}000 + 0 + 9{,}000 + 100 + 10 + 7 \\ &= 6{,}209{,}117 \end{aligned}$$

Your Turn 4.0.3

Convert the following to Hindu-Arabic Numerals. 1. $6 \times 10^2 + 2 \times 10^1 + 1 \times 10^0$ 2. $3 \times 10^3 + 2 \times 10^2 + 0 \times 10^1 + 3 \times 10^0$ 3. $4 \times 10^7 + 0 \times 10^6 + 6 \times 10^5 + 3 \times 10^4 + 0 \times 10^3 + 8 \times 10^2 + 9 \times 10^1 + 1 \times 10^0$

Check Your Understanding

1. What is meant by a place value system?
2. Evaluate the following exponential expression: $4 \times 8^2 + 2 \times 8^1 + 7 \times 8^0$.
3. Express the following number in expanded form: 45,209.
4. What number provides the value of a digit in our system of numeration?
5. How are numerals and numbers different?
6. Express as a Hindu-Arabic number: $6 \times 10^5 + 0 \times 10^4 + 1 \times 10^3 + 9 \times 10^2 + 4 \times 10^1 + 7 \times 10^0$.

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4.0.0: Exercises

? Exercise 4.0.0.1

What does it mean for a system to be a place value system?

? Exercise 4.0.0.2

In the system we use today, what number are the place values based on?

? Exercise 4.0.0.3

How are numerals and numbers different?

? Exercise 4.0.0.4

What relates numerals to numbers?

For the following exercises, evaluate the exponential expression.

? Exercise 4.0.0.5

$3 \times 4^2 + 5 \times 2^3$

? Exercise 4.0.0.6

$5 \times 6^3 + 7 \times 3^2$

? Exercise 4.0.0.7

$7 \times 5^2 + 2 \times 4^5$

? Exercise 4.0.0.8

$10 \times 11^2 + 7 \times 3^4$

? Exercise 4.0.0.9

$5 \times 6^2 + 3 \times 6^1 + 4 \times 6^0$

? Exercise 4.0.0.10

$4 \times 12^2 + 11 \times 12^1 + 2 \times 12^0$

? Exercise 4.0.0.11

$14 \times 8^3 + 19 \times 5^4 + 2 \times 3^1$

? Exercise 4.0.0.12

$8 \times 10^4 + 3 \times 5^3 + 9 \times 4^2$

Exercise 4.0.0.13

$1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$

Exercise 4.0.0.14

$5 \times 8^4 + 1 \times 8^3 + 0 \times 8^2 + 7 \times 8^1 + 3 \times 8^0$

For the following exercises, express the Hindu-Arabic number in expanded form.

Exercise 4.0.0.15

13

Exercise 4.0.0.16

25

Exercise 4.0.0.17

82

Exercise 4.0.0.18

99

Exercise 4.0.0.19

131

Exercise 4.0.0.20

408

Exercise 4.0.0.21

651

Exercise 4.0.0.22

3,901

Exercise 4.0.0.23

5,098

Exercise 4.0.0.24

12,430

For the following exercises, express the expanded number as a Hindu-Arabic number.

? Exercise 4.0.0.25

$3 \times 10^1 + 2 \times 10^0$

? Exercise 4.0.0.26

$5 \times 10^1 + 7 \times 10^0$

? Exercise 4.0.0.27

$2 \times 10^2 + 4 \times 10^1 + 9 \times 10^0$

? Exercise 4.0.0.28

$6 \times 10^2 + 0 \times 10^1 + 1 \times 10^0$

? Exercise 4.0.0.29

$1 \times 10^3 + 4 \times 10^2 + 4 \times 10^1 + 0 \times 10^0$

? Exercise 4.0.0.30

$7 \times 10^3 + 0 \times 10^2 + 1 \times 10^1 + 8 \times 10^0$

? Exercise 4.0.0.31

$6 \times 10^4 + 7 \times 10^3 + 0 \times 10^2 + 0 \times 10^1 + 0 \times 10^0$

? Exercise 4.0.0.32

$9 \times 10^4 + 8 \times 10^3 + 7 \times 10^2 + 3 \times 10^1 + 4 \times 10^0$

? Exercise 4.0.0.33

$7 \times 10^7 + 3 \times 10^6 + 4 \times 10^5 + 0 \times 10^4 + 4 \times 10^3 + 1 \times 10^2 + 5 \times 10^1 + 1 \times 10^0$

? Exercise 4.0.0.34

$8 \times 10^8 + 0 \times 10^7 + 4 \times 10^6 + 9 \times 10^5 + 9 \times 10^4 + 2 \times 10^3 + 2 \times 10^2 + 6 \times 10^1 + 0 \times 10^0$

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4.1: Early Numeration Systems

Figure 4.1.2: Babylonians used clay tablets for writing and record keeping. (credit: modification of work by Osama Shukir Muhammed Amin FRCP(Glasg), CC BY 4.0 International)

Learning Objectives

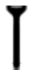
1. Understand and convert Babylonian numerals to Hindu-Arabic numerals.
2. Understand and convert Mayan numerals to Hindu-Arabic numerals.
3. Understand and convert between Roman numerals and Hindu-Arabic numerals.

Each culture throughout history had to develop its own method of counting and recording quantity. The system used in Australia would necessarily differ from the system developed in Babylon that would, in turn, differ from the system developed in sub-Saharan Africa. These differences arose due to cultural differences. In nearly all societies, knowing the difference between one and two would be useful. But it might not be useful to know the difference between 145 and 167, as those quantities never had a practical use. For example, a shepherd likely didn't manage more than 100 sheep, so quantities larger than 100 might never have been encountered. This can even be seen in our use of the term few, which is an inexact quantity that most would agree means more than two. However, as societies became more complex, as commerce arose, as military bodies developed, so did the need for a system to handle large numbers. No matter the system, the issues of representing multiple values and how many symbols to use had to be addressed. In this section, we explore how the Babylonians, Mayans, and Romans addressed these issues.

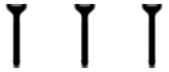
Understand and Convert Babylonian Numerals to Hindu-Arabic Numerals

The Babylonians used a mix of an additive system of numbers and a positional system of numbers. An additive system is a number system where the value of repeated instances of a symbol is added the number of times the symbol appears. A positional system is a system of numbers that multiplies a "digit" by a number raised to a power, based on the position of the "digit."

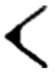
The Babylonian place values didn't use powers of 10, but instead powers of 60. They didn't use 60 different symbols though. For the value 1, they used the following symbol:



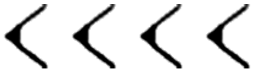
For values up to 9, that symbol would be repeated, so three would be written as



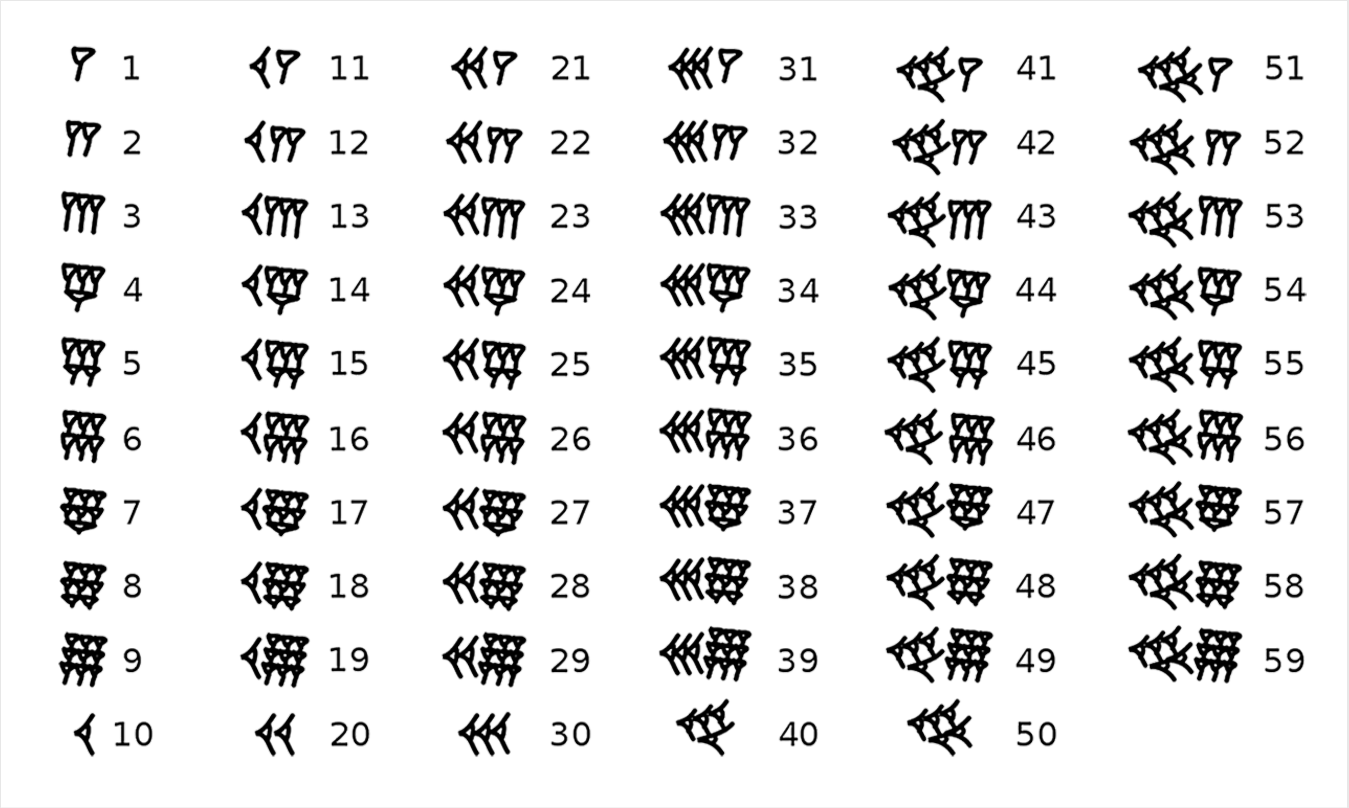
To represent the quantity 10, they used



For 20, 30, 40, and 50, they repeated the symbol for 10 however many times it was needed, so 40 would be written



When they reached 60, they moved to the next place value. The complete list of the Babylonian numerals up to 59 is in Table 4.1.



1	11	21	31	41	51
2	12	22	32	42	52
3	13	23	33	43	53
4	14	24	34	44	54
5	15	25	35	45	55
6	16	26	36	46	56
7	17	27	37	47	57
8	18	28	38	48	58
9	19	29	39	49	59
10	20	30	40	50	

Table 4.1.1 Babylonian Numerals

You can see how Babylonians repeated the symbols to indicate multiples of a value. The number 6 is 6 of the symbol for 1 grouped together. The symbol for 30 is three of the symbols for 10 grouped together. However, their system doesn't go past 59. To go past 59, they used place values. As opposed to the Hindu-Arabic system, which was based on powers of 10, the Babylonian positional system was based on powers of 60. You should also notice there is no symbol for 0, which has some impact on the number system. Since the Babylonian number system lacked a 0, they didn't have a placeholder when a power of 60 was absent. Without a 0, 101, 110, and 11 all look the same. However, there is some evidence that the Babylonians left a small space between "digits" where we would use a 0, allowing them to represent the absence of that place value. To summarize, the **Babylonian system of numbers** used repeating a symbol to indicate more than one, used place values, and lacked a 0.

Who Knew?: Invention of 0

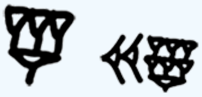
The idea of 0 is not a natural one. Most cultures failed to recognize the need for a 0. If someone asked a farmer in 300 B.C.E. how many cows they had, but they had none, they would not answer "zero." They'd say "I don't have any" and be done with it. It wasn't until roughly 3 B.C.E. that 0 appeared in Mesopotamia. It was independently discovered (or invented!) in the Mayan culture around 4 C.E. it made its appearance in India in the 400s C.E., and began to spread at that point. It wasn't developed earlier mostly because positional systems were not yet fully developed. Once positional systems arose, the need to represent a missing power had to be addressed.

So how do we convert from Babylonian numbers to Hindu-Arabic numbers? To do so, we need to use the symbols from Table 4.1, and then place values based on powers of 60. If you have n digits in the Babylonian number, you multiply the first "digit" by 60 raised to one less than the number of "digits." You then continue through the "digits," multiplying each by 60 raised to a power that is one smaller. However, be careful of spaces, since they represent a zero in that place.

✓ Example

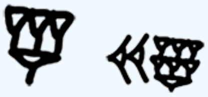
✓ $4.1.1$: Converting Two-Digit Babylonian Numbers to Hindu-Arabic Numbers

Convert the Babylonian number

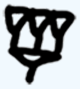


into a Hindu-Arabic number.

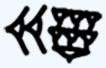
Answer



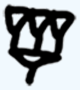
has two digits:



and

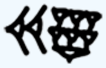


Step 1: So the first symbol,



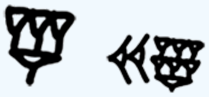
represents 4 in the Babylonian system. This is multiplied by 60 to the first power (just as would happen in a two digit number), which gives us 4×60^1.

Step 2: The next symbol is



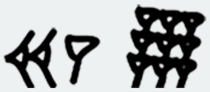
which represents 27 in the Babylonian system. This is multiplied by 60 raised to 0, which gives $4 \times 60^1 + 27 \times 60^0$.

Step 3: Calculating that yields $4 \times 60^1 + 27 \times 60^0 = 240 + 27 = 267$. So the Babylonian number



equals 267 in the Hindu-Arabic number system.

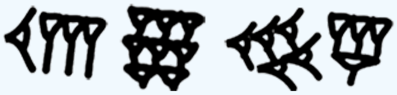
Your Turn 4.1.1

Convert the Babylonian number  into a Hindu-Arabic number.

Example

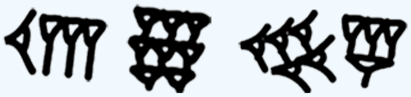
4.1.2: Converting Three-Digit Babylonian Numbers to Hindu-Arabic Numbers

Convert the Babylonian number

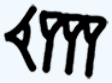


into a Hindu-Arabic number.

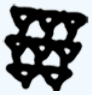
Answer



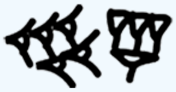
has three digits:



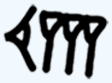
and



and

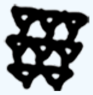


Step 1: So the first symbol,



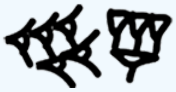
represents 13 in the Babylonian system. This is multiplied by 60 to the second power (since there are 3 digits), which gives us 13×60^2.

Step 2: The next symbol is



which represents 8 in the Babylonian system, is multiplied by 60 raised to the first power, which gives us $13 \times 60^2 + 8 \times 60^1$.

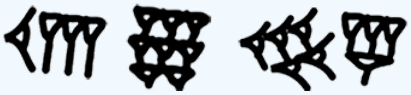
Step 3: The last digit is



representing 54, which is multiplied by 60 raised to 0, which gives $13 \times 60^2 + 8 \times 60^1 + 54 \times 60^0$.

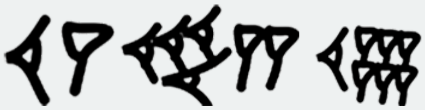
Step 4: Calculating that yields $13 \times 60^2 + 8 \times 60^1 + 54 \times 60^0 = 13 \times 3{,}600 + 8 \times 60 + 54 \times 1 = 46{,}800 + 480 + 54 = 47{,}334$.

So, the Babylonian number



equals 47,334 in the Hindu-Arabic number system.

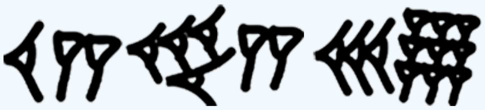
Your Turn 4.1.2

Convert the Babylonian number  into a Hindu-Arabic number.

Example

4.1.3: Converting Four-Digit Babylonian Numbers to Hindu-Arabic Numbers

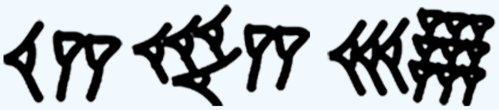
Convert the Babylonian number



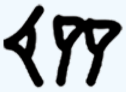
into a Hindu-Arabic number.

Answer

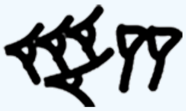
It appears that



has three digits, but there is a space in between

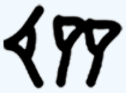


and



Remember, the Babylonian system has no 0, it instead employs a space where we expect a zero. This means this is a four digit number.

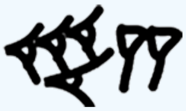
Step 1: The first symbol,



represents 12 in the Babylonian system. This is multiplied by 60 to the third power since there are four digits, which gives us 12×60^3.

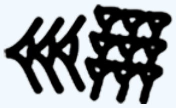
Step 2: The next symbol is a blank, which for us is a 0, representing 0×10^2, giving us $12 \times 60^3 + 0 \times 10^2$.

Step 3: The next symbol is



which represents 42 in the Babylonian system, is multiplied by 60 raised to the first power, which gives us $12 \times 60^3 + 0 \times 10^2 + 42 \times 60^1$.

Step 4: The last Babylonian digit,

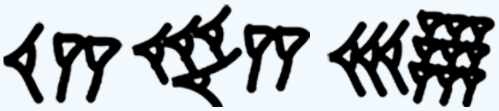


represents 39 in the Babylonian system. This is multiplied by 60 raised to 0, which gives $12 \times 60^3 + 0 \times 10^2 + 42 \times 60^1 + 39 \times 60^0$.

Step 5: Calculating that yields

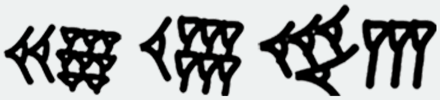
$$\begin{aligned}&12 \times 60^3 + 0 \times 10^2 + 42 \times 60^1 + 39 \times 60^0\\&= 12 \times 216{,}000 + 0 \times 10^2 + 42 \times 60 + 39 \times 1\\&= 2{,}592{,}000 + 0 + 2{,}520 + 39\\&= 2{,}594{,}559\end{aligned}$$

So the Babylonian number



equals 2,594,559 in the Hindu-Arabic number system.

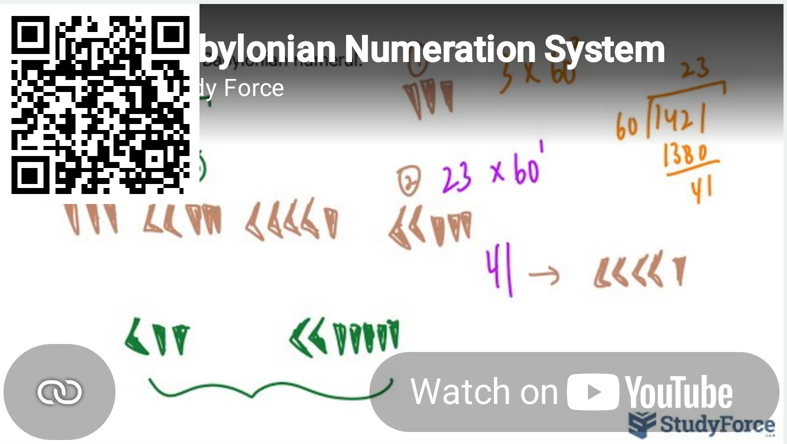
Your Turn 4.1.3

Convert the Babylonian number  into a Hindu-Arabic number.

Who Knew?: The Legacy of Babylonian System

The Babylonian system can still be seen today. An hour is 60 minutes, and a minute is 60 seconds. Additionally, when measuring angles in degrees, each degree can be split into 60 minutes (1/60th of a degree) and 60 seconds (1/60th of a minute).

Video



Converting Between Babylonian and Hindu-Arabic Numbers

Understand and Convert Mayan Numerals to Hindu-Arabic Numerals

The Mayans employed a positional system just as we do and the Babylonians did, but they based their position values on powers of 20 and they had a dedicated symbol for zero. Similar to the Babylonians, the Mayans would repeat symbols to indicate certain values. A single dot was a 1, two dots were a 2, up to four dots. Then a five was a horizontal bar. The horizontal bars could be used three times, since the fourth horizontal bar would make a 20, which was a new position in the number. The 0 was a special picture, which appears like a turtle lying on its back. The shell would then be "empty," so maybe that’s why the symbol was 0. The complete list is provided in Table 4.2. Another feature of Mayan numbers was that they were written vertically. The powers of 20 increased from bottom to top.

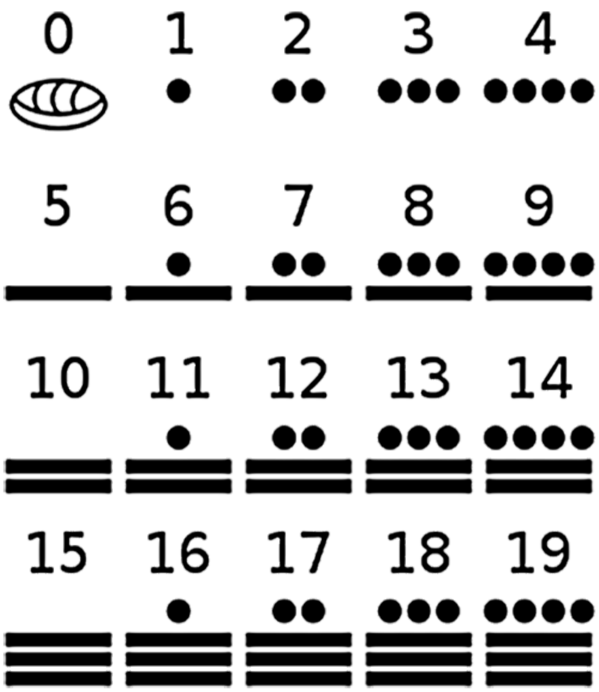

0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19

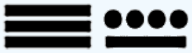
Table 4.1.2 Mayan Numerals

To summarize, the **Mayan system of numbers** used repeating symbol to indicate more than one, used place values, and employed a 0. So how do we convert from Mayan numbers to Hindu-Arabic numbers? To do so, we need to use the symbols from Table 4.2 and then place values based on powers of 20. If you have n digits in the Mayan number, you multiply the first "digit" by 20 raised to one less than the number of "digits." You then continue through the "digits," multiplying each by 20 raised to a power that is one smaller than the previous power. Fortunately, there is an explicit 0, so there is no ambiguity about numbers like 110, 101, and 11.

✓ Example

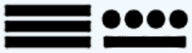
✓ 4.1.4: Converting Two-Digit Mayan Numbers to Hindu-Arabic Numbers

Convert the Mayan number

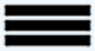


into a Hindu-Arabic number.

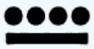
Answer



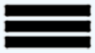
has two digits:



and

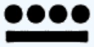


Step 1: So, the first symbol,



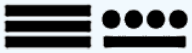
represents 15 in the Mayan system. This is multiplied by 20 to the first power, which gives us 15×20^1.

Step 2: The next symbol is



which represents 9 in the Mayan system. This is multiplied by 20 raised to 0, which gives $15 \times 20^1 + 9 \times 20^0$.

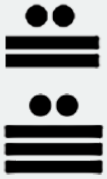
Step 3: Calculating that yields $15 \times 20^1 + 9 \times 20^0 = 300 + 9 = 309$. So



equals 309 in the Hindu-Arabic number system.

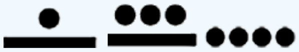
Your Turn 4.1.4

Convert the Mayan number into a Hindu-Arabic number.



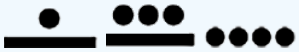
Example 4.1.5: Converting Three-Digit Mayan Numbers to Hindu-Arabic Numbers

Convert the Mayan number

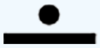


into a Hindu-Arabic number.

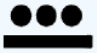
Answer



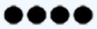
has three digits:



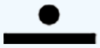
and



and

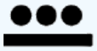


Step 1: So the first symbol,



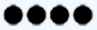
represents 6 in the Mayan system. This is multiplied by 20 to the second power (since there are 3 digits), which gives us 6×20^2.

Step 2: The next symbol is



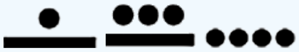
which represents 8 in the Mayan system, is multiplied by 20 raised to the first power, which gives us $6 \times 20^2 + 8 \times 20^1$.

Step 3: The last digit is



representing 4, which is multiplied by 20 raised to 0, which gives $6 \times 20^2 + 8 \times 20^1 + 4 \times 20^0$.

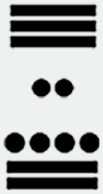
Step 4: Calculating that yields $6 \times 20^2 + 8 \times 20^1 + 4 \times 20^0 = 6 \times 400 + 8 \times 20 + 4 \times 1 = 2{,}400 + 160 + 4 = 2{,}564$. So the Mayan number



equals 2,564 in the Hindu-Arabic number system.

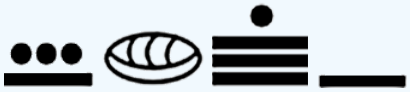
Your Turn 4.1.5

Convert the Mayan number into a Hindu-Arabic number.



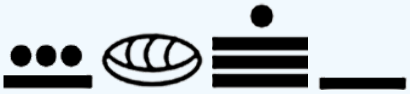
Example 4.1.6: Converting Four-Digit Mayan Numbers to Hindu-Arabic Numbers

Convert the Mayan number



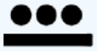
into a Hindu-Arabic number.

Answer



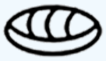
has four digits, so the first power of 20 that is used is 3.

Step 1: The first symbol,



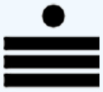
represents 8 in the Mayan system. This is multiplied by 20 to the third power (since there are four digits), which gives us 8×20^3.

Step 2: The next symbol is



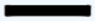
which is a 0, representing 0×20^2, giving us $8 \times 20^3 + 0 \times 20^2$.

Step 3: The next symbol is



which represents 16 in the Mayan system, is multiplied by 20 raised to the first power, which gives us $8 \times 20^3 + 0 \times 20^2 + 16 \times 20^1$.

Step 4: The last Mayan digit,

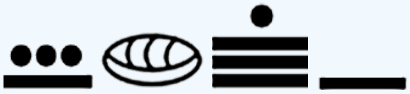


represents 5 in the Mayan system. This is multiplied by 20 raised to 0, which gives $8 \times 20^3 + 0 \times 20^2 + 16 \times 20^1 + 5 \times 20^0$.

Step 5: Calculating that yields

$$\begin{aligned} &8 \times 20^3 + 0 \times 20^2 + 16 \times 20^1 + 5 \times 20^0 \\ &= 8 \times 8000 + 0 \times 400 + 16 \times 20 + 5 \times 1 \\ &= 64{,}000 + 0 + 320 + 5 \\ &= 64{,}325 \end{aligned}$$

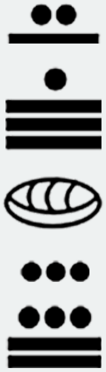
So the Mayan number



equals 64,325 in the Hindu-Arabic number system.

Your Turn 4.1.6

Convert the Mayan number into a Hindu-Arabic number.

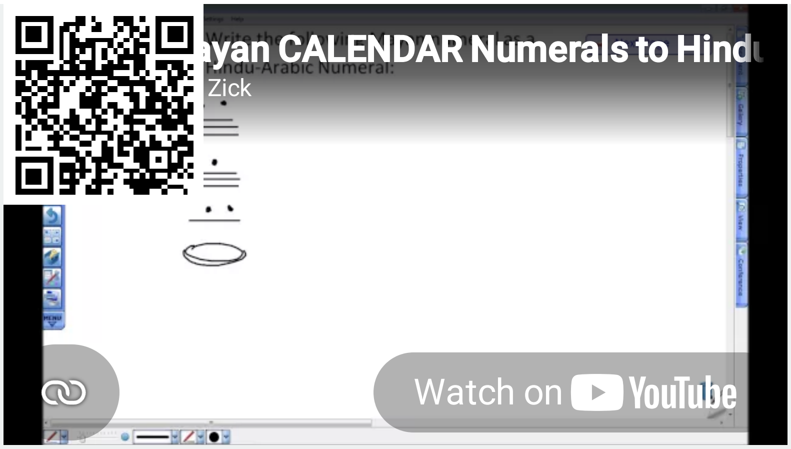


Who Knew?: The Mayan Calendar

The Mayans used this base 20 system for everyday situations. But their culturally important, and extremely accurate, calendar system used a slightly different system. For their calendars, they used a system where the place values were 1, 20, then 20*18, then 20*18*18. The reason for this is 20*18 is 360, which is closer to the number of days in a year. Had they used a purely base 20 system for their calendar, they'd be very far off with 400 days in a year.

Three hundred sixty days still left the Mayans a bit short, as there are 365 days in a year (ignoring leap years). The Mayan calendar also included 5 days, called Wayeb days, which brings their calendar to 365 days. As it happens, Wayeb is the Mayan god of misfortune, so these 5 days were considered the bad luck days.

Video


ayan CALENDAR Numerals to Hindu
Zick
Write the following numbers as a
Hindu-Arabic Numeral:
Watch on YouTube

Converting Mayan Numbers to Hindu-Arabic Numbers

Understand and Convert Between Roman Numerals and Hindu-Arabic Numerals

The Mayan and Babylonian systems shared two features, one of which we are familiar with (place value) and one that we don't use (repeated symbols). The **Roman system of numbers** used repeated symbols, but does not employ a place value. It also lacks a 0. The Roman system is built on the following symbols in Table 4.3.

Table 4.3 Roman Numerals

Roman Numeral	Hindu-Arabic Value
I	1
V	5
X	10
L	50
C	100
D	500
M	1,000

As in the Mayan and Babylonian systems, a symbol may be repeated to indicate a larger value. However, at 4, they did not use IIII. They instead used IV. Since the I came before the V, the number stands for "one before five." A similar process was used for 9, which was written IX, or "one before ten." The value 40 was written XL, or "ten before fifty," while 49 was written XLIX, or "forty plus nine."

The following are the rules for writing and reading Roman numerals.

- The representations for bigger values precede those for smaller values.
- Up to three symbols may be grouped together; for example, III for 3, or XXX for 30, or CC for 200.
- A larger value followed by a smaller value indicated addition; for example, VII for 7, XIII for 13, LV for 55, and MCC for 1200.
- I can be placed before V to indicate 4, or before X, to indicate 9. These are the only ways I is used as a subtraction.
- X can be placed before L to indicate 40, and before C to indicate 90. These are the only ways X is used as a subtraction.
- C can be placed before D to indicate 400, and before M to indicate 900. These are the only ways C is used as a subtraction.
- If multiple symbols are used, and a subtraction involving that symbol, the subtraction part comes after the multiple symbols. For example, XXIX for 29 and CCXC for 290.

Who Knew?: Legacy of Roman Numerals

The Roman numbering system is still used today in some situations. Many cornerstones of buildings have the year written in Roman numerals. Movie titles often represent the year the movie was produced as Roman numerals. The most recognizable might be that the Super Bowl is numbered using Roman numerals.

✓ Example 4.1.4: Converting Roman Numerals to Hindu-Arabic Numbers

Convert the following Roman numerals into Hindu-Arabic numerals.

1. XXVII
2. XXXIV
3. MMCMXLVIII

Answer

1. The numeral XXVII begins with two X’s, which is then followed by a V. So, the two X’s combine to be 20. The V is followed by two I’s, so the V indicates the addition of 5. The two I’s that follow indicate addition of two. That ends the symbols, so the value is 20 plus 5 plus 2, or 27 in Hindu-Arabic numerals.
2. The numeral XXXIV begins with three X’s, which is then followed by an I. So, the three X’s combine to be 30. The I is followed by a V, which indicates 4. That ends the symbols, so the value is 30 plus 4, or 34 in Hindu-Arabic numerals.
3. The numeral MMCMXLVIII begins with two M’s, which is then followed by a C. So, the two M’s combine to make 2000. The C is followed by an M, which indicates 900. The CM is followed by XL, which indicates 40. The L is followed by V, which indicates 5. The V is followed by three I’s, indicating 3. Adding those values yields 2,948.

Your Turn 4.1.7

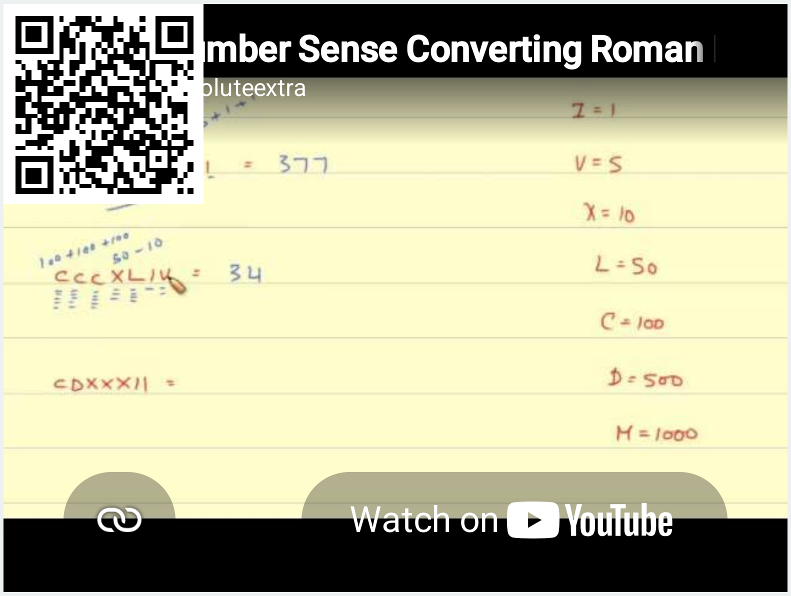
Convert the following Roman numerals into Hindu-Arabic numerals.

1. LXXVII

CCXL

MMMCDXLVII

Video



Converting From Roman Numbers to Hindu-Arabic Numbers

Of course, we can convert from Hindu-Arabic numerals, to Roman numerals, too.

✓ Example 4.1.8: Converting Hindu-Arabic Numbers to Roman Numerals

Convert the following Hindu-Arabic numerals into Roman numerals.

1. 38
2. 94
3. 846
4. 2,987

Answer

1. Thirty is represented as three X's, and the 8 is represented with VIII, so 38 in Roman numerals is XXXVIII.
2. Ninety is represented by XC, and four is represented by IV, so 94 in Roman numerals is XCIV.
3. The number is less than 900 and more than 500, so the first symbol to be used is D, which is 500. To get to 800, we need 300 more, which is represented with three C's. Forty is represented with XL, and the six. The Roman numerals are DCCCXLVI.
4. The two thousand is represented by two M's. The 900 is represented by CM. The 80 is represented by LXXX (50 plus 30). Finally, the 7 is represented by VII. We have that 2,987 in Roman numerals is MMCMLXXXVII.

Your Turn 4.1.8

Convert the following Hindu-Arabic numerals into Roman numerals.

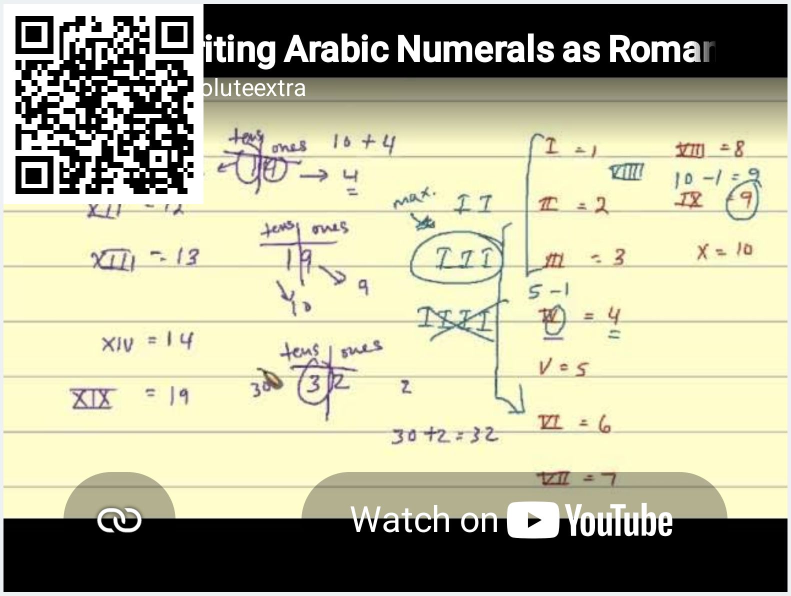
27

49

739

3,647

Video

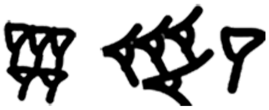


Converting From Hindu-Arabic Numbers to Roman Numbers

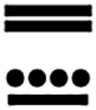
Check Your Understanding

1. What is the place value for Babylonian numerals?
2. What place value is used in the Mayan numeration system?
3. What place value is used for Roman numerals?

4.

Convert the Babylonian numeral  into a Hindu-Arabic numeral.

5. Convert the Mayan numeral into a Hindu-Arabic numeral.



6. Convert the Roman numeral CCXLVII into a Hindu-Arabic numeral.

7. Convert 479 into a Roman numeral.

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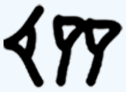
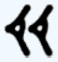
4.1.0: Exercises

For the following exercises, convert the Babylonian numeral into a Hindu-Arabic numeral.

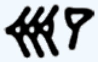
Exercise 4.1.0.1



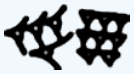
Exercise 4.1.0.2



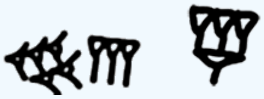
Exercise 4.1.0.3



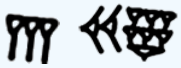
Exercise 4.1.0.4



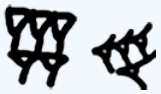
Exercise 4.1.0.5



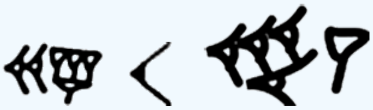
Exercise 4.1.0.6



Exercise 4.1.0.7

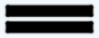


Exercise 4.1.0.8

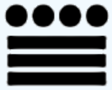


For the following exercises, express the Mayan numeral as a Hindu-Arabic numeral. Use the common system, which is based on powers of 20 only.

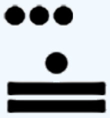
? Exercise 4.1.0.9



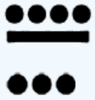
? Exercise 4.1.0.10



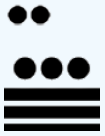
? Exercise 4.1.0.11



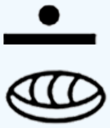
? Exercise 4.1.0.12



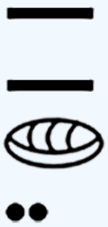
? Exercise 4.1.0.13



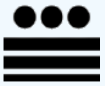
? Exercise 4.1.0.14

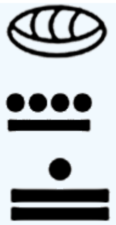


? Exercise 4.1.0.15



? Exercise 4.1.0.16





For the following exercises, express the Roman numeral as a Hindu-Arabic numeral.

? Exercise 4.1.0.17

VII

? Exercise 4.1.0.18

XI

? Exercise 4.1.0.19

IX

? Exercise 4.1.0.20

XXIV

? Exercise 4.1.0.21

MCXLII

? Exercise 4.1.0.22

CXXII

? Exercise 4.1.0.23

DCCXLIV

? Exercise 4.1.0.24

MCMLIX

For the following exercises, express the Hindu-Arabic numeral as a Roman numeral.

? Exercise 4.1.0.25

8

? Exercise 4.1.0.26

14

? Exercise 4.1.0.27

27

? Exercise 4.1.0.28

94

? Exercise 4.1.0.29

274

? Exercise 4.1.0.30

487

? Exercise 4.1.0.31

936

? Exercise 4.1.0.32

2,481

? Exercise 4.1.0.33

What uses a place value system for numbers: Roman, Babylonian, Egyptian, Greek?

? Exercise 4.1.0.34

What uses an additive system: Roman, Mayan, Egyptian, Greek?

? Exercise 4.1.0.35

What uses a 0: Roman, Mayan, Egyptian, Greek?

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4.2: Converting with Base Systems

Figure $4.2.1$: Computers use Base 2, which only uses 0's and 1's, to represent quantity. (credit: modification of work "IMAG0933" by yvanhou/Flickr, CC BY 2.0)

Learning Objectives

1. Convert another base to base 10.
2. Write numbers in different base systems.
3. Convert base 10 to other bases.
4. Determine errors in converting between bases.

In our system of numbers, we use base 10, but using base 10 was not a given within other systems. There were other systems that used bases other than 10, as we saw with the Mayans and the Babylonians. The base 10 system comes down to grouping objects in sets of 10, but grouping in sets of 10 only happens if the culture values grouping by that many. We feel 10 is natural because we have 10 fingers. There are other systems using other grouping values, such as 4 or 20.

One good reason for examining other bases is to remind ourselves how we had to learn arithmetic when we were young, memorizing rules for our base 10 system. We had to learn why those arithmetic rules made sense, such as why 1+1=2 and 1+2=3. Another good reason for learning other base systems is due to computers; their circuitry instead uses base 2.

In this section, we explore other base systems and how to convert between them.

Conversion of Another Base into Base 10 and Other Bases

We saw in Hindu-Arabic Positional System that our Hindu-Arabic system uses base 10, which is a system using place values of digits that depend on powers of 10 (or, are based on powers of 10). We've already worked with bases other than base 10: The Babylonian system was base 60, while the Mayan system was base 20.

To explore how our base 10 system is used, answer the following question: What's the following quantity: 4,572? You probably said four thousand five hundred seventy-two (no, there is no "and" between hundred and seventy). But why do you think that 4 means four thousand? A very young person when learning their numbers might say that's a four five seven and two. But you added the context of thousands to the four. Why?

Place value, that's why. You learned early on that where the numeral was gave it different meanings. Ten thousands, thousands, hundreds, tens, and ones. So, you translate that symbol string (4,572) into "four thousand five hundred seventy-two." As we saw in Hindu-Arabic Positional System, expanding a Hindu-Arabic number involved writing the number using each digit times its appropriate power of 10. So, we could write 4,572 as $4 \times 10^3 + 5 \times 10^2 + 7 \times 10^1 + 2 \times 10^0 = 4 \times 1000 + 5 \times 100 + 7 \times 10 + 2 \times 1$.

One possible reason we use base 10 is that we have 10 fingers, and in the cultures where the Hindu-Arabic system developed, that became the standard. Other cultures may have used other ways of organizing numbers, perhaps using 20 by including toes, or using 60 because 60 has many divisors. Mathematically though, base 10 is an awkward base to work in since 10 has limited divisors. But we think it is easy and simple because that's what we've been taught to use.

Using a base 10 system means we need 10 symbols to make our numbering system work: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

Now imagine that we all only had 6 fingers instead of 10 and our counting system was based on those 6 fingers. We would be counting in groups of 6, not groups of 10. How would this change how we work with quantity?

First, we'd need only six symbols. Let's use 0, 1, 2, 3, 4, 5. Second, our place values would be based on powers of 6, not powers of 10. For instance, the number 3,024 in base 6 would be $3 \times 6^3 + 0 \times 6^2 + 2 \times 6^1 + 4 \times 6^0$. That is how you can translate a base 6 number into a base 10 number. When we calculate that expression we get $3 \times 6^3 + 0 \times 6^2 + 2 \times 6^1 + 4 \times 6^0 = 3 \times 216 + 0 \times 36 + 2 \times 6 + 4 \times 1 = 648 + 0 + 12 + 4 = 664$.

This means the base 6 number 3,024 is equal to the base 10 number 664.

From now on, if we are using a base 6 number, we will follow it with the subscript 6, like the following: $3{,}024_6$ means the number is in base 6.

A base 10 number gets no subscript (it's the standard). So, 3,024 is a base 10 number. A base 13 number would be $4{,}672_{13}$.

So, a base 6 system uses only the symbols 0, 1, 2, 3, 4, and 5. Also, the place values use powers of 6. However, we still don't know how to count in base 6. In order to do so, we'd have to know how to represent the quantities larger than five in base 6. Let's review how our base 10 system works by counting from 0 to 100, which shows how larger values are represented.

In writing the base 10 numbers, you start with these first 10 values:

0	1	2	3	4	5	6	7	8	9

But you've run out of symbols. So, we use two digits:

10	11	12	13	14	15	16	17	18	19

The 1 out front means you've run out of digits one time.

But now you've run out twice. Continuing with those numbers gives:

20	21	22	23	24	25	26	27	28	29

And so on,

30	31	32	33	34	etc.…

Eventually, you hit the 90s,

90	91	92	93	94	95	96	97	98	99

And you've run out of the digits again! So, we say we've run out of digits in the tens place one time, hence:

100	101	102	103	104	105	106	107	108	109

That's the pattern we use in base 10. We write out the symbols until we've used all the symbols, then add a digit in front that counts how many times we've used the digits. Knowing the numbers, or being able to count higher and higher, is necessary to understand how all the arithmetic works, as it all goes back to counting.

The counting pattern is the same for any other base, including base 6. So, let's start:

0	1	2	3	4	5

But we've run out of symbols! Just like in base 10, we use a second digit, where the first digit will tell us we've run out of symbols one time.

10	11	12	13	14	15

And we use the same pattern:

20	21	22	23	24	25

30	31	32	33	34	35
40	41	42	43	44	45
50	51	52	53	54	55

But we’ve run out of symbols for that front digit. So, we indicate it the same way as in base 10…by adding a third digit in front, indicating we’ve run out of symbols once in the second place:

100	101	102	103	104	105
110	111	etc.			

The symbol pattern is the same, but truncated. We only use the six symbols. So that is how we represent base 6. Being able to write out these numbers is important when working with addition in the base.

When using a base larger than 10, though, we need more symbols. Instead of creating new symbols, we use capital letters, with A representing the digit for "10," B representing the digit for "11," and so on.

Example 4.2.1 2: Determining Digits of a Base with Less Than 10 Digits

What are the digits used for base 7?

Answer

Since this is base 7, we need only 7 symbols: 0, 1, 2, 3, 4, 5, 6.

Your Turn 4.2.1

What are the digits for base 4?

Example 4.2.2: Determining Digits of a Base with More Than 10 Digits

What are the digits used for base 14?

Answer

Since this is base 14, we need 14 symbols. We don’t have single character numbers for 10, 11, 12, and 13, so, in a fit of inspired creativity, we use capital letters A, B, C, D to represent those quantities. So, the digits in base 14 are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D.

Your Turn 4.2.2

What are the digits used for base 12?

Who Knew?: Using Base 12

As mentioned in the text, working in base 10 is mathematically awkward. Ten has only two natural number divisors: 2 and 5. This means dividing into groups is not easy. However, 12, or a dozen, has more divisors: 2, 3, 4, and 6. The Dozenal Society recognizes this more mathematically pleasant detail. It advocates for a switch to using base 12 for numbers. Their argument is based on the divisibility of the number 12. But has there ever been a society that used such a system? The answer is yes. A dialect of the Gwandara language in Nigeria uses the base 12 system. It is unlikely, though, that the Dozenal Society will achieve their goal, as the base 10 system is so entrenched in our society.

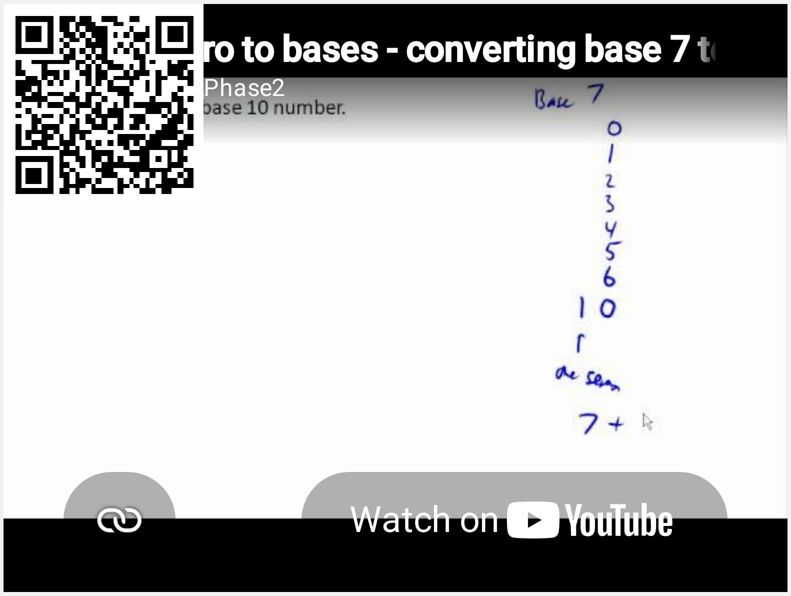
✓ Example ✓ 4.2.3: Converting from One Base into Another

Convert $3{,}601_7$ into base 10.

Answer

In base 7, the place values are powers of 7. Since there are four digits, the highest power of 7 that is used is 3. This yields $3{,}601_7 = 3 \times 7^3 + 6 \times 7^2 + 0 \times 7^1 + 1 \times 7^0 = 3 \times 343 + 6 \times 49 + 0 \times 7 + 1 = 1{,}029 + 294 + 0 + 1 = 1{,}324$.

Video



Convert Base 7 to Base 10

Your Turn 4.2.3

Convert 421_6 into base 10.

✓ Example 4.2.4: Converting from Base 14 to Base 10

Convert $4B7_{14}$ into base 10.

Answer

In base 14, the place values are powers of 14. Since there are three digits, the highest power of 14 is 2. Also recall that in base 14, 10 is represented by A, 11 is represented by B, 12 is represented by C, and 13 is represented by D. Using that, we convert to base 10:

4B714=4×142+B×141+7×140=4×196+11×14+7×1=784+154+7=9454B714=4×142+B×141+7×140=4×196+11×14+7×1=784+154+7=945.

Your Turn 4.2.4

Convert $A3C_{14}$ into base 10.

Example 4.2.5: Converting from Base 12 to Base 10

Convert $A16_{12}$ into base 10.

Answer

In base 12, the place values are powers of 12. Since there are three digits, the highest power of 12 is 2. Also recall that in base 12, 10 is represented by A and 11 is represented by B. Using that, we convert to base 10:

A1614=10×122+1×121+6×120=10×144+1×12+6×1=1,440+12+6=1,458A1614=10×122+1×121+6×120=10×144+1×12+6×1=1,440+12+6=1,458.

Your Turn 4.2.5

Convert $5AB_{12}$ into base 10.

Example 4.17: Converting from Base 2 to Base 10

Convert 1011_2 into base 10.

Answer

In base 2, the place values are powers of 2. Since there are four digits, the highest power of 2 is 3. Using that, we convert to base 10:

$$1011_2 = 1\times 2^3 + 0\times 2^2 + 1\times 2^1 + 1\times 2^0 = 1\times 8 + 0\times 4 + 1\times 2 + 1\times 1 = 8+0+2+1 = 11$$

.

Your Turn 4.2.5

Convert 11011_2 into base 10.

Who Knew?: Before Napoleon

Before Napoleon's France, which adopted the base 10 system, a modified base 12 system was often used in Europe. Twelve is easily divisible into groups of 2, 3, 4, and 6, which makes it easier to work with. Even our numbering system retains a bit of this. You have likely noticed that we use the words *thirteen*, *fourteen*, *fifteen*, and so on to indicate 10 and 3, 10 and 4, 10 and 5, and so one. Even the 20s reinforce this idea, as in twenty-one, and twenty-two. However, two numbers don't follow this pattern, namely 11 and 12. If they followed the same rules, they'd be one teen and two teen. We even have a special word for 12; that is, a dozen. However, etymologically speaking, the words *eleven* and *twelve* are likely derived by referencing the number 10. These two numbers may date back to the Old English words *endleofan* and *twelf*, which can be traced back further to *ain lif* and *twa lif*. The word *lif* here may be the base word for "to leave." This would suggest *ain lif* is one left after 10, and *twa lif* is two left after 10, or, 11 and 12.

Example 4.2.6: Writing Numbers in Base Systems Other Than Base 10

Write the numbers in base 7 up to 100_7.

Answer

Step 1: Using the patterns we indicated earlier, we begin with the first seven digits.

0, 1, 2, 3, 4, 5, 6

Step 2: Since we've run out of digits, we start with 10, indicating we've run out of symbols once.

10, 11, 12, 13, 14, 15, 16

Step 3: Continuing in the same way, we get:

20, 21, 22, 23, 24, 25, 26

30, 31, 32, 33, 34, 35, 36

40, 41, 42, 43, 44, 45, 46

50, 51, 52, 53, 54, 55, 56

60, 61, 62, 63, 64, 65, 66

Now, all the digits have been used in the leading digits. Since the digits have all been used in that leading digit, we use 100, as in base 10.

100

Your Turn $4.2.6$

Write the numbers of base 4 up to 100_4.

Example $4.2.7$: Writing Numbers in Bases with More Than 10 Symbols

Write the numbers in base 14 up to 100_{14}.

Answer

Step 1: Using the patterns we indicated earlier, we begin with the first 14 digits.

0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D

Step 2: Since we've run out of digits, we start with 10, indicating we've run out of symbols once.

10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 1A, 1B, 1C, 1D

Step 3: Continuing in the same way, we get:

20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 2A, 2B, 2C, 2D

30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 3A, 3B, 3C, 3D

40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 4A, 4B, 4C, 4D

50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 5A, 5B, 5C, 5D

60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 6A, 6B, 6C, 6D

70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 7A, 7B, 7C, 7D

80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 8A, 8B, 8C, 8D

90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 9A, 9B, 9C, 9D

A0, A1, A2, A3, A4, A5, A6, A7, A8, A9, AA, AB, AC, AD

B0, B1, B2, B3, B4, B5, B6, B7, B8, B9, BA, BB, BC, BD

C0, C1, C2, C3, C4, C5, C6, C7, C8, C9, CA, CB, CC, CD

D0, D1, D2, D3, D4, D5, D6, D7, D8, D9, DA, DB, DC, DD

100

Your Turn 4.2.7

Write the numbers of base 12 up to 100_4.

Base 2 is important in the digital age, as it is the system used by computers. It is the simplest base to work with, but has the drawback that the numbers in base 2 may use many, many digits. In Addition and Subtraction in Base Systems and Multiplication and Division in Base Systems, we will look at base 2 in each situation.

Example 4.2.8: Writing Numbers in Base 2

Write the numbers in base 2 up to 100_2.

Answer

Base 2 uses only two symbols: 0 and 1. Following the pattern established previously, the numbers in base 2 up to 100_2 are 0, 1, 10, 11, and 100.

Your Turn 4.2.8

Write the numbers in base 3 up to 100_3.

Who Knew?: Early Hawaiian Numeration System

Before the British arrived in Hawaii, people there used a system that combined two different bases. Objects were initially grouped into collections of four, and a collection of four was referred to as *kauna*. A person could have two *kauna* and three “ones” (in Hindu-Arabic, 11). Or they could have eight *kauna* and one “ones” (In Hindu-Arabic, 33). However, sets of four were grouped in collections of 10. A set of 10 *kauna* was *ka’au*. The collections of *ka’au* were grouped by 10 also. Which meant that 10 *ka’au* (this is 40 in Hindu-Arabic) would be *lau* (or 400 in Hindu-Arabic). What this shows is that the Hawaiian culture developed a system that used base 4 combined with base 10.

Conversion of Base 10 into Another Base

Converting from base 10 into another base uses repeated division, recording the remainder at each step. Then, the number in the new base is the remainder starting from the last remainder found. To be accurate in what we’re saying, we need to remind ourselves of some terminology associated with division. When integers are divided, the one being divided is the dividend, and the one that is dividing the dividend is the divisor. The quotient is the largest natural number that can be multiplied by the divisor where the product is less than the dividend.

When the integer n is divided by the integer d, n is called the **dividend** and d is the **divisor.**

To convert a base 10 number n into base d, we divide n by d, recording the remainder. Then we divide the quotient from that step by the base d, and record the remainder again. We continue this process until the quotient is 0. Then, the base d number has digits that start with the last remainder and use each remainder in reverse order.

Example 4.2.9: Converting from Base 10 into a Lower Base

Convert 298 to base 6.

Answer

We divide 298 by 6, and record the remainder. Then we divide the quotient from that step by 6, and record the remainder again. We continue this process until the quotient is 0. Then, the base 6 number has digits that start with the last remainder and use each remainder in reverse order.

Step 1: When we divide 298 by 6, we get 49r4629849r46298. The quotient is 49 and the remainder is 4.

Step 2: Now we divide the quotient, 49, by 6. This gives 8r16498r1649. The quotient is 8 and the remainder is 1.

Step 3: Repeating, we get 1r2681r268. The quotient is 1 and the remainder is 2.

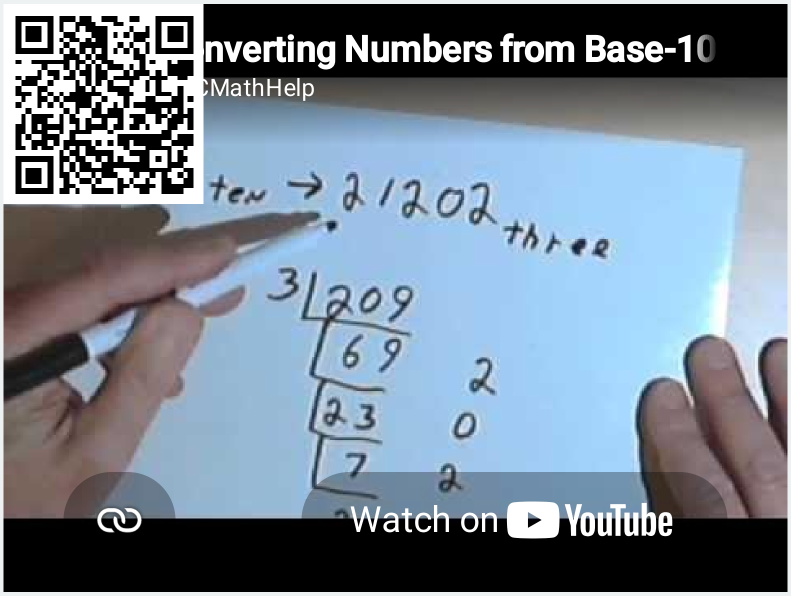
Step 4: Finally, we perform the operation on the quotient 1, 0r1610r161 giving us a quotient of 0 and a remainder of 1.

Step 5: The base 6 number has digits equal to the remainders in reverse order, 1214_6. So, 298 in base 10 when converted to base 6 is 1214_6.

Your Turn 4.2.9

Convert 693 to base 7.

Video



Converting from Base 10 to Another Base

Example 4.2.10: Converting from Base 10 into a Higher Base

Convert 45,134 to base 13.

Answer

We divide 45,134 by 13, and record the remainder. Then we divide the quotient from that step by 13, and record the remainder again. We continue this process until the quotient is 0. Then, the base 13 number has digits that start with the last remainder and use each remainder in reverse order. It is at this step that we'll convert to the base 13 digits, which are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C.

Step 1: When we divide 45,134 by 13, we get 3,471r111345,134 3,471r111345,134. The quotient is 3,471 and the remainder is 11.

Step 2: Now we divide the quotient, 3,471, by 13. This gives 267r0133,471267r0133,471. The quotient is 267 and the remainder is 0.

Step 3: Repeating, we get 20r71326720r713267. The quotient is 20 and the remainder is 7.

Step 4: Again, and we get 1r713201r71320. The quotient is 1 and the remainder is 7.

Step 5: Finally, we get 0r11310r1131, with quotient 0 and a remainder 1.

Step 6: The base 13 number has digits equal to the remainders in reverse order, which were 1, 7, 7, 0, and 11. The 11 is written as B in base 13. So, 45,134 in base 10 when converted to base 13 is $1770B_{13}$.

Your Turn 4.2.10

Convert 9,275 to base 12.

Example 4.2.11: Converting from Base 10 into Base 2

Convert 100 to base 2.

Answer

Following the pattern above:

Step 1: We divide 100 by 2, and record the remainder.

Step 2: Then we divide the quotient from that step by 2, and record the remainder again.

Step 3: We continue this process until the quotient is 0.

Step 4: Following this process, the remainders are, in order, 0, 0, 1, 0, 0, 1, 1. Writing those in reverse order gives the number in base 2, 1100100_2.

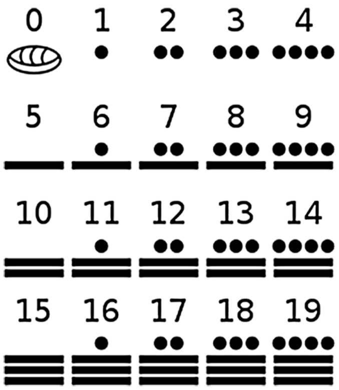
Notice that 100 in base 2 took seven digits.

Your Turn 4.2.11

Convert 137 to base 2.

Converting from Hindu-Arabic Numbers to Mayan Numbers

To convert from a Hindu-Arabic number to a Mayan number involves two distinct processes. First, the number must be converted to base 20, using the process described and demonstrated previously. Next, that base 20 number has to be written using Mayan numerals. For reference, the Mayan numerals and their values are below.



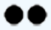
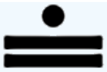
Example 4.2.12: Converting from Base 10 into the Mayan System

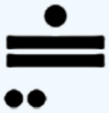
Convert the following into Mayan numbers.

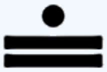
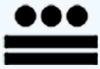
1. 51
2. 653

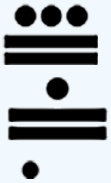
Answer

1. The Mayan system is base 20, so we must use 20 in the process from above. The first division has a quotient of 2 and remainder of 11. The 11 serves as the “ones” digit. Dividing that quotient, 2, by 20 has a quotient of 0 with a remainder of 2. The 2 becomes the “twenties” digit of the number. So, in base 20, the number would be 2 followed by 11. The

Mayan symbols for 2 and 11 are  and . Writing these vertically, with the “ones” digit on top, as appropriate for Mayan numbers, results in:



2. The Mayan system is base 20, so we must use 20 in the process from above. The first division, 673 divided by 20, has a quotient of 32 and remainder of 13. Dividing that quotient, 32, by 20 has a quotient of 1 with a remainder of 12. Dividing that quotient, 1, by 20 has a quotient of 0 and a remainder of 1. Since there are three remainders here, this is a three-digit number. The 1 is the “20-squared” digit, the 12 is the “twenties” digit, and the 13 is the “ones” digit. So, in base 20, the number would be 1 followed by 12 followed by 13.. The Mayan symbols for 1, 12 and 13 are , , and . Writing these vertically, as appropriate for Mayan numbers, would result in:



Your Turn 4.2.12

Convert the following into Mayan numbers.

1372.

2,171

Who Knew?: Other Languages, Other Bases

There have been base systems that use bases other than 10. Some bases used were 20, 12, and 27! Visit this site to see more on the languages that used other bases.

Errors in Converting Between Bases

There are some common errors that are made when converting between bases. Often, it comes down to using an “illegal” symbol in the new base.

Example 4.2.13: Detecting an Illegal Symbol When Converting Between Bases

A base 10 number is converted to base 7 and the result was 2081_7. Was an error committed? How do you know?

Answer

The result has the digit 8 in it. In base 7, 8 is an illegal symbol. Based on that, an error was committed.

Your Turn 4.2.13

A base 10 number is converted to base 4 and the result was 3702_4. Was an error committed? How do you know?

When converting from base 10 to another base, an illegal symbol will be used if a mistake was made in the division process used to find the number in the new base. Since the digits are based on the remainders, any remainder that is an illegal symbol would indicate an error.

✓ Example 4.2.14: Detecting an Error in Division When Converting Between Bases

When changing from base 10 to base 8, the division process resulted in the following remainders: 1, 0, 9, 2, 4. Was an error committed? How do you know?

Answer

The remainders include 9, which in base 8 is an illegal symbol.

Your Turn 4.2.14

When changing from base 10 to base 6, the division process resulted in the following remainders: 5, 0, 0, 10. Was an error committed? How do you know?

Another possible way to detect an error in converting between bases is to count the number of digits. When converting from a higher base to a lower base, the number of digits cannot get smaller. Similarly, when converting from a lower base to a higher base, the number of digits cannot get bigger. So, if a base 10 number is converted to a base 3 number, the number of digits in the new base 3 numbers cannot be less than the number of digits in the base 10 number. Similarly, if a base 7 number is converted to base 10, the number of digits in the base 10 number cannot be more than the number of digits in the original base 7 number.

✓ Example 4.2.15: Detecting an Error in Number of Digits When Converting Between Bases

A five-digit base 10 number is converted to a base 5 number. The base 5 number has four digits. Was an error committed? How do you know?

Answer

Since 10 is larger than 5, the base 5 number cannot have less digits than the base 10 number. Since it did, we know an error has been made.

Your Turn 4.2.15

A six-digit base 12 number is converted into a base 10 number. The base 10 number has five digits. Was an error committed? How do you know?

Check Your Understanding

1. In base 25, how many symbols would be necessary?
2. In base 18, what would the place value of the 4 be in the number 348_{18} be?
3. Convert 2304_5 into base 10.
4. When counting in base 9, what number would follow 38_9?
5. Convert 329 into base 8.
6. Convert ABC_{14} into base 10.
7. How do you know a mistake was made when converting from base 10 to base 4 and the result is 152_4?

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4.2.0: Exercises

? Exercise $4.2.0.1$

What does it mean when we say a number is written in base 7?

? Exercise $4.2.0.2$

What does it mean when we say a number is written in base 12?

? Exercise $4.2.0.3$

How many symbols are there in a base 3 system? What are they?

? Exercise $4.2.0.4$

How many symbols are there in a base 15 system? What are they?

? Exercise $4.2.0.5$

List the numbers, up to 100, in the base 5 system.

? Exercise $4.2.0.6$

List the numbers, up to 100, in the base 2 system.

For the following exercises, convert the number into a base 10 number.

? Exercise $4.2.0.7$

14_5

? Exercise $4.2.0.8$

21_6

? Exercise $4.2.0.9$

12_3

? Exercise $4.2.0.10$

34_5

? Exercise $4.2.0.11$

14_8

? Exercise $4.2.0.12$

78_9

? Exercise 4.2.0.13

$3B_{12}$

? Exercise 4.2.0.14

241_6

? Exercise 4.2.0.15

101_2

? Exercise 4.2.0.16

$4A7_{14}$

? Exercise 4.2.0.17

804_9

? Exercise 4.2.0.18

101001_2

? Exercise 4.2.0.19

3223_6

? Exercise 4.2.0.20

1436_7

? Exercise 4.2.0.21

$8A0BD_{15}$

? Exercise 4.2.0.22

110202_3

? Exercise 4.2.0.23

$100A4_{12}$

? Exercise 4.2.0.24

1010000101_2

For the following exercises, convert the base 10 number into the given base.

? Exercise 4.2.0.25

12 into base 4

? Exercise 4.2.0.26

25 into base 2

? Exercise 4.2.0.27

43 into base 12

? Exercise 4.2.0.28

153 into base 5

? Exercise 4.2.0.29

203 into base 2

? Exercise 4.2.0.30

431 into base 4

? Exercise 4.2.0.31

543 into base 12

? Exercise 4.2.0.32

1,023 into base 2

? Exercise 4.2.0.33

2,876 into base 4

? Exercise 4.2.0.34

1,765 into base 5

? Exercise 4.2.0.35

1,993 into base 7

? Exercise 4.2.0.36

2,000 into base 2

? Exercise 4.2.0.37

4,368 into base 12

Exercise 4.2.0.38

12,562 into base 16 (Hint: Base 16 uses the symbols 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F.)

Exercise 4.2.0.39

When converting from base 10 to base 6, a student finds the following remainders: 3, 1, 3, 6. How do you know that a mistake was made?

Exercise 4.2.0.40

When converting from base 10 to base 4, a student finds the following remainders: 0, 0, 3, 7, 2. How do you know a mistake was made?

Exercise 4.2.0.41

Suppose a base 12 number is converted into a base 10 number, and one of the digits is A. Was an error committed? How do you know?

Exercise 4.2.0.42

Suppose a base 10 number is converted into a base 5 number and one of the digits is 6. Was an error committed? How do you know?

Exercise 4.2.0.43

Suppose a base 2 number is converted into a base 10 number, and the base 10 number has more digits that the base 2 number. Was an error committed? How do you know?

Exercise 4.2.0.44

Suppose a base 16 number is converted to base 2, and the base 2 number has fewer digits than the base 16 number. Was an error committed? How do you know?

For the following exercises, convert the Hindu-Arabic number into a Mayan number.

Exercise 4.2.0.45

25

Exercise 4.2.0.46

71

Exercise 4.2.0.47

400

Exercise 4.2.0.48

723

The Babylonian system used base 60. To convert from Hindu-Arabic numbers into Babylonian numbers, the process for converting from base 10 to a different base would be done first. Then, the results found in the conversion process would be changed to

Babylonian numerals. This process is similar to the one for Mayan numbers. For the following exercises, convert the Hindu-Arabic number to a Babylonian number.

? Exercise $4.2.0.49$

67

? Exercise $4.2.0.50$

135

? Exercise $4.2.0.51$

781

? Exercise $4.2.0.52$

10,952

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4.3: Addition and Subtraction in Base Systems

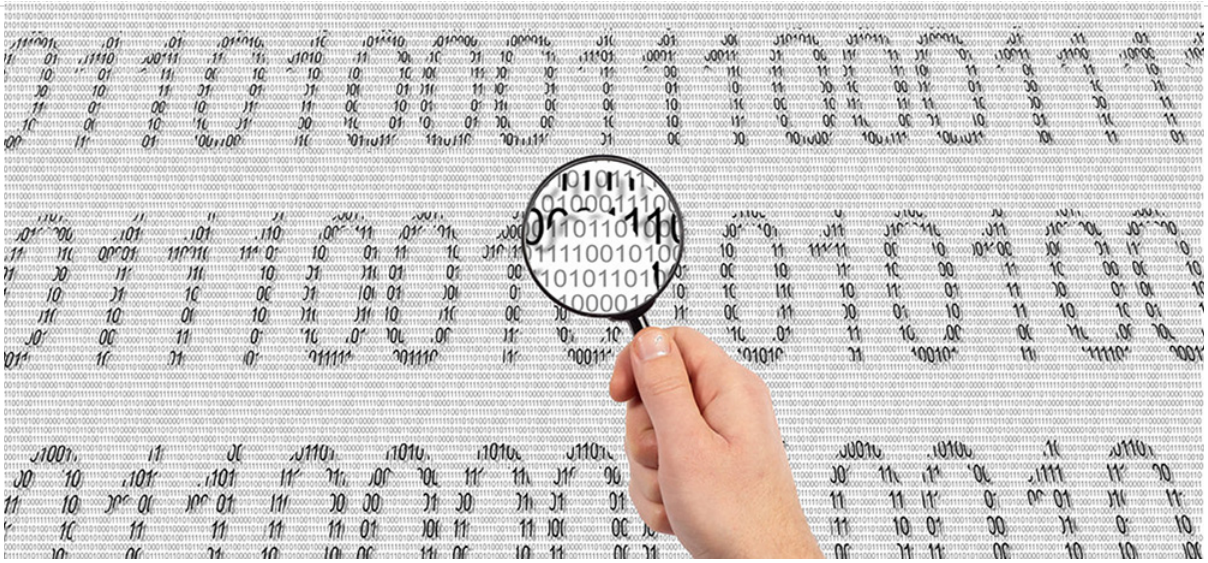


Figure 4.3.1: All information in computers is represented by 0's and 1's, including quantity, which means computers use Base 2 for arithmetic. (credit: modification of work “Magnifying glass and binary code” by Marco Verch Professional Photographer/Flickr, CC BY 2.0)

Learning Objectives

1. Add and subtract in bases 2–9 and 12.
2. Identify errors in adding and subtracting in bases 2–9 and 12.

Once we decide on a system for counting, we need to establish rules for combining the numbers we’re using. This begins with the rules for addition and subtraction. We are familiar with base 10 arithmetic, such as $2+5=7$ or $3\times 5=15$. How does that change if we instead use a different base? A larger base? A smaller one? In particular, computers use base 2 for all number representation. When your calculator adds or subtracts, multiplies or divides, it uses base 2. This is because the circuitry recognizes only two things, high current and low current, which means the system is uses only has two symbols. Which is what base 2 is.

In this section, we use addition and subtraction in bases other than 10 by referencing the processes of base 10, but applied to a new base system.

Addition in Bases Other Than Base 10

Now that we understand what it means for numbers to be expressed in a base other than 10, we can look at arithmetic using other bases, starting with addition. When you think back to when you first learned addition, it is very likely you learned the addition table. Once you knew the addition table, you moved on to addition of numbers with more than one digit. The same process holds for addition in other bases. We begin with an addition table, and then move on to adding numbers with two or more digits.

We worked with base 6 earlier, and have the numbers in base 6 up to 100_6. Using that table of values, we can create the base 6 addition table.

Here’s the beginning of the base 6 addition table:

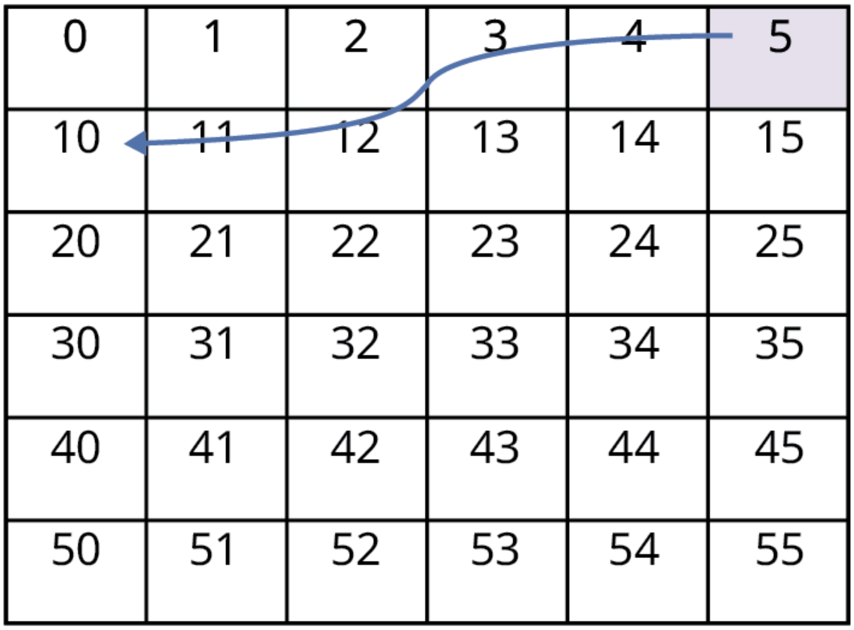
+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	?
2	2	3	4	5	?	?
3	3	4	5	?	?	?
4	4	5	?	?	?	?

5	5	?	?	?	?	?

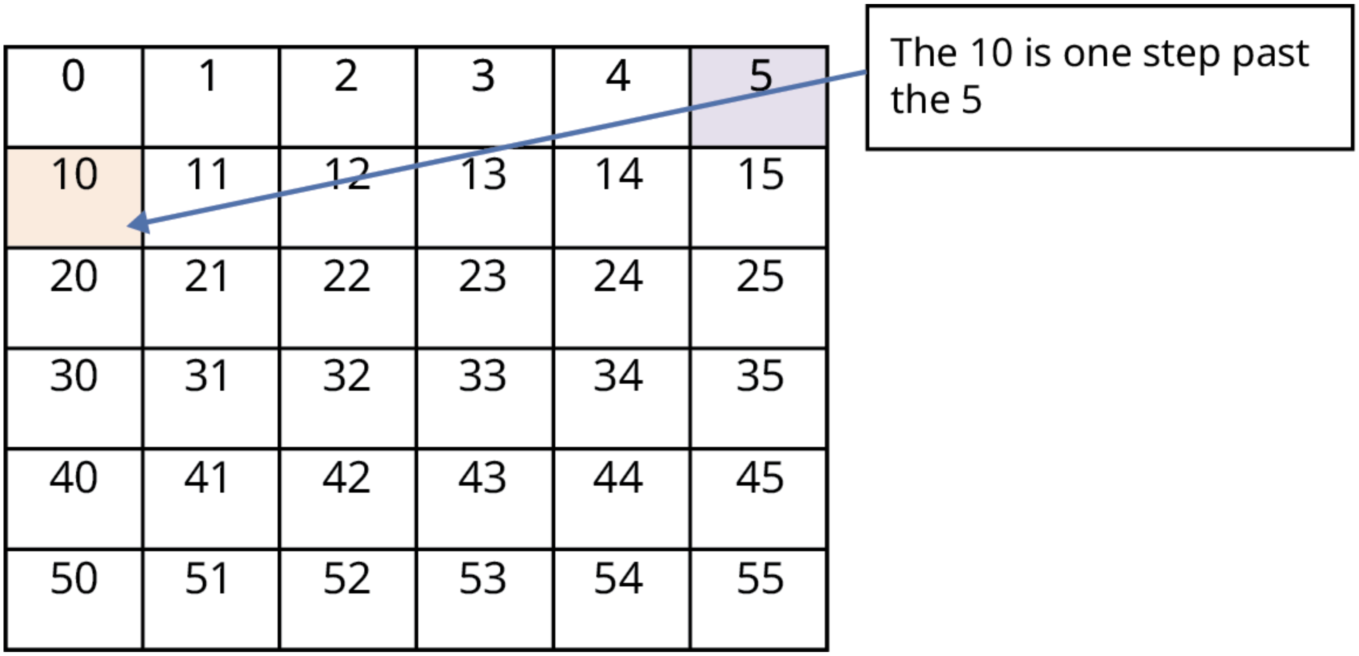
Many of the cells are not filled out. The ones filled in are values that never get past 5, which is the largest legal symbol in base 6, so they are acceptable symbols. But what do we do with 5 + 3 in base 6? We can't represent the answer as "8" since "8" is not a symbol available to us. Let's go back to the list of numbers we have for base 6.

0	1	2	3	4	5
10	11	12	13	14	15
20	21	22	23	24	25
30	31	32	33	34	35
40	41	42	43	44	45
50	51	52	53	54	55

So, what is 5 + 1 equal to in base 6? Well, start at the 5, and jump ahead one step. You land on 10.



0	1	2	3	4	5
10	11	12	13	14	15
20	21	22	23	24	25
30	31	32	33	34	35
40	41	42	43	44	45
50	51	52	53	54	55

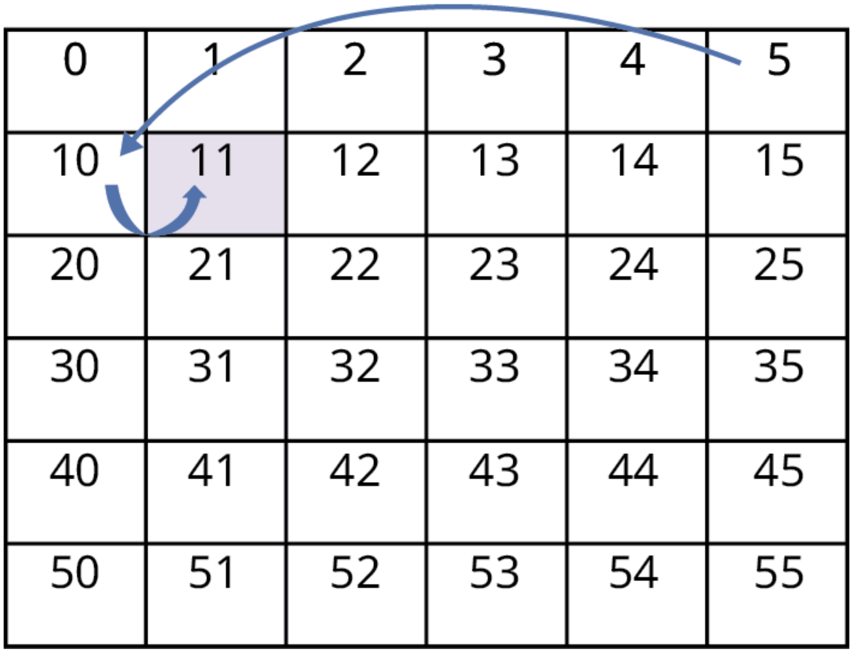


0	1	2	3	4	5
10	11	12	13	14	15
20	21	22	23	24	25
30	31	32	33	34	35
40	41	42	43	44	45
50	51	52	53	54	55

The 10 is one step past the 5

This means that, in base 6, 5 + 1 = 10.

So, what is 5 + 2 in base 6? Well, 5 + 2 = 5 + 1 + 1, so 10 + 1…jump one more space and you land on 11. So, 5 + 2 = 11 in base 6.



0	1	2	3	4	5
10	11	12	13	14	15
20	21	22	23	24	25
30	31	32	33	34	35
40	41	42	43	44	45
50	51	52	53	54	55

And so it goes. Using that process, stepping one more along the list, we can fill in the remainder of the base 6 addition table (Table 4.4).

Table 4.4 Base 6 Addition Table

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	**10**
2	2	3	4	5	**10**	**11**
3	3	4	5	**10**	**11**	**12**

4	4	5	**10**	**11**	**12**	**13**
5	5	**10**	**11**	**12**	**13**	**14**

With this table, and with our understanding of “carrying the one,” we can then use the addition table to do addition in base 6 for numbers with two or more digits, using the same processes you learned for addition when you did it by hand.

✓ Example ✓ 4.3.1: Adding in Base 6

Calculate $251_6 + 133_6$.

Answer

Step 1: Let’s set up the addition using columns.

	2	5	1
+	1	3	3

Step 2: Let’s do the one’s place first. According to the base 6 addition table (Table 4.4), 1 + 3 = 4.

	2	5	1
+	1	3	3
			4

Step 3: Now, we do the “tens” place (it’s really the sixes place). According to the base 6 addition table (Table 4.4), we have 5 + 3 = 12. So, like in base 10, we use the 2 and carry the 1.

	1		
	2	5	1
+	1	3	3
		2	4

Step 4: Now the “hundreds” place (really, thirty-sixes place). There, we have 1 + 2 + 1 = 3 + 1 = 4.

	1		
	2	5	1
+	1	3	3
	4	2	4

So, $251_6 + 133_6 = 424_6$.

As you can see, the process is the same as when you learned base 10 addition, just a different symbol set.

Your Turn 4.3.1

Calculate $453_6 + 345_6$.

✓ Example ✓ 4.3.2: Creating an Addition Table for a Base Lower Than 10

1. Create the addition table for base 7.
2. Create the addition table for base 2.

Answer

1. We begin with the table below.

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	
2	2	3	4	5	6		
3	3	4	5	6			
4	4	5	6				
5	5	6					
6	6						

In base 7, the number that follows 6 is 10 (since we've run out of symbols!). So, $6_7 + 1_7 = 10_7$. Once that is established, $6_7 + 2_7$ will be two numbers past 6, which is 11 in base 7.

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	**10**
2	2	3	4	5	6		**11**
3	3	4	5	6			
4	4	5	6				
5	5	6					
6	6	**10**	**11**				

Continuing, we can fill in the rows as we would in base 10, but being aware that we are working in base 7 (Table 4.5).

Table 4.5 Base 7 Addition Table

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	10
2	2	3	4	5	6	10	11
3	3	4	5	6	10	11	12
4	4	5	6	10	11	12	13
5	5	6	10	11	12	13	14
6	6	10	11	12	13	14	15

- We revisit base 2 here. Begin with the table:

Table 4.6 Base 2 Addition Table

+	0	1
0	0	1
1	1	

In base 2, the number that follows 1 is 10 (since we've run out of symbols!). So, $1_2 + 1_2 = 10_2$. The complete table for base two then is below.

+	0	1
0	0	1
1	1	10

This demonstrates that the rules necessary for base 2 addition are as small as possible: four rules.

Your Turn 4.3.2

Create the addition table for base 4.

To summarize the creation of the addition tables for a given base, do the following.

Step 1: Set up the table.

Step 2: Fill in all the additions that use the "legal" symbols for the base. The diagonal that goes from upper left to lower right that is immediately next to the filled boxes all get the value 10, regardless of base.

Step 3: Enter the values that are in the "teens." This can all be done on one table without creating multiple copies of previously done work.

Example 4.3.3: Adding in Base 7

Calculate $536_7 + 433_7$.

Answer

Step 1: Let's set up the addition using columns.

	5	3	6
+	4	3	3

Step 2: Let's do the one's place first. According to the base 7 addition table in the solution for Example 4.29, 6 + 3 = 12. We will carry the 1.

		1	
	5	3	6
+	4	3	3
			2

Step 3: Now, we do the "tens" place (it's really the sevens place). According to the base 7 addition table in the solution for Example 4.29, we have 1 + 3 + 3 = 10. So, like in base 10, we use the 0 and carry the 1.

	1		
	5	3	6

+	4	3	3
		0	2

Step 4: Now the “hundreds” place (really, forty-ninths place). There, we have 1 + 5 + 4 = 6 + 4 = 13.

	1		
	5	3	6
+	4	3	3
1	**3**	0	2

So, $536_7 + 333_7 = 1302_7$.

Your Turn 4.3.3

Calculate $461_7 + 142_7$.

As seen previously, when performing addition in another base, set up the problem exactly as you would for addition in base 10. At each step, check the addition table for the base. As in base 10 addition, move right to left, adding down the columns using the rules in the addition table. When necessary and just as in base 10, be sure to carry the 1.

Example 4.3.4: Creating an Addition Table for a Base Higher Than 10

Create the addition table for base 12.

Answer

Step 1: Recall, in base 12, the symbol set is 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, and B. So, the addition table begins as shown below.

+	0	1	2	3	4	5	6	7	8	9	A	B
0	0	1	2	3	4	5	6	7	8	9	A	B
1	1	2	3	4	5	6	7	8	9	A	B	
2	2	3	4	5	6	7	8	9	A	B		
3	3	4	5	6	7	8	9	A	B			
4	4	5	6	7	8	9	A	B				
5	5	6	7	8	9	A	B					
6	6	7	8	9	A	B						
7	7	8	9	A	B							
8	8	9	A	B								
9	9	A	B									
A	A	B										
B	B											

Step 2: The diagonal immediately to the right of the filled in boxes is where the 10 goes for this base.

+	0	1	2	3	4	5	6	7	8	9	A	B

0	0	1	2	3	4	5	6	7	8	9	A	B
1	1	2	3	4	5	6	7	8	9	A	B	10
2	2	3	4	5	6	7	8	9	A	B	10	
3	3	4	5	6	7	8	9	A	B	10		
4	4	5	6	7	8	9	A	B	10			
5	5	6	7	8	9	A	B	10				
6	6	7	8	9	A	B	10					
7	7	8	9	A	B	10						
8	8	9	A	B	10							
9	9	A	B	10								
A	A	B	10									
B	B	10										

Step 3: Using the pattern we're familiar with, and counting in base 12, we can fill in the other cells.

Table 4.7 Base 12 addition table

+	0	1	2	3	4	5	6	7	8	9	A	B
0	0	1	2	3	4	5	6	7	8	9	A	B
1	1	2	3	4	5	6	7	8	9	A	B	10
2	2	3	4	5	6	7	8	9	A	B	10	11
3	3	4	5	6	7	8	9	A	B	10	11	12
4	4	5	6	7	8	9	A	B	10	11	12	13
5	5	6	7	8	9	A	B	10	11	12	13	14
6	6	7	8	9	A	B	10	11	12	13	14	15
7	7	8	9	A	B	10	11	12	13	14	15	16
8	8	9	A	B	10	11	12	13	14	15	16	17
9	9	A	B	10	11	12	13	14	15	16	17	18
A	A	B	10	11	12	13	14	15	16	17	18	19
B	B	10	11	12	13	14	15	16	17	18	19	1A

Notice that the lower-right entry is $1A_{12}$, as this is the number one past 19_{12}.

Your Turn 4.3.4

Create the addition table for base 14.

Example 4.3.5: Adding in Base 12

Calculate $3A7_{12} + 9BA_{12}$.

Answer

Step 1: Using the process established in the earlier addition problem, set up the columns.

	3	A	7
+	9	B	A

Step 2: Using the rules from the base 12 addition table in the solution for Example 4.31, and being careful to carry the 1 when necessary, we get the following:

	1	1	
	3	A	7
+	9	B	A
1	**1**	A	5

The ones that were carried are located over the columns.

So, $3A7_{12}$ + $9BA_{12}$= $11A5_{12}$.

Your Turn 4.3.5

Calculate $4B3_{12}$ + $B06_{12}$.

Example 4.3.6: Adding in Base 2

We again return to base 2, the base used by computers. Calculate 1001_2 + 11011_2.

Answer

Step 1: Using the process established in the earlier addition problem, set up the columns.

		1	0	0	1
+	1	1	0	1	1

Step 2: Using the rules from the base 2 addition table in the solution for Example 4.29, and being careful to carry the 1 when necessary (and shown at the top of the grid), we get the following:

	1		1	1	
		1	0	0	1
+	1	1	0	1	1
1	0	0	1	0	0

Step 3: Calculate 1001_2 + 11011_2 = 100100_2.

So, 1001_2 + 11011_2 = 100100_2.

Your Turn 4.3.6

Calculate 101111_2 + 1100011_2.

Subtraction in Bases Other Than Base 10

Subtraction in bases other than base 10 follow the same processes as base 10 subtraction, but, as with addition, using the addition table for the base.

✓ Example ✓ 4.3.7: Subtracting in Base 6

Calculate $52_6 - 34_6$.

Answer

Step 1: Let’s set up the subtraction using columns.

	5	2
−	3	4

Step 2: Just as we might do in base 10, we borrow a 1 from the 5 for the ones digit.

	~~5~~4	**12**
−	3	4

Step 3: Referring to the base 6 addition table (Table 4.4), we see that 4 + 4 = 12, so $12_6 - 4_6$ is 4_6.

	~~5~~4	**12**
−	3	4
		4

Step 4: Now we deal with the “tens” (really, sixes) digit, $4_6 - 3_6$, which equals 1_6 according to the base 6 addition table (Table 4.4).

	~~5~~ 4	**12**
−	3	4
	1	4

So, $52_6 - 34_6 = 14_6$.

Your Turn 4.3.71

Calculate $115_6 - 43_6$.

✓ Example ✓ 4.3.8: Subtracting in Base 12

Calculate $A17_{12} - 4B3_{12}$.

Answer

Step 1: Let’s set up the subtraction using columns.

	A	1	7
−	4	B	3

Step 2: Even in base 12, $7_{12} - 3_{12} = 4_{12}$.

	A	1	7

−	4	B	3
			4

Step 3: Moving to the “tens” digit, we have $1_{12} - B_{12}$. Since 1 is less than B in base 12, we need to borrow a 1 from the A, just as we would for subtraction in base 10.

	~~A~~ **9**	**11**	7
−	4	B	3
			4

Step 4: According to the base 12 addition table in the solution for Example 4.31, $B_{12} + 2_{12} = 11_{12}$, so $11_{12} - B_{12} = 2_{12}$.

	~~A~~ 9	11	7
−	4	B	3
		2	4

Step 5: Finally, we deal with the “hundreds” digit. According to the base 12 addition table in the solution for Example 4.31, $4_{12} + 5_{12} = 9_{12}$, so $9_{12} - 4_{12} = 5_{12}$.

	~~A~~ 9	11	7
−	4	B	3
	5	2	4

So, $A17_{12} - 4B3_{12} = 524_{12}$.

Your Turn 4.3.8

Calculate $716_{12} - 4AB_{12}$.

Errors When Adding and Subtracting in Bases Other Than Base 10

Errors when computing in bases other than 10 often involve applying base 10 rules or symbols to an arithmetic problem in a base other than base 10. The first type of error is using a symbol that is not in the symbol set for the base. For instance, if a 9 shows up when working in base 7, you know an error has happened because 9 is not a legal symbol in base 7.

Example 4.3.9: Identifying an Illegal Symbol in Arithmetic in a Base Other Than Base 10

Explain the error in the following calculation:

$$15_6 + 34_6 = 49_6$$

Answer

Since the problem is in base 6, the symbol set available is 0, 1, 2, 3, 4 and 5. The 9 in the answer is clearly not a legal symbol for base 6. Looking back to the base 6 addition table (Table 4.4), we see that $5_6 + 4_6 = 13_6$. Correcting the error, we see the sum is $15_6 + 34_6 = 53_6$.

Your Turn 4.3.9

Explain the error in the following calculation and correct the problem:

$133_4 + 112_4 = 245_4$

The second type of error is using a base 10 rule when the numbers are not in base 10. For instance, if you are working in base 13, then $9_{13} + 9_{13}$ is not 18_{13}, even though 18 is the correct answer in base 10.

Example 4.3.10: Identifying an Arithmetic Error in a Base Other Than Base 10

Explain the error in the following calculation, and correct the error:

$$89_{12} + 76_{12} = 165_{12}$$

Answer

If this problem was a base 10 problem, this would be the correct answer. However, in base 12, 9 + 6 is not 15, but is instead 13. To correct this error, carefully use the addition table for base 12. If properly used, the correct answer would be 143_{12}, as seen below:

		8	9
+		7	6
	1	4	3

Your Turn 4.3.10

Explain the error in the following calculation, and correct the error:

$149_{14} + 19_{14} = 168_{14}$

Check Your Understanding

1. Determine the addition table for base 8.
2. Compute $24_6 + 53_6$.
3. Compute $35_8 - 26_8$.
4. Compute $3B_{14} + 45_{14}$.
5. Compute $A4_{12} - 9B_{12}$.
6. How do you know an error has occurred in a base 8 addition question if the answer obtained was 28_8?
7. What is one common error made in calculating in base 14?

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4.3.0: Exercises

For the following exercises, create the addition table for the given base.

Exercise $4.3.0.1$

base 5

Exercise $4.3.0.2$

base 3

Exercise $4.3.0.3$

base 16 (Hint: Use the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, and F.)

Exercise $4.3.0.4$

base 2

For the following exercises, perform the indicated base 6 operation.

Exercise $4.3.0.5$

$4_6 + 3_6$

Exercise $4.3.0.6$

$14_6 + 25_6$

Exercise $4.3.0.7$

$31_6 + 3_6$

Exercise $4.3.0.8$

$43_6 + 34_6$

Exercise $4.3.0.9$

$532_6 + 23_6$

Exercise $4.3.0.10$

$254_6 + 143_6$

Exercise $4.3.0.11$

$20_6 - 3_6$

? Exercise 4.3.0.12

$23_6 - 5_6$

For the following exercises, perform the indicated base 12 operation.

? Exercise 4.3.0.13

$5_{12} + 6_{12}$

? Exercise 4.3.0.14

$3_{12} + A_{12}$

? Exercise 4.3.0.15

$34_{12} + 7_{12}$

? Exercise 4.3.0.16

$76_{12} + B_{12}$

? Exercise 4.3.0.17

$59_{12} + 1A_{12}$

? Exercise 4.3.0.18

$A1_{12} + 36_{12}$

? Exercise 4.3.0.19

$53_{12} - 9_{12}$

? Exercise 4.3.0.20

$2B_{12} - 7_{12}$

? Exercise 4.3.0.21

Explain two ways to detect an error in arithmetic in bases other than base 10.

? Exercise 4.3.0.22

Explain the error in the following calculation: $28_{13} + 47_{13} = 75_{13}$.

? Exercise 4.3.0.23

Explain the error in the following calculation: $36_7 + 23_7 = 59_7$.

Exercise 4.3.0.24

In base 10 addition, there are 100 addition rules plus a rule for carrying a 1. How many addition rules are there for base 6?

Exercise 4.3.0.25

In base 10 addition, there are 100 addition rules plus a rule for carrying a 1. How many addition rules are there for base 14?

Exercise 4.3.0.26

In base 10 addition, there are 100 addition rules plus a rule for carrying a 1. How many addition rules are there for base 2?

For the following exercises, use the addition table that you created from Exercise 4 to perform the indicated base 2 operations.

Exercise 4.3.0.27

$101_2 + 111_2$

Exercise 4.3.0.28

$1011_2 + 10011_2$

Exercise 4.3.0.29

$11111_2 + 11111_2$

Exercise 4.3.0.30

$1010101_2 + 1010101_2$

For the following exercises, use the addition table that you created from Exercise 3 to perform the indicated base 16 operations.

Exercise 4.3.0.31

$29_{16} + 38_{16}$

Exercise 4.3.0.32

$4D_{16} + 89_{16}$

Exercise 4.3.0.33

$927_{16} + 438_{16}$

Exercise 4.3.0.34

$BFA_{16} - 78E_{16}$

For the following exercises, tell how you know an error was committed without performing the operation in the given base.

Exercise 4.3.0.35

$/43_5 + 32_5 = 75_5$

Exercise $4.3.0.36$

$/15_{14}+19_{14}=34_{14}$

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4.4: Multiplication and Division in Base Systems

Figure 4.4.1: The processes for multiplication and division are the same for arithmetic in any bases. (credit: modification of work "NCTR Intern Claire Boyle" by Danny Tucker/U.S. Food and Drug Administration, Public Domain)

Learning Objectives

1. Multiply and divide in bases other than 10.
2. Identify errors in multiplying and dividing in bases other than 10.

Just as in Addition and Subtraction in Base Systems, once we decide on a system for counting, we need to establish rules for combining the numbers we're using. This includes the rules for multiplication and division. We are familiar with those operations in base 10. How do they change if we instead use a different base? A larger base? A smaller one?

In this section, we use multiplication and division in bases other than 10 by referencing the processes of base 10, but applied to a new base system.

Multiplication in Bases Other Than 10

Multiplication is a way of representing repeated additions, regardless of what base is being used. However, different bases have different addition rules. In order to create the multiplication tables for a base other than 10, we need to rely on addition and the addition table for the base. So let's look at multiplication in base 6.

Multiplication still has the same meaning as it does in base 10, in that 4×6 is 4 added to itself six times, $4 \times 6 = 4 + 4 + 4 + 4 + 4 + 4$.

So, let's apply that to base 6. It should be clear that 0 multiplied by anything, regardless of base, will give 0, and that 1 multiplied by anything, regardless of base, will be the value of "anything."

Step 1: So, we start with the table below:

*	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4			
3	0	3				
4	0	4				
5	0	5				

Step 2: Notice Math input error

*	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	10		
3	0	3	10			
4	0	4				
5	0	5				

Step 3: Next, we need to fill in 2×4. Using repeated addition, $2 \times 4 = 2 + 2 + 2 + 2 = 10 + 2 = 12$ (if we use our base 6 addition rules). So, we add that to our table:

*	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	10	12	
3	0	3	10			
4	0	4	12			
5	0	5				

Step 4: Finally, $2 \times 5 = 2 + 2 + 2 + 2 + 2 = 12 + 2 = 14$. And so we add that to our table:

*	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	10	12	14
3	0	3	10			
4	0	4	12			
5	0	5	14			

Step 5: A similar analysis will give us the remainder of the entries. Here is 4×5 demonstrated: $4 \times 5 = 4 + 4 + 4 + 4 + 4 = 12 + 12 + 4 = 24 + 4 = 32$.

This is done by using the addition rules from Addition and Subtraction in Base Systems, namely that Math input error

Table 4.8 Base 6 Multiplication Table

*	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	10	12	14
3	0	3	10	13	20	23
4	0	4	12	20	24	32

5	0	5	14	23	32	41

Notice anything about that bottom line? Is that similar to what happens in base 10?

To summarize the creation of a multiplication in a base other than base 10, you need the addition table of the base with which you are working. Create the table, and calculate the entries of the multiplication table by performing repeated addition in that base. The table needs to be drawn only the one time.

✓ Example ✓ 4.4.1: Creating a Multiplication Table for a Base Lower Than 10

Create the multiplication table for base 7.

Answer

Step 1: Let’s apply the process demonstrated and outlined above to find the base 7 multiplication table. It should be clear that 0 multiplied by anything, regardless of base, will give 0, and that 1 multiplied by anything, regardless of base, will be the value of “anything.” So, we start with the table below:

*	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6			
3	0	3	6				
4	0	4					
5	0	5					
6	0	6					

Step 2: Notice Math input error

*	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	**11**		
3	0	3	6				
4	0	4	**11**				
5	0	5					
6	0	6					

Step 3: Next, we need to fill in 2×5. Using repeated addition, $2 \times 5 = 2 + 2 + 2 + 2 + 2 = 11 + 2 = 13$ if we use our base 7 addition rules. So, we add that to our table:

*	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	11	**13**	
3	0	3	6				

4	0	4	11				
5	0	5	**13**				
6	0	6					

Step 4: Finally, $2 \times 6 = 2+2+2+2+2+2 = 13+2 = 15$. And so we add that to our table:

*	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	11	13	**15**
3	0	3	6				
4	0	4	11				
5	0	5	13				
6	0	6	**15**				

Step 5: A similar analysis will give us the remainder of the entries. Here is $4_7 \times 5_7$ demonstrated:

$$4_7 \times 5_7 = 4_7+4_7+4_7+4_7+4_7 = 11_7+11_7+4_7 = 22_7+4_7 = 26_7$$

This is done by using the addition rules from Addition and Subtraction in Base Systems, namely that $4_7+4_7=$117Example 4.29. Using those addition rules, the rest of the table is given below:

*	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	11	13	15
3	0	3	6	12	15	21	24
4	0	4	11	15	22	26	33
5	0	5	13	21	26	34	42
6	0	6	15	24	33	42	51

Your Turn 4.4.1

Create the multiplication table for base 4.

Example 4.4.2: Creating a Multiplication Table for a Base Higher Than 10

Create the multiplication table for base 12.

Answer

Let’s apply the repeated addition to base 12. Here is $7_{12} \times 9_{12}$ demonstrated:

$$7_{12} \times 9_{12} = 7_{12}+7_{12}+7_{12}+7_{12}+7_{12}+7_{12}+7_{12}+7_{12}+7_{12} = 12_{12}+12_{12}+12_{12}+12_{12}+7_{12} = 48_{12}+7_{12} = 53_{12}$$

This is done by using the addition rules from Addition and Subtraction in Base Systems, namely that Math input error

Table 4.9 Base 12 Multiplication Table

*	0	1	2	3	4	5	6	7	8	9	A	B
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	A	B
2	0	2	4	6	8	A	10	12	14	16	18	1A
3	0	3	6	9	10	13	16	19	20	23	26	29
4	0	4	8	10	14	18	20	24	28	30	34	38
5	0	5	A	13	18	21	26	2B	34	39	42	47
6	0	6	10	16	20	26	30	36	40	46	50	56
7	0	7	12	19	24	2B	36	41	48	53	5A	65
8	0	8	14	20	28	34	40	48	54	60	68	74
9	0	9	16	23	30	39	46	53	60	69	76	83
A	0	A	18	26	34	42	50	5A	68	76	84	92
B	0	B	1A	29	38	47	56	65	74	83	92	A1

Your Turn 4.4.2

Create the multiplication table for base 14.

The multiplication table in base 2 below is as minimal as the addition table in the solution for Table 4.6. Since the product of 1 with anything is itself, the following multiplication table is found.

Table 4.10 Base 2 Multiplication Table

*	0	1
0	0	0
1	0	1

As with the addition table, we can use the multiplication tables and the addition tables to perform multiplication of two numbers in bases other than base 10. The process is the same, with the same carry rules and placeholder rules.

Example 4.4.3: Multiplying in a Base Lower Than 10

1. Calculate $45_6 \times 24_6$.
2. Calculate $101_2 \times 110_2$.

Answer

1. **Step 1:** Use the base 6 multiplication table (Table 4.8) and, when necessary, the base 6 addition table (Table 4.4).

   Set up this calculation using columns:

	4	5
x	2	4

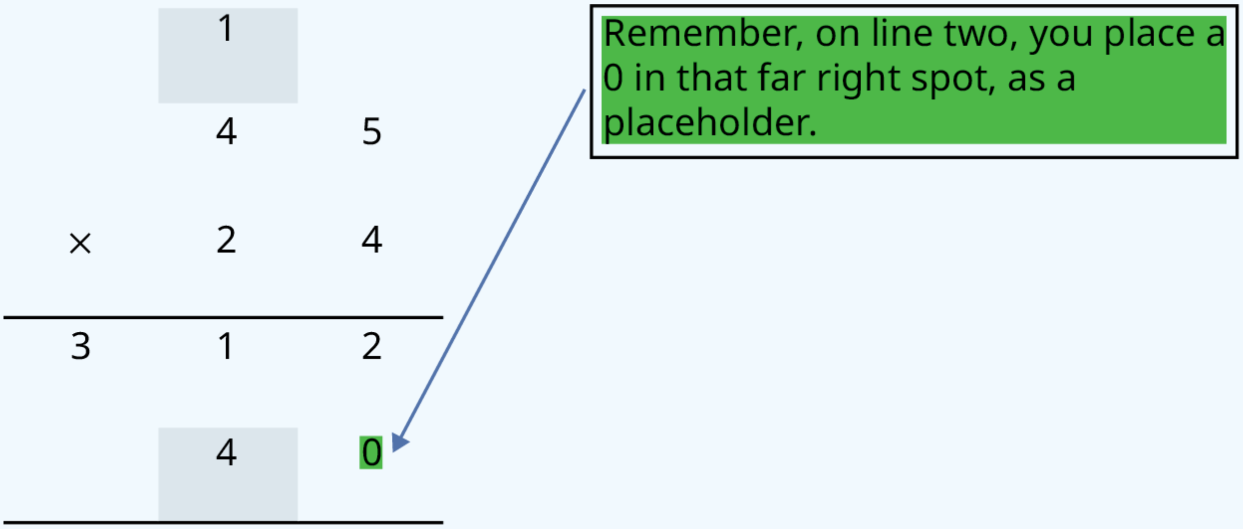
   Step 2: Multiply the 1s digits, 5 and 4, using the base 6 multiplication table (Table 4.8). There we see the result is 32_6. So, we enter the 2 and carry the 3.

Table 4.8]), then $24_6 + 3_6 = 31_6$. So, we enter the 31.

	3	
	4	5
x	2	4
		2

	3	
	4	5
x	2	4
3	**1**	2

Step 4: Now we move on to the 2 in the “tens” place in the bottom value. We multiply the 2_6 and the 5_6, and we get 14_6. So, we enter the 4 and carry the 1.



Remember, on line two, you place a 0 in that far right spot, as a placeholder.

Step 5: Next up, we multiply the 2 and the 4, and then add 1. This gives us $12_6 + 1_6 = 13_6$. We enter those on that second line.

		1	
		4	5
	x	2	4
	3	1	2
1	**3**	4	0

Step 6: Now we add down the columns.

		1	
		4	5
	x	2	4
1	3	1	2
1	3	4	0

2	0	5	2

Step 7: The 3 and the 3 add to 10 in base 6, so we enter the 0 and carry the 1. We now have the result:

45 6 × 24 6 = 2052 6 45 6 × 24 6 = 2052 6

2.

Step 1: Use the base 2 multiplication table (Table 4.10) and, when necessary, the base 2 addition table in the solution for Example 4.29. Set up this calculation using columns:

	1	0	1
x	1	1	0

Step 2: Using the pattern established above, and the processes from multiplication from base 10, we find the following:

			1	0	1
x			1	1	0
			0	0	0
		1	0	1	
	1	0	1		

Step 3: Adding down the columns results in the following:

			1	0	1
x			1	1	0
			0	0	0
		1	0	1	
	1	0	1		
	1	1	1	1	0

So, $101_2 \times 110_2 = 11110_2$.

Your Turn 4.4.3

1. Calculate $43_6 \times 52_6$.
2. Calculate $11101_2 \times 11_2$.

Summarizing the process of multiplying two numbers in different bases, the multiplication table is referenced. Using that table, the multiplication is carried out in the same manner as it is in base 10. The addition rules for the base will also be referenced when carrying a 1 or when adding the results for each digit's multiplication line.

Example 4.4.4: Multiplying in a Base Higher Than 10

Calculate $3A_{12} \times 74_{12}$.

Answer

Step 1: Use the base 12 multiplication table in the solution for Example 4.39 and, when necessary, the base 12 addition table in the solution for Table 4.7. Set up this calculation using columns:

	3	A
×	7	4

Step 2: First, the 4 is multiplied by 3A, resulting in the first line.

	3	A
×	7	4
1	**3**	**4**

Step 3: Now we move on to the 7 in the “tens” place in the bottom value.

		5	
		3	A
	x	7	4
	1	3	4
2	**2**	**A**	**0**

Step 4: Now we add down the columns.

		3	A
	x	7	4
	1	3	4
2	2	A	0
2	**4**	**1**	**4**

Step 5: The 3 and the A add to 11 in base 12, so we enter the 1 and carry the 1.

We now have the result: 3A12×7412=2414123A12×7412=2414123 $A_{12} \times 74_{12} = 2414_{12}$

Your Turn 4.4.4

Calculate $B3_{12} \times 47_{12}$.

Division in Bases Other Than 10

Just as with the other operations, division in a base other than 10, the process of division in a base other than 10 is the same as the process when working in base 10. For instance, $72 \div 9 = 8$ because, we know that $9 \times 8 = 72$. So, for many division problems, we are simply looking to the multiplication table to identify the appropriate multiplication rule.

Example 4.4.5: Dividing with a Base Other Than 10

1. Calculate $14_6 \div 5_6$.
2. Calculate $5A_{12} \div 7_{12}$

Answer

1. Looking at the multiplication table for base 6 (Table 4.8), we see that $5_6 \times 2_6 = 12_6$. Using that, we know that $14_6 \div 5_6 = 2_6$.
2. Looking at the multiplication table for base 12 in the solution for Example 4.39, we see that $7_{12} \times A_{12} =$. Using that, we know that $5A_{12} \div 7_{12} =$.

Your Turn 4.4.5

Calculate $10_6 \div 3_6$

Calculate $50_{12} \div \mathrm{A}_{12}$.

Errors in Multiplying and Dividing in Bases Other Than Base 10

The types of errors encountered when multiplying and dividing in bases other than base 10 are the same as when adding and subtracting. They often involve applying base 10 rules or symbols to an arithmetic problem in a base other than base 10. The first type of error is using a symbol that is not in the symbol set for the base.

✓ Example ✓ 4.4.6: Identifying an Illegal Symbol in a Base Other Than Base 10

Explain the error in the following calculation, and determine the correct answer:

$$4_6 \times 2_6 = 8_6$$

Answer

Since the problem is in base 6, the symbol set available is 0, 1, 2, 3, 4, and 5. The 8 in the answer is clearly not a legal symbol for base 6. Looking back to the base 6 multiplication table (Table 4.8), we see that $4_6 \times 2_6 = 12_6$.

Your Turn 4.4.6

Explain the error in the following calculation and determine the correct answer: $13_4 \times 21_4 = 54_4$

The second type of error is using a base 10 rule when the numbers are not in base 10. For instance, in base 17, $6_{17} \times 9_{17} = 54_{17}$ would be incorrect, even though in base 10, $6 \times 9 = 54$. That rule doesn't apply in base 17.

✓ Example

✓ 4.4.7: Identifying an Error in Arithmetic in a Base Other Than Base 10

Explain the error in the following calculation. Determine the correct answer:

$$18_{12} \times 7_{12} = 126_{12}$$

Answer

If this problem was a base 10 problem, this would be the correct answer. However, in base 12, $8_{12} \times 7_{12}$ is not 56 , but is instead 48 . To correct this error, carefully use the multiplication table for base 12. If properly used, the correct answer would be $18_{12} \times 7_{12} = \mathrm{B}8_{12}$.

msub

Math input error

Your Turn 4.4.7

Explain the error in the following calculation. Determine the correct answer: $49_{14} \times 9_{14} = 441_{14}$

Check Your Understanding

1. To create the multiplication table for a given base, what should be used?
2. What are the differences between multiplying in base 10 and multiplying in a different base?
3. When dividing in a base other than base 10 , what table is referenced?
4. Compute $24_6 \times 53_6$.

5. Compute $32_{14} \div 4_{14}$.
6. How do you know an error has occurred in a base 5 multiplication question if the answer obtained was 285 ?
7. What are two common ways to determine an error is committed when computing in s base other than base 10?

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4.4.0: Exercises

For the following exercises, create the multiplication table for the given base.

Exercise $4.4.0.1$

base 5

Exercise $4.4.0.2$

base 3

Exercise $4.4.0.3$

base 16 (Hint: Use the digits $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F$.

Exercise $4.4.0.4$

base 2

For the following exercises, perform the indicated base 6 operation.

Exercise $4.4.0.5$

$4_6 \times 3_6$

Exercise $4.4.0.6$

$14_6 \times 5_6$

Exercise $4.4.0.7$

$31_6 \times 3_6$

Exercise $4.4.0.8$

$43_6 \times 34_6$

Exercise $4.4.0.9$

$532_6 \times 23_6$

Exercise $4.4.0.10$

$254_6 \times 143_6$

Exercise $4.4.0.11$

$20_6 \div 36$

? Exercise 4.4.0.12

$23_5 \div 5_6$

For the following exercises, perform the indicated base 12 operation.

? Exercise 4.4.0.13

$5_{12} \times 6_{12}$

? Exercise 4.4.0.14

$3_{12} \times \mathrm{A}_{12}$

? Exercise 4.4.0.15

$34_{12} \times 7_{12}$

? Exercise 4.4.0.16

$76_{12} \times \mathrm{B}_{12}$

? Exercise 4.4.0.17

$59_{12} \times 1\ \mathrm{A}_{12}$

? Exercise 4.4.0.18

$\mathrm{A}1_{12} \times 36_{12}$

? Exercise 4.4.0.19

$53_{12} \div 9_{12}$

? Exercise 4.4.0.20

$2\ \mathrm{B}_{12} \div 7_{12}$

? Exercise 4.4.0.21

Explain two ways to detect an error in arithmetic in bases other than base 10.

? Exercise 4.4.0.22

Explain the error in the following calculation: $15_{12} \times 7_{12} = 105_{12}$

? Exercise 4.4.0.23

Explain the error in the following calculation: $45_8 \times 6_8 = 94_8$.

Exercise $4.4.0.24$

In base 10 multiplication, there are 100 multiplication rules plus a rule for carrying a number. How many multiplication rules are there for base 6 ?

Exercise $4.4.0.25$

In base 10 multiplication, there are 100 multiplication rules plus a rule for carrying a number. How many multiplication rules are there for base 14 ?

Exercise $4.4.0.26$

In base 10 multiplication, there are 100 multiplication rules plus a rule for carrying a number. How many multiplication rules are there for base 2?

Exercise $4.4.0.27$

Consider the answers from Exercise 24 and Exercise 26. Which base do you think would be more efficient: base 10, base 6, or base 2?

For the following exercises, use the multiplication table that you created from Exercise 4 to perform the indicated base 2 operations.

Exercise $4.4.0.28$

$11_2 \times 11_2$

Exercise $4.4.0.29$

$101_2 \times 10_2$

Exercise $4.4.0.30$

$11011_2 \times 1011_2$

Exercise $4.4.0.31$

$1011_2 \times 1010101_2$

Exercise $4.4.0.32$

Convert 1011_2 and 1010101_2 to base 10. Then multiply those base 10 numbers. Next, convert the answer you got for Exercise 31 to base 10. Do these numbers match?

For the following exercises, use the multiplication table that you created from Exercise 3 to perform the indicated base 16 operations.

Exercise $4.4.0.33$

$19_{16} \times 5_{16}$

? Exercise 4.4.0.34

$3\ \mathrm{B}_{16} \times \mathrm{A}_{16}$

? Exercise 4.4.0.35

$25_{16} \times 16_{16}$

? Exercise 4.4.0.36

$\mathrm{C}_{16} \div 4_{16}$

For the following exercises, explain how you know an error was committed without performing the operation in the given base.

? Exercise 4.4.0.37

$43_5 \times 32_5 = 126_5$

? Exercise 4.4.0.38

$5_{14} \times 9_{14} = 45_{14}$

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SECTION OVERVIEW

4.5: Chapter Summary

4.5.0: Key Terms

4.5.1: Key Concepts

4.5.2: Videos

4.5.3: Projects

4.5.4: Chapter Review

4.5.5: Chapter Test

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4.5.0: Key Terms

4.1 Hindu-Arabic Positional System

- numeral
- number
- exponential expression
- base
- exponent
- place value
- base 10 system
- Hindu-Arabic numeration system
- expanded form

4.2 Early Numeration Systems

- additive system of numbers
- positional system of numbers
- Babylonian system of numbers
- Mayan system of numbers
- Roman system of numbers

4.3 Converting with Base Systems

- base 10
- remainder
- dividend
- divisor
- quotient

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4.5.1: Key Concepts

Key Concepts

4.1 Hindu-Arabic Positional System

Exponents are used to represent repeated multiplication of a base.

- In arithmetic, exponents are computed before multiplication, division, addition, and subtraction. Computing an exponent is done by multiplying the base by itself the number of times equal to the exponent.
- The system of numbers currently used is the Hindu-Arabic system. Digits in this system take on values based on their place in the number. The place values are determined by multiplying the digit by 10 raised to the appropriate power.
- The expanded form of a Hindu-Arabic number is the sum of each digit times 10 raised to the exponent for that place value.

4.2 Early Numeration Systems

Historically, there have been many systems for numbering. One system is an additive system, in which symbols are repeated to express larger numbers. Another system is a positional system, in which the digits and their positions determine the quantity being represented.

- The Babylonian system was a combination of a positional and additive system. It used 60 as its base. Using that in the positional system makes it possible to convert between Babylonian and Hindu-Arabic numbers.
- The Mayan system was a combination of a positional and additive system. It used 20 as its base. Using that in the positional system makes it possible to convert between Mayan and Hindu-Arabic numbers.
- The Roman system was an additive system. Knowing what each symbol represents makes it possible to convert between Roman and Hindu-Arabic numbers.

4.3 Converting with Base Systems

The system we use is the base 10 system. Base 10 is not the only base that can be used. To use another base, one could start with a list of numbers in that base.

- To indicate that a number is written in a base other than 10, a subscript is appended to the end of the number. That subscript indicates the base for the number.
- Numbers written in a base smaller than 10 use the same symbols as base 10. However, when using bases larger than 10, the symbols A, B, C, … are used to represent digits larger than 9.
- To convert from a number written in a base other than 10 into a base 10 number, the number is written in expanded form and then that expression is computed.
- To convert a number from base 10 into another base, the base 10 number is repeatedly divided by the new base. The remainders when performing these divisions become the digits for the number in the new base.
- Common errors can be detected when performing base conversions.

4.4 Addition and Subtraction in Base Systems

Addition tables for bases other than 10 can be built using the same processes that are used in base 10, including using a number line.

- Addition in bases other than base 10 use the same processes as addition in base 10, but use the addition table for that base.
- Subtraction in bases other than base 10 use the same processes as subtraction in base 10, but use the addition table for that base.

4.5 Multiplication and Division in Base Systems

Multiplication tables for bases other than 10 can be built using the same processes that are used in base 10, including using repeated addition and the addition table for the base.

- Multiplication in bases other than base 10 use the same processes as multiplication in base 10, but use the multiplication table for that base.
- Basic division in bases other than base 10 use the same processes as basic division in base 10, where the missing factor process is used.

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4.5.2: Videos

4.1 Hindu-Arabic Positional System

- Exponential Notation

4.2 Early Numeration Systems

- Converting Between Babylonian and Hindu-Arabic numbers
- Converting Mayan Numbers to Hindu-Arabic Numbers
- Converting From Roman Numbers to Hindu-Arabic Numbers
- Converting From Hindu-Arabic Numbers to Roman Numbers

4.3 Converting with Base Systems

- Convert Base 7 to Base 10
- Converting from Base 10 to Another Base

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4.5.3: Projects

Additive Systems

Go online. Google “additive number systems.”What system comes up?

Describe the additive system you found.

Using Google, identify three more additive systems of numbers.

- Compare and contrast the systems you found. For instance, how many times can a symbol be used before a new symbol is used.
- Identify three situations where additive systems are still used.

Computers and Bases

Use Google to determine what base computers use.

Were other bases attempted for use in computers?

Determine why the base used in computers is appropriate.

Determine how the base used in computers is related to the circuitry in computers.

Determine how Boolean logic and the base used in computers are related, and might be identical.

There is research into using quibits in computers. Find out what quibits are and how can they improve computing speed.

Cultures Using Base Systems Other Than 10

Using Google, find three cultures, other than Babylonian or Mayan, that use base systems other than 10.

- Tell what base is used for each system.
- If possible, determine why the culture used that base system.
- Choose one of those systems. Explain that base system. Be sure to address whether the system is additive, place-value based, a blend of the two, and if it employs a zero.

History of Zero

Using any resources available to you, determine the history of 0 in at least three different numbering systems. Address at least when and why such a development occurred and why a 0 is vital to the use of a positional system.

Numbering Systems from Other Global Regions

Using any resources available to you, find at least three numbering systems from sub-Saharan Africa, Australia, China, or the Pacific Islands. Explore if they are positional or additive systems (or combinations!), the terminology of the system, if they used a 0, and what base they employed (if positional).

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4.5.4: Chapter Review

Hindu-Arabic Positional System

Exercise 4.5.4.1

What is the base of 5^7 ?

Exercise 4.5.4.2

What is the exponent of 5^7 ?

Exercise 4.5.4.3

Compute $4^5 + 3^2$.

Exercise 4.5.4.4

Convert the Hindu-Arabic number into expanded form: 4,201.

Exercise 4.5.4.5

Convert the expression to a Hindu-Arabic numeral: $6 \times 10^3 + 3 \times 10^2 + 7 \times 10^1 + 5 \times 10^0$.

Early Numeration Systems

Exercise 4.5.4.6

Which systems—Hindu-Arabic, Roman, Mayan, or Babylonian—are additive systems?

Exercise 4.5.4.7

Which systems—Hindu-Arabic, Roman, Mayan, or Babylonian—are positional systems?

Exercise 4.5.4.8

Which systems—Hindu-Arabic, Roman, Mayan, or Babylonian—use a 0?

Exercise 4.5.4.9

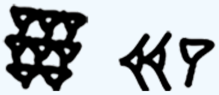
In the Babylonian system, what are the place values based on?

Exercise 4.5.4.10

In the Mayan system, what are the place values based on?

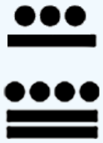
Exercise 4.5.4.11

Convert the Babylonian numeral to a Hindu-Arabic numeral.



Exercise 4.5.4.12

Convert the Mayan numeral to a Hindu-Arabic numeral.



Exercise 4.5.4.13

Convert the Roman numeral MMCDXLVII into a Hindu-Arabic numeral.

Exercise 4.5.4.14

Convert the Hindu-Arabic numeral 394 to a Roman numeral.

Converting with Base Systems

Exercise 4.5.4.15

List the numbers from 0 to 100 in base 5.

Exercise 4.5.4.16

In base 8, what is the place value of the 3 in the number 638_8?

Exercise 4.5.4.17

How many symbols are needed for a base 17 system?

Exercise 4.5.4.18

What does it mean for a number to be in base 6?

Exercise 4.5.4.19

What symbols are used in a base 12 system?

Exercise 4.5.4.20

When converting from a base 10 number to a base 2 number, would the number of digits decrease?

Exercise 4.5.4.21

Convert 311_5 to base 10.

Exercise 4.5.4.22

Convert 45_{12} to base 10.

? Exercise 4.5.4.23

Convert 1001_2 to base 10.

? Exercise 4.5.4.24

Convert 459 to base 8.

? Exercise 4.5.4.25

Convert 1198 to base 12.

? Exercise 4.5.4.26

Convert 38 to base 2.

? Exercise 4.5.4.27

When converting from base 10 to base 4, the result obtained was 142_4. How can you tell an error was made?

Addition and Subtraction in Base Systems

? Exercise 4.5.4.28

Create the addition table for base 5.

? Exercise 4.5.4.29

How many addition rules are there for a base 7 system?

? Exercise 4.5.4.30

Calculate $34_5 + 44_5$.

? Exercise 4.5.4.31

Calculate $A7_{12} + 88_{12}$.

? Exercise 4.5.4.32

Calculate $541_6 - 233_6$.

? Exercise 4.5.4.33

Calculate $5B_{12} - 1A_{12}$.

? Exercise 4.5.4.34

When adding in base 8, the result 911_8 is found. How do we know a mistake was made?

Multiplication and Division in Base Systems

? Exercise 4.5.4.35

What is the process for creating the multiplication table for a base other than 10 ?

? Exercise 4.5.4.36

Create the multiplication table for base 7.

? Exercise 4.5.4.37

How many multiplication rules are there in a base 3 system?

? Exercise 4.5.4.38

Calculate $14_7 \times 25_7$.

? Exercise 4.5.4.39

Calculate $67_{12} \times 3B_{12}$.

? Exercise 4.5.4.40

Calculate $42_7 \div 5_7$.

? Exercise 4.5.4.41

Calculate $38_{12} \div \mathrm{B}_{12}$.

? Exercise 4.5.4.42

When multiplying $14_{12} \times 10_{12}$, the result 140 is found. How do we know a mistake was made?

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4.5.5: Chapter Test

1. Expand the Hindu-Arabic numeral 5,789 .

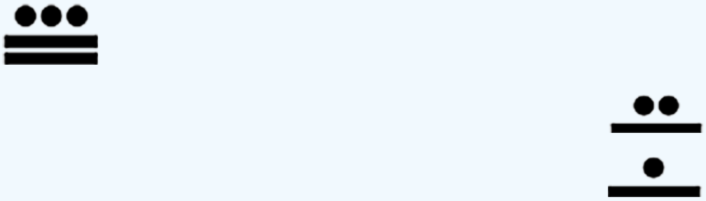
2. Evaluate the expression $4 \times 7^3 + 5 \times 2^5$.

3. Rewrite $6 \times 10^5 + 0 \times 10^4 + 8 \times 10^3 + 0 \times 10^2 + 1 \times 10^1 + 9 \times 10^0$ in Hindu-Arabic form.

4. Convert the Babylonian numeral to a Hindu-Arabic numeral.

? Exercise 4.5.5.5

Convert the Mayan number to a Hindu Arabic numeral.



6. What base system did the Babylonians use?

? Exercise 4.5.5.7

Which system—Roman, Babylonian, Mayan—used place values?

? Exercise 4.5.5.8

Convert the Roman numeral MDXLVII to a Hindu-Arabic numeral.

? Exercise 4.5.5.9

How many symbols are needed for a base 9 system?

? Exercise 4.5.5.10

For a system in a base larger than 10, what symbols are used as digits representing more than 10?

? Exercise 4.5.5.11

Convert 132_8 to a base 10 number.

? Exercise 4.5.5.12

List the numbers in base 4 up to 100_4.

? Exercise 4.5.5.13

Convert 74 to a base 12 number.

? Exercise 4.5.5.14

Create the addition table for base 4.

? Exercise 4.5.5.15

Calculate $314_6 + 453_6$.

? Exercise 4.5.5.16

Calculate $4B1_{12} - 2A6_{12}$.

? Exercise 4.5.5.17

When calculating $23_6 + 53_6$, a student obtains 76_6 . How do you know an error was made?

? Exercise 4.5.5.18

Create the multiplication table for base 4.

? Exercise 4.5.5.19

Calculate $323_4 \times 132_4$.

? Exercise 4.5.5.20

Calculate $21_4 \div 3_4$.

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CHAPTER OVERVIEW

5: Algebra

5.0: Introduction
5.1: Algebraic Expressions
  5.1.1: Exercises
5.2: Linear Equations in One Variable with Applications
  5.2.1: Exercises
5.3: Linear Inequalities in One Variable with Applications
  5.3.1: Exercises
5.4: Ratios and Proportions
  5.4.1: Exercises
5.5: Graphing Linear Equations and Inequalities
  5.5.1: Exercises
5.6: Quadratic Equations in One Variable With Applications
  5.6.0: Exercises
5.7: Functions
  5.7.1: Exercises
5.8: Graphing Functions
  5.8.1: Exercises
5.9: Systems of Linear Equations in Two Variables
  5.9.1: Exercises
5.10: Systems of Linear Inequalities in Two Variables
  5.10.0: Exercises
5.11: Linear Programming
  5.11.1: Exercises
5.12: Chapter Summary
  5.12.0: Key Terms
  5.12.1: Key Concepts
  5.12.2: Videos
  5.12.3: Formula Review
  5.12.4: Projects
  5.12.5: Chapter Review
  5.12.6: Chapter Test

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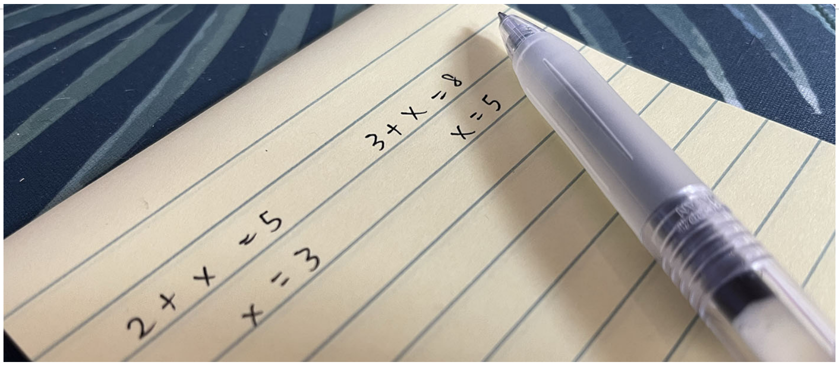
5.0: Introduction



In these algebraic equations, the x represents different numbers. (credit: Larissa Chu, CC BY 4.0)

The jump from arithmetic to algebra can be a difficult one for many students. Many students struggle with the idea that mathematics can include situations that aren't static and do change. In elementary arithmetic, a situation such as:

$$5+3=\_\_\_$$

is a static situation and will yield the answer of 8 every time. However, a situation such as:

$$5x+3=\_\_\_$$

can yield many different answers because the answer depends on what amount (number) that x represents. Since the value of x can vary (represent different values), it is known as a variable.

Algebra is useful to better model real life situations. In the first equation shown, $5+3=\_\_\_$ can only model situations where you add those two numbers together. For example, if your uncle gives you five dollars and your aunt gives you three dollars, then you will always receive eight dollars. The second equation $5x+3=\_\_\_$ can model more complex situations. For example, you wish to buy a game that costs $38 but you only have three dollars. Your uncle will pay you five dollars an hour to work for him. If you've worked five hours, have you earned enough money? If not, how many hours will you have to work?

Algebra and algebraic thinking open up a world of possibilities that arithmetic alone cannot do.

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5.1: Algebraic Expressions

Learning Objectives

1. Convert between written and symbolic algebraic expressions and equations.
2. Simplify and evaluate algebraic expressions.
3. Add and subtract algebraic expressions.
4. Multiply and divide algebraic expressions.

Algebraic expressions are the building blocks of algebra. While a numerical expression (also known as an arithmetic expression) like 5 + 3 can represent only a single number, an algebraic expression such as 5x+ can represent many different numbers. This section will introduce you to algebraic expressions, how to create them, simplify them, and perform arithmetic operations on them.

Figure 5.1.1: Two college graduates! (credit: modification of work UC Davis College of Engineering/Flickr, CC BY 2.0)

Algebraic Expressions and Equations

Xavier and Yasenia have the same birthday, but they were born in different years. This year Xavier is 20 years old and Yasenia is 23, so Yasenia is three years older than Xavier. When Xavier was 15, Yasenia was 18. When Xavier will be 33, Yasenia will be 36. No matter what Xavier’s age is, Yasenia’s age will always be 3 years more.

In the language of algebra, we say that Xavier's age and Yasenia's age are variable and the 3 is a constant. The ages change, or vary, so age is a variable. The 3 years between them always stays the same or has the same value, so the age difference is the constant. In algebra, letters of the alphabet are used to represent variables. The letters most often used for variables are x, y, z, a, b, and c. Suppose we call Xavier's age x. Then we could use $x+3$ to represent Yasenia's age, as shown in the table below.

Xavier’s Age	Yasenia’s Age
15	18
20	23
33	36
x	$x+3$

To write algebraically, we need some symbols as well as numbers and variables. The symbols for the four basic arithmetic operations: addition, subtraction, multiplication, and division are summarized in Table 5.1, along with words we use for the operations and the result.

Table 5.1.1 Symbols for Operations

Operation	Notation	Say:	The result is…
Addition	$a+b$	a plus b	The sum of a and b

Operation	Notation	Say:	The result is…
Subtraction	$a - b$	a minus b	The difference of a and b
Multiplication	$a \cdot b$, $(a)(b)$, $(a)b$, $a(b)$, semanticsab, ba	a times b	The product of a and b
Division	$a \div b$, a/b	a divided by b	The quotient of a and b

Checkpoint

In algebra, the cross symbol (x) is normally not used to show multiplication because that symbol could cause confusion. For example, does semantics3xy3xy mean semantics3×y3×y (three times y) or $3 \bullet x \bullet y$ (three times x times y)? To make it clear, use $\bullet$ or parentheses for multiplication.

We perform these operations on two numbers. When translating from symbolic form to words, or from words to symbolic form, pay attention to the words *of* or *and* to help you find the numbers.

- The *sum* ***of*** 5 ***and*** 3 means add 5 plus 3, which we write as $5 + 3$.
- The *difference* ***of*** 9 ***and*** 2 means subtract 9 minus 2, which we write as $9 - 2$.
- The *product* ***of*** 4 ***and*** 8 means multiply 4 times 8, which we can write as $4 \bullet 8$.
- The *quotient* ***of*** 20 ***and*** 5 means divide 20 by 5, which we can write as $20 \div 5$.

Example 5.1.1: Translating from Algebra to Words

Translate the following algebraic expressions from algebra into words.

1. $12 + 14$
2. $(30)(5)$
3. $64 \div 8$
4. $x - y$

Answer

1. According to Table 5.1, this could be translated as 12 plus 14 OR the sum of 12 and 14.
2. According to Table 5.1, this could be translated as 30 times 5 OR the product of 30 and 5.
3. According to Table 5.1, this could be translated as 64 divided by 8 OR the quotient of 64 and 8.
4. According to Table 5.1, this could be translated as x minus y OR the difference of x and y.

Your Turn 5.1.1

Translate the following algebraic expressions from algebra into words.

$18 + 11$

$(27)(9)$

$84 \div 7$

$p - q$

Example 5.1.2: Translating from Words to Algebra

Translate the following phrases from words into algebraic expressions.

1. 43 plus 67
2. The product of 45 and 3
3. The quotient of 45 and 3
4. 89 minus 42

Answer

1. According to Table 5.1, these words could be translated as $47-19$.
2. According to Table 5.1, these words could be translated as $72 \div 9$.
3. According to Table 5.1, these words could be translated as $m+n$.
4. According to Table 5.1, these words could be translated as (13) (7).

Your Turn 5.1.2

Translate the following phrases from words into algebraic expressions.

43 plus 67

The product of 45 and 3

The quotient of 45 and 3

89 minus 42

What is the difference in English between a phrase and a sentence? A phrase expresses a single thought that is incomplete by itself, but a sentence makes a complete statement. “Running very fast” is a phrase, but “The football player was running very fast” is a sentence. A sentence has a subject and a verb. In algebra, we have expressions and equations. Example 5.1 and Example 5.2 used expressions. An expression is like an English phrase. Notice that the English phrases do not form a complete sentence because the phrase does not have a verb. The following table has examples of expressions, which are numbers, variables, or combinations of numbers and variables using operation symbols.

Expression	Words	English Phrase
$3+5$	3 plus 5	The sum of three and five
$n-1$	n minus one	The difference of n and one
$6 \bullet 7$	6 times 7	The product of six and seven
$x \div y$Math input error	x divided by y	The quotient of x and y

Example 5.1.3: Translating from an English Phrase to an Expression

Translate the following phrases from words into algebraic expressions.

1. Seven more than a number n.
2. A number n times itself.
3. Six times a number n, plus two more.
4. The cost of postage is a flat rate of 10 cents for every parcel, plus 34 cents per ounce x.

Answer

1. $n+7$
2. $n \bullet n$ or n^2
3. $6n+2$
4. $10+34x$

Your Turn 5.1.3

Translate the following phrases from words into algebraic expressions.

Twenty less than a number n. (Hint: you have a number n and you want 20 less than it.)

Add two to a number n, then multiply it by six.

A number n to the third power minus five.

A plumber charges $\$60$ per hour h, plus a $\$40$ flat fee for every job.

An equation is two expressions linked with an equal sign (the symbol =). When two quantities have the same value, we say they are equal and connect them with an equal sign. When you read the words the symbols represent in an equation, you have a complete sentence in English. The equal sign gives the verb. So, $a=b$ is read "a is equal to b." The following table has some examples of equations.

Equation	English Sentence
$3+5=8$	The sum of three and five is equal to eight.
\(n-1=14\)	n minus one equals fourteen.
$6 \bullet 7=42$	The product of six and seven is equal to forty-two.
\(x=53\)	x is equal to fifty-three.
\(y+9=2 y-3\)	y plus nine is equal to two times y minus three.

Example 5.1.4: Translating from an English Sentence to an Equation

Translate the following sentences from words into algebraic equations.

1. Two times x is 6.
2. n plus 2 is equal to n times 3.
3. The quotient of 35 and 7 is 5.
4. Sixty-seven minus x is 56.

Answer

1. $2x=6$
2. $n+2=3n$
3. $35 \div 7=5$
4. $67-x=56$

Your Turn 5.1.4

Translate the following sentences from words into algebraic equations.

1. Five times y is 50.

2. Half of a number n is 30.

3. The difference of three times a number n and 7 is 2.

4. Two times x plus 7 is 21.

Who Knew?: The Use of Variables

French philosopher and mathematician René Descartes (1596–1650) is usually given credit for the use of the letters x, y, and z to represent unknown quantities in algebra. He introduced these ideas in his publication of *La Geometrie*, which was printed in 1637. In this publication, he also used the letters a, b, and c to represent known quantities. There is a (possibly fictitious) story that, when the book was being printed for the first time, the printer began to run short of the last three letters of the alphabet. So the printer asked Descartes if it mattered which of x, y, or z were used for the mathematical equations in the book. Descartes decided it made no difference to him; so the printer decided to use x predominantly for the mathematics in the book, because the letters y and z would occur more often in the body of the text (written in French) than the letter x would! This might explain why the letter x is still used today as the most common variable to represent unknown quantities in algebra.

Simplifying and Evaluating Algebraic Expressions

To simplify an expression means to do all the math possible. For example, to simplify $4 \bullet 2 + 1$ we would first multiply $4 \bullet 2$ to get 8 and then add 1 to get 9. We have introduced most of the symbols and notation used in algebra, but now we need to clarify the order of operations. Otherwise, expressions may have different meanings, and they may result in different values. Consider $2 + 7 \bullet 3$. Do you add first or multiply first? Do you get different answers?

Add first: $9 \bullet 3 = 27$	Multiply first: $2 + 21 = 23$	Which one is correct?

Early on, mathematicians realized the need to establish some guidelines when performing arithmetic operations to ensure that everyone would get the same answer. Those guidelines are called the order of operations and are listed in the table below.

Step 1: Parentheses and Other Grouping Symbols	Simplify all expressions inside the parentheses or other grouping symbols, working on the innermost parentheses first.
Step 2: Exponents	Simplify all expressions with exponents.
Step 3: Multiplication and Division	Perform all multiplication and division in order from left to right. These operations have equal priority.
Step 4: Addition and Subtraction	Perform all addition and subtraction in order from left to right. These operations have equal priority.

Checkpoint

You may have heard about Please Excuse My Dear Aunt Sally or PEMDAS. Be careful to notice in Steps 3 and 4 in the table above that multiplication and division, as well as addition and subtraction, happen in order from LEFT to RIGHT. It is possible, for example, to have PEDMAS or PEMDSA. The PEMDAS trick can be misleading if not fully understood!

✓ Example $5.1.5$: Making a Numerical Equation True Using the Order of Operations

Use parentheses to make the following statements true.

1. $17 - 10 + 3 = 10$
2. $2 \bullet 26 - 7 = 38$
3. $8 + 12 \div 5 - 3 = 14$
4. $5 + 2^3 \bullet 7 = 91$

Answer

1. Add the parentheses around the $17 - 10$. Then you have $(17 - 10) + 3 = 7 + 3 = 10$.
2. Add the parentheses around the $26 - 7$. Then you have $2 \bullet (26 - 7) = 2 \bullet 19 = 38$.
3. Add the parentheses around the $5 - 3$. Then you have

$$8 + 12 \div (5 - 3) = 8 + 12 \div 2 = 8 + 6 = 14$$

4. Add the parentheses around the $5 + 2^3$. Then you have

$$\left(5 + 2^3\right) \bullet 7 = (5 + 8) \bullet 7 = 13 \bullet 7 = 91.$$

Your Turn $5.1.5$

$5.1.1$

Use parentheses and the order of operations to make each equation true.

$24 - 17 - 6 = 13$

$3 \bullet 6+13=31$

$12-6 \div 5-3=3$

$5 \bullet 3^2+5=70$

In the last example, we simplified expressions using the order of operations. Now we'll evaluate some expressions—again following the order of operations. To evaluate an expression means to find the value of the expression when the variable is replaced by a given number.

✓ Example 5.1.6: Evaluating and Simplifying an Expression

1. Evaluate $3x+5$ when $x=2$.
2. Evaluate x^2+3x+1 when $x=2$..

Answer

1. Evaluate $5x-6$ when $x=3$.
2. Evaluate x^2-6x+3 when $x=3$.

Your Turn 5.1.6

Evaluate $5x-6$ when $x=3$.

Evaluate x^2-6x+3 when $x=3$.

Operations of Algebraic Expressions

Algebraic expressions are made up of terms. A term is a constant or the product of a constant and one or more variables. Examples of terms are 7, y, $5x^2$, $9a$, and semanticsb5b5. The constant that multiplies the variable is called the coefficient. Think of the coefficient as the number in front of the variable. Consider the algebraic expressions $5x^2$, which has a coefficient of 5, and $9a$, which has a coefficient of 9. If there is no number listed in front of the variable, then the coefficient is 1 since $x=1 \bullet x$.

Some terms share common traits. When two terms are constants or have the same variable and exponent, we say they are **like terms**. If there are like terms in an expression, you can simplify the expression by combining the like terms. We add the coefficients and keep the same variable.

✓ Example 5.1.7: Adding Algebraic Expressions

Add $(x^2+4x-9)+(3x^2-x+12)$.

Answer

Step 1: Add the terms in any order and get the same result (think: $2+3=3+2$) and drop the parentheses:

$$x^2+4x-9+3x^2-x+12$$

Step 2: Group like terms together:

$$x^2+3x^2+4x-x-9+12$$

Step 3: Combine the like terms:

$$4x^2+3x+3$$

Your Turn $5.1.7$

Add $(2x^2-4x+5)+(3x^2+x-12)$.

Example $5.1.8$: Subtracting Algebraic Expressions

Subtract $(5x2+4x-9)-(3x2-x+12)(5x2+4x-9)-(3\times 2-x+12)$.

Answer

Step 1: Distribute the negative inside the parentheses (think: $2-(3-4)\ =2-3+4=-1\ +4\ =3$, which is the correct answer). You cannot just drop the parentheses (for example, $2-3-4=-1-4\ =-5$, which is not correct as we have already verified the answer is 3):

$$5x^2+4x-9-3x^2+x-12$$

Step 2: Group like terms together:

$$5x^2-3x^2+4x+x-9-12$$

Step 3: Combine the like terms:

$$2x^2+x-21$$

Your Turn $5.1.8$

Subtract $(x^2-4x+8)-(x^2+5x+12)$.

Before looking at multiplying algebraic expressions we look at the Distributive Property, which says that to multiply a sum, first you multiply each term in the sum and then you add the products. For example, $5(4+3)\ =5(4)+5(3)\ =20\ +15\ =35$ can also be solved as $5(4+3)\ =5(7)=35$. If we use a variable, then $5(x+3)\ =5x+15$.

We can extended this example to $(5+2)(4+3)=(5)(4)+(5)(3)+(2)(4)+(2)(3)=20\ +15\ +8+6=49$, which can also be solved as $(5+2)(4+3)=(7)(7)=49$. If we use variables, then

$$(x+5)(x+4)=(x)(x)+(x)(4)+(5)(x)+(5)(4)=x^2+4x+5x+20=x^2+9x+20.$$

FORMULA

Distributive Property: $a(b+c)=ab+ac$

Example$5.1.9$: Simplifying an Expression Using the Order of Operations

Simplify each expression.

1. $(x-3)5$
2. $(-3)(x+y-2)$
3. $5^2(7+3)(x)$
4. $4+x\bullet 5$
5. $(4+x)\bullet 5$

Answer

1. $5x-3\bullet 5=5x-15$
2. $(-3)\bullet x+(-3)\bullet y-(-3)\bullet 2=-3x-3y+6$
3. $25(7+3)(x)=25(10)(x)=250x$

4. $4+5x$
5. $(4)\bullet(5)+(x)\bullet(5)=20+5x$

Your Turn $5.1.9$

Simplify each expression.

$2(y+5)$

$(-2)(a+b-4)$

$4^2(47-40+x)$

$(18\div 3)(x+7-4)$

$2(3a+5)+(-3)(a+2)$

✓ Example $5.1.10$: Multiplying Algebraic Expressions

Multiply $(4x-9)(x+2)$.

Answer

Step 1: Use the Distributive Property:

$$(4x)(x)+(4x)(2)-(9)(x)-(9)(2)$$

Step 2: Multiply:

$$4x^2+8x-9x-18$$

Step 3: Combine the like terms:

$$4x^2-x-18$$

Your Turn $5.1.10$

Multiply $(x-4)(2x-3)$.

Checkpoint

You may have heard the term FOIL which stands for: First, Outer, Inner, Last. FOIL essentially describes a way to use the Distributive Property if you multiply a two-term expression by another two-term expression, but FOIL only works in that specific situation. For example, suppose you have a two-term expression multiplied by a three-term expression, such as $(x+2)\,(x+y-5)$. *What terms qualify as inner terms and what terms qualify as outer terms? In this particular situation, FOIL cannot possibly work; the multiplication of* $(x+2)\,(x+y-5)$ *should yield six terms, where FOIL is designed to only give you four! The Distributive Property works regardless of how many terms there are. FOIL can be misleading and applied inappropriately if not fully understood!*

✓ Example $5.1.11$: Dividing Algebraic Expressions

Divide $(8x^2+4x-16)\div(4x)$

Answer

Divide EACH term by $4x$:

$$\left(8x^2\div 4x\right)+\left(4x\div 4x\right)-\left(16\div 4x\right)=2x+1-\frac{4}{x}$$

mfrac

Your Turn 5.1.11

Divide $(16x^2+4x-8)\div(4)$.

Checkpoint

Be careful how you divide! Sometimes students incorrectly divide only one term on top by the bottom term. For example, $\frac{8x^2+6x-3}{2x}$ *might turn into* $4x+3x-3=7x-3$ *if done incorrectly. When we divide expressions, EACH term is divided by the divisor. So,* $\frac{8x^2+6x-3}{2x}=\frac{8x^2}{2x}+\frac{6x}{2x}-\frac{3}{2x}=4x+3-\frac{3}{2x}$*. If you forget, it is always a good idea to check these rules by creating an example using numerical expressions. For example,* $\frac{9+6+3}{3}=\frac{18}{3}=6$*. Dividing each term on top by 3 would yield* $\frac{9+6+3}{3}=\frac{9}{3}+\frac{6}{3}+\frac{3}{3}=3+2+1=6$*, which is the correct answer. However, if you just divided the 9 on top by the 3 on the bottom, getting* $\frac{9+6+3}{3}=3+6+3=12$*, this does not result in the correct answer.*

People in Mathematics: Al-Khwarizmi

Abu Ja'far Muhammad ibn Musa Al-Khwarizmi was born around 780 AD, probably in or around the region of Khwarizm, which is now part of modern-day Uzbekistan. For most of his adult life, he worked as a scholar at the House of Wisdom in Baghdad, Iraq. He wrote many mathematical works during his life, but is probably most famous for his book *Al-kitab al-muhtasar fi hisab al-jabr w'al'muqabalah*, which translates to *The Condensed Book on the Calculation of al-Jabr* (*completion*) *and al'muqabalah* (*balancing*). The word *al-jabr* would eventually become the word we use to describe the topic that he was writing about in this book: *algebra*. From another book of his, with the Latin title *Algoritmi de numero Indorum* (*Al-Khwarizmi on the Hindu Art of Reckoning*), our word *algorithm* is derived. In addition to writing on mathematics, Al-Khwarizmi wrote works on astronomy, geography, the sundial, and the calendar.

Figure 5.3: Al-Khwarizmi

In 2012, Andrew Hacker wrote an opinion piece in the *New York Times Magazine* suggesting that teaching algebra in high school was a waste of time. Keith Devlin, a British mathematician, was asked to comment on Hacker's article by his students in his Stanford University Continuing Studies course "Mathematics: Making the Invisible Visible" on iTunes University. Devlin concludes that Hacker was displaying his ignorance of what algebra is.

Video

A: Why we teach algebra
h Devlin
Watch on YouTube

Q&A: Why We Teach Algebra

Check Your Understanding

1. Juliette is 2 inches taller than her friend Vivian. Which algebraic equations represent their height? Use J for Juliette's height and V for Vivian's height.

$$J = V + 2$$
$$V = J - 2$$
$$J + 2 = V$$
$$J = V - 2$$

2. Which options represent algebraic expressions?

$$2x^2 + 3x - 1 = 0$$
$$5x + 8$$
$$2n + 3m$$
$$5x - 7 = 3x + 1$$

3. Which expression equals $10x$?

$$(8x + 12x) \div 4x - 2x$$
$$8x + (12x \div 4x) - 2x$$
$$8x + 12x \div (4x - 2x)$$
$$(8x + 12x) \div (4x - 2x)$$

4. Using the expression $3x^2 - 7x + 2$, when a certain number is put in for x, the result is 50 . What is the value of x ?

-2

-3

2

3

5. Which expression equals $(x - y)(x - y)$? Hint: Use the Distributive Property.

$$x^2 - y^2$$
$$x^2 + y^2$$
$$x^2 - 2xy - y^2$$
$$x^2 - 2xy + y^2$$

6. Given the expression $9x^3 + 3x^2 - 6x$, the Distributive Property allows it to be rewritten as:

$$3x\left(3x^2 + x - 2\right)$$
$$3x^2 + x - 2$$
$$27x^5 - 54x^4$$
$$27x^6 - 54x^3$$

7. Given the two algebraic expressions $(x + 2)$ and $(x + y - 5)$, the solution is $x^2 + xy - 3x + 2y - 10$. What mathematical operation was performed on the two algebraic expressions?

8. Given the two algebraic expressions $8x^2 - 9x + 6$ and $6x$, the solution is $3x - 1.5 + \frac{1}{x}$. What mathematical operation was performed on the two algebraic expressions?

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5.1.1: Exercises

For the following exercises, translate from algebra to words.

? Exercise $5.1.1.1$

$50 - 15$

? Exercise $5.1.1.2$

(10) (x)

? Exercise $5.1.1.3$

$2a - b$

? Exercise $5.1.1.4$

$100 \div 33$

? Exercise $5.1.1.5$

$3x + 5$

For the following exercises, translate from words to algebra.

? Exercise $5.1.1.6$

15 divided by 3 .

? Exercise $5.1.1.7$

The sum of 13 and 13.

? Exercise $5.1.1.8$

120 minus 12 .

? Exercise $5.1.1.9$

The product of 5 and 4.

? Exercise $5.1.1.10$

The sum of double x and 5 .

For the following exercises, translate from an English phrase to an expression.

? Exercise $5.1.1.11$

Three times y minus 7 .

? Exercise 5.1.1.12

a divided by 2 ; then add 4.

? Exercise 5.1.1.13

x squared minus 3 .

? Exercise 5.1.1.14

A rental car company charges $\$0.15$ per mile m, plus a $\$40$ flat fee for the rental.

? Exercise 5.1.1.15

A parking garage in New York City charges $\$20$ for the first hour, then $\$5$ per hour h.

For the following exercises, use parentheses to make the statements true.

? Exercise 5.1.1.16

$16 \div 4 \bullet 2 + 5 = 13$

? Exercise 5.1.1.17

$2^2 - 5 + 3 \bullet 2 = 5$

? Exercise 5.1.1.18

$x - 3 \bullet x - 2 = x^2 - 5x + 6$

? Exercise 5.1.1.19

$20x \div 5 - 1 - 5x = 0$

? Exercise 5.1.1.20

$5x + 3x \div 3 - 7x + 1 \bullet x = 0$

For the following exercises, evaluate and simplify the expression.

? Exercise 5.1.1.21

x^2 when $x = 9$

? Exercise 5.1.1.22

$2x + 5$ when $x = 3$

? Exercise 5.1.1.23

$(3x + 1)(4x - 6)$ when $x = 2$

? Exercise 5.1.1.24

x^2+3x+8 when $x=3$

? Exercise 5.1.1.25

$(x^2+5x-4)(2x)$ when $x=4$

? Exercise 5.1.1.26

$4a+5-2a-8$ when $a=6$

? Exercise 5.1.1.27

$8a^2+4a+9-a^2-1$ when $a=5$

? Exercise 5.1.1.28

Yasenia is 3 years older than Xavier. How old is Yasenia when Xavier is 18 years old?

? Exercise 5.1.1.29

A rental car company charges $\$0.15$ per mile m, plus a $\$40$ flat fee for the rental. What is the cost of the car rental if one drives 100 miles?

? Exercise 5.1.1.30

A parking garage in New York City charges $\$20$ for the first hour, then $\$5$ per hour h. What is the cost of parking for 10 hours?

For the following exercises, perform the indicated operation for the expressions.

? Exercise 5.1.1.31

Add $(4x-9)+(x+12)$.

? Exercise 5.1.1.32

Add $(3x^2+2x+1)+(x^2-2x+2)$.

? Exercise 5.1.1.33

Subtract $(4x-9)-(-x+2)$.

? Exercise 5.1.1.34

Subtract $(3x^2+5x)-(x^2-3x+11)$.

? Exercise 5.1.1.35

Multiply $4(x+2)$.

? Exercise 5.1.1.36

Multiply $2\left(3x^2-2x+1\right)$.

? Exercise 5.1.1.37

Multiply $(3x)(x-1)$.

? Exercise 5.1.1.38

Multiply $(2x-1)(x+3)$.

? Exercise 5.1.1.39

$\left(125x^2+35x-5\right)\div(5)$.

? Exercise 5.1.1.40

$\left(9x^2+18x-27\right)\div(3x)$.

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5.2: Linear Equations in One Variable with Applications

Learning Objectives

1. Solve linear equations in one variable using properties of equations.
2. Construct a linear equation to solve applications.
3. Determine equations with no solution or infinitely many solutions.
4. Solve a formula for a given variable.

In this section, we will study linear equations in one variable. There are several real-world scenarios that can be represented by linear equations: taxi rentals with a flat fee and a rate per mile; cell phone bills that charge a monthly fee plus a separate rate per text; gym memberships with a monthly fee plus a rate per class taken; etc. For example, if you join your local gym at \$10 per month and pay \$5 per class, how many classes can you take if your gym budget is \$75 per month?

Figure 5.2.1: Most gyms have a monthly membership fee. (credit: modification of work "Morning PT after the Holidays 2021" by Fort Drum & 10th Mountain Division (LI)/Flickr, Public Domain Mark 1.0)

Linear Equations and Applications

Solving any equation is like discovering the answer to a puzzle. The purpose of solving an equation is to find the value or values of the variable that makes the equation a true statement. Any value of the variable that makes the equation true is called a solution to the equation. It is the answer to the puzzle! There are many types of equations that we will learn to solve. In this section, we will focus on a linear equation, which is an equation in one variable that can be written as

$$ax + b = 0$$

where a and b are real numbers and $a \neq 0$, such that a is the coefficient of x and b is the constant.

To solve a linear equation, it is a good idea to have an overall strategy that can be used to solve any linear equation. In the Example 5.12, we will give the steps of a general strategy for solving any linear equation. Simplifying each side of the equation as much as possible first makes the rest of the steps easier.

✓ Example ✓ 5.2.1: Solving a Linear Equation Using a General Strategy

Solve $7(n-3)-8=-15$

Answer

Step 1: Simplify each side of the equation as much as possible.	Use the Distributive Property. Notice that each side of the equation is now simplified as much as possible.	$7(n-3)-8=-15$ $7n-21-8=-15$ $7n-29=-15$
Step 2: Collect all variable terms on one side of the equation.	Nothing to do; all terms are on the left side.	$7n-29=-15$
Step 3: Collect constant terms on the other side of the equation.	To get constants only on the right, add 29 to each side. Simplify.	$7n-\mathbf{29}+29=-15+\mathbf{29}$ $7n=14$
Step 4: Make the coefficient of the variable term equal to 1.	Divide each side by 7. Simplify.	$\frac{7n}{7}=\frac{14}{7}$ $n=2$
Step 5: Check the solution.	Let $n=2$ Subtract.	Check: $7(n-3)-8=-15$ $7(2-3)-8\overset{?}{=}-15$ $7(-1)-8\overset{?}{=}-15$ $-7-8\overset{?}{=}-15$ $-15=-15$

Your Turn 5.2.1

Solve $2(x+1)-3=5$

In Example 5.2.1, we used both the addition and division property of equations. All the properties of equations are summarized in table below. Basically, what you do to one side of the equation, you must do to the other side of the equation to preserve equality.

Operation	Property	Example
Addition	If $a=b$ Then $a+c=b+c$	$2=2$ $2+3=2+3$ $5=5$
Subtraction	If $a=b$ Then $a-c=b-c$	$5=5$ $5-2=5-2$ $3=3$
Multiplication	If $a=b$ Then $a\bullet c=b\bullet c$	$3=3$ $3\bullet 4=3\bullet 4$ $12=12$

Operation	Property	Example
Division	If $a = b$ Then $a \div c = b \div c$ for $c \neq 0$	$8 = 8$ $8 \div 2 = 8 \div 2$ $4 = 4$

Checkpoint

Be careful to multiply and divide every term on each side of the equation. For example, $2 + x = \frac{x}{3}$ *is solved by multiplying BOTH sides of the equation by 3 to get* $3(2+x) = 3\left(\frac{x}{3}\right)$ *which gives* $6 + 3x = x$. *Using parentheses will help you remember to use the distributive property! A division example, such as* $3(x+2) = 6x+9$, *can be solved by dividing BOTH sides of the equation by 3 to get* $\frac{3(x+2)}{3} = \frac{6x+9}{3}$, *which then will lead to* $x + 2 = 2x + 3$.

Example 5.2.2: Solving a Linear Equation Using Properties of Equations

Solve $9(y-2) - y = 16 + 7y$.

Answer

Step 1: Simplify each side.

$$\begin{aligned} 9(y-2) - y &= 16 + 7y \\ 9y - 18 - y &= 16 + 7y \\ 8y - 18 &= 16 + 7y \end{aligned}$$

Step 2: Collect all variables on one side.

$$\begin{aligned} 8y - 18 - 7y &= 16 + 7y - 7y \\ y - 18 &= 16 \end{aligned}$$

Step 3: Collect constant terms on one side.

$$\begin{aligned} y - 18 + \mathbf{18} &= 16 + \mathbf{18} \\ y &= 34 \end{aligned}$$

Step 4: Make the coefficient of the variable 1. Already done!

Step 5: Check.

$$\begin{aligned} 9(34) - 18 - (34) &\overset{?}{=} 16 + 7(34) \\ 306 - 18 - 34 &\overset{?}{=} 16 + 238 \\ 288 - 34 &\overset{?}{=} 254 \\ 254 &= 254\checkmark \end{aligned}$$

Your Turn 5.2.2

Solve $6(y-2) - 5y = 4(y+3) - 4(y-1)$.

Who Knew?: Who Invented the Symbol for Equals ?

Before the creation of a symbol for equality, it was usually expressed with a word that meant equals, such as *aequales* (Latin), *esgale* (French), or *gleich* (German). Welsh mathematician and physician Robert Recorde is given credit for inventing the modern sign. It first appears in writing in *The Whetstone of Witte*, a book Recorde wrote about algebra, which was published in 1557. In this book, Recorde states, "I will set as I do often in work use, a pair of parallels, or Gemowe (twin) lines of one length, thus: ===, because no two things can be more equal." Although his version of the sign was a bit longer than the one we use today, his idea stuck and "=" is used throughout the world to indicate equality in mathematics.

In Algebraic Expressions, you translated an English sentence into an equation. In this section, we take that one step further and translate an English paragraph into an equation, and then we solve the equation. We can go back to the opening question in this section: *If you join your local gym at $10 per month and pay $5 per class, how many classes can you take if your gym budget is $75 per month?* We can create an equation for this scenario and then solve the equation (see Example 5.15).

✓ Example 5.2.3: Constructing a Linear Equation to Solve an Application

The Beaudrie family has two cats, Basil and Max. Together, they weigh 23 pounds. Basil weighs 16 pounds. How much does Max weigh?

Answer

Let $b =$ Basil's weight and $m =$ Max's weight.

$$b + m = 23$$

We also know that Basil weighs 16 pounds so:

Steps 1 and 2: $16 + m = 23$

Since both sides are simplified, the variable is on one side of the equation, we start in Step 3 and collect the constants on one side:

Step 3:

$$\begin{aligned} 16 + m - \mathbf{16} &= 23 - \mathbf{16} \\ m &= 7 \end{aligned}$$

Step 4: is already done so we go to Step 5:

Step 5:

$$\begin{aligned} 16 + 7 &\overset{?}{=} 23 \\ 23 &= 23 \checkmark \end{aligned}$$

Basil weighs 16 pounds and Max weighs 7 pounds.

Your Turn 5.2.3

Sam and Henry are roommates. Together, they have 68 books. Sam has 26 books. How many books does Henry have?

✓ Example 5.2.4: Constructing a Linear Equation to Solve Another Application

If you join your local gym at $10 per month and pay $5 per class, how many classes can you take if your gym budget is $75 per month?

Answer

If we let $x =$ number of classes, the expression $5x + 10$ would represent what you pay per month if each class is $\$5$ and there's a $\$10$ monthly fee per class. $\$10$ is your constant. If you want to know how many classes you can take if you have a $\$75$ monthly gym budget, set the equation equal to 75 . Then solve the equation $5x + 10 = 75$ for x.

Steps 1 and 2:

$$5x + 10 = 75$$

Step 3:

$$\begin{aligned} 5x + 10 - 10 &= 75 - 10 \\ 5x &= 65 \end{aligned}$$

Step 4:

$$\begin{aligned}\frac{5x}{5} &= \frac{65}{5}\\ x &= 13\end{aligned}$$

Step 5:

$$\begin{aligned}5(13)+10 &\overset{?}{=} 75\\ 65+10 &\overset{?}{=} 75\\ 75 &= 75\end{aligned}$$

The solution is 13 classes. You can take 13 classes on a $\$75$ monthly gym budget.

Your Turn 5.2.4

On June 7, 2021, the national average price for regular gasoline was $3.053 per gallon. If Aiko fills up his car with 16 gallons, how much is the total cost? Round to the nearest cent.

Example 5.2.5: Constructing an Application from a Linear Equation

Write an application that can be solved using the equation 50x+35=18550x+35=185. Then solve your application.

Answer

Answers will vary. Let's say you want to rent a snowblower for a huge winter storm coming up. If $x =$ the number of days you rent a snowblower, then the expression $50x + 35$ represents what you pay if, for each day, it costs $\$50$ to rent the snowblower and there is a $\$35$ flat rental fee. $\$35$ is the constant. To find out how many days you can rent a snowblower for $\$185$, set the expression equal to 185.
Then solve the equation $50x + 35 = 185$ for x.

Steps 1 and 2:

$$50x+35=185$$

Step 3:

$$\begin{aligned}50x+35-\mathbf{35} &= 185-\mathbf{35}\\ 50x &= 150\end{aligned}$$

Step 4:

$$\begin{aligned}\frac{50x}{\mathbf{50}} &= \frac{150}{\mathbf{50}}\\ x &= 3\end{aligned}$$

Step 5:

$$\begin{aligned}50(3)+35 &\overset{?}{=} 185\\ 150+35 &\overset{?}{=} 185\\ 185 &= 185\checkmark\end{aligned}$$

The equation is $50x + 35 = 185$ and the solution is 3 days. You can rent a snowblower for 3 days on a $\$185$ budget.

Your Turn 5.2.5

Write an application that can be solved using the equation $25x + 75 = 200$. Then solve your application.

Linear Equations with No Solutions or Infinitely Many Solutions

Every linear equation we have solved thus far has given us one numerical solution. Now we'll look at linear equations for which there are no solutions or infinitely many solutions.

✓ Example $5.2.6$: Solving a Linear Equation with No Solution

Solve $3(x+4)=4x+8-x$.

Answer

Step 1: Simplify each side. $3(x+4)=4x+8-x$

$$3x+12-\mathbf{3}\boldsymbol{x}=3x+8-\mathbf{3}\boldsymbol{x}$$

Step 2: Collect all variables to one side. $3x+12-\mathbf{3x}=3x+8-\mathbf{3}\boldsymbol{x}$

$$12=8$$

The variable x disappeared! When this happens, you need to examine what remains. In this particular case, we have $12=8$, which is not a true statement. When you have a false statement, then you know the equation has no solution; there does not exist a value for x that can be put into the equation that will make it true.

Your Turn $5.2.6$

Solve $2(x+6)=3x+4-(x+5)$.

✓ Example $5.2.7$: Solving a Linear Equation with Infinitely Many Solutions

Solve $2(x+5)=4(x+3)-2x-2$.

Answer

Step 1:

$$\begin{aligned}2(x+5)&=4(x+3)-2x-2\\2x+10&=4x+12-2x-2\\2x+10&=2x+10\end{aligned}$$

Step 2:

$$\begin{aligned}2x+10-\mathbf{2x}&=2x+10-\mathbf{2x}\\10&=10\end{aligned}$$

As with the previous example, the variable disappeared. In this case, however, we have a true statement $(10=10)$. When this occurs we say there are infinitely many solutions; any value for x will make this statement true.

Your Turn $5.2.7$

Solve $3x-7-(x+5)=2(x-6)$.

Solving a Formula for a Given Variable

You are probably familiar with some geometry formulas. A formula is a mathematical description of the relationship between variables. Formulas are also used in the sciences, such as chemistry, physics, and biology. In medicine they are used for calculations for dispensing medicine or determining body mass index. Spreadsheet programs rely on formulas to make calculations. It is important to be able to manipulate formulas and solve for specific variables.

To solve a formula for a specific variable means to isolate that variable on one side of the equal sign with a coefficient of 1. All other variables and constants are on the other side of the equal sign. To see how to solve a formula for a specific variable, we will start with the distance, rate, and time formula.

Example 5.2.8: Solving for a Given Variable with Distance, Rate, and Time

Solve the formula $d = rt$ for t. This is the distance formula where $d =$ distance, $r =$ rate, and $t =$ time.

Answer

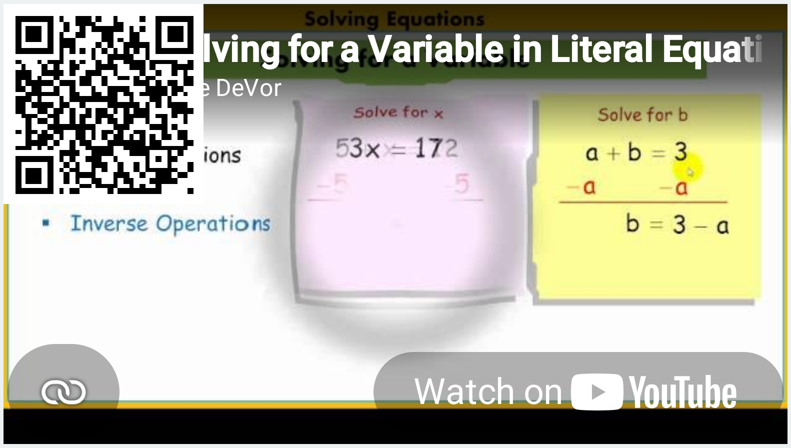
Divide both sides by $r : d/\boldsymbol{r} = rt/\boldsymbol{r}$

$$d/r = t$$

Your Turn 5.2.8

Solve the formula $I = Prt$ for t. This formula is used to calculate simple interest I, for a principal P, invested at a rate r, for t years.

Video


Solving Equations
lving for a Variable in Literal Equati
e DeVor
ions
Solve for x
53x = 172
−5 −5
Solve for b
a + b = 3
−a −a
b = 3 − a
Inverse Operations
Watch on YouTube

Solving for a Variable in an Equation

Example 5.2.9: Solving for a Given Variable in the Area Formula for a Triangle

Solve the formula $A = 1/2bh$ for h. This is the area formula of a triangle where $A =$ area, $b =$ base,

Step 1: Multiply both sides by 2.

$$2A = 2(1/2bh)$$
$$2A = bh$$

Step 2: Divide both sides by b.

$$\frac{2A}{b} = \frac{bh}{b}$$
$$\frac{2A}{b} = h$$
$$h = \frac{2A}{b}$$

mfrac

Your Turn 5.2.9

Solve the formula $V = \frac{1}{3}\pi r^2 h$ for h. This formula is used to calculate the volume V of a right circular cone with radius r and height h.

WORK IT OUT: Using Algebra to Understand Card Tricks

You will need to perform this card trick with another person. Before you begin, the two people must first decide which of the two will be the *Dealer* and which will be the *Partner*, as each will do something different. Once you have decided upon that, follow the steps here:

Step 1: Dealer and Partner: Take a regular deck of 52 cards, and remove the face cards and the 10s.

Step 2: Dealer and Partner: Shuffle the remaining cards

Step 3: Dealer and Partner: Select one card each, but keep them face down and don't look at them yet.

Step 4: Dealer: Look at your card (just the Dealer!). Multiply its value by 2 (Aces = 1).

Step 5: Dealer: Add 2 to this result.

Step 6: Dealer: Multiply your answer by 5.

Step 7: Partner: Look at your card.

Step 8: Partner: Calculate: 10 - your card, and tell this information to the dealer.

Step 9: Dealer: Subtract the value the Partner tells you from your total to get a final answer.

Step 10: Dealer: verbally state the final answer.

Step 11: Dealer and Partner: Turn over your cards. Now, answer the following questions

1. Did the trick work? How do you know?
2. Why did this occur? In other words, how does this trick work?

Check Your Understanding

1. Is the solution strategy used in solving the linear equation correct? If it is correct, show the final step (check the solution). If it is not correct, explain why.

$$\begin{gathered}
8(x-2)=6(x+10)\\
8x-16=6x+60\\
8x-16-6x=6x+60-6x\\
2x-16+16=60+16\\
2x=76\\
x=38
\end{gathered}$$

2. Is the solution strategy used in solving the linear equation correct? If it is correct, show the final step (check the solution). If it is not correct, explain why.

$$\begin{gathered}
7+4(2+5x)=3(6x+7)-(13x+36)\\
7+8+20x=18x+21-13x-36\\
15+20x=5x-15\\
15+20x-\mathbf{5x}=5x-15-\mathbf{5x}\\
15+15x-\mathbf{15}=-15-\mathbf{15}\\
15x=-30\\
x=-2
\end{gathered}$$

3. Is the solution strategy used in solving the linear equation correct? If it is correct, show the final step (check the solution). If it is not correct, explain why.

$$\begin{gathered}
8x+7-(2x-9)=22-(4x-4)\\
8x+7-2x-9=22-4x-4\\
6x-2=18-4x\\
6x-2+4x=18-4x+4x
\end{gathered}$$

$$10x - 2 + 2 = 18 + 2$$
$$10x = 20$$
$$x = 2$$

For the following exercises, use this scenario: The Nice Cab Company charges a flat rate of $\$3.00$ for each fare, plus $\$1.70$ per mile. A competing taxi service, the

Enjoyable Cab Company, charges a flat rate of $\$5.00$ for each fare, plus $\$1.60$ per mile.

4. Using the variable x for number of miles, write the equation that would allow you to find the total fare (T) using the Nice Cab Company.

5. It is 22 miles from the airport to your hotel. What would be your total fare using the Nice Cab Company?

6. Using the variable y for number of miles, write the equation that would allow you to find the total fare (T) using the Enjoyable Cab Company.

7. Using the same 22-mile trip from the airport to the hotel, how much would the total fare be for using the Enjoyable Cab Company?

8. Based on the cost of each cab ride, which cab company should you use for the trip from the airport to the hotel? Why?

9. After solving the linear equation $3(2x-3) = 12(x-3) - 3(2x-9)$, Nancy says there is no solution. Luis believes there are infinitely many solutions. Who is right?

10. The conversion formula between the Fahrenheit temperature scale and the Celsius temperature scale is given by this formula: $C = \frac{5}{9}(F-32)$, where C is the temperature in degrees Celsius and F is the temperature in degrees Fahrenheit. What is the correct formula when solved for F ?
a. $F = \frac{5}{9}C - 32$
b. $F = \frac{9}{5}C - 32$
c. $F = \frac{5}{9}C + 32$
d. $F = \frac{9}{5}C + 32$

11. To find a temperature on the Kelvin temperature scale, add 273 degrees to the temperature in Celsius. Which formula illustrates this?
a. $C = K + 273$
b. $K = C + 273$
c. $K = C - 273$
d. $C = K - 273$

12.

To find a temperature on the Kelvin temperature scale, add 273 degrees to the temperature in Celsius. Which formula illustrates this?

a. $C = K + 273$
b. $K = C + 273$
c. $K = C - 273$
d. $C = K - 273$

13. Using the information from exercise 18 and exercise 19, which conversion formula would you use to find degrees Kelvin when given degrees Fahrenheit?
a. $K = \frac{5}{9}(F-32) + 273$
b. $K = \frac{5}{9}F + 241$
c. $K = \frac{9}{5}(F-32) + 273$
d. $K = \frac{9}{5}F + 241$
14. There is a fourth temperature scale, although it is not used much today. The Rankin temperature scale varies from the Fahrenheit scale by about 460 degrees. So given a temperature in Fahrenheit, add 460 degrees to get the temperature in Rankin. Which formula represents a formula to find degrees Rankin when given degrees Celsius?
a. $R = \frac{5}{9}C - 492$
b. $R = \frac{9}{5}C + 492$
c. $R = C + 492$
d. $R = \frac{5}{9}(C - 492)$

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5.2.1: Exercises

For the following exercises, solve the linear equations using a general strategy.

? Exercise $5.2.1.2$

$-(t-19)=28$

? Exercise $5.2.1.2$

$51+5(4-q)=56$

? Exercise $5.2.1.3$

$-6+6(5-k)=15$

? Exercise $5.2.1.4$

$3(10-2x)+54=0$

? Exercise $5.2.1.5$

$-2(11-7x)+54=4$

For the following exercises, solve the linear equations using properties of equations.

? Exercise $5.2.1.6$

$-12+8(x-5)=-4+3(5x-2)$

? Exercise $5.2.1.7$

$4(p-4)-(p+7)=5(p-3)$

? Exercise $5.2.1.8$

$3(a-2)-(a+6)=4(a-1)$

? Exercise $5.2.1.9$

$4[5-8(4c-3)]=12(1-13c)-8$

? Exercise $5.2.1.10$

$5[9-2(6d-1)]=11(4-10d)-139$

For the following exercises, construct a linear equation to solve an application.

? Exercise $5.2.1.11$

It costs $\$0.55$ to mail one first class letter. Construct a linear equation and solve to find how much it costs to mail 13 letters.

? Exercise 5.2.1.12

Normal yearly snowfall at the local ski resort is 12 inches more than twice the amount it received last season. The normal yearly snowfall is 62 inches. Construct a linear equation and solve to find what the snowfall was last season.

? Exercise 5.2.1.13

Guillermo bought textbooks and notebooks at the bookstore. The number of textbooks was three more than twice the number of notebooks. He bought seven textbooks. Construct a linear equation and solve to find how many notebooks he bought.

? Exercise 5.2.1.14

Gerry worked Sudoku puzzles and crossword puzzles this week. The number of Sudoku puzzles he completed is eight more than twice the number of crossword puzzles. He completed 22 Sudoku puzzles. Construct a linear equation and solve to find how many crossword puzzles he did.

? Exercise 5.2.1.15

Laurie has $\$46,000$ invested in stocks. The amount invested in stocks is $\$8,000$ less than three times the amount invested in bonds. Construct a linear equation and solve to find how much Laurie invested in bonds.

For the following exercises, construct an application from a linear equation.

? Exercise 5.2.1.16

$1,000x + 2,500 = 16,500$.

? Exercise 5.2.1.17

$0.36t$ for $t = 333$.

? Exercise 5.2.1.18

$150n + 120 = 570$.

? Exercise 5.2.1.19

$4c + 2c$ for $c = 5$.

? Exercise 5.2.1.20

$2s + 10 = 24$.

For the following exercises, state whether each equation has exactly one solution, no solution, or infinitely many solutions.

? Exercise 5.2.1.21

$23z + 19 = 3(5z - 9) + 8z + 46$

? Exercise 5.2.1.22

$18(5j - 11) = 47j + 17$

? Exercise 5.2.1.23

$22(3m+4)=17(4m-6)$

? Exercise 5.2.1.24

$7v+42=11(3v+8)-2(13v-1)$

? Exercise 5.2.1.25

$45(3y-2)=9(15y-6)$

? Exercise 5.2.1.26

$9(14d+9)+4d=13(10d+6)+3$

For the following exercises, solve the given formula for the specified variable.

? Exercise 5.2.1.27

Solve the formula $C=\pi d$ for d.

? Exercise 5.2.1.28

Solve the formula $V=LWH$ for L.

? Exercise 5.2.1.29

Solve the formula $A=(1/2)bh$ for b.

? Exercise 5.2.1.30

Solve the formula $A=(1/2)d_1d_2$ for d_1.

? Exercise 5.2.1.31

Solve the formula $A=(1/2)h\left(b_1+b_2\right)$ for b_1.

? Exercise 5.2.1.32

Solve the formula $h=54t+(1/2)a$ for a.

? Exercise 5.2.1.33

Solve $180=a+b+c$ for a.

? Exercise 5.2.1.34

Solve the formula $A=(1/2)pl+B$ for p.

? Exercise $5.2.1.35$

Solve the formula: $P=2L+2W$ for L.

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5.3: Linear Inequalities in One Variable with Applications

Learning Objectives

After completing this section, you should be able to:

- Graph inequalities in one variable.
- Solve linear inequalities in one variable.
- Construct a linear inequality to solve applications.

In this section, we will study linear inequalities in one variable. Inequalities can be used when the possible values (answers) in a certain situation are numerous, not just a few, or when the exact value (answer) is not known but it is known to be within a range of possible values. There are many real-world scenarios that can be represented by linear inequalities. For example, consider the survey of the mayoral election in Figure $5.3.1$. Surveys and polls are usually conducted with only a small group of people. The margin of error indicates a range of how the actual group of voters would vote given the results of the survey. This range can be expressed using inequalities.

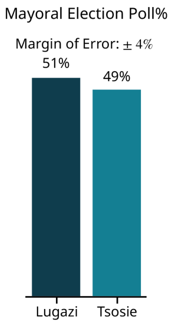


Figure $5.3.1$: These poll results, showing a margin of error at 4 percent, are an example of a real-world scenario that can be represented by linear inequalities.

Another example involves college tuition. Say a local community college charges $113 per credit hour. You budget $1,500 for tuition this fall semester. What are the number of credit hours that you could take this fall? Since this answer could be many different values, it can be expressed as an inequality.

Graphing Inequalities on the Number Line

In Algebraic Expressions, we introduced equality and the $=$ symbol. In this section, we look at inequality and the symbols $<$, $>$, $\leq$, and $\geq$. The table below summarizes the symbols and their meaning.

Symbol	Meaning
$<$	less than
$>$	greater than
$\leq$	less than or equal to
$\geq$	greater than or equal to

Suppose you had the inequality statement $x > 3$. What possible number or numbers would make the inequality $x > 3$ true? If you are thinking, "x could be 4," that's correct, but x could also be 5, 6, 37, 1 million, or even 3.001. The number of solutions is infinite; any number greater than 3 is a solution to the inequality $x > 3$.

Rather than trying to list all possible solutions, we show all the solutions to the inequality $x > 3$ on the number line. All the numbers to the right of 3 on the number line are shaded, to show that all numbers greater than 3 are solutions. At the number 3 itself, an open parenthesis is drawn, since the number 3 is not part of the solutions of $x > 3$.

We can also represent inequalities using interval notation. There is no upper end to the solution to this inequality. In interval notation, we express $x > 3$ as $(3, \infty)$. The symbol ∞ is read as "infinity." Infinity is not an actual number. Figure $5.3.21$ shows both the number line and the interval notation for $x > 3$.

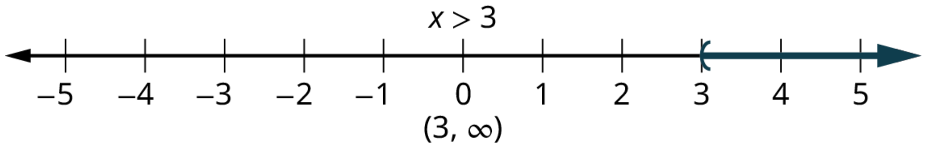


Figure 5.3.2: The inequality $x > 3$ is graphed on this number line and written in interval notation.

We used the left parenthesis symbol to show that the endpoint of the inequality is not included. Parentheses are used when the endpoints are not included as a possible answer to the inequality. The notation for inequalities on a number line and in interval notation use the same symbols to express the endpoints of intervals.

The inequality $x \leq 1$ means all numbers less than or equal to 1 . To illustrate that solution on a number line, we first put a bracket at $x = 1$; brackets are used when the endpoint is included. We then shade in all the numbers to the left of 1 , to show that all numbers less than one are solutions. There is no lower end to those numbers. We write $x \leq 1$ in interval notation as $(-\infty, 1]$. The symbol $-\infty$ is read as "negative infinity." Figure 5.3.3 shows both the number line and interval notation for $x = 1$.

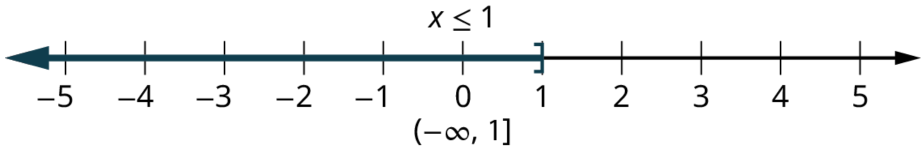


Figure 5.3.3: The inequality $x \leq 1$ is graphed on this number line and written in interval notation.

Figure 5.3.4 summarizes the general representations in both number line form and interval notation of solutions for $x > a$, $x < a$, $x \geq a$, and $x \leq a$.

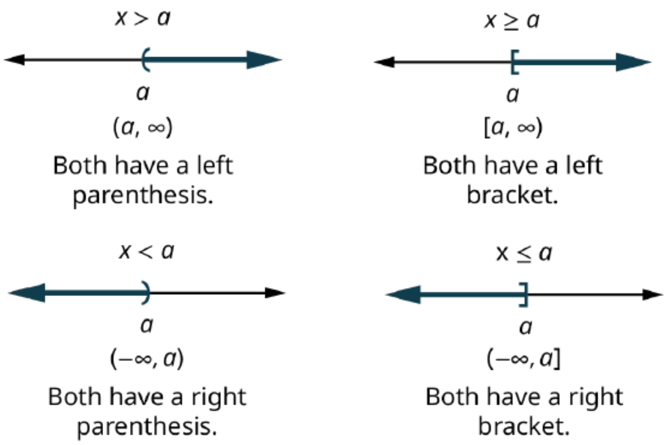

x > a
a
(a, ∞)
Both have a left parenthesis.
x ≥ a
a
[a, ∞)
Both have a left bracket.
x < a
a
(−∞, a)
Both have a right parenthesis.
x ≤ a
a
(−∞, a]
Both have a right bracket.

Figure 5.3.4: Summary of representations in number line form and interval notation.

✓ Example ✓ 5.3.1: Graphing an Inequality

Graph the inequality $x \geq -3$ and write the solution in interval notation.

Answer

Shade to the right of -3 (Figure ✓ 5.3.5)

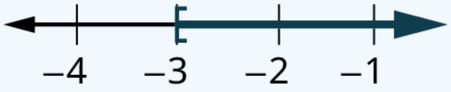


Figure ✓ 5.3.5 Write in interval notation starting at −3 with a bracket to show that −3 is included in the solution and then infinity because the solution includes all the numbers greater than or equal to −3:[−3,∞)

Your Turn 5.3.1

Graph the inequality $x < 2.5$ and write the solution in interval notation.

✓ Example 5.3.2: Graphing a Compound Inequality

Graph the inequality $x > -3$ and $x < 4$ and write the solution in interval notation.

Answer

Step 1: Graph $x > -3$ (Figure ✓ 5.3.6).

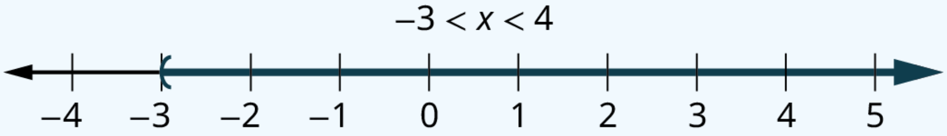


Figure ✓ 5.3.6

Step 2: Graph x <4 Figure ✓ 5.3.7.

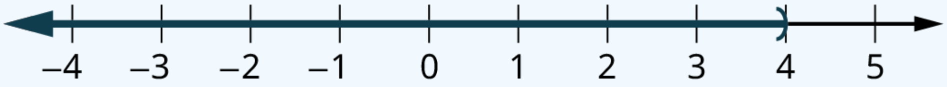


Figure ✓ 5.3.8

Step 3: Graph both on the same number line and think of where the solutions are to BOTH inequalities Figure 5.12. This will be where BOTH are shaded.

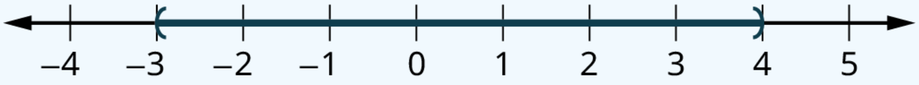


Figure ✓ 5.3.9

$(-3, 4)$

Your Turn 5.3.2

1. Graph the inequality $x \geq 0$ and $x \leq 2.5$ and write the solution in interval notation.

Who Knew?: Where Did the Inequality Symbols Come From?

The first use of the $<$ symbol to represent "less than" and $>$ to represent "greater than" appeared in a mathematics book written by Englishman Thomas Harriot that was published in 1631. However, Harriot did not invent the symbols…the editor of the book did! Harriot used triangular symbols to represent less than and greater than; the editor, for reasons unknown, changed to symbols that are similar to the ones we use today. The symbols used to represent less than or equal to, and greater than or equal to ($\leq$ and $\geq$) were first used in 1731 by French hydrologist and surveyor Pierre Bouguer. Interestingly, English mathematician John Wallis had used similar symbols as early as 1670, but he put the bar above the less than and greater than symbols instead of below them.

Solving Linear Inequalities

A **linear inequality** is much like a linear equation—but the equal sign is replaced with an inequality sign. A linear inequality is an inequality in one variable that can be written in one of the forms $ax + b < c$, $ax + b \leq c$, $ax + b \geq c$, or $ax + b > c$, where a, b, and c are all real numbers.

When we solved linear equations, we were able to use the properties of equality to add, subtract, multiply, or divide both sides and still keep the equality. Similar properties hold true for inequalities. We can add or subtract the same quantity from both sides of an inequality and still keep the inequality. For example, we know that 2 is less than 4 , i.e., $2<4$. If we add 6 to both sides of this inequality, we still have a true statement:

$$\begin{array}{c}2+6<4+6\\ 8<10\end{array}$$

The same would happen if we subtracted 6 from both sides of the inequality; the statement would stay true:

$$\begin{array}{c}2-6<4-6\\ -4<-2\end{array}$$

Notice that the inequality signs stayed the same. This leads us to the Addition and Subtraction Properties of Inequality.

FORMULA

For any numbers a, b, and c, if $a<b$, then $a+c<b+c$ and $a-c<b-c$.

For any numbers a, b, and c, if $a>b$, then $a+c>b+c$ and $a-c>b-c$.

We can add or subtract the same quantity from both sides of an inequality and still keep the inequality the same. But what happens to an inequality when we divide or multiply both sides by a number? Let's first multiply and divide both sides by a positive number, starting with an inequality we know is true, $10<15$. We will multiply and divide this inequality by 5 :

$$\begin{array}{ll}10<15 & 10<15\\ 10(5)\ ?\ 15(5) & \frac{10}{5} ? \frac{15}{5}\\ 50\ ?\ 75 & 2\ ?\ 3\\ 50<75\ (\text{true}) & 2<3\ (\text{true})\end{array}$$

The inequality signs stayed the same. Does the inequality stay the same when we divide or multiply by a negative number? Let's use our inequality $10<15$ to find out, multiplying it and dividing it by -5 :

$$\begin{array}{rll}10 & <15 & 10<15\\ 10(-5) & ?15(-5) & \frac{10}{-5} ? \frac{15}{-5}\\ -50 & ?-75 & -2\end{array}$$

Notice that when we filled in the inequality signs, the inequality signs reversed their direction in order to make it true! To summarize, when we divide or multiply an inequality by a positive number, the inequality sign stays the same. When we divide or multiply an inequality by a negative number, the inequality sign reverses. This gives us the Multiplication and Division Property of Inequality.

FORMULA

For any numbers a, b, and c,

multiply or divide by a positive:	if $a<b$ and $c>0$, then $ac<bc$ and $\frac{a}{c}<\frac{b}{c}$
	if $a>b$ and $c>0$, then $ac>bc$ and $\frac{a}{c}>\frac{b}{c}$
multiply or divide by a negative:	if $a<b$ and $c<0$, then $ac>bc$ and $\frac{a}{c}>\frac{b}{c}$
	if $a>b$ and $c<0$, then $ac<bc$ and $\frac{a}{c}<\frac{b}{c}$

To summarize, when we divide or multiply an inequality by:

- a positive number, the inequality sign stays the same.
- a negative number, the inequality sign reverses.

Checkpoint

Be careful to only reverse the inequality sign when you are multiplying and dividing by a negative. You do NOT reverse the inequality sign when you add or subtract a negative. For example, $2x<-4$ *is solved by dividing both sides of the inequality by 2 to get* $x<-2$. *You do NOT reverse the inequality sign because there is a negative 4. As another example,* $-2x+5<3x$ *is solved by adding* $-2x$ *to both sides to get* $5<5x$. *This does not reverse the inequality sign because we were not multiplying or dividing by a negative. We then divide both sides by 5 and get* $1<x$.

✓ Example 5.3.3: Solving a Linear Inequality Using One Operation

Solve $9y<54$, graph the solution on the number line, and write the solution in interval notation.

Answer

$$\begin{gathered} 9y<54 \\ \frac{9y}{9}<\frac{54}{9} \\ y<6 \end{gathered}$$

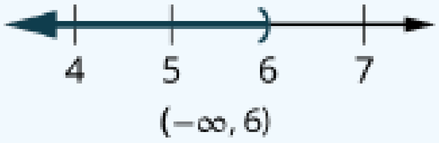


Figure ✓ 5.3.10

Your Turn 5.3.3

1. Solve $-13m\geq 65$, graph the solution on the number line, and write the solution in interval notation.

✓ Example 5.3.4: Solving a Linear Inequality Using Multiple Operations

Solve the inequality $6y\leq 11y+17$, graph the solution on the number line, and write the solution in interval notation.

$$\begin{gathered} 6y\leq 11y+17 \\ 6y-\mathbf{11y}\leq \mathbf{11y}+17-11y \\ -5y\leq 17 \\ \frac{-5y}{-5}\geq -\frac{17}{5} \\ y\geq -\frac{17}{5} \end{gathered}$$

Answer

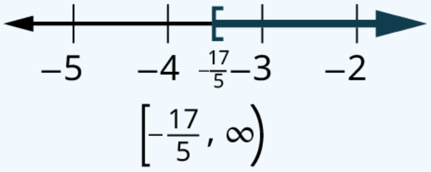


Figure ✓ 5.3.11

Your Turn $5.3.4$

Solve the inequality $8p+3(p-12)\geq 7p-28$, graph the solution on the number line, and write the solution in interval notation.

Solving Applications with Linear Inequalities

Many real-life situations require us to solve inequalities. The method we will use to solve applications with linear inequalities is very much like the one we used when we solved applications with equations. We will read the problem and make sure all the words are understood. Next, we will identify what we are looking for and assign a variable to represent it. We will restate the problem in one sentence to make it easy to translate into an inequality. Then, we will solve the inequality.

Sometimes an application requires the solution to be a whole number, but the algebraic solution to the inequality is not a whole number. In that case, we must round the algebraic solution to a whole number. The context of the application will determine whether we round up or down.

Example $5.3.5$: Constructing a Linear Inequality to Solve an Application with Tablet Computers

A teacher won a mini grant of \$4,000 to buy tablet computers for their classroom. The tablets they would like to buy cost \$254.12 each, including tax and delivery. What is the maximum number of tablets the teacher can buy?

Answer

Let $t=\text{the number of tablets.}$

t times \$254.12 has to be less than \$4,000, so $254.12t\leq 4,000$.

Solve for t:

$$\begin{aligned}\frac{254.12t}{254.12} &\leq \frac{4{,}000}{254.12}\\ t &\leq 15.74\end{aligned}$$

The teacher can buy 15 tablets and stay under \$4,000.

Your Turn $5.3.5$

Taleisha’s phone plan costs her \$28.80 per month plus \$0.20 per text message. How many text messages can she send/receive and keep her monthly phone bill no more than \$50?

Example $5.3.6$: Constructing a Linear Inequality to Solve a Tuition Application

The local community college charges \$113 per credit hour. Your budget is \$1,500 for tuition this fall semester. What number of credit hours could you take this fall?

Answer

Let $c=$ the number of credit hours you could take.

c times \$113 has to be less than \$1,500, so 113c≤1,500113c≤1,500.

Solve for c:

$$\begin{aligned}\frac{113c}{113} &\leq \frac{1500}{113}\\ c &\leq 13.27\end{aligned}$$

You can take up to 13 credits and stay under \$1,500.

Your Turn 5.3.6

You are awarded a $500 scholarship! In addition to the $1,500 you have saved for tuition, you now have an additional $500 to spend on credit hours for fall semester. Now, how many credit hours could you take this fall semester? Assume the cost is still $113 per credit hour.

✓ Example 5.3.7: Constructing a Linear Inequality to Solve an Application with Travel Costs

Brenda’s best friend is having a destination wedding and the event will last 3 days and 3 nights. Brenda has $500 in savings and can earn $15 an hour babysitting. She expects to pay $350 for airfare, $375 for food and entertainment, and $60 a night for her share of a hotel room. How many hours must she babysit to have enough money to pay for the trip?

Answer

Let $b =$ number of babysitting hours.

b times $15 plus $500 has to be more than $\$350 + \$375 + \$60/\text{night}$, so 15b+500≥350+375+60(3)15b+500≥350+375+60(3).

Solve for b:

$$\begin{aligned} 15b + 500 - \mathbf{500} &\geq 905 - \mathbf{500} \\ 15b &\geq 405 \\ \frac{15b}{15} &\geq \frac{405}{15} \\ b &\geq 27 \end{aligned}$$

Brenda must babysit at least 27 hours.

Your Turn 5.3.7

Malik is planning a 6-day summer vacation trip. He has $840 in savings, and he earns $45 per hour for tutoring. The trip will cost him $525 for airfare, $780 for food and sightseeing, and $95 per night for the hotel. How many hours must he tutor to have enough money to pay for the trip?

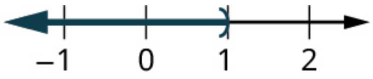
Tech Check

The Desmos activities called "Inequalities on a Number Line" and "Compound Inequalities on a Number Line" are ways for students to develop and deepen their understanding of inequalities. Teachers will need a Desmos account to assign the activity for student use. Once they have assigned the activity to their students, teachers need to share the code for the activity with their students. Students will input the code to work on the activity.

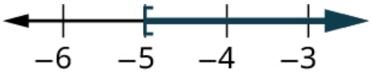
Check Your Understanding

For the following exercises, choose the correct interval notation for the graph.

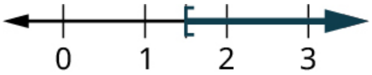
1.



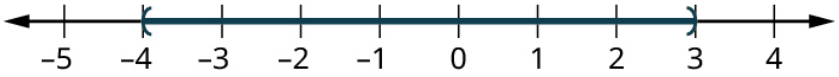
a. $[-1, \infty)$
b. $(-1, 1)$
c. $(\infty, 1)$
d. $(-\infty, 1)$
e. $(-\infty, -1)$



a. $(-5, \infty)$
b. $[-5, \infty)$
c. $[-5, \infty)$
d. $[-5, -3)$
e. $[-5, -3]$



a. $(1, \infty)$
b. $[1, \infty)$
c. $\left[\frac{3}{2}, \infty\right)$
d. $\left(\frac{3}{2}, \infty\right)$
e. $\left(\infty, \frac{3}{2}\right)$



2.
a. $(-4, 3)$
b. $(3, -4)$
c. $[-\infty, \infty)$
d. $[-4, 3]$
e. $[3, -4]$

3. $[4, \infty)$ is the solution for which inequality?
a. $4x \geq 0$
b. $4x \leq 0$
c. $6x < 24$
d. $6x > 24$
e. $6x \geq 24$

4. $(-\infty, -3)$ is the solution for which inequality?
a. $-6x < 18$
b. $-6x > 18$
c. $-6x \leq 18$
d. $-6x \geq 18$
e. $-6x \leq -18$

5. $(-2, \infty)$ is the solution for which inequality?
a. $-4x > -8$
b. $4x + 3 > -11$
c. $4x - 3 > -11$
d. $-4x \leq -8$
e. $-4x + 3 \leq 5$

6. $(-\infty, 9)$ is the solution for which inequality?
a. $9x < 0$
b. $-3x \geq 27$
c. $-3x + 14 > -13$
d. $-3x - 14 > -13$
e. $-3x \geq 27$

For the following exercises, choose the equation that best models the situation.

7. Renaldo is hauling boxes of lawn chairs. Each box is the same size, 8 cubic feet. Renaldo's truck has a capacity of 764 cubic feet. How many boxes of lawn chairs can Renaldo put in his truck?

a. $8 < 764x$
b. $8x < 764$
c. $8 > 764x$
d. $8x > 764$
e. None of these

8. Bernadette babysits the neighbor's kids, making on average $\$50$ a night. How many nights will she have to babysit in order to earn enough money to buy a used car, whose cost is $\$8,120$?

a. $50 < 8,120\text{x}$
b. $50x < 8,120$
c. $50 < 8,120\text{x}$
d. $50x > 8,120$
e. None of these

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5.3.1: Exercises

For the following exercises, graph the inequality on a number line and write the interval notation.

? Exercise $5.3.1.1$

$x > 3$

? Exercise $5.3.1.2$

$x \leq -0.5$

? Exercise $5.3.1.3$

$x \geq \frac{1}{3}$

? Exercise $5.3.1.4$

$x < -\frac{7}{3}$

? Exercise $5.3.1.5$

$-2 < x < 0$

? Exercise $5.3.1.6$

$-5 \leq x < -3$

? Exercise $5.3.1.7$

$0 \leq x \leq 3.5$

? Exercise $5.3.1.8$

$-4 < x < 2$

? Exercise $5.3.1.9$

$-5 < x \leq -2$

? Exercise $5.3.1.10$

$-3.75 \leq x \leq 0$

For the following exercises, solve the inequality, graph the solution on the number line, and write the solution in interval notation.

? Exercise $5.3.1.11$

$6y < 48$

? Exercise 5.3.1.12

$40 < \frac{5}{8}k$

? Exercise 5.3.1.13

$7s < -28$

? Exercise 5.3.1.14

$\frac{9}{4}g \leq 36$

? Exercise 5.3.1.15

$-8v \leq 96$

? Exercise 5.3.1.16

$\frac{b}{-10} \geq 30$

? Exercise 5.3.1.17

$-7d > 105$

? Exercise 5.3.1.18

$-18 < \frac{q}{-6}$

? Exercise 5.3.1.19

$5u \leq 8u - 21$

? Exercise 5.3.1.20

$9p > 14p - 18$

? Exercise 5.3.1.21

$9y + 5(y + 3) < 4y - 35$

? Exercise 5.3.1.22

$4k - (k - 2) \geq 7k - 26$

For the following exercises, construct a linear inequality to solve the application.

? Exercise 5.3.1.23

The elevator in Yehire's apartment building has a sign that says the maximum weight is 2,100 pounds. If the average weight of one person is 150 pounds, how many people can safely ride the elevator?

? Exercise 5.3.1.24

Arleen got a $\$20$ gift card for the coffee shop. Her favorite iced drink costs $\$3.79$. What is the maximum number of drinks she can buy with the gift card?

? Exercise 5.3.1.25

Ryan charges his neighbors $\$17.50$ to wash their car. How many cars must he wash next summer if his goal is to earn at least $\$1,500$?

? Exercise 5.3.1.26

Kimuyen needs to earn $\$4,150$ per month in order to pay all her expenses. Her sales job pays her $\$3,475$ per month plus 4 percent of her total sales. What is the minimum Kimuyen's total sales must be in order for her to pay all her expenses?

? Exercise 5.3.1.27

Nataly is considering two job offers. The first job would pay her $\$83,000$ per year. The second would pay her $\$66,500$ plus 15 percent of her total sales. What would her total sales need to be for her salary on the second offer to be higher than the first?

? Exercise 5.3.1.28

Kiyoshi's phone plan costs $\$17.50$ per month plus $\$0.15$ per text message. What is the maximum number of text messages Kiyoshi can use so the phone bill is no more than $\$56.60$?

? Exercise 5.3.1.29

Kellen wants to rent a banquet room in a restaurant for her cousin's baby shower. The restaurant charges $\$350$ for the banquet room plus $\$32.50$ per person for lunch. How many people can Kellen have at the shower if she wants the maximum cost to be $\$1,500$?

? Exercise 5.3.1.30

Noe installs and configures software on home computers. He charges $\$125$ per job. His monthly expenses are $\$1,600$. How many jobs must he work in order to make a profit of at least $\$2,400$?

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5.4: Ratios and Proportions

Learning Objectives

1. Construct ratios to express comparison of two quantities.
2. Use and apply proportional relationships to solve problems.
3. Determine and apply a constant of proportionality.
4. Use proportions to solve scaling problems.

Ratios and proportions are used in a wide variety of situations to make comparisons. For example, using the information from Figure 5.15, we can see that the number of Facebook users compared to the number of Twitter users is 2,006 M to 328 M. Note that the "M" stands for million, so 2,006 million is actually 2,006,000,000 and 328 million is 328,000,000. Similarly, the number of Qzone users compared to the number of Pinterest users is in a ratio of 632 million to 175 million. These types of comparisons are ratios.

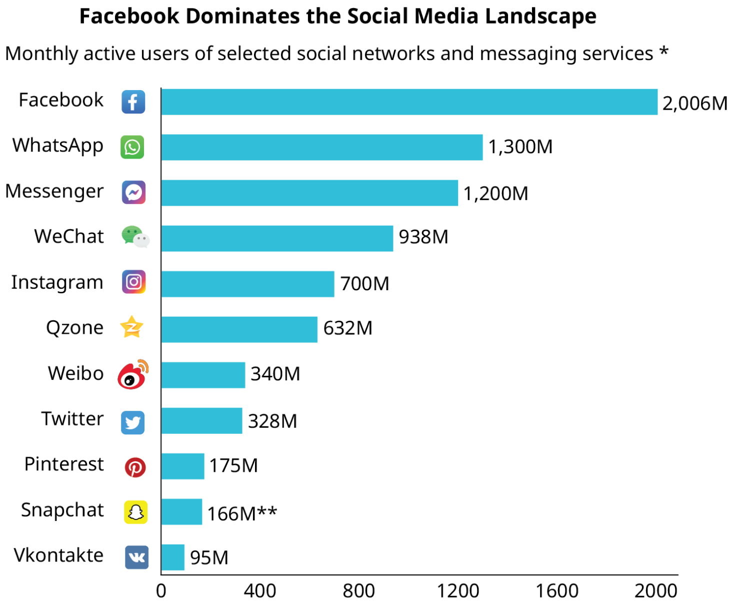


Figure $5.4.1$: This bar graph shows popular social media app usage. (Source)

Constructing Ratios to Express Comparison of Two Quantities

Note there are three different ways to write a ratio, which is a comparison of two numbers that can be written as: a to b OR $a : b$ OR the fraction a/b. Which method you use often depends upon the situation. For the most part, we will want to write our ratios using the fraction notation. Note that, while all ratios are fractions, not all fractions are ratios. Ratios make part to part, part to whole, and whole to part comparisons. Fractions make part to whole comparisons only.

Example $5.4.1$

The Euro (€) is the most common currency used in Europe. Twenty-two nations, including Italy, France, Germany, Spain, Portugal, and the Netherlands use it. On June 9, 2021, 1 U.S. dollar was worth 0.82 Euros. Write this comparison as a ratio.

Solution

written as either 1 to 0.82 ; or $1 : 0.82$; or $\frac{1}{0.82}$.

Your Turn $5.4.1$

On June 9, 2021, 1 U.S. dollar was worth 1.21 Canadian dollars. Write this comparison as a ratio.

✓ Example ✓ 5.4.2

The gravitational pull on various planetary bodies in our solar system varies. Because weight is the force of gravity acting upon a mass, the weights of objects is different on various planetary bodies than they are on Earth. For example, a person who weighs 200 pounds on Earth would weigh only 33 pounds on the moon! Write this comparison as a ratio.

Solution

Using the definition of ratio, let $a = 200$ pounds on Earth and let $b = 33$ pounds on the moon. Then the ratio can be written as either 200 to 33 ; or 200:33; or $\frac{200}{33}$.

Your Turn 5.4.2

A person who weighs 170 pounds on Earth would weigh 64 pounds on Mars. Write this comparison as a ratio.

Using and Applying Proportional Relationships to Solve Problems

Using proportions to solve problems is a very useful method. It is usually used when you know three parts of the proportion, and one part is unknown. Proportions are often solved by setting up like ratios. If $\frac{a}{b}$ and $\frac{c}{d}$ are two ratios such that $\frac{a}{b} = \frac{c}{d}$, then the fractions are said to be proportional. Also, two fractions $\frac{a}{b}$ and $\frac{c}{d}$ are proportional $(\frac{a}{b} = \frac{c}{d})$ if and only if $a \times d = b \times c$.

✓ Example 5.4.3

You are going to take a trip to France. You have \$520 U.S. dollars that you wish to convert to Euros. You know that 1 U.S. dollar is worth 0.82 Euros. How much money in Euros can you get in exchange for \$520?

Solution

Step 1: Set up the two ratios into a proportion; let x be the variable that represents the unknown. Notice that U.S. dollar amounts are in both numerators and Euro amounts are in both denominators.

$$\frac{1}{0.82} = \frac{520}{x}$$

Step 2: Cross multiply, since the ratios $\frac{a}{b}$ and $\frac{c}{d}$ are proportional, then $a \times d = b \times c$.

$$\begin{aligned} 520(0.82) &= 1(x) \\ 426.4 &= x \end{aligned}$$

You should receive 426.4 Euros (426.4€).

Your Turn 5.4.3

After your trip to France, you have 180 Euros remaining. You wish to convert them back into U.S. dollars. Assuming the exchange rate is the same $(\$1 = 0.82\text{E})$, how many dollars should you receive? Round to the nearest cent if necessary.

✓ Example 5.4.4

A person who weighs 170 pounds on Earth would weigh 64 pounds on Mars. How much would a typical racehorse (1,000 pounds) weigh on Mars? Round your answer to the nearest tenth.

Solution

Step 1: Set up the two ratios into a proportion. Notice the Earth weights are both in the numerator and the Mars weights are both in the denominator.

$$\frac{170}{64} = \frac{1,000}{x}$$

Step 2: Cross multiply, and then divide to solve.

$$\begin{aligned} 170x &= 1,000(64) \\ 170x &= 64,000 \end{aligned}$$

$$\frac{170x}{170} = \frac{64,000}{170}$$
$$x = 376.5$$

So the 1,000-pound horse would weigh about 376.5 pounds on Mars.

Your Turn 5.4.4

A person who weighs 200 pounds on Earth would weigh only 33 pounds on the moon. A 2021 Toyota Prius weighs 3,040 pounds on Earth; how much would it weigh on the moon? Round to the nearest tenth if necessary.

✓ Example 5.4.5

A cookie recipe needs $2\frac{1}{4}$ cups of flour to make 60 cookies. Jackie is baking cookies for a large fundraiser; she is told she needs to bake 1,020 cookies! How many cups of flour will she need?

Solution

Step 1: Set up the two ratios into a proportion. Notice that the cups of flour are both in the numerator and the amounts of cookies are both in the denominator. To make the calculations more efficient, the cups of flour $(2\frac{1}{4})$ is converted to a decimal number (2.25).

$$\frac{2.25}{60} = \frac{x}{1020}$$

Step 2: Cross multiply, and then simplify to solve.

$$\begin{aligned} 2.25(1,020) &= 60x \\ 2,295 &= 60x \\ 38.25 &= x \end{aligned}$$

Jackie will need 38.25 , or $38\frac{1}{4}$, cups of flour to bake 1,020 cookies.

Your Turn 5.4.5

oing to bake cookies, using the same recipe as above. You find out that you have 27 cups of flour in your pantry. Assuming you have all the other ingredients necessary, how many cookies can you make with 27 cups of flour?

Checkpoint

Part of the definition of proportion states that two fractions $\frac{a}{b}$ and $\frac{c}{d}$ are proportional if $a \times d = b \times c$. This is the "cross multiplication" rule that students often use (and unfortunately, often use incorrectly). The only time cross multiplication can be used is if you have two ratios (and only two ratios) set up in a proportion. For example, you cannot use cross multiplication to solve for x in an equation such as $\frac{2}{5} = \frac{x}{8} + 3x$ because you do not have just the two ratios. Of course, you could use the rules of algebra to change it to be just two ratios and then you could use cross multiplication, but in its present form, cross multiplication cannot be used.

People in Mathematics

Eudoxus was born around 408 BCE in Cnidus (now known as Knidos) in modern-day Turkey. As a young man, he traveled to Italy to study under Archytas, one of the followers of Pythagoras. He also traveled to Athens to hear lectures by Plato and to Egypt to study astronomy. He eventually founded a school and had many students.

Eudoxus made many contributions to the field of mathematics. In mathematics, he is probably best known for his work with the idea of proportions. He created a definition of proportions that allowed for the comparison of any numbers, even irrational ones. His definition concerning the equality of ratios was similar to the idea of cross multiplying that is used today. From his work on proportions, he devised what could be described as a method of integration, roughly 2000 years before calculus (which includes integration) would be fully developed by Isaac Newton and Gottfried Leibniz. Through this technique, Eudoxus became the first person to rigorously prove various theorems involving the volumes of certain objects. He also developed a planetary theory, made a sundial still usable today, and wrote a seven volume book on geography called *Tour of the Earth*, in which he wrote about all the civilizations on the Earth, and their political systems, that were known at the time. While this book has been lost to history, over 100 references to it by different ancient writers attest to its usefulness and popularity.

Determining and Applying a Constant of Proportionality

In the last example, we were given that $2\frac{1}{4}$ cups of flour could make 60 cookies; we then calculated that $38\frac{1}{4}$ cups of flour would make 1,020 cookies, and 720 cookies could be made from 27 cups of flour. Each of those three ratios is written as a fraction below (with the fractions converted to decimals). What happens if you divide the numerator by the denominator in each?

$$\frac{2.25}{60}=0.0375 \quad \frac{38.25}{1,020}=0.0375 \quad \frac{27}{720}=0.0375$$

The quotients in each are exactly the same! This number, determined from the ratio of cups of flour to cookies, is called the constant of proportionality. If the values a and b are related by the equality $\frac{a}{b}=k$, then k is the constant of proportionality between a and b. Note since $\frac{a}{b}=k$, then $b=\frac{a}{k}$. and $b=\frac{a}{k}$.

One piece of information that we can derive from the constant of proportionality is a unit rate. In our example (cups of flour divided by cookies), the constant of proportionality is telling us that it takes 0.0375 cups of flour to make one cookie. What if we had performed the calculation the other way (cookies divided by cups of flour)?

$$\frac{60}{2.25}=26.66666\ldots \quad \frac{1,020}{38.25}=26.66666\ldots \quad \frac{720}{27}=26.66666\ldots$$

In this case, the constant of proportionality $\left(26.66666\ldots=26\frac{2}{3}\right)$ is telling us that $26\frac{2}{3}$ cookies can be made with one cup of flour. Notice in both cases, the "one" unit is associated with the denominator. The constant of proportionality is also useful in calculations if you only know one part of the ratio and wish to find the other.

? Exercise 5.4.6

Finding a Constant of Proportionality

Isabelle has a part-time job. She kept track of her pay and the number of hours she worked on four different days, and recorded it in the table below. What is the constant of proportionality, or pay divided by hours? What does the constant of proportionality tell you in this situation?

Pay	\$87.50	\$50.00	\$37.50	\$100.00
Hours	7	4	3	8

Answer

To find the constant of proportionality, divide the pay by hours using the information from any of the four columns. For example, $\frac{87.5}{7}=12.5$. The constant of proportionality is 12.5 , or \$12.50. This tells you Isabelle's hourly pay: For every hour she works, she gets paid \$12.50.

Your Turn 5.4.6

The following table contains the lengths of four objects in both inches and centimeters. What is the constant of proportionality (centimeters divided by inches)? What does the constant of proportionality tell you in this situation?

Object	floor tile	book	table	pencil
Length (in.)	24	13	60	7.5
Length (cm)	60.96	33.02	152.4	19.05

Exercise 5.4.7

Zac runs at a constant speed: 4 miles per hour (mph). One day, Zac left his house at exactly noon (12:00 PM) to begin running; when he returned, his clock said 4:30 PM. How many miles did he run?

Answer

The constant of proportionality in this problem is 4 miles per hour (or 4 miles in 1 hour). Since $\frac{a}{b} = k$, where k is the constant of proportionality, we have

$$\frac{a \text{ miles}}{b \text{ hours}} = k$$

$\frac{a}{4.5} = 4$ (30 minutes is $1/2$, or 0.5 , hours)
$a = 4(4.5)$, since from the definition we know $a = kb$

$$a = 18$$

Zac ran 18 miles.

Your Turn 5.4.7

One week, Zac ran a total of 122 miles. How much time did he spend running in that week?

Exercise 5.4.8

Joe had a job where every time he filled a bucket with dirt, he was paid $2.50. One day Joe was paid $337.50. How many buckets did he fill that day?

Answer

The constant of proportionality in this situation is $\$2.50$ per bucket (or $\$2.50$ for 1 bucket). Since $\frac{a}{b} = k$, where k is the constant of proportionality, we have

$$\frac{a \text{ dollars}}{b \text{ buckets}} = k$$
$$\frac{337.50}{b} = 2.50$$

Since we are solving for b, and we know from the definition that

$$b = \frac{a}{k}$$
$$b = \frac{337.50}{2.50}$$
$$b = 135$$

Joe filled 135 buckets.

Your Turn 5.4.8

Suppose one day Joe filled 83 buckets; how much money would he make on that day?

Exercise 5.4.9

While driving in Canada, Mabel quickly noticed the distances on the road signs were in kilometers, not miles. She knew the constant of proportionality for converting kilometers to miles was about 0.62—that is, there are about 0.62 miles in 1 kilometer. If the last road sign she saw stated that Montreal is 104 kilometers away, about how many more miles does Mabel have to drive? Round your answer to the nearest tenth.

Answer

The constant of proportionality in this situation is 0.62 miles per 1 kilometer. Since $\frac{a}{b} = k$, where k is the constant of proportionality, we have

$$\begin{aligned} \frac{a \text{ miles}}{b \text{ kilometers}} &= k \\ \frac{a}{104} &= 0.62 \\ a &= 0.62(104) \\ a &= 64.48 \end{aligned}$$

Rounding the answer to the nearest tenth, Mabel has to drive 64.5 miles.

Your Turn 5.4.9

Later in her trip, Mabel decides to drive to the capital of Canada, Ottawa. As she left Montreal, she saw a road sign that read that Ottawa is 203 kilometers away. About how many miles is that? Round your answer to the nearest tenth.

Using Proportions to Solve Scaling Problems

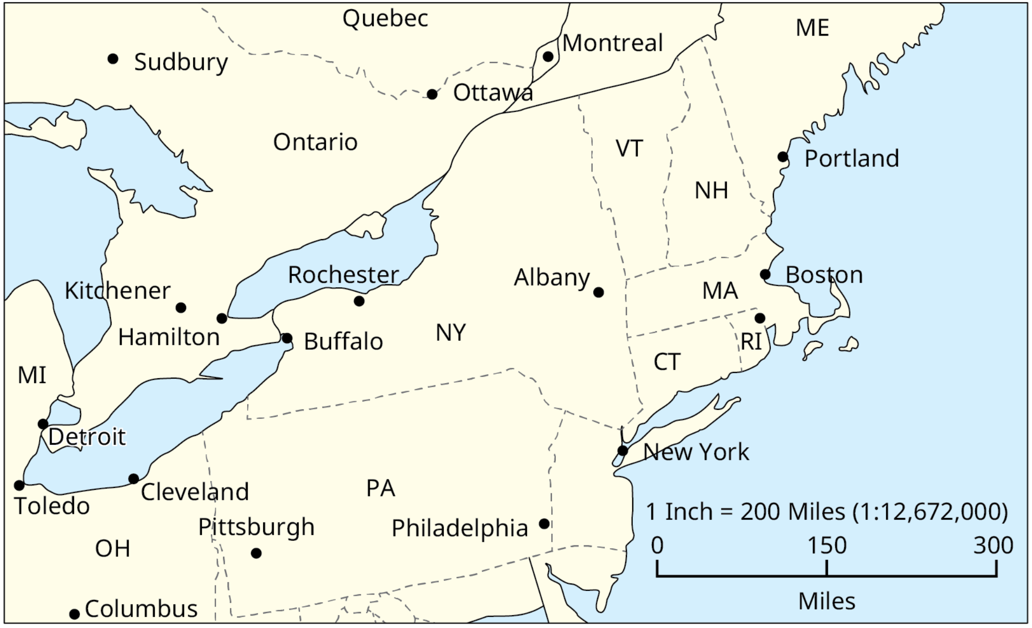

Quebec
Sudbury
Montreal
ME
Ottawa
Ontario
VT
Portland
NH
Rochester
Albany
Boston
Kitchener
MA
Hamilton
Buffalo
NY
RI
MI
CT
Detroit
New York
Cleveland
PA
Toledo
1 Inch = 200 Miles (1:12,672,000)
Pittsburgh
Philadelphia
0
150
300
OH
Miles
Columbus

Figure Exercise 5.4.20 A map of the northeastern United States

Ratio and proportions are used to solve problems involving **scale**. One common place you see a scale is on a map (as represented in Figure 5.16). In this image, 1 inch is equal to 200 miles. This is the scale. This means that 1 inch on the map corresponds to 200 miles on the surface of Earth. Another place where scales are used is with models: model cars, trucks, airplanes, trains, and so on. A common ratio given for model cars is 1:24—that means that 1 inch in length on the model car is equal to 24 inches (2 feet) on an actual automobile. Although these are two common places that scale is used, it is used in a variety of other ways as well.

Exercise 5.4.10

Figure 5.17 is an outline map of the state of Colorado and its counties. If the distance of the southern border is 380 miles, determine the scale (i.e., 1 inch = how many miles). Then use that scale to determine the approximate lengths of the other borders of the state of Colorado.

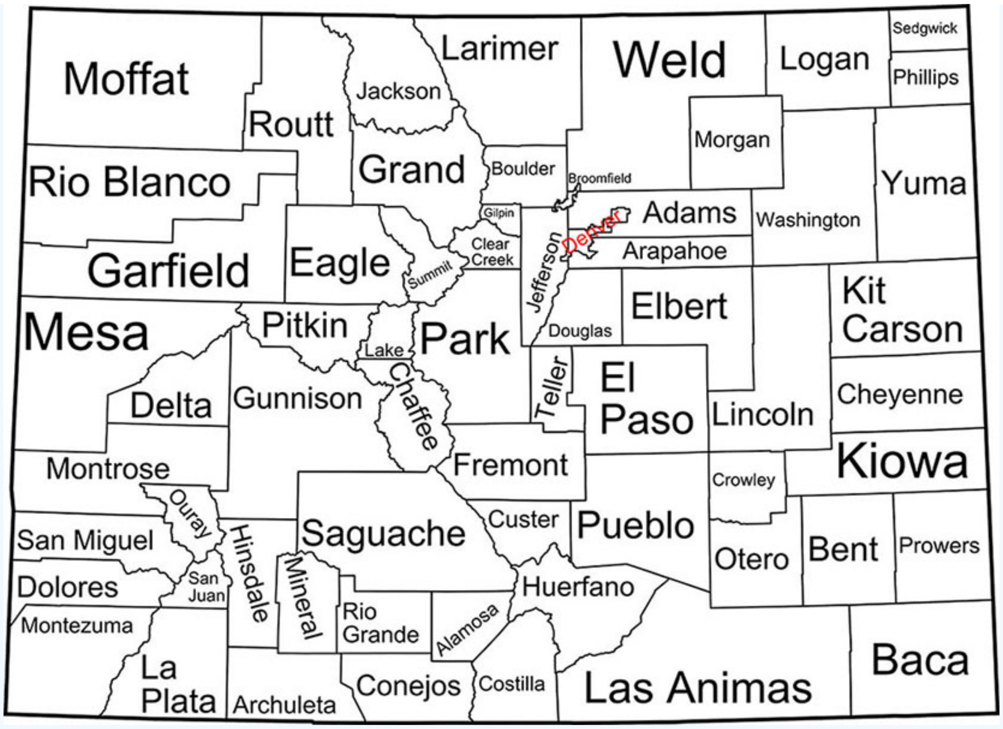

Moffat
Routt
Jackson
Larimer
Weld
Logan
Sedgwick
Phillips
Rio Blanco
Grand
Boulder
Broomfield
Morgan
Yuma
Gilpin
Adams
Washington
Denver
Clear Creek
Arapahoe
Garfield
Eagle
Summit
Jefferson
Elbert
Kit Carson
Mesa
Pitkin
Lake
Park
Douglas
Delta
Gunnison
Chaffee
Teller
El Paso
Lincoln
Cheyenne
Montrose
Fremont
Crowley
Kiowa
Ouray
San Miguel
Hinsdale
Saguache
Custer
Pueblo
Otero
Bent
Prowers
Dolores
San Juan
Mineral
Huerfano
Montezuma
Rio Grande
Alamosa
La Plata
Archuleta
Conejos
Costilla
Las Animas
Baca

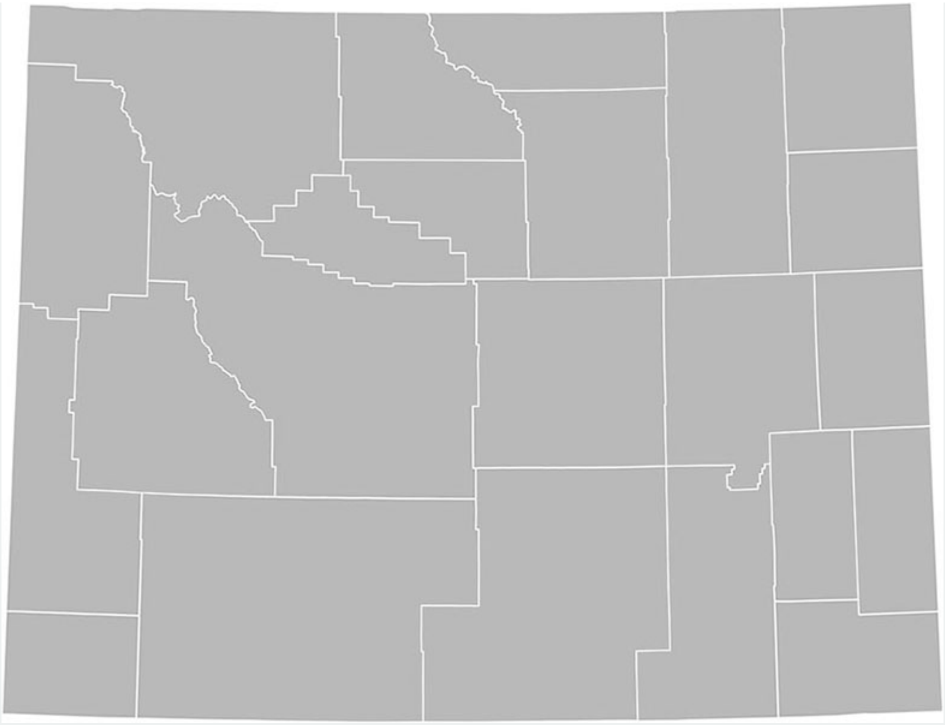
Figure ❓ Exercise $5.4.3$ Outline Map of Colorado (credit: "Map of Colorado Counties" by David Benbennick/Wikimedia Commons, Public Domain)

Answer

When the southern border is measured with a ruler, the length is 4 inches. Since the length of the border in real life is 380 miles, our scale is 1 inch =95 miles.

The eastern and western borders both measure 3 inches, so their lengths are about 285 miles. The northern border measures the same as the southern border, so it has a length of 380 miles.

Your Turn Exercise $5.4.10$



Outline Map of Wyoming (credit: "Blank map of Wyoming showing counties" by David Benbennick/Wikimedia Commons, Public Domain)

Consider the outline map of the state of Wyoming and its counties. If the distance of the southern border is 365 miles, determine the scale (i.e., 1 inch=1 inch= how many miles). Then use that scale to determine the approximate lengths of the other borders of the state of Wyoming.

Example Exercise 5.4.11

Solving a Scaling Problem Involving Model Cars

Die-cast NASCAR model cars are said to be built on a scale of 1:24 when compared to the actual car. If a model car is 9 inches long, how long is a real NASCAR automobile? Write your answer in feet.

Answer

The scale tells us that 1 inch of the model car is equal to 24 inches (2 feet) on the real automobile. So set up the two ratios into a proportion. Notice that the model lengths are both in the numerator and the NASCAR automobile lengths are both in the denominator.

$$\begin{aligned} \frac{1}{24} &= \frac{9}{x} \\ 24(9) &= x \\ 216 &= x \end{aligned}$$

This amount (216) is in inches. To convert to feet, divide by 12, because there are 12 inches in a foot (this conversion from inches to feet is really another proportion!). The final answer is:

$$\frac{216}{12} = 18$$

The NASCAR automobile is 18 feet long.

Your Turn Exercise 5.4.11

A toy Jeep is built on a 1:16 scale. The website for the toy Jeep says the toy is 11.5 inches long. Based on this, how long is the real Jeep?

Check Your Understanding

1. If $a:b=c:d$, then $b:a=d:c$ for all non-zero whole numbers a, b, c, and d.
a. True
b. False
2. If the ratio of wolves to rabbits in a national park is $3:5$, then the ratio of rabbits to (wolves and rabbits) is 5:8.
a. True
b. False
3. All fractions are ratios but not all ratios are fractions.
a. True
b. False
4. In the following equation, $\frac{x}{46}=\frac{20}{90}+4$, cross multiplication can be used as the first step towards solving for x.
a. True
b. False
5. All fractions are ratios but not all ratios are fractions.
a. True
b. False
6. There are 16 math majors and 12 non-math majors in Ms. Kraft's class. What is not a correct way to express the ratio of math majors to non-math majors?

/16:12	12:16	4:3

7. There are 16 math majors and 12 non-math majors in Ms. Kraft’s class. What shows the ratio of math majors to all the students in Ms. Kraft’s class?

16:12	12:16	16:28	28:12	None of these

8. One U.S. dollar is worth 0.72 British pounds. Damon is traveling to Great Britain and wishes to exchange $450 U.S. dollars for British pounds. How many British pounds should Damon get in return?

625	6,250	3,456	345.6	None of these

9. The HO scale for model trains is the most common size of model trains. This scale is 1:87. If a real locomotive is 73 feet long, how long should the model locomotive be (in inches)? Round your answer to the nearest inch.

10. Albert’s Honda Civic gets 37 miles per gallon of gasoline. The gas tank on the Civic can hold 13.5 gallons of gas. Albert is driving from Tucson, Arizona to Los Angeles, California, a distance of 485 miles. Albert thinks he can make it on one full tank of gasoline. Can he? Explain.

11. The average price of a gallon of regular gasoline in the California on July 1, 2021 was $4.28 per gallon. Albert stops at a gas station in California and puts 9.5 gallons of gasoline into his Civic. How much did he pay for the gas?

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5.4.1: Exercises

For the following exercises, use this scenario: Kelly opened a bag of colored chocolate coated candies and counted the number of each color of candy. She found she had 9 green, 4 yellow, 13 black, 11 orange, 8 blue, and 7 red. What is the ratio of the following candy colors?

? Exercise $5.4.1.1$

Red candies to green candies

? Exercise $5.4.1.2$

Green candies to black candies

? Exercise $5.4.1.3$

Yellow candies to black candies

? Exercise $5.4.1.4$

Black candies to blue candies

? Exercise $5.4.1.5$

Orange candies to non-orange candies

? Exercise $5.4.1.6$

Yellow candies to non-yellow candies

? Exercise $5.4.1.7$

Red candies to all candies

? Exercise $5.4.1.8$

Pink candies to all candies

? Exercise $5.4.1.9$

Candies with the letter ‘r’ in their name to all candies

? Exercise $5.4.1.10$

Candies with the letter ‘r’ in their name to candies without the letter ‘r’ in their name

For the following exercises, solve each proportion for the unknown variable.

? Exercise $5.4.1.11$

$\frac{x}{46} = \frac{4}{8}$

? Exercise 5.4.1.12

$\frac{27}{x} = \frac{3.5}{14}$

? Exercise 5.4.1.13

$\frac{16}{7} = \frac{x}{14}$

? Exercise 5.4.1.14

$\frac{1}{0.8} = \frac{x}{514}$

? Exercise 5.4.1.15

$\frac{203}{g} = \frac{10}{22}$

? Exercise 5.4.1.16

$\frac{1}{29.5} = \frac{16}{m}$

? Exercise 5.4.1.17

$\frac{14}{95} = \frac{a}{19}$

? Exercise 5.4.1.18

$\frac{13}{140} = \frac{s}{2961}$

? Exercise 5.4.1.19

$\frac{1}{k} = \frac{69}{111}$ (Round answer to the nearest hundredth.)

? Exercise 5.4.1.20

$\frac{p}{1} = \frac{22}{7}$ (Round answer to the nearest hundredth.)

? Exercise 5.4.1.21

Pet Paradise has 20 cats and 16 dogs. Animal Acres has 15 cats. How many dogs must be at Animal Acres so that Pet Paradise and Animal Acres have the same ratio of cats to dogs?

? Exercise 5.4.1.22

Pet Paradise has 20 cats and 16 dogs. Critter Corral has 28 dogs. How many cats must be at Critter Corral so that Pet Paradise and Critter Corral have the same ratio of cats to dogs?

? Exercise 5.4.1.23

A high school has 960 students. The ratio of students to high school teachers is $16:1$. How many high school teachers are at the school?

Exercise 5.4.1.24

A high school has 960 students. The ratio of students to high school teachers is $16 : 1$. How many more teachers are needed to have a $12 : 1$ ratio at the high school of students to teachers?

Exercise 5.4.1.25

One U.S. dollar is worth $1.23 Canadian dollars. Bernice is traveling to Canada and wants to convert $\$550$ U.S. to Canadian money. How much in Canadian money should she receive?

Exercise 5.4.1.26

One U.S. dollar is worth $1.23 Canadian dollars. Rene is traveling from Canada to the United States and wants to convert $\$550$ of Canadian money to U.S. money. How much in U.S. money should he receive? Round your answer to the nearest cent.

Exercise 5.4.1.27

One U.S. dollar is worth $1.23 Canadian dollars. What is one Canadian dollar worth in U.S. funds? Round your answer to the nearest cent.

Exercise 5.4.1.28

A salad recipe needs one cup of crushed almonds. It will serve eight people. Rashida needs to make a salad to serve 20 people. How many cups of crushed almonds does she need?

Exercise 5.4.1.29

A salad recipe needs one cup of crushed almonds. It will serve eight people. Elmer has 4.75 cups of crushed almonds. If he uses all of the crushed almonds he has to make this salad, how many people will it serve?

Exercise 5.4.1.30

Jorge is 6 feet tall and casts a 7-foot shadow. At the same time, a nearby tree has a shadow of 56 feet. How tall is the tree?

Exercise 5.4.1.31

Tony can run 4 kilometers in 30 minutes. At that rate, how far could he run in 1 hour, 45 minutes?

Exercise 5.4.1.32

Kara's parent owns a restaurant. When she came in one day, they asked her to figure out how much they were spending per ounce on steak they were buying from a vendor. They had their last four receipts, but unfortunately they spilled liquid on them and some parts were unreadable. Find out how much Kara's parent is spending per ounce on steak; then use that information to fill in the unreadable parts of the receipts (labeled a, b, and c below).

Receipt	1	2	3	4
Ounces	128	460	b	541
Cost	$163.84	a	$277.76	c

? Exercise $5.4.1.33$

The scale for a map reads " 1 inch $=250$ miles." You measure the distance on the map from Fargo, North Dakota to Winnipeg, Manitoba and get 1.44 inches. How far is it from Fargo to Winnipeg?

? Exercise $5.4.1.34$

Hot Wheels toy cars are said to be built on a scale of $1:64$ when compared to the actual car. If a real car is 18 feet long, how long should the Hot Wheels toy car be (in inches)?

? Exercise $5.4.1.35$

The Eiffel Tower in Paris, France, is 1,067 feet tall. The replica Eiffel Tower in Las Vegas, Nevada, is built on the scale of 1.976:1. How tall is the replica Eiffel Tower in Las Vegas? Round your answer to the nearest foot.

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5.5: Graphing Linear Equations and Inequalities

Learning Objectives

1. Graph linear equations and inequalities in two variables.
2. Solve applications of linear equations and inequalities.

In this section, we will learn how to graph linear equations and inequalities. There are several real-world scenarios that can be represented by graphs of linear inequalities. Think of filling your car up with gasoline. If gasoline is $3.99 per gallon and you put 10 gallons in your car, you will pay $39.90. Your friend buys 15 gallons of gasoline and pays $59.85. You can plot these points on a coordinate system and connect the points with a line to create the graph of a line. You'll learn to do both in this section.

UNLEADED
MID-GRADE
PREMIUM
3.99 9
4.09 9
4.19 9

Figure Exercise $5.5.1$: How much would it cost to fill up your gas tank? (credit: "Gas Under 4 Bucks" by Mark Turnauckas, Flickr/CC BY 2.0)

Plotting Points on a Rectangular Coordinate System

Just like maps use a grid system to identify locations, a grid system is used in algebra to show a relationship between two variables in a rectangular coordinate system. The rectangular coordinate system is also called the xy-plane or the “coordinate plane.”

The rectangular coordinate system is formed by two intersecting number lines, one horizontal and one vertical. The horizontal number line is called the semanticsx axis.

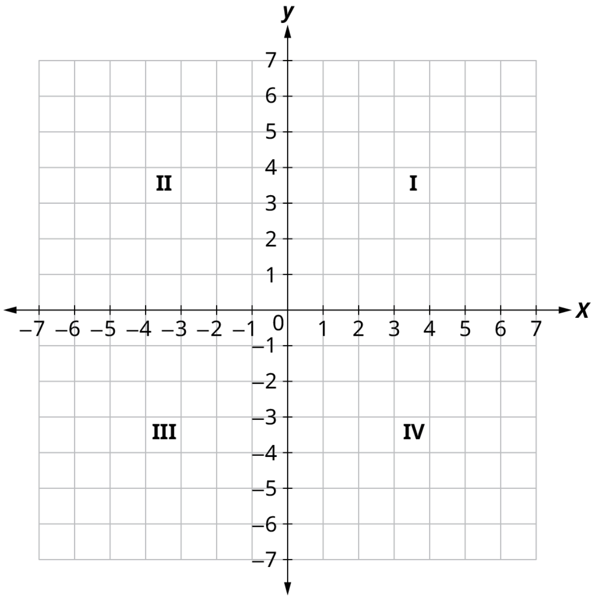


Figure $5.5.1$: Quadrants on the Coordinate Plane

In the rectangular coordinate system, every point is represented by an **ordered pair** ($5.5.2$). The first number in the ordered pair is the x-coordinate of the point, and the second number is the y-coordinate of the point. The phrase "ordered pair" means that the order is important. At the point where the axes cross and where both coordinates are zero, the ordered pair is $(0, 0)$. The point $(0, 0)$ has a special name. It is called the **origin**.

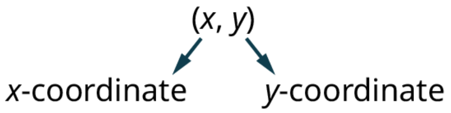


Figure $5.5.2$: Ordered Pair

We use the coordinates to locate a point on the semanticsxyFigure $5.5.3$.

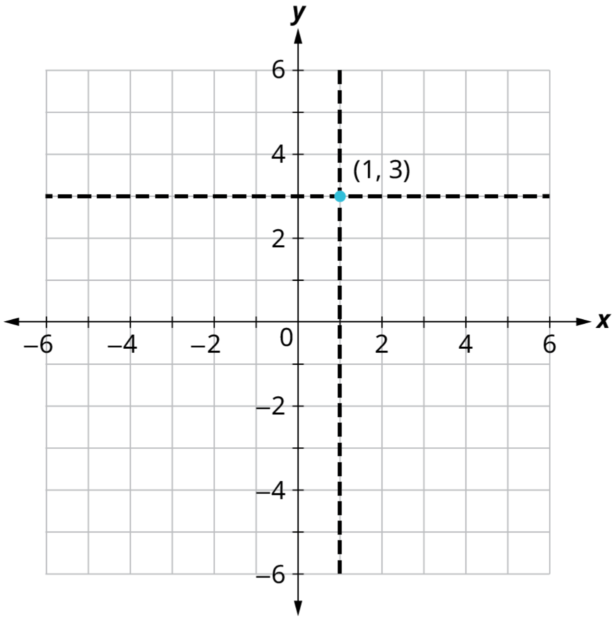


Figure $5.5.3$: Point $(1, 3)$ Plotted on the Coordinate Plane

Notice that the vertical line through $x =$1Figure $5.5.4$, the point $(0, 4)$ is on the y-axis and the point (−2, 0) is on the x-axis.

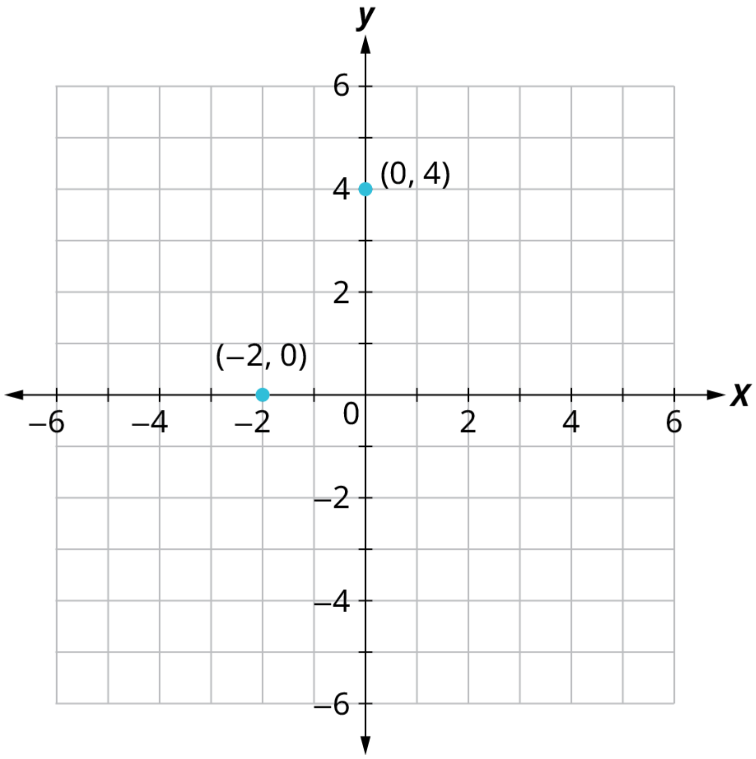


Figure $5.5.4$: Points $(-2, 0)$ and $(0, 4)$ Plotted on the Coordinate Plane

✓ Example 5.5.1: Plotting Points on a Coordinate System

Plot the following points in the rectangular coordinate system and identify the quadrant in which the point is located:

1. $(-5, 4)$
2. $(-3, -4)$
3. $(2, -3)$
4. $(0, -1)$
5. $(3, \frac{5}{2})$

Answer

The first number of the coordinate pair is the semanticsxFigure 5.24). Their intersection is the point.

1. Since $x = -5$, the point is to the left of the y-axis. Also, since $y = 4$, the point is above the x-axis. The point $(-5, 4)$ is in quadrant II.
2. Since $x = -3$, the point is to the left of the y-axis. Also, since $y = -4$, the point is below the x-axis. The point $(-3, -4)$ is in quadrant III.
3. Since $x = 2$, the point is to the right of the y-axis. Since $y = -3$, the point is below the x-axis. The point $(2, -3)$ is in quadrant IV.
4. Since $x = 0$, the point whose coordinates are $(0, -1)$ is on the y-axis.
5. Since $x = 3$, the point is to the right of the y-axis. Since $y =$52y=52, which is equal to 2.5, the point is above the x-axis. The point $(3, \frac{5}{2})$ is in quadrant I.

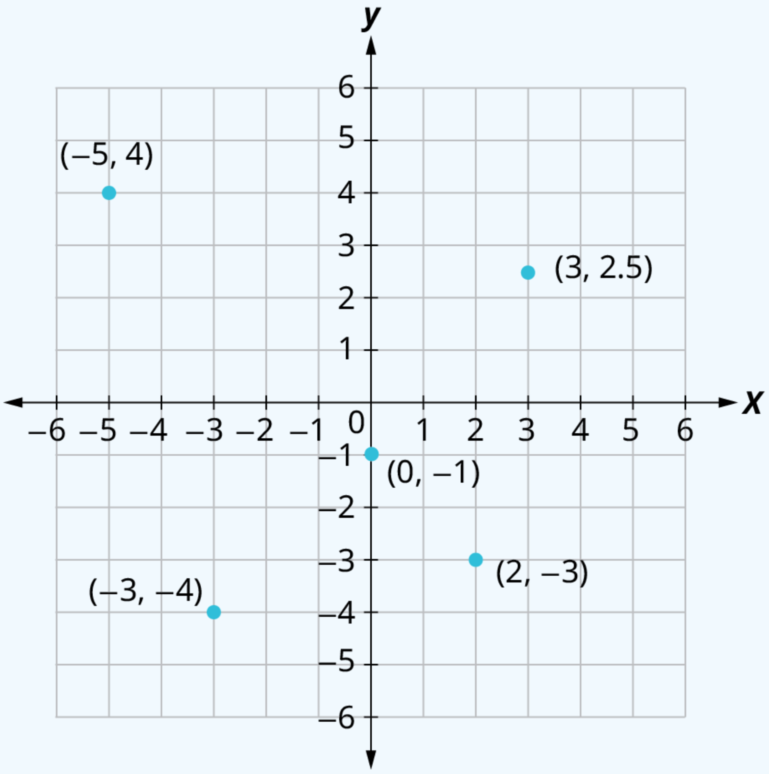


Figure ✓ 5.5.5

Your Turn 5.5.1

Plot the following points in the rectangular coordinate system and identify the quadrant in which the point is located:

a. $(-4, 2)$
b. $(-1, -2)$
c. $(3, -5)$

d. $(-3,0)$
e. $(\frac{5}{3},2)$

Graphing Linear Equations in Two Variables

Up to now, all the equations you have solved were equations with just one variable. In almost every case, when you solved the equation, you got exactly one solution. But equations can have more than one variable. Equations with two variables may be of the form $Ax+By=C$. An equation of this form, where A and B are both not zero, is called a **linear equation in two variables**. Here is an example of a linear equation in two variables, x and y.

$$\begin{array}{l} \boldsymbol{A}x+\boldsymbol{B}y=C \\ \boldsymbol{A}+4y=\boldsymbol{8} \\ \boldsymbol{A}=\boldsymbol{1},\boldsymbol{B}=4,\boldsymbol{C}=\boldsymbol{8} \end{array}$$

The equation $y=-3x+5$ is also a linear equation. But it does not appear to be in the form $Ax+By=C$. We can use the addition property of equality and rewrite it in $Ax+By=C$ form.

Step 1: Add $3x$ to both sides. $y+3x=-3x+5+3x$

Step 2: Simplify. $y+3x=5$

Step 3: Put it in $Ax+By=C$ form. $3x+y=5$

By rewriting $y=-3x+5$ as $3x+y=5$, we can easily see that it is a linear equation in two variables because it is of the form $Ax+By=C$. When an equation is in the form $Ax+By=C$, we say it is in **standard form of a linear equation**. Most people prefer to have A, B, and C be integers and $A\geq 0$ when writing a linear equation in standard form, although it is not strictly necessary.

Linear equations have infinitely many solutions. For every number that is substituted for x there is a corresponding y value. This pair of values is a **solution** to the linear equation and is represented by the ordered pair (x,y). When we substitute these values of x and y into the equation, the result is a true statement, because the value on the left side is equal to the value on the right side.

We can plot these solutions in the rectangular coordinate system. The points will line up perfectly in a straight line. We connect the points with a straight line to get the graph of the linear equation. We put arrows on the ends of each side of the line to indicate that the line continues in both directions.

A graph is a visual representation of all the solutions of a linear equation. The line shows you all the solutions to that linear equation. Every point on the line is a solution of that linear equation. And every solution of the linear equation is on this line. This line is called the graph of the equation. Points not on the line are not solutions! The graph of a linear equation $Ax+By=C$ is a straight line.

- Every point on the line is a solution of the equation.
- Every solution of this equation is a point on this line.

✓ Example 5.5.2: Determining Points on a Line

Figure ✓ 5.5.6 is the graph of $y=2x-3$.

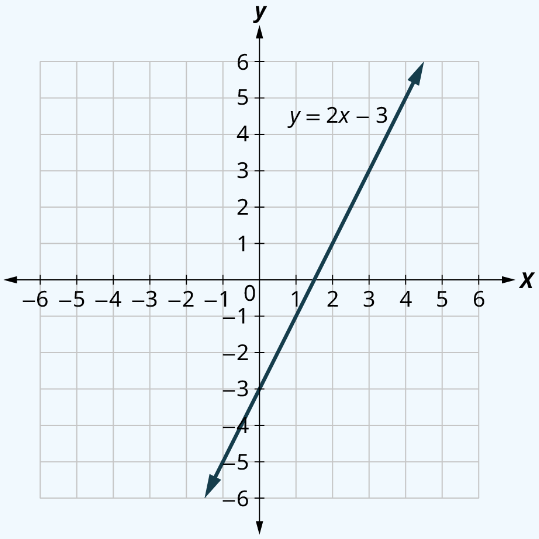


Figure ✓ 5.5.6 Graph of $y = 2x – 3$

For each ordered pair, decide:

I. Is the ordered pair a solution to the equation?
II. Is the point on the line?

A: $(0, –3)$ B: $(3, 3)$ C: $(2, –3)$ D: $(–1, –5)$

Answer

Substitute the x- and y-values into the equation to check if the ordered pair is a solution to the equation.

$\text{A} :(0,-3)$ $y=2x-3$ $-3 \overset{?}{=} 2(0)-3$ $-3=-3\checkmark$ $(0,-3)$ is a solution.	$\text{B} : (3,3)$ $y=2x-3$ $3 \overset{?}{=} 2(3)-3$ $3=3\checkmark$ $(3,3)$ is a solution.	$\text{C} : (2,-3)$ $y=2x-3$ $-3 \overset{?}{=} 2(2)-3$ $-3 \neq 1$ $(2,-3)$ is not a solution.	$\text{D} : (-1,-5)$ $y=2x-3$ $-5 \overset{?}{=} 2(-1)-3$ $-5=-5\checkmark$ $(-1,-5)$ is a solution.

I. Plot the points $(0, −3)$, $(3, 3)$, $(2, −3)$, and $(−1, −5)$.

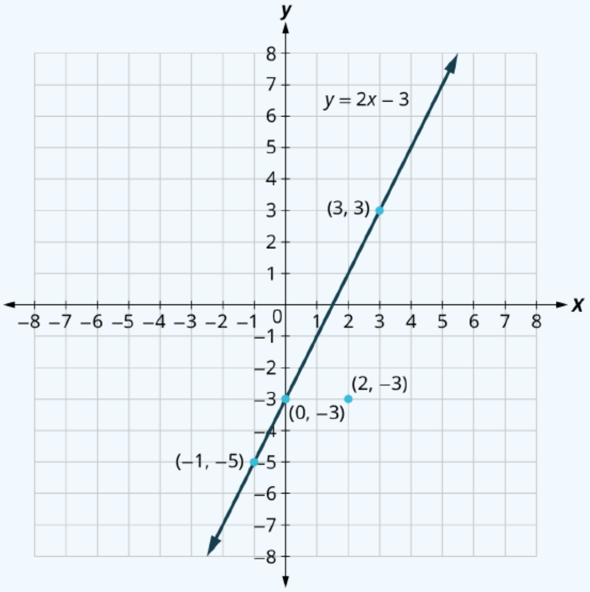


Figure ✓ 5.5.7

In Figure ✓ 5.5.7, the points (0 , 3) (0 , 3) , (3 , - 3) (3 , - 3) , and (− 1 , − 5) (− 1 , − 5) are on the line y = 2 x − 3 y = 2 x − 3 , and the point (2 , − 3) (2 , − 3) is not on the line. The points that are solutions to y = 2 x − 3 y = 2 x − 3 are on the line, but the point that is not a solution is not on the line.

📌 Your Turn 5.5.2

The given figure is the graph of $y = x + 2$.

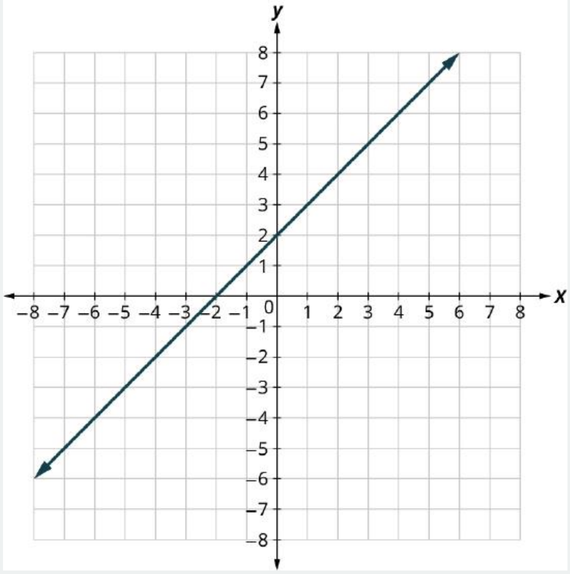


Figure 📌 5.5.8: Graph of $y = x + 2$

For each ordered pair below, decide:

I. Is the ordered pair a solution to the equation?
II. Is the point on the line?

$(0, 2)$

$(1, 2)$

$(-1,1)$

$(-3,-1)$

The steps to take when graphing a linear equation by plotting points are:

- **Step 1:** Find three points whose coordinates are solutions to the equation. Organize them in a table.
- **Step 2:** Plot the points in a rectangular coordinate system. Check that the points line up. If they do not, carefully check your work.
- **Step 3:** Draw the line through the three points. Extend the line to fill the grid and put arrows on both ends of the line.

It is true that it only takes two points to determine a line, but it is a good habit to use three points. If you only plot two points and one of them is incorrect, you can still draw a line, but it will not represent the solutions to the equation. It will be the wrong line. If you use three points, and one is incorrect, the points will not line up. This tells you something is wrong, and you need to check your work.

✓ Example 5.5.3: Graphing a Line by Plotting Points

Graph the equation: $y=12x+3$.

Answer

Find three points that are solutions to the equation. Since this equation has the fraction $\frac{1}{2}$ as a coefficient of x, we will choose values of x carefully. We will use zero as one choice and multiples of 2 for the other choices. Why are multiples of two a good choice for values of x? By choosing multiples of 2, the multiplication by $\frac{1}{2}$ simplifies to a whole number.

$$\begin{array}{lll}
x=0 & x=2 & x=4 \\
y=\frac{1}{2}x+3 & y=\frac{1}{2}x+3 & y=\frac{1}{2}x+3 \\
y=\frac{1}{2}(0)+3 & y=\frac{1}{2}(2)+3 & y=\frac{1}{2}(4)+3 \\
y=0+3 & y=1+3 & y=2+3 \\
y=3 & y=4 & y=5
\end{array}$$

x	y	$(x,\ y)$
0	3	$(0,3)$
2	4	$(2,4)$
4	5	$(4,5)$

Plot the points, check that they line up, and draw the line (Figure 5.28).

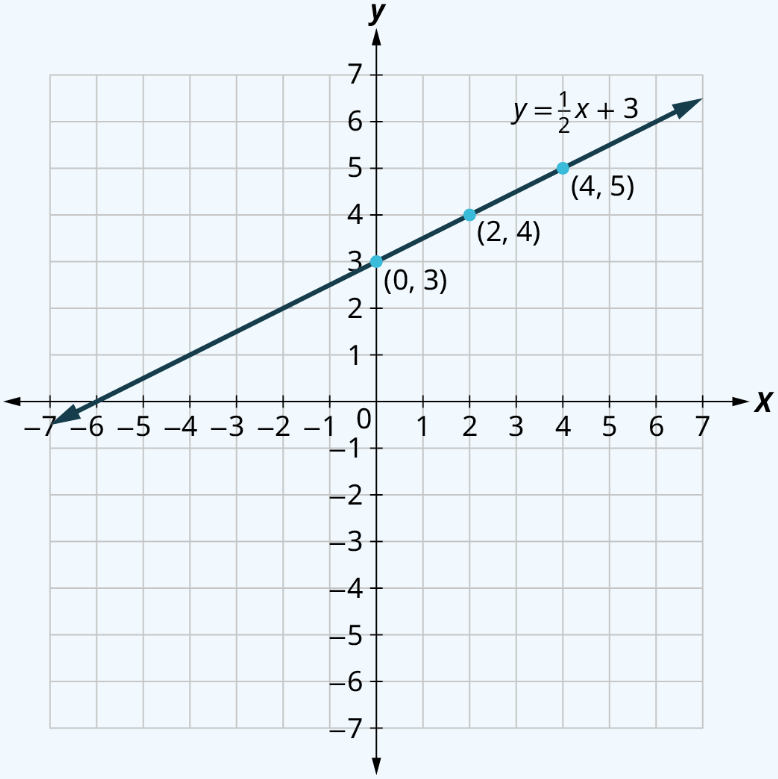


Figure ✓ 5.5.9

Your Turn 5.5.3

Graph the equation $y = 3x - 1$ by plotting points.

Solving Applications Using Linear Equations in Two Variables

Many fields use linear equalities to model a problem. While our examples may be about simple situations, they give us an opportunity to build our skills and to get a feel for how they might be used.

✓ Example 5.5.4: Pumping Gas

Gasoline costs $3.53 per gallon. You put 10 gallons of gasoline in your car, and pay $35.30. Your friend puts 15 gallons of gasoline in their car and pays $52.95. Your neighbor needs 5 gallons of gasoline, how much will they pay?

Answer

Let $x = \text{the number of gallons of}$gasolineFigure 5.29).

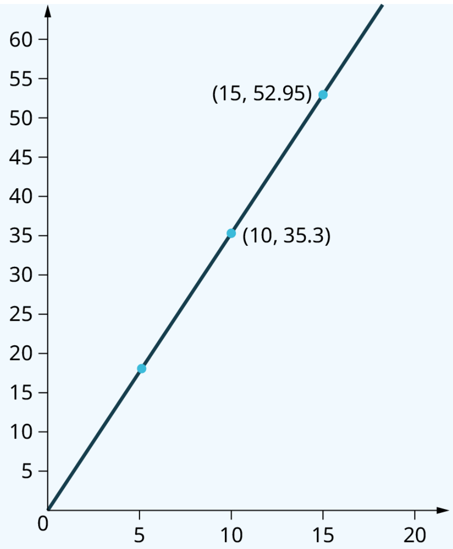


Figure ✓ 5.5.10

We can see the point at $x = 5$. The y -value is found by multiplying 5 by \$3.53 to get \$17.65. Your neighbor will pay \$17.65.

Your Turn 5.5.4

If a stamp costs \$0.55 and you buy a book of 20 stamps, then you pay \$11. If you want to mail 100 letters, you can buy a roll of stamps for \$55. Your friend only needs 3 stamps, how much will they pay?

People in Mathematics: René Descartes

René Descartes was born in 1596 in La Haye, France. He was sickly as a child, so much so that he was allowed to stay in bed until 11:00 AM rather than get up at 5:00 AM like the other school children. He kept this habit of rising late for most of the rest of his life.

Figure 5.5.11: René Descartes (credit: Flickr, Public Domain)

After his primary schooling, Descartes attended the University of Poitiers, receiving a law degree in 1616. He then embarked on a myriad of journeys, joining two different militaries (one in the Netherlands, the other in Bavaria) and generally travelling around Europe until 1628, when he settled in the Netherlands. It was here that he began to delve deeply into his ideas of science, mathematics, and philosophy.

> In 1637, at the urging of his friends, Descartes published *Discourse on the Method for Conducting One's Reason Well and Seeking the Truth in the Sciences*. The book had three appendices: *La Dioptrique*, a work on optics; *Les Météores*, which pertained to meteorology; and *La Géométrie*, a work on mathematics. It was in this appendix that he proposed a geometric way of representing many different algebraic expressions and equations. It is this system of representation that almost all mathematical textbooks use today.
>
> These publications (along with several others) brought much fame to Descartes. So renowned was his reputation that late in 1649, Queen Christina of Sweden asked Descartes to come to Sweden to tutor her. However, she wished to do her studies at 5:00 in the morning; Descartes had to break his lifelong habit of sleeping in late. A few months later, in February 1650, Descartes died of pneumonia.

Graphing Linear Inequalities

Previously we learned to solve inequalities with only one variable. We will now learn about inequalities containing two variables that can be written in one of the following forms: $Ax + By \geq C, Ax + By > C, Ax + By \leq C$, and $Ax + By < C$ where A and B are not both zero. We will look at **linear inequalities in two variables**, which are very similar to linear equations in two variables.

Like linear equations, linear inequalities in two variables have many solutions. Any ordered pair (x, y) that makes an inequality true when we substitute in the values is a solution to a linear inequality.

Like linear equations, linear inequalities in two variables have many solutions. Any ordered pair (x, y) that makes an inequality true when we substitute in the values is a **solution to a linear inequality**.

✓ Example $5.5.5$: Determining Solutions to an Inequality

Determine whether each ordered pair is a solution to the inequality $y > x + 4$:

1. $(0, 0)$
2. $(1, 6)$
3. $(2, 6)$
4. $(-5, -15)$
5. $(-8, 12)$

Answer

1	$(0, 0)$ Substitute 0 for x and 0 for y Substitute 0 for x and 0 for y	$y > x + 4$ $0 \overset{?}{>} 0 + 4$ $0 \not 4$ $(0, 0)$ is not a solution to $y > x + 4$.	
2	$(1, 6)$ Substitute 1 for x and 6 for y Simplify.	$y > x + 4$ $6 \overset{?}{>}_{>} 1 + 4$ $6 > 5$ $(1, 6)$ is a solution to $y > x + 4$.	
3	$(2, 6)$ Substitute 2 for x and 6 for y Simplify.	$y > x + 4$ $6 \overset{?}{>} 2 + 4$ $6 \not 6$ $(2, 6)$ is not a solution to $y > x + 4$.	
4	$(-5, -15)$ Substitute -5 for x and -15 for \(y) Simplify.	$y > x + 4$ $-15 \overset{?}{>} -5 + 4$ $-15 \not> -1$ So, $(-5, -15)$ is not a solution to $y > x + 4$.	

5	$(-8,-12)$ Substitute -8 for x and 12 for y Simplify.	$y > x+4$ $12 \overset{?}{>} -8+4$ $12 > -4$ So, $(-8,12)$ is a solution to $y > x+4$.	

Your Turn 5.5.5

Determine whether each ordered pair is a solution to the inequality $y > x-1$:

1. $(0,1)$
2. $(-4,-1)$
3. $(4,2)$
4. $(3,0)$
5. $(-2,-3)$

Let us think about $x > 3$. The point $x = 3$ separated that number line into two parts. On one side of 3 are all the numbers less than 3 . On the other side of 3 all the numbers are greater than 3 . See Figure $5.5.12$.

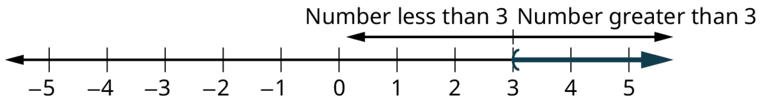


Figure $5.5.12$: Solution to $x > 3$ on a Number Line

Similarly, the line $y = x+4$ separates the plane into two regions. On one side of the line are points with $y < x+4$. On the other side of the line are the points with $y > x+4$. We call the line $y = x+4$ a **boundary line**.

For an inequality in one variable, the endpoint is shown with a parenthesis (Figure $5.5.13$) or a bracket (Figure $5.5.14$) depending on whether or not a is included in the solution:

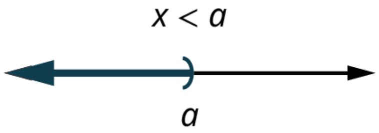


Figure $5.5.13$: Endpoint with Parenthesis

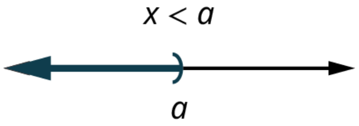


Figure $5.5.14$: Endpoint with Bracket

Similarly, for an inequality in two variables, the boundary line is shown with a solid or dashed line to show whether or not it the line is included in the solution.

$Ax + By$ <C	$Ax + By$ ≤C
$Ax + By$ >C	$Ax + By$ ≥C
Boundary line is $Ax + By$ =C	Boundary line is $Ax + By$ =C
Boundary line is not included in solution.	Boundary line is included in solution.
Boundary line is dashed.	Boundary line is solid.

Now, let us take a look at what we found in Example 5.43. We will start by graphing the line $y = x+4$, and then we will plot the five points we tested, as graphed in Figure 5.32. We found that some of the points were solutions to the inequality $y > x+4$ and some were not. Which of the points we plotted are solutions to the inequality $y > x + 4$? The points $(1, 6)$ and $(-8, 12)$ are solutions to the inequality $y > x + 4$. Notice that they are both on the same side of the boundary line $y = x + 4$. The two points $(0, 0)$ and $(-5, -15)$ are on the other side of the boundary line $y = x + 4$, and they are not solutions to the inequality $y > x + 4$. For those two points, $y < x + 4$. What about the point $(2, 6)$? Because $6 = 2 + 4$, the point is a solution to the equation $y = x + 4$, but not a solution to the inequality $y > x + 4$. So, the point $(2, 6)$ is on the boundary line.

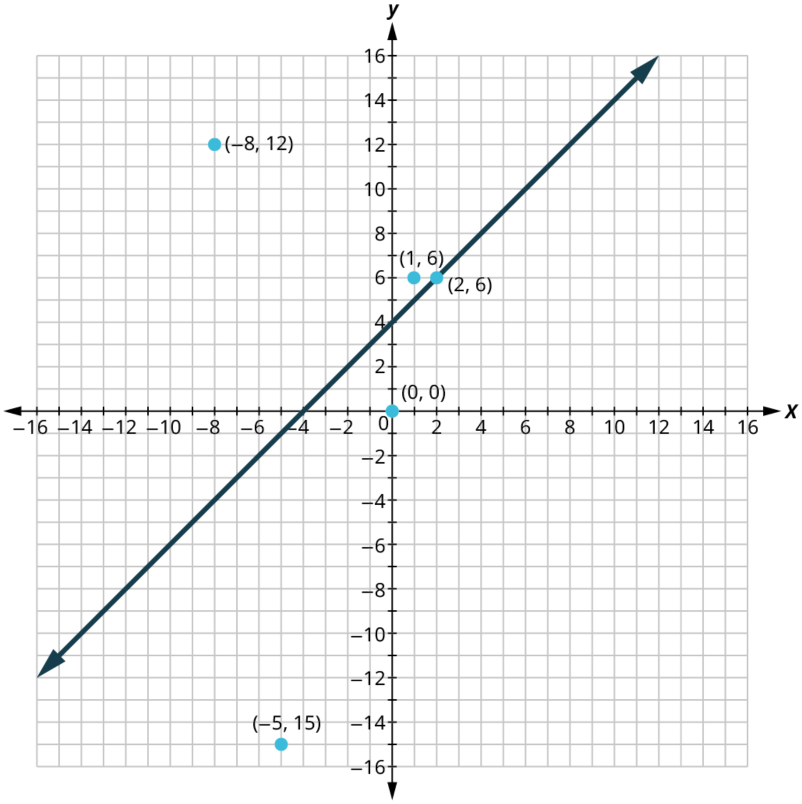


Figure $5.5.15$: Graph of $y = x + 4$

Let us take another point above the boundary line and test whether or not it is a solution to the inequality $y > x + 4$. The point $(0, 10)$ clearly looks to be above the boundary line, doesn't it? Is it a solution to the inequality?

$$\begin{gathered} y > x + 4 \\ 10 \overset{?}{>} 0 + 4 \\ 10 > 4 \checkmark \end{gathered}$$

Yes, $(0, 10)$ is a solution to $y > x + 4$. Any point you choose above the boundary line is a solution to the inequality $y > x + 4$. All points above the boundary line are solutions. Similarly, all points below the boundary line, the side with $(0, 0)$ and $(-5, -15)$, are not solutions to $y > x + 4$, as shown in Figure $5.5.16$.

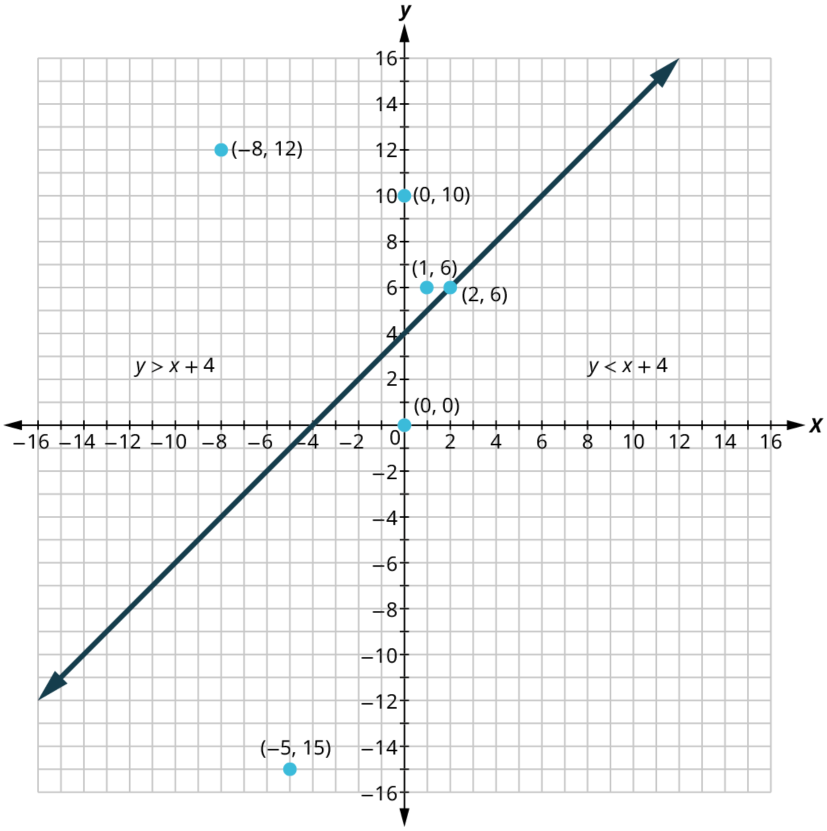


Figure 5.5.16: Graph of $y=x+4$, with $y>x+4$ Above the Boundary Line and $y<x+4$ Below the Boundary Line

The graph of the inequality $y>x$+4 (Figure 5.5.17). The line $y=x+4$ divides the plane into two regions. The shaded side shows the solutions to the inequality $y>x+4$. The points on the boundary line, those where $y=x+4$, are not solutions to the inequality $y>x+4$, so the line itself is not part of the solution. We show that by making the boundary line dashed, not solid.

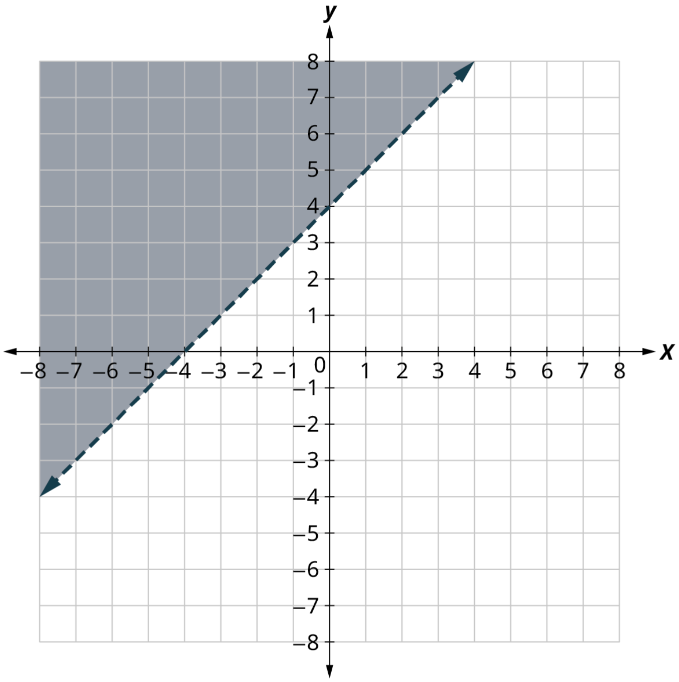


Figure 5.5.17: Graph of $y>x+4$

✓ Example 5.5.6: Writing a Linear Inequality Shown by a Graph

The boundary line shown in this graph is $y = 2x - 1$ Figure ✓ 5.5.18.

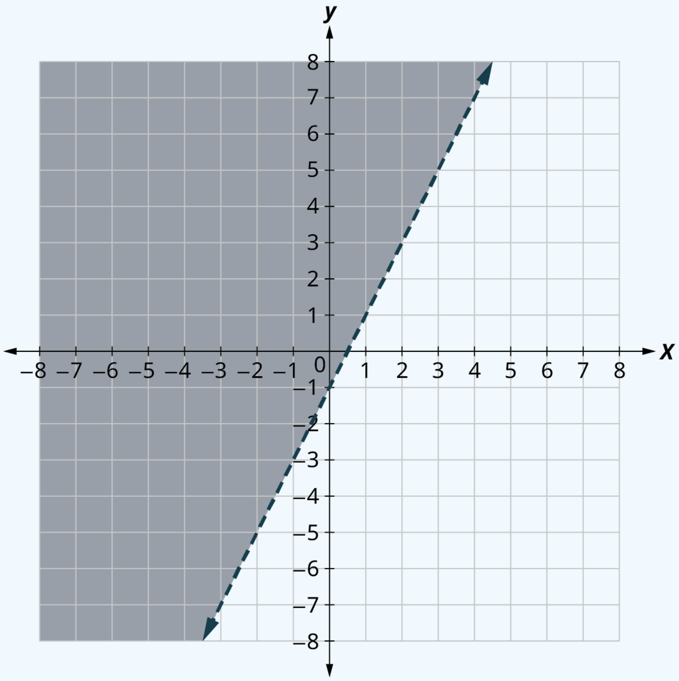


Figure ✓ 5.5.18

Answer

The line $y = 2x - 1$ is the boundary line. On one side of the line are the points with $y > 2x - 1$ and on the other side of the line are the points with $y < 2x - 1$. Let us test the point $(0, 0)$ and see which inequality describes its position relative to the boundary line. At $(0, 0)$, which inequality is true: $y > 2x - 1$ or $y < 2x - 1$?

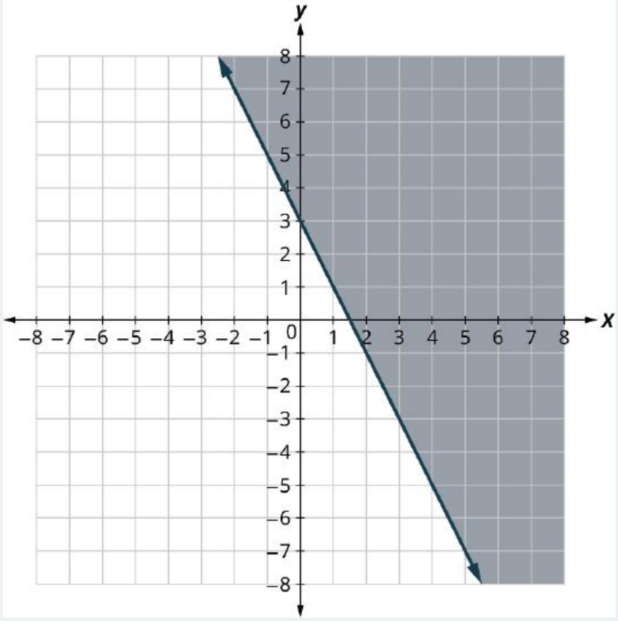
$0 > 2(0) - 1$	$0 < 2(0) - 1$
$0 > 0 - 1$	$0 < 0 - 1$
$0 > -1$	$0 < -1$
True	False

Since $y > 2x - 1$ is true, the side of the line with $(0, 0)$, is the solution. The shaded region shows the solution of the inequality $y > 2x - 1$. Since the boundary line is graphed with a dashed line, the inequality does not include the equal sign. The graph shows the inequality $y > 2x - 1$.

We could use an y point as a test point, provided it is not on the line. Why did we choose $(0, 0)$? Because it is the easiest to evaluate. You may want to pick a point on the other side of the boundary line and check that $y < 2x - 1$.

Your Turn 5.5.6

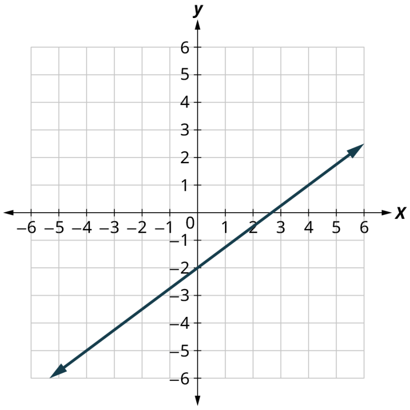
Write the inequality shown by the graph with the boundary line $y = -2x + 3$.



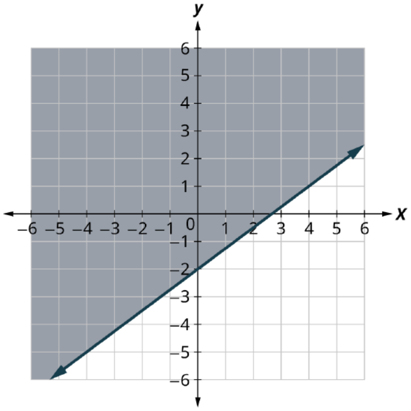
✓ Example $5.5.7$: Graphing a Linear Inequality

Graph the linear inequality $y \geq\geq 34x - 2$.

Answer

Step 1. Identify and graph the boundary line (Figure 5.38). - If the inequality is ≤ or ≥, the boundary line is solid. - If the inequality is < or >, the boundary line is dashed.	Replace the inequality sign with an equal sign to find the boundary line. Graph the boundary line y =34x−2y=34x−2. The inequality sign is ≥, so we draw a solid line.	 Figure ✓ $5.5.19$

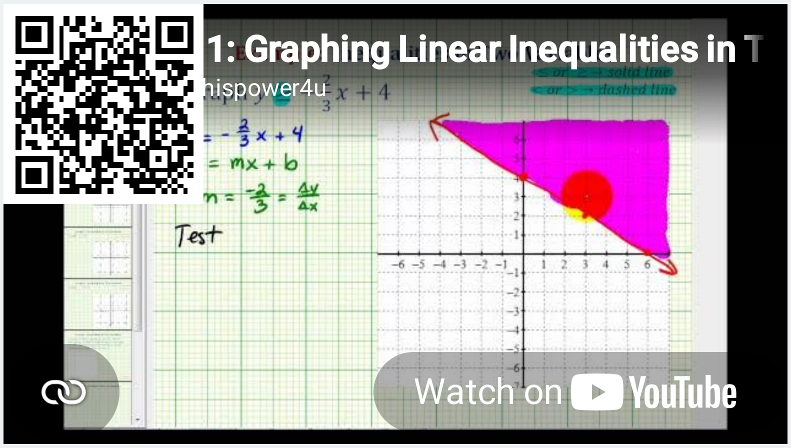
Step 2. Test a point that is not on the boundary line. Is it a solution of the inequality?	We'll test (0 , 0) (0 , 0) . Is it a solution of the inequality?	At $(0, 0)$, is y ≥34x−2y≥34x−2? $0 \overset{?}{\geq} \frac{3}{4}(0) - 2$ $0 \geq -2$ So, $(0, 0)$ is a solution.

Step 3. Shade in one side of the boundary line (Figure ✓ 5.5.20). • If the test point is a solution, shade in the side that includes the point. • If the test point is not a solution, shade in the opposite side.	The test point $(0,0)$ is a solution to $y \geq \frac{3}{4}x - 2$. So we shade in that side.	 Figure ✓ 5.5.20 All points in the shaded region and on the boundary line represent the solution to $y \geq \frac{3}{4}x - 2$.

Your Turn 5.5.7

Graph the linear inequality: $y > \frac{2x}{3} - 1$

Video



Graphing Linear Inequalities in Two Variables

Solving Applications Using Linear Inequalities in Two Variables

Many fields use linear inequalities to model a problem. While our examples may be about simple situations, they give us an opportunity to build our skills and to get a feel for how they might be used.

Example 5.5.8U: Working Multiple Jobs

Hilaria works two part time jobs to earn enough money to meet her obligations of at least $240 a week. Her job in food service pays $10 an hour and her tutoring job on campus pays $15 an hour. How many hours does Hilaria need to work at each job to earn at least $240?

1. Let x be the number of hours she works at the job in food service and let y be the number of hours she works tutoring. Write an inequality that would model this situation.
2. Graph the inequality.
3. Find three ordered pairs (x,y) that would be solutions to the inequality. Then, explain what that means for Hilaria.

Answer

1. Let x be the number of hours she works at the job in food service and let y be the number of hours she works tutoring. She earns $10 per hour at the job in food service and $15 an hour tutoring. At each job, the number of hours multiplied by the hourly wage will give the amount earned at that job.

$$\left(\underbrace{\text{Amount earned at the food service}}_{10x}\underset{+}{\text{plus}}\underbrace{\text{the amount earned tutoring}}_{15y}\underbrace{\text{is at leaste}}_{\geq 240}\right)$$

\(\underset{10x}{\underbrace{\text{Amount earned at the food service}} \underset{\dotplus }{\text{plus}}\underset{15y}{\underbrace{\text{the amount earned tutoring }}\underset{\ge 240}{\underbrace{\text{is at leaste}}\)

2. Graph the inequality:

   Step 1: Graph the boundary line $10x+15y=240$

   Create a table of values

x	y
0	$10(0)+15y=240->y=16$
6	$10(6)+15y=240->y=12$
12	$10(12)+15y=240->y=8$

   Step 2: Pick a test point. Let us pick $(0,0)$ again:

   $10\,(0)+15\,(0)\geq 240?$

   0 ≥240 (Figure ✔ \(\PageIndex{21}\)).

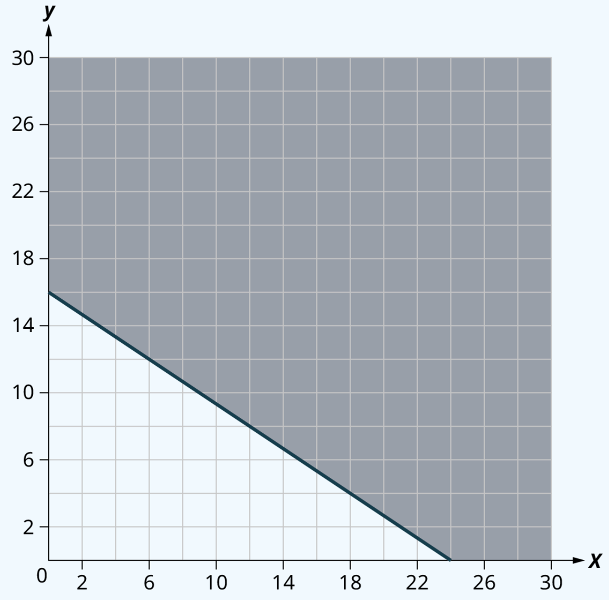


Figure ✓ 5.5.21

From the graph, we see that the ordered pairs (15 , 10) (15 , 10) , (0 , 16) (0 , 16) , (24 , 0) (24 , 0) represent three of infinitely many solutions. Check the values in the inequality.

$(15, 10)$ $10x + 15y \geq 240$ $10(15) + 15(10) \overset{?}{\geq} 240$ $300 \geq 240$ True	$(0, 16)$ $10x + 15y \geq 240$ $10(0) + 15(16) \overset{?}{\geq} 240$ $240 \geq 240$ True	$(24, 0)$ $10x + 15y \geq 240$ $10(24) + 15(0) \overset{?}{\geq} 240$ $240 \geq 240$ True

1For Hilaria, it means that to earn at least \$240, she can work 15 hours tutoring and 10 hours at her food service job, earn all her money tutoring for 16 hours, or earn all her money while working 24 hours at the job in food service.

Your Turn 5.5.8

Harrison works two part time jobs. One at a gas station that pays \$11 an hour and the other is as an IT consultant for \$16.50 an hour. Between the two jobs, Harrison wants to earn at least \$330 a week. How many hours does Harrison need to work at each job to earn at least \$330?

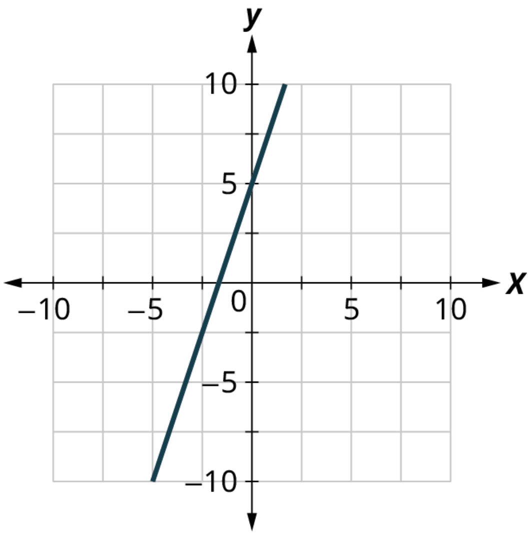
1. Let x be the number of hours he works at the gas station and let y be the number of hours he works as an IT consultant. Write an inequality that would model this situation.
2. Graph the inequality.
3. Find three ordered pairs (x, y) that would be solutions to the inequality. Then, explain what that means for Harrison.

Check Your Understanding

1. Choose the correct solution to the equation $6y + 10 = 12y$.

a. $y = 5$
b. $y = -1$
c. $y = \frac{1}{2}$
d. $y = \frac{5}{3}$

2. Choose the correct graph for $y = 3x + 5$.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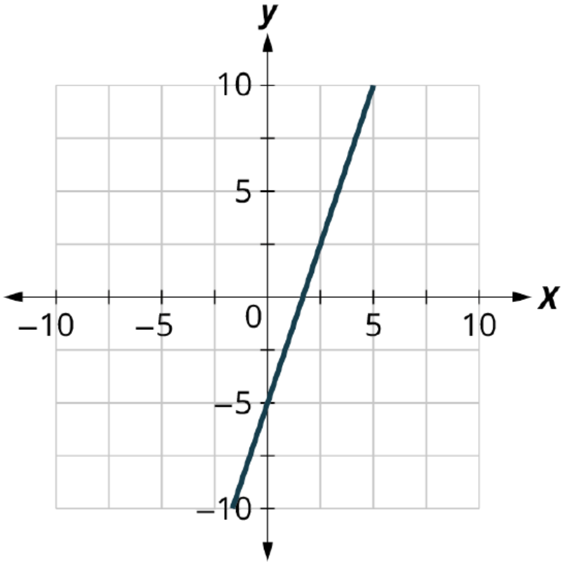
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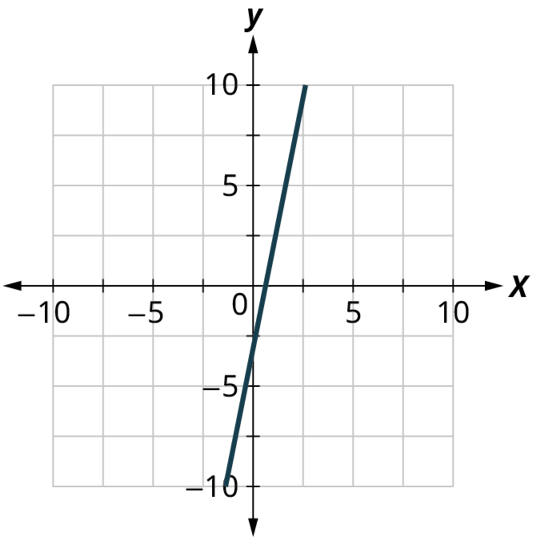
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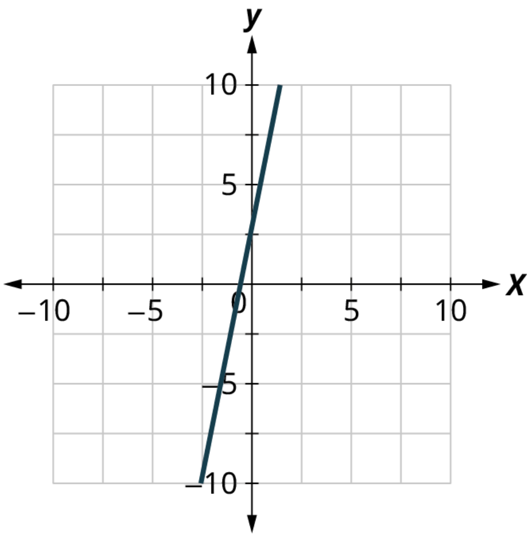
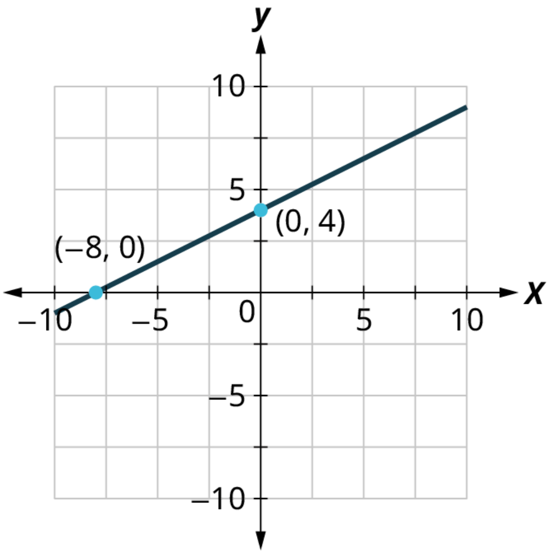
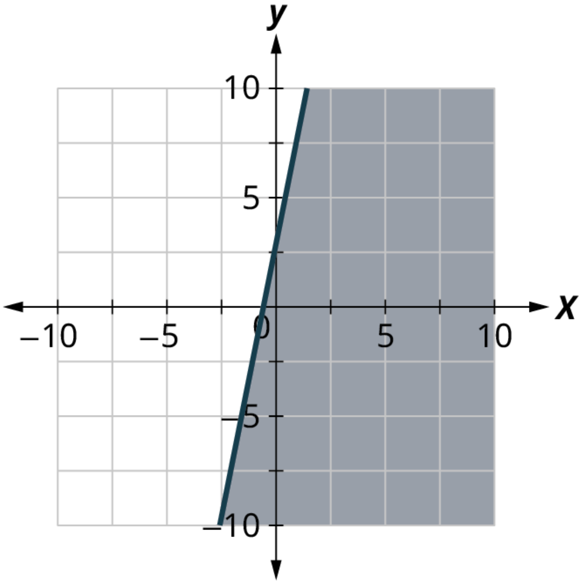


3. Choose the correct equation for the graph shown:



a. $y=2x+4$
b. $y=\frac{1}{2}x+4$
c. $y=-2x+4$
d. $y=\frac{1}{2}x+4$

46. Choose the correct graph for $y>3x+5$.


y
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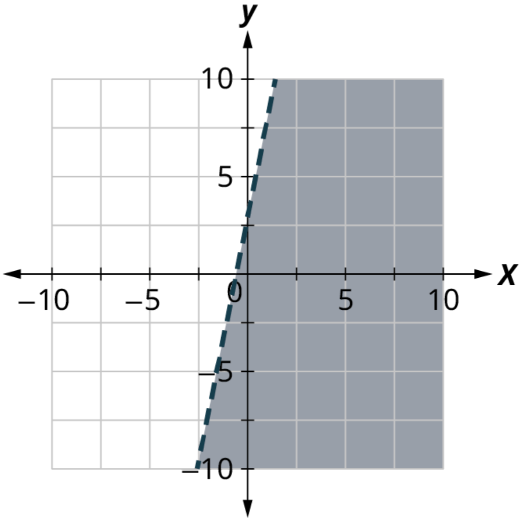
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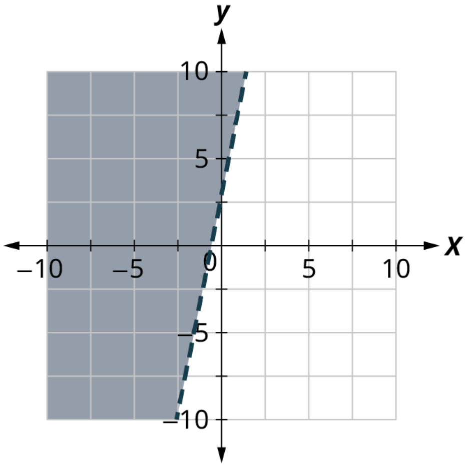
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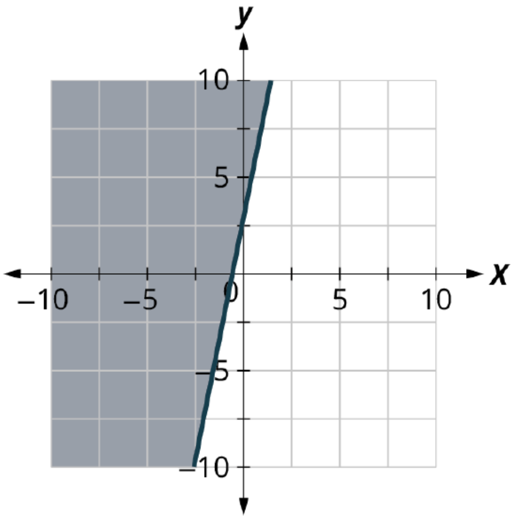
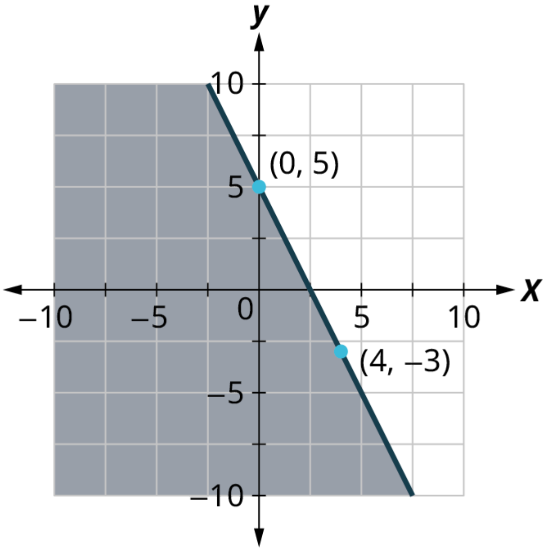


5. Choose the correct inequality for the graph shown.



a. $y=-2x+5$
b. $y\leq -2x+5$
c. $y\geq -2x+5$
d. $y<-2x+5$

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5.5.1: Exercises

Exercise $5.5.1.1$

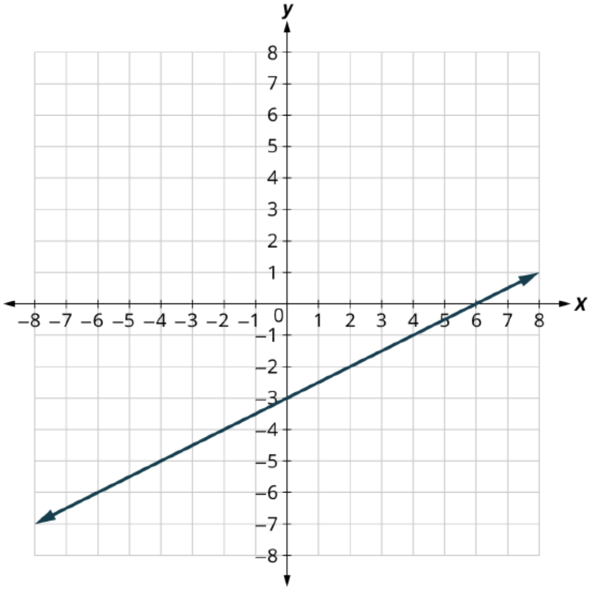
Plot each point in a rectangular coordinate system and identify the quadrant in which the point is located.

a. $(3, -1)$
b. $(-3, 1)$
c. $(-2, 0)$
d. $(-4, -3)$
e. $(1, \frac{1}{4}5)$

For each ordered pair below, decide:

I. Is the ordered pair a solution to the equation?
II. Is the point on the line in the given graph?

$y = \frac{1}{2}x - 3$



Exercise $5.5.1.2$

$(0, -3)$

Exercise $5.5.1.3$

$(2, -2)$

Exercise $5.5.1.4$

$(-2, -4)$

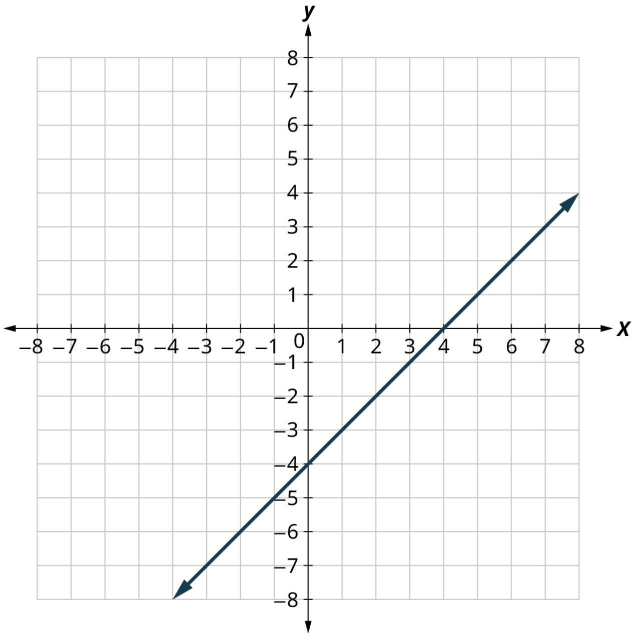
Exercise 5.5.1.5

$(4, 1)$

For each ordered pair below, decide:

I. Is the ordered pair a solution to the equation?
II. Is the point on the line in the given graph?

$y = x - 4$



Exercise 5.5.1.6

$(0, -4)$

Exercise 5.5.1.7

$(3, -1)$

Exercise 5.5.1.8

$(2, 2)$

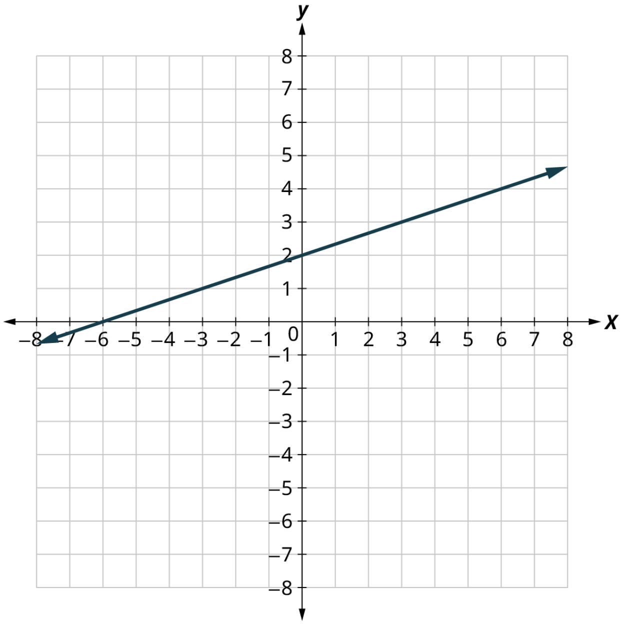
Exercise 5.5.1.9

$(1, -5)$

For each ordered pair below, decide:

I. Is the ordered pair a solution to the equation?
II. Is the point on the line in the given graph?

$y = \frac{1}{3}x + 2$



Exercise 5.5.1.10

$(0,2)$

Exercise 5.5.1.11

$(3,3)$

Exercise 5.5.1.12

$(-3,2)$

Exercise 5.5.1.13

$(-6,0)$

For the following exercises, graph by plotting points.

Exercise 5.5.1.14

$y=3x-1$

Exercise 5.5.1.15

$y=-x-3$

Exercise 5.5.1.16

$y=2x$

Exercise 5.5.1.17

$y=\frac{1}{2}x+2$

? Exercise 5.5.1.18

$y=\frac{4}{3}x-5$

? Exercise 5.5.1.19

$y=\frac{2}{5}x+1$

? Exercise 5.5.1.20

$y=\frac{3}{2}x+2$

For the following exercises, determine whether each ordered pair is a solution to the inequality.

? Exercise 5.5.1.21

$y>x-3$

A: $(0,0)$ B: $(2,1)$ C: $(-1,-5)$ D: $(-6,-3)$ E: $(1,0)$

? Exercise 5.5.1.22

$y<3x+2$

A: $(0,3)$ B: $(-3,-2)$ C: $(-2,0)$ D: $(0,0)$ E: $(-1,4)$

? Exercise 5.5.1.23

$y<-2x+5$
$\mathrm{A}:(-3,0)\quad \mathrm{B}:(1,6)\quad \mathrm{C}:(-6,-2)\quad \mathrm{D}:(0,1)\quad \mathrm{E}:(5,-4)$

? Exercise 5.5.1.24

$3x-4y>4$

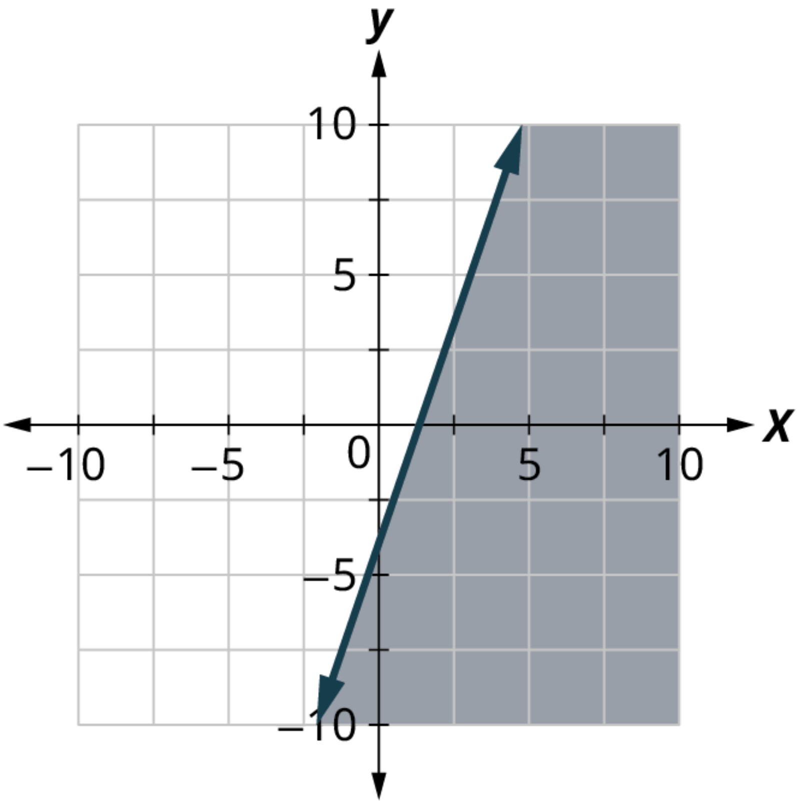
A: $(5,1)\quad \mathrm{B}:(-2,6)\quad \mathrm{C}:(3,2)\quad \mathrm{D}:(10,-5)\quad \mathrm{E}:(0,0)$

? Exercise 5.5.1.25

$2x+3y>2$
$\mathrm{A}:(1,1)\quad \mathrm{B}:(4,-3)\quad \mathrm{C}:(0,0)\quad \mathrm{D}:(-8,12)\quad \mathrm{E}:(3,0)$

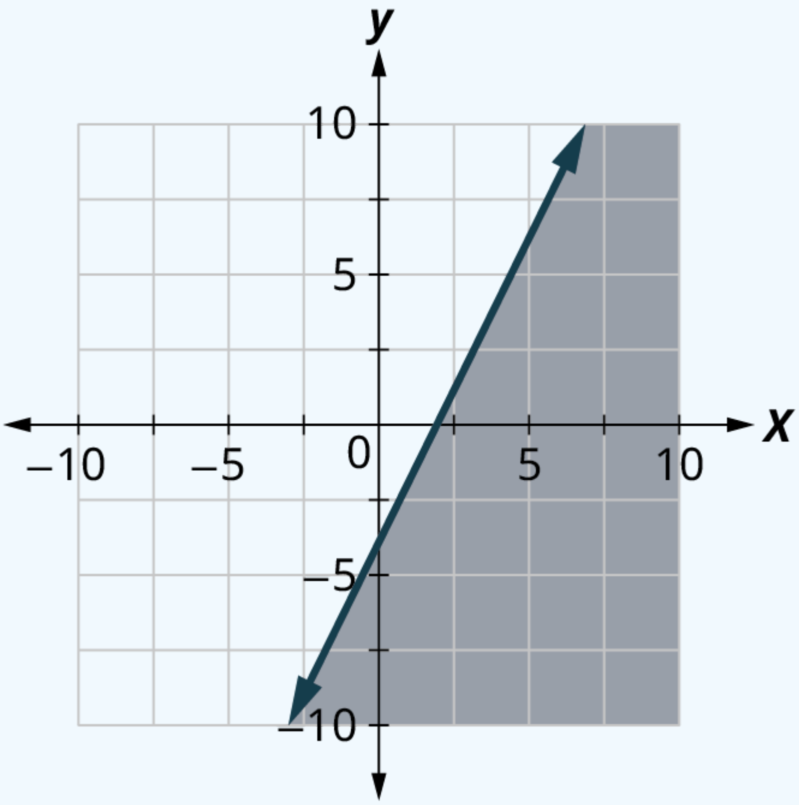
? Exercise 5.5.1.26

Write the inequality shown by the graph with the boundary line $y=3x-4$.



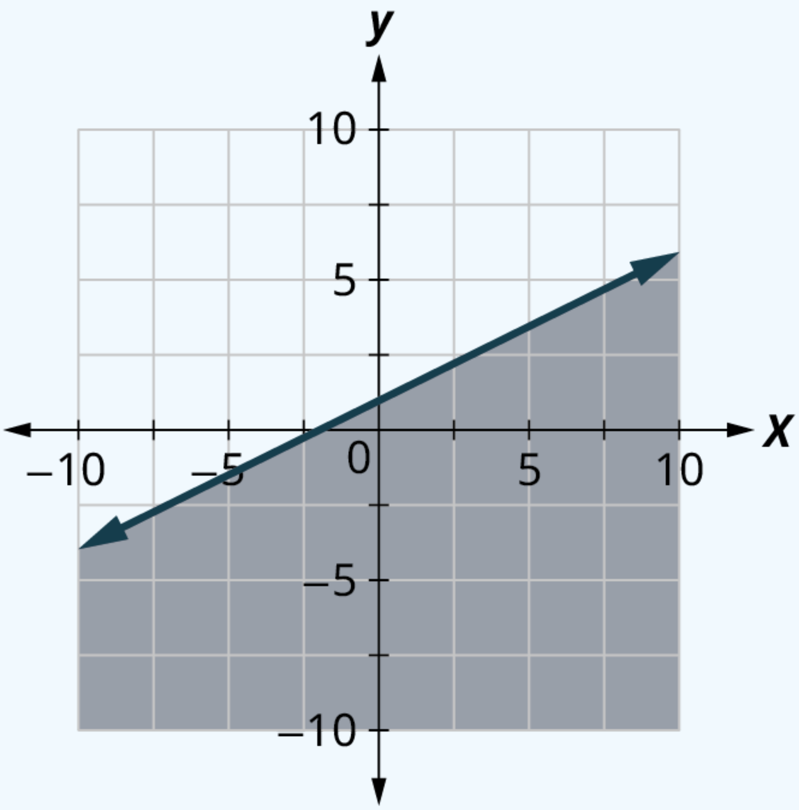
Exercise 5.5.1.27

Write the inequality shown by the graph with the boundary line $y = 2x - 4$.



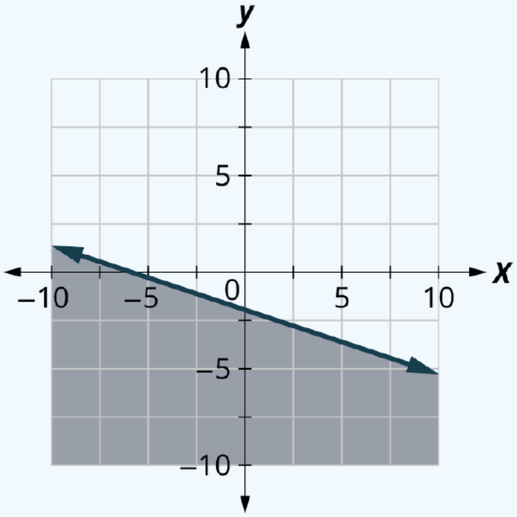
? Exercise 5.5.1.28

Write the inequality shown by the graph with the boundary line $y = \frac{1}{2}x + 1$.



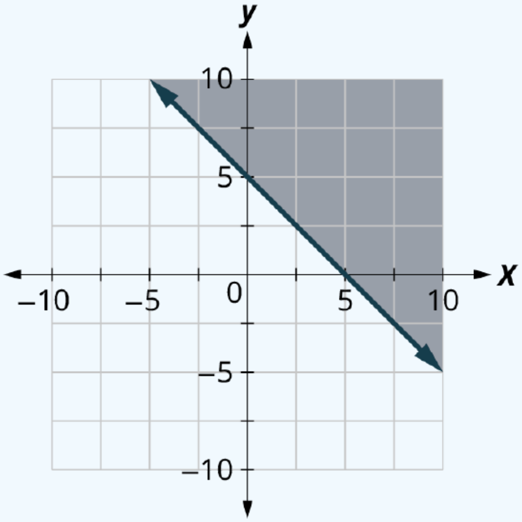
? Exercise 5.5.1.29

Write the inequality shown by the graph with the boundary line $y = -\frac{1}{3}x - 2$.



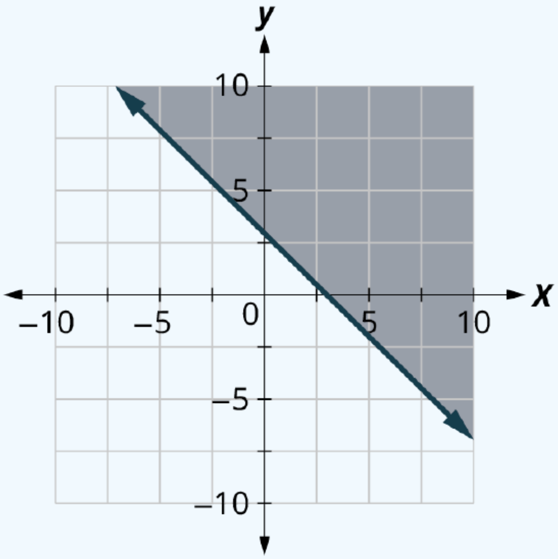
Exercise 5.5.1.30

Write the inequality shown by the shaded region in the graph with the boundary line $x + y = 5$.



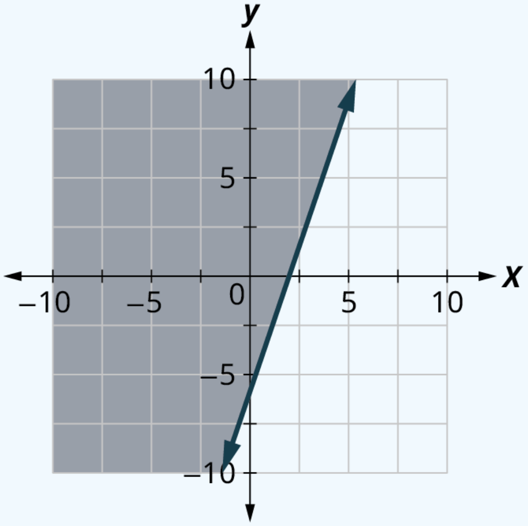
Exercise 5.5.1.31

Write the inequality shown by the shaded region in the graph with the boundary line $x + y = 3$.



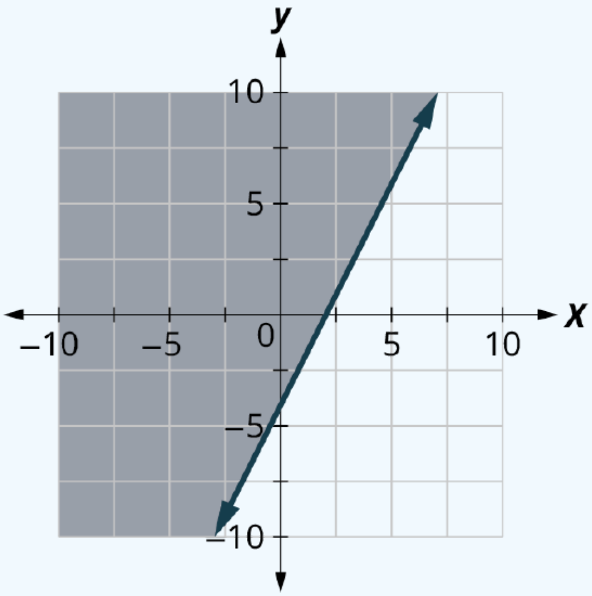
Exercise 5.5.1.32

Write the inequality shown by the shaded region in the graph with the boundary line $3x - y = 6$.



Exercise 5.5.1.33

Write the inequality shown by the shaded region in the graph with the boundary line $2x - y = 4$.



For the following exercises, graph the linear inequality.

Exercise 5.5.1.34

$y < \frac{3}{5}x + 2$

Exercise 5.5.1.35

$y \leq -\frac{1}{2}x + 4$

? Exercise 5.5.1.36

$y \geq -\frac{1}{3}x - 2$

? Exercise 5.5.1.37

$x - y \leq 3$

? Exercise 5.5.1.38

$x - y \geq -2$

? Exercise 5.5.1.39

$4x + y > -4$

? Exercise 5.5.1.40

$x + 5y < -5$

? Exercise 5.5.1.41

$3x + 2y \geq -6$

? Exercise 5.5.1.42

$4x + 2y \geq -8$

? Exercise 5.5.1.43

$y > 4x$

? Exercise 5.5.1.44

$y \leq -3x$

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5.6: Quadratic Equations in One Variable With Applications

Learning Objectives

1. Multiply binomials.
2. Factor trinomials.
3. Solve quadratic equations by graphing.
4. Solve quadratic equations by factoring.
5. Solve quadratic equations using square root method.
6. Solve quadratic equations using the quadratic formula.
7. Solve real world applications modeled by quadratic equations.

In this section, we will discuss quadratic equations. There are several real-world scenarios that can be represented by the graph of a quadratic equation. Think of the Gateway Arch in St. Louis, Missouri. Both ends of the arch are 630 feet apart and the arch is 630 feet tall. You can plot these points on a coordinate system and create a parabola to graph the quadratic equation.

Figure 5.6.1: The Gateway Arch in St. Louis, Missouri (credit: modification of work "Gateway Arch - St. Louis - Missouri" by Sam valadi/Flickr, CC BY 2.0)

Identify Polynomials, Monomials, Binomials and Trinomials

You have learned that a term is a constant, or the product of a constant and one or more variables. When it is of the form ax^m, where a is a constant and x^m is a positive whole number, it is called a monomial. Some examples of monomial are $8, -2x^2, 4y^3$, and $11z$.

A monomial or two or more monomials combined by addition or subtraction is a polynomial. Some examples include: $b+11, 4y^2-7y+2$, and $4x^4+x^3+8x^2-9x+1$. Some polynomials have special names, based on the number of terms. A monomial is a polynomial with exactly one term (examples: $14, 8y^2, -9x^3y^5$, and -13). A binomial has exactly two terms (examples: $a+7, 4b-5, y^2-16$, and $3x^3-9x^2$), and a trinomial has exactly three terms (examples: $x^2-7x+12, 9y^2+2y-8, 6m^4-m^3+8m$, and x^4+3x^2-1).

Notice that every monomial, binomial, and trinomial is also a polynomial. They are just special members of the "family" of polynomials and so they have special names. We use the words monomial, binomial, and trinomial when referring to these special polynomials and just call all the rest polynomials.

Multiply Binomials

Recall multiplying algebraic expressions from Algebraic Expressions. In this section, we will continue that work and multiply binomials as well. We can use an area model to do multiplication.

Exercise 5.6.1: Multiply Binomials

Multiply $(x+2)(x+3)$.

Answer

Step 1: Use the distributive property:

$$(x)(x)+(x)(3)+(2)(x)+(2)(3)$$

In the area model (Figure 5.6.2) multiply each term on the side by each term on the top (think of it as a multiplication table).

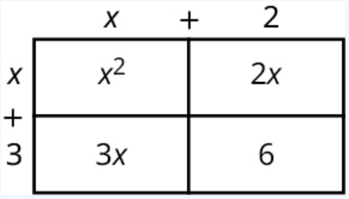


Figure 5.6.2

Step 2: After we multiply, we get the following equation:

$$x^2+3x+2x+6$$

Step 3: Combine the like terms to arrive at:

$$x^2+5x+6$$

Your Turn 5.6.1

Multiply $(x-3)(2x+1)$.

Who Knew?: They Are Teaching Multiplication of Binomials in Elementary School

Manipulatives are often used in elementary school for students to experience a hands-on way to experience the mathematics they are learning. Base Ten Blocks, or Dienes Blocks, are often used to introduce place value and the operation of whole numbers. When multiplying two-digit numbers, students can make an array to visualize the Distributive Property. Figure 5.6.3 shows the value of each Base Ten Block and Figure 5.6.4 shows how to multiply 17 and 23 using an area model and Base Ten Blocks. You can see how this helps students visualize the multiplication using the Distributive Property. Consider how $(10+7)(20+3)$ can easily extend to $(10+x)(20+y)$ in algebra!

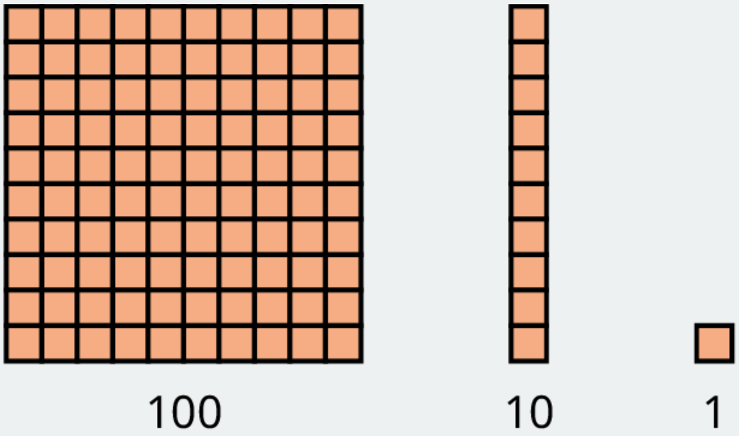


Figure 5.6.3 The Value of Each Base Ten Block

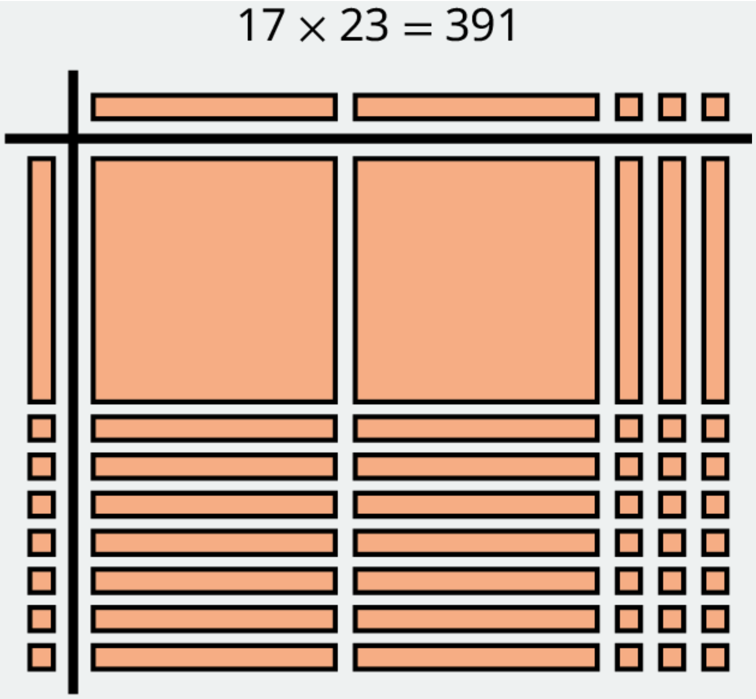


Figure 5.6.4 How to Multiply 17 and 23 Using an Area Model and Base Ten Blocks

Factoring Trinomials

We've just covered how to multiply binomials. Now you will need to "undo" this multiplication-to start with the product and end up with the factors. Let us review an example of multiplying binomials to refresh your memory.

$$(x+2)(x+3)=x^2+5x+6$$

To factor the trinomial means to start with the product, x^2+5x+6, and end with the factors, $(x+2)(x+3)$. You need to think about where each of the terms in the trinomial came from. The first term came from multiplying the first term in each binomial. So, to get x^2 in the product, each binomial must start with an x.

$$\begin{array}{l} x^2+5x+6 \\ (x)(x) \end{array}$$

The last term in the trinomial came from multiplying the last term in each binomial. So, the last terms must multiply to 6 . What two numbers multiply to 6 ? The factors of 6 could be 1 and 6 , or 2 and 3 . How do you know which pair to use? Consider the middle term. It came from adding the outer and inner terms. So the numbers that must have a product of 6 will need a sum of 5 .

We'll test both possibilities and summarize the results in the following table, which will be very helpful when you work with numbers that can be factored in many different ways.

Factors of 6	Sum of Factors
1, 6	1 + 6 = 7
2, 3	2 + 3 = 5

We see that 2 and 3 are the numbers that multiply to 6 and add to 5 . We have the factors of x^2+5x+6. They are $(x+2)(x+3)$.

$$\begin{array}{cc} x^2+5x+6 & \text{product} \\ (x+2)(x+3) & \text{factors} \end{array}$$

You can check if the factors are correct by multiplying. Looking back, we started with x^2+5x+6, which is of the form x^2+bx+c, where $b=5$ and $c=6$. We factored it into two binomials of the form $(x+m)$ and $(x+n)$.

$$\begin{array}{cc} x^2+5x+6 & x^2+bx+c \\ (x+2)(x+3) & (x+m)(x+n) \end{array}$$

To get the correct factors, we found two number m and n whose product is c and sum is b. With the area model (Figure $5.6.5$), start with an empty box and then put in the x^2 term and c.

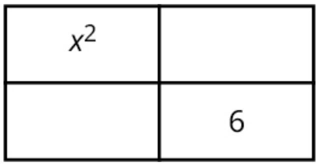


Figure $5.6.5$

Continue by putting in two terms that add up to $5x$: $2x$ and $3x$ (Figure $5.6.6$):

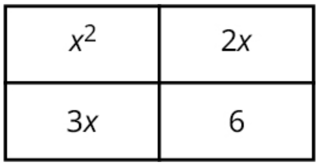


Figure $5.6.6$

Then you find the terms of the binomials on the top and side (Figure$5.6.7$):

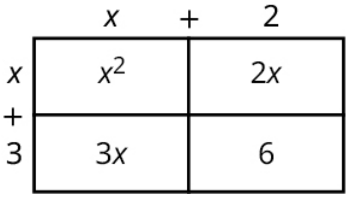


Figure $5.6.7$

Exercise $5.6.2$: Factoring Trinomials

Factor $x^2+7x+12$

Answer

The numbers that must have a product of 12 will need a sum of 7 . We will summarize the results in a table below.

Factors of 12	**Sum of Factors**
1, 12	1 + 12 = 13
2, 6	2 + 6 = 8
3, 4	3 + 4 = 7

We see that 3 and 4 are the numbers that multiply to 12 and add to 7 . The factors of $x^2+7x+12$ are $(x+3)(x+4)$.

Your Turn $5.6.2$

Factor x^2+6x+8

? Exercise 5.6.3: Factoring More Complex Trinomials

Factor $x^2 - 11x + 28$

Answer

The numbers that must have a product of 28 will need a sum of -11 . We will summarize the results in a table.

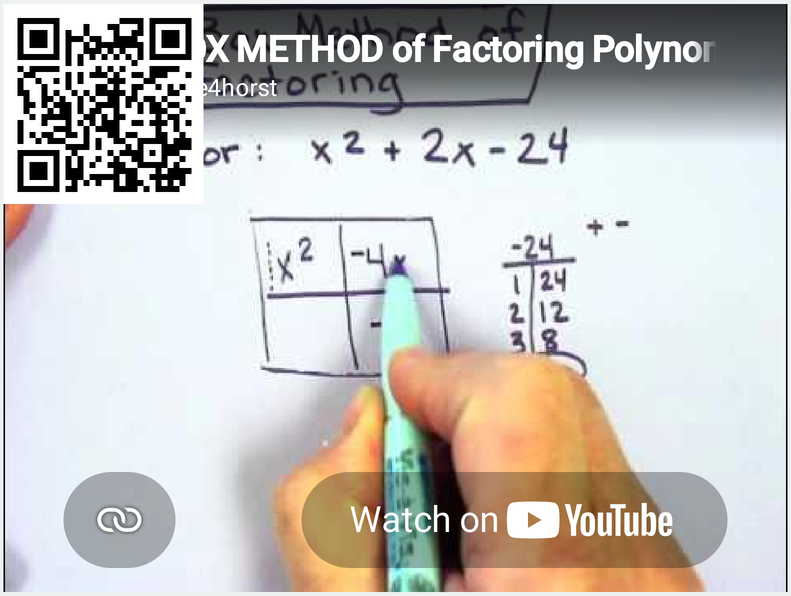
Factors of 28	Sum of Factors
1, 28	1 + 28 = 29
2, 14	2 + 14 = 16
4, 7	4 + 7 = 11

We see that 4 and 7 are the numbers that multiply to 28 and add to 11 . But we needed -11 , so we will need to use -4 and -7 because $(-4)(-7) = 28$ and $(-4) + (-7) = -11$. The factors of $x^2 - 11x + 28$ are $(x - 4)(x - 7)$.

Your Turn 5.6.3

Factor: $x^2 - 16x + 63$.

Video



Factoring with the Box Method (Area Model)

Solving Quadratic Equations by Graphing

We have already solved and graphed linear equations in Graphing Linear Equations and Inequalities, equations of the form $Ax + By = C$. In linear equations, the variables have no exponents. Quadratic equations are equations in which the variable is squared. The following are some examples of quadratic equations:

$$x^2 + 5x + 6 = 0 \quad 3y^2 + 4y = 106 \quad 4u^2 - 81 = 0 \quad n(n+1) = 42$$

The last equation does not appear to have the variable squared, but when we simplify the expression on the left, we will get $n^2 + n$. The general form of a quadratic equation is $ax^2 + bx + c = 0$, where a, b, and c are real numbers, with $a \neq 0$. Remember that a solution of an equation is a value of a variable that makes a true statement when substituted into the equation. The solutions of quadratic equations are the values of the variables that make the quadratic equation $ax^2 + bx + c = 0$ true.

To solve quadratic equations, we need methods different than the ones we used in solving linear equations. We will start by solving a quadratic equation from its graph. Just like we started graphing linear equations by plotting points, we will do the same for quadratic equations. Let us look first at graphing the quadratic equation $y=x^2$. We will choose integer values of x between -2 and 2 and find their y values, as shown in the table below.

$y=x^2$

x	y
0	0
1	1
-1	1
2	4
-2	4

Notice when we let $x=1$ and $x=-1$, we got the same value for y.

$$\begin{array}{lll} y=x^2 & y=1^2 & y=1 \\ y=x^2 & y=(-1)^2 & y=1 \end{array}$$

The same thing happened when we let $x=2$ and $x=-2$. Now, we will plot the points to show the graph of $y=x^2$. See Figure $5.6.8$.

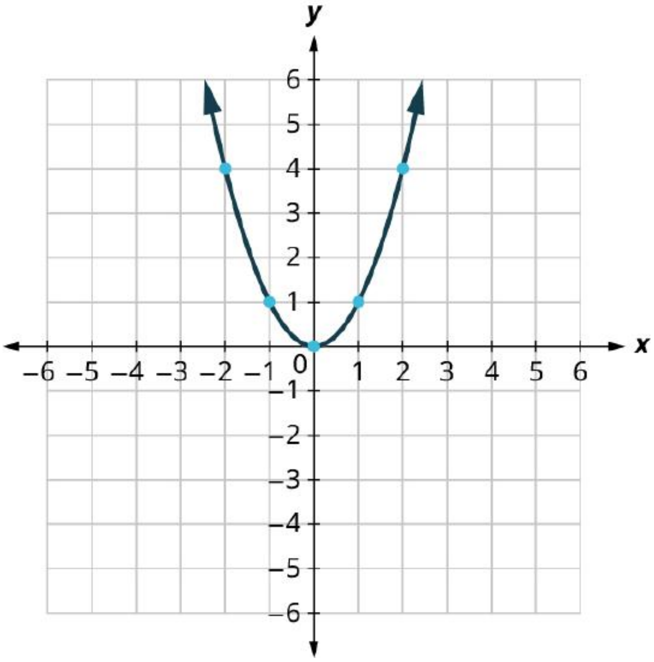


Figure $5.6.8$:

The graph is not a line. This figure is called a parabola. Every quadratic equation has a graph that looks like this. When $y=0$ the solution to the quadratic $y=x^2$ is 0 because $x^2=0$ at $x=0$.

? Exercise $5.6.4$: Graphing a Quadratic Equation

Graph $y=x^2-1$ and list the solutions to the quadratic equation.

Answer

We will graph the equation by plotting points.

Step 1: Choose integer values for x, substitute them into the equation, and solve for y.

Step 2: Record the values of the ordered pairs in the chart.Add texts here. Do not delete this text first.

x	y
0	−1
1	0
−1	0
2	3
−2	3

Step 3: Plot the points and then connect them with a smooth curve. The result will be the graph of the equation $y = x^2 - 1$ (Figure

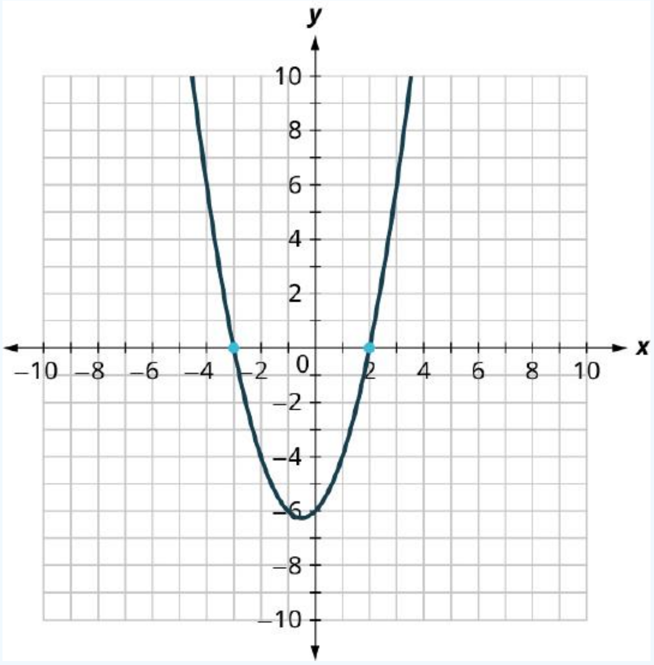


Figure \(\PageIndex{$5.6.9$}\):

Your Turn $5.6.4$

Graph $y = -x^2$.

❓ Exercise $5.6.5$: Solving a Quadratic Equation From Its Graph

Find the solutions of $y = x^2 + 5x + 4$ from its graph (Figure ❓ $5.6.10$).

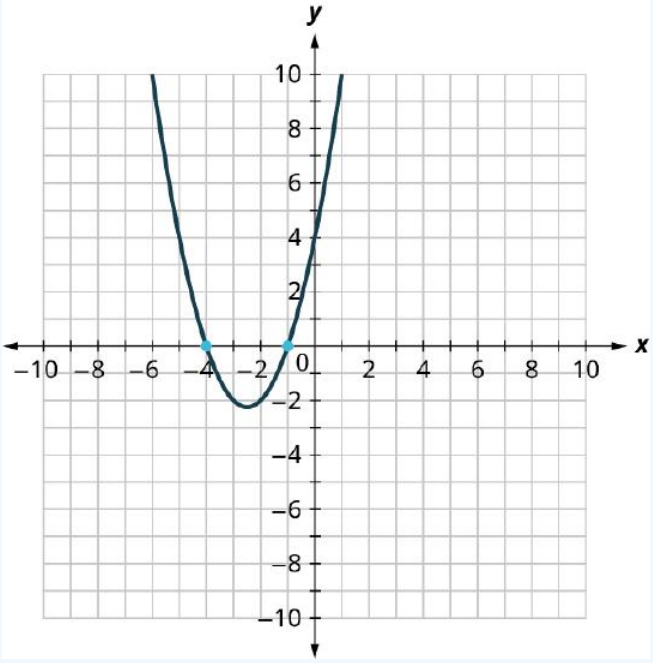


Figure 5.6.10:

Answer

The solutions of a quadratic equations are the values of x that make the equation a true statement when set equal to zero (i.e. when $y = 0$). $x^2 + 5x + 4 = 0$ at $x = -4$ and $x = -1$.

Your Turn 5.6.5

Find the solutions of $y = x^2 + x - 6$ from its graph.

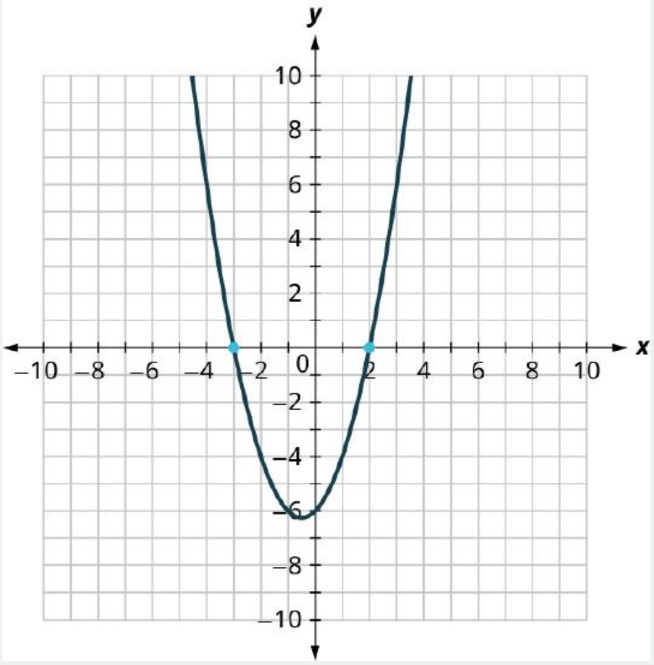


Figure 5.6.11:

Solving Quadratic Equations by Factoring

Another way of solving quadratic equations is by factoring. We will use the **Zero Product Property** that says that if the product of two quantities is zero, it must be that at least one of the quantities is zero. The only way to get a product equal to zero is to multiply by zero itself.

Exercise 5.6.6: Solving a Quadratic Equation by Factoring

Solve $(x+1)(x-4)=0$.

Answer

Step 1. Set each factor equal to zero.	The product equals zero, so at least one factor must equal zero.	$(x+1)(x-4)=0$ $x+1=0 \quad \text{or} \quad x-4=0$
Step 2. Solve the linear equations.	Solve each equation.	$x=-1 \quad \text{or} \quad x=4$
Step 3. Check.	Substitute each solution separately into the original equation.	$x=\mathbf{-1}$ $(x+1)(x-4)=0$ $(\mathbf{-1}+1)(\mathbf{-1}-4)\stackrel{?}{=}0$ $(0)(-5)\stackrel{?}{=}0$ $0=0$ $x=\mathbf{4}$ $(x+1)(x-4)=0$ $(4+1)(4-4)\stackrel{?}{=}0$ $(5)(0)\stackrel{?}{=}0$ $0=0\checkmark$

Your Turn 5.6.6

Solve $(x-2)(x+5)=0$

Exercise 5.6.7: Solving Another Quadratic Equation by Factoring

Solve $x^2+2x-8=0$.

Answer

Step 1. Write the quadratic equation in standard form, $ax^2+bx+c=0$.	The equation is already in standard form.	$x^2+2x-8=0$
Step 2. Factor the quadratic expression.	Factor x^2+2x-8 $(x+4)(x-2)$	$(x+4)(x-2)=0$
Step 3. Use the Zero Product Property.	Set each factor equal to zero.	$x+4=0 \quad \text{or} \quad x-2=0$
Step 4. Solve the linear equations.	We have two linear equations.	$x=-4 \quad \text{or} \quad x=2$ $x^2+2x-8=0$

Step 5. Check.	Substitute each solution separately into the original equation.	$x^2+2x-8=0$ $x=-4$ $(-4)^2-2(-4)-8\overset{?}{=}0$ $16+(-8)-8\overset{?}{=}0$ $0=0\checkmark$ $x^2+2x-8=0$ $x=\mathbf{2}$ $\mathbf{2}^2-2(\mathbf{2})-8\overset{?}{=}0$ $4+4-8\overset{?}{=}0$ $0=0\checkmark$

Your Turn 5.6.7

Solve $x^2+2x-15=0$

Checkpoint

Be careful to write the quadratic equation in standard form first. The equation must be set equal to zero in order for you to use the Zero Product Property! Often students start in Step 2 resulting in an incorrect solution. For example, $x^2+2x-15=-7$ *cannot be factored to* $(x-3)(x+5)=-7$ *and then solved by setting each factor equal to -7 .*

The correct way to solve this quadratic equation is to set it equal to zero FIRST: $x^2+2x-15+7=-7+7$ *which becomes* $x^2+2x-8=0$*, then continue to factor. See the table below for the correct way to apply the Zero Product Property.*

		$x^2+2x-15=-7$	$x^2+2x-15=-7$	
Step 1	Skipped		$x^2+2x-15+7=-7+7$ $x^2+2x-8=0$	
Step 2	$(x-3)(x+5)=-7$		$(x-2)(x+4)=0$	
Step 3	$x-3=-7$	$x+5=-7$	$x-2=0$	$x+4=0$
Step 4	$x=-4$	$x=-12$	$x=2$	$x=2$
Step 5	$(-4)^2+2(-4)-15$ $=16-8-15$ $=-7$	$(-12)^2+2(-12)-15$ $=144-24-1=105$ $\neq -7$	$(2)^2+2(2)-8$ $=4+4-8$ $=0$	$(-4)^2+2(-4)-18$ $=16-8-8$ $=0$

Solving Quadratic Equations Using the Square Root Property

We just solved some quadratic equations by factoring. Let us use factoring to solve the quadratic equation $x^2=9$.

Step 1: Put the equation in standard form.	$x^2-9=0$	
Step 2: Factor the left side.	$(x+3)(x-3)=0$	
Step 3: Use the Zero Product Property.	$x+3=0$	$x-3=0$
Step 4: Solve each equation.	$x=-3$	$x=3$
Step 5: Combine the two solutions into ±	$x=\pm 3$	

The solution is read as " x is equal to positive or negative 3."

What happens when we have an equation like $x^2=7$? Since 7 is not a perfect square, we cannot solve the equation by factoring. These equations are all of the form $x^2=k$. We define the square root of a number in this way: If $n^2=m$, then n is a square root of m. This leads to the **Square Root Property.**

Exercise 5.6.8: Using the Square Root Property to Solve a Quadratic Equation

Solve using the square Root Property: $x^2 = 169$.

Answer

Step 1: Use the Square Root Property.	$x = \pm\sqrt{169}$	
Step 2: Simplify the radical.	$x = \pm 13$	
Step 3: Rewrite to show the two solutions.	$x = 13$	$x = -13$

Your Turn 5.6.8

Solve using the Square Root Property: $x^2 = 25$.

Exercise 5.6.9: Using the Square Root Property to Solve Another Quadratic Equation

Solve using the Square Root Property: $144q^2 = 25$.

Answer

Step 1: Solve for q	\(q^2 = \dfrac{25}{144}	
Step 2: Use the Square Root Property.	\(q= \pm \sqrt {\dfrac{25}{144}\)}	
Step 3: Simplify the radical.	$q = \pm dfrac512$	
Step 4: Rewrite to show the two solutions.	$q = dfrac512$	$q = -dfrac512$

Your Turn \(\PageIndex{9}\

Solve using the Square Root Property: $25p^2 = 49$.

Solving Quadratic Equations Using the Quadratic Formula

This last method we will look at for solving quadratic equations is the quadratic formula. This method works for all quadratic equations, even the quadratic equations we could not factor! To use the quadratic formula, we substitute the values of a, b, and c into the expression on the right side of the formula. Then, we do all the math to simplify the expression. The result gives the solution(s) to the quadratic equation.

Exercise 5.6.10: Solving a Quadratic Equation Using the Quadratic Formula

Solve using the quadratic formula: $x^2 - 6x + 5 = 0$.

Answer

$$x^2 - 6x + 5 = 0$$

This equation is in standard form.

$$\begin{gathered} ax^2 + bx + c = 0 \\ x^2 - 6x + 5 = 0 \end{gathered}$$

Step 1: Identify the a, b, and c values.

$$a = 1, b = -6, c = 5$$

Step 2: Write the quadratic formula.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Step 3: Substitute in the values of a, b, c.

$$x=\frac{(-6)\pm\sqrt{(-6)^2-4(1)(5)}}{2.1}$$

Step 4: Simplify.

$$x=\frac{6\pm\sqrt{36-20}}{2}$$
$$x=\frac{6\pm\sqrt{16}}{2}$$
$$x=\frac{6\pm 4}{2}$$

Step 5: Rewrite to show two solutions.

$$x=\frac{6+4}{2}, x=\frac{6-4}{2}$$

Step 6: Simplify.

$$x=\frac{10}{2}, x=\frac{2}{2}$$
$$x=5, x=1$$

$x^2-6x+5=0$ $5^2-6\cdot 5+5\overset{?}{=}0$ $25-30+5\overset{?}{=}0$ $0=0\checkmark$	$x^2-6x+5=0$ $1^2-6\cdot 1+5\overset{?}{=}0$ $1-6+5\overset{?}{=}0$ $0=0\checkmark$

Your Turn 5.6.10

Solve using the quadratic formula: $a^2-2a-15=0$.

Exercise 5.6.11: Solving Another Quadratic Equation Using the Quadratic Formula

Solve using the quadratic formula: $2x^2+9x-5=0$.

Answer

Step 1. Write the quadratic equation in standard form. Identify the a, b, c values.	This equation is in standard form.	$\boldsymbol{ax^2+bx+c=0}$ $2x^2+9x-5=0$ $a=2, b=9, c=-5$
Step 2. Write the quadratic formula. Then substitute in the values of a, b, c.	Substitute in $\boldsymbol{a=2}, b=9, c=-5$	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ $x=\frac{-9\pm\sqrt{9^2-4(2)(-5)}}{2\cdot 2}$ $x=\frac{-9\pm\sqrt{81-(-40)}}{4}$ $x=\frac{-9\pm\sqrt{121}}{4}$ $x=\frac{-9\pm 11}{4}$ $x=\frac{-9+11}{4}\quad x=\frac{-9-11}{4}$
Step 3. Simplify the fraction, and solve for x		$x=\frac{2}{4}\quad x=\frac{-20}{4}$ $x=\frac{1}{2}\quad x=-5$

Step 4. Check the solutions.	Put each answer in the original equation to check. Substitute $x=\frac{1}{2}$. Substitute $x=-5$.	$2x^2+9x-5=0$ $2\left(\frac{1}{2}\right)^2+9\cdot\frac{1}{2}-5\overset{?}{=}0$ $2\cdot\frac{1}{4}+9\cdot\frac{1}{2}-5\overset{?}{=}0$ $\frac{1}{2}+\frac{9}{2}-5\overset{?}{=}0$ $\frac{10}{2}-5\overset{?}{=}0$ $5-5\overset{?}{=}0$ $0=0$ $2x^2+9x-5=0$ $2(-5)^2+9(-5)-5\overset{?}{=}0$ $2\cdot 25-45-5\overset{?}{=}0$ $50-45-5\overset{?}{=}0$ $0=0\checkmark$

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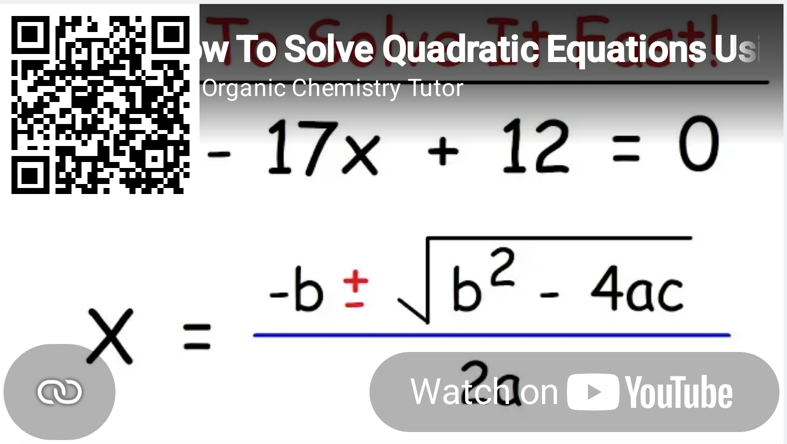
Your Turn 5.6.11

\text { Solve using the quadratic formula: } 3 y^2-5 y+2=0 \text {. }

Video



Solving Quadratics with the Quadratic Formula

Solving Real-World Applications Modeled by Quadratic Equations

There are problem solving strategies that will work well for applications that translate to quadratic equations. Here's a problem-solving strategy to solve word problems:

Step 1: Read the problem. Make sure all the words and ideas are understood.

Step 2: Identify what we are looking for.

Step 3: Name what we are looking for. Choose a variable to represent that quantity.

Step 4: Translate into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.

Step 5: Solve the equation using good algebra techniques.

Step 6: Check the answer in the problem and make sure it makes sense.

Step 7: Answer the question with a complete sentence.

✓ Example $5.6.12$: Finding Consecutive Integers

The product of two consecutive integers is 132. Find the integers.

Answer

Step 1: Read the problem.

Step 2: Identify what we are looking for.

We are looking for two consecutive integers.

Step 3: Name what we are looking for.

Let $n =$ the first integer

Let $n+1 =$ the next consecutive integer.

Step 4: Translate into an equation. Restate the problem in a sentence.

$$n(n+1)=132$$

The product of the two consecutive integers is 132. The first integer times the next integer is 132 .

Step 5: Solve the equation.

$$n^2+n=132$$

Bring all the terms to one side.

$$n^2+n-132=0$$

Factor the trinomial.

$$(n-11)(n+12)=0$$

Use the zero product property.

$$n-11=0 \text{ or } n+12=0$$

Solve the equations.

$$n=11, n=-12$$

There are two values for n that are solutions to this problem. So, there are two sets of consecutive integers that will work.

If the first integer is $n=11$, then the next integer is 12 . If the first integer is $n=-12$, then the next integer is -11 .

Step 6: Check the answer.

The consecutive integers are 11, 12 and $-11, -12$. The product of 11 and $12=132$ and the product of $-11(-12)=132$. Both pairs of consecutive integers are solutions.

Step 7: Answer the question.

The consecutive integers are 11,12 , and $-11, -12$.

Your Turn $5.6.12$

The product of two consecutive odd integers is 240. Find the integers.

? Exercise \(\PageIndex{13}\: Finding Length and Width of a Garden

A rectangular garden has an area 15 square feet. The length of the garden is 2 feet more than the width. Find the length and width of the garden.

Answer

Step 1: Read the problem. In problems involving geometric figures, a sketch can help you visualize the situation (Figure ? $5.6.12$).Add texts here. Do not delete this text first.

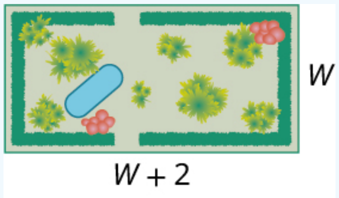


Figure $5.6.12$:

Step 2: Identify what you are looking for.

We are looking for the length and width.

Step 3: Name what you are looking for. The length is 2 feet more than width.

Let $W =$ the width of the garden.

$W + 2 =$ the length of the garden.

Step 4: Translate into an equation.

Restate the important information in a sentence.

The area of the rectangular garden is 15 square feet.

Use the formula for the area of a rectangle.

$$A = L \times W$$

Substitute in the variables.

$$15 = (W + 2)W$$

Step 5: Solve the equation. Distribute first.

$$15 = W^2 + 2W$$

Get zero on one side.

$$0 = W^2 + 2W - 15$$

Factor the trinomial.

$$0 = (W + 5)(W - 3)$$

Use the Zero Product Property.

$$\begin{gathered} 0 = W + 5 \\ 0 = W - 3 \end{gathered}$$

Solve each equation.

$$\begin{gathered} -5 = W \\ 3 = W \end{gathered}$$

Since W is the width of the garden, it does not make sense for it to be negative. We eliminate that value for W.

$$W = 3$$

Width is 3 feet.

Find the value of the length.

$W + 2 =$ length.

$$\begin{gathered}3+2\\5\end{gathered}$$

Length is 5 feet.

Step 6: Check the answer (Figure 5.6.13).

Does the answer make sense?

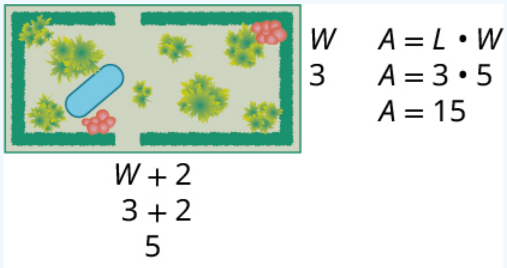


Figure 5.6.13:

Yes, this makes sense.

Step 7: Answer the question.

The width of the garden is 3 feet and the length is 5 feet.

Your Turn 5.6.14

A rectangular sign has an area of 30 square feet. The length of the sign is 1 foot more than the width. Find the length and width of the sign.

Work It Out

Completing the Square

Recall the two methods used to solve quadratic equations of the form $ax^2 + bx + c$: by factoring and by using the quadratic formula. There are, however, many different methods for solving quadratic equations that were developed throughout history. Egyptian, Mesopotamian, Chinese, Indian, and Greek mathematicians all solved various types of quadratic equations, as did Arab mathematicians of the ninth through the twelfth centuries. It is one of these Arab mathematicians' methods that we wish to investigate with this activity.

Muhammad ibn Musa al-Khwarizmi was employed as a scholar at the House of Wisdom in Baghdad, located in present day Iraq. One of the many accomplishments of Al-Khwarizmi was his book on the topic of algebra. In that book, he asks, "What must be the square which, when increased by ten of its own roots, amounts to 39?" Al-Khwarizmi, like many Arab mathematicians of his time, was well versed in Euclid's Elements. Like Euclid, he viewed algebra very geometrically, and thus had a geometric approach to solving a problem like the one above. In his approach, he used a method which today we refer to as completing the square.

His description of the solution method for the above problem is: halve the number of roots, which in the present instance yields 5 . This you multiply by itself; the product is 25 . Add this to 39 ; the sum is 64 . Now take the root of this which is 8 , and subtract from it half the number of the roots, which is 5 ; the remainder is 3 . This is the root of the square which you sought for. Thus the square is 9 .

So, what does all of this mean? Al-Khwarizmi would start with a square of unknown length of side (we will label the side length x). See Figure 5.6.14 So, this square has area x^2

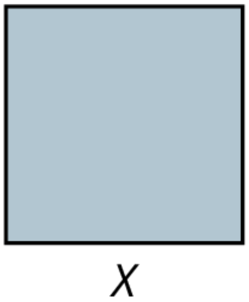


Figure 5.6.14

He would then proceed to halve the number of roots (i.e., there are 10 roots by which the square is increasing) to get 5 ; this he would add to the first square. See Figure 5.6.15 The area of the two new pieces added into the original square are both $5x$. At this point, we have $x^2+10x=39$.

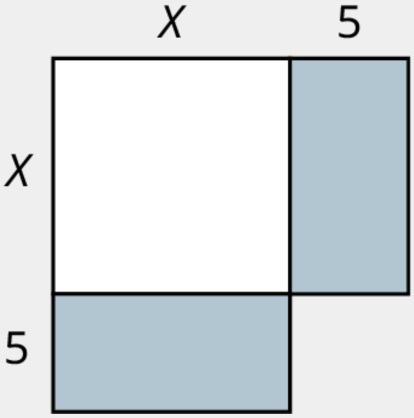


Figure 5.6.15

Now Al-Khwarizmi needed to "complete the square" by adding into the drawing a small square. See Figure 5.6.16 This square has an area of 25.

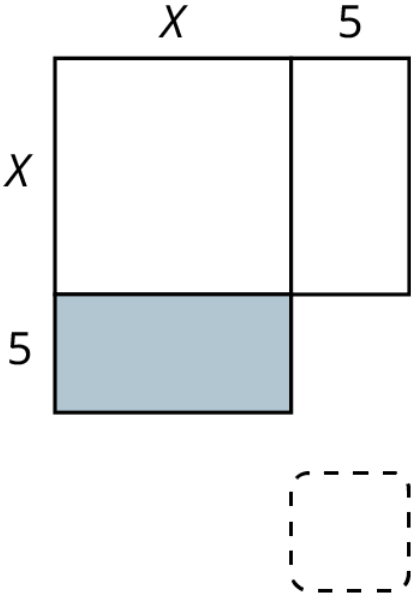


Figure 5.6.16

$$x^2+10x+25=39+25, \text{ or } x^2+10x+25=64.$$

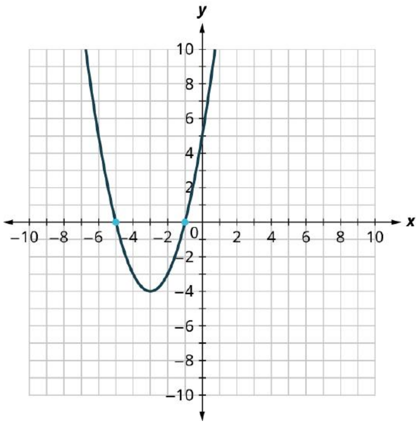
Notice that the completed square has side length $x+5$, so the large square has area $(x+5)^2$. (Notice algebraically that the left half of the equation $x^2+10x+25=64$ factors to $(x+5)^2=64$.) This means the area of large square equals 64 . If $(x+5)^2=64$, then $x+5=8$; so x must be equal to 3 or -13 to make this true. Note that AI-Kwarimi would not have considered the possibility of a negative solution, since he approached the solution geometrically, and negative distances do not exist.

Check Your Understanding

1. Which quadratic equation equals $(x-3)(x+5)$?
a. x^2-15
b. $x^2-3x+15$
c. $x^2-2x-15$
d. $x^2+2x-15$

2. Which product is equal to $x^2-8x+15$?
a. $(x-3)(x+5)$
b. $(x-3)(x-5)$
c. $(x+3)(x+5)$
d. $(x+3)(x-5)$

3. The graph shown is the graph of which quadratic equation?



a. $x^2-x+5=0$
b. $x^2-5x+1=0$
c. $x^2+6x+5=0$
d. $x^2-4x+5=0$

4. What is the solution to $x^2-49=0$?
a. $x=7$
b. $x=-7$
c. $x=\pm 49$
d. $x=\pm 7$

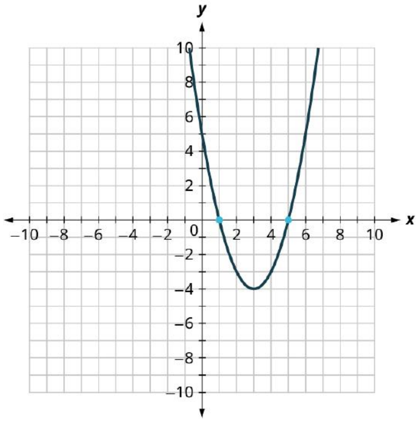
5. $x^2-5x+5=0$ can be factored to $(x-1)(x-5)$.
a. True
b. False

6. $x^2-5x+5=0$ can be solved using the square root method.
a. True

b. False

7. $x^2 - 5x + 5 = 0$ can be solved using the quadratic formula.
a. True
b. False

8. $x^2 - 5x + 5 = 0$ can be graphed as:



a. True
b. False

9. Using the square root method, find the solutions to $x^2 - 5x + 5 = 0$.

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5.6.0: Exercises

For the following exercises, multiply the binomials.

? Exercise $5.6.0.1$

$(w+5)(w+7)$

? Exercise $5.6.0.2$

$(y+9)(y+3)$

? Exercise $5.6.0.3$

$(p+11)(p-4)$

? Exercise $5.6.0.4$

$(q+4)(q-8)$

? Exercise $5.6.0.5$

$(x+8)(x+3)$

? Exercise $5.6.0.6$

$(2x-1)(x+6)$

For the following exercises, factor the trinomials.

? Exercise $5.6.0.7$

$a^2-5a-14$

? Exercise $5.6.0.8$

$n^2+10n+24$

? Exercise $5.6.0.9$

u^2-16

? Exercise $5.6.0.10$

$x^2+4x-21$

? Exercise $5.6.0.11$

$y^2-8y+15$

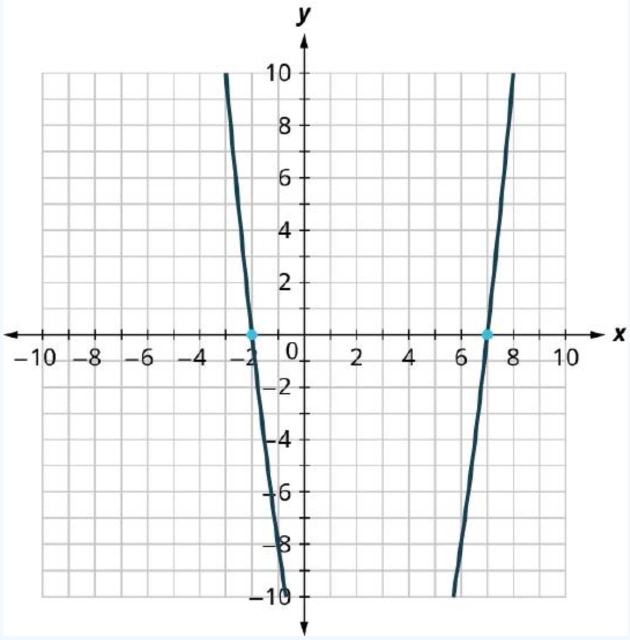
? Exercise 5.6.0.12

$x^2 - 25$

For the following exercises, solve the quadratic equations by graphing.

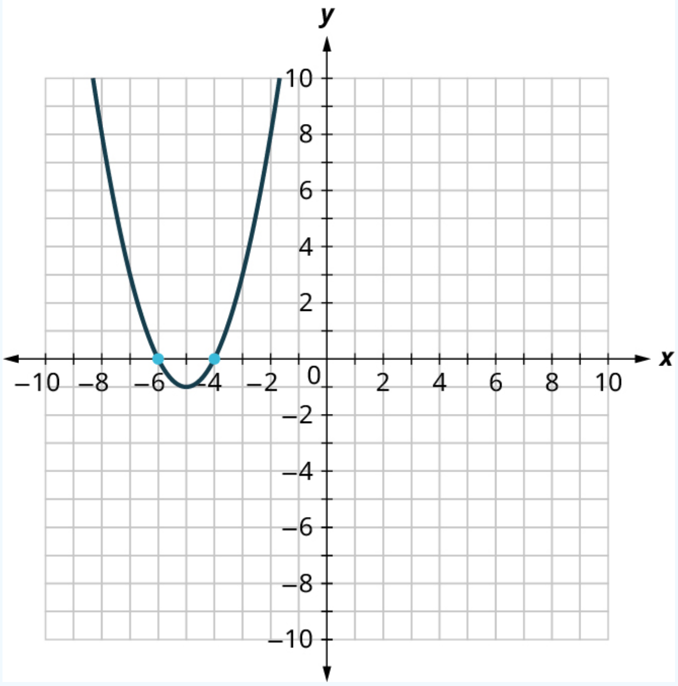
? Exercise 5.6.0.13

Graph and list the solutions to the quadratic equation $x^2 - 5x - 14 = 0$.



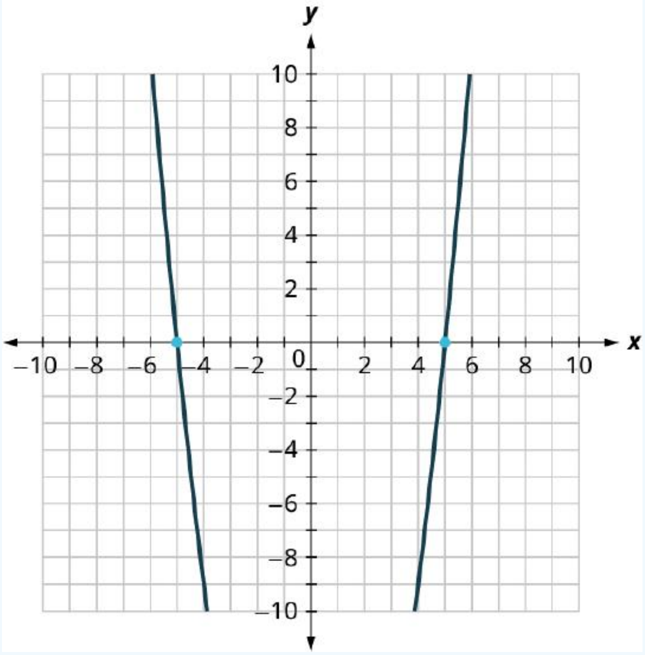
? Exercise 5.6.0.14

Graph and list the solutions to the quadratic equation $x^2 + 10x + 24 = 0$.



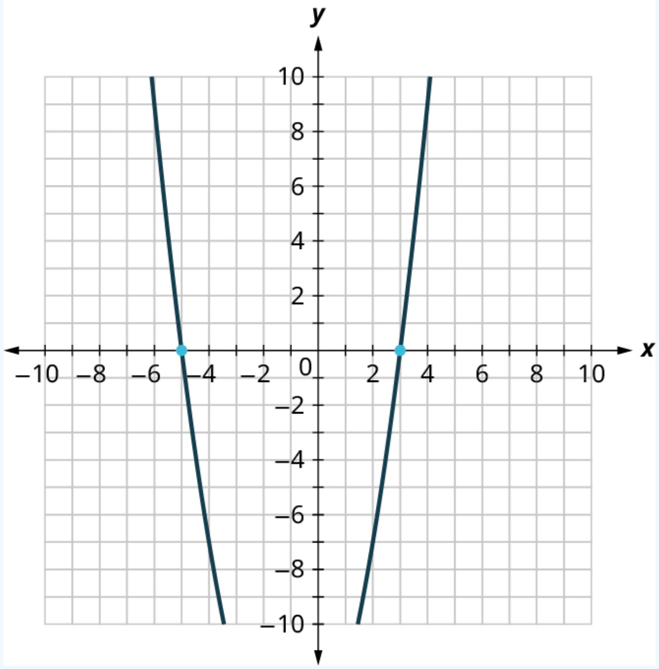
Exercise $5.6.0.15$

Graph and list the solutions to the quadratic equation $x^2 - 25 = 0$.



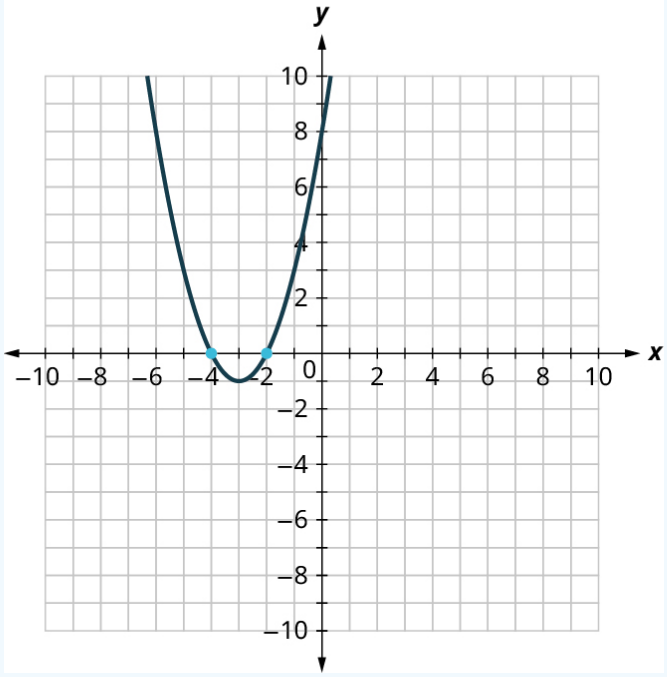
Exercise $5.6.0.16$

Find the solutions of $y = x^2 + 2x - 15$ from its graph.



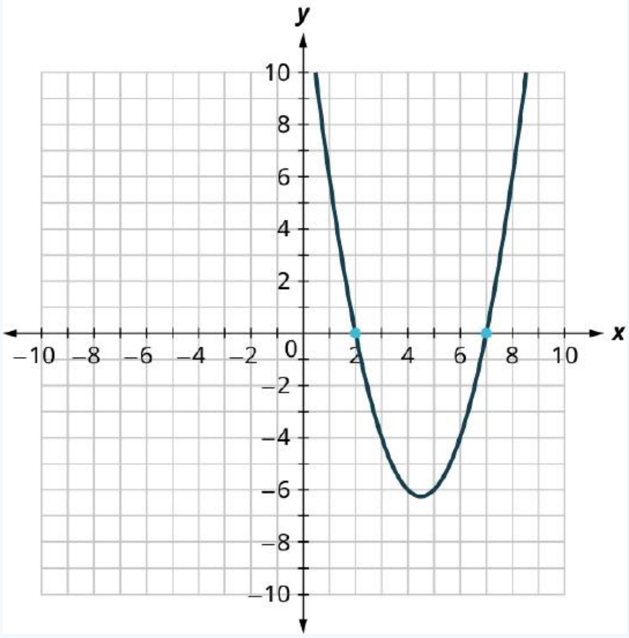
Exercise $5.6.0.17$

Find the solutions of $y = x^2 + 6x + 8$ from its graph.



Exercise 5.6.0.18

Find the solutions of $y = x^2 - 9x + 14$ from its graph.



For the following exercises, solve the quadratic equation by factoring.

Exercise 5.6.0.19

$x^2 + 7x + 12 = 0$

? Exercise 5.6.0.20

$y^2 - 8y + 15 = 0$

? Exercise 5.6.0.21

$n^2 - 5n + 6 = 0$

? Exercise 5.6.0.22

$a^2 + 2a = a$

? Exercise 5.6.0.23

$b^2 = -b$

? Exercise 5.6.0.24

$x^2 - 9x + 14 = 0$

For the following exercises, solve the quadratic equation using the square root method.

? Exercise 5.6.0.25

$x^2 = 16$

? Exercise 5.6.0.26

$49m^2 = 144$

? Exercise 5.6.0.27

$625 = x^2$

For the following exercises, solve the quadratic equation using the quadratic formula.

? Exercise 5.6.0.28

$16p^2 = 24p - 9$

? Exercise 5.6.0.29

$m^2 - 2m = 1$

? Exercise 5.6.0.30

$4x^2 - 2x - 3 = 0$

? Exercise 5.6.0.31

$y^2 - 6y + 7 = 0$

? Exercise 5.6.0.32

$x^2 - 3x - 7 = 0$

? Exercise 5.6.0.33

$2x^2 - 4x - 11 = 0$

? Exercise 5.6.0.34

The product of two consecutive odd integers is 99. Find the integers.

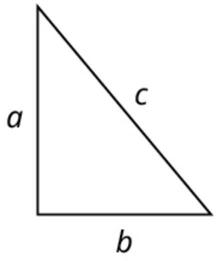
? Exercise 5.6.0.35

The product of two consecutive even integers is 168. Find the integers.

? Exercise 5.6.0.36

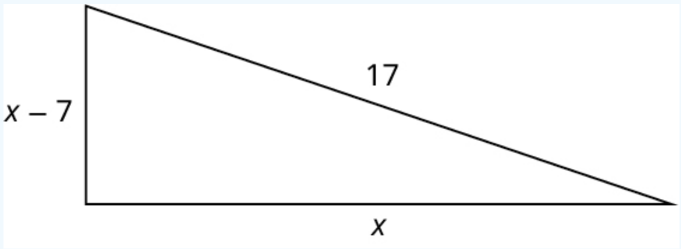
A rectangular patio has an area of 180 square feet. The width of the patio is three feet less than the length. Find the length and width of the patio.

For the following exercises, use the Pythagorean Theorem: In any right triangle, where a and b are the lengths of the legs and c is the length of the hypotenuse, as shown in the given figure, $a^2 + b^2 = c^2$.



? Exercise 5.6.0.37

Justine wants to put a deck in the corner of her backyard in the shape of a right triangle, as shown in the given figure. The hypotenuse will be 17 feet long. The length of one side will be 7 feet less than the length of the other side. Find the lengths of the sides of the deck.



? Exercise 5.6.0.38

A boat's sail is a right triangle. The length of one side of the sail is 7 feet more than the other side. The hypotenuse is 13. Find the lengths of the two sides of the sail.

? Exercise $5.6.0.39$

The sun casts a shadow from a flagpole. The height of the flagpole is three times the length of its shadow. The distance between the end of the shadow and the top of the flagpole is 21 feet. Find the length of the shadow and the length of the flagpole. Round to the nearest tenth of a foot.

? Exercise $5.6.0.40$

Rene is setting up a holiday light display. He wants to make a “tree” in the shape of two right triangles and has two 10-foot strings of lights to use for the sides. He will attach the lights to the top of a pole and to two stakes on the ground. He wants the height of the pole to be the same as the distance from the base of the pole to each stake. How tall should the pole be? Round to the nearest tenth of a foot.

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5.7: Functions

Learning Objectives

1. Use function notation.
2. Determine if a relation is a function with different representations.
3. Apply the vertical line test.
4. Determine the domain and range of a function.

In this section, we will learn about relations and functions. As we go about our daily lives, we have many data items or quantities that are paired to our names. Our social security number, student ID number, email address, phone number, and our birthday are matched to our name. There is a relationship between our name and each of those items. When your teacher gets their class roster, the names of all the students in the class are listed in one column and then the student ID number is likely to be in the next column. If we think of the correspondence as a set of ordered pairs, where the first element is a student name and the second element is that student's ID number, we call this a relation.

(Student name, Student ID #)

The set of all the names of the students in the class is called the domain of the relation and the set of all student ID numbers paired with these students is the range of the relation. In general terms, a relation is any set of ordered pairs, (x, y). All the x-values in the ordered pairs together make up the domain. All the y-values in the ordered pairs together make up the range.

Figure $5.7.1$: A small group of elementary students learning from their teacher. (credit: modification of work "Our school" by Woodleywonderworks/Flickr, CC BY 2.0)

There are many situations similar to the student's name and student ID # where one variable is paired or matched with another. The set of ordered pairs that records this matching is a relation. A special type of relation, called a function, occurs extensively in mathematics. A function is a relation that assigns to each element in its domain exactly one element in the range. For each ordered pair in the relation, each x-value is matched with only one y-value.

Let us look at the relation between your friends and their birthdays in Figure $5.7.2$. Every friend has a birthday, but no one has two birthdays. It is okay for two people to share a birthday. It is okay that Danny and Stephen share July 24 as their birthday and that June and Liz share August 2. Since each person has exactly one birthday, the relation is a function.

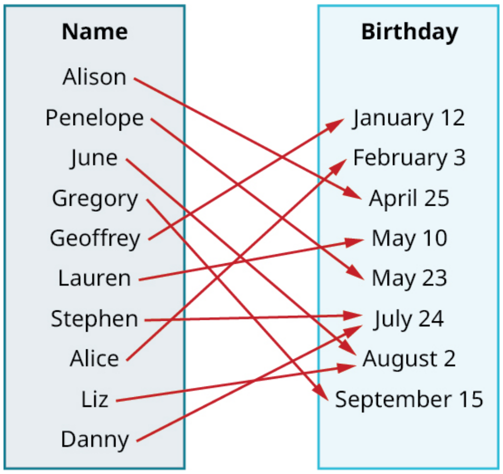


Figure $5.7.2$: Birthday Mapping

Use Function Notation

It is very convenient to name a function; most often functions are named f, g, h, F, G, or H. In any function, for each x-value from the domain, we get a corresponding y-value in the range. In the function f, we write this range value y as $f(x)$. This notation $f(x)$ is called function notation and is read "f of x " or "the value of f at x." In this case the parentheses do not indicate multiplication.

We call x the independent variable as it can be any value in the domain. We call y the dependent variable as its value depends on x. Much like when you first encountered the variable x, function notation may be rather unsettling. But the more you use the notation, the more familiar you become with the notation, and the more comfortable you will be with it.
Let's review the equation $y=4x-5$. To find the value of y when $x=2$, we know to substitute $x=2$ into the equation and then simplify.

	$y=4x-5$
Let $x=2$.	$y=4\cdot 2-5$ $y=3$

The value of the function at $x=2$ is 3 . We do the same thing using function notation, the equation $y=4x-5$ can be written as $f(x)=4x-5$. To find the value when $x=2$, we write:

	$f(x)=4x-5$
Let $x=2$.	$f(2)=4\cdot 2-5$ $f(2)=3$

The value of the function at $x=2$ is 3. This process of finding the value of $f(x)$ for a given value of x is called evaluating the function.

✓ Example $5.7.1$: Evaluating the Function

For the function $f(x)=2x^2+3x-1$, evaluate the function.

1. $f(3)$
2. $f(-2)$
3. $f(a)$

Answer

1. To evaluate $f(3)$, substitute 3 , for x.

Simplify.

$$f(x)=2x^2+3x-1$$
$$f(\mathbf{3})=2(\mathbf{3})^2+3\cdot\mathbf{3}-1$$

$$\begin{aligned}f(3)&=2\cdot 9+3\cdot 3-1\\ f(3)&=18+9-1\\ f(3)&=26\end{aligned}$$

2. To evaluate $f(-2)$, substitute -2 for x.

Simplify.

$$\begin{aligned}f(x)&=2x^2+3x-1\\ f(-2)&=2(-2)^2+3(-2)-1\\ f(-2)&=2\cdot 4+(-6)-1\\ f(-2)&=8+(-6)-1\\ f(-2)&=1\end{aligned}$$

3. To evaluate $f(a)$, substitute a for x.

Simplify.

$$\begin{aligned}f(x)&=2x^2+3x-1\\ f(\boldsymbol{a})&=2(\boldsymbol{a})^2+3\cdot\boldsymbol{a}-1\\ f(a)&=2a^2+3a-1\end{aligned}$$

Your Turn 5.7.1

For the function $f(x)=3x^2-2x+1$, evaluate the function.

1. $f(3)$
2. $f(-1)$
3. $f(t)$

Example 5.7.2: Evaluating the Function in an Application

The number of unread emails in Sylvia's inbox is 75 . This number grows by 10 unread emails a day. The function $N(t)=75+10t$ represents the relation between the number of emails, N, and the time, t, measured in days. Find $N(5)$. Explain what this result means.

Answer

Find $N(5)$. Explain what this result means.
Substitute in $t=5$.
Simplify.

$$\begin{aligned}N(\mathbf{5})&=75+10\cdot\mathbf{5}\\ N(t)&=75+10t\\ N(5)&=75+50\\ N(5)&=125\end{aligned}$$

If 5 is the number of days, $N(5)$ is the number of unread emails after 5 days. After 5 days, there are 125 unread emails in Sylvia's inbox.

Your Turn 5.7.2

The number of unread emails in Bryan’s account is 100. This number grows by 15 unread emails a day. The function $N(t)=100+15t$ represents the relation between the number of emails, N, and the time, t, measured in days. Find $N(7)$. Explain what the result means.

Determining If a Relation Is a Function with Different Representations

We can determine whether a relation is a function by identifying the input and the output values. If each input value leads to only one output value, classify the relation as a function. If any input value leads to two or more outputs, do not classify the relation as a function.

We will review three different representations of relations and determine if they are functions: ordered pairs, mapping, and equations.

✓ Example $5.7.3$: Determining If a Relation Is a Function with a Set of Ordered Pairs

Use the set of ordered pairs to determine whether the relation is a function.

1. $\{(-3, 27), (-2, 8), (-1, 1), (0, 0), (1, 1), (2, 8), (3, 27)\}$
2. $\{(9, -3), (4, -2), (1, -1), (0, 0), (1, 1), (4, 2), (9, 3)\}$

Answer

1. Each x-value is matched with only one y-value. This relation is a function.
2. The x-value 9 is matched with two y-values, both 3 and -3. This relation is not a function.

Your Turn $5.7.3$

Use the set of ordered pairs to determine whether the relation is a function.

1. $\{\left(-3, -6\right), \left(-2, -4\right), \left(-1, -2\right), \left(0, 0\right), \left(1, 2\right), \left(2, 4\right), \left(3, 6\right)\}$
2. $\{(8, -4), (4, -2), (2, -1), (0, 0), (2, 1), (4, 2), (8, 4)\}$

A mapping is sometimes used to show a relation. The arrows show the pairing of the elements of the domain with the elements of the range. Consider the example of the relation between your friends and their birthdays used in Figure 5.60. In this particular example, the domain is the set of people's names, and the range is the set of their birthdays. This mapping was a function because everybody's name maps to exactly one birthday.

✓ Example $5.7.4$: Determining If a Relation Is a Function with Mapping

Use the mapping in Figure ✓ $5.7.3$ to determine whether the relation is a function.

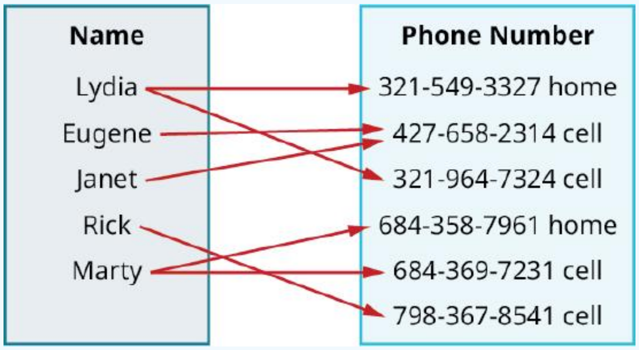


Figure $5.7.3$

Answer

Both Lydia and Marty have two phone numbers. Each x-value is not matched with only one y-value. This relation is not a function.

Your Turn 5.7.5

Use the mapping in the given figure to determine whether the relation is a function.

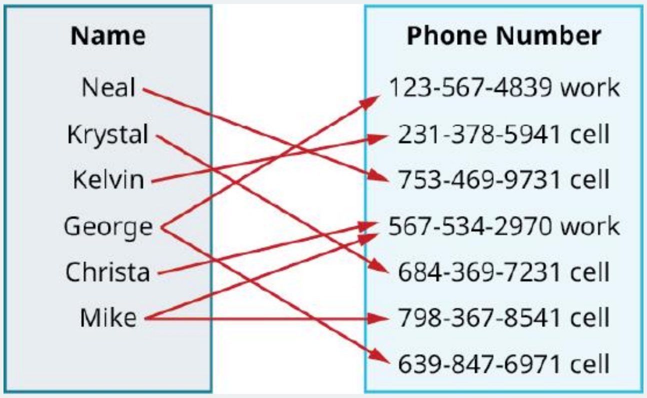

Name
Neal
Krystal
Kelvin
George
Christa
Mike
Phone Number
123-567-4839 work
231-378-5941 cell
753-469-9731 cell
567-534-2970 work
684-369-7231 cell
798-367-8541 cell
639-847-6971 cell

Figure 5.7.4

In algebra functions will usually be represented by an equation. It is easiest to see if the equation is a function when it is solved for y. If each value of x results in only one value of y, then the equation defines a function.

Exercise 5.7.6

Determine whether each equation is a function. Assume x is the independent variable.

1. $2x + y = 7$
2. $y = x^2 + 1$
3. $x + y^2 = 3$

Answer

1. $2x + y = 7$

For each value of x, we multiply it by -2 and then add 7 to get the y-value.
For example, if $x = 3$:

$$\begin{aligned}y &= -2x + 7\\ y &= -2 \cdot \mathbf{3} + 7\\ y &= 1\end{aligned}$$

We have that when $x = 3$, then $y = 1$. It would work similarly for any value of x. Since each value of x, corresponds to only one value of y the equation defines a function.

2. $y = x^2 + 1$

For each value of x, we square it and then add 1 to get the y-value.
For example, if $x = 2$

$$\begin{aligned}y &= x^2 + 1\\ y &= \mathbf{2}^2 + 1\\ y &= 5\end{aligned}$$

We have that when $x = 2$, then $y = 5$. It would work similarly for any value of x. Since each value of x corresponds to only one value of y, the equation defines a function.

3. $x + y^2 = 3$

$$x + y^2 = 3$$

Isolate the y term.

$$y^2 = -x + 3$$

Let us substitute $x=2$.

$$y^2=-\mathbf{2}+3$$
$$y^2=1$$

This gives us two values for y.

$$y=1, y=-1$$

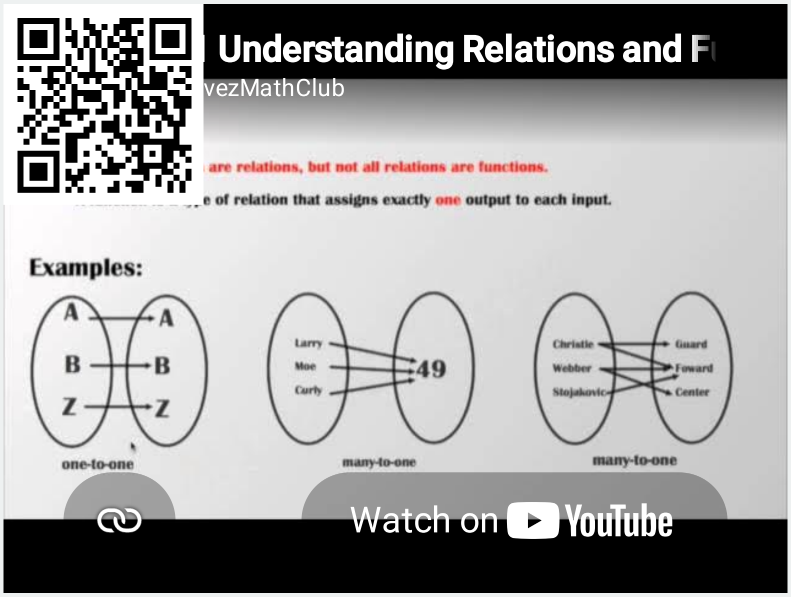
We have shown that when $x=2$, then $y=1$ and $y=-1$. It would work similarly for any value of x. Since each value of x does not corresponds to only one value of y the equation does not define a function.

Your Turn 5.7.6

Determine whether each equation is a function.

1. \($4x+y=-3$\)
2. \ $(x+\{y^{\wedge}2\}=1\backslash)$
3. \ $(y-\{x^{\wedge}2\}=2\backslash)$

Video

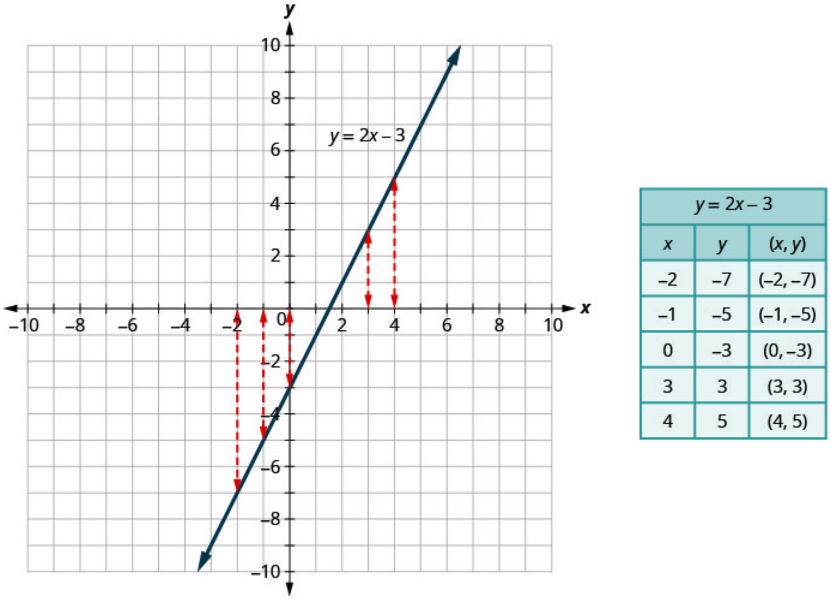

Understanding Relations and F
vezMathClub
are relations, but not all relations are functions.
e of relation that assigns exactly one output to each input.
Examples:
A
B
Z
A
B
Z
one-to-one
Larry
Moe
Curly
49
many-to-one
Christie
Webber
Stojakovic
Guard
Foward
Center
many-to-one
Watch on YouTube

Relations and Functions

Applying the Vertical Line Test

We reviewed how to determine if a relation is a function. The relations we looked at were expressed as a set of ordered pairs, a mapping, or an equation. We will now cover how to tell if a graph is that of a function.

An ordered pair (x, y) is a solution of a linear equation, if the equation is a true statement when the x-values and y -values of the ordered pair are substituted into the equation. The graph of a linear equation is a straight line where every point on the line is a solution of the equation, and every solution of this equation is a point on this line. Figure



$y = 2x - 3$		
x	y	(x, y)
–2	–7	(–2, –7)
–1	–5	(–1, –5)
0	–3	(0, –3)
3	3	(3, 3)
4	5	(4, 5)

Figure 5.58: Graph of te Equation $y = 2x - 3$

A relation is a function if every element of the domain has exactly one value in the range. The relation defined by the equation Math input errorvertical line test, which states that a set of points in a rectangular coordinate system is the graph of a function if every vertical line intersects the graph in at most one point. If any vertical line intersects the graph in more than one point, the graph does not represent a function.

✓ Example 5.7.7: Applying the Vertical Line Test

Determine whether the graph (Figure ✓ 5.7.6) is the graph of a function applying the vertical line test.

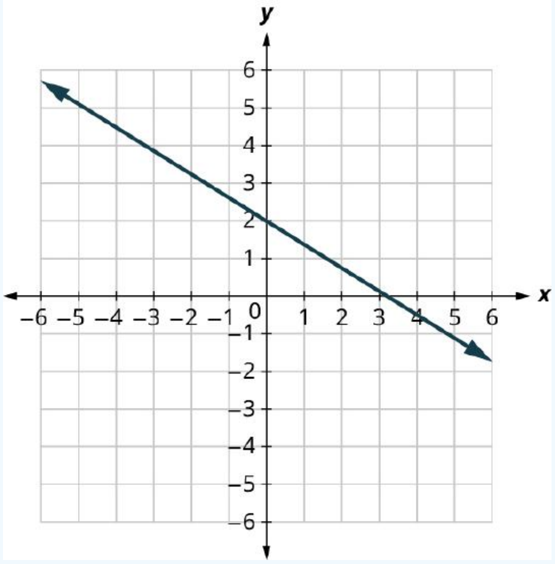


Figure ✓ 5.7.6

Answer

On the graph (Figure ✓ 5.7.7), only three vertical dashed lines are drawn. However, it can be determined that any vertical dashed line that is drawn will intersect the solid line at exactly one point. It is the graph of a function.

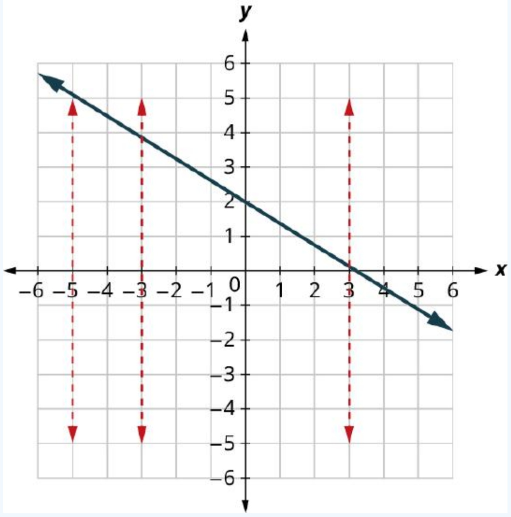


Figure ✓ 5.7.7

🖈 Your Turn 5.7.7

Determine whether the graph is the graph of a function.

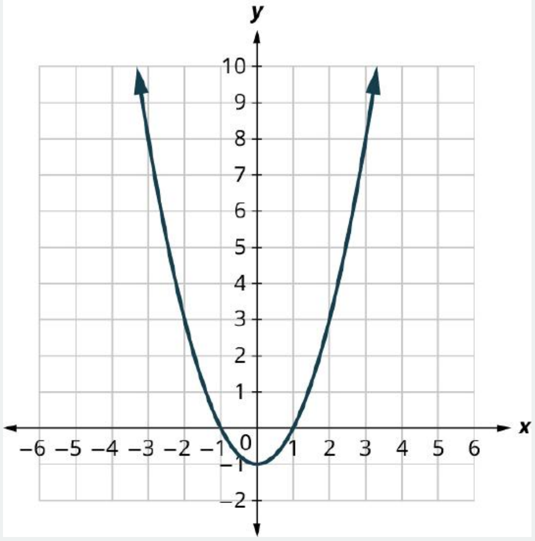


Figure 🖈 5.7.8

✓ Example 5.7.8: Applying the Vertical Line Test to a Parabola

Determine whether the graph is the graph of a function (Figure ✓ 5.7.9).

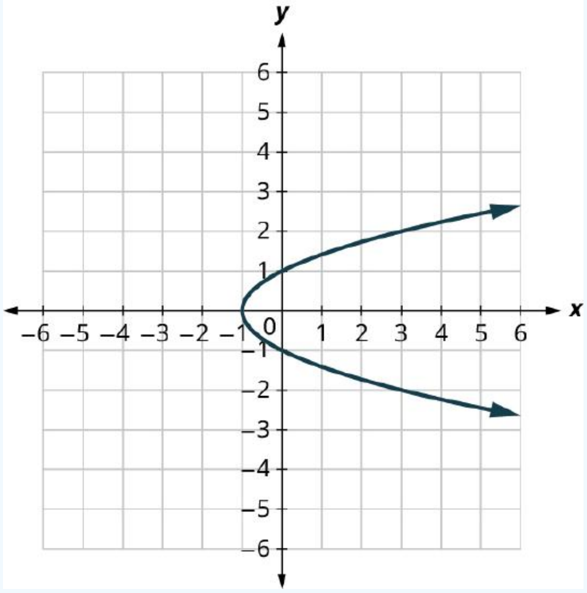


Figure ✓ 5.7.9

Your Turn 5.7.8

Determine whether the graph is the graph of a function.

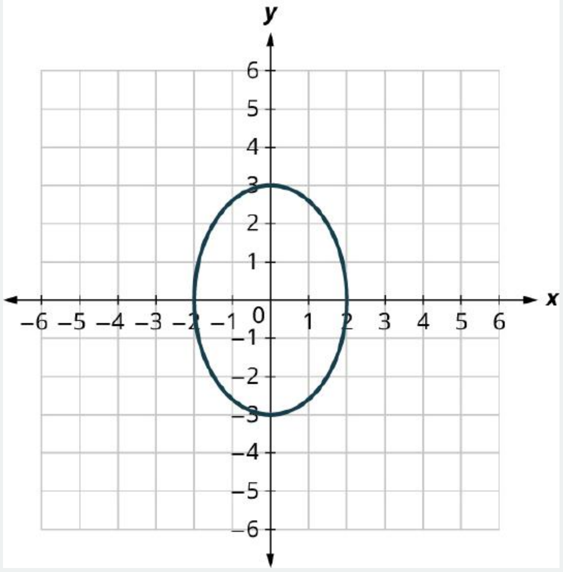


Figure 5.7.10

Determining the Domain and Range of a Function

For the function $y=f(x)$, x is the independent variable as it can be any value in the domain, and y is the dependent variable since its value depends on x. For the function $y=f(x)$, the values of x make up the domain and the values of y make up the range.

✓ Example 5.7.9: Finding the Domain and Range of Ordered Pairs

For $\{(1,1),(2,4),(3,9),(4,16),(5,25)\}$:

1. Find the domain of the relation.
2. Find the range of the relation.

Answer

1. The domain is the set of all x-values of the relation: $\{1, 2, 3, 4, 5\}$
2. The range is the set of all y-values of the relation: $\{1, 4, 9, 16, 25\}$

Your Turn $5.7.9$

For the relation $\{(1,1), (2,8), (3,27), (4,64), (5,125)\}$:

1. Find the domain of the relation.
2. Find the range of the relation.

✓ Example $5.7.10$: Finding the Domain and Range on a Graph

Use Figure $5.7.11$ to:

1. List the ordered pairs of the relation.
2. Find the domain of the relation.
3. Find the range of the relation.

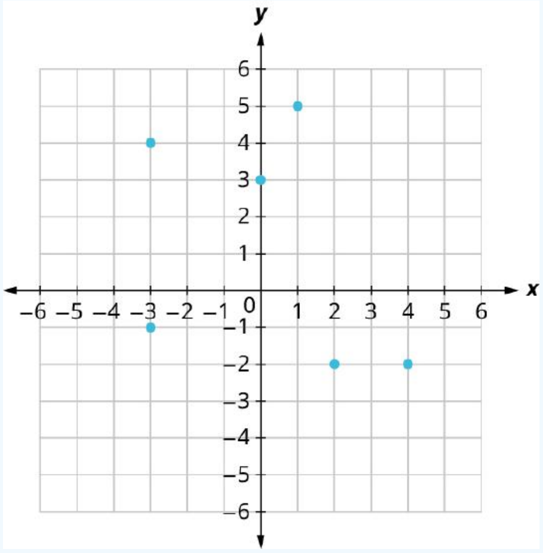


Figure ✓ $5.7.11$

Answer

1. The ordered pairs of the relation are: $\{(1,5), (-3,-1), (4,-2), (0,3), (2,-2), (-3,4)\}$.
2. The domain is the set of all x-values of the relation: $\{-3, 0, 1, 2, 4\}$. Notice that while -3 repeats, it is only listed once.
3. The range is the set of all y-values of the relation: $\{-2, -1, 3, 4, 5\}$. Notice that while -2 repeats, it is only listed once.

Your Turn $5.7.10$

Use the given figure to:

1. List the ordered pairs of the relation.
2. Find the domain of the relation.
3. Find the range of the relation.

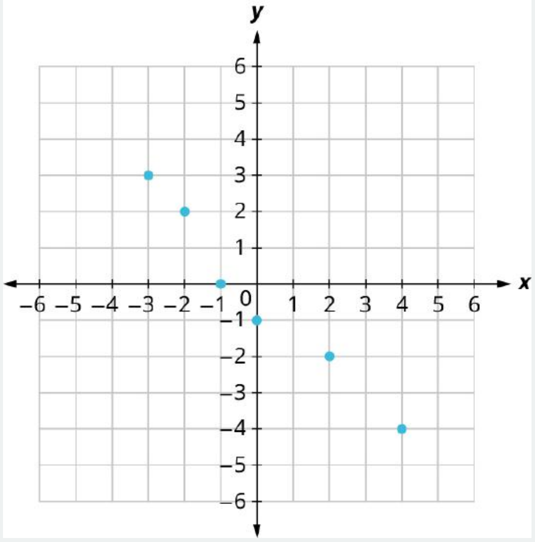


Figure 5.7.12

Video

termining Domain and Range
hispower4u
troduction to
termining the
Domain and Range of
Functions
Watch on YouTube

Domain and Range on Graphs

Who Knew?: Function and Function Notation

In 1673, Gottfried Leibniz, the German mathematician who co-invented calculus, seems to be the first person to use the word *function* in a mathematical sense, although his use of it does not exactly fit with the modern use and definition. The person who is credited with the modern definition of function is Swiss mathematician Johann Bernoulli, who wrote about it in a letter to Leibniz in 1698. Supposedly, Leibniz wrote Bernoulli back, approving of this use of the word. In 1734, the use of the notation $f(x)$ for a function was first used by Swiss mathematician Leonhard Euler (pronounced "Oiler"). Euler had a knack for inventing notation. He also introduced the notation e for the base of natural logs (1727), i for the square root of -1 (1777), $\sum$ for summation (1755), and many others. Euler also introduced many other ideas associated with functions. Euler defined exponential functions and defined logarithmic functions as their inverse; he also introduced the beta and gamma functions, and was the first person to consider the trigonometric identities (sine, cosine, etc.) as functions.

Check Your Understanding

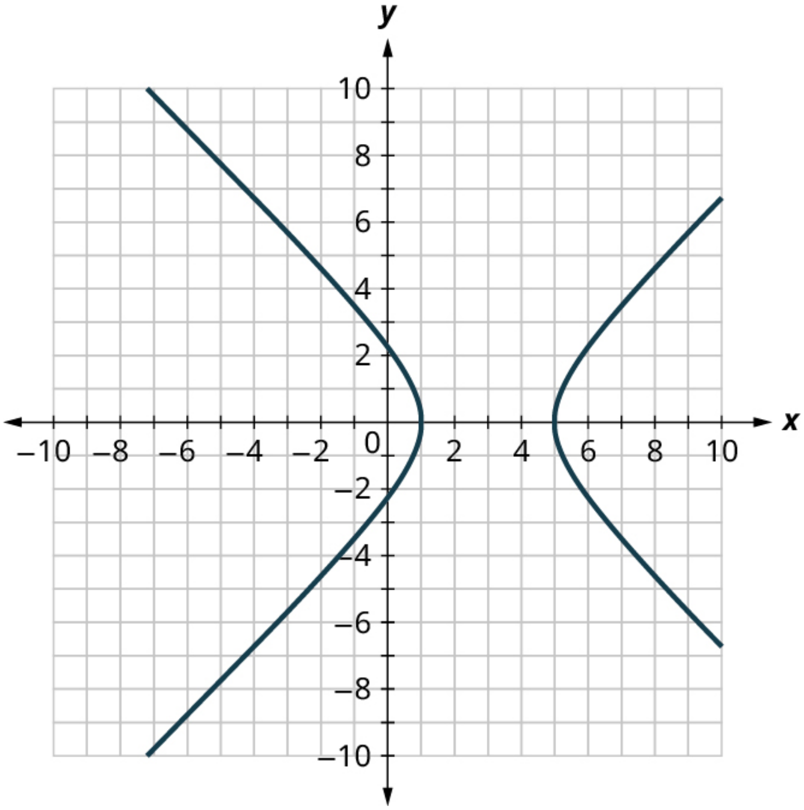
1.. If $f(x) = 2x - 8$ then $f(3) = -2$.

a. True
b. False

2.. $\{(1,2), (2,3), (3,4), (2,1), (3,2), (4,3)\}$ represent the ordered pairs of a function.

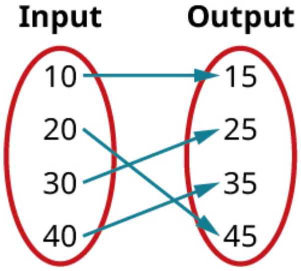
a. True
b. False

3.. The graph shown represents the graph of a function:



a. True
b. False

4.. The figure shown represents the mapping of a function.



a. True
b. False

5.. The domain of the mapping in the figure is $\{15, 25, 35, 45\}$.

a. True
b. False

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5.7.1: Exercises

For the following exercises, evaluate the functions at the values $f\left(-2\right), f\left(-1\right), f\left(0\right), f\left(1\right)$, and $f\left(2\right)$.

Exercise 5.7.1.1

$f\left(x\right)=4-2x$

Exercise 5.7.1.2

$f\left(x\right)=8-3x$

Exercise 5.7.1.3

$f\left(x\right)=8x^2-7x+3$

Exercise 5.7.1.4

$f\left(x\right)=3+\sqrt{x+3}$

Exercise 5.7.1.5

$f(x)=\frac{x-2}{x+3}$

Exercise 5.7.1.6

$f(x)=x^3$

For the following exercises, determine whether the ordered pairs represent a function.

Exercise 5.7.1.7

$\left\{\left(-1,-1\right), \left(-2,-2\right), \left(-3,-3\right)\right\}$

Exercise 5.7.1.8

$\left\{\left(3,4\right), \left(4,5\right), \left(5,6\right)\right\}$

Exercise 5.7.1.9

$\left\{\left(2,5\right), \left(7,11\right), \left(15,8\right), \left(7,9\right)\right\}$

Exercise 5.7.1.10

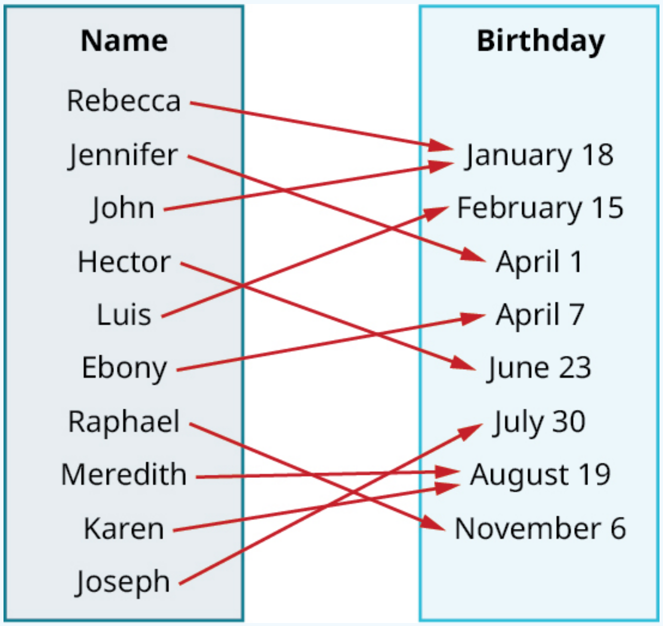
$\left\{\left(-3\ ,9\right), \left(-2,4\right), \left(-1,1\right), \left(0,0\right), \left(1,1\right), \left(2,4\right), \left(3,9\right)\right\}$

Exercise 5.7.1.11

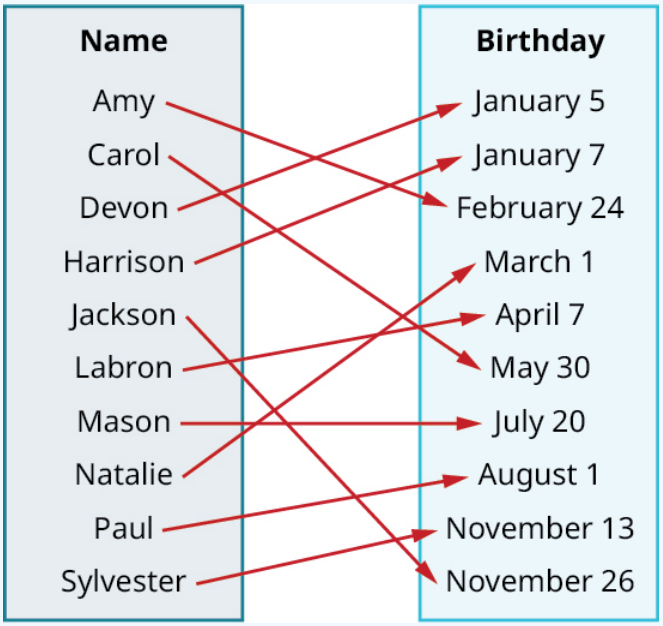
$\left\{\left(9,-3\right), \left(4,-2\right), \left(1,-1\right), \left(0,0\right), \left(1,1\right), \left(4,\ 2\right), \left(9,3\right)\right\}$

For the following exercises, determine whether the mapping represented a function.

Exercise 5.7.1.12


Name
Rebecca
Jennifer
John
Hector
Luis
Ebony
Raphael
Meredith
Karen
Joseph
Birthday
January 18
February 15
April 1
April 7
June 23
July 30
August 19
November 6

Exercise 5.7.1.12


Name
Amy
Carol
Devon
Harrison
Jackson
Labron
Mason
Natalie
Paul
Sylvester
Birthday
January 5
January 7
February 24
March 1
April 7
May 30
July 20
August 1
November 13
November 26

For the following exercises, determine whether the equations represent y as a function of x.

Exercise 5.7.1.14

$5x+2y=10$

? Exercise 5.7.1.15

$y=x^2$

? Exercise 5.7.1.16

$x=y^2$

? Exercise 5.7.1.17

$3x^2+y=14$

? Exercise 5.7.1.18

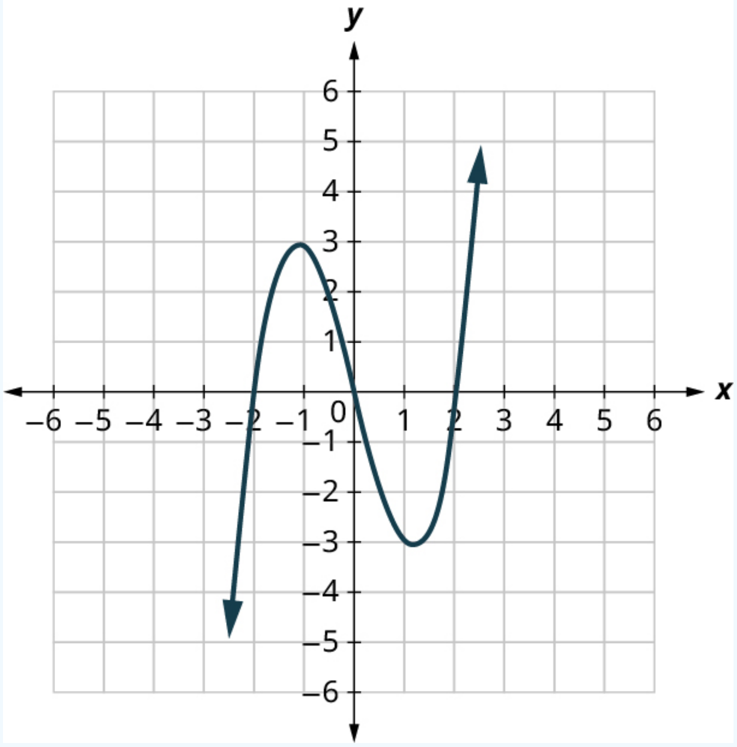
$2x+y^2=6$

? Exercise 5.7.1.19

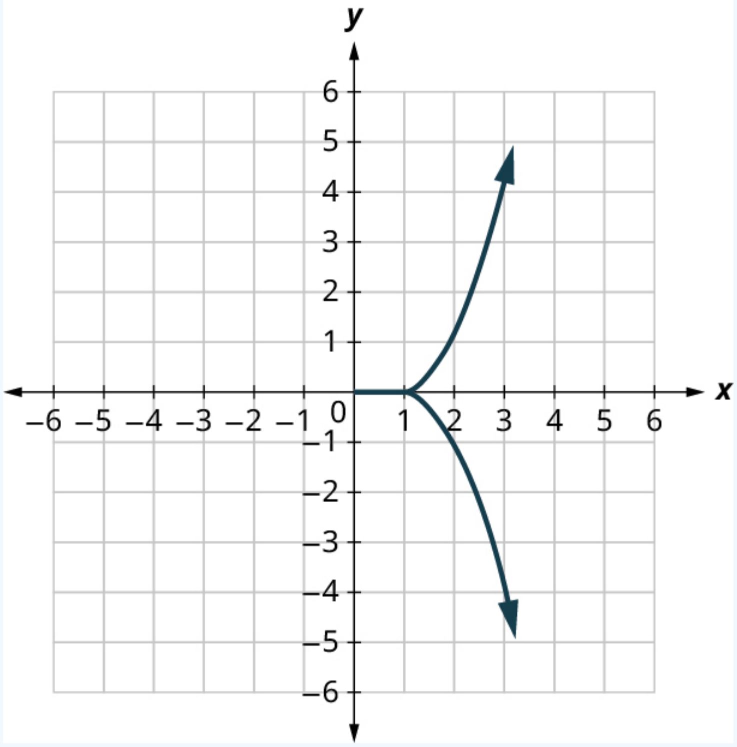
$y=-2x^2+40x$

For the following exercises, use the vertical line test to determine which graph represents a function.

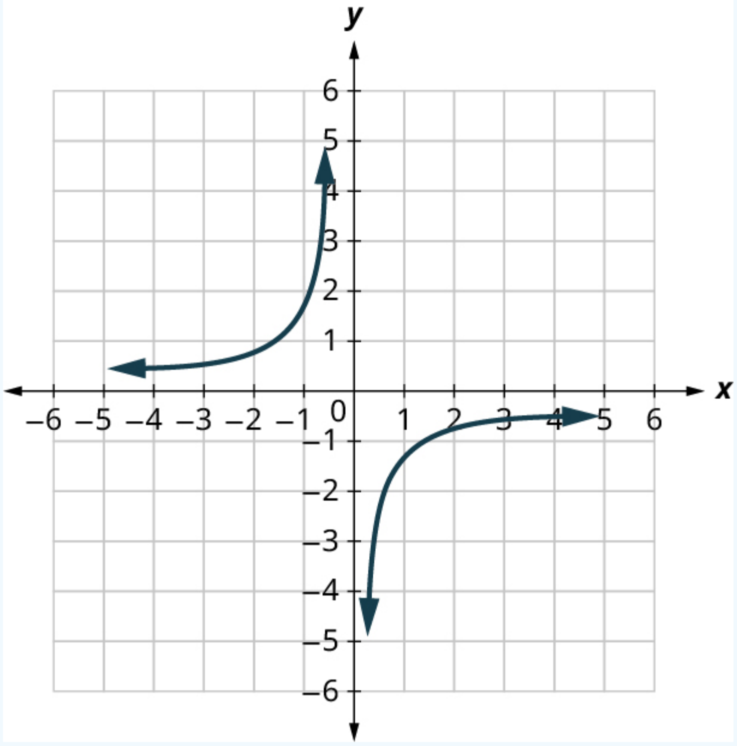
? Exercise 5.7.1.20



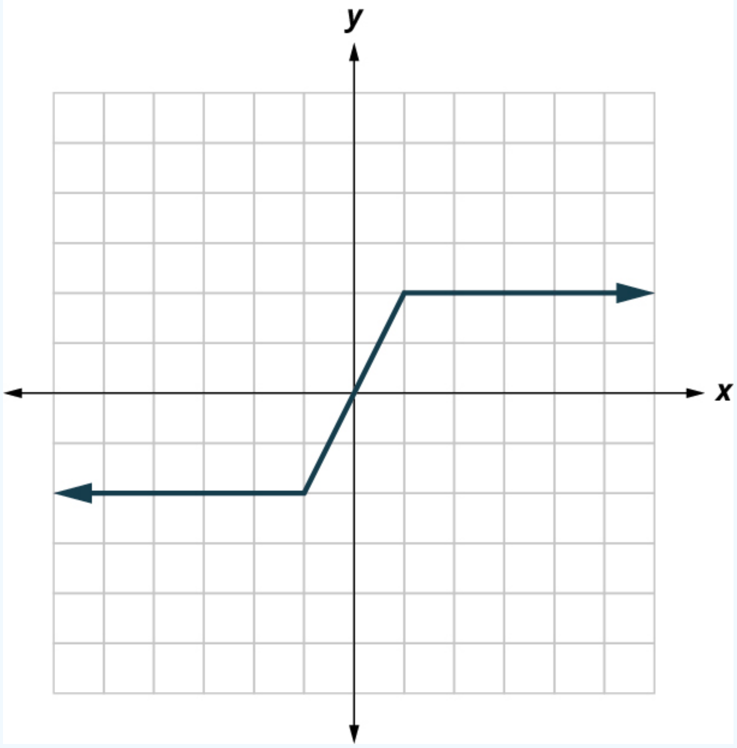
? Exercise 5.7.1.21



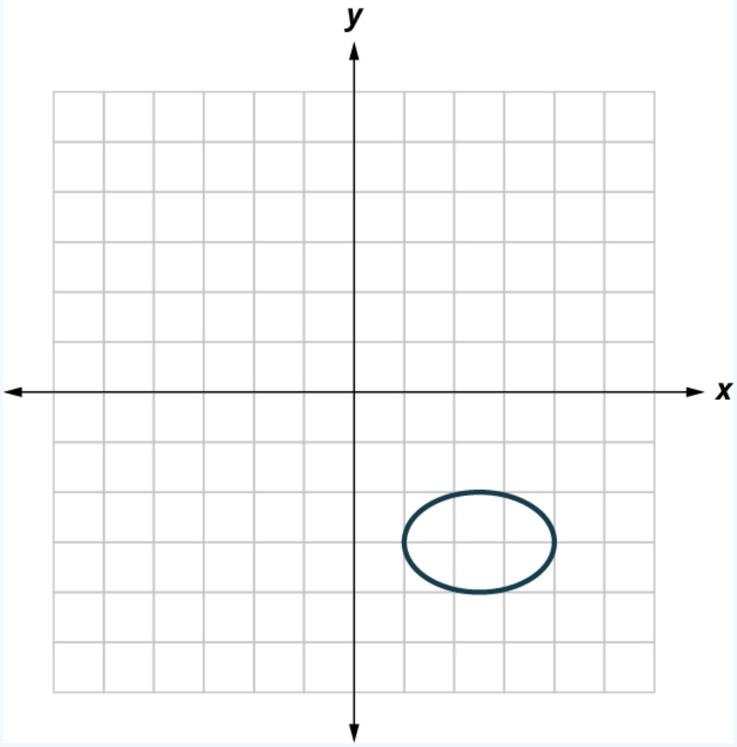
Exercise $5.7.1.22$



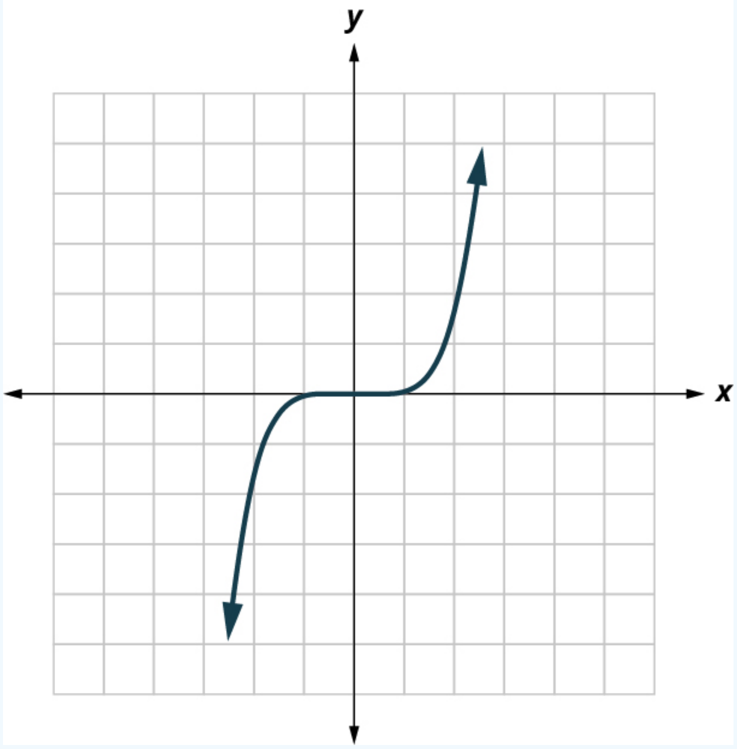
Exercise 5.7.1.23



Exercise 5.7.1.24



Exercise 5.7.1.25



For the following exercises, use the set of ordered pairs to find the domain and the range.

Exercise 5.7.1.26

$\{(-3,9),(-2,4),(-1,1),(0,0),(1,1),(2,4),(3,9)\}$

Exercise 5.7.1.27

$\{(-9,-3),(-4,-2),(-1,-1),(0,0),(1,1),(4,2),(9,3)\}$

Exercise 5.7.1.28

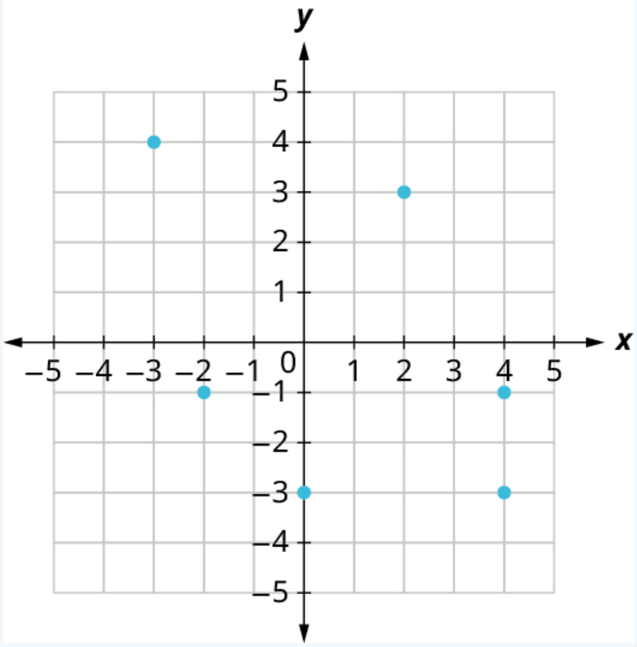
$\{(-3,27),(-2,8),(-1,1),(0,0),(1,1),(2,8),(3,27)\}$

Exercise 5.7.1.29

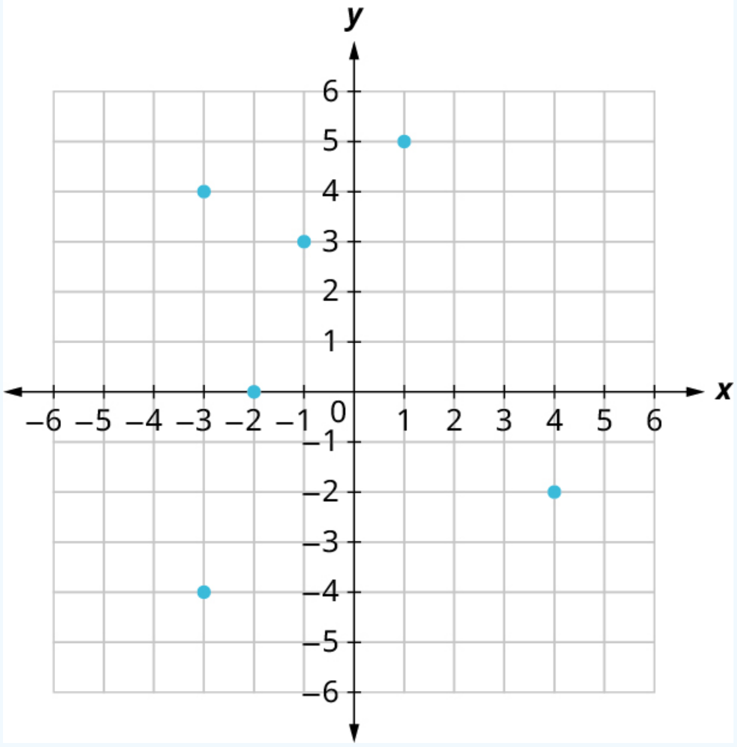
$\{(-3,-27),(-2,-8),(-1,-1),(0,0),(1,1),(2,8),(3,27)\}$

For the following exercises, use the graph to find the domain and the range.

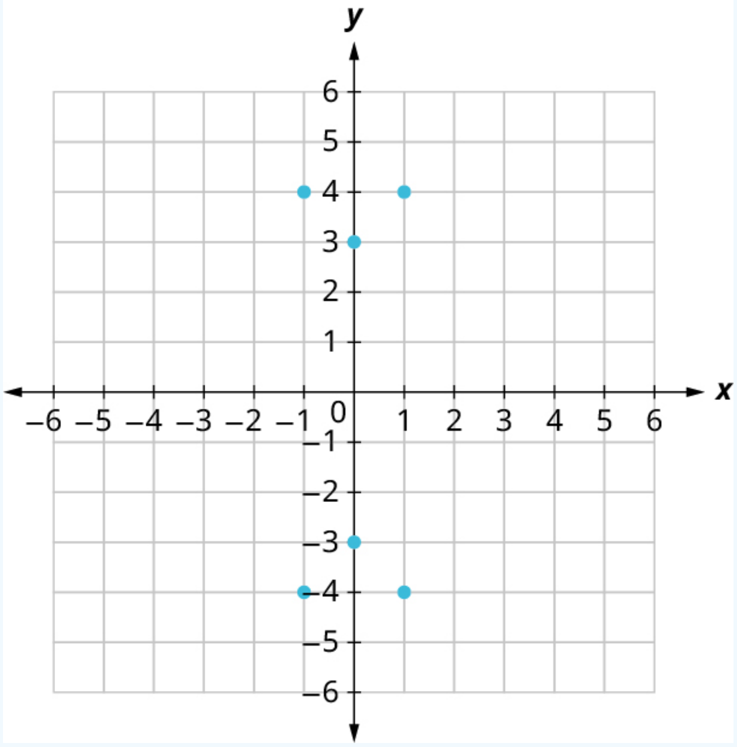
Exercise 5.7.1.30



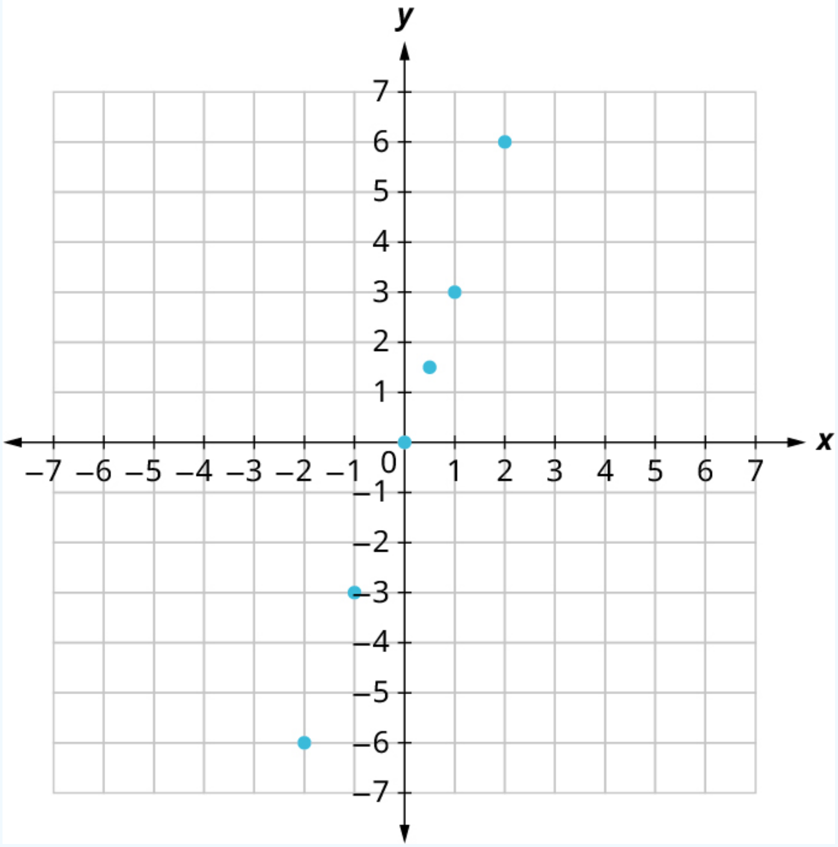
? Exercise 5.7.1.31



? Exercise 5.7.1.32



Exercise 5.7.1.33



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5.8: Graphing Functions

Learning Objectives

After completing this module, you should be able to:

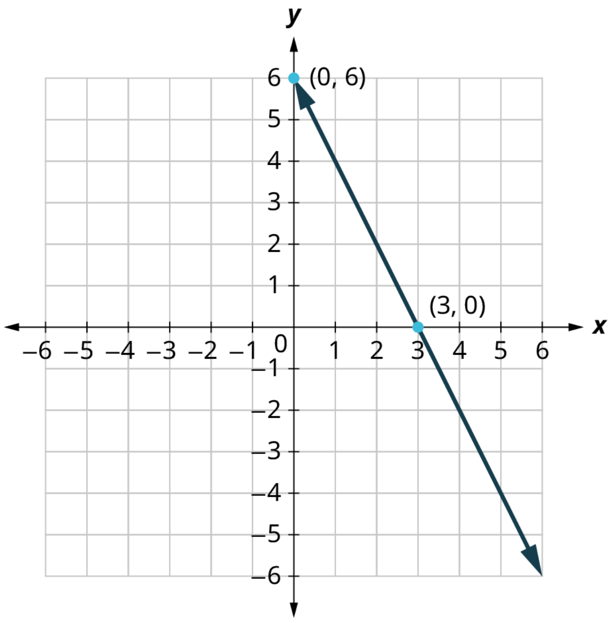
- Graph functions using intercepts.
- Compute slope.
- Graph functions using slope and y-intercept.
- Graph horizontal and vertical lines.
- Interpret graphs of functions.
- Model applications using slope and y-intercept.

In this section, we will expand our knowledge of graphing by graphing linear functions. There are many real-world scenarios that can be represented by graphs of linear functions. Imagine a chairlift going up at a ski resort. The journey a skier takes travelling up the chairlift could be represented as a linear function with a positive slope. The journey a skier takes down the slopes could be represented by a linear function with a negative slope.

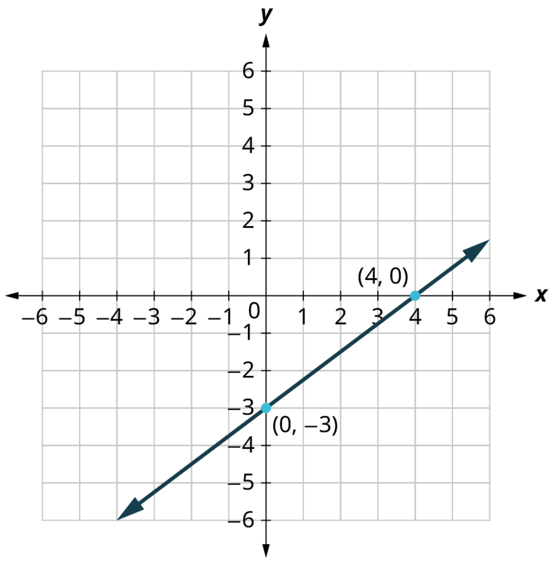
Figure $5.8.1$: The ski lifts and the mountain both have a slope. (credit: modification of work "colorado springs zoo tram" by woodleywonderworks/Flickr, CC BY 2.0)

Graphing Functions Using Intercepts

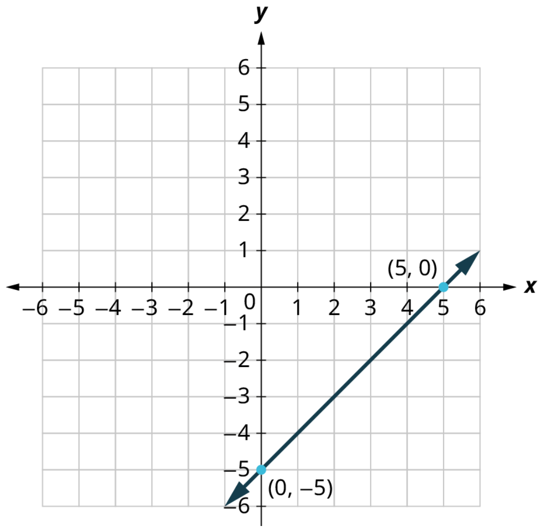
Every linear equation can be represented by a unique line that shows all the solutions of the equation. We have seen that when graphing a line by plotting points, you can use any three solutions to graph. This means that two people graphing the line might use different sets of three points. At first glance, their two lines might not appear to be the same, since they would have different points labeled. But if all the work was done correctly, the lines should be exactly the same. One way to recognize that they are indeed the same line is to look at where the line crosses the $\bar{x}$-axis and the y-axis. These points are called the intercepts of a line. Let us review the graphs of the lines in Figure $5.8.2$.



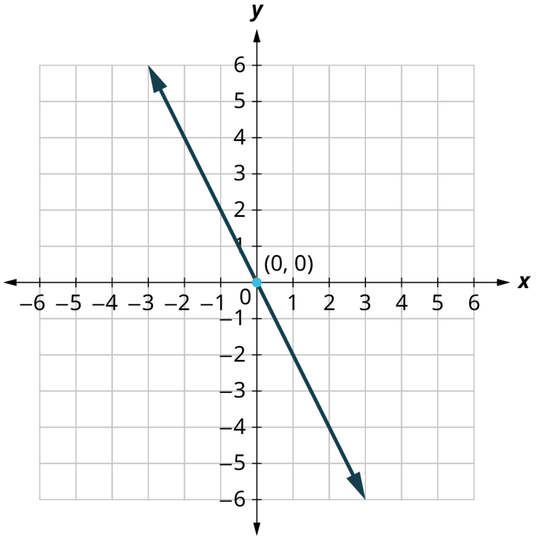
(a) $2x + y = 6$



(b) $3x - 4y = 12$



(c) $x - y = 5$



(d) $y = -2x$

Figure $5.8.2$

The table below lists where each of these lines crosses the x- and y-axis. Do you see a pattern? For each line, the y-coordinate of the point where the line crosses the x-axis is zero. The point where the line crosses the x-axis has the form $(a, 0)$ and is called the x-intercept of the line. The x-intercept occurs when y is zero. In each line, the x-coordinate of the point where the line crosses the y-axis is zero. The point where the line crosses the y-axis has the form $(0, b)$ and is called the y-intercept of the line. The y-intercept occurs when x is zero.

Figure	**The line crosses the x-axisx-axis at:**	**Ordered Pair for this Point**	**The line crosses the y-axisy-axis at:**	**Ordered Pair for This Point**
Figure (a)	3	$(3, 0)$	6	$(0, 6)$
Figure (b)	4	$(4, 0)$	-3	$(0, -3)$
Figure (c)	5	$(5, 0)$	-5	$(0, -5)$
Figure (d)	0	$(0, 0)$	0	$(0, 0)$
General Figure	a	$(a, 0)$	b	$(0, b)$

✓ Example $5.8.1$: Finding x- and y-Intercepts

Find the x-intercept and y-intercept on the (a) and (b) graphs in Figure $5.8.3$.

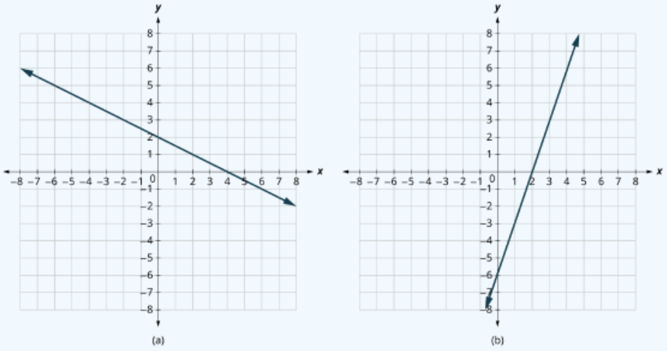


Figure $5.8.3$

Answer

In Figure $5.8.3$, the graph crosses the x-axis at the point $(4, 0)$. The x-intercept is $(4, 0)$. The graph crosses the y-axis at the point $(0, 2)$. The y-intercept is $(0, 2)$. In Figure 5.66 , the graph crosses the x-axis at the point $(2, 0)$. The x-intercept is $(2, 0)$. The graph crosses the y-axis at the point $(0, -6)$. The y-intercept is $(0, -6)$.

Your Turn $5.8.1$

Find the x-intercept and y-intercept on the given graph.

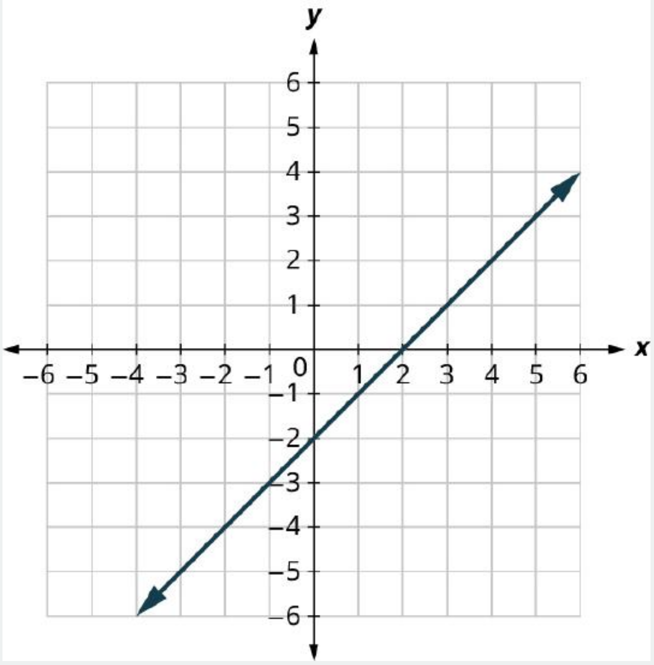


Figure $5.8.4$

Example $5.8.2$: Graphing a Function Using Intercepts

Find the intercepts of $2x + y = 8$. Then graph the function using the intercepts.

Answer

Let to find the x-intercept, and let to find the-intercept.

	$2x + y = 8$		$2x + y = 8$
To find the x-intercept, let $y = 0$.	$2x + 0 = 8$	To find the y intercept let $x = 0$.	$2(0) + y = 8$
Simplify.	$2x = 8$ $x = 4$	Simplify.	$y = 8$
The x-intercept is:	$(4, 0)$	The y-intercept is:	$(0, 8)$

Plot the intercepts to get the graph in Figure $5.8.5$.

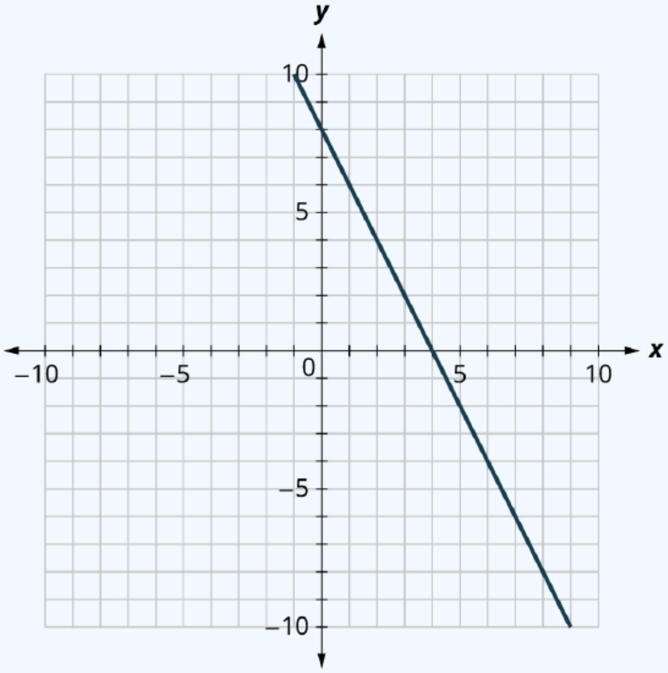


Figure $5.8.5$

Your Turn $5.8.2$

Find the intercepts of $3x+y=12$ and use them to graph the equation.

Computing Slope

When graphing linear equations, you may notice that some lines tilt up as they go from left to right and some lines tilt down. Some lines are very steep and some lines are flatter. In mathematics, the measure of the steepness of a line is called the slope of the line. To find the slope of a line, we locate two points on the line whose coordinates are integers. Then we sketch a right triangle where the two points are vertices of the triangle and one side is horizontal and one side is vertical. Next, we measure or calculate the distance along the vertical and horizontal sides of the triangle. The vertical distance is called the rise and the horizontal distance is called the run.

We can assign a numerical value to the slope of a line by finding the ratio of the rise and run. The rise is the amount the vertical distance changes while the run measures the horizontal change, as shown in this illustration. Slope (Figure $5.8.6$) is a rate of change.

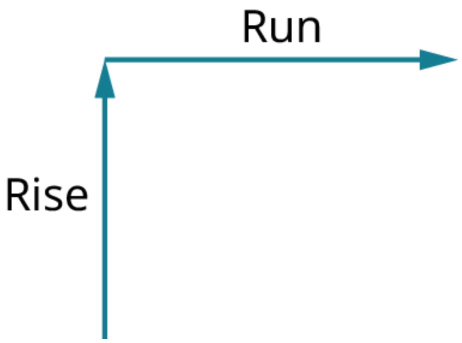


Figure $5.8.6$:

FORMULA

To calculate slope (m), use the formula

$$m=\frac{rise}{run}$$

> where the rise measures the vertical change and the run measures the horizontal change.

The concept of slope has many applications in the real world. In construction, the pitch of a roof, the slant of plumbing pipes, and the steepness of stairs are all applications of slope. As you ski or jog down a hill, you definitely experience slope.

✓ Example $5.8.3$: Finding the Slope from a Graph

Find the slope of the line shown in Figure $5.8.7$.

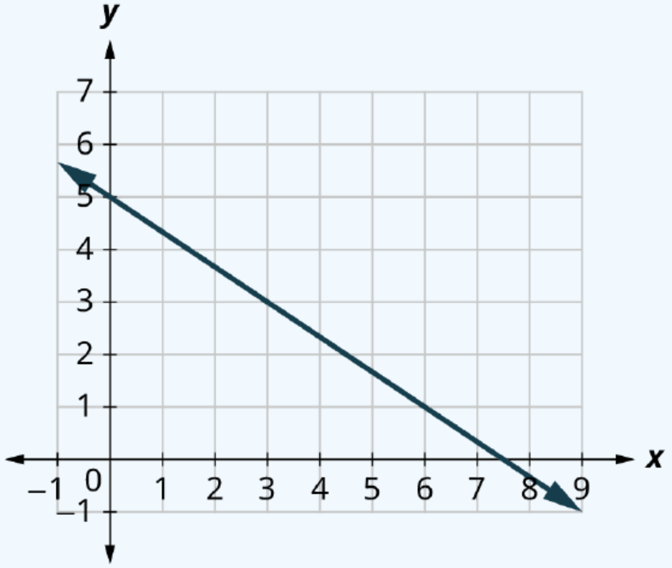


Figure $5.8.7$

Answer

Step 1: Locate two points on the graph whose coordinates are integers, such as (0,5) and (3,3). Starting at (0,5), sketch a right triangle to (3,3) as shown in Figure $5.8.8$.

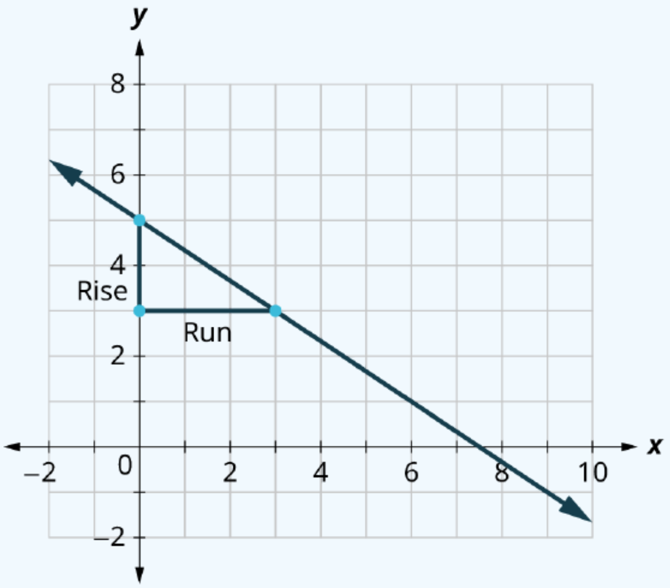


Figure $5.8.8$

Step 2: Count the rise; since it goes down, it is negative. The rise is −2.

Step 3: Count the run. The run is 3.

Step 4: Use the slope formula $m = \frac{rise}{run}$ substitute the values of the rise and run. $m = \frac{-2}{3}$

The slope of the line is $-\frac{2}{3}$.

The solution is y decreases by 2 units as x increases by 3 units.

Your Turn $5.8.3$

Find the slope of the line shown in the graph.

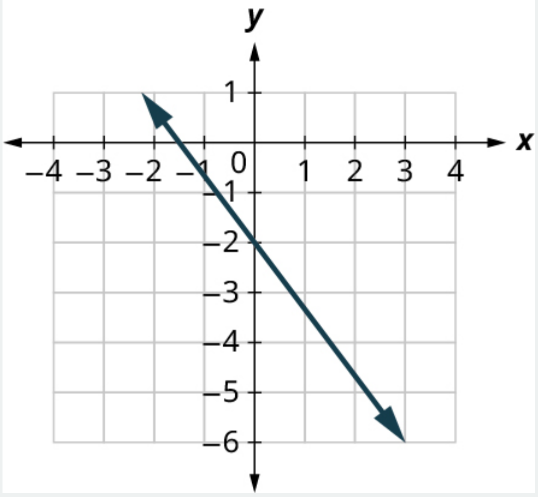


Figure $5.8.9$

Sometimes we will need to find the slope of a line between two points when we don't have a graph to measure the rise and the run. We could plot the points on grid paper, then count out the rise and the run, but there is a way to find the slope without graphing. First, we need to introduce some algebraic notation.

We have seen that an ordered pair (x, y) gives the coordinates of a point. But when we work with slopes, we use two points. How can the same symbol (x, y) be used to represent two different points? Mathematicians use subscripts to distinguish such points. For example, (x_1, y_1) would be said aloud as " x sub $1, y$ sub 1 " and (x_2, y_2) read " x sub 2, y sub 2." The "sub" is a short way of saying "subscript." We will use (x_1, y_1) to identify the first point and (x_2, y_2) to identify the second point in our slope equation. If we had more than two points, (if we were finding more than one slope), we could use ($x_3, y_3)$, (x_4, y_4), and so on.

Let's review how the rise and run relate to the coordinates of the two points by taking another look at the slope of the line between the points $(2, 3)$ and $(7, 6)$, as shown in Figure $5.8.10$.

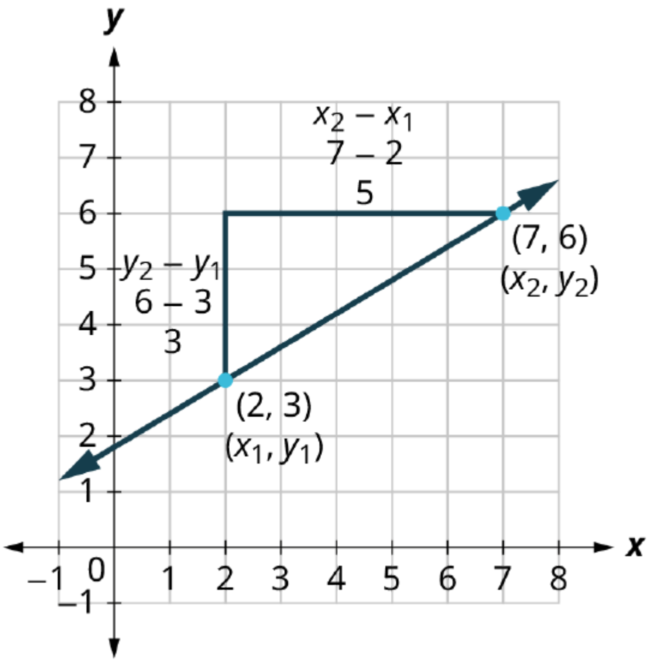


Figure 5.8.10

On the graph, we count the rise of 3 and the run of 5 . Notice on the graph that that (x_1, y_1) is the point $(2,3)$ and ($x_2, y_2)$ is the point $(7,6)$. The rise can be found by subtracting the y-coordinates, 6 and 3 , and the run can be found by subtracting the x-coordinates 7 and 2 .

$$m=\frac{rise}{run}=\frac{6-3}{7-2}=\frac{3}{5}$$

We have shown that $m=\frac{y_2-y_1}{x_2-x_1}$ is really another version of $m=\frac{rise}{run}$. We can use this formula to find the slope of a line.

FORMULA

To find the slope of the line between two points (x_1, y_1) and (x_2, y_2), use the formula

$$m=\frac{y_2-y_1}{x_2-x_1}$$

Exercise 5.8.4

Use the slope formula to find the slope of the line through the points (−2, −3) and (−7, 4).

Answer

We'll call $(-2,-3)$ point 1 and $(-7,4)$ point 2.

Step 1: Use the slope formula: $m=\frac{y_2-y_1}{x_2-x_1}$

Step 2: Substitute the values: $m=\frac{4-(-3)}{-7-(-2)}$

Step 3: Simplify: $m=\frac{7}{-5}=-\frac{7}{5}$

Step 4: Verify the slope on the graph shown in Figure 5.8.11.

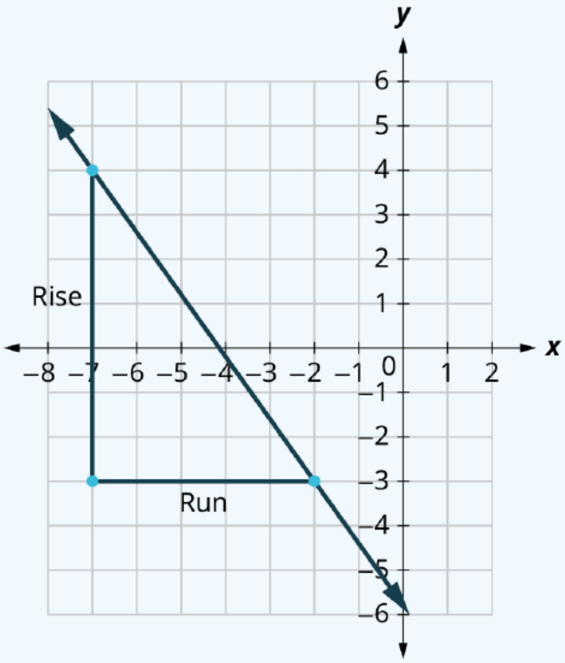


Figure $5.8.11$: Copy and Paste Caption here. (Copyright; author via source)

Your Turn 5.8.4

Use the slope formula to find the slope of the line through the pair of points $(-3, 4)$ and $(2, -1)$.

Graphing Functions Using Slope and y-Intercept

We have graphed linear equations by plotting points and using intercepts. Once we see how an equation in slopeintercept form and its graph are related, we will have one more method we can use to graph lines. Review the graph of the equation $y = \frac{1}{2}x + 3$ in Figure 5.73 and find its slope and y-intercept.

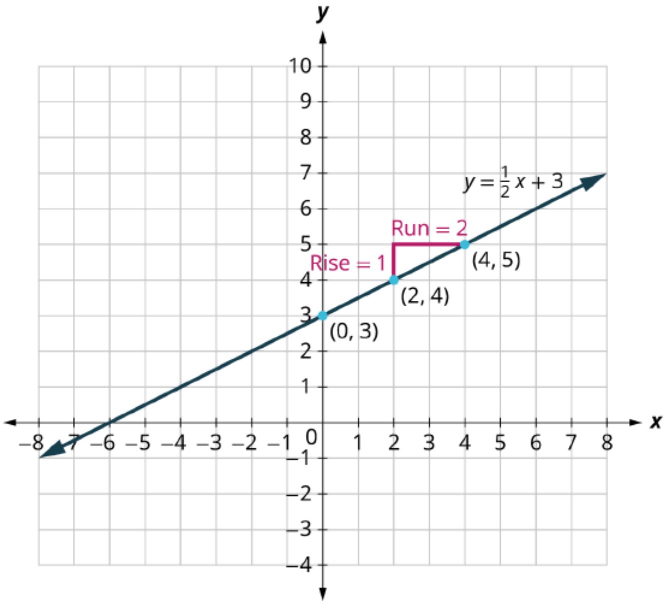


Figure $5.8.12$: Graph of the equation $y = \frac{1}{2}x + 3$

The vertical and horizontal lines in the graph show us the rise is 1 and the run is 2 , respectively.

Substituting into the slope formula: $m=\frac{rise}{run}=\frac{1}{2}$

The y-intercept is $(0,3)$. Look at the equation of this line.

$$y=\frac{1}{2}x+3$$

Look at the slope and y-intercept.

$$\text{slope } m=\frac{1}{2} \text{ and } y\text{-intercept } (0,3)$$

When a linear equation is solved for y, the coefficient of the x term is the slope and the constant term is the y coordinate of the y-intercept. We say that the equation $y=\frac{1}{2}x+3$ is in slope-intercept form. Sometimes the slope-intercept form is called the y-form.

✓ Example $5.8.5$: Finding the Slope and y-Intercept of a Line

Identify the slope and y-intercept of the line from the equation:

1. $y=-\frac{4}{7}x-2$
2. $x+3y=9$

Answer

1. We compare our equation to the slope-intercept form of the equation.

Step 1: Write the slope-intercept form of the equation of the line.

$$y=mx+b$$

Step 2: Write the equation of the line.

$$y=-\frac{4}{7}x-2$$

Step 3: Identify the slope.

$$m=-\frac{4}{7}$$

Step 4: Identify the y-intercept.

$$y\text{-intercept is } (0,-2)$$

2. When an equation of a line is not given in slope-intercept form, our first step will be to solve the equation for y.

Step 1: Solve for y.

$$x+3y=9$$

Step 2: Subtract x from each side.

$$3y=-x+9$$

Step 3: Divide both sides by 3.

$$\frac{3y}{3}=\frac{-x+9}{3}$$

Step 4: Simplify.

$$y=-\frac{1}{3}x+3$$

Step 5: Write the slope-intercept form of the equation of the line.

$$y=mx+b$$

Step 6: Write the equation of the line.

$$y=-\frac{1}{3}x+3$$

Step 7: Identify the slope.

$$m=-\frac{1}{3}$$

Step 8: Identify the y-intercept.

$$y\text{-intercept is } (0,3)$$

Your Turn 5.8.5

Identify the slope and y-intercept from the equation of the line.

1. $y=2x-1$
2. $x+4y=16$

Example 5.8.6: Graphing the Slope and y-Intercept

Graph the line of the equation $y=-x+4$ using its slope and y-intercept.

Answer

The equation is in slope-intercept form $y=mx+b$.

$$y=-x+4.$$

Step 1: Identify the slope and y-intercept.
$m=-1, y$-intercept is $(0,4)$.

Step 2: Plot the y-intercept on the coordinate system (Figure 5.74).

1. Identify the rise over the run.

$$m=-1=\frac{-1}{1}$$

2. Count out the rise and run to mark the second point. rise -1 , run 1

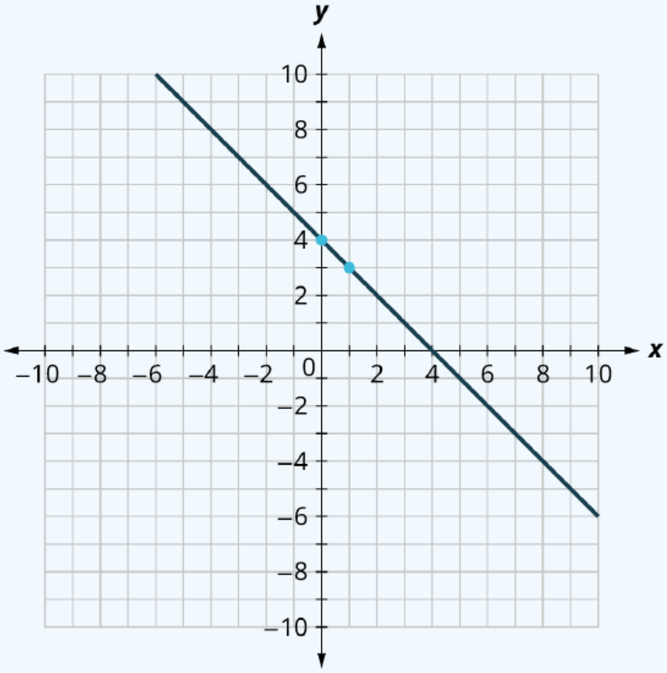


Figure 5.8.13

Your Turn 5.8.6

Graph the line of the equation $y = -x - 3$ using its slope and y-intercept.

Graphing Horizontal and Vertical Lines

Some linear equations have only one variable. They may have just x without the y, or just y without an x. This changes how we make a table of values to get the points to plot. Let us consider the equation $x = -3$. This equation has only one variable, x. The equation says that x is always equal to -3, so its value does not depend on y. No matter what the value of y is, the value of x is always -3. To make a table of values, write -3 in for all the x-values. Then choose any values for y. Since x does not depend on y, you can choose any numbers you like. But to fit the points on our coordinate graph, we will use 1, 2, and 3 for the y-coordinates in the table below.

$x = -3$

x	y	(x, y)
−3	1	$(-3, 1)$
-3	2	$(-3, 2)$
-3	3	$(-3, 3)$

Plot the points from the table and connect them with a straight line (Figure 5.8.14). Notice that we have graphed a vertical line.

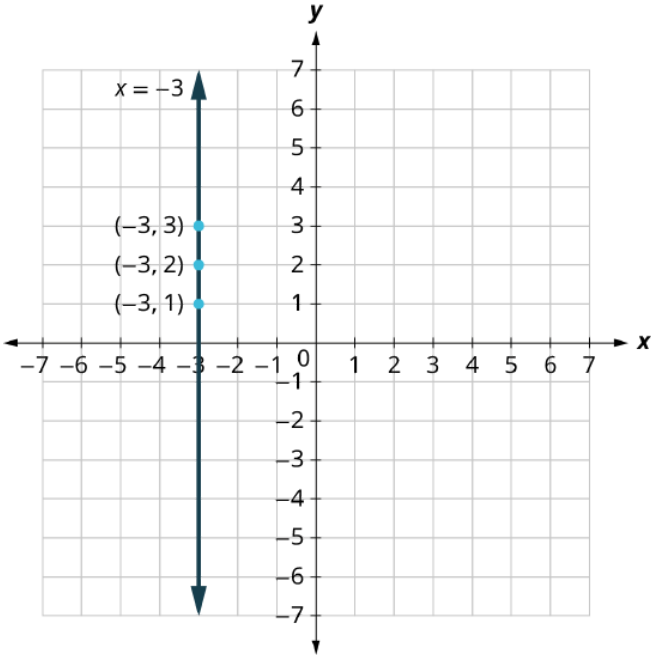


Figure $5.8.14$: Graph of $x=-3$

What is the slope? If we take the two points $(-3,3)$ and $(-3,1)$ then the rise is 2 and the run is 0 .

Using the slope formula we get: $m=\frac{rise}{run}=\frac{2}{0}$

The slope is undefined since division by zero is undefined. We say that the slope of the vertical line $x=-3$ is undefined. The slope of any vertical line $x=a$ (where a is any number) will be undefined.

What if the equation has y but no x ? Let's graph the equation $y=4$. This time the y-value is a constant, so in this equation, y does not depend on x. Fill in 4 for all the y values in the table below and then choose any values for x. We will use 0,2 , and 4 for the x-coordinates.

$y=4$		
x	y	(x, y)
0	4	$(0,4)$
2	4	$(2,4)$
4	4	$(4,4)$

In Figure $5.8.152$, we have graphed a horizontal line passing through the y-axis at 4.

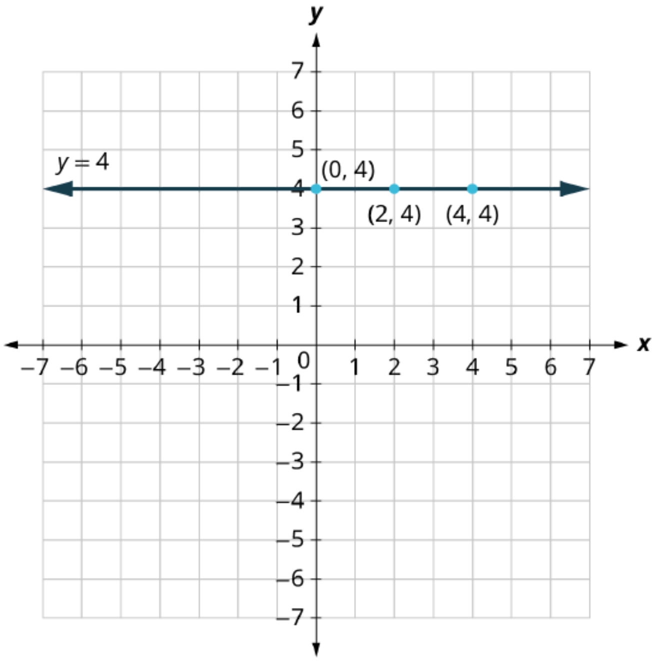


Figure $5.8.15$: Graph of $y=4$

What is the slope? If we take the two points $(2,4)$ and $(4,4)$ then the rise is 0 and the run is 2 . Using the slope formula, we get $m=\frac{rise}{run}=\frac{0}{2}=0$. The slope of the horizontal line $y=4$ is 0 . The slope of any horizontal line $y=b$ (where b is any number) will be 0 . When the y-coordinates are the same, the rise is 0 .

Example $5.8.7$: Graphing A Vertical Line

Graph: $x=2$.

Answer

The equation has only one variable, x, and x is always equal to 2 . We create a table where x is always 2 and then put in any values for y. The graph is a vertical line passing through the x-axis at 2 (Figure $5.8.16$).

$x=2$

x	y	(x, y)
2	1	$(2,1)$
2	2	$(2,2)$
2	3	$(2,3)$

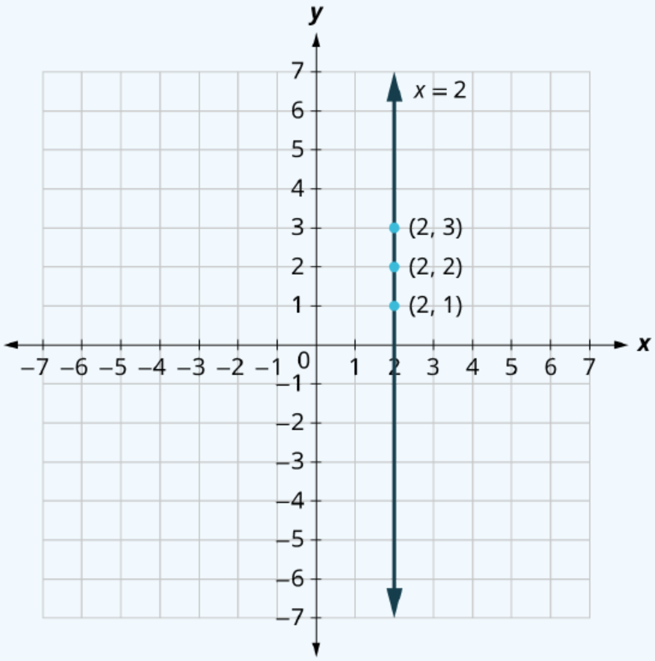


Figure $5.8.16$

Your Turn $5.8.7$

Graph $x=5$.

Example $5.8.8$: Graphing A Horizontal Line

Graph: $y=-1$.

Answer

The equation $y=-1$ has only one variable, y. The value of y is constant. All the ordered pairs in the next table have the same y-coordinate. The graph is a horizontal line passing through the y-axis at -1 (Figure $5.8.17$).

$y=-1$		
x	y	(x, y)
0	-1	$(0,-1)$
3	-1	$(3,-1)$
-3	-1	$(-3,-1)$

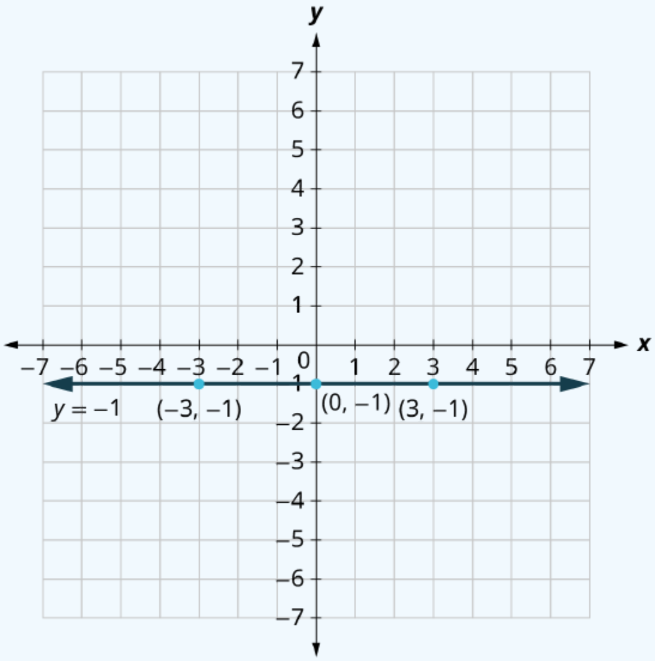
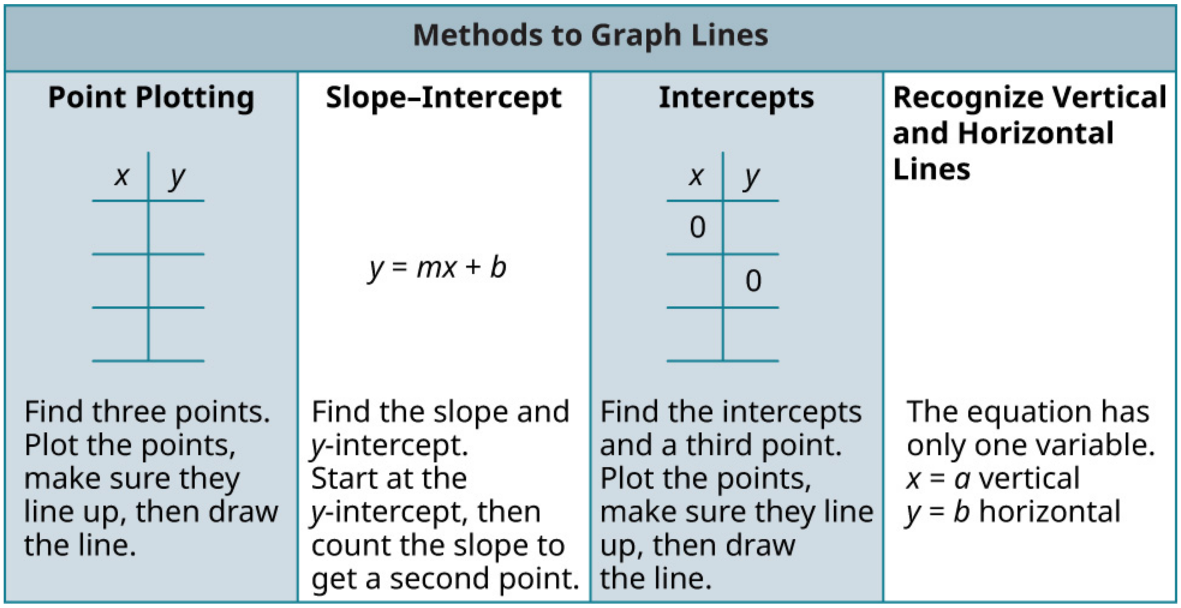


Figure 5.8.17

Your Turn 5.8.8

Graph the equation $y = -4$.

The table below summarizes all the methods we have used to graph lines.



Methods to Graph Lines

Point Plotting	Slope–Intercept	Intercepts	Recognize Vertical and Horizontal Lines
x | *y*	$y = mx + b$	*x* | *y* 0 | | 0	
Find three points. Plot the points, make sure they line up, then draw the line.	Find the slope and *y*-intercept. Start at the *y*-intercept, then count the slope to get a second point.	Find the intercepts and a third point. Plot the points, make sure they line up, then draw the line.	The equation has only one variable. $x = a$ vertical $y = b$ horizontal

Interpreting Graphs of Functions

An important yet often overlooked area in algebra involves interpreting graphs. Oftentimes in math classes, students are given mathematical functions and can make graphs to represent them. But the interpretation of graphs is a more applicable skill to the real world. Being able to “read” a graph—understanding its domain and range, what the intercepts mean, and what the slope (or curve) means— that's a real-world skill.

Example 5.8.9: Interpreting a Graph

In Figure $5.8.18$ the x-axis on the graph represents the 120-minute bike ride Juan went on. The y-axis represents how far away he was from his home.

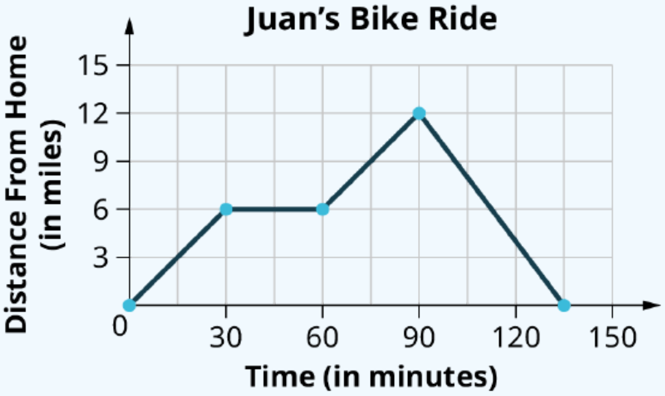


Figure $5.8.18$

1. Interpret the x- and y-intercept.
2. For each segment, find the slope.
3. Create an interpretation of this graph (i.e., make up a story that goes with it).

Answer

1. $(0, 0)$ is the x- and y-intercept and represents Juan at home before his bike ride. The distance from home is 0 miles and 0 minutes have passed.
2. In the first 30 minutes, the slope is $\frac{1}{5}$ and indicates Juan is traveling 1 mile for every 5 minutes. Between 30 and 60 minutes, the slope is 0 and indicates that he's not riding the bike (the distance is not increasing). Then between 60 and 90 minutes, the slope is $\frac{1}{5}$ again. Finally, after 90 minutes the slope is $-\frac{4}{15}$, meaning Juan is getting 4 miles closer to home every 15 minutes.
3. Answers will vary. Juan left his house for a bike ride. After 30 minutes, he was 6 miles from home and he stopped for ice cream at his local ice-cream truck. He enjoyed his ice cream for 30 minutes. He then jumped back on his bike and rode to his friend's house. He arrived there 30 minutes later. His friend's house was 12 miles from his home. His friend was not home so he immediately turned around and quickly rode home in 45 minutes.

Your Turn $5.8.9$

In the given figure the x-axis on the graph represents the years. The y-axis represents the number of teachers at Jones High School.

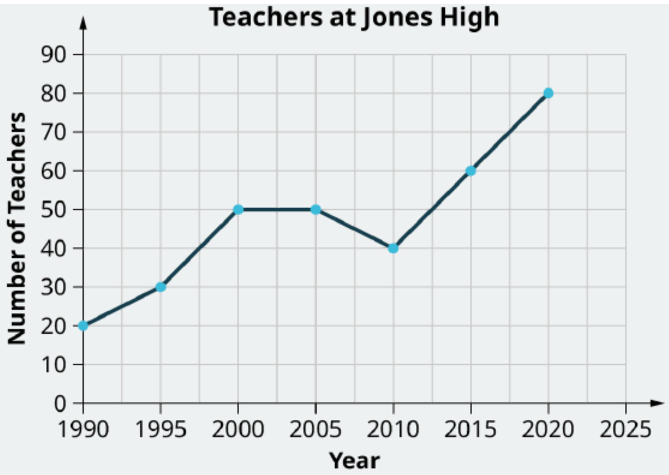


Figure $5.8.19$: Teachers at Jones High

Interpret the x- and y-intercept.

For each segment, find the slope.

Create an interpretation of this graph (i.e., make up a story that goes with it).

Modeling Applications Using Slope and y-Intercept

Many real-world applications are modeled by linear equations. We will review a few applications here so you can understand how equations written in slope-intercept form relate to real-world situations. Usually when a linear equation model uses real-world data, different letters are used for the variables instead of using only x and y. The variable names often remind us of what quantities are being measured. Also, we often need to extend the axes in our rectangular coordinate system to bigger positive and negative numbers to accommodate the data in the application.

✓ Example $5.8.10$: Converting Temperature

The equation F =95C+32 is used to convert temperatures from degrees Celsius (C) to degrees Fahrenheit (F).

1. Find the Fahrenheit temperature for a Celsius temperature of 0°.
2. Find the Fahrenheit temperature for a Celsius temperature of 20°.
3. Interpret the slope and F-intercept of the equation.
4. Graph the equation.

Answer

1. Find the Fahrenheit temperature for a Celsius temperature of 0°.

Find F when \C=0\)	$F=\frac{9}{5}(0)+32$
Simplify.	$F=32$

2. Find the Fahrenheit temperature for a Celsius temperature of 20°.

Find F when \C=20\)	$F=\frac{9}{5}(20)+32$
Simplify.	$F=36+32$
Simplify.	$F=68$

3. Interpret the slope and F-intercept of the equation.

Even though this equation uses F and C, it is still in slope-intercept form.

$$y=mx+b$$
$$F=mC+b$$

$$F=\frac{9}{5}C+32$$

The slope $\frac{9}{5}$ means that the temperature Fahrenheit (F) increases 9 degrees when the temperature Celsius (C) increases 5 degrees.

The F-intercept means that when the temperature is 0° on the Celsius scale, it is 32° on the Fahrenheit scale.

4. Graph the equation.

We will need to use a larger scale than our usual. Start at the F-intercept $(0,32)$, and then count out the rise of 9 and the run of 5 to get a second point as shown in Figure $5.8.20$.

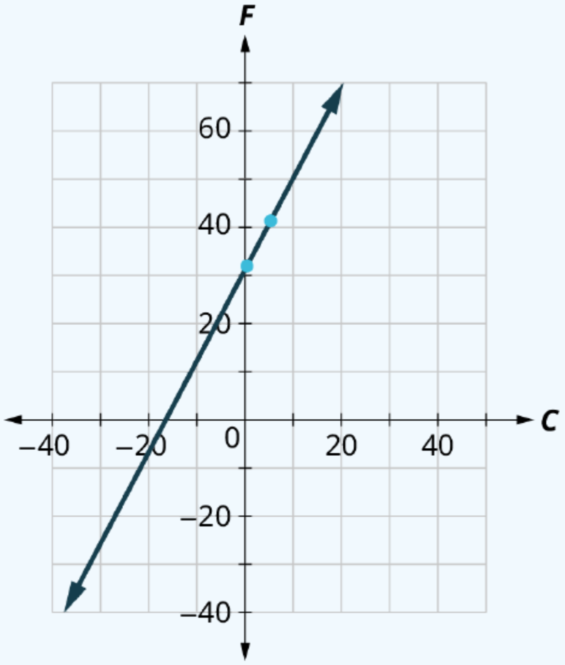


Figure $5.8.20$: Copy and Paste Caption here. (Copyright; author via source)

Your Turn $5.8.10$

The equation $h=2s+50$ is used to estimate a person’s height in inches, h, based on women’s shoe size, s.

1. Estimate the height of a child who wears women’s shoe size 0.
2. Estimate the height of a woman with shoe size 8.
3. Interpret the slope and h-intercept of the equation.
4. Graph the equation.

Example $5.8.11$: Calculating Driving Costs

Sam drives a delivery van. The equation $C=0.5d+60$ models the relation between his weekly cost, C, in dollars and the number of miles, d, that he drives.

1. Find Sam’s cost for a week when he drives 0 miles.
2. Find the cost for a week when he drives 250 miles.
3. Interpret the slope and C-intercept of the equation.
4. Graph the equation.

Answer

1. Find Sam’s cost for a week when he drives 0 miles.

Find C when $d=0$.	\(C=0.5(0)+60\)

Simplify.	\(C=60\)

Sam’s costs are $60 when he drives 0 miles.

2. Find the cost for a week when he drives 250 miles.

Find C when $d=250$.	\(C=0.5(250)+60\)
Simplify.	\(C=185\)

Sam’s costs are $185 when he drives 250 miles.

3. Interpret the slope and C-intercept of the equation.

$$\begin{gathered} y=mx+b \\ C=0.5d+60 \end{gathered}$$

The slope, 0.5 , means that the weekly cost, C, increases by $\$0.50$ when the number of miles driven, d, increases by 1 . The C-intercept means that when the number of miles driven is 0 , the weekly cost is $\$60$.
4. Graph the equation (Figure $5.8.21$).

We'll need to use a larger scale than usual. Start at the C-intercept (0,60). To count out the slope $m=0.5$, we rewrite it as an equivalent fraction that will make our graphing easier.

$$m=0.5=\frac{5}{10}=\frac{50}{100}$$

So to graph the next point go up 50 from the intercept of 60 and then to the right 100 . The second point will be $(100,110)$.

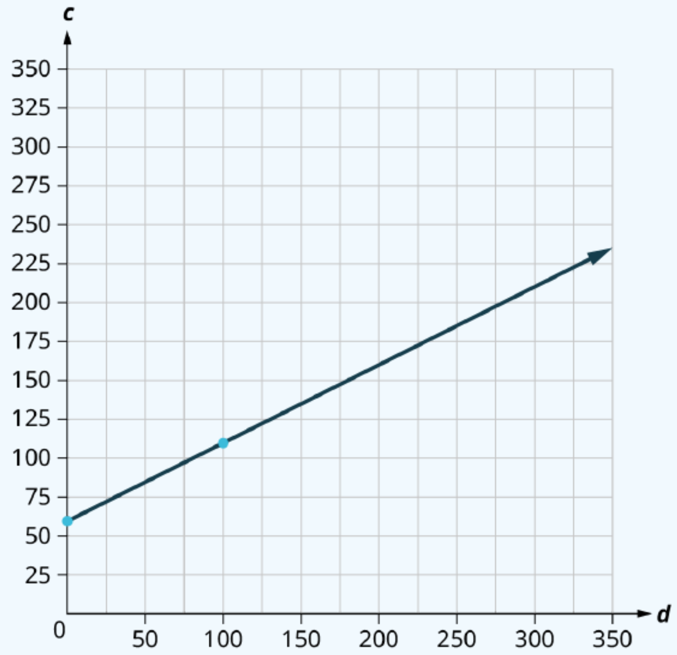


Figure $5.8.21$: Copy and Paste Caption here. (Copyright; author via source)

Your Turn $5.8.11$

Stella has a home business selling gourmet pizzas. The equation $C=4p+25$ models the relation between her weekly cost, C, in dollars and the number of pizzas, p, that she sells.

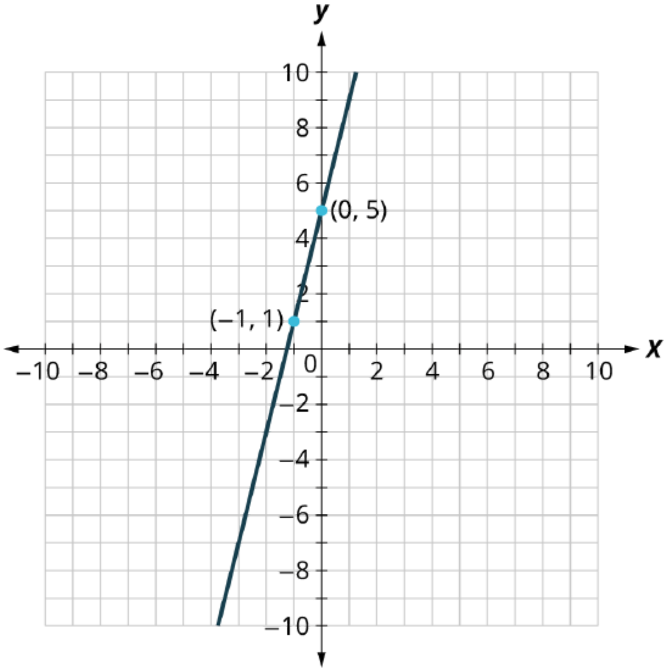
1. Find Stella’s cost for a week when she sells no pizzas.
2. Find the cost for a week when she sells 15 pizzas.

3. Interpret the slope and C-intercept of the equation.
4. Graph the equation.

Check Your Understanding

1. True or False. The x-intercept of $y = 2x - 8$ is $(0, -8)$.
2. True or False. The slope of the line containing the points (1, 2) and (2, 4) is 1.

For the following exercises, use the graph shown.



3. True or False. This graph has a slope of 5.

4. True or False. This is the graph of the equation $y = 4x + 5$.

5. True or False. All vertical lines have a slope of zero.

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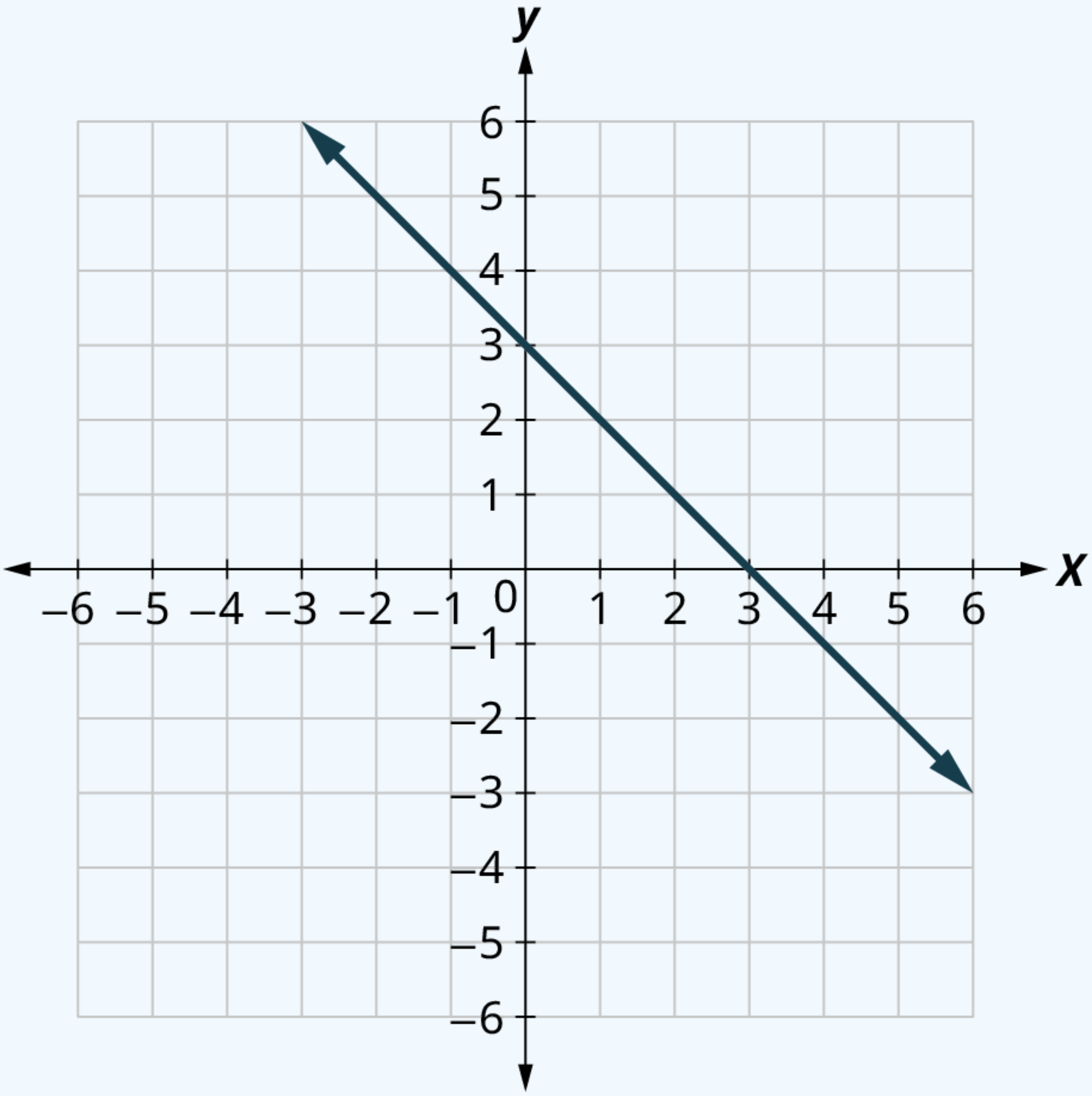
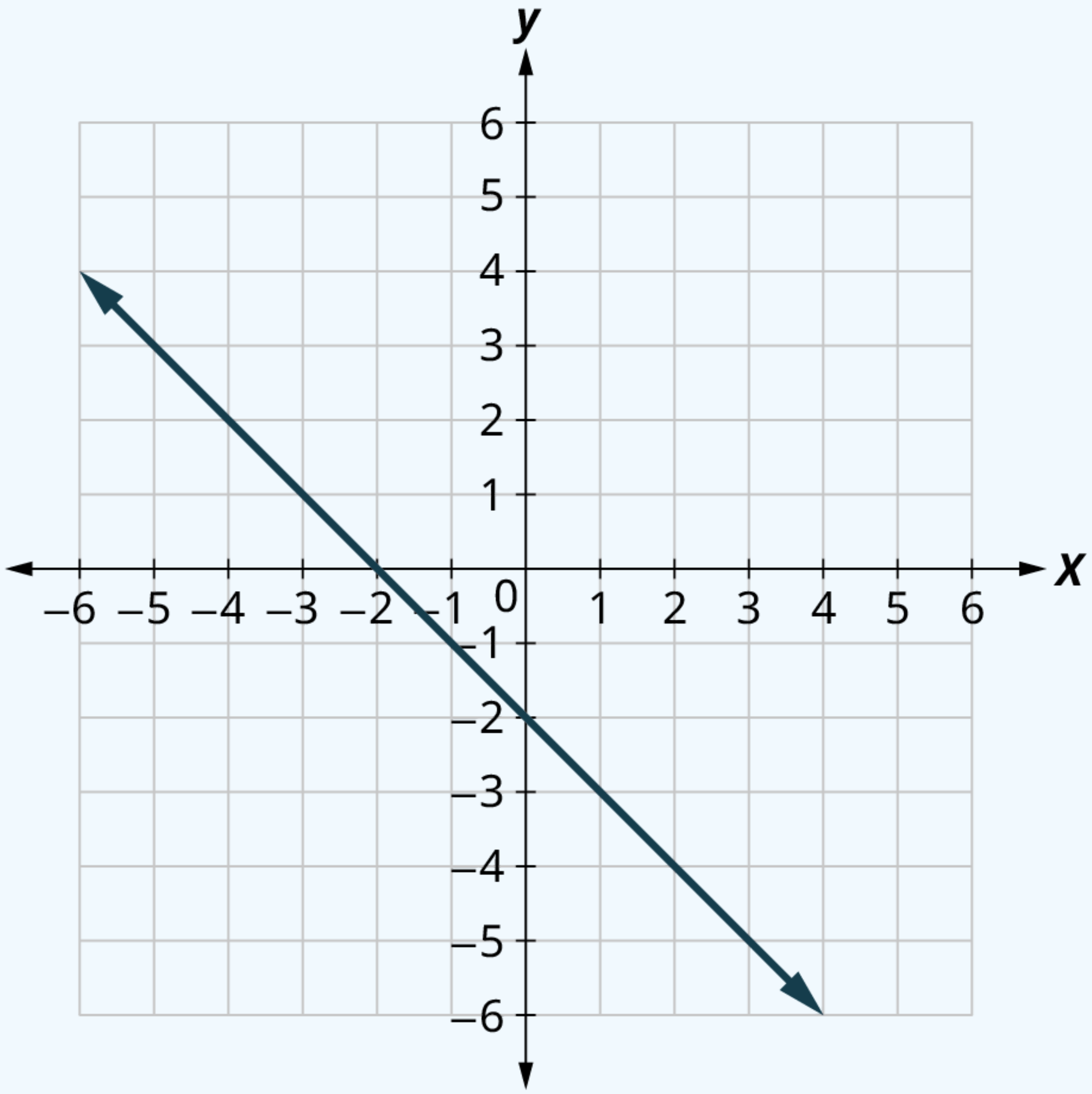
5.8.1: Exercises

For the following exercises, find the x- and y-intercepts on the graph.

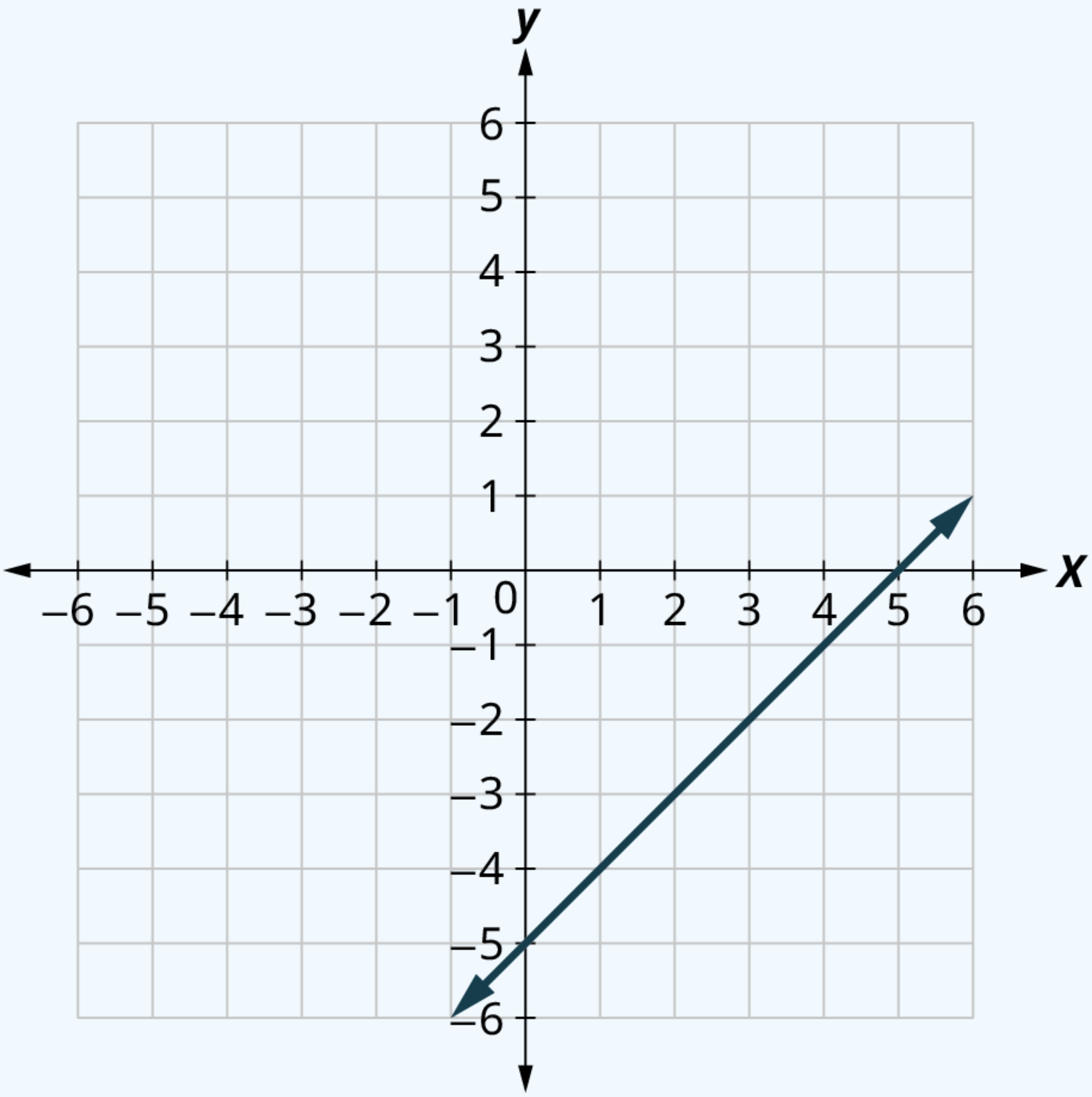
? Exercise 5.8.1.1



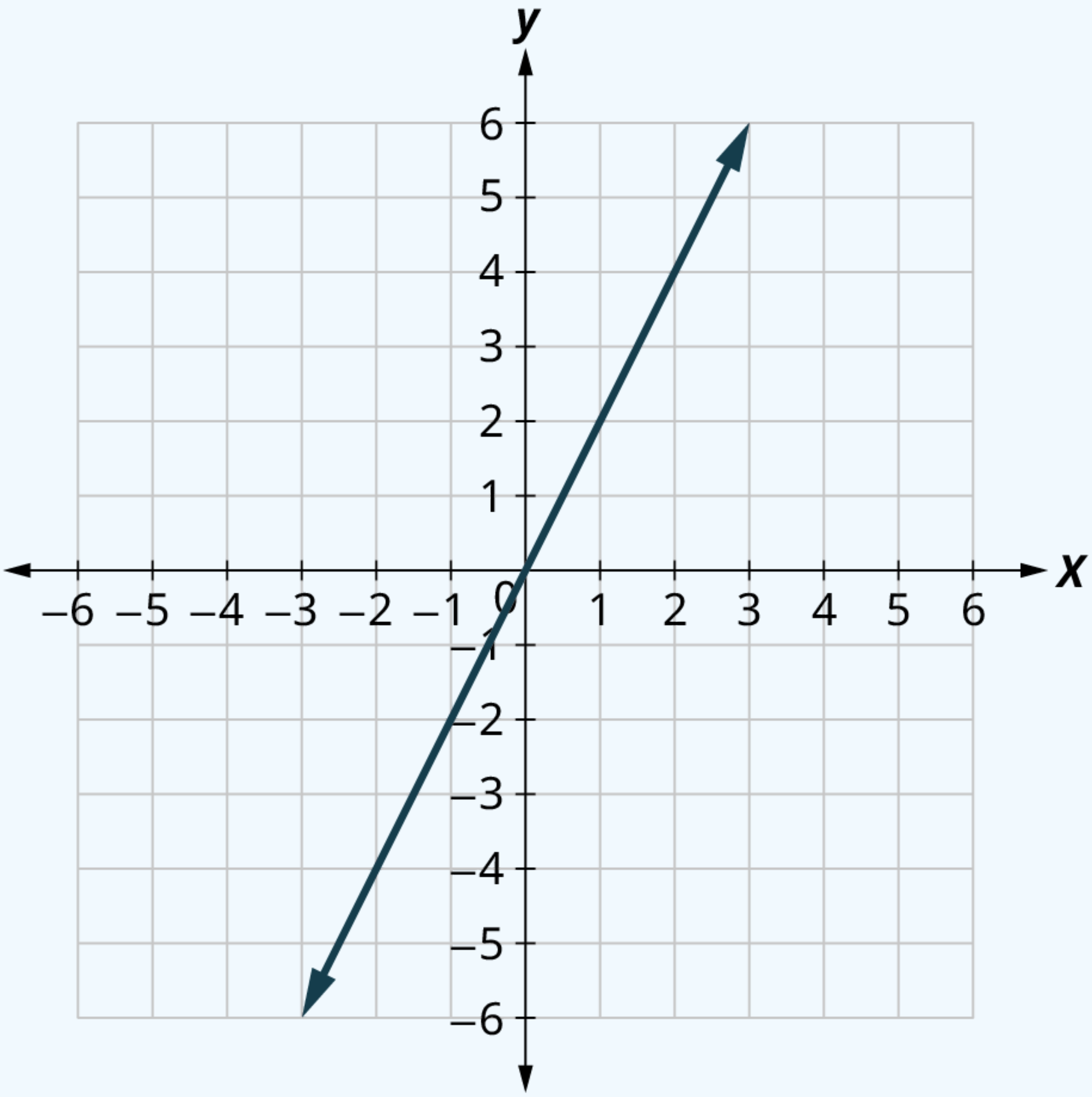
? Exercise 5.8.1.2



? Exercise $5.8.1.3$



? Exercise $5.8.1.4$



For the following exercises, graph using the intercepts.

Exercise 5.8.1.5

$-x+4y=8$

Exercise 5.8.1.6

$x+y=-3$

Exercise 5.8.1.7

$4x+y=4$

Exercise 5.8.1.8

$3x-y=-6$

? Exercise 5.8.1.9

$2x + 4y = 12$

? Exercise 5.8.1.10

$2x - 5y = -20$

? Exercise 5.8.1.11

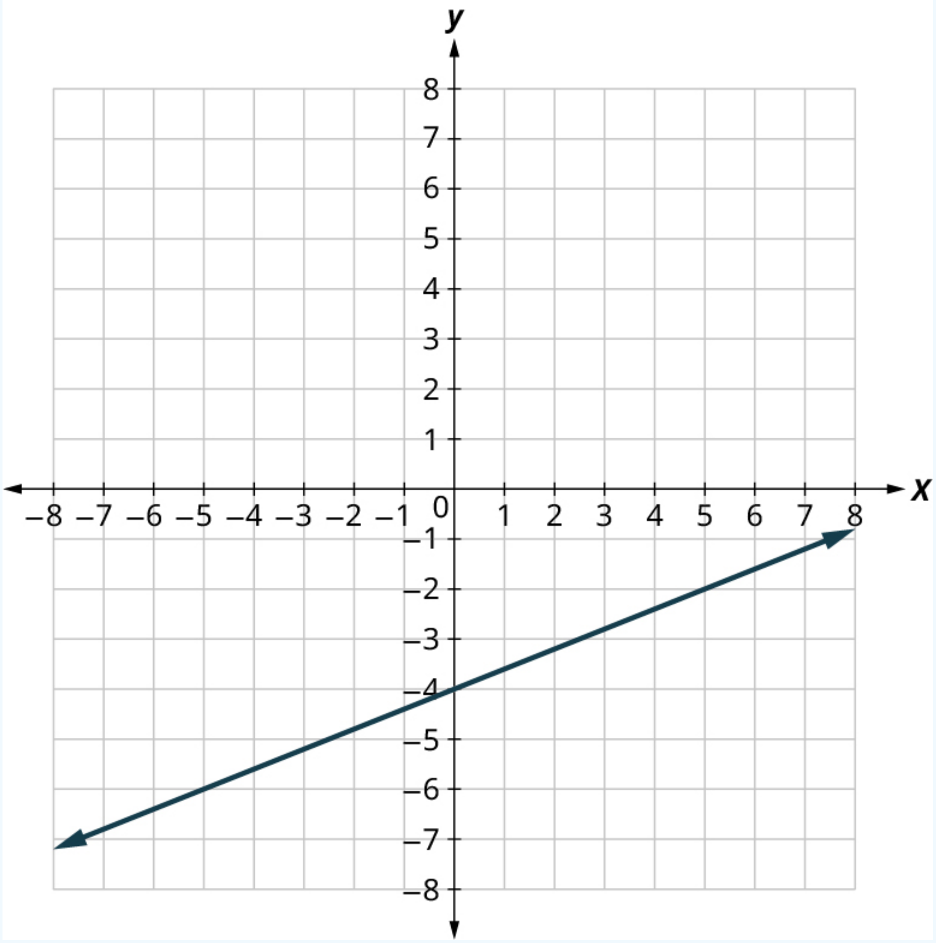
$y = -2x$

? Exercise 5.8.1.12

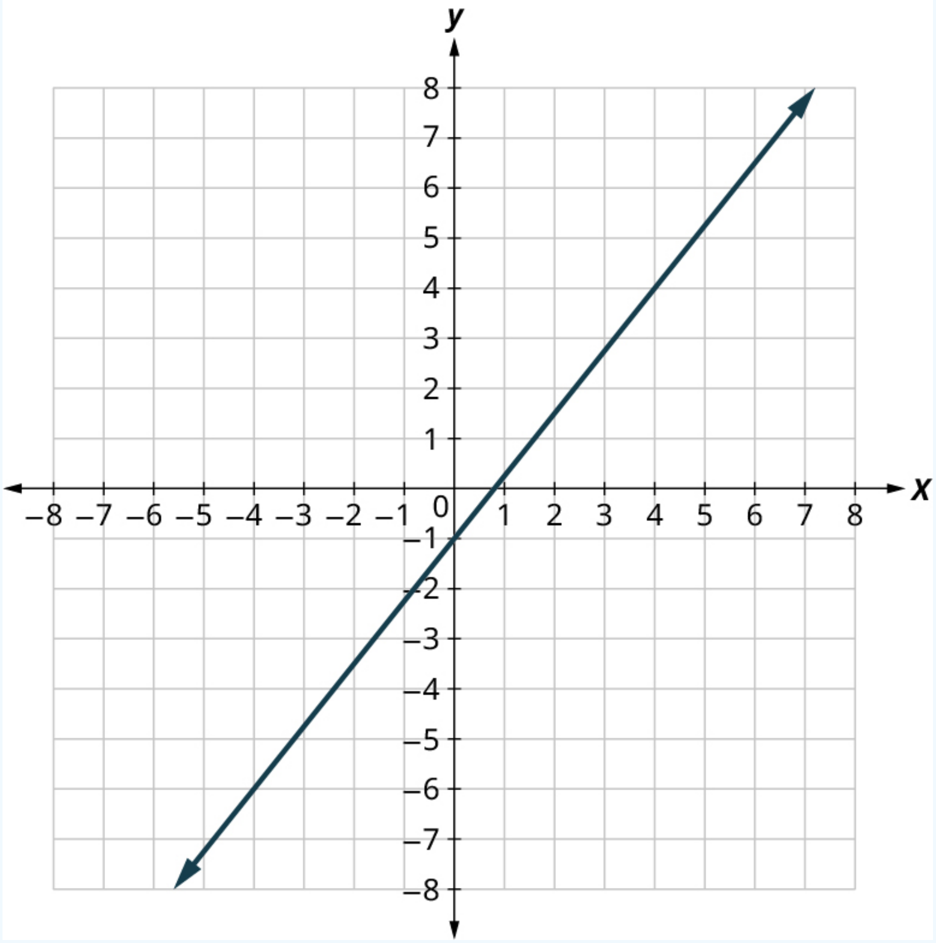
$y = x$

For the following exercises, find the slope of the line.

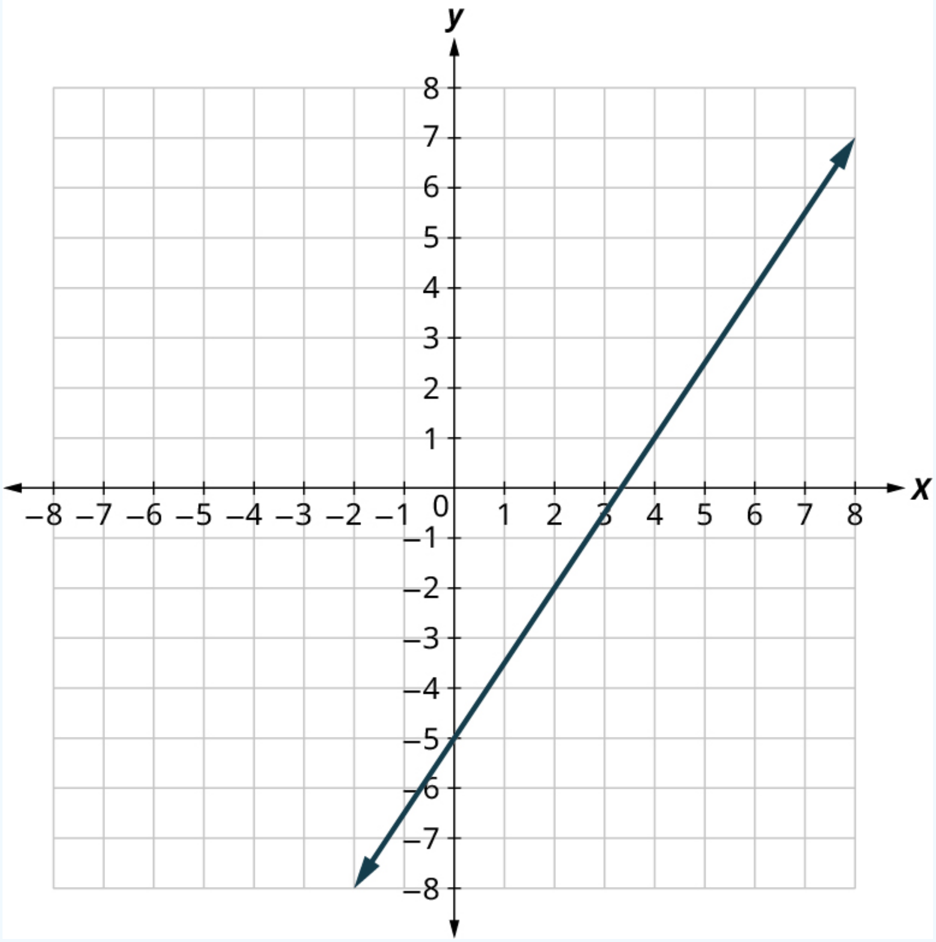
? Exercise 5.8.1.13



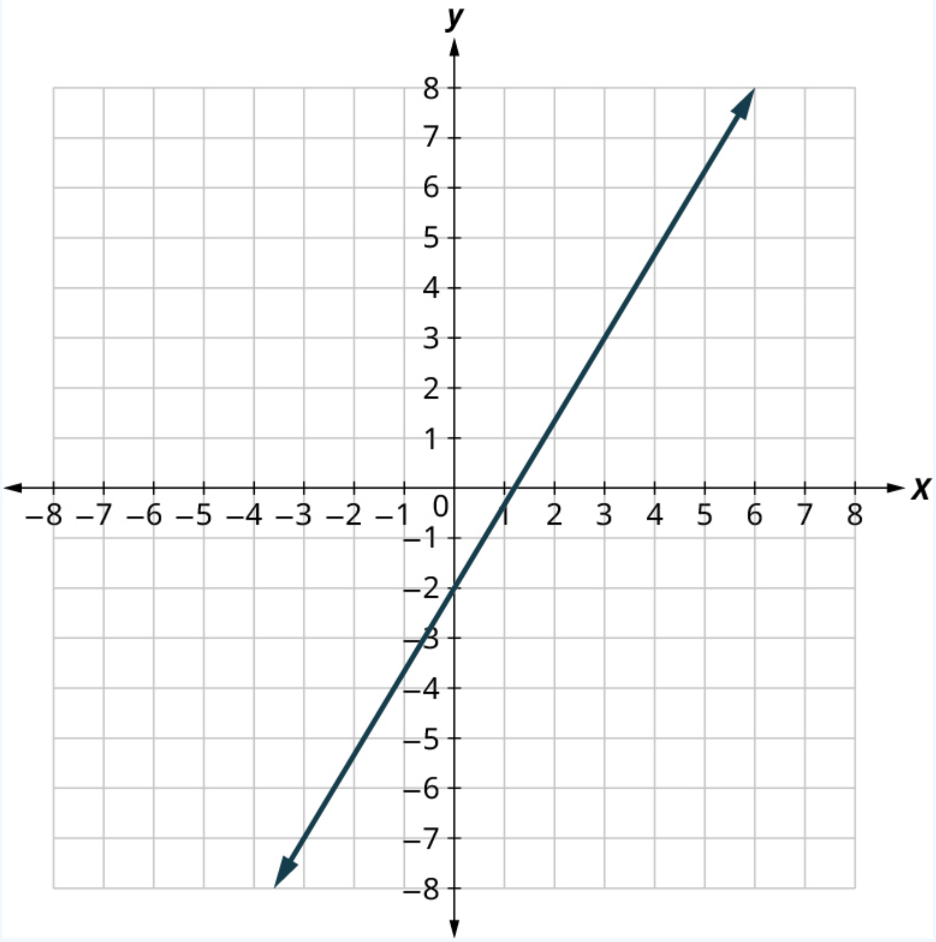
? Exercise 5.8.1.14



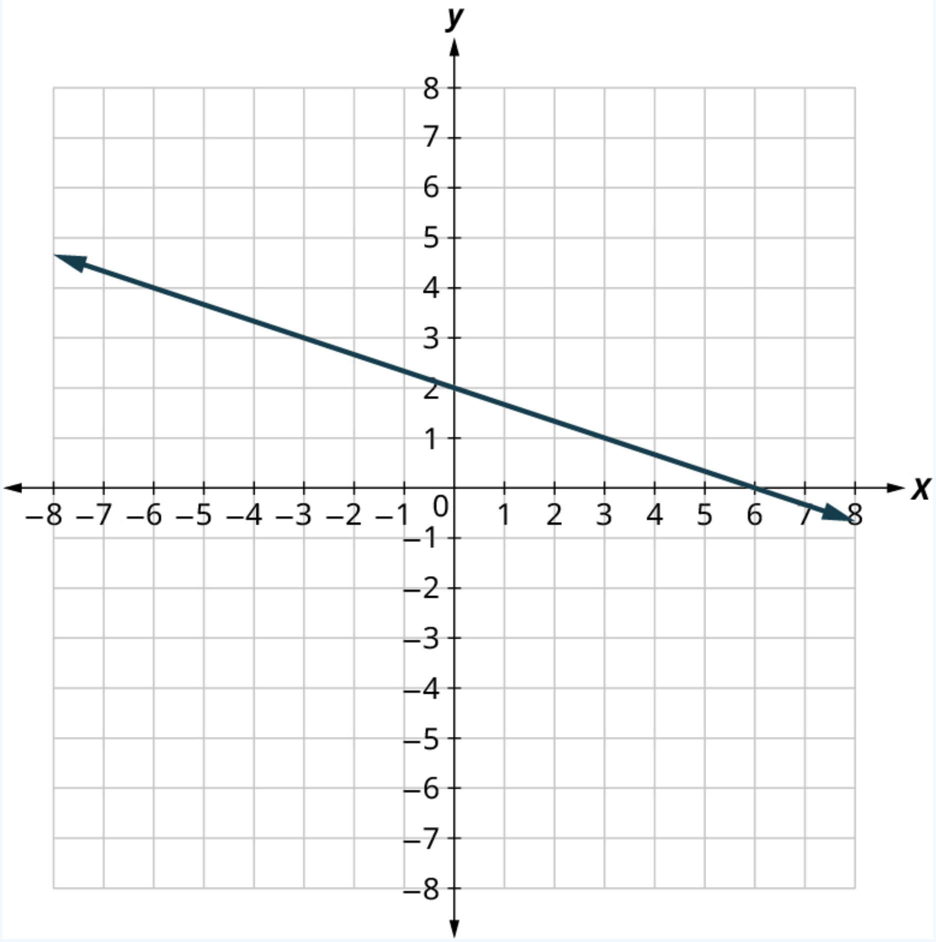
Exercise $5.8.1.15$



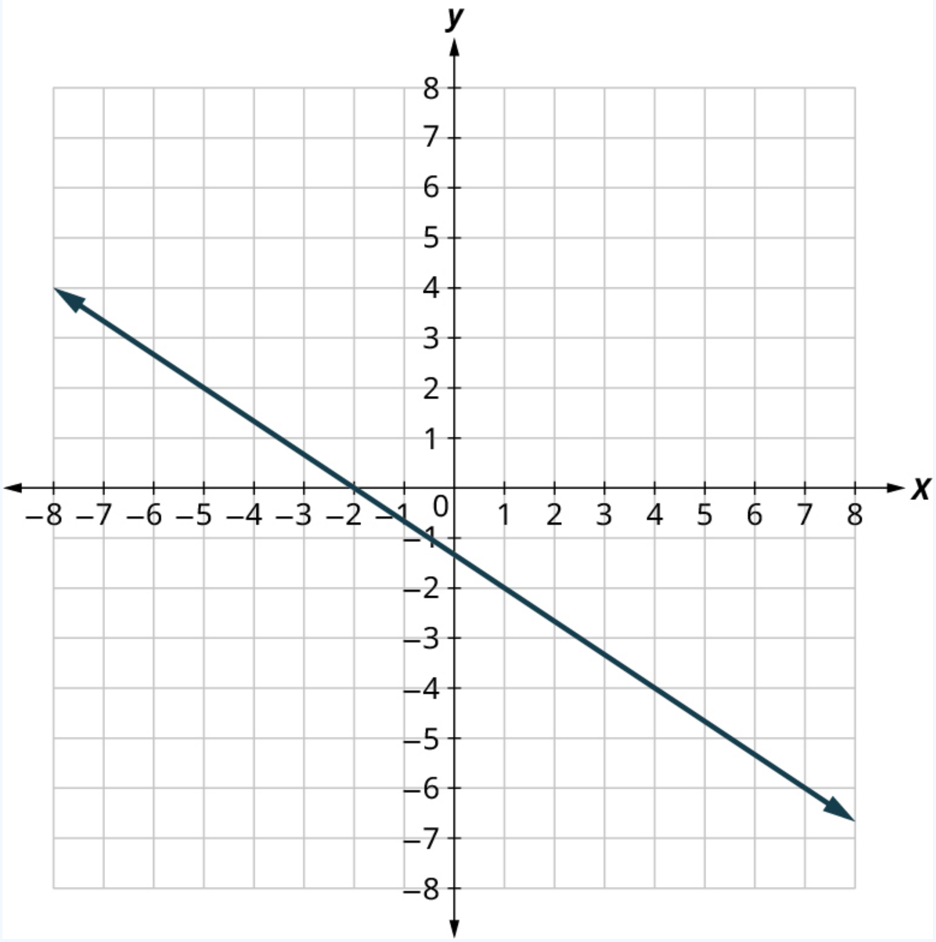
Exercise 5.8.1.16



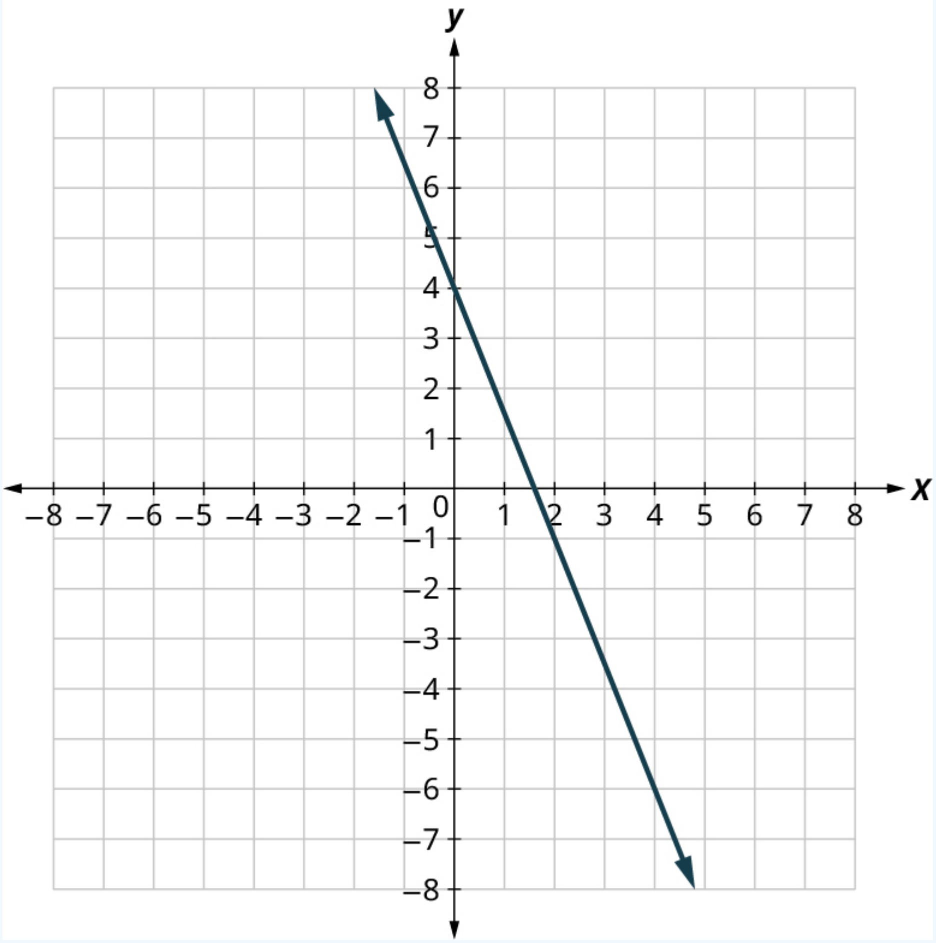
Exercise 5.8.1.17



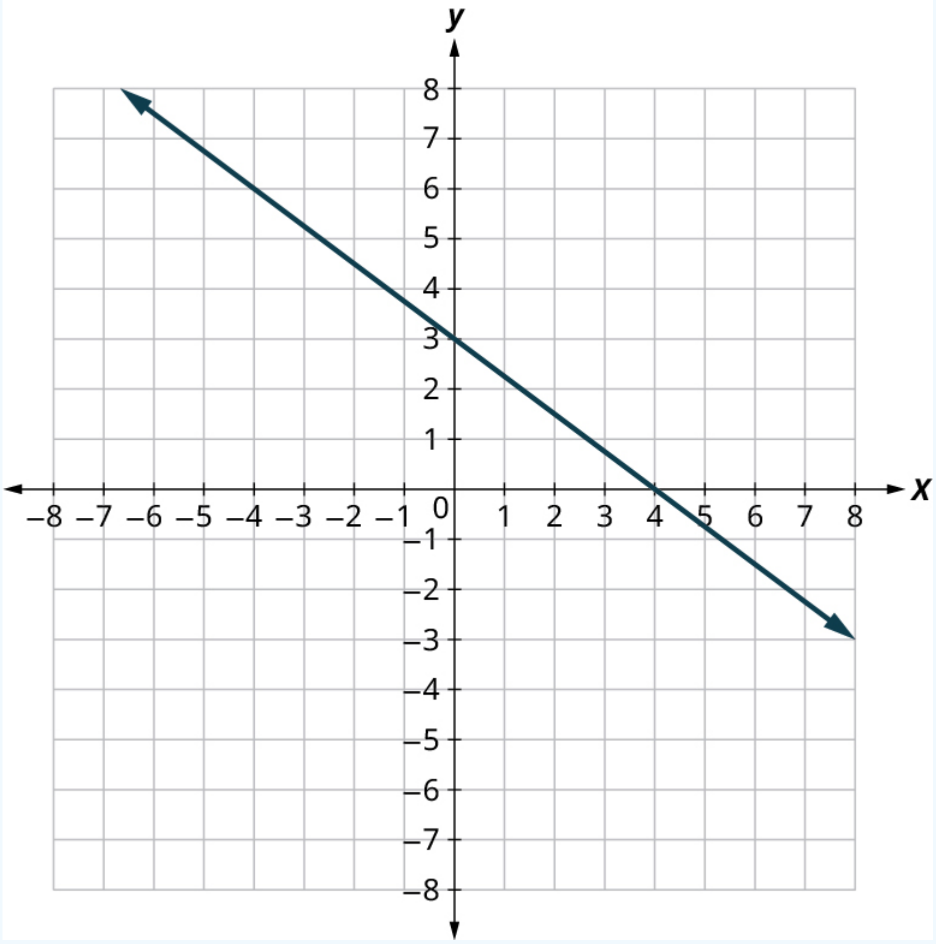
? Exercise $5.8.1.18$



Exercise $5.8.1.19$



Exercise $5.8.1.20$



For the following exercises, use the slope formula to find the slope of the line between each pair of points.

Exercise $5.8.1.21$

(2, 5), (4, 0)

Exercise $5.8.1.22$

(−3, 3), (4, −5)

Exercise $5.8.1.23$

(−1, −2), (2, 5)

Exercise $5.8.1.24$

(4, −5), (1, −2)

For the following exercises, identify the slope and y-intercept of each line.

Exercise $5.8.1.25$

$y=-7x+3$

? Exercise 5.8.1.26

$y = 4x - 10$

? Exercise 5.8.1.27

$3x + y = 5$

? Exercise 5.8.1.28

$4x + y = 8$

? Exercise 5.8.1.29

$6x + 4y = 12$

? Exercise 5.8.1.30

$8x + 3y = 12$

? Exercise 5.8.1.31

$5x - 2y = 6$

? Exercise 5.8.1.32

$7x - 3y = 9$

For the following exercises, graph the line of each equation using its slope and y-intercept.

? Exercise 5.8.1.33

$y = 3x - 1$

? Exercise 5.8.1.34

$y = 2x - 3$

? Exercise 5.8.1.35

$y = -x + 3$

? Exercise 5.8.1.36

$y = -x - 4$

? Exercise 5.8.1.37

$y = -2x - 3$

Exercise 5.8.1.38

$y=-3x+2$

Exercise 5.8.1.39

$3x-2y=4$

Exercise 5.8.1.40

$3x-4y=8$

For the following exercises, find the slope of each line and graph.

Exercise 5.8.1.41

$y=3$

Exercise 5.8.1.42

$y=-2$

Exercise 5.8.1.43

$x=-5$

Exercise 5.8.1.44

$x=4$

For the following exercises, graph and interpret applications of slope-intercept.
The equation $P=31+1.75w$ models the relation between the amount of Tuyet's monthly water bill payment, P, in dollars, and the number of units of water, w, used.

Exercise 5.8.1.45

Find Tuyet's payment for a month when 0 units of water are used.

Exercise 5.8.1.46

Find Tuyet's payment for a month when 12 units of water are used.

Exercise 5.8.1.47

Interpret the slope and P-intercept of the equation.

Exercise 5.8.1.48

Graph the equation.

For the following exercises, graph and interpret applications of slope-intercept.
Bruce drives his car for his job. The equation $R=0.575m+42$ models the relation between the amount in dollars, R, that he is reimbursed and the number of miles, m, he drives in one day.

Exercise $5.8.1.49$

Find the amount Bruce is reimbursed on a day when he drives 0 miles.

Exercise $5.8.1.50$

Find the amount Bruce is reimbursed on a day when he drives 220 miles.

Exercise $5.8.1.51$

Interpret the slope and R-intercept of the equation.

Exercise $5.8.1.52$

Graph the equation.

For the following exercises, graph and interpret applications of slope-intercept.
Cherie works in retail and her weekly salary includes commission for the amount she sells. The equation $S = 400 + 0.15c$ models the relation between her weekly salary, S, in dollars and the amount of her sales, c, in dollars.

Exercise $5.8.1.53$

Find Cherie’s salary for a week when her sales were $0.

Exercise $5.8.1.54$

Find Cherie’s salary for a week when her sales were $3,600.

Exercise $5.8.1.55$

Interpret the slope and S-intercept of the equation.

Exercise $5.8.1.56$

Graph the equation.

For the following exercises, graph and interpret applications of slope-intercept.
Costa is planning a lunch banquet. The equation $C = 450 + 28g$ models the relation between the cost in dollars, C, of the banquet and the number of guests, g.

Exercise $5.8.1.57$

Find the cost if the number of guests is 40.

Exercise $5.8.1.58$

Find the cost if the number of guests is 80.

Exercise $5.8.1.59$

Interpret the slope and C-intercept of the equation.

Exercise $5.8.1.60$

Graph the equation.

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5.9: Systems of Linear Equations in Two Variables

Learning Objectives

After completing this section, you should be able to:

- Determine and show whether an ordered pair is a solution to a system of equations.
- Solve systems of linear equations using graphical methods.
- Solve systems of linear equations using substitution.
- Solve systems of linear equations using elimination.
- Identify systems with no solution or infinitely many solutions.
- Solve applications of systems of linear equations.

In this section, we will learn how to solve systems of linear equations in two variables. There are several real-world scenarios that can be represented by systems of linear equalities. Suppose two friends, Andrea and Bart, go shopping at a farmers market to buy some vegetables. Andrea buys 2 tomatoes and 4 cucumbers and spends \$2.00. Bart buys 4 tomatoes and 5 cucumbers and spends \$2.95. What is the price of each vegetable?

Figure $5.9.1$: Fruits and vegetables at a farmer's market. (credit: "California Ave. Farmers' Market" by Jun Seita/Flickr, CC BY 2.0)

Determining If an Ordered Pair Is a Solution to a System of Equations

When we solved linear equations in Linear Equations in One Variable with Applications and Linear Inequalities in One Variable with Applications, we learned how to solve linear equations with one variable. Now we will work with two or more linear equations grouped together, which is known as a system of linear equations.

In this section, we will focus our work on systems of two linear equations in two unknowns (variables) and applications of systems of linear equations. An example of a system of two linear equations is shown below. We use a brace to show the two equations are grouped together to form a system of equations.

$$\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$$

A linear equation in two variables, such as $2x+y=7$, has an infinite number of solutions. Its graph is a line. Remember, every point on the line is a solution to the equation and every solution to the equation is a point on the line. To solve a system of two linear equations, we want to find the values of the variables that are solutions to both equations. In other words, we are looking for the ordered pairs (x, y) that make both equations true. These are called the solutions of a system of equations.

To determine if an ordered pair is a solution to a system of two equations, we substitute the values of the variables into each equation. If the ordered pair makes both equations true, it is a solution to the system.

✓ Example ✓ $5.9.1$: Determining Whether an Ordered Pair Is a Solution to the System

Determine whether the ordered pair is a solution to the system.

$$\begin{cases} x-y=-1 \\ 2x-y=-5 \end{cases}$$

1. $(-2,-1)$
2. $(-4,-3)$

Answer

1.

$$\begin{cases} x-y=-1 \\ 2x-y=-5 \end{cases}$$

$x-y=-1$ $-2-(-1)\overset{?}{=}-1$ $-1=-1\checkmark$	$2x-y=-5$ $2(-2)-(-1)\overset{?}{=}-5$ $-3\neq-5$

$(-2,-1)$ does not make both equations true. $(-2,-1)$ is not a solution.

2. We substitute $x=-4$ and $y=-3$ into both equations.

$x-y=-1$ $-4-(-3)\overset{?}{=}-1$ $-1=-1\checkmark$	$2x-y=-5$ $2\bullet(-4)-(-3)\overset{?}{=}-5$ $-5=-5\checkmark$

$(-4,-3)$ makes both equations true. $(-4,-3)$ is a solution.

Your Turn 5.9.1

Determine whether the ordered pair is a solution to the system.

$$\begin{cases} x-3y=-8 \\ -3x-y=4 \end{cases}$$

$(2,-2)$

$(-2,2)$

Example 5.9.2: Determining Whether an Ordered Pair Is a Solution to the System

Determine whether the ordered pair is a solution to the system

$$\begin{cases} y=\frac{3}{2}x+1 \\ 2x-3y=7 \end{cases}$$

1. $(-4,-5)$
2. $(-4,5)$

Answer

1.

$$\begin{cases} y=\frac{3}{2}x+1 \\ 2x-3y=7 \end{cases}$$

Substitute -4 for x and -5 for y into both equations.

$-5\overset{?}{=}\frac{3}{2}(-4)+1$ $-5\overset{?}{=}3(-2)+1$ $-5\overset{?}{=}-6+1$ $-5=-5\checkmark$	$2(-4)-3(-5)\overset{?}{=}7$ $(-8)-(-15)\overset{?}{=}7$ $-8+15\overset{?}{=}7$ $7=7\checkmark$

$(-4,-5)$ is a solution.

2.

$$\begin{cases} y=\frac{3}{2}x+1 \\ 2x-3y=7 \end{cases}$$

Substitute -4 for x and 5 for y into both equations.

$$\begin{aligned} 5 &\overset{?}{\overset{3}{2}} (-4)+1 \\ 5 &\overset{?}{=} 3(-2)+1 \\ 5 &\overset{?}{=} -6+1 \\ 5 &\neq -5 \end{aligned}$$

$$\begin{aligned} 2(-4)-3(5) &\overset{?}{=} 7 \\ (-8)-(15) &\overset{?}{=} 7 \\ -8-15 &\overset{?}{=} 7 \\ -23 &\neq 7 \end{aligned}$$

$(-4,5)$ is not a solution.

Your Turn 5.9.2

Determine whether the ordered pair is a solution to the system.

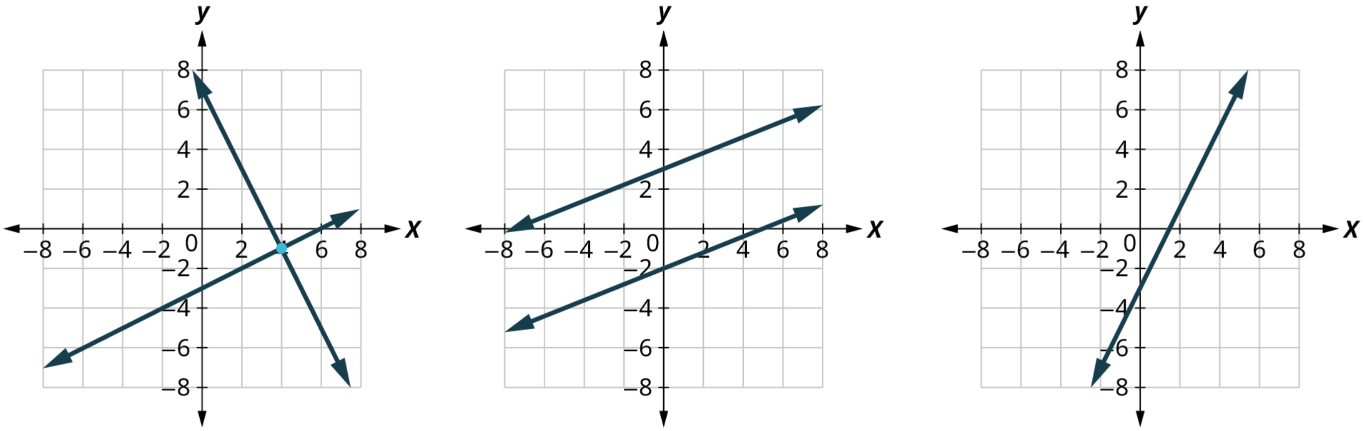
$$\begin{cases} y=\frac{2}{3}x+1 \\ -2x+3y=5 \end{cases}$$

1. $(3,3)$
2. $(-3,3)$

Solving Systems of Linear Equations Using Graphical Methods

We will use three methods to solve a system of linear equations. The first method we will use is graphing. The graph of a linear equation is a line. Each point on the line is a solution to the equation. For a system of two equations, we will graph two lines. Then we can see all the points that are solutions to each equation. And, by finding what points the lines have in common, we will find the solution to the system.

Most linear equations in one variable have one solution; but for some equations called **contradictions**, there are no solutions, and for other equations called **identities**, all numbers are solutions. Similarly, when we solve a system of two linear equations represented by a graph of two lines in the same plane, there are three possible cases, as shown in Figure 5.9.2 .



The lines intersect.
Intersecting lines have one point in common.
There is one solution to this system.

The lines are parallel.
Parallel lines have no points in common.
There is no solution to this system.

Both equations give the same line.
Because we have just one line, there are infinitely many solutions.

Figure 5.9.2

Each time we demonstrate a new method, we will use it on the same system of linear equations. At the end you will decide which method was the most convenient way to solve this system.

The steps to use to solve a system of linear equations by graphing are shown here.

Step 1: Graph the first equation.

Step 2: Graph the second equation on the same rectangular coordinate system.

Step 3: Determine whether the lines intersect, are parallel, or are the same line.

Step 4: Identify the solution to the system.

If the lines intersect, identify the point of intersection. This is the solution to the system.

If the lines are parallel, the system has no solution.

If the lines are the same, the system has an infinite number of solutions.

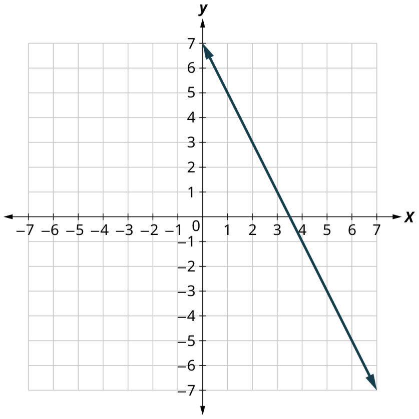
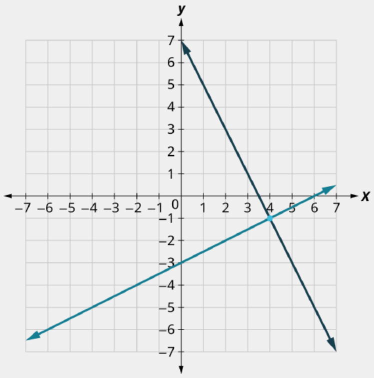
Step 5: Check the solution in both equations.

Example 5.9.3: Solving a System of Linear Equations by Graphing

Solve this system of linear equations by graphing.

$$\left\{\begin{array}{l}2x+y=7 \\ x-2y=6\end{array}\right.$$

Answer

Step 1: Graph the first equation.	To graph the first line, write the equation in slope-intercept form. $2x+y=7$ $y=-2x+7$ $m=-2 \quad b=7$	\(\left\{\begin{array}{l}2 x+y=7 \\ x-2 y=6\end{arr {2x+y=7x-2y=6 
Step 2: Graph the second equation on the same rectangular coordinate system.	To graph the second line, use intercepts. $\boldsymbol{x-2y=6}$ $(0,-3) \quad (6,0)$	
Step 3: Determine whether the lines intersect, are parallel, or are the same line. Look at the graph of the lines. The lines intersect. **Step 4 and Step 5:** Identify the solution to the system.	Look at the graph of the lines.	The lines intersect.

Step 4 and Step 5: Identify the solution to the system. If the lines intersect, identify the point of intersection. Check to make sure it is a solution to both equations. This is the solution to the system. If the lines are parallel, the system has no solution. If the lines are the same, the system has an infinite number of solutions.	Since the lines intersect, find the point of intersection. Check the point in both equations.	The lines intersect at $(4,-1)$. $2x+y=7$ $2(4)+(-1)\overset{?}{=}7$ $8-1\overset{?}{=}7$ $7=7\checkmark$ $x-2y=6$ $4-2(-1)\overset{?}{=}6$ $6=6\checkmark$ The solution is $(4,-1)$.

Your Turn 5.9.3

1. Solve this system of linear equations by graphing.

$$\begin{cases} x-3y=-3 \\ x+y=5 \end{cases}$$

Solving Systems of Linear Equations Using Substitution

We will now solve systems of linear equations by the substitution method. We will use the same system we used for graphing.

$$\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$$

We will first solve one of the equations for either x or y. We can choose either equation and solve for either variable—but we'll try to make a choice that will keep the work easy. Then, we substitute that expression into the other equation. The result is an equation with just one variable—and we know how to solve those!

After we find the value of one variable, we will substitute that value into one of the original equations and solve for the other variable. Finally, we check our solution and make sure it makes both equations true. This process is summarized here:

Step 1: Solve one of the equations for either variable.

Step 2: Substitute the expression from Step 1 into the other equation.

Step 3: Solve the resulting equation.

Step 4: Substitute the solution in Step 3 into either of the original equations to find the other variable.

Step 5: Write the solution as an ordered pair.

Step 6: Check that the ordered pair is a solution to both original equations.

Example 5.9.4: Solving a System of Linear Equations Using Substitution

Solve this system of linear equations by substitution:

$$\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$$

Answer

Step 1: Solve one of the equations for either variable.	We'll solve the first equation for y.	$\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$ > $2x+y=7$ $y=7-2x$
Step 2: Substitute the expression from Step 1 into the other equation.	We replace y in the second equation with the expression$7-2x$	$x-2y=6$ $x-2(7-2x)=6$
Step 3: Solve the resulting equation.	Now we have an equation with just 1 variable. We know how to solve this!	$x-2(7-2x)=6$ $x-14+4x=6$ $5x=20$ $x=4$
Step 4: Substitute the solution from Step 3 into one of the original equations to find the other variable.	We'll use the first equation and replace x with 4.	$2x+y=7$ $2(4)+y=7$ $8+y=7$ $y=-1$

Step 5: Write the solution as an ordered pair.	The ordered pair is $(x, y$	$(4, -1)$
Step 6: Check that the ordered pair is a solution to **both** original equations.	Substitute	$\begin{array}{rlrl} 2x+y & =7 & x-2y & =6 \\ 2(4)+(-1) & \overset{?}{=}7 & 4-2(-1) & \overset{?}{=}6 \\ 7 & =7\checkmark & & 6 \end{array}$ Both equations are true. $(4, -1)$ is the solution to the system.

Your Turn 5.9.4

Solve this system of linear equations by substitution:

$$\begin{cases} x-3y=-3 \\ x+y=5 \end{cases}$$

Solving Systems of Linear Equations Using Elimination

We have solved systems of linear equations by graphing and by substitution. Graphing works well when the variable coefficients are small, and the solution has integer values. Substitution works well when we can easily solve one equation for one of the variables and not have too many fractions in the resulting expression.

The third method of solving systems of linear equations is called the elimination method. When we solved a system by substitution, we started with two equations and two variables and reduced it to one equation with one variable. This is what we'll do with the elimination method, too, but we'll have a different way to get there.

The elimination method is based on the Addition Property of Equality. The Addition Property of Equality says that when you add the same quantity to both sides of an equation, you still have equality. We will extend the Addition Property of Equality to say that when you add equal quantities to both sides of an equation, the results are equal. For any expressions a, b, c, and d:

if $a=b$

and $c=d$

then $a+c=b+d$.

To solve a system of equations by elimination, we start with both equations in standard form. Then we decide which variable will be easiest to eliminate. How do we decide? We want to have the coefficients of one variable be opposites, so that we can add the equations together and eliminate that variable. Notice how that works when we add these two equations together:

$$\begin{array}{r} \begin{cases} 3x+y=5 \\ 2x-y=0 \end{cases} \\ \hline 5x=5 \end{array}$$

The y 's add to zero and we have one equation with one variable. Let us try another one:

$$\begin{cases} x+4y=-2 \\ 2x+5y=-2 \end{cases}$$

This time we do not see a variable that can be immediately eliminated if we add the equations. But if we multiply the first equation by -2 , we will make the coefficients of x opposites. We must multiply every term on both sides of the equation by -2 .

$$\begin{cases} -2(x+4y)=-2(-2) \\ 2x+5y=-2 \end{cases}$$

Then rewrite the system of equations.

$$\begin{cases} -2x-8y=4 \\ 2x+5y=-2 \end{cases}$$

Now we see that the coefficients of the x terms are opposites, so x will be eliminated when we add these two equations.

$$\left\{ \begin{array}{r} -2x-8y=4 \\ 2x+5y=-2 \\ \hline -3y=2 \end{array} \right.$$

Once we get an equation with just one variable, we solve it. Then we substitute that value into one of the original equations to solve for the remaining variable. And, as always, we check our answer to make sure it is a solution to both of the original equations. Here's a summary of using the elimination method:

Step 1: Write both equations in standard form. If any coefficients are fractions, clear them.

Step 2: Make the coefficients of one variable opposites.

Decide which variable you will eliminate.

Multiply one or both equations so that the coefficients of that variable are opposites.

Step 3: Add the equations resulting from Step 2 to eliminate one variable.

Step 4: Solve for the remaining variable.

Step 5: Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable.

Step 6: Write the solution as an ordered pair.

Step 7: Check that the ordered pair is a solution to both original equations.

Exercise 5.9.5

Solve this system of linear equations by elimination:

$$\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$$

Answer

Step 1: Write both equations in standard form. If any coefficients are fractions, clear them.	Both equations are in standard form, $Ax+By=C$. There are no fractions.	$\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$
Step 2: Make the coefficients of one variable opposites. Decide which variable you will eliminate. Multiply one or both equations so that the coefficients of that variable are opposites.	We can eliminate the y 's by multiplying the first equation by 2. Multiply both sides of $2x+y=7$ by 2	$\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$ $\begin{cases} 2(2x+y)=\mathbf{2}(7) \\ x-2y=6 \end{cases}$
Step 3: Add the equations resulting from Step 2 to eliminate one variable.	Solve for x.	$\begin{cases} 4x+2y=14 \\ x-2y=6 \end{cases}$ $5x \quad =20$
Step 4: Solve for the remaining variable.	Substitute $x=4$ into the second equation, $x-2y=6$. Then solve for y.	$x=4$
Step 5: Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable.	Substitute $x=4$ into the second equation, $x-2y=6$. Then solve for y.	$x-2y=6$ $4-2y=6$ $-2y=2$ $y=-1$
Step 6: Write the solution as an ordered pair.	Write it as (x,y)	$(4,-1)$
Step 7: Check that the ordered pair is a solution to **both** original equations.	Substitute $x=4$, $y=-1$ into $2x+y=7$ and $x-2y=6$. Do they make both equations true? Yes!	$\begin{aligned} 2x+y &= 7 \\ 2(4)+(-1) &\overset{?}{=} 7 \\ 7 &= 7\checkmark \end{aligned} \quad \begin{aligned} x-2y &= 6 \\ 4-2(-1) &\overset{?}{=} 6 \\ 6 &= 6\checkmark \end{aligned}$ The solution is $(4,-1)$

Your Turn 5.9.5

Solve this system of linear equations by elimination:

$$\begin{cases} x-3y=-3 \\ x+y=5 \end{cases}$$

Identifying Systems with No Solution or Infinitely Many Solutions

In all the systems of linear equations so far, the lines intersected, and the solution was one point. In Example 5.86 and Example 5.87, we will look at a system of equations that has no solution and at a system of equations that has an infinite number of solutions.

Example 5.9.6: Solving a System of Linear Equations with No Solution

Solve the system by a method of your choice:

$$\begin{cases} y=\frac{1}{2}x-3 \\ x-2y=4 \end{cases}$$

$x - 2y = 4$

Answer

Let us solve the system of linear equations by graphing.

$$\begin{cases} y = \frac{1}{2}x - 3 \\ x - 2y = 4 \end{cases}$$

To graph the first equation, we will use its slope and y-intercept.

$$\begin{aligned} \mathbf{y = \frac{1}{2}x - 3} \\ m = \frac{1}{2} \\ b = -3 \end{aligned}$$

To graph the second equation, we will use the intercepts.

$x - 2y = 4$

x	y
0	−2
4	0

Graph the lines (Figure ✓ 5.9.3).

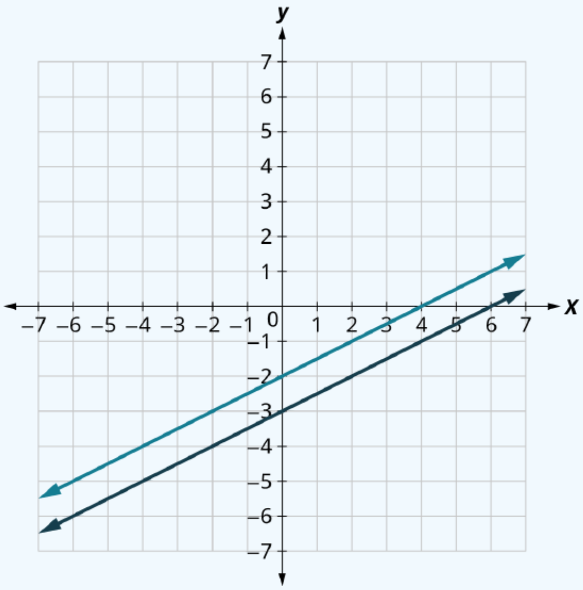


Figure ✓ 5.9.3

Determine the points of intersection. The lines are parallel. Since no point is on both lines, there is no ordered pair that makes both equations true. There is no solution to this system.

Your Turn 5.9.6

Solve the system by a method of your choice:

$$\begin{cases} y & = & \frac{1}{3}x + 2 \\ x - 3y & = & 7 \end{cases}$$

✓ Example 5.9.7: Solving a System of Linear Equations with Infinite Solutions

Solve the system by a method of your choice:

$$\begin{cases} y=2x-3 \\ -6x+3y=-9 \end{cases}$$

Answer

Let us solve the system of linear equations by graphing.

$$\begin{cases} y=2x-3 \\ -6x+3y=-9 \end{cases}$$

Find the slope and y-intercept of the first equation.

$$\begin{aligned} y&=2x-3 \\ m&=2 \\ b&=-3 \end{aligned}$$

Find the intercepts of the second equation.

$$-6x+3y=-9$$

x	y
0	−3
$\frac{3}{2}$	0

Graph the lines (Figure ✓ 5.9.4).

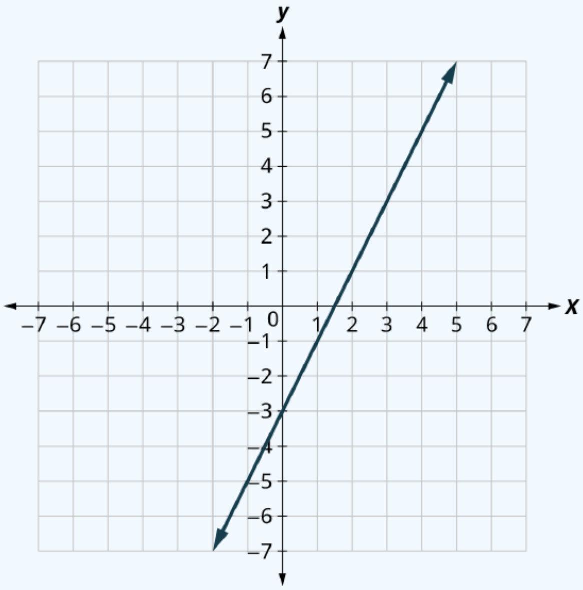


Figure ✓ 5.9.4

The lines are the same! Since every point on the line makes both equations true, there are infinitely many ordered pairs that make both equations true.

There are infinitely many solutions to this system.

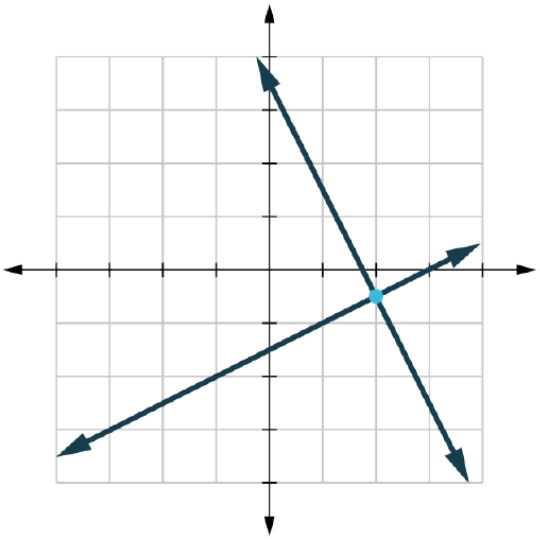
Your Turn 5.9.7

Solve the system by a method of your choice:

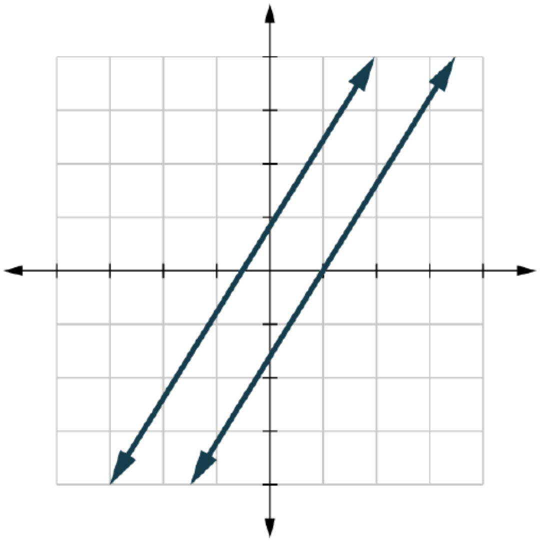
$$\begin{cases} y &= 3x-5 \\ -18x+6y &= -30 \end{cases}$$

In the previous example, if you write the second equation in slope-intercept form, you may recognize that the equations have the same slope and same y-intercept. Since every point on the line makes both equations true, there are infinitely many ordered pairs that make both equations true. There are infinitely many solutions to the system. We say the two lines are coincident. **Coincident lines** have the same slope and same y-intercept. A system of equations that has at least one solution is called a **consistent system**. A system with parallel lines has no solution. We call a system of equations like this an **inconsistent system**. It has no solution.

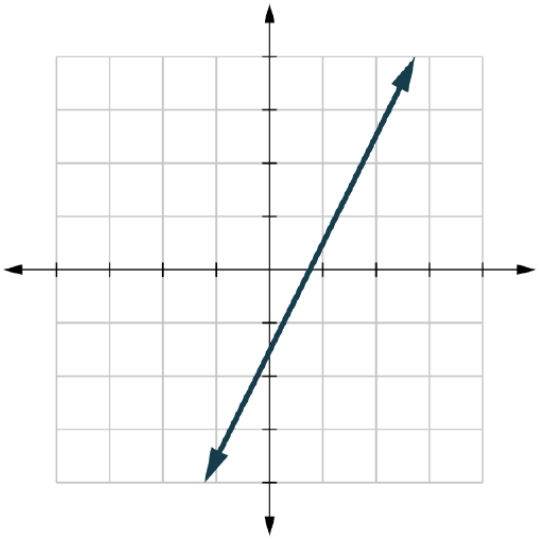
We also categorize the equations in a system of equations by calling the equations independent or dependent. If two equations are independent, they each have their own set of solutions. Intersecting lines and parallel lines are independent. If two equations are dependent, all the solutions of one equation are also solutions of the other equation. When we graph two dependent equations, we get coincident lines. Let us sum this up by looking at the graphs of the three types of systems. See Figure $5.9.5$ and the table that follows



Intersecting



Parallel



Coincident

Figure $5.9.5$

Lines	Intersecting	Parallel	Coincident
Number of Solutions	1 point	No solution	Infinitely many
Consistent/Inconsistent	Consistent	Inconsistent	Consistent
Dependent/Independent	Independent	Independent	Dependent

WORK IT OUT: Using Matrices and Cramer’s Rule to Solve Systems of Linear Equations

An m by n matrix is an array with m rows and n columns, where each item in the matrix is a number. Matrices are used for many things, but one thing they can be used for is to represent systems of linear equations. For example, the system of linear equations

$$\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$$

can be represented by the following matrix:

$$\begin{bmatrix} 2 & 1 & 7 \\ 1 & -2 & 6 \end{bmatrix}$$

To use Cramer’s Rule, you need to be able to take the determinant of a matrix. The determinant of a 2 by 2 matrix A, denoted $|A|$, is

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = (a_{11} \times a_{22}) + (a_{21} \times a_{12})$$

For example, the determinant of the matrix $\begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} = (2 \times -2) - (3 \times 1) = -4 - 3 = -7.$

Cramer’s Rule involves taking three determinants:

1. The determinant of the first two columns, denoted $|D|$;
2. The determinant of the first column and the third column, denoted $|D_y|$;
3. The determinant of the third column and the first column, denoted $|D_x|$.

Going back to the original matrix $\begin{bmatrix} 2 & 1 & 7 \\ 1 & -2 & 6 \end{bmatrix}$

$$|D| = \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} = -4 - 1 = -5$$

$$|D_y| = \begin{vmatrix} 2 & 7 \\ 1 & 6 \end{vmatrix} = 12 - 7 = 5$$

$$|D_x| = \begin{vmatrix} 7 & 1 \\ 6 & -2 \end{vmatrix} = -14 - 6 = -20$$

Now Cramer’s Rule for the solution of the system will be:

$$x = \frac{|D_x|}{|D|}, y = \frac{|D_y|}{|D|}$$

Putting in the values for these determinants, we have $x = \frac{-20}{-5} = 4; y = \frac{5}{-5} = -1$. The solution to the system is the ordered pair $(4, -1)$.

Solving Applications of Systems of Linear Equations

Systems of linear equations are very useful for solving applications. Some people find setting up word problems with two variables easier than setting them up with just one variable. To solve an application, we will first translate the words into a system of linear equations. Then we will decide the most convenient method to use, and then solve the system.

Step 1: Read the problem. Make sure all the words and ideas are understood.

Step 2: Identify what we are looking for.

Step 3: Name what we are looking for. Choose variables to represent those quantities.

Step 4: Translate into a system of equations.

Step 5: Solve the system of equations using good algebra techniques.

Step 6: Check the answer in the problem and make sure it makes sense.

Step 7: Answer the question with a complete sentence.

Example 5.9.8: Applying System to a Real-World Application

Heather has been offered two options for her salary as a trainer at the gym. Option A would pay her $25,000 a year plus $15 for each training session. Option B would pay her $10,000 a year plus $40 for each training session. How many training sessions would make the salary options equal?

Answer

Step 1: Read the problem.

Step 2: Identify what we are looking for.

We are looking for the number of training sessions that would make the pay equal.

Step 3: Name what we are looking for.

Let $s = \text{Heather's}$salarys=Heather’s salary, and $n = \text{the number of training}$sessionsn=the number of training sessions

Step 4: Translate into a system of equations.

Option A would pay her $25,000 plus $15 for each training session.

$$s = 25{,}000 + 15n$$

Option B would pay her $10,000 + $40 for each training session.

$$s = 10{,}000 + 40n$$

The system is shown.

$$\begin{cases} s = 25{,}000 + 15n \\ s = 10{,}000 + 40n \end{cases}$$

Step 5: Solve the system of equations.

We will use substitution.

Substitute $25{,}000+$15n25,000+15n for s in the second equation

$$\begin{aligned} s &= 25{,}000 + 15n \\ s &= 10{,}000 + 40n \end{aligned}$$

Solve for n.

$$\begin{aligned} \mathbf{25{,}000 + 15}\boldsymbol{n} &= 10{,}000 + 40n \\ 25{,}000 &= 10{,}000 + 25n \\ 15{,}000 &= 25n \\ 600 &= n \end{aligned}$$

Step 6: Check the answer.

Are 600 training sessions a year reasonable?

Are the two options equal when n =600n=600?

Substitute into each equation.

$s = 25,000 + 15\,(600) = 34,000$
$s = 10,000 + 40\,(600) = 34,000$

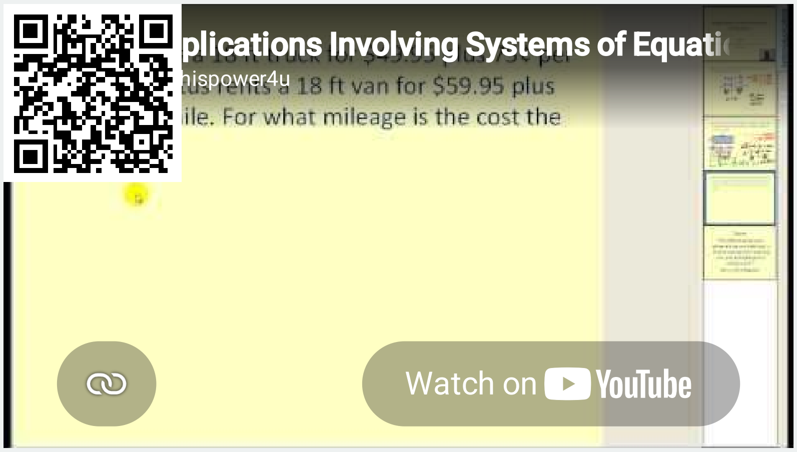
Step 7: Answer the question.

The salary options would be equal for 600 training sessions.

Your Turn 5.9.8

Translate to a system of equations and then solve.
When Jenna spent 10 minutes on the elliptical trainer and then did circuit training for 20 minutes, her fitness app says she burned 278 calories. When she spent 20 minutes on the elliptical trainer and 30 minutes circuit training, she burned 473 calories. How many calories does she burn for each minute on the elliptical trainer? How many calories for each minute of circuit training?

Video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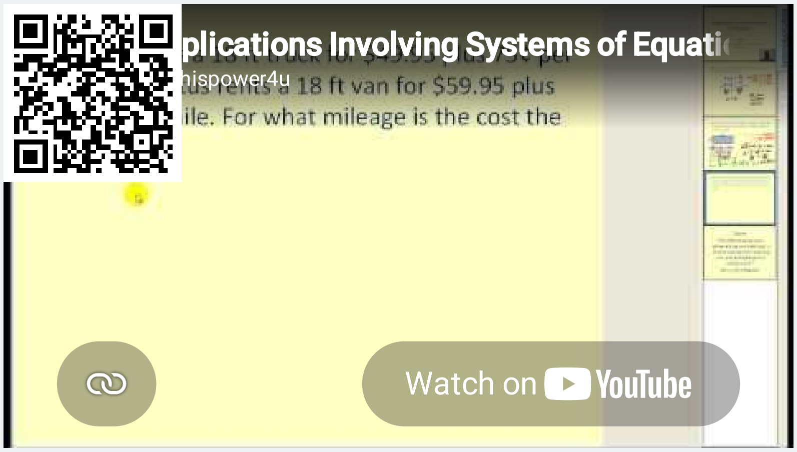
Practice with Solving Applications of Systems of Equations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Applications of Systems of Linear Equations

Check Your Understanding

Decide whether it would be more convenient to solve the system of equations by substitution or elimination.

1. $\left\{\begin{aligned} 8x-15y &= -32 \\ 6x+3y &= -5 \end{aligned}\right.$
2. $\left\{\begin{aligned} x &= 4y-3 \\ 4x-2y &= -6 \end{aligned}\right.$
3. $\left\{\begin{aligned} y &= 7x-5 \\ 3x-2y &= 16 \end{aligned}\right.$
4. $\left\{\begin{aligned} 12x-5y &= -42 \\ 3x+7y &= -15 \end{aligned}\right.$
5. $\left\{\begin{aligned} y &= 4x+9 \\ 5x-2y &= -21 \end{aligned}\right.$
6. $\left\{\begin{aligned} 9x-4y &= 24 \\ 3x+5y &= 14 \end{aligned}\right.$
7. $\left\{\begin{aligned} 14x-15y &= -30 \\ 7x+2y &= 10 \end{aligned}\right.$
8. $\left\{\begin{aligned} x &= 9y-11 \\ 2x-7y &= -27 \end{aligned}\right.$

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5.9.1: Exercises

For the following exercises, determine if the points are solutions to the given system of equations.

Exercise 5.9.1.1

$$\begin{cases} 2x-6y &= 0 \\ 3x-4y &= 5 \end{cases} \qquad \begin{array}{l} \text{A: } (3,1) \\ \text{B: } (-3,4) \end{array}$$

Exercise 5.9.1.2

$$\begin{cases} -3x+y &= 8 \\ -x+2y &= -9 \end{cases} \qquad \begin{array}{l} \text{A: } (-5,-7) \\ \text{B: } (-5,7) \end{array}$$

Exercise 5.9.1.3

$$\begin{cases} x+y &= 2 \\ y &= \frac{3}{4}x \end{cases} \qquad \begin{array}{l} \text{A: } \left(\frac{8}{7},\frac{6}{7}\right) \\ \text{B: } \left(1,\frac{3}{4}\right) \end{array}$$

Exercise 5.9.1.4

$$\begin{cases} 2x+3y &= 6 \\ y &= \frac{2}{3}x+2 \end{cases} \qquad \begin{array}{l} \text{A: } (-6,2) \\ \text{B: } (-3,4) \end{array}$$

For the following exercises, solve the following systems of equations by graphing.

Exercise 5.9.1.5

$$\begin{cases} -x+y &= 2 \\ 2x+y &= -4 \end{cases}$$

Exercise 5.9.1.6

$$\begin{cases} y &= x-2 \\ y &= -3x+2 \end{cases}$$

Exercise 5.9.1.7

$$\begin{cases} y &= \frac{2}{3}x-2 \\ y &= -\frac{1}{3}x-5 \end{cases}$$

Exercise 5.9.1.8

$$\begin{cases} -x+3y &= 3 \\ x+3y &= 3 \end{cases}$$

Exercise 5.9.1.9

$$\begin{cases} 2x-y &= 4 \\ 2x+3y &= 12 \end{cases}$$

Exercise 5.9.1.10

$$\left\{\begin{array}{rcl} -x+2y & = & -6 \\ y & = & -\frac{1}{2}x-1 \end{array}\right.$$

Exercise 5.9.1.11

$$\left\{\begin{array}{rcl} 3x+5y & = & 10 \\ y & = & -\frac{3}{5}x+1 \end{array}\right.$$

Exercise 5.9.1.12

$$\left\{\begin{array}{rcl} 4x & = & 3y+7 \\ 8x-6y & = & 14 \end{array}\right.$$

For the following exercises, solve the systems of equations by substitution.

Exercise 5.9.1.13

$$\left\{\begin{array}{rcl} 2x+y & = & -4 \\ 3x-2y & = & -6 \end{array}\right.$$

Exercise 5.9.1.14

$$\left\{\begin{array}{rcl} 2x+y & = & -2 \\ 3x-y & = & 7 \end{array}\right.$$

Exercise 5.9.1.15

$$\left\{\begin{array}{lcl} x-2y & = & -5 \\ 2x-3y & = & -4 \end{array}\right.$$

Exercise 5.9.1.16

$$\left\{\begin{array}{lcl} x-3y & = & -9 \\ 2x+5y & = & 4 \end{array}\right.$$

Exercise 5.9.1.17

$$\left\{\begin{array}{rcl} -2x+2y & = & 6 \\ y & = & -3x+1 \end{array}\right.$$

Exercise 5.9.1.18

$$\left\{\begin{array}{rcl} 3x+4y & = & 1 \\ y & = & -\frac{2}{5}x+2 \end{array}\right.$$

Exercise 5.9.1.19

$$\left\{\begin{array}{rcl} x & = & 2y \\ 4x-8y & = & 0 \end{array}\right.$$

Exercise 5.9.1.20

$$\begin{cases} y & = & \frac{7}{8}x+4 \\ -7x+8y & = & 6 \end{cases}$$

For the following exercises, solve the systems of equations by elimination.

Exercise 5.9.1.21

$$\begin{cases} 5x+2y & = & 2 \\ -3x-y & = & 0 \end{cases}$$

Exercise 5.9.1.22

$$\begin{cases} 6x-5y & = & -1 \\ 2x+y & = & 13 \end{cases}$$

Exercise 5.9.1.23

$$\begin{cases} 2x-5y & = & 7 \\ 3x-y & = & 17 \end{cases}$$

Exercise 5.9.1.24

$$\begin{cases} 5x-3y & = & -1 \\ 2x-y & = & 2 \end{cases}$$

Exercise 5.9.1.25

$$\begin{cases} 3x-5y & = & -9 \\ 5x+2y & = & 16 \end{cases}$$

Exercise 5.9.1.26

$$\begin{cases} \frac{1}{3}x-y & = & -3 \\ x+\frac{5}{2}y & = & 2 \end{cases}$$

Exercise 5.9.1.27

$$\begin{cases} 6x+3y & = & 9 \\ 2x+y & = & 3 \end{cases}$$

Exercise 5.9.1.28

$$\begin{cases} \frac{2}{3}x+y & = & 5 \\ 2x+3y & = & 11 \end{cases}$$

For the following exercises, solve the system of equations by graphing, substitution, or elimination.

Exercise 5.9.1.29

$$\begin{cases} 3x+y & = & -3 \\ 2x+3y & = & 5 \end{cases}$$

Exercise 5.9.1.30

$$\begin{cases} y &= x+2 \\ y &= -2x+2 \end{cases}$$

Exercise 5.9.1.31

$$\begin{cases} y &= \frac{3}{2}x+1 \\ y &= -\frac{1}{2}x+5 \end{cases}$$

Exercise 5.9.1.32

$$\begin{cases} x+y &= -4 \\ -x+2y &= -2 \end{cases}$$

Exercise 5.9.1.33

$$\begin{cases} -2x+3y &= 3 \\ x+3y &= 12 \end{cases}$$

Exercise 5.9.1.34

$$\begin{cases} x+3y &= -6 \\ y &= -\frac{4}{3}x+4 \end{cases}$$

Exercise 5.9.1.35

$$\begin{cases} 4x-3y &= 8 \\ 8x-6y &= 14 \end{cases}$$

Exercise 5.9.1.36

$$\begin{cases} x &= -3y+4 \\ 2x+6y &= 8 \end{cases}$$

For the following exercises, translate to a system of equations and solve.

Exercise 5.9.1.37

Jackie has been offered positions by two cable companies. The first company pays a salary of $14,000 plus a commission of $100 for each cable package sold. The second pays a salary of $20,000 plus a commission of $25 for each cable package sold. How many cable packages would need to be sold to make the total pay the same?

Exercise 5.9.1.38

Drew burned 1,800 calories Friday playing 1 hour of basketball and canoeing for 2 hours. Saturday she spent 2 hours playing basketball and 3 hours canoeing and burned 3,200 calories. How many calories did she burn per hour when playing basketball? How many calories did she burn per hour when canoeing?

Exercise 5.9.1.39

Mitchell currently sells stoves for company A at a salary of $12,000 plus a $150 commission for each stove he sells. Company B offers him a position with a salary of $24,000 plus a $50 commission for each stove he sells. How many stoves would Mitchell need to sell for the options to equal?

? Exercise $5.9.1.40$

The total number of calories in 2 hot dogs and 3 cups of cottage cheese is 960 calories. The total number of calories in 5 hot dogs and 2 cups of cottage cheese is 1,190 calories. How many calories are in a hot dog? How many calories are in a cup of cottage cheese?

? Exercise $5.9.1.41$

Andrea and Bart go to the local farmers market to purchase some fruit. Andrea buys 4 apples and 5 oranges, which cost $3.10; Bart buys 4 apples and 6 oranges, which cost $3.40. What is the cost of an orange? What is the cost of an apple?

? Exercise $5.9.1.42$

Jack and Jill go to a local farmers market to purchase some fruit. Jack buys 3 peaches and 2 limes, which cost $1.50; Jill buys 6 peaches and 5 limes, which cost $3.45. What is the cost of a peach? What is the cost of a lime?

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5.10: Systems of Linear Inequalities in Two Variables

Learning Objectives

After completing this section, you should be able to:

- Demonstrate whether an ordered pair is a solution to a system of linear inequalities.
- Solve systems of linear inequalities using graphical methods.
- Graph systems of linear inequalities.
- Interpret and solve applications of linear inequalities.

In this section, we will learn how to solve systems of linear inequalities in two variables. In Systems of Linear Equations in Two Variables, we learned how to solve for systems of linear equations in two variables and found a solution that would work in both equations. We can solve systems of inequalities by graphing each inequality (as discussed in Graphing Linear Equations and Inequalities) and putting these on the same coordinate system. The double-shaded part will be our solution to the system. There are many real-life examples for solving systems of linear inequalities.

TULLY'S COFFEE

Figure $5.10.4$: Many college students find part-time jobs at places such as coffee shops to help pay for college. (credit: modification of work “TULLY’s COFFEE” by MIKI Yoshihito/Flickr, CC BY 2.0)

Consider Ming who has two jobs to help her pay for college. She works at a local coffee shop for \$7.50 per hour and at a research lab on campus for \$12 per hour. Due to her busy class schedule, she cannot work more than 15 hours per week. If she needs to make at least \$150 per week, can she work seven hours at the coffee shop and eight hours in the lab?

Determining If an Ordered Pair Is a Solution of a System of Linear Inequalities

The definition of a system of linear inequalities is similar to the definition of a system of linear equations. A system of linear inequalities looks like a system of linear equations, but it has inequalities instead of equations. A system of two linear inequalities is shown here.

$$\begin{cases} x+4y \geq 10 \\ 3x-2y<12 \end{cases}$$

To solve a system of linear inequalities, we will find values of the variables that are solutions to both inequalities. We solve the system by using the graphs of each inequality and show the solution as a graph. We will find the region on the plane that contains all ordered pairs (x, y) that make both inequalities true. The solution of a system of linear inequalities is shown as a shaded region in the xy-coordinate system that includes all the points whose ordered pairs make the inequalities true.

To determine if an ordered pair is a solution to a system of two inequalities, substitute the values of the variables into each inequality. If the ordered pair makes both inequalities true, it is a solution to the system.

✓ Example 5.10.1: Determining Whether an Ordered Pair Is a Solution to a System

Determine whether the ordered pair is a solution to the system:

$$\begin{cases} x+4y \geq 10 \\ 3x-2y < 12 \end{cases}$$

1. $(-2,4)$
2. $(3,1)$

Answer

1. Is the ordered pair $(-2,4)$ a solution?

We substitute $x = -2$ and $y = 4$ into both inequalities.

$$\begin{aligned} x+4y &\geq 10 \\ \mathbf{-2}+4(\mathbf{4}) &\overset{?}{\geq} 10 \\ 14 &\geq 10 \text{ true} \end{aligned} \tag{5.10.1}$$

$$\begin{aligned} 3x-2y &< 12 \\ 3(\mathbf{-2})-2(\mathbf{4}) &\overset{?}{<} 12 \\ -14 &< 12 \text{ true} \end{aligned} \tag{5.10.2}$$

The ordered pair $(-2,4)$ made both inequalities true. Therefore $(-2,4)$ is a solution to this system.

2. Is the ordered pair $(3,1)$ a solution?

We substitute $x = 3$ and $y = 1$ into both inequalities.

$$\begin{aligned} x+4y &\geq 10 \\ \mathbf{3}+4(\mathbf{1}) &\overset{?}{\geq} 10 \\ 7 &\geq 10 \text{ false} \end{aligned} \tag{5.10.3}$$

$$\begin{aligned} 3x-2y &< 12 \\ 3(\mathbf{3})-2(\mathbf{1}) &\overset{?}{<} 12 \\ 7 &< 12 \text{ true} \end{aligned} \tag{5.10.4}$$

The ordered pair $(3,1)$ made one inequality true, but the other one false. Therefore $(3,1)$ is not a solution to this system.

Your Turn 5.10.1

Determine whether the ordered pair is a solution to the system:

1. $(3,-1)$
2. $(6,-3)$

Solving Systems of Linear Inequalities Using Graphical Methods

The solution to a single linear inequality was the region on one side of the boundary line that contains all the points that make the inequality true. The solution to a system of two linear inequalities is a region that contains the solutions to both inequalities. We will review graphs of linear inequalities and solve the linear inequality from its graph.

✓ Example 5.10.2: Solving a System of Linear Inequalities by Graphing

Use Figure ✓ 5.10.2 to solve the system of linear inequalities:

$$\begin{cases} y \geq 2x-1 \\ y < x+1 \end{cases}$$

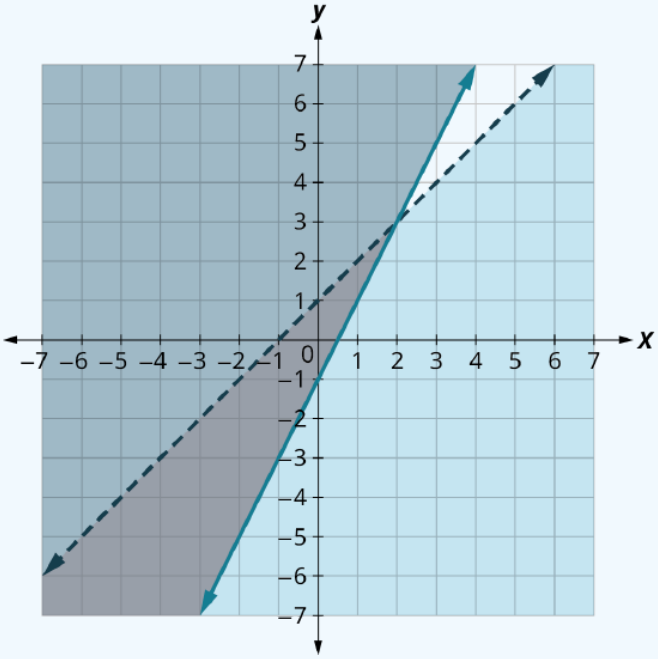


Figure 5.10.2

Answer

To solve the system of linear inequalities we look at the graph and find the region that satisfies BOTH inequalities. To do this we pick a test point and check. Let's us pick $(-1,-1)$.

Is $(-1,-1)$ a solution to $y \geq 2x-1$?

$$\begin{array}{l}-1 \overset{?}{\geq} 2(-1)-1 \\ -1 \geq -3 \text{ true}\end{array}$$

Is $(\mathbf{-1},\mathbf{-1})$ a solution to $y < x+1$?

$$\begin{array}{l}\mathbf{-1} \overset{?}{<} \mathbf{-1}+1 \\ -1 < 0 \text{ true}\end{array}$$

The region containing $(-1,-1)$ is the solution to the system of linear inequalities. Notice that the solution is all the points in the area shaded twice, which appears as the darkest shaded region.

Your Turn 5.10.2

Use the graph shown to solve the system of linear inequalities:

$$\begin{cases} y < 3x+2 \\ y > -x-1 \end{cases}$$

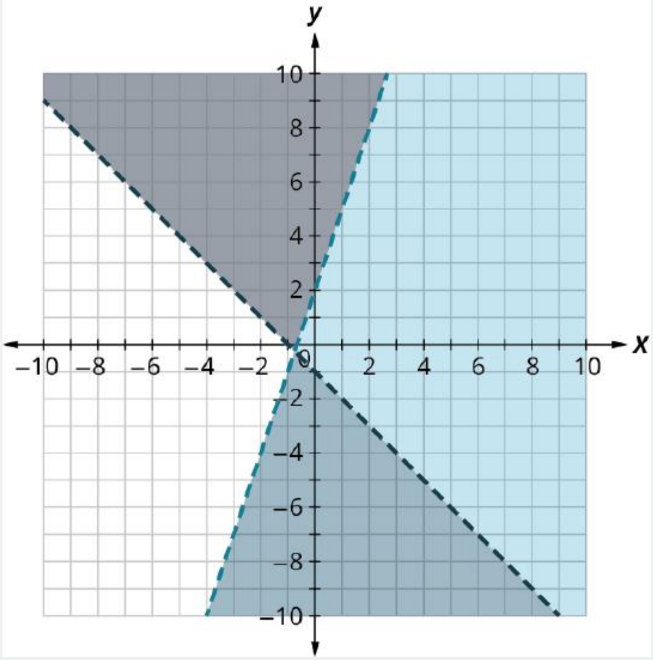


Figure 5.10.3

Graphing Systems of Linear Inequalities

We learned that the solution to a system of two linear inequalities is a region that contains the solutions to both inequalities. To find this region by graphing, we will graph each inequality separately and then locate the region where they are both true. The solution is always shown as a graph.

Step 1: Graph the first inequality.

Graph the boundary line.

Shade in the side of the boundary line where the inequality is true.

Step 2: On the same grid, graph the second inequality.

Graph the boundary line.

Shade in the side of that boundary line where the inequality is true.

Step 3: The solution is the region where the shading overlaps.

Step 4: Check by choosing a test point.

✓ Example 5.10.3: Solving a System of Linear Inequalities by Graphing

Solve the system by graphing:

$$\begin{cases} x - y > 3 \\ y < -\frac{1}{5}x + 4 \end{cases}$$

Answer

Graph $x - y > 3$ by graphing $x - y = 3$ and testing a point (Figure

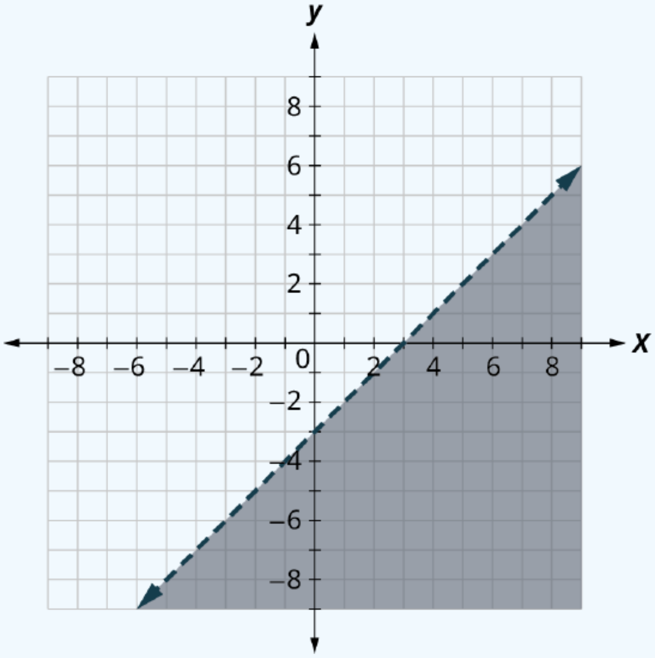


Figure ✓ 5.10.4

Graph $y< -\frac{1}{5}x+4$ by graphing $y= -\frac{1}{5}x+4$ using the slope $m= -\frac{1}{5}$ and y-intercept $b= 4$ (Figure ✓ 5.10.5). The boundary line will be dashed. Test $(0,0)$ which makes the inequality true, so shade the side that contains $(0,0)$.

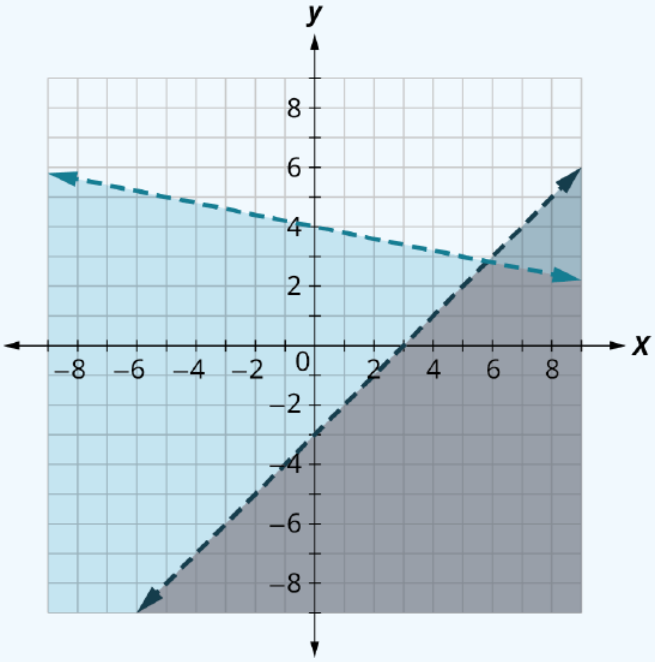


Figure ✓ 5.10.5

Choose a test point in the solution and verify that it is a solution to both inequalities. The point of intersection of the two lines is not included as both boundary lines were dashed. The solution is the area shaded twice—which appears as the darkest shaded region.

Your Turn 5.10.3

Solve the system by graphing:

$$\begin{cases} x+y\leq 2 \\ y\geq \frac{2}{3}x-1 \end{cases}$$

✓ Example 5.10.4: Graphing a System of Linear Inequalities

Solve the system by graphing:

$$\begin{cases} x-2y<5 \\ y>-4 \end{cases}$$

Answer

Graph $x-2y<5$ by graphing $x-2y=5$ (Figure

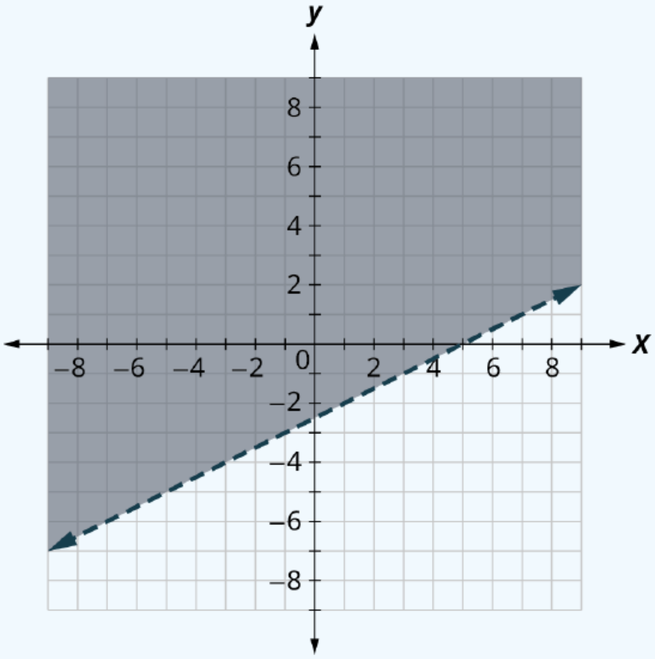


Figure ✓ 5.10.6

Graph

$y > -4$

by graphing

$y = -4$

and recognizing that it is a horizontal line through

$y = -4$

(Figure

✓ 5.10.7

). The boundary line will be dashed. Test

$(0, 0)$

, which makes the inequality true so shade the side that contains

$(0, 0)$

.

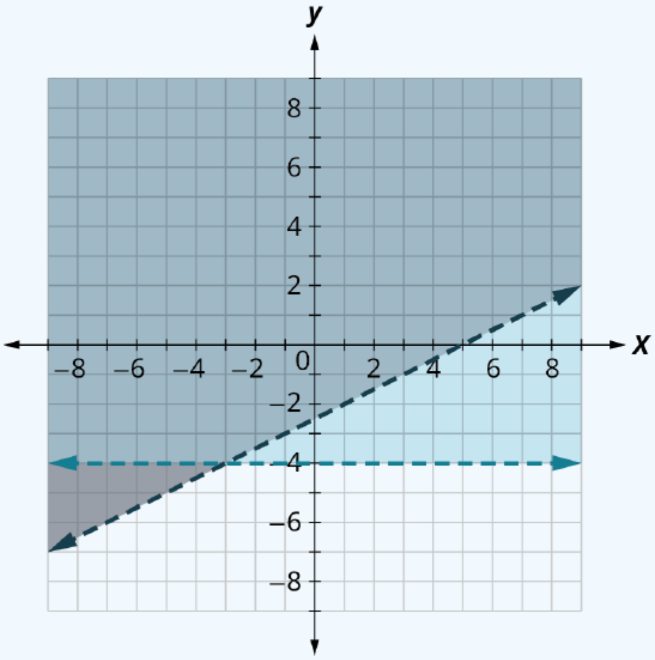


Figure ✓ 5.10.7

The point $(0, 0)$ is in the solution, and we have already found it to be a solution of each inequality. The point of intersection of the two lines is not included as both boundary lines were dashed. The solution is the area shaded twice, which appears as the darkest shaded region.

Your Turn 5.10.4

Solve the system by graphing:

$$\begin{cases} y \geq 3x - 2 \\ y < -1 \end{cases}$$

Systems of linear inequalities where the boundary lines are parallel might have no solution. We will see this in the next example.

✓ Example 5.10.5: Graphing Parallel Boundary Lines with No Solution

Solve the system by graphing:

$$\begin{cases} 4x + 3y \geq 12 \\ y < -\frac{4}{3}x + 1 \end{cases}$$

Answer

Graph $4x+3y\geq 12$, by graphing $4x+3y=12$ (Figure ✔ $5.10.8$) and testing a point. The intercepts are $x=3$ and $y=4$ and the boundary line will be solid. Test $(0,0)$, which makes the inequality false, so shade the side that does not contain $(0,0)$.

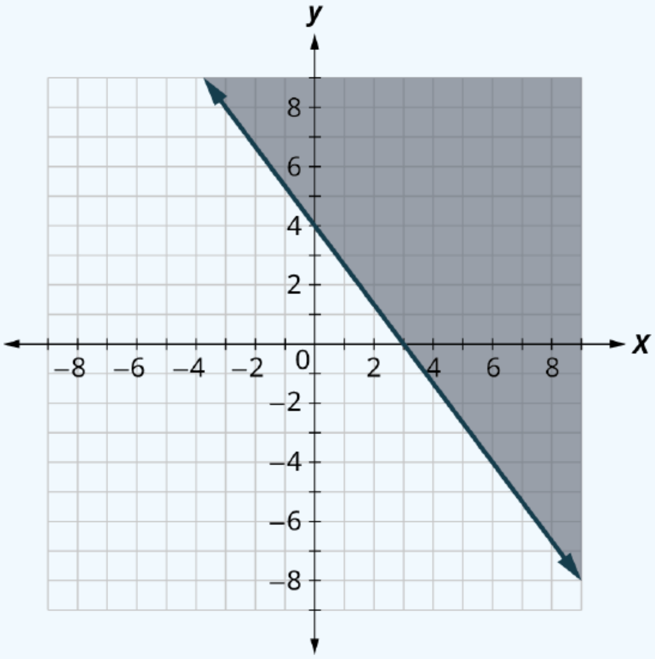
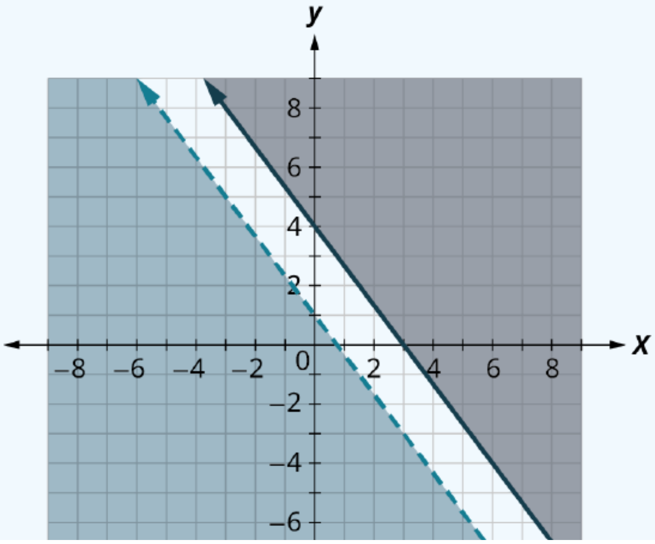


Figure ✓ 5.10.8

Graph $y < -\frac{4}{3}x + 1$ by graphing $y = -\frac{4}{3}x + 1$ using the slope $m = -\frac{4}{3}$ and y -intercept $b = 1$ (Figure ✓ 5.10.9). The boundary line will be dashed. Test $(0, 0)$, which makes the inequality true, so shade the side that contains $(0, 0)$.



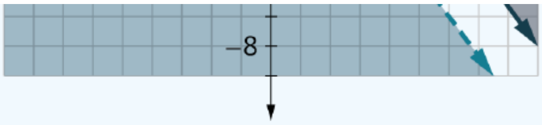


Figure ✓ 5.10.9

No shared point exists in both shaded regions, so the system has no solution.

Your Turn 5.10.5

Solve the system by graphing:
$\begin{cases} 3x-2y\geq 12 \\ y\geq \frac{3}{2}x+1 \end{cases}$

Some systems of linear inequalities where the boundary lines are parallel will have a solution. We will see this in the next example.

✓ Example 5.10.6: Graphing Parallel Boundary Lines with a Solution

Solve the system by graphing:

$$\begin{cases} y>\frac{1}{2}x-4 \\ x-2y<-4 \end{cases}$$

Answer

Graph $y>\frac{1}{2}x-4$ by graphing $y=\frac{1}{2}x-4$ using the slope $m=\frac{1}{2}$ and the y-intercept $b=-4$ (Figure ✓ 5.10.10). The boundary line will be dashed. Test $(0,0)$, which makes the inequality true, so shade the side that contains $(0,0)$.

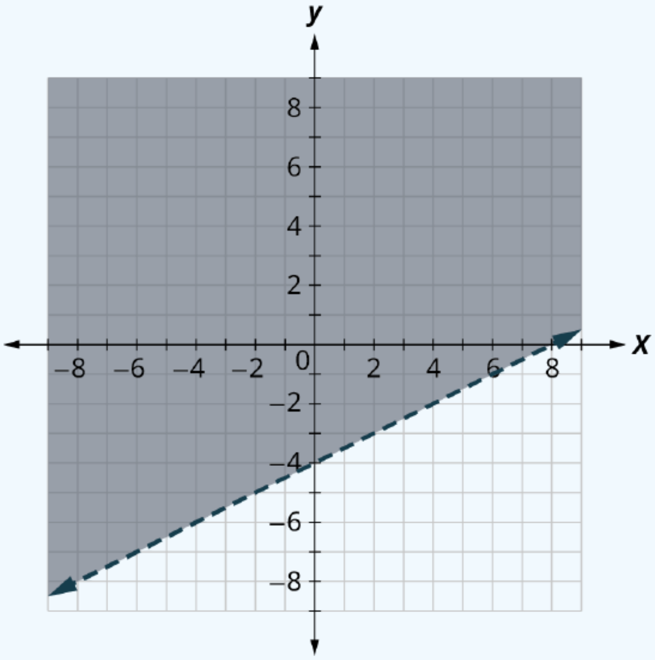


Figure ✓ 5.10.10

Graph

$x - 2y < -4$

by graphing

$x - 2y = -4$

(Figure

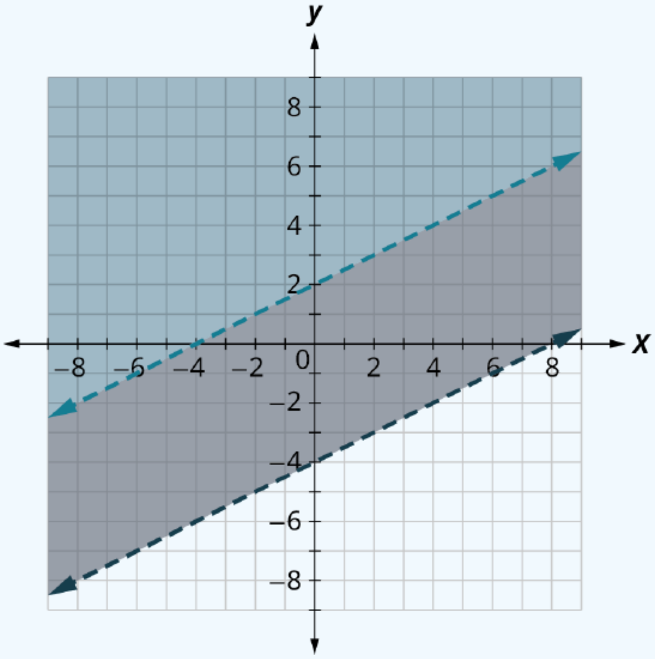


Figure ✓ 5.10.11

No point on the boundary lines is included in the solution as both lines are dashed. The solution is the region that is shaded twice which is also the solution to x − 2 y < − 4 x − 2 y < − 4 .

Your Turn 5.10.6

Solve the system by graphing:

$$\begin{cases} y \geq 3x+1 \\ -3x+y \geq -4 \end{cases}$$

Interpreting and Solving Applications of Linear Inequalities

When solving applications of systems of inequalities, first translate each condition into an inequality. Then graph the system, as we did above, to see the region that contains the solutions. Many situations will be realistic only if both variables are positive, so add inequalities to the system as additional requirements.

✓ Example 5.10.7: Applying Linear Inequalities to Calculating Photo Costs

A photographer sells their prints at a booth at a street fair. At the start of the day, they want to have at least 25 photos to display at their booth. Each small photo they display costs $4 and each large photo costs $10. They do not want to spend more than $200 on photos to display.

1. Write a system of inequalities to model this situation.
2. Graph the system.
3. Could they display 10 small and 20 large photos?
4. Could they display 20 large and 10 small photos?

Answer

1. Let $x=$ the number of small photos and $y=$ the number of large photos. To find the system of equations translate the information. They want to have at least 25 photos.

The number of small plus the number of large should be at least 25.	$x+y\geq 25$
$4 for each small and $10 for each large must be no more than $200	$4x+10y\leq 200$
The number of small photos must be greater than or equal to 0.	$x\geq 0$
The number of large photos must be greater than or equal to 0.	$y\geq 0$
We have our system of equations.	$\begin{cases} x+y\geq 25 \\ 4x+10y\leq 200 \\ x\geq 0 \\ y\geq 0 \end{cases}$

2. Since $x\geq 0$ and $y\geq 0$ (both are greater than or equal to) all solutions will be in the first quadrant. As a result, our graph shows only Quadrant I. To graph $x+y\geq 25$, graph $x+y=25$ as a solid line. Choose $(0,0)$ as a test point. Since it does not make the inequality true, shade the side that does not include the point $(0,0)$.

To graph $4x+10y\leq 200$, graph $4x+10y=200$ as a solid line. Choose $(0,0)$ as a test point. Since it does make the inequality true, shade (bottom left) the side that include the point $(0,0)$.

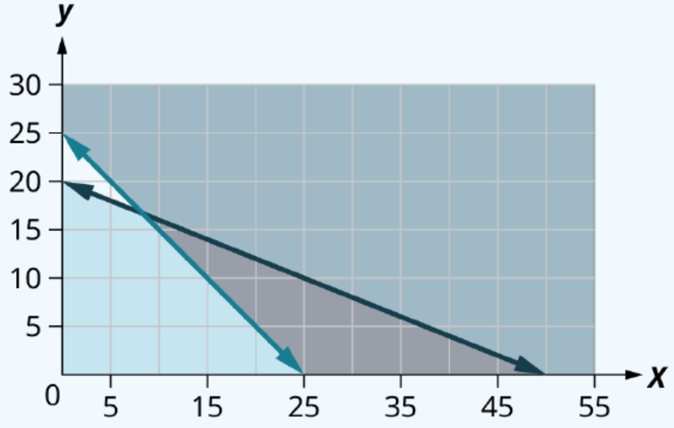


Figure $5.10.12$:

The solution of the system is the region of Figure ✓ $5.10.12$ that is shaded the darkest. The boundary line sections that border the darkly shaded section are included in the solution as are the points on the x-axis from $(25, 0)$ to $(55, 0)$.

3. To determine if 10 small and 20 large photos would work, we look at the graph to see if the point $(10, 20)$ is in the solution region. We could also test the point to see if it is a solution of both equations. It is not, so the photographer would not display 10 small and 20 large photos.

4. To determine if 20 small and 10 large photos would work, we look at the graph to see if the point $(20, 10)$ is in the solution region. We could also test the point to see if it is a solution of both equations. It is, so the photographer could choose to display 20 small and 10 large photos. Notice that we could also test the possible solutions by substituting the values into each inequality.

Your Turn $5.10.7$

Omar needs to eat at least 800 calories before going to his team practice. All he wants is hamburgers and cookies, and he doesn’t want to spend more than $5. At the hamburger restaurant near his college, each hamburger has 240 calories and costs $1.40. Each cookie has 160 calories and costs $0.50.

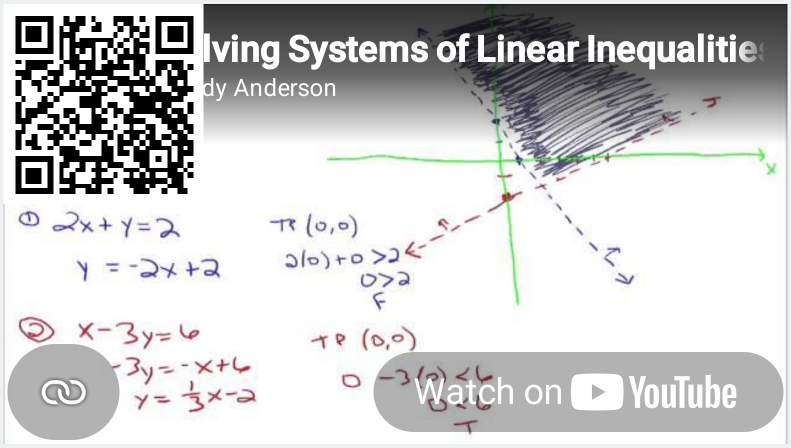
Write a system of inequalities to model this situation.

Graph the system.

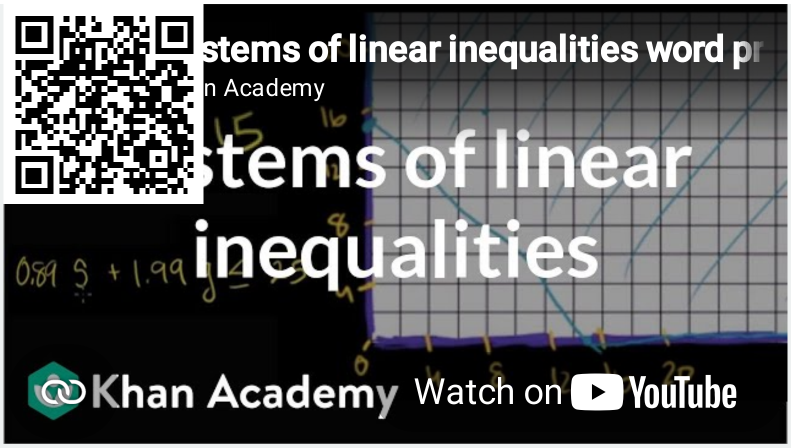
Could he eat 3 hamburgers and 2 cookies?

Could he eat 2 hamburgers and 4 cookies?

Video



Solving Systems of Linear Inequalities by Graphing


stems of linear inequalities word pr
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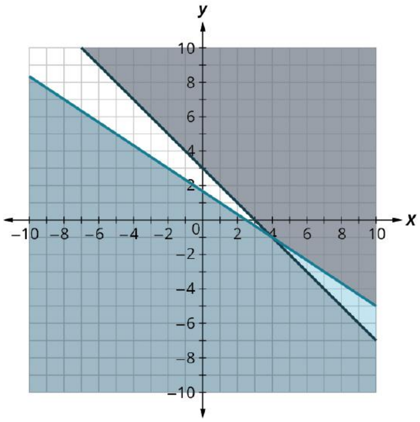
Watch on YouTube

Systems of Linear Inequalities

Check Your Understanding

Match the correct graph to its system of inequalities.

1. $\begin{cases} 2x + 3y < 5 \\ x - y > 3 \end{cases}$



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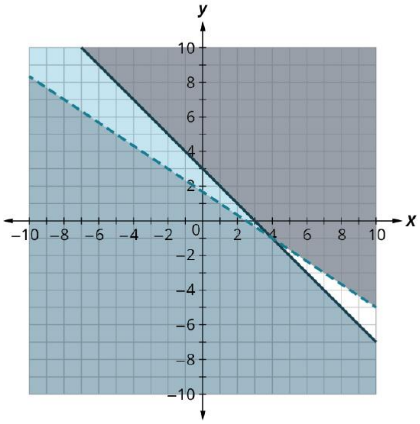
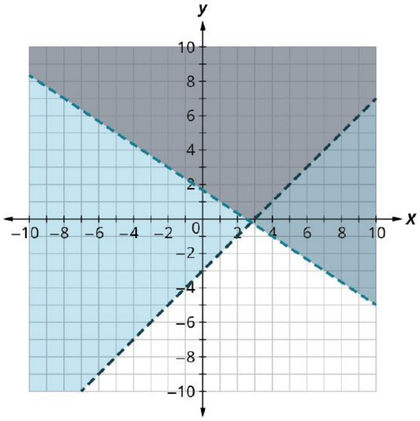


b.



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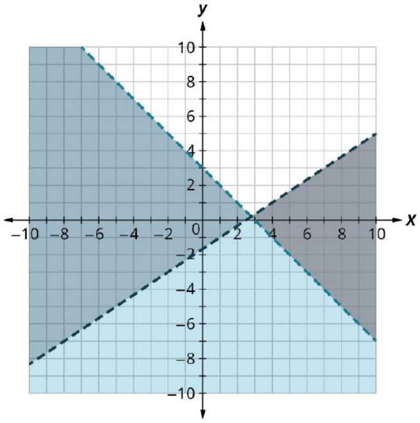
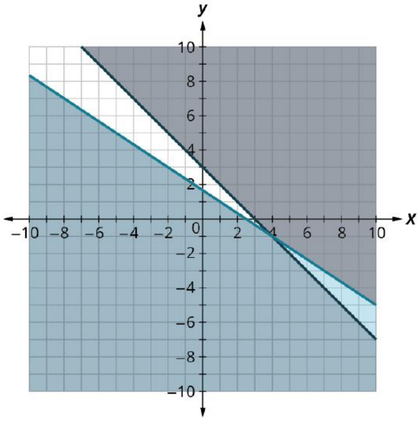
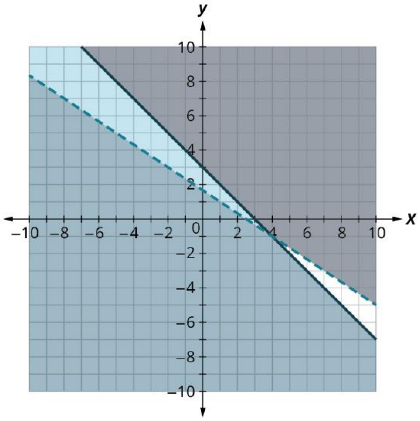


d.



e.

2. $\begin{cases} 2x - 3y \leq 5 \\ x - y \geq 3 \end{cases}$



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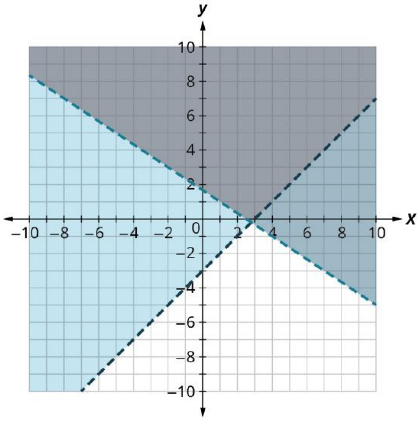
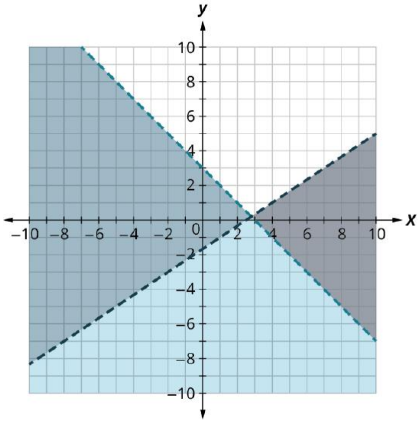
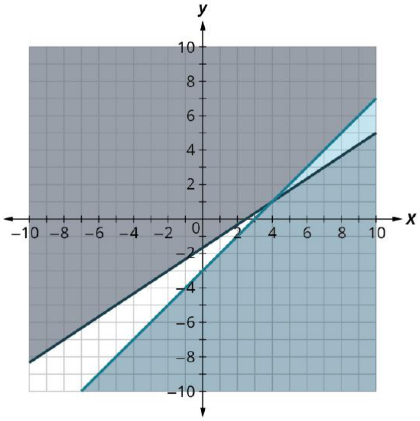
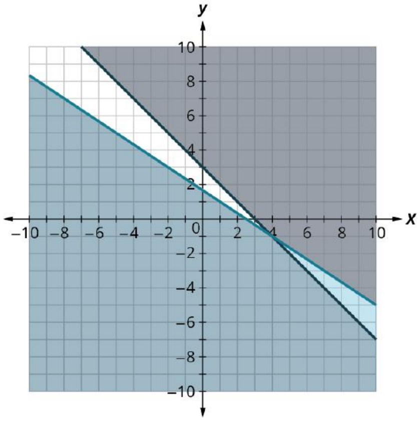
b. 

c. 

d. 

3. $\begin{cases} 2x-3y>5 \\ x+y<3 \end{cases}$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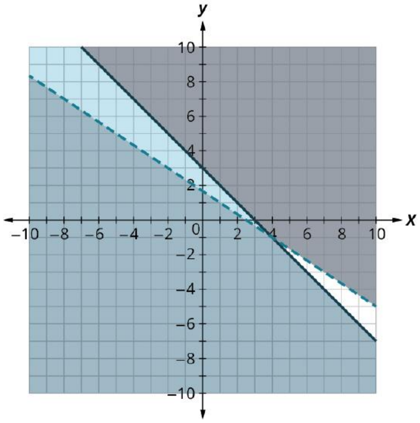
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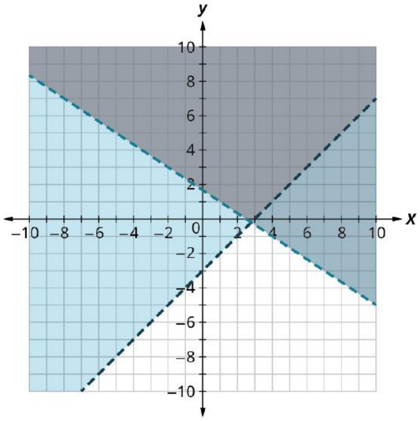
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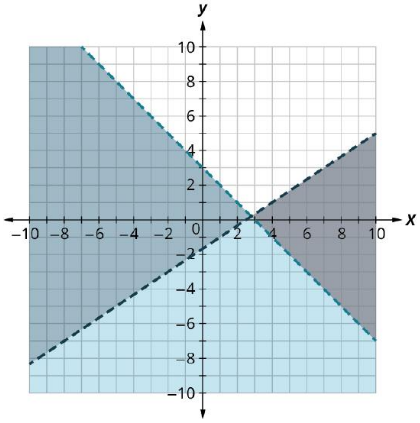
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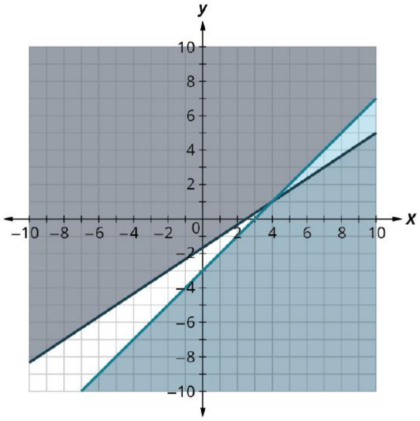
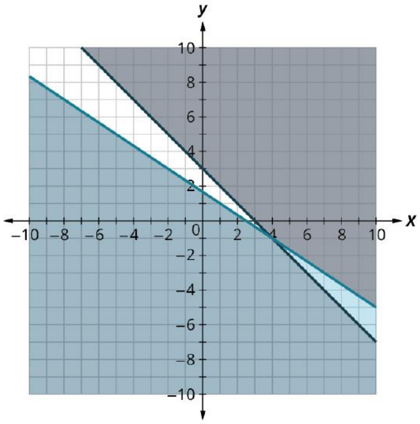
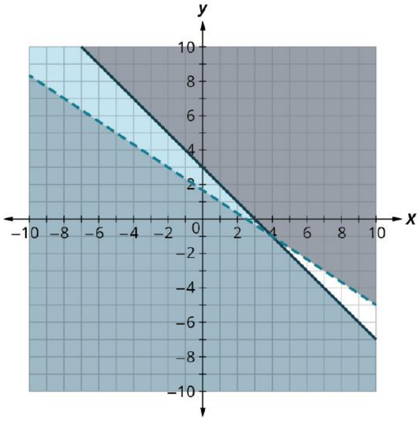


e.

4. $\begin{cases} 2x + 3y > 5 \\ x + y \leq 3 \end{cases}$



a.



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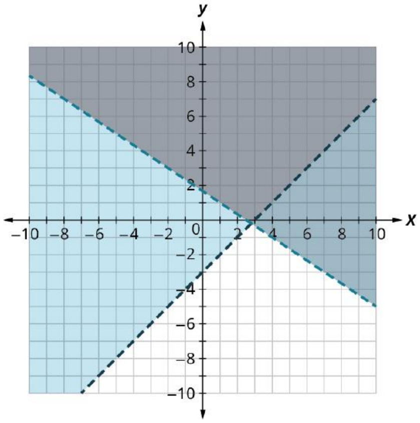

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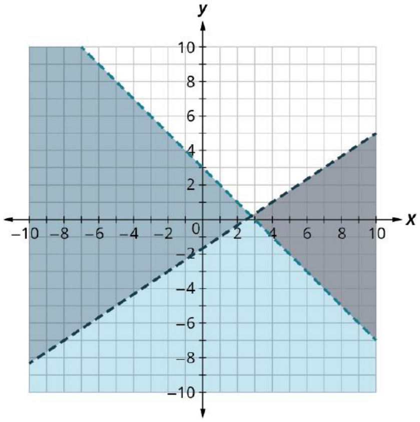

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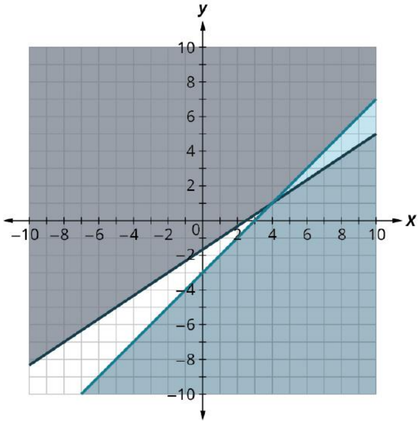

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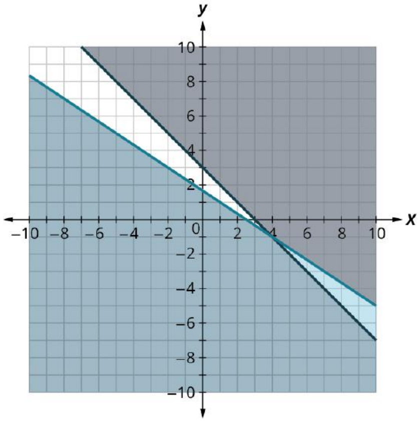
5. $\begin{cases} 2x+3y \leq 5 \\ x+y \geq 3 \end{cases}$



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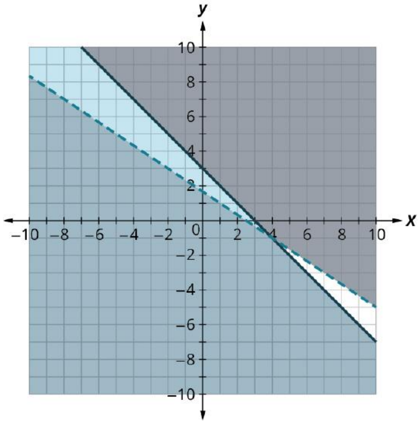

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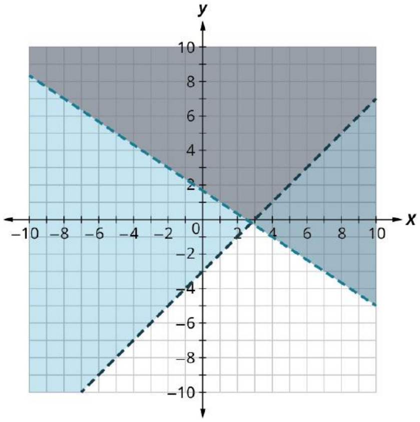

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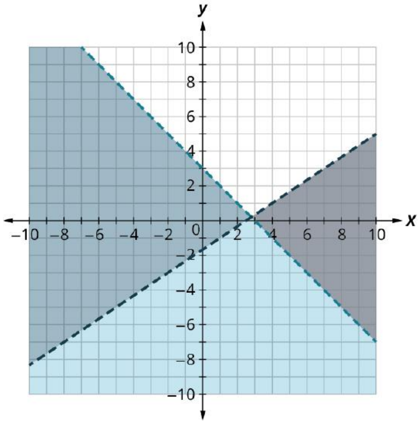

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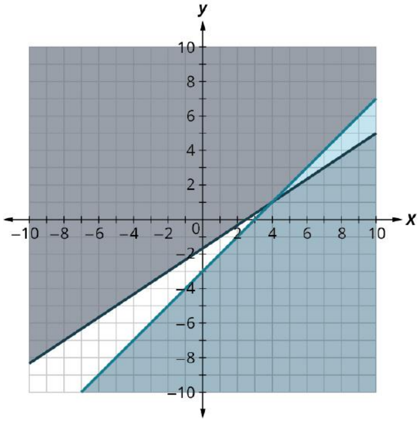

e.

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5.10.0: Exercises

For the following exercises, determine whether each ordered pair is a solution to the system.

Exercise 5.10.0.1

$$\begin{cases} 3x+y>5 \\ 2x-y\leq 10 \end{cases}$$

A. $(3,-3)$
B. $(7,1)$

Exercise 5.10.0.2

$$\begin{cases} y<\frac{3}{2}x+3 \\ \frac{3}{4}x-2y<5 \end{cases}$$

A. $(-4,-1)$
B. $(8,3)$

Exercise 5.10.0.3

$$\begin{cases} 4x-y<10 \\ -2x+2y>-8 \end{cases}$$

A. $(5,-2)$
B. $(-1,3)$

Exercise 5.10.0.4

$$\begin{cases} y>\frac{2}{3}x-5 \\ x+\frac{1}{2}y\leq 4 \end{cases}$$

A. $(6,-4)$
B. $(3,0)$

Exercise 5.10.0.5

$$\begin{cases} 6x-5y<20 \\ -2x+7y>-8 \end{cases}$$

A. $(1,-3)$
B. $(-4,4)$

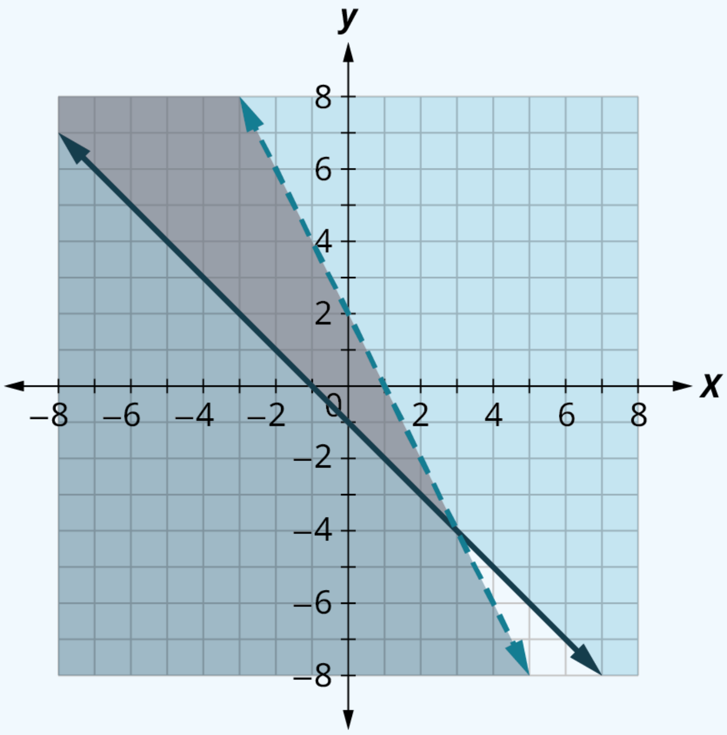
Exercise 5.10.0.6

$$\begin{cases} 7x+2y>14 \\ 5x-y\leq 8 \end{cases}$$

A. $(2,3)$
B. $(7,-1)$

For the following exercises, determine whether each ordered pair is a solution to the darkest shaded region of the graph.

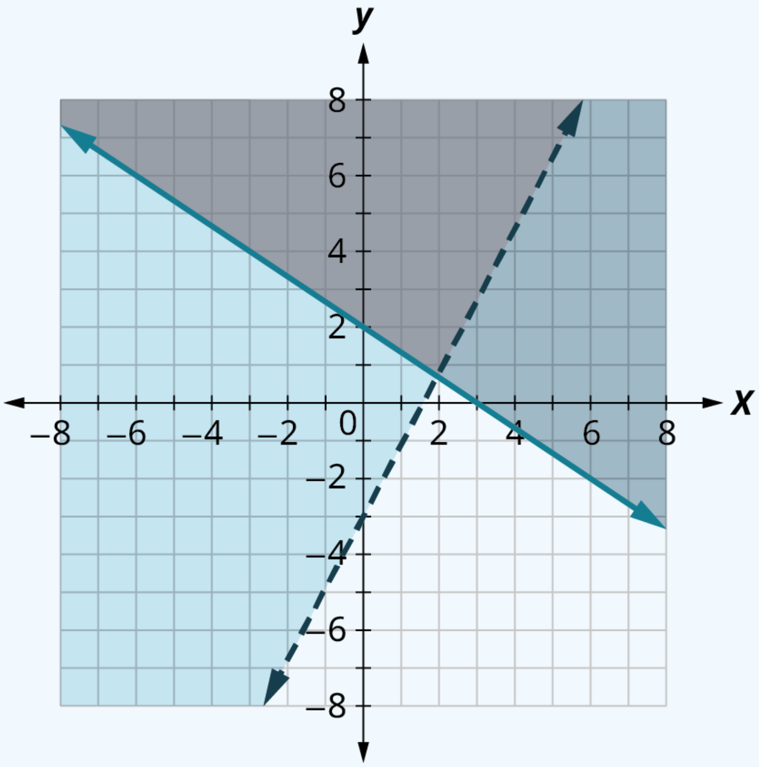
? Exercise 5.10.0.7



A. $(0,0)$

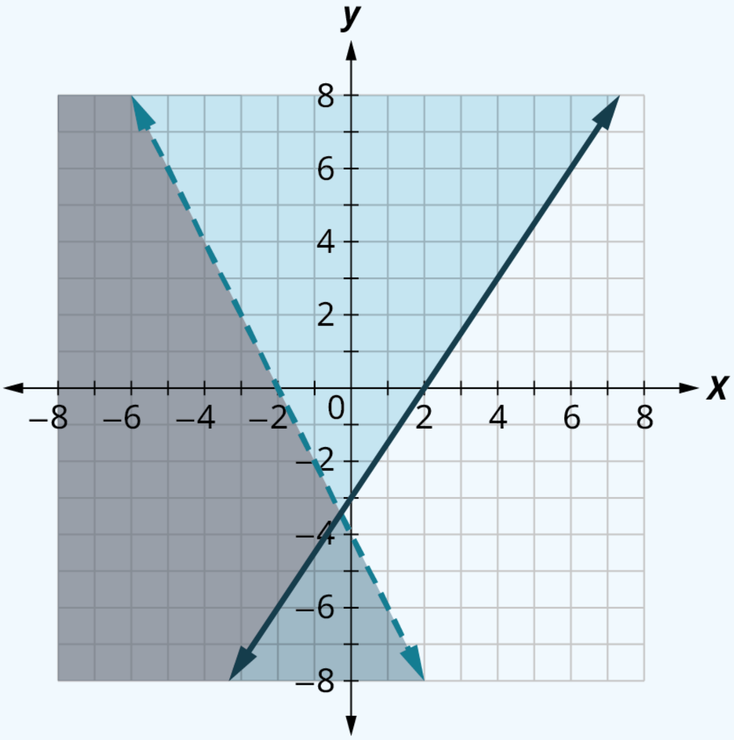
B. $(6,-8)$

? Exercise 5.10.0.8



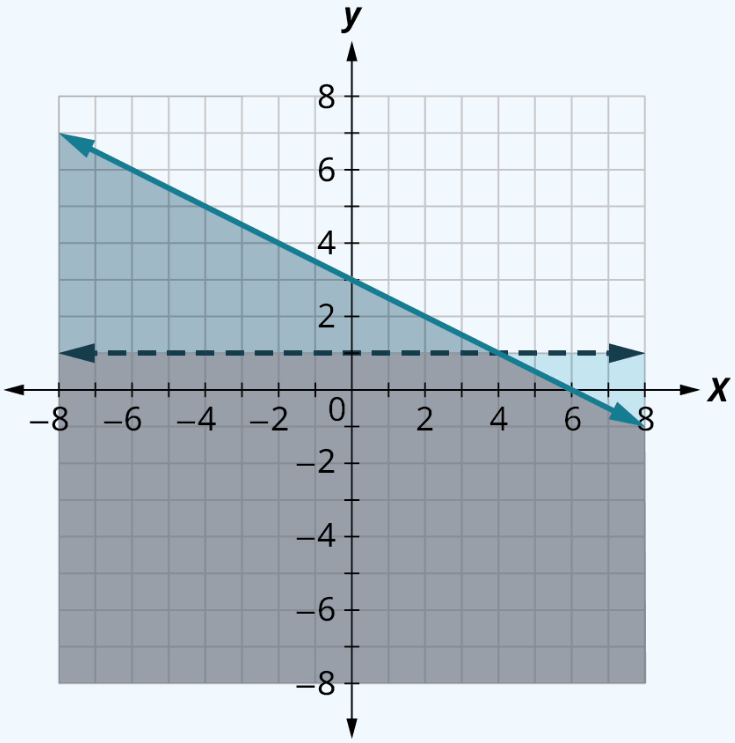
A. $(0,0)$
B. $(2,-4)$

Exercise 5.10.0.9



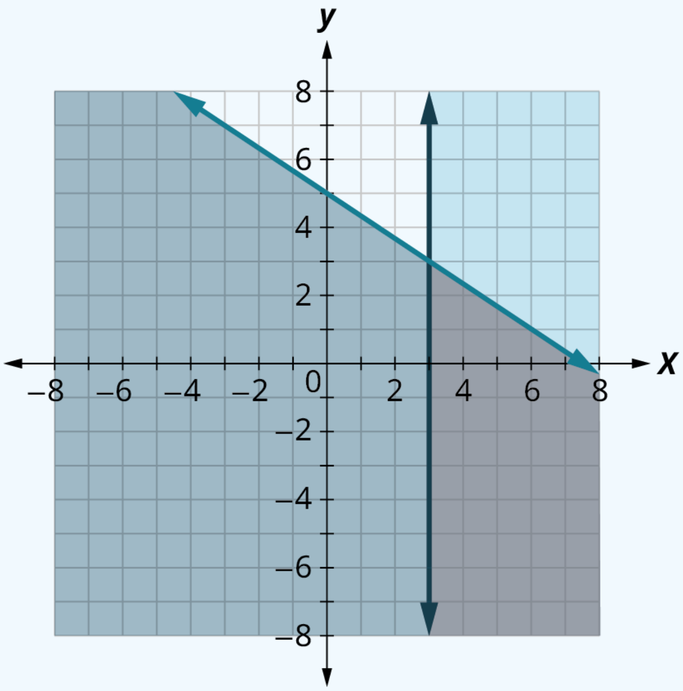
a. $(0,0)$
b. $(2,2)$

Exercise 5.10.0.10



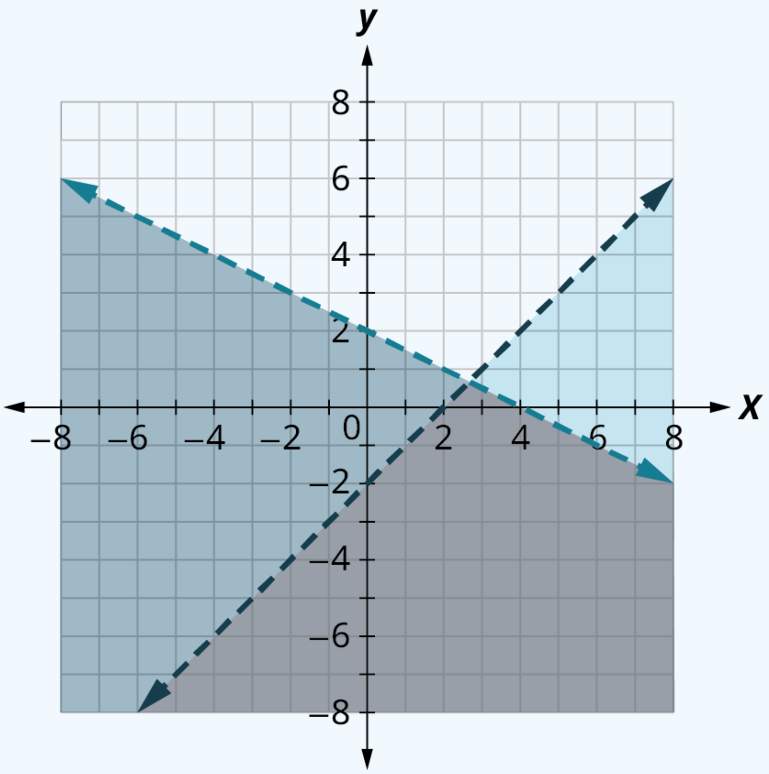
A. $(0,0)$
B. $(2,3)$

Exercise 5.10.0.11



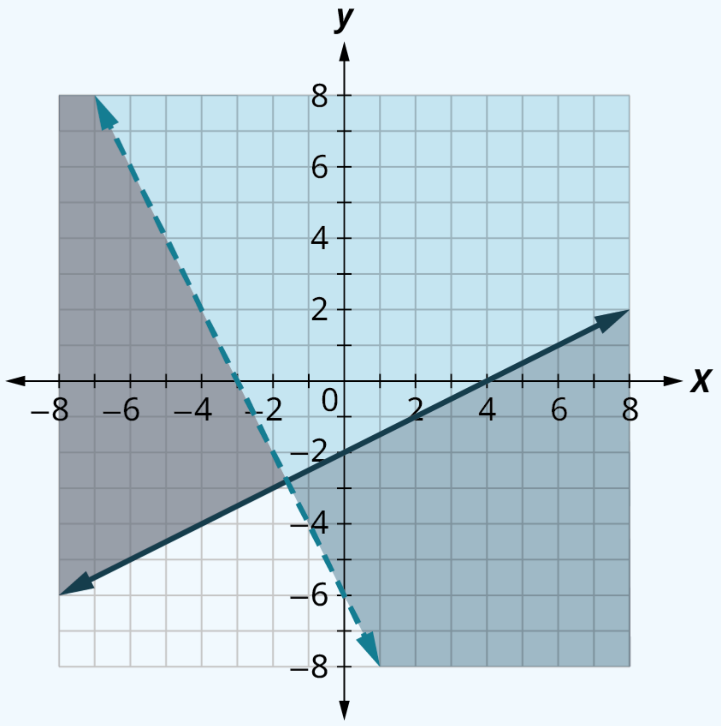
A. $(0,0)$
B. $(6,0)$

Exercise 5.10.0.12



A. $(0, 0)$
B. $(6, 2)$

Exercise 5.10.0.13



A. $(0,0)$
B. $(2,2)$

For the following exercises, solve the systems of linear equations by graphing.

? Exercise 5.10.0.14

$$\begin{cases} y & \leq & 3x+2 \\ y & > & x-1 \end{cases}$$

? Exercise 5.10.0.15

$$\begin{cases} y & < & 2x-1 \\ y & \leq & -\frac{1}{2}x+4 \end{cases}$$

? Exercise 5.10.0.16

$$\begin{cases} x-y & > & 1 \\ y & < & -\frac{1}{4}x+3 \end{cases}$$

? Exercise 5.10.0.17

$$\begin{cases} 3x-y & \geq & 6 \\ y & \geq & -\frac{1}{2}x \end{cases}$$

? Exercise 5.10.0.18

$$\begin{cases} 2x+4y & \geq & 8 \\ y & \leq & \frac{3}{4}x \end{cases}$$

? Exercise 5.10.0.19

$$\begin{cases} 2x-5y & < & 10 \\ 3x+4y & \geq & 12 \end{cases}$$

? Exercise 5.10.0.20

$$\begin{cases} 2x+2y & > & -4 \\ -x+3y & \geq & 9 \end{cases}$$

? Exercise 5.10.0.21

$$\begin{cases} x-2y & < & 3 \\ y & \leq & 1 \end{cases}$$

? Exercise 5.10.0.22

$$\begin{cases} x-3y & > & 4 \\ y & \leq & -1 \end{cases}$$

? Exercise 5.10.0.23

$$\begin{cases} y & \geq & -\frac{1}{2}x-3 \\ x & \leq & 2 \end{cases}$$

Exercise 5.10.0.24

$$\begin{cases} x-3y & \geq & 6 \\ y & > & \frac{1}{3}x+1 \end{cases}$$

Exercise 5.10.0.25

$$\begin{cases} y & \geq & \frac{3}{4}x-2 \\ y & < & 2 \end{cases}$$

Exercise 5.10.0.26

$$\begin{cases} 3x-4y & < & 8 \\ x & < & 1 \end{cases}$$

Exercise 5.10.0.27

$$\begin{cases} -3x+5y & > & 10 \\ x & > & -1 \end{cases}$$

Exercise 5.10.0.28

$$\begin{cases} x & \geq & 3 \\ y & \leq & 2 \end{cases}$$

Exercise 5.10.0.29

$$\begin{cases} x & \leq & -1 \\ y & \geq & 3 \end{cases}$$

Exercise 5.10.0.30

$$\begin{cases} 2x+4y & > & 4 \\ y & \leq & -\frac{1}{2}x-2 \end{cases}$$

For the following exercises, translate to a system of inequalities and solve.

A gardener does not want to spend more than \$50 on bags of fertilizer and peat moss for their garden. Fertilizer costs \$2 a bag and peat moss costs \$5 a bag. The gardener’s van can hold at most 20 bags.

Exercise 5.10.0.31

Write a system of inequalities to model this situation.

Exercise 5.10.0.32

Graph the system.

Exercise 5.10.0.33

Can they buy 15 bags of fertilizer and 4 bags of peat moss?

? Exercise 5.10.0.34

Can they buy 10 bags of fertilizer and 10 bags of peat moss?

For the following exercises, translate to a system of inequalities and solve.

A student is studying for their final exams in chemistry and algebra. They only have 24 hours to study, and it will take them at least 3 times as long to study for algebra than chemistry.

? Exercise 5.10.0.35

Write a system of inequalities to model this situation.

? Exercise 5.10.0.36

Graph the system.

? Exercise 5.10.0.37

Can they spend 4 hours on chemistry and 20 hours on algebra?

? Exercise 5.10.0.38

Can they spend 6 hours on chemistry and 18 hours on algebra?

For the following exercises, translate to a system of inequalities and solve.

Mara is attempting to build muscle mass. To do this, she needs to eat an additional 80 grams of protein or more in a day. A bottle of protein water costs $3.20 and a protein bar costs $1.75. The protein water supplies 27 grams of protein and the bar supplies 16 grams. Let w be the number of water bottles Mara can buy, and let b be the number of protein bars she can buy. If Mara has $10 dollars to spend:

? Exercise 5.10.0.39

Write a system of inequalities to model this situation.

? Exercise 5.10.0.40

Graph the system.

? Exercise 5.10.0.41

Could she buy 3 bottles of protein water and 1 protein bar?

? Exercise 5.10.0.42

Could she buy no bottles of protein water and 5 protein bars?

For the following exercises, translate to a system of inequalities and solve.

Mark is increasing his exercise routine by running and walking at least 4 miles each day. His goal is to burn a minimum of 1,500 calories from this exercise. Walking burns 270 calories/mile and running burns 650 calories/mile.

? Exercise 5.10.0.43

Write a system of inequalities to model this situation.

? Exercise 5.10.0.44

Graph the system.

? Exercise 5.10.0.45

Could he meet his goal by walking 3 miles and running 1 mile?

? Exercise 5.10.0.46

Could he meet his goal by walking 2 miles and running 2 miles?

For the following exercises, translate to a system of inequalities and solve.

Tension needs to eat at least an extra 1,000 calories a day to prepare for running a marathon. He has only $25 to spend on the extra food he needs and will spend it on $0.75 donuts, which have 360 calories each, and $2 energy drinks, which have 110 calories.

? Exercise 5.10.0.47

Write a system of inequalities that models this situation.

? Exercise 5.10.0.48

Graph the system.

? Exercise 5.10.0.49

Can he buy 8 donuts and 4 energy drinks and satisfy his caloric needs?

? Exercise 5.10.0.50

Can he buy 1 donut and 3 energy drinks and satisfy his caloric needs?

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5.11: Linear Programming

Figure 5.11.1: The aftermath of an earthquake and tsunami. (credit: modification of work "Earthquake and Tsunami Japan" by Climate and Ecosystems Change Adaptation Research University Network/Flickr, CC BY 2.0)

Learning Objectives

1. Compose an objective function to be minimized or maximized.
2. Compose inequalities representing a system application.
3. Apply linear programming to solve application problems.

Imagine you hear about some natural disaster striking a far-away country; it could be an earthquake, a fire, a tsunami, a tornado, a hurricane, or any other type of natural disaster. The survivors of this disaster need help—they especially need food, water, and medical supplies. You work for a company that has these supplies, and your company has decided to help by flying the needed supplies into the disaster area. They want to maximize the number of people they can help. However, there are practical constraints that need to be taken into consideration; the size of the airplanes, how much weight each airplane can carry, and so on. How do you solve this dilemma? This is where linear programming comes into play. Linear programming is a mathematical technique to solve problems involving finding maximums or minimums where a linear function is limited by various constraints.

As a field, linear programming began in the late 1930s and early 1940s. It was used by many countries during World War II; countries used linear programming to solve problems such as maximizing troop effectiveness, minimizing their own casualties, and maximizing the damage they could inflict upon the enemy. Later, businesses began to realize they could use the concept of linear programming to maximize output, minimize expenses, and so on. In short, linear programming is a method to solve problems that involve finding a maximum or minimum where a linear function is constrained by various factors.

Who Knew?: A Mathematician Invents a “Tsunami Cannon”

On December 26, 2004, a massive earthquake occurred in the Indian Ocean. This earthquake, which scientists estimate had a magnitude of 9.0 or 9.1 on the Richter Scale, set off a wave of tsunamis across the Indian Ocean. The waves of the tsunami averaged over 30 feet (10 meters) high, and caused massive damage and loss of life across the coastal regions bordering the Indian Ocean.

Usama Kadri works as an applied mathematician at Cardiff University in Wales. His areas of research include fluid dynamics and non-linear phenomena. Lately, he has been focusing his research on the early detection and easing of the effects of tsunamis. One of his theories involves deploying a series of devices along coastlines which would fire acoustic-gravity waves (AGWs) into an oncoming tsunami, which in theory would lessen the force of the tsunami. Of course, this is all in theory, but Kadri believes it will work. There are issues with creating such a device: they would take a tremendous amount of electricity to generate an AGW, for instance, but if it would save lives, it may well be worth it.

Compose an Objective Function to Be Minimized or Maximized

An **objective function** is a linear function in two or more variables that describes the quantity that needs to be maximized or minimized.

✓ Example

✓ $5.11.1$: Composing an Objective Function for Selling Two Products

Miriam starts her own business, where she knits and sells scarves and sweaters out of high-quality wool. She can make a profit of $8 per scarf and $10 per sweater. Write an objective function that describes her profit.

Answer

Let x represent the number of scarves sold, and let y represent the number of sweaters sold. Let P represent profit. Since each scarf has a profit of $\$8$ and each sweater has a profit of $\$10$, the objective function is $P = 8x + 10y$.

Your Turn $5.11.1$

For a fundraiser at school, the Robotics Club is selling bags of apples and bunches of bananas during lunch. They will make a profit of $4 per bag of apples and $6 per bunch of bananas. Write an objective function that describes the profit the Robotics Club will make.

✓ Example $5.11.2$: Composing an Objective Function for Production

William's factory produces two products, widgets and wadgets. It takes 24 minutes for his factory to make 1 widget, and 32 minutes for his factory to make 1 wadget. Write an objective function that describes the time it takes to make the products.

Answer

Let x equal the number of widgets made; let y equal the number of wadgets made; let T represent total time. The objective function is $T = 24x + 32y$.

Your Turn $5.11.2$

Suppose William has a second factory that can make widgets in 20 minutes and wadgets in 28 minutes. Write an objective function that describes the time it takes to make the products.

Composing Inequalities Representing a System Application

For our two examples of profit and production, in an ideal world the profit a person makes and/or the number of products a company produces would have no restrictions. After all, who wouldn't want to have an unrestricted profit? However in reality this is not the case; there are usually several variables that can restrict how much profit a person can make or how many products a company can produce. These restrictions are called constraints.

Many different variables can be constraints. When making or selling a product, the time available, the cost of manufacturing and the amount of raw materials are all constraints. In the opening scenario with the tsunami, the maximum weight on an airplane and the volume of cargo it can carry would be constraints. Constraints are expressed as linear inequalities; the list of constraints defined by the problem forms a system of linear inequalities that, along with the objective function, represent a system application.

✓ Example 5.11.3: Representing the Constraints for Selling Two Products

Two friends start their own business, where they knit and sell scarves and sweaters out of high-quality wool. They can make a profit of \$8 per scarf and \$10 per sweater. To make a scarf, 3 bags of knitting wool are needed; to make a sweater, 4 bags of knitting wool are needed. The friends can only make 8 items per day, and can use not more than 27 bags of knitting wool per day. Write the inequalities that represent the constraints. Then summarize what has been described thus far by writing the objective function for profit and the two constraints.

Answer

Let x represent the number of scarves sold, and let y represent the number of sweaters sold. There are two constraints: the number of items the business can make in a day (a maximum of 8) and the number of bags of knitting wool they can use per day (a maximum of 27). The first constraint (total number of items in a day) is written as:

$$x+y \leq 8$$

Since each scarf takes 3 bags of knitting wool and each sweater takes 4 bags of knitting wool, the second constraint, total bags of knitting wool per day, is written as:

$$3x+4y \leq 27$$

In summary, here are the equations that represent the new business: $P=8x+10y$; This is the profit equation: The business makes $\$8$ per scarf and $\$10$ per sweater.

$$\begin{gathered} x+y \leq 8 \\ 3x+4y \leq 27 \end{gathered}$$

Your Turn 5.11.3

For a fudraiser at school, the Robotics Club is selling bags of apples and bunches of bananas during lunch. They will make a profit of \$4 per bag of apples and \$6 per bunch of bananas. Due to school health regulations, the club is allowed to have only 20 bags and bunches of fruit on school grounds each day to sell. Another regulation: the container where the Robotics Club keeps the fruit has a maximum weight capacity of 70 pounds. Each bag of apples weighs 3 pounds, while each bunch of bananas weighs 5 pounds. Write the inequalities that represent these constraints. Then summarize the equations that represent this system.

✓ Example 5.11.4: Representing Constraints for Production

A factory produces two products, widgets and wadgets. It takes 24 minutes for the factory to make 1 widget, and 32 minutes for the factory to make 1 wadget. Research indicates that long-term demand for products from the factory will result in average sales of 12 widgets per day and 10 wadgets per day. Because of limitations on storage at the factory, no more than 20 widgets or 17 wadgets can be made each day. Write the inequalities that represent the constraints. Then summarize what has been described thus far by writing the objective function for time and the two constraints.

Answer

Let x equal the number of widgets made; let y equal the number of wadgets made. Based on the long-term demand, we know the factory must produce a minimum of 12 widgets and 10 wadgets per day. We also know because of storage limitations, the factory cannot produce more than 20 widgets per day or 17 wadgets per day. Writing those as inequalities, we have:

$$\begin{gathered} x \geq 12 \\ y \geq 10 \\ x \leq 20 \\ y \leq 17 \end{gathered}$$

The number of widgets made per day must be between 12 and 20 , and the number of wadgets made per day must be between 10 and 17. Therefore, we have:

$$\begin{gathered}12 \leq x \leq 20 \\ 10 \leq y \leq 17\end{gathered}$$

The system is:

$$\begin{aligned}&T=24x+32y \\ &12 \leq x \leq 20 \\ &10 \leq y \leq 17\end{aligned}$$

T is the variable for time; it takes 24 minutes to make a widget and 32 minutes to make a wadget.

Your Turn $5.11.4$

Suppose a second factory can make widgets in 20 minutes and wadgets in 28 minutes. Research for this factory indicates that long-term demand for products from this second factory will result in average sales of 15 widgets per day and 13 wadgets per day. Because of limitations on storage at his factory, no more than 22 widgets or 19 wadgets can be made each day. Write the inequalities that represent the constraints. Then summarize what has been described thus far by writing the objective function for time and the two constraints.

Applying Linear Programming to Solve Application Problems

There are four steps that need to be completed when solving a problem using linear programming. They are as follows:

Step 1: Compose an objective function to be minimized or maximized.

Step 2: Compose inequalities representing the constraints of the system.

Step 3: Graph the system of inequalities representing the constraints.

Step 4: Find the value of the objective function at each corner point of the graphed region.

The first two steps you have already learned. Let’s continue to use the same examples to illustrate Steps 3 and 4.

✓ Example $5.11.5$: Solving a Linear Programming Problem for Two Products

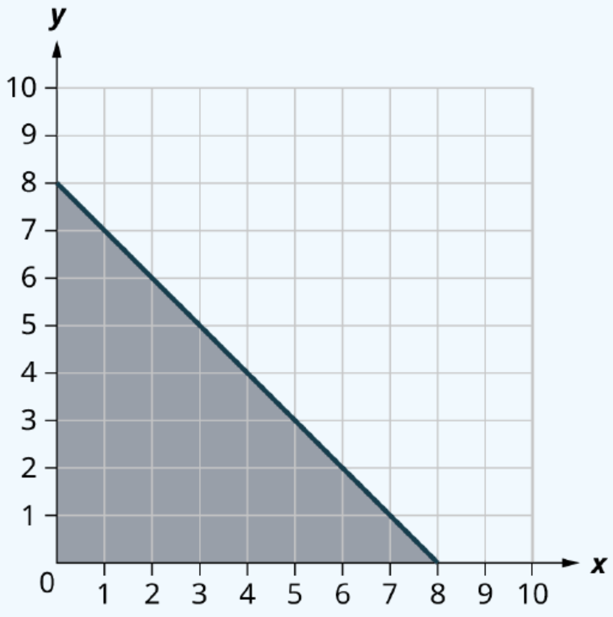
Three friends start their own business, where they knit and sell scarves and sweaters out of high-quality wool. They can make a profit of $8 per scarf and $10 per sweater. To make a scarf, 3 bags of knitting wool are needed; to make a sweater, 4 bags of knitting wool are needed. The friends can only make 8 items per day, and can use not more than 27 bags of knitting wool per day. Determine the number of scarves and sweaters they should make each day to maximize their profit.

Answer

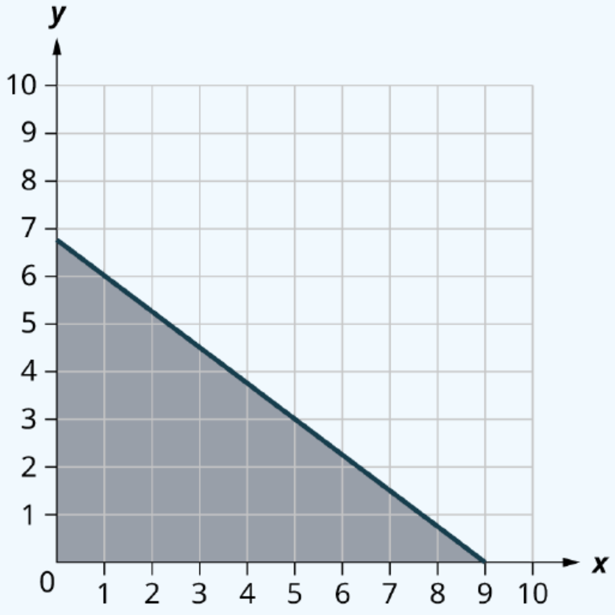
Step 1: Compose an objective function to be minimized or maximized. From Example 5.98, the objective function is $P=8x+10y$.

Step 2: Compose inequalities representing the constraints of the system. From Example 5.98, the constraints are $x+y \leq 8$ and $3x+4y \leq 27$.

Step 3: Graph the system of inequalities representing the constraints. Using methods discussed in Graphing Linear Equations and Inequalities, the graphs of the constraints are shown below. Because the number of scarves (x) and the number of sweaters (y) both must be non-negative numbers (i.e., $x \geq 0$ and $y \geq 0$), we need to graph the system of inequalities in Quadrant I only. Figure ✓ $5.11.2$ shows each constraint graphed on its own axes, while Figure ✓ $5.11.3$ shows the graph of the system of inequalities (the two constraints graphed together). In Figure ✓ $5.11.4$, the large shaded region represents the area where the two constraints intersect. If you are unsure how to graph these regions, refer back to Graphing Linear Equations and Inequalities.



$x + y \leq 8$



$3x + 4y \leq 27$

Figure 5.✓ 5.11.2 Graphs of each constraint

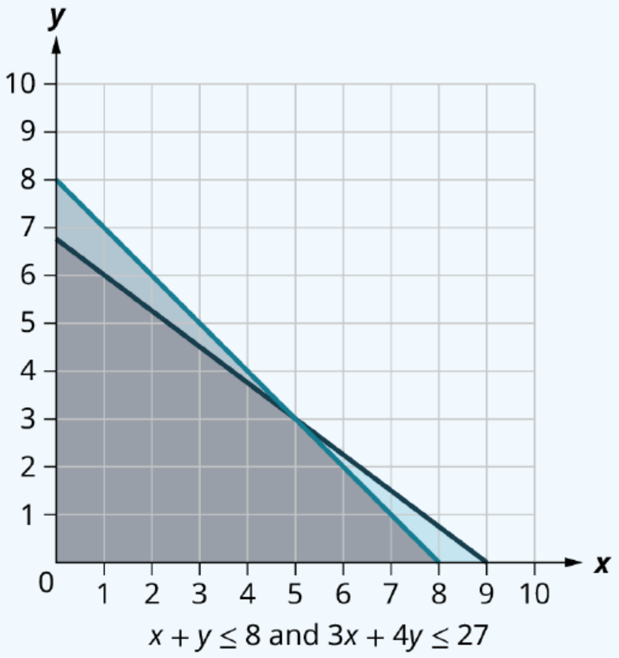


Figure 5.11.3: Graph of the System of Inequalities

Step 4: Find the value of the objective function at each corner point of the graphed region. The "graphed region" is the area where both of the regions intersect; in Figure 5.112, it is the large shaded area. The "corner points" refer to each vertex of the shaded area. Why the corner points? Because the maximum and minimum of every objective function will occur at one (or more) of the corner points. Figure 5.112 shows the location and coordinates of each corner point.

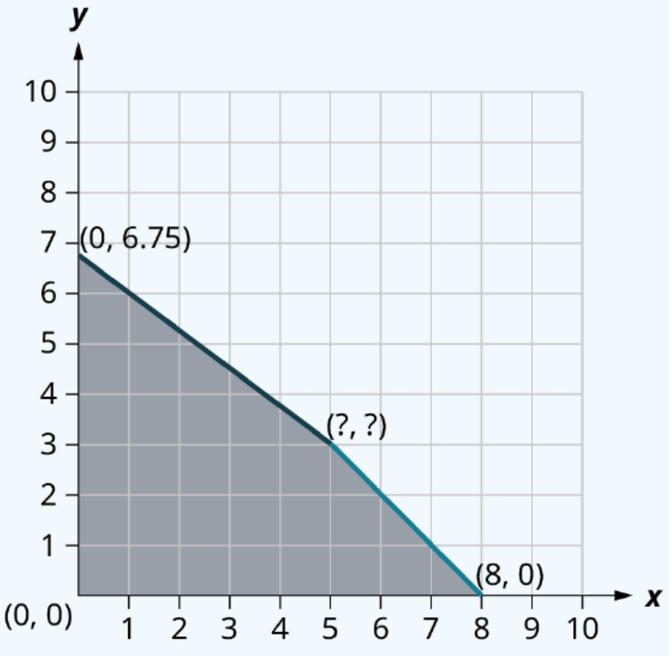


Figure 5.11.4: Graph of Region with Corner Points

Three of the four points are readily found, as we used them to graph the regions; the fourth point, the intersection point of the two constraint lines, will have to be found using methods discussed in Systems of Linear Equations in Two Variables, either using substitution or elimination. As a reminder, set up the two equations of the constraint lines:

$$\begin{aligned} 3x + 4y &= 27 \\ x + y &= 8 \end{aligned}$$

For this example, substitution will be used.

$$\begin{aligned} x + y &= 8 \\ y &= 8 - x. \end{aligned}$$

Substituting $8 - x$ into the first equation for y, we have

$$\begin{aligned} 3x + 4(8 - x) &= 27 \\ 3x + 32 - 4x &= 27 \\ -x &= -5 \\ x &= 5 \end{aligned}$$

Now, substituting the 5 in for x in either equation to solve for y. Choosing the second equation, we have:

$$\begin{aligned} 5 + y &= 8 \\ y &= 3 \end{aligned}$$

Therefore, $x = 5$, and $y = 3$.

To find the value of the objective function, $P = 8x + 10y$, put the coordinates for each corner point into the equation and solve. The largest solution found when doing this will be the maximum value, and thus will be the answer to the question originally posed: determining the number of scarves and sweaters the new business should make each day to maximize their profit.

	Objective Function $P = 8x + 10y$
$(0, 0)$	$P = 8(0) + 10(0) = 0$
$(0, 6.75)$	$P = 8(0) + 10(6.75) = 40 + 30 = 70$
$(5, 3)$	$P = 8(5) + 10(3) = 67.5$
$(8, 0)$	$P = 8(8) + 10(0) = 64$

The maximum value for the profit P occurs when $x = 5$ and $y = 3$. This means that to maximize their profit, the new every day.

Your Turn 5.11.5

For a fundraiser at school, the Robotics Club is selling bags of apples and bunches of bananas during lunch. They will make a profit of $4 per bag of apples and $6 per bunch of bananas. Due to school health regulations, the club is allowed to have only 20 bags and bunches of fruit on school grounds each day to sell. Another regulation: the container where the Robotics Club keeps the fruit has a maximum weight capacity of 70 pounds. Each bag of apples weighs 3 pounds, while each bunch of bananas weighs 5 pounds. Determine the number of bags of apples and the number of bags of bananas the Robotics Club should sell each day to maximize their profit.

People in Mathematics: Leonid Kantorovich

Leonid Vitalyevich Kantorovich was born January 19, 1912, in St. Petersburg, Russia. Two major events affected young Leonid's life: when he was five, the Russian Revolution began, making life in St. Petersburg very difficult; so much so that Leonid's family fled to Belarus for a year. When Leonid was 10, his father died, leaving his mother to raise five children on her own.

Despite the hardships, Leonid showed incredible mathematical ability at a young age. When he was only 14, he enrolled in Leningrad State University to study mathematics. Four years later, at age 18, he graduated with what would be equivalent to a Ph.D. in mathematics.

Although his primary interests were in pure mathematics, in 1938 he began working on problems in economics. Supposedly, he was approached by a local plywood manufacturer with the following question: how to come up with a work schedule for eight lathes to maximize output, given the five different kinds of plywood they had at the factory. By July 1939, Leonid had come up with a solution, not only to the lathe scheduling problem but to other areas as well, such as an optimal crop rotation schedule for farmers, minimizing waste material in manufacturing, and finding optimal routes for transporting goods. The technique he discovered to solve these problems eventually became known as linear programming. He continued to use this technique for solving many other problems involving optimization, which resulted in the book *The Best Use of Economic Resources*, which was published in 1959. His continued work in linear programming would ultimately result in him winning the Nobel Prize of Economics in 1975.

Check Your Understanding

1. Kellie makes tables and chairs. Kellie profits $20 from a table ($t$), and $10 from a chair ($c$). The objective function for profit in this situation is:

a. $P = 20t - 10c$
b. $P = 20c + 10t$
c. $P = 20t + 10c$
d. $P = 20c - 10t$

2. Dave grows wheat (w) and barley (b) on a farm. Dave expects to profit $150 per acre for wheat and $180 per acre for barley. The objective function for profit in this situation is:

a. $P = 150w + 180b$
b. $P = 150b + 180w$
c. $P = 180w + 150b$
d. $P = 150w - 180b$

3. An antique music store sells two types of vinyl records; 45 rpm records (f) and 33 rpm records (t). It makes a profit of $2.50 for each 45 rpm record and $6.75 for each 33 rpm record. The objective function for profit in this situation is:

a. $P = 2.50f + 6.75c$
b. $P = 2.50f + 6.75t$
c. $P = 2.50t + 6.75f$
d. None of these

4. Kellie makes tables and chairs. Kellie profits $20 from a table ($t$), and $10 from a chair ($c$). A table requires 15 board feet of wood, while a chair requires 4 board feet of wood. Kellie has 70 board feet available. What is the constraint inequality in this situation?

a. $20t + 10c \leq 70$
b. $t + c \leq 70$
c. $15t + 4c \geq 70$
d. $15t + 4c \leq 70$

5. Kellie makes tables and chairs. Kellie profits $20 from a table ($t$), and $10 from a chair ($c$). The maximum number of tables and chairs Kellie can make in any one day is 12. What is the constraint inequality in this situation?

a. $t + c \leq 12$
b. $20t + 10c \leq 12$
c. $20t + 10c \geq 12$
d. $20t + 10c \leq 70$

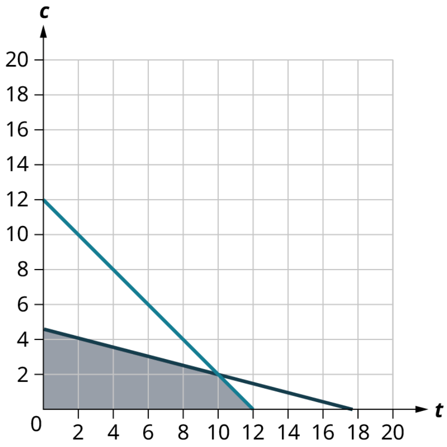
6. Dave grows wheat (w) and barley (b) on a farm. Dave expects to profit $150 per acre for wheat and $180 per acre for barley. The cost of seed is $10 per acre for wheat and $15 per acre for barley. Dave can only afford to spend $945 on seed. What is the constraint inequality in this situation?

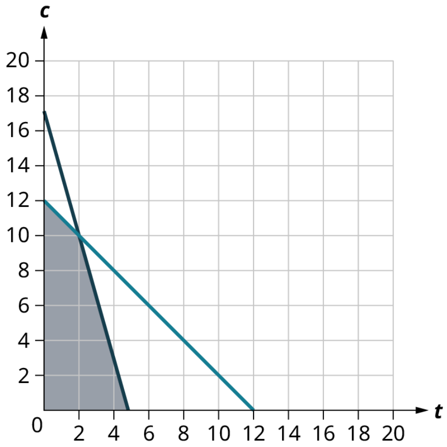
a. $w + b \leq 945$
b. $10w + 15b \geq 945$
c. $10w + 15b \leq 945$
d. $150w + 180w \leq 945$

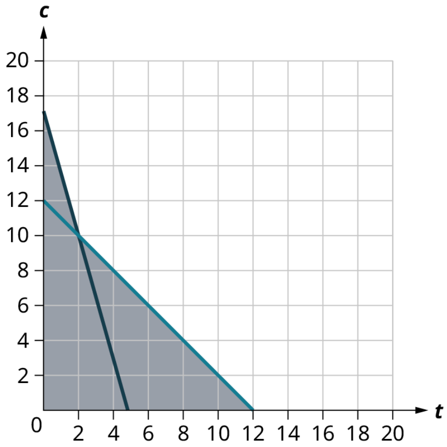
7. Dave grows wheat (w) and barley (b) on a farm. Dave expects to profit $150 per acre for wheat and $180 per acre for barley. The cost of raising each crop is $30 per acre for wheat and $25 per acre for barley. Dave budgets $1,635 for the raising of both crops. What is the constraint inequality in this situation?

a. $150w + 180b \leq 1{,}635$
b. $25w + 30b \leq 1{,}635$
c. $w + b \leq 1{,}635$
d. $30w + 25b \leq 1{,}635$

8. Kellie makes tables and chairs. Kellie profits $20 from a table ($t$), and $10 from a chair ($c$). A table requires 15 board feet of wood, while a chair requires 4 board feet of wood. Kellie has 70 board feet available. The maximum number of tables and chairs Kellie can make in any one day is 12. The graph of the system of inequalities representing the constraints is:







9. Kellie makes tables and chairs. Kellie profits \$20 from a table (t), and \$10 from a chair (c). A table requires 15 board feet of wood, while a chair requires 4 board feet of wood. Kellie has 70 board feet available. The maximum number of tables and chairs Kellie can make in any one day is 12. The four corner points of the system are:

a. $(0,0), (0,12), (10,2), (12,0)$
b. $(0,0), (0,12), (2,10), (4\frac{2}{3},0)$
c. $(0,0), (17.5,0), (2,10), (12,0)$

10. Kellie makes tables and chairs. Kellie profits \$20 from a table (t), and \$10 from a chair (c). A table requires 15 board feet of wood, while a chair requires 4 board feet of wood. Kellie has 70 board feet available. The maximum number of tables and chairs Kellie can make in any one day is 12. The maximum profit Kellie can make in one day is:
\$220 \$120 \$140 \$24

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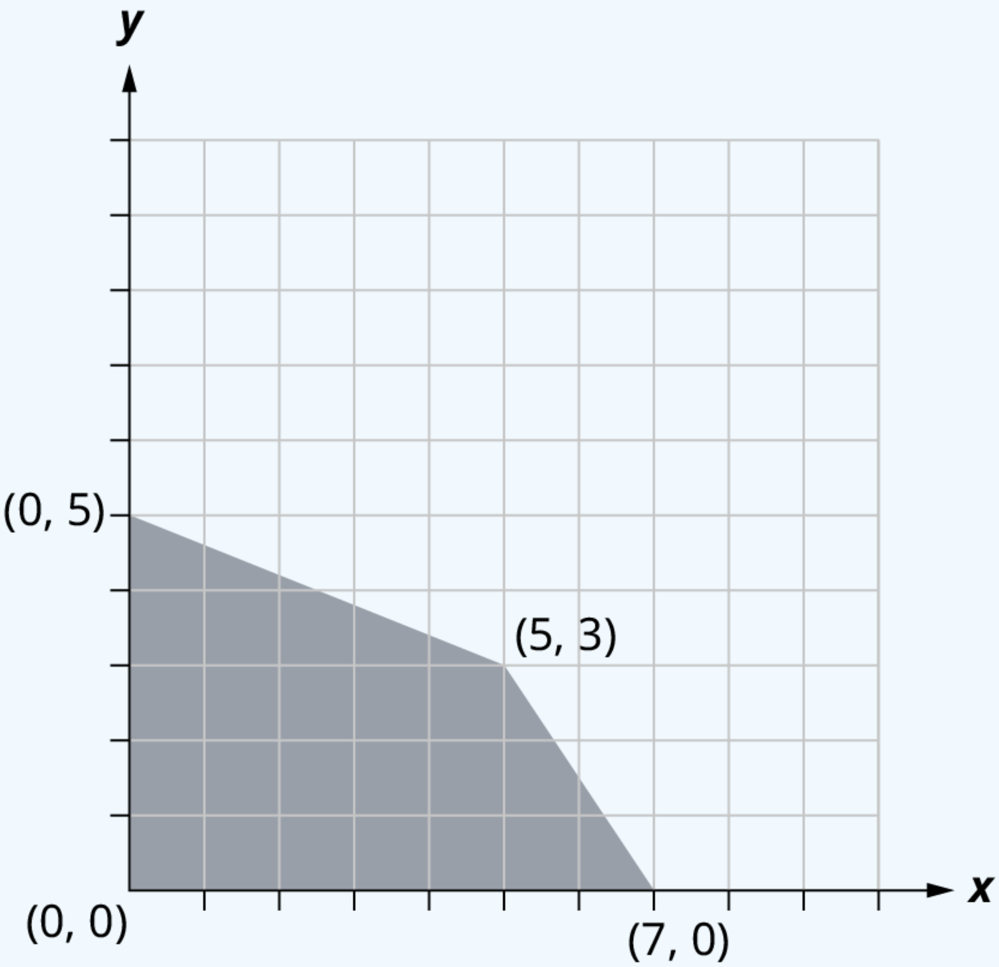
5.11.1: Exercises

For the following exercises, find the value of the objective function at each corner of the graphed region.

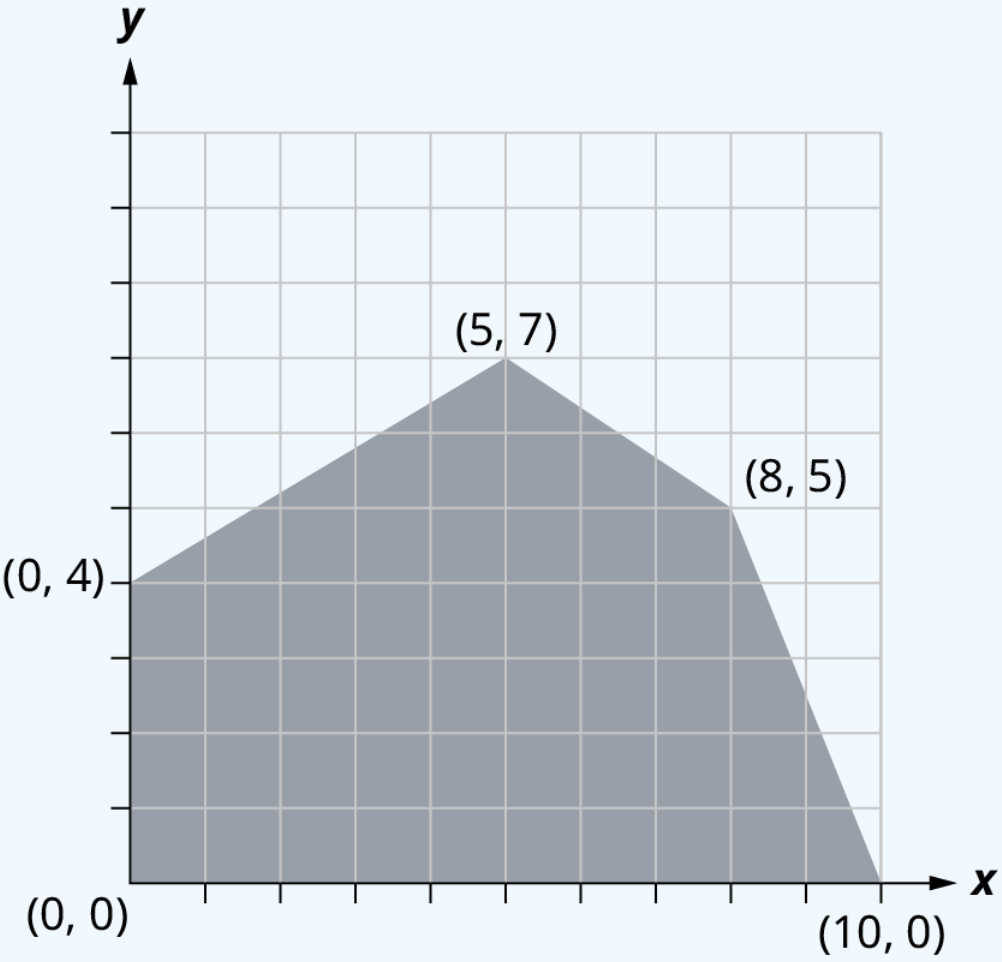
? Exercise 5.11.1.1

Objective Function $P = 6x + 11y$.



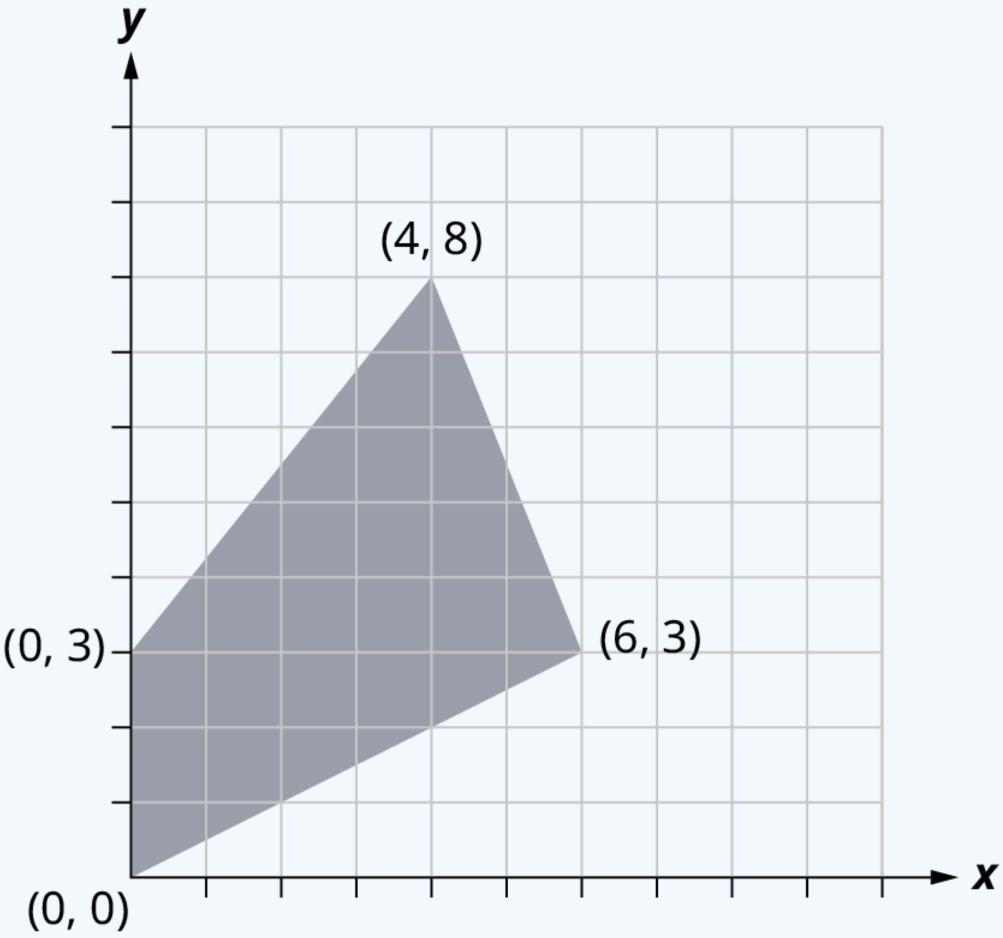
? Exercise 5.11.1.2

Objective Function $T = 5x + 3y$



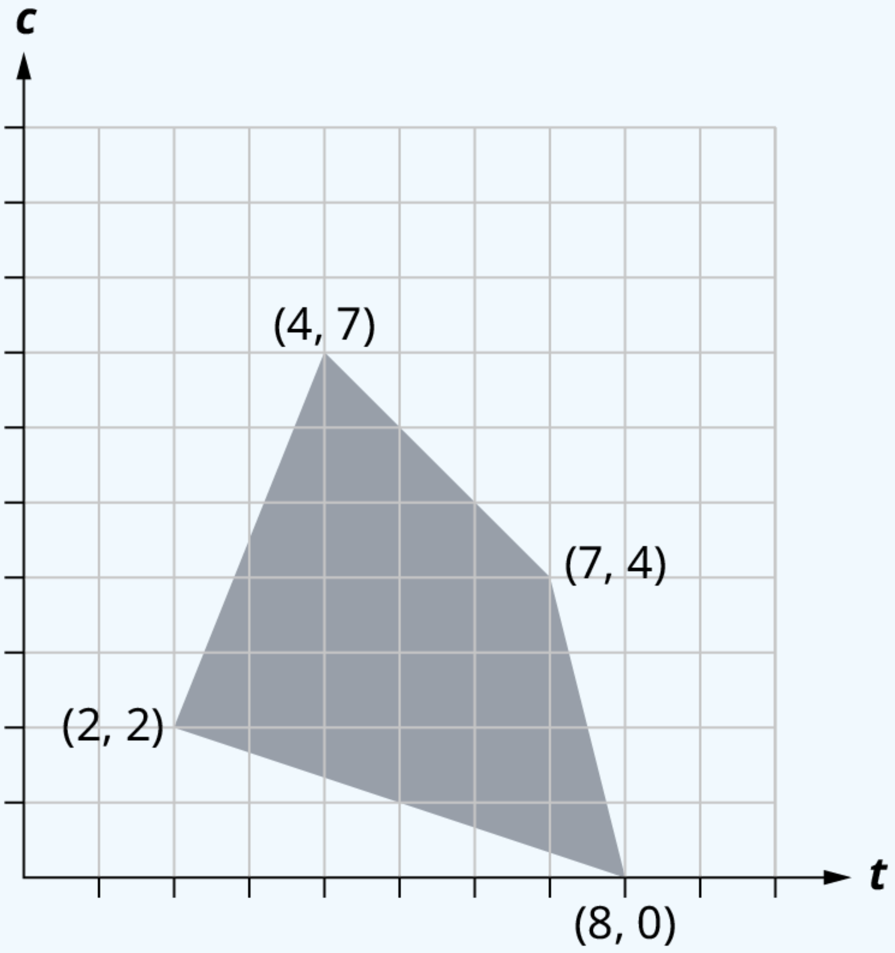
? Exercise 5.11.1.3

Objective Function $L = 33x + 45y$



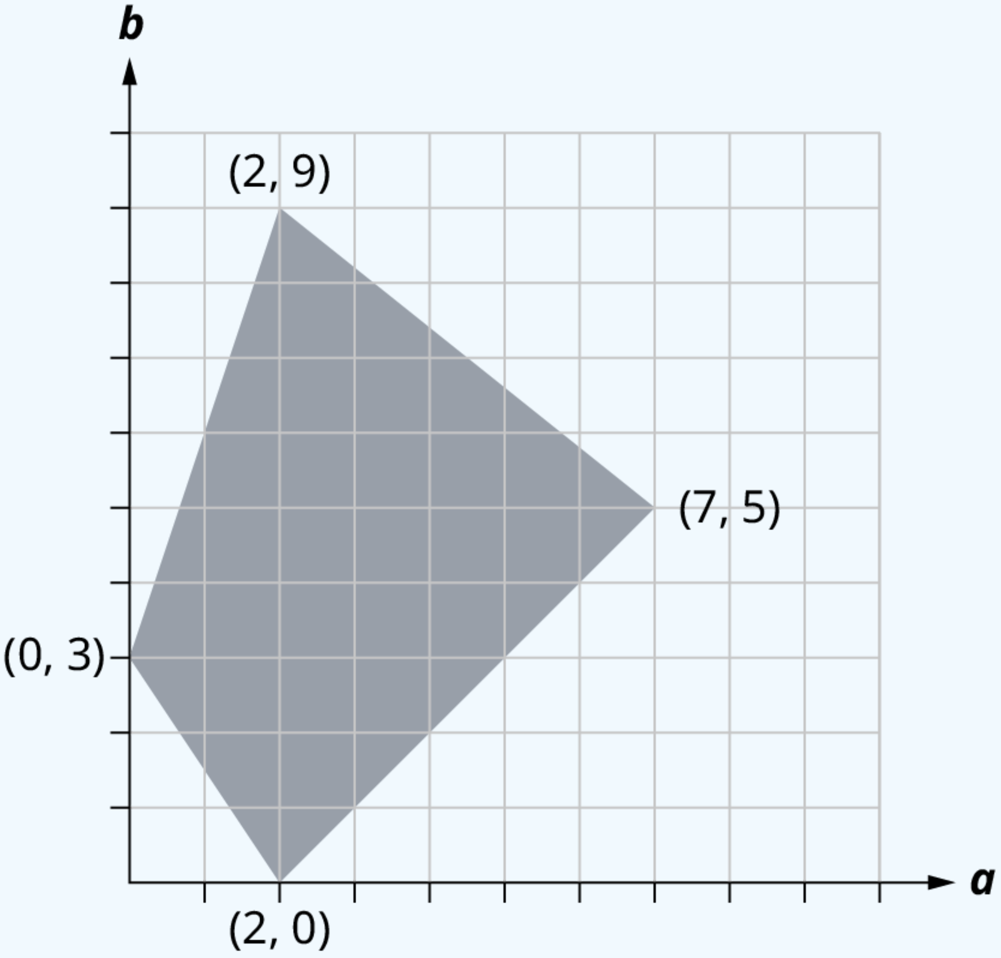
Exercise 5.11.1.4

Objective Function $P = 2t + 4c$



Exercise 5.11.1.5

Objective Function $P = 2.5a + 3.75b$



For the following exercises, write the constraint inequalities. The variables to use are given in parentheses.

? Exercise 5.11.1.6

Fernando builds birdbaths (x) and birdhouses (y). Fernando can make a total of 7 birdbaths and birdhouses per day. A birdbath costs $8 to make, while a birdhouse costs $6 to make. Fernando has $48 to spend on building materials for the day. When he sells them, Fernando makes $12 in profit on a birdbath and $9 in profit on a birdhouse.

? Exercise 5.11.1.7

A fruit pie (p) requires 12 ounces of fruit and 15 ounces of dough; a fruit tart (t) requires 4 ounces of fruit and 3 ounces of dough. There are 72 ounces of fruit and 60 ounces of dough.

? Exercise 5.11.1.8

One recipe for chocolate cake (c) calls for 9 ounces of chocolate chips and 4 eggs; a recipe for dark chocolate cake (d) requires 12 ounces of chocolate chips but only 3 eggs. There are 90 ounces of chocolate chips and 36 eggs.

? Exercise 5.11.1.9

To build an outdoor bench (b), a carpenter needs 10 pieces of wood and 26 nails; to build an outdoor chair (c), the carpenter need 8 pieces of wood and 33 nails. There are 92 pieces of wood and 286 nails.

For the following exercises, graph each of the system of inequalities from Exercises 6–9. Assume all graphs are in the first quadrant.

? Exercise 5.11.1.10

Graph of Exercise 6

? Exercise 5.11.1.11

Graph of Exercise 7

? Exercise 5.11.1.12

Graph of Exercise 8

? Exercise 5.11.1.13

Graph of Exercise 9

For the following exercises, use the four steps for solving linear programming problems to solve.

? Exercise 5.11.1.14

A restaurant sells both regular milk and chocolate milk. To make a glass of regular milk (x), it takes 16 ounces of, well, milk. To make a glass of chocolate milk (y), it takes 15 ounces of milk and 1 ounce of chocolate flavoring. The restaurant makes a profit of $1.50 per glass on regular milk and $1.00 per glass on chocolate milk. At the beginning of the day, the restaurant has 600 ounces of milk and 24 ounces of chocolate flavoring. To maximize profits, how much of each should they sell that day?

? Exercise 5.11.1.15

To make a package of all-beef hot dogs (x), a factory uses one pound of beef; to make their regular all-meat hot dogs (y), they use ½ pound of beef and ½ pound of pork. The profit on the package of all-beef hot dogs is $2.40 per pack; the profit on the all-meat hot dogs is $3.20 per pack. If there are 400 pounds of beef and 250 pounds of pork available, how many of each product should the factory make to maximize their profit?

? Exercise 5.11.1.16

A toy maker makes two plastic toys, the Ring (x) and the Stick (y). The toy maker makes $5 per Ring and $4 per Stick. The Ring uses 4 feet of plastic, while the Stick uses 3 feet of plastic. Today the toy maker has 36 feet of plastic available. The toy maker also only makes 10 plastic toys per day. To maximize profit, how many of each toy should the toy maker make?

? Exercise 5.11.1.17

The toy maker also makes exactly two toys out of wood, the Box (x) and the Bat (y). The toy maker makes $6 per Box and $7 per Bat. Each Box requires 25 ounces of wood, and each Bat requires 40 ounces of wood. Today the toy maker has 260 ounces of wood available. The toy maker also only makes 8 wooden toys per day. To maximize profit, how many of each wooden toy should the toy maker make?

? Exercise 5.11.1.18

Sara makes two kinds of kites out of fabric and popsicle sticks. Her Famous Flyer (x) needs 2 yards of fabric and 9 popsicle sticks; her Gallant Glider (y) needs 3 yards of fabric and 18 popsicle sticks. She makes a profit of $4 on the Famous Flyer and $6 on the Gallant Glider. Today she has 30 yards of fabric and 153 popsicle sticks. How many of each kite should she make to maximize her profit?

? Exercise 5.11.1.19

Randy's RV Storage stores two types of Recreational Vehicles (RVs), The Xtra RV (x) takes up 400 square feet of space, while the Yosemite RV (y) takes up 600 square feet of space. Randy has 55,000 square feet of storage space. By local law, he is only allowed to have a maximum of 100 RVs on his property at any one time. He charges $60 a month to store an Xtra RV, and $80 a month to store a Yosemite RV. How many of each should he store in order to maximize his profit?

? Exercise 5.11.1.20

A Belgian chocolatier wants to introduce two new chocolate bar creations. The first chocolate bar is called Super Dark (x), and it consists of 90 grams of chocolate and 10 grams of sugar. The second chocolate bar is called Special Dark (y), containing 80 grams of chocolate and 20 grams of sugar. She calculates that her company will make 1 Euro per bar of Super Dark, and 2 Euros per bar on Special Dark. She first will create some samples to sell out of 1,260 grams of chocolate and 240 grams of sugar. How many of each bar should the chocolatier create to maximize profit?

? Exercise 5.11.1.21

A juice bottler makes two kinds of specialty juices using different mixtures of pineapple (x) and orange (y) juices. A 16-ounce bottle of Island Delight has 10 ounces of pineapple juice and 6 ounces of orange juice. A 16-ounce bottle of Sun Fun has 4 ounces of pineapple juice and 12 ounces of orange juice. The bottler makes $1.60 per bottle on Island Delight and $1.20 per bottle on Sun Fun. The amounts of juice available today are 640 ounces of pineapple juice and 768 ounces of orange juice. To maximize profit, how many of each bottle of juice should the juice bottler make?

? Exercise 5.11.1.22

Fernando builds birdbaths (x) and birdhouses (y). Fernando can make a total of 7 birdbaths and birdhouses per day. A birdbath costs $8 to make, while a birdhouse costs $6 to make. Fernando has $48 to spend on building materials for the day. When he sells them, Fernando makes $12 in profit on a birdbath and $9 in profit on a birdhouse. Determine how many of each Fernando should make to maximize his profit for the day.

? Exercise 5.11.1.23

A farmer grows wheat (x) and barley (y) on his 500 acres of cropland. He expects to profit $150 per acre for wheat and $180 per acre for barley. The cost of raising each crop (seed, pesticide, etc.) is $60 per acre for wheat and $90 per acre for barley. The farmer can budget $36,000 for the growing of the crops. To maximize his profit, how many acres of each crop should be grown?

? Exercise 5.11.1.24

A company is going to ship food (x) and water (y) to the victims of a tsunami. Each container of food will feed 8 people for a day, and each container of water will give 12 people their daily water. The food containers each weigh 30 pounds and take up 8 cubic feet of space; each container of water weighs 120 pounds, but takes up only 2 cubic feet of space. The airplanes lined up to carry the supplies to the victims cannot have its cargo exceed 24,000 pounds; also, the total cargo area in the airplanes is 4,000 cubic feet. How many containers of food and water can be sent with each plane shipment that maximizes the shipment?

? Exercise 5.11.1.25

Another company will send clothing (x) and medical supplies (y) to the victims of the tsunami. Each container of clothing contains enough clothing for 12 people; each container of medical supplies can aid 8 people. The clothing containers each weigh 50 pounds and take up 6 cubic feet of space; each container of medical supplies weighs 20 pounds, and takes up 4 cubic feet of space. The airplanes lined up to carry the supplies to the victims cannot have its cargo exceed 24,000 pounds; also, the total cargo area in the airplanes is 3,000 cubic feet. How many containers of clothing and medical supplies can be sent with each plane shipment that maximizes the shipment?

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SECTION OVERVIEW

5.12: Chapter Summary

5.12.0: Key Terms

5.12.1: Key Concepts

5.12.2: Videos

5.12.3: Formula Review

5.12.4: Projects

5.12.5: Chapter Review

5.12.6: Chapter Test

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5.12.0: Key Terms

5.1 Algebraic Expressions

- variable
- constant
- expression
- equation
- equal sign
- term
- coefficient
- like terms
- Distributive Property

5.2 Linear Equations in One Variable with Applications

- linear equation

5.3 Linear Inequalities in One Variable with Applications

- linear inequality
- Addition and Subtraction Property of Linear Inequalities
- Multiplication and Division Property of Linear Inequalities

5.4 Ratios and Proportions

- ratio
- proportion
- constant of proportionality
- scale
- construct ratios
- solve proportions
- use proportions to solve scaling problems

5.5 Graphing Linear Equations and Inequalities

- ordered pair
- origin
- points on the axes
- linear equation in two variables
- standards form of a linear equation
- solution
- linear inequality in two variables
- solution to a linear inequality
- boundary line

5.6 Quadratic Equations with Two Variables with Applications

- monomial
- polynomial
- binomial
- trinomial
- quadratic equation
- Zero Product Property

5.7 Functions

- relation
- domain
- function
- mapping
- vertical line test

5.8 Graphing Functions

- intercepts of a line
- slope
- slope-intercept form

5.9 Systems of Linear Equations in Two Variables

- system of linear equations
- solutions of a system of equations
- contradictions
- identities
- coincident lines
- consistent system of linear equations
- inconsistent system of linear equations

5.10 Systems of Linear Inequalities in Two Variables

- system of linear inequalities

5.11 Linear Programming

- linear programming
- objective function
- constraint

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5.12.1: Key Concepts

5.1 Algebraic Expressions

- Algebra is useful because it allows us to understand many situations in real life by modeling them with expressions.
- Algebraic expressions are the building blocks of algebra. From algebraic expressions we can create algebraic equations.
- Algebraic expressions are the building blocks of algebra. From algebraic expressions we can create algebraic equations.
- Algebraic expressions are often simplified and evaluated using the four arithmetic operations.

5.2 Linear Equations in One Variable with Applications

- Solving linear equations means discovering what the value of the variable in a linear equation represents in the given conditions.
- When solving a linear equation, most often you will have one solution; however, a linear equation may have no solutions or infinitely many solutions.

5.3 Linear Inequalities in One Variable with Applications

- Inequalities can be used when the possible values (answers) in a certain situation are numerous, or when the exact value (answer) is not known, but it is known to be within a range of possible values.
- Linear inequalities can be represented using a number line or using interval notation.

5.4 Ratios and Proportions

- A ratio is a comparison of two numbers. The ratio of two numbers a and b can be written as: a to b OR a:b OR the fraction a/b.
- All fractions are ratios, but not all ratios are fractions. Ratios make part to part, part to whole, and whole to part comparisons. Fractions make part to whole comparisons only.
- When two ratios are equal, we say they are in proportion or are proportional.
- Setting up proportions allows us to solve many various situations where three of the four values of the proportion are known.

5.5 Graphing Linear Equations and Inequalities

- Linear equations can be represented graphically on a rectangular coordinate system.
- Solving linear equations in two variables means finding the point where two lines intersect. There are three possibilities: The lines intersect at exactly one point; the lines do not intersect (they are parallel); or the lines intersect everywhere (they are the same line).
- Solving linear inequalities in two variables means finding a region of possible answers. Every point in this region will make both inequalities true statements.
- Plotting points is a standard way to help graph linear equations and linear inequalities.

5.6 Quadratic Equations with Two Variables with Applications

- A quadratic equation is an algebraic equation where the highest power (degree) of the equation is two.
- To solve a quadratic equation is to find the value(s) that when substituted in for the variables, will make the equation equal to zero.
- There can be two, one, or no solutions to any quadratic equation.
- There are several methods to solve a quadratic equation. These methods include factoring quadratic equations, graphic quadratic equations, using the square root method, and using the quadratic formula.

5.7 Functions

- A relation is any set of ordered pairs (x, y). All of the x-values of the set are the domain, and all of the y-values of the set are the range.
- A relation is a function if each x-value in the domain is assigned to exactly one element in the range. A y-value in the range can have more than one x-value assigned to it; but each x-value can only be assigned to one y-value.
- For the function $y = f(x)$, f is the name of the function, x is the domain value variable, and $y = f(x)$ is the range value variable.
- The vertical line test is a test that can be done on the graph of a relation to determine if it is a function.

5.8 Graphing Functions

- Every linear function can be graphically represented by a unique line that shows all the solutions of the equation.
- The points where the graph of a line intersects the x-axis and y-axis are called the intercepts of the line.
- Most lines will have one x-intercept and one y-intercept. Only if the line is straight vertical (no y-intercept) or straight horizontal (no x-intercept) will it not have both intercepts. Note that a line that is straight vertical is not a function, but a line that is straight horizontal is a function.
- Since any two points determine a straight line, any linear function can be graphed if both intercepts are known.
- The slope of a linear function is the ratio of the vertical change divided by the horizontal change. It is often referred to as $\frac{rise}{run}$.
- A formula for finding the slope of linear functions is mfrac, for any two points of the linear function (x_1, y_1) and (x_2, y_2).

5.9 Systems of Linear Equations in Two Variables

- To solve a system of linear equations means finding the point or points where the two linear equations intersect.
- Two lines can intersect at one point, no points if they are parallel, or every point if they are the same equation.
- Systems of linear equations can be solved by graphing, by using substitution, or by using the elimination method.

5.10 Systems of Linear Inequalities in Two Variables

- To solve a system of linear inequalities means to find the area(s) where the points in that area make all the linear inequalities true.
- Systems of linear inequalities can be solved by graphing the linear equations associated with the inequalities, then 'testing' points to see whether the values of the point make the equation true or not.

5.11 Linear Programming

- Linear programming is a mathematical technique to solve problems involving finding maximums or minimums where a linear function is limited by various constraints.
- An objective function is a linear function in two or more variables that describes the quantity that needs to be maximized or minimized.
- In linear programming, a constraint is a restriction that affects the maximum or minimum values of an objective function.
- Through the creation of objective functions and restraints, a linear system can be developed and solved through linear programming.

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5.12.2: Videos

5.1 Algebraic Expressions

- Q&A: Why We Teach Algebra

5.2 Linear Equations in One Variable with Applications

- Solving for a Variable in an Equation

5.5 Graphing Linear Equations and Inequalities

- Graphing Linear Inequalities in Two Variables

5.6 Quadratic Equations with Two Variables with Applications

- Factoring with the Box Method (Area Model)
- Solving Quadratics with the Zero Property
- Solving Quadratics with the Quadratic Formula

5.7 Functions

- Relations and Functions
- Domain and Range on Graphs

5.9 Systems of Linear Equations in Two Variables

- Practice with Solving Applications of Systems of Equations
- Applications of Systems of Linear Equations

5.10 Systems of Linear Inequalities in Two Variables

- Solving Systems of Linear Inequalities by Graphing
- Systems of Linear Inequalities

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5.12.3: Formula Review

5.1 Algebraic Expressions

- Distributive Property: $a(b+c)=ab+ac$

5.3 Linear Inequalities in One Variable with Applications

- For any numbers a, b, and, if $a<b$, then $a+c<b+c$ and $a-c<b-c$.
- For any numbers a, b, and c, if $a>b$, then $a+c>b+c$ and $a-c>b-c$.
- For any numbers a, b, and c, multiply or divide by a positive:

if $a<b$ and $c>0$, then $ac<bc$ and $\frac{a}{c}<\frac{b}{c}$ if $a>b$ and $c>0$, then $ac>bc$ and $\frac{a}{c}>\frac{b}{c}$ multiply or divide by a negative:

if $a<b$ and $c<0$, then $ac>bc$ and $\frac{a}{c}>\frac{b}{c}$

if $a>b$ and $c<0$, then $ac<bc$ and $\frac{c}{a}<\frac{c}{c}$

5.8 Graphing Functions

- To calculate slope (m), use the formula

$$m=\frac{rise}{run}$$

  where the rise measures the vertical change and the run measures the horizontal change.

- To find the slope of the line between two points (x_1, y_1) and (x_2, y_2), use the formula $m=\frac{y2-y1}{x2-x1}$

mfrac

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5.12.4: Projects

Ratio and Proportion—Comparing Prices, Part 1

Go to your favorite coffee shops and find out what a same sized drink costs at each. You can do something similar for pizza as well. Find the unit rate (i.e., price per ounce or price per square inch). For example, go to your favorite coffee place and find the price per units on all their large coffee drinks. Or go to your favorite pizza place and compare prices of all their extra-large pizzas (by price per square inch). Write a report on the best deals.

Ratio and Proportion—Comparing Prices, Part 2

Rather than comparing prices of different, but same sized drinks (or pizzas), compare unit prices of the same drinks but of different sizes. Find out what the best bargain is based on price per ounce, price per square inch, etc. For example, compare the prices of your favorite soft drink sold at a local store, but in various sizes (i.e., 12-ounce can, 16-ounce bottle, 20-ounce bottle, 1-liter bottle, and multipacks). Or go to a pizza place and find out what the best bargain is on their menu, based on price per square inch of pizza. Write a report on the best deals.

Systems of Linear Inequalities—Comparing Cell Phone Plans

Go to the websites of different cell phone companies and compare their plans. Write a report on “the best deals. "Best Deals” doesn’t necessarily mean “cheapest.” You will need to look at what each company provides concerning restrictions (constraints) on minutes to talk. What are the constraints on the cell phone coverage for each company? Do they cover your area of the country well? Do they cover the entire United States well, or at least areas where you will be travelling? Is this coverage 5G, or is it less? Can you add a phone easily? Can you bring your previous phone number to this plan? The possibilities of constraints affecting each plan are several. So your task is to determine which plan is best, based on not only cost but also all constraints you deem important.

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5.12.5: Chapter Review

Algebraic Expressions

? Exercise 5.12.5.1

Translate from algebra to words: $2x+3$

? Exercise 5.12.5.2

Translate from words to algebra: the quotient of $5x$ and 7.

? Exercise 5.12.5.3

Translate from an English phrase to an expression: A gym charges \$5.00 per class c and a \$20 membership fee.

? Exercise 5.12.5.4

Use parentheses to make the following statement true: $2^2 \div +5-4\times 3=-5$

? Exercise 5.12.5.5

Evaluate and simplify x^2-5x when $x=5$.

? Exercise 5.12.5.6

Perform the indicated operation for the expression: $(3x^2+2x+1)+(x^2-2x+2-(x^2-3x+11)$

Linear Equations in One Variable with Applications

? Exercise 5.12.5.7

Solve the linear equations using a general strategy: $3x-7=8$

? Exercise 5.12.5.8

Solve the linear equations using properties of equations: $(5x-3)+7=9x-2(x+7)$

? Exercise 5.12.5.9

It costs 30 cents for an ear of corn. Construct a linear equation and solve how much it costs to buy 23 ears of corn.

? Exercise 5.12.5.10

State whether the following equation has exactly one solution, no solution, or infinitely many solutions $5x+4=7x-14$

? Exercise 5.12.5.11

Solve the formula $A=\frac{(b_1+b_2)}{2}h$ for b_1

Linear Inequalities in One Variable with Applications

? Exercise $5.12.5.12$

Graph the inequality $x > 8$ on a number line and write the interval notation.

? Exercise $5.12.5.13$

Solve the inequality $-2x < -6$, graph the solution on the number line, and write the solution in interval notation.

? Exercise $5.12.5.14$

Construct a linear inequality to solve the application: Daniel wants to surprise his girlfriend with a birthday party at her favorite restaurant. It will cost \$42.75 per person for dinner, including tip and tax. His budget for the party is \$500. What is the maximum number of people Daniel can have at the party?

Ratio and Proportions

Christer opened a bag of marbles and counted the number of each color. They found they had 9 green, 4 yellow, 13 black, 11 orange, 8 blue, and 7 red.

? Exercise $5.12.5.15$

What is the ratio of green marbles to orange marbles?

? Exercise $5.12.5.16$

What is the ratio of red marbles to blue marbles?

? Exercise $5.12.5.17$

What is the ratio of sum of the marbles with an odd number of marbles to the sum of the marbles with an even number of marbles?

? Exercise $5.12.5.18$

What is the ratio of black marbles to all marbles?

? Exercise $5.12.5.19$

Solve: $\frac{7}{2}1 = \frac{2}{1}x$

? Exercise $5.12.5.20$

Basil the cat is 17 pounds and 24 inches long from head to tail. In his new movie *Claws*, he is supersized to 50 pounds. How many inches long will he be? Round your answer to the nearest tenth.

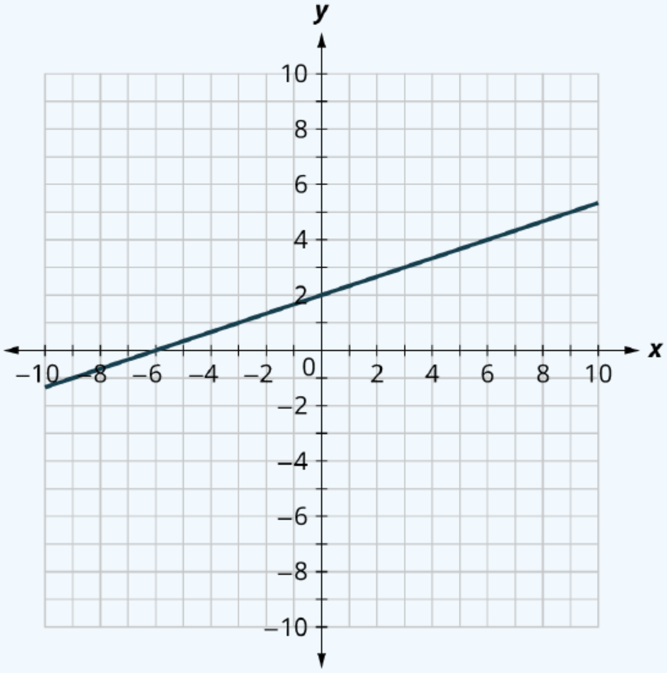
Graphing Linear Equations and Inequalities

? Exercise $5.12.5.21$

For each ordered pair, decide:

I. Is the ordered pair a solution to the equation?
II. Is the point on the line in the graph shown?

$y=\frac{1}{3}x+2$



A: $(0,-3)$ B: $(-3,1)$; C: $(-2,-4)$; D: $(0,2)$

? Exercise 5.12.5.22

Graph $y=\frac{1}{2}x+5$ by plotting points.

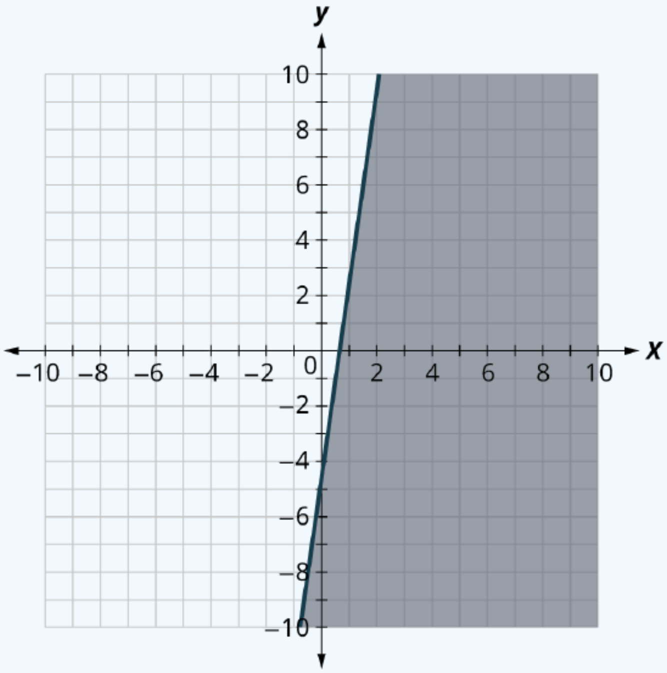
? Exercise 5.12.5.23

Determine whether each ordered pair is a solution to the inequality $y>2x+3$.

A: $(0,0)$ B: $(2,1)$ C: $(-1,-5)$ D: $(-6,-3)$ E: $(1,0)$

? Exercise 5.12.5.24

Write the inequality shown by the graph with the boundary line $y=7x-5$.



Exercise 5.12.5.25

Graph the linear inequality: $y < \frac{1}{3}x + 7$.

Quadratic Equations with Two Variables with Applications

Exercise 5.12.5.26

Multiply $(3x - 1)(x + 7)$.

Exercise 5.12.5.27

Factor $x^2 + 8x - 9$.

Exercise 5.12.5.28

Graph and list the solutions to the quadratic equation $x^2 + 11x + 18 = 0$.

Exercise 5.12.5.29

Solve $x^2 + 7x + 12 = 15x - 3$ by factoring.

Exercise 5.12.5.30

Solve $169x^2 = 25$ using the square root method.

Exercise 5.12.5.31

Solve $x^2 + 7x + 3 = 0$ using the quadratic formula.

Exercise 5.12.5.32

The height in feet, h, of an object shot upwards into the air with initial velocity, v_0, after t seconds is given by the formula $h = -16t^2 + v_0 t$. A firework is shot upwards with initial velocity 130 feet per second. How many seconds will it take to reach a height of 260 feet? Round to the nearest tenth of a second.

Functions

Exercise 5.12.5.33

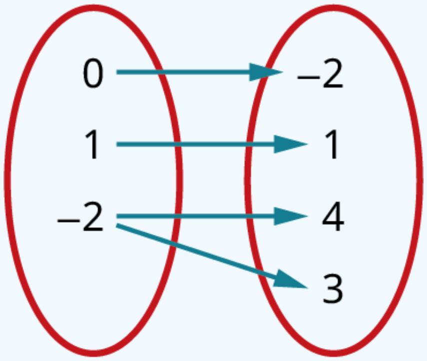
Evaluate the function $f(x) = x^2 - 3x + 21$ at the values $f(-2)$, $f(-1)$, $f(0)$, (1), and $f(2)$.

Exercise 5.12.5.34

Determine whether $\{(-2,31),\ (-1,25),\ (1,19),\ (2,19)\}$ represents a function.

Exercise 5.12.5.35

Determine whether the mappings represent a function:

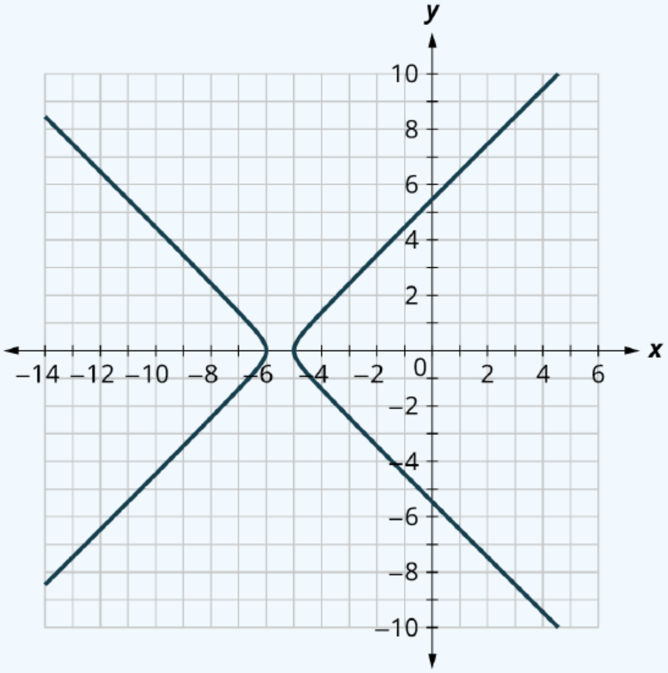


Exercise 5.12.5.36

Determine whether $5x^2 + 2y^2 = 10$ represent y as a function of x.

Exercise 5.12.5.37

Use the vertical line test to determine if the graph represents a function.



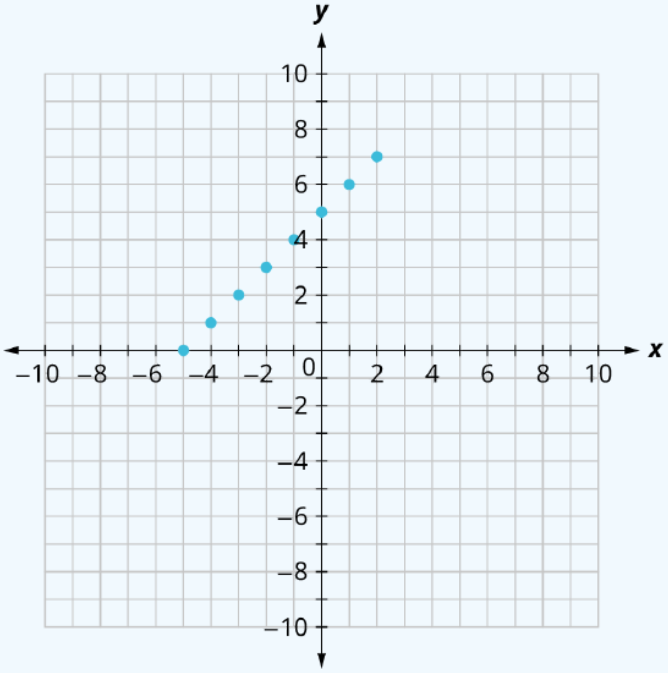
Exercise 5.12.5.38

Use the set of ordered pairs to find the domain and the range.

$\{(-3,9),(-2,4),(-1,1),(0,0),(1,1),(2,4),(3,9)\}$

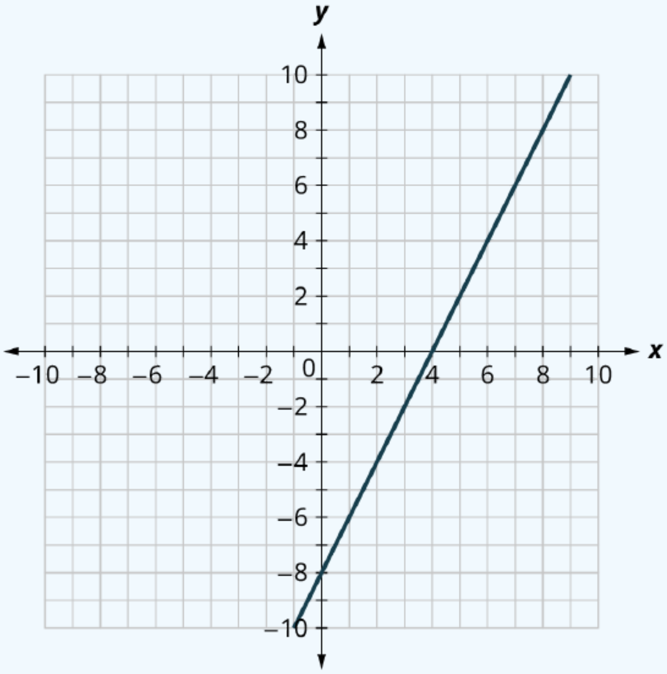
Exercise 5.12.5.39

Use the graph shown to find the domain and the range.



Exercise $5.12.5.40$

Find the x- and y-intercepts on the graph.

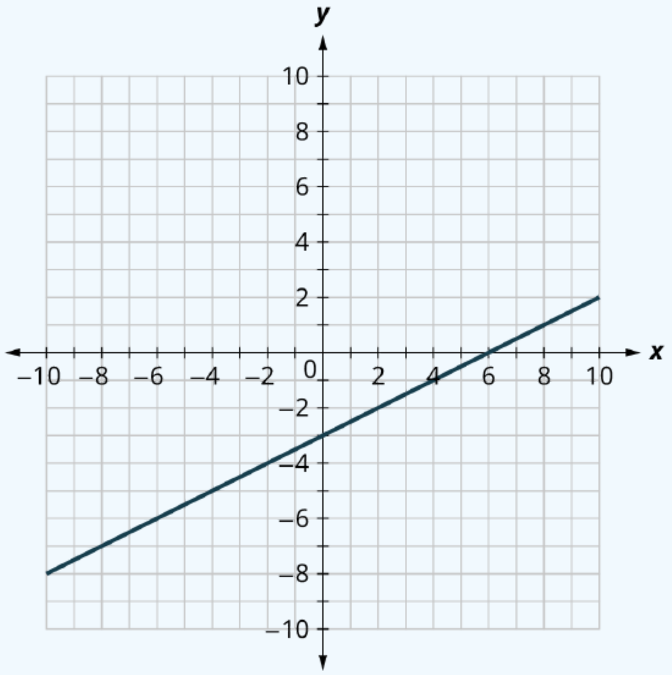


Exercise $5.12.5.41$

Graph $y=3x-3$ using the intercepts.

Exercise $5.12.5.42$

Find the slope of the line in the graph shown.



? Exercise 5.12.5.43

Use the slope formula to find the slope of the line between (2, 4) and (5, 7).

? Exercise 5.12.5.44

Identify the slope and y-intercept of $2y = -\frac{2}{3}x + 10$.

? Exercise 5.12.5.45

Graph the line of $2y = -\frac{2}{3}x - 4$ using its slope and y-intercept.

The equation $P = 28 + 2.54w$ models the relation between the monthly water bill payment, P, in dollars, and the number of units of water, w, used.

? Exercise 5.12.5.46

Find the payment for a month when 0 units of water were used.

? Exercise 5.12.5.47

Find the payment for a month when 15 units of water were used.

? Exercise 5.12.5.48

Interpret the slope and P-intercept of the equation.

? Exercise 5.12.5.49

Graph the equation.

System of Linear Equations in Two Variables

? Exercise 5.12.5.50

Determine if (0, 1) and (2, 3) are solutions to the given system of equations.

$$\begin{cases} x-2y=0 \\ 3x-y=5 \end{cases}$$

? Exercise 5.12.5.51

Solve the system of equations by graphing.

$$\begin{cases} y=3x+1 \\ 6x-y=2 \end{cases}$$

? Exercise 5.12.5.52

Solve the system of equations by substitution.

$$\begin{cases} y=7x+2 \\ 3x-y=-6 \end{cases}$$

? Exercise 5.12.5.53

Solve the systems of equations by elimination.

$$\begin{cases} -5x+y=-15 \\ 3x+y=17 \end{cases}$$

? Exercise 5.12.5.54

Kenneth currently sells suits for company A at a salary of $22,000 plus a $10 commission for each suit sold. Company B offers him a position with a salary of $28,000 plus a $4 commission for each suit sold. How many suits would Kenneth need to sell for the options to be equal?

Systems of Linear Inequalities in Two Variables

? Exercise 5.12.5.55

Determine whether (0, 0) and (2, 3) are solutions to the system.

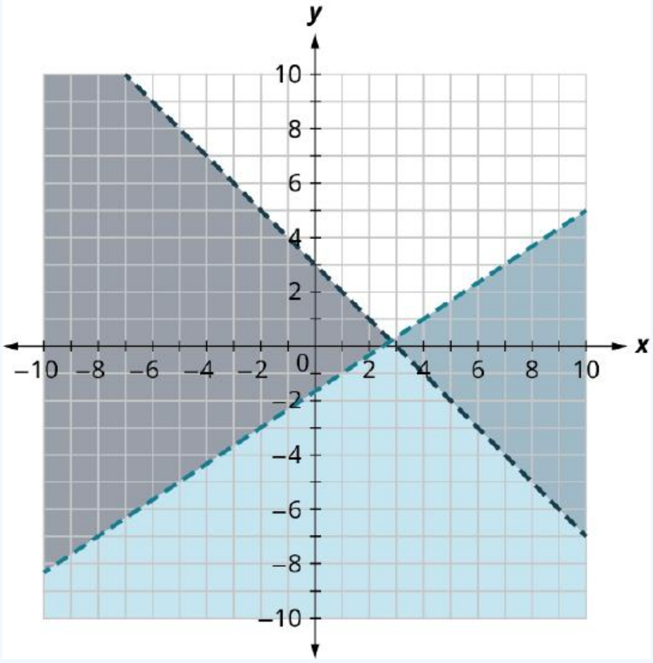
5\ 2x - y \le 10 \end{array} \right." class=" math-rendered">/**/\left\{

$$\begin{array}{c} 3x+y>5 \\ 2x-y\le 10 \end{array} \tag{5.12.5.1}$$

\right./**/

? Exercise 5.12.5.56

Determine whether (0, 0) and (6, –8) are solutions to the darkest shaded region of the graph.



Exercise 5.12.5.57

Solve the systems of linear equations by graphing.

$$\begin{cases} y < 3x + 1 \\ 6x - y \geq 2 \end{cases}$$

Jocelyn desires to increase both her protein consumption and caloric intake. She desires to have at least 35 more grams of protein each day and no more than an additional 200 calories daily. An ounce of cheddar cheese has 7 grams of protein and 110 calories. An ounce of parmesan cheese has 11 grams of protein and 22 calories.

Exercise 5.12.5.58

Write a system of inequalities to model this situation.

Exercise 5.12.5.59

Graph the system.

Exercise 5.12.5.60

Could she eat 1 ounce of cheddar cheese and 3 ounces of parmesan cheese?

Exercise 5.12.5.61

Could she eat 2 ounces of cheddar cheese and 1 ounce of parmesan cheese?

Linear Programming

A toy maker makes two plastic toys, the Ring (x) and the Stick (y). The toy maker makes $4 per Ring and $6 per Stick. The Ring uses 3 feet of plastic, while the Stick uses 5 feet of plastic. Today the toy maker has 40 feet of plastic available. The toy maker also only makes 10 plastic toys per day. To maximize profit, how many of each toy should the toy maker make?

? Exercise $5.12.5.62$

Find the objective function.

? Exercise $5.12.5.63$

Write the constraints as a system of inequalities.

? Exercise $5.12.5.64$

Graph of the system of inequalities.

? Exercise $5.12.5.65$

Find the value of the objective function at each corner point of the graphed region.

? Exercise $5.12.5.66$

To maximize his profit, how many of each toy should the toymaker make?

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5.12.6: Chapter Test

Exercise 5.12.6.1

Perform the indicated operation for the expression: $(3x^2+2x+1)-(x^2-2+2)+x(3x+11)$

Exercise 5.12.6.2

Solve the linear equations using properties of equations: $x(x-2)+=9x+x(x-6)$

Exercise 5.12.6.3

It costs 55 cents for a stamp. Construct a linear equation and solve how much it cost to buy 50 stamps.

Exercise 5.12.6.4

Solve the formula \(A = \frac

{2}h\) for h.

Exercise 5.12.6.5

Solve the inequality $-2x<-6$, graph the solution on the number line, and write the solution in interval notation.

Exercise 5.12.6.6

Construct a linear inequality to solve the application: Bella wants to buy a round of shakes for her friends. It will cost $4.75 per shake, including tip and tax. Her budget is $50. What is the maximum number of friends Bella can buy shakes for?

Exercise 5.12.6.7

Manneken Pis is a famous statue in Brussels, Belgium. It is 24 inches tall and weighs 37.5 pounds. The average man is 69 inches tall and weighs 198 pounds. Is *Manneken Pis* proportional to the average male?

Exercise 5.12.6.8

Graph $y=\frac{1}{4}x+1$ by plotting points.

Exercise 5.12.6.9

Graph the linear inequality: $y\geq 2x+7$

Exercise 5.12.6.10

Graph and list the solutions to the quadratic equation $x^2-36=0$.

Exercise 5.12.6.11

Solve $x^2-49=0$ by factoring.

Exercise 5.12.6.12

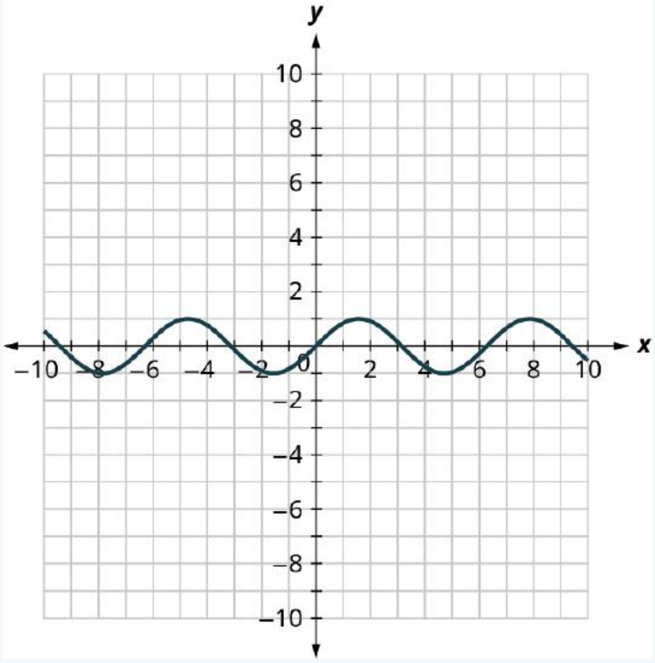
Solve $x^2 + 5x + 2 = 0$ using the quadratic formula.

Exercise 5.12.6.13

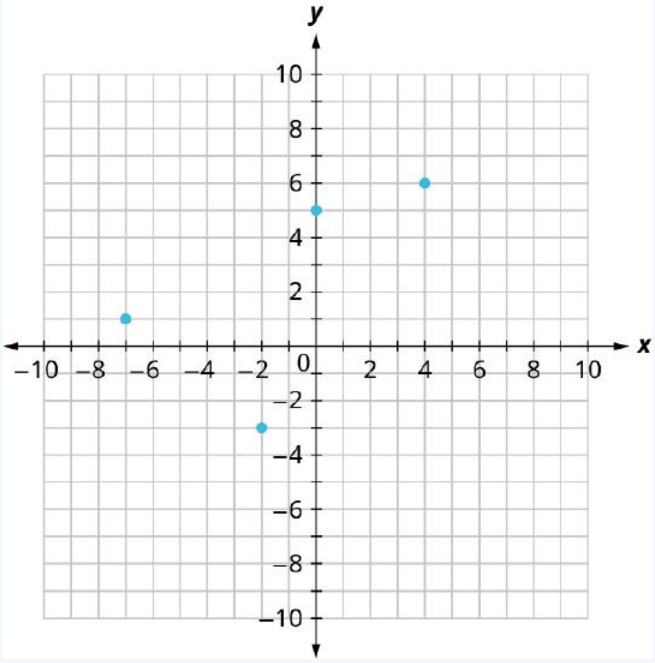
Evaluate the function $f(x) = -3x + 21$ at the values $f(-2)$, $f(-1)$, $f(0)$, $f(1)$, and $f(2)$.

Exercise 5.12.6.14

Use the vertical line test to determine if the given graph represents a function.



Exercise 5.12.6.15



Use the graph shown to find the domain and the range.

? Exercise 5.12.6.16

Graph $2y = x - 4$ using the intercepts.

? Exercise 5.12.6.17

Use the slope formula to find the slope of the line between (1, 4) and (3, 5).

? Exercise 5.12.6.18

Identify the slope and y-intercept of $y - \frac{2}{3}x = 1$.

? Exercise 5.12.6.19

Graph the line of $y = \frac{2}{3}x + 1$ using its slope and y-intercept.

The equation $C = 4.50 + 15p$, models the cost of visiting the Cat Café in San Diego for one hour. C, in dollars, is the total cost and the cost per person, p, is $15 plus a $4.50 reservation fee.

? Exercise 5.12.6.20

Find the payment for two people.

? Exercise 5.12.6.21

Find the payment for five people.

? Exercise 5.12.6.22

Interpret the slope and C-intercept of the equation.

? Exercise 5.12.6.23

Graph the equation.

? Exercise 5.12.6.24

Solve the system of equations by graphing.

$$\begin{cases} y-3x=3 \\ y=-6x+12 \end{cases}$$

? Exercise 5.12.6.25

Solve the system of equations by substitution.

$$\begin{cases} y=5x-5 \\ -3x+y=-3 \end{cases}$$

? Exercise 5.12.6.26

Solve the systems of equations by elimination.

$$\begin{cases} y=\frac{1}{3}x-6 \\ y=x-3 \end{cases}$$

? Exercise 5.12.6.27

Anna goes to the concession stand at a movie theater. She buys 5 popcorns and 4 large sodas and pays a total of \$60. During intermission, Isabelle goes to the concession stand. She buys 1 popcorn and 2 large sodas and pays a total of \$18. What is the cost of one popcorn, and the cost of one large soda?

? Exercise 5.12.6.28

Solve the systems of linear equations by graphing.

$$\begin{cases} y>\frac{1}{3}x-1 \\ y<x-3 \end{cases}$$

Juliette is selling fresh lemonade and cupcakes. She sells a cup of lemonade for \$2 and a cupcake for \$3. She needs to make at least \$100 to donate to the local cat sanctuary. She needs to sell at least 20 cups of lemonade.

? Exercise 5.12.6.29

Write a system of inequalities to model this situation.

? Exercise 5.12.6.30

Graph the system.

? Exercise 5.12.6.31

Could she sell 30 cups of lemonade and 10 cupcakes and make \$100?

? Exercise 5.12.6.32

Could she sell 20 cups of lemonade and 30 cupcakes and make \$100?

A toy maker makes exactly two toys out of wood; the Box (x) and the Bat (y). He makes \$5 per Box and \$6 per Bat. Each Box requires 30 ounces of wood, and each Bat requires 45 ounces of wood. Today the toy maker has 270 ounces of wood available. The toy maker also only makes 8 wooden toys per day. To maximize profit, how many of each wooden toy should the toy maker make?

? Exercise 5.12.6.33

Find the objective function.

? Exercise 5.12.6.34

Write the constraints as a system of inequalities.

? Exercise 5.12.6.35

Graph of the system of inequalities.

? Exercise 5.12.6.36

Find the value of the objective function at each corner point of the graphed region.

? Exercise 5.12.6.37

To maximize profit, how many of each toy should the toymaker make?

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CHAPTER OVERVIEW

6: Money Management

6.0: Introduction
6.1: Understanding Percent
  6.1.0: Exercises
6.2: Discounts, Markups, and Sales Tax
  6.2.0: Exercises
6.3: Simple Interest
  6.3.0: Exercises
6.4: Compound Interest
  6.4.0: Exercises
6.5: Making a Personal Budget
  6.5.0: Exercises
6.6: Methods of Savings
  6.6.0: Exercises
6.7: Investments
  6.7.0: Exercises
6.8: The Basics of Loans
  6.8.0: Exercises
6.9: Understanding Student Loans
  6.9.0: Exercises
6.10: Credit Cards
  6.10.0: Exercises
6.11: Buying or Leasing a Car
  6.11.0: Exercises
6.12: Renting and Homeownership
  6.12.0: Exercises
6.13: Income Tax
  6.13.0: Exercises
6.14: Chapter Summary
  6.14.0: Key Terms
  6.14.1: Key Concepts
  6.14.2: Videos
  6.14.3: Formula Review
  6.14.4: Projects
  6.14.5: Chapter Review
  6.14.6: Chapter Test

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6.0: Introduction

The topic of money management is a broad and sometimes complex one. Ultimately, personal money management involves managing both our debt and also our savings and investments.

In 2021, the average American had consumer debt balance of $96,371. Nearly $100,000 per person. And less than 25% of Americans are debt free. Consumer debt can include mortgages, credit cards, as well as student loans. A key question all consumers should consider is how to manage debt and not become overburdened by it. The first step is to create a budget, which puts earnings into perspective, indicating what we can, and cannot, afford. A budget also entails setting aside certain funds for savings and investment, which help us achieve our short- and long-term goals.

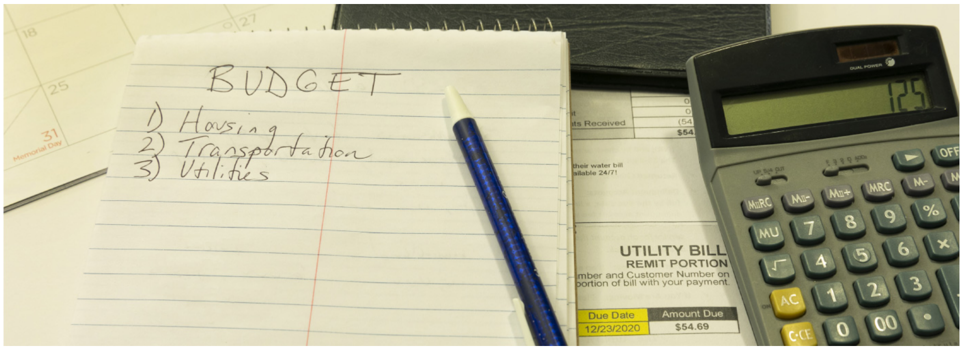

Figure 6.0.1 Financial health helps you realize your goals. (credit: modification of work "Budget and Bills" by Alabama Extension/Flickr, Public Domain)

Creating a budget requires an understanding of how money—debt and savings—works. Initially, percentages and interest need to be understood. They drive most of what happens with debt and savings. With that understanding, discussions of buying a house, a car, or incurring credit card debt can be addressed from a financial perspective. All the while, retirement is waiting. Preparing for retirement involves saving and saving earlier rather than later. The power of compound interest is on full display when saving early.

This chapter covers some of the basics of money management: percentages, interest, budgeting, debt (student loans, mortgage, car, credit cards), savings, investments, and taxes.

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6.1: Understanding Percent

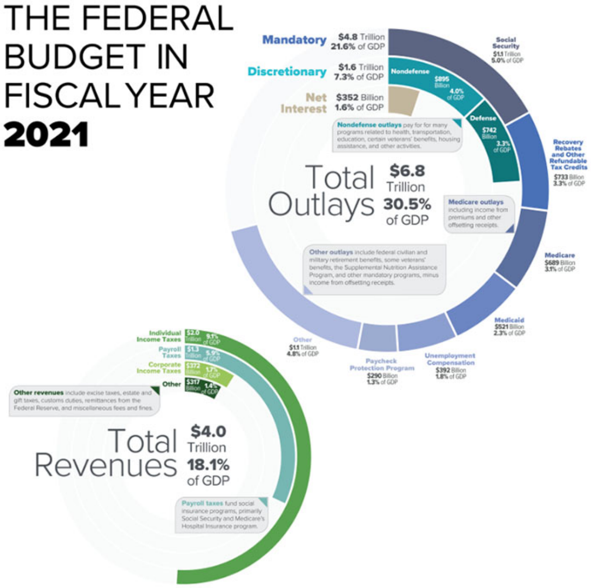


Figure 6.1.1: The federal budget describes how money is spent and how money is earned. (credit: "Breakdown of revenues and outlays in 2021 US Federal budget" Wikimedia Commons, Public Domain)

Learning Objectives

1. Define and calculate percent.
2. Convert between percent, decimal, and fractional values.
3. Calculate the total, percent, or part.
4. Solve application problems involving percents.

In 2020, the U.S. federal government budgeted $3.5 billion for the National Park Service, which appears to be a very large number (and is!) and a large portion of the total federal budget. However, the total outlays from the U.S. federal government in 2020 was $6.6 trillion. So, the amount budgeted for the National Park Service was less than one-tenth of 1 percent, or 1/10%, of the total outlays. This **percent** describes a specific number. Understanding that ratio puts the $3.5 billion budgeted to the National Park Service in perspective.

This chapter focuses on percent as a primary tool for understanding money management. The interest paid on debt, the interest earned through investments, and even taxes are entirely determined using percent. This section introduces the basics of working with this invaluable tool.

Define and Calculate Percent

The word percent comes from the Latin phrase per centum, which means "by the hundred." So any percent is a number divided by 100. Changing a percent to a fraction is to write the percent in its fractional form. To write $n\%$ in its fractional form is to write the percent as the fraction $\frac{n}{100}$.

Checkpoint

A percent need not be an integer and does not have to be less than 100.

Example 6.1.1: Rewriting a Percent as a Fraction

Rewrite the following as fractions:

1. 18%
2. 84%
3. 38.7%

4. 213%

Answer

1. Using the definition and $n = 18$, 18% in fractional form is $\frac{18}{100}$.
2. Using the definition and $n = 84$, 84% in fractional form is $\frac{84}{100}$.
3. Using the definition and $n = 38.7$, 38.7% in fractional form is $\frac{38.7}{100}$.
4. Using the definition and $n = 213$, 213% in fractional form is $\frac{213}{100}$.

Your Turn $6.1.1$

Rewrite the following as fractions:

1. 3%
2. 94%
3. 67.2%
4. 670%

Convert Between Percent, Decimal, and Fractional Values

When any calculation with a percent is to be performed, the form of the percent must be changed, either to its fractional form or its **decimal form**. We can change a percent into decimal form by dividing the percent by 100 and representing the result as a decimal.

FORMULA

The decimal form of n% is found by calculating the decimal value of semanticsn÷100.

Example $6.1.2$: Converting a Percent to Decimal Form

Convert the following percents to decimal form:

1. 17%
2. 7%
3. 18.45%

Answer

1. To convert 17% to its decimal form, divide 17 by 100. This moves the decimal two places to the left, resulting in 0.17. The decimal form of 17% is 0.17.
2. To convert 7% to its decimal form, divide 7 by 100. This moves the decimal two places to the left, resulting in 0.07. The decimal form of 7% is 0.07.
3. To convert 18.45% to its decimal form, divide 18.45 by 100. This moves the decimal two places to the left, resulting in 0.1845. The decimal form of 18.45% is 0.1845.

Your Turn $6.1.2$

Convert the following percents to decimal form:

1. 9%
2. 24%
3. 2.18%

You should notice that, to convert from percent to decimal form, you can simply move the decimal two places to the left without performing the division.

FORMULA

To convert the number x from decimal form to percent, multiply x by 100 and place a percent sign, %, after the number, $(x \times 100)\%$.

Example 6.1.3: Converting the Decimal Form of a Percent to Percent

Convert each of the following to percent:

1. 0.34
2. 4.15
3. 0.0391

Answer

1. Using the formula and x = 0.34, we calculate $(0.34 \times 100)\%$, which gives us 34%.
2. Using the formula and x = 4.15, we calculate $(4.15 \times 100)\%$, which gives us 415%.
3. Using the formula and x = 0.0391, we calculate $(0.0391 \times 100)\%$, which gives us 3.91%.

Your Turn 6.1.3

Convert the following to percent:

1. 0.41
2. 0.02
3. 9.2481

You should notice that, to convert from decimal form to percent form, you can simply move the decimal two places to the right without performing the multiplication.

Calculate the Total, Percent, or Part

The word "of" is used to indicate multiplication using fractions, as in "one-fourth of 56." To find "one-fourth of 56" we would multiply 56 by one-fourth. We can think of percents as fractions with a specific denominator—100. So, to calculate "25% of 52," we multiply 52 by 25%. But, first we need to convert the percent to either fractional form (25/100) or decimal form. Using the decimal form of 25% we have 0.25 × 52, which equals 13.

In this problem, 52 is the **total** or **base**, 25 is the **percentage**, and 13 is **the percentage of** 52, or the **part** of 52. This is sometimes referred to as the **amount**.

FORMULA

The mathematical formula relating the total (base), the percent in decimal form, and the part (amount) is $part$ =percent×totalpart=percent×total, or, $amount$ =percent×baseamount=percent×base.

Checkpoint

In all calculations, the percent is expressed in decimal form.

Knowing any two of the values in our formula allows us to calculate the third value. In the following example, we know the total and the percent, and are asked to find the percentage of the total.

Example 6.1.4: Finding the Percent of a Total

1. Determine 70% of 3,500
2. Determine 156% of 720

Answer

1. The total is $x = 3,500$, and the percent is $n = 70$. The decimal form of 70% is 0.70 . To find the part, or percent of the total, substitute those values into the formula and calculate.

$$\begin{aligned} \text{part} &= \text{percent} \times \text{total} \\ &= 0.70 \times 3500 \\ &= 2450 \end{aligned}$$

From this, we say that 70% of 3,500 is 2,450 .
2. The total is $x = 720$, and the percent is $n = 156$. The decimal form of 156% is 1.56 . To find the part, or percent of the total, substitute those values into the formula and calculate.

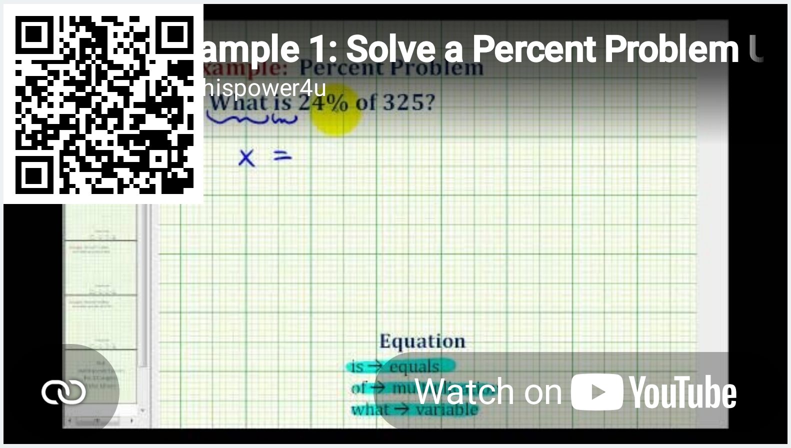
$$\begin{aligned} \text{part} &= \text{percent} \times \text{total} \\ &= 1.56 \times 720 \\ &= 1,123.2 \end{aligned}$$

From this, we say that 156% of 720 is $1,123.2$.

Your Turn 6.1.4

1. Determine 26% of 1,300.
2. Determine 225% of 915.

Video



Finding Percent of a Total

In the previous example, we knew the total and the percent and found the part using our formula. We may instead know the percent and the part, but not the total. We can use our formula again to solve for the total.

Example 6.1.5: Finding the Total from the Percent and the Part

1. What is the total if 35% of the total is 70?
2. What is the total if 10% of the total is 4,000?

Answer

1. **Step 1**: The percent is 35 , which in decimal form is 0.35 . We were given that 35% of the total is 70 , so the part is 70 . We are to find the total. Substituting into the formula, we have

<

$$\begin{aligned} \text{part} &= \text{percent} \times \text{total} \\ 70 &= 0.35 \times \text{total} \end{aligned}$$

Step 2: To find the total, we solve the equation for the total.

$$70 = 0.35 \times \text{total}$$

$$\begin{aligned} 70 &= 0.35 \times \text{ total} \\ \frac{70}{0.35} &= \frac{0.35 \times \text{ total}}{0.35} \\ 200 &= \frac{0.35 \times \text{ total}}{0.35} \\ 200 &= \text{ total} \end{aligned}$$

From this we see that 200 is the total, or, that 35% of 200 is 70.

2. **Step 1**: The percent is 10 , which in decimal form is 0.1 . We were given that 10% of the total is 4,000 , so the part is 4,000 . Substituting into the formula, we have

$$\begin{aligned} \text{part } &= \text{ percent } \times \text{ total} \\ 4,000 &= 0.1 \times \text{ total} \end{aligned}$$

Step 2: To find the total, we solve the equation for the total.

Misplaced &

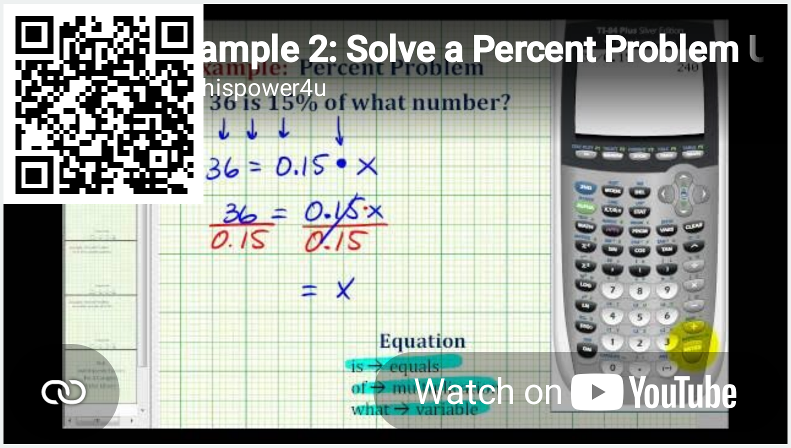
From this we see that 40,000 is the total, or that 10% of 40,000 is 4,000 .

Your Turn $6.1.5$

What is the total if 18% of the total is 45?

What is the total if 15% of the total is 900?

Video


ample 2: Solve a Percent Problem U
hispower4u
Example: Percent Problem
36 is 15% of what number?
36 = 0.15 • x
36/0.15 = 0.15x/0.15
= x
Equation
is → equals
of → mu
what → variable
Watch on YouTube

Finding the Total from the Percent and the Part

Similarly, the percent can be found if the total and the percent of the total (the part) are known. This will result in the decimal form of the percent, so it must be converted to percent form.

Example $6.1.6$: Finding the Percent from the Total and the Part

1. What percent of 500 is 175?
2. What percent of 228 is 155?

Answer

1. Step 1: The total is 500 , the percent of the total is 175 . Substituting into the formula, we have

$$\begin{aligned} \text{part } &= \text{ percent } \times \text{ total} \\ 175 &= \text{ percent } \times 500 \end{aligned}$$

Step 2: To find the percent, we solve the equation for the percent.

$$\begin{aligned}175 &= \text{percent } \times 500\\ \frac{175}{500} &= \frac{\text{percent } \times 500}{500}\\ 0.35 &= \frac{\text{percent } \times 500}{500}\\ 0.35 &= \text{percent}\end{aligned}$$

We see the percent in decimal form is 0.35 . Converting from the decimal form yields 35%. We say that 175 is 35% of 500 .

2. Step 1: The total is 228 , the percent of the total is 155 . Substituting into the formula, we have

$$\begin{aligned}\text{part } &= \text{ percent } \times \text{ total}\\ 155 &= \text{percent } \times 228\end{aligned}$$

Step 2: To find the percent, we solve the equation for the percent.

$$\begin{aligned}155 &= \text{percent } \times 228\\ \frac{155}{228} &= \frac{\text{percent } \times 228}{228}\\ 0.6798 &= \frac{\text{percent } \times 228}{228}\\ 0.6798 &= \text{percent}\end{aligned}$$

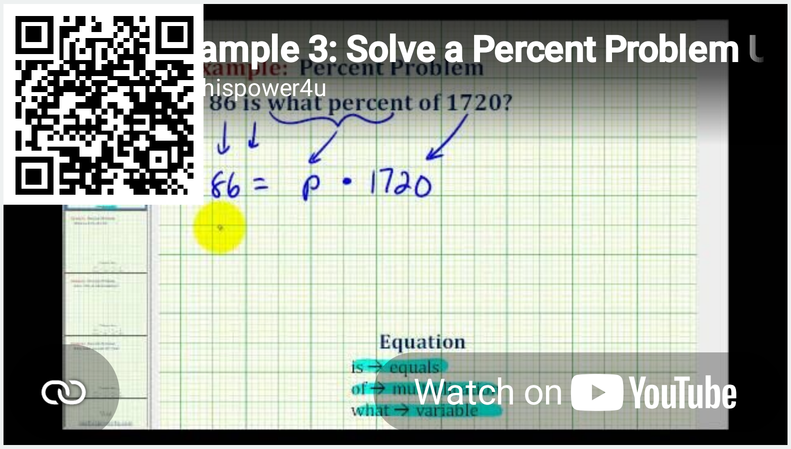
We see the percent is 0.6798 (rounded to four decimal places). Converting from the decimal form yields 67.98%. We say that 155 is 67.98% of 228.

Your Turn 6.1.6

Find the percent in the following:

1. Total is 40, percent of the total is 25
2. Total is 730, percent of the total is 292

Video


ample 3: Solve a Percent Problem U
hispower4u
xample: Percent Problem
86 is what percent of 1720?
86 = p • 1720
Equation
is → equals
of → mu
what → variable
Watch on YouTube

Finding the Percent When the Total and the Part Are Known

Solve Application Problems Involving Percents

Percents are frequently used in finance, research, science experiments, and even casual conversation. Understanding these types of values helps when consuming media or discussing finances, for instance. Effectively working with and interpreting numbers and percents will help you become an informed consumer of this information.

In most cases, working through what is presented requires you to identify that you are indeed working with a question of percents, which two of the three values that are related through percents are known, and which of the three values you need to find.

✓ Example 6.1.7: Retention Rate at College

Justine applies to a medium size university outside her hometown and finds out that the retention rate (percent of students who return for their sophomore year) for the 2021 academic year at the university was 84%. During a visit to the registrar's office, she finds out that 1,350 people had enrolled in academic year 2021. How many students from the academic year 2021 are returning for the 2022 academic year?

Answer

The percent of students who will return for the 2022 academic year (the retention rate) is 84%. The total number of students who enrolled in the 2021 academic year was 1,350. This means the percent is known and the total is known. From this, we can determine the number of students who will return (percent of the total) for the 2022 academic year using the formula

$$\text{part} =$$

percent×totalpart=percent×total. Substituting into the formula and calculating, we find that the number of students that are returning is

$$\begin{aligned}\text{part} &= \text{percent} \times \text{total}\\ &= 0.84 \times 1{,}350\\ &= 1{,}134\end{aligned}$$

So 1,134 students will return for the 2022 academic year.

Your Turn 6.1.7

Harris works the bookstore in their hometown. During one particular day, the store had total sales of $1,765, of which Harris sold 30%. What were Harris's total sales that day?

✓ Example 6.1.8: Percent of Chemistry Majors

In this situation, the percent is to be determined. We know the total number of students, 45 , and the part of the students that are chemistry majors, 18 . Using that information and the formula

$$\text{part} \quad = \quad \text{percent} \quad \times \quad \text{total}$$

, the percent can be found. Substituting and solving, we have

$$\begin{aligned}18 &= \text{percent} \ \times 45\\ \frac{18}{45} &= \frac{\text{percent} \ \times 45}{45}\\ 0.4 &= \frac{\text{percent} \ \times 45}{45}\\ 0.4 &= \text{percent}\end{aligned}$$

Converting the 0.4 from decimal form, we find that 40% of the students in the calculus class are chemistry majors.

Your Turn 6.1.8

At the Fremont County fair, there were 2,532 adult visitors. Of these, 1,679 purchased the Adult Mega Pass. What percent of the adult visitors purchased the Adult Mega Pass?

✓ Example 6.1.9: Total Sales and Commission

Mariel makes a 20% commission on every sale she makes. One week, her commission check is for $153.00. What were her total sales that week?

Answer

In this problem, Mariel's total sales is to be determined. We know the percent she earns is 20%. We also know that her sales commission was $\$153.00$, which is the percent of the total. Using this information and the formula

$$\text{part} \ = \ \text{percent} \ \times \ \text{total}$$

we can find Mariel's total sales. The decimal form of 20% is 0.2 . The part, or percent of the total, is 153. Substituting and solving, we obtain

$$\begin{aligned} \text{part} \ &= \ \text{percent} \ \times \ \text{total} \\ 153 &= 0.2 \times \ \text{total} \\ \frac{153}{0.2} &= \frac{0.2 \times \ \text{total}}{0.2} \\ 765 &= \frac{0.2 \times \ \text{total}}{0.2} \end{aligned}$$

Mariel’s total sales were $765.00.

Your Turn 6.1.9

Mina’s family has replaced 65% of their home’s older light bulbs with LED bulbs. If they now have 52 LED bulbs, how many total lightbulbs are in Mina’s house?

Who Knew?: LED Lightbulbs

According to the energy website from the U.S. government, LED lightbulbs use at least 75% less energy than incandescent bulbs. They also last up to 25 times as long as an incandescent bulb. If lighting is a significant percent of your electrical use, replacing all incandescent bulbs with LED bulbs will significantly reduce your electric bill.

Check Your Understanding

1. What is the denominator for any percent?

2. Convert 38.7% to decimal form.

3. What is 68% of 280?

4. Find the total if 41% of the total is 342. If necessary, round to two decimal places.

5. TikTok has an estimated 80,000,000 (80 million) registered users in the United States. The population of the United States is 332,403,650. What percent of the U.S. population are registered TikTok users? If necessary, round to two decimal places.

6. An Amazon fulfillment center needs to hire 20% more drivers. If there are currently 80 drivers, how many more drivers will be hired?

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6.1.0: Exercises

For any answer, round to two decimal places, if necessary.

In the following exercises, rewrite the percent as a fraction

? Exercise 6.1.0.1

45%

? Exercise 6.1.0.2

9.1%

? Exercise 6.1.0.3

8%

? Exercise 6.1.0.4

673%

In the following exercises, rewrite the percent in decimal form.

? Exercise 6.1.0.5

18%

? Exercise 6.1.0.6

9%

? Exercise 6.1.0.7

71.2%

? Exercise 6.1.0.8

934%

? Exercise 6.1.0.9

Find 35% of 250

? Exercise 6.1.0.10

Calculate 83.1% of 390

? Exercise 6.1.0.11

Calculate 3.1% of 500

Exercise 6.1.0.12

Calculate 750% of 620

Exercise 6.1.0.13

If 40% of the total is 32, how much is the total?

Exercise 6.1.0.14

If 3% of the total is 6.32, how much is the total?

Exercise 6.1.0.15

If 150% of the total is 61.9, how much is the total?

Exercise 6.1.0.16

If 18.1% of the total is 18.5, how much is the total?

Exercise 6.1.0.17

13 is what percent of 40?

Exercise 6.1.0.18

89 is what percent of 500?

Exercise 6.1.0.19

31 is what percent of 73?

Exercise 6.1.0.20

593.2 is what percent of 184.5?

Exercise 6.1.0.21

36 people in a village of 150 want to install a new splashpad at the local playground. What percent of the village wants to install the new splashpad?

Exercise 6.1.0.22

Mitena is enrolled in a movie appreciation course. There are 84 students (including Mitena) in the course. After having the students fill out a survey, the professor informs the students that 45.2% chose horror as their favorite movie genre. How many students in Mitena’s class chose horror as their favorite movie genre? Round off to the nearest integer.

Exercise 6.1.0.23

Jadyn’s dorm has a “Rick and Morty night” every Wednesday during the semester. One Wednesday, 27 students from the dorm come to watch the TV show Rick and Morty. Jadyn knows this is 30% of the dorm’s residents. How many students reside in the dorm?

24. **Percent Error.** When performing a scientific experiment that results in quantities of some sort, such as mass in chemistry or momentum in physics, the percent error is often computed. Percent error, $\%E$, is the percent by which the value obtained in an experiment, the observed value O, is different than the value that was expected, the expected value E, in the experiment. The formula is below.

\(\% E = \frac

{E}\)
Jim and Kelly are working on a chemistry experiment and expect the result to be 50 grams. However, their result was 48.7 grams. Find Jim and Kelly's percent error.

25. Percent Error. See Exercise 24 for the definition of percent error.

Hailey and Elsbeth are using an experiment to determine Earth's gravity. The expected value is $9.807 \mathrm{~m}/\mathrm{s}^2$. Their experiment gives them a value of $9.457 \mathrm{~m}/\mathrm{s}^2$. Find the percent error for Hailey and Elsbeth's experiment.

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6.2: Discounts, Markups, and Sales Tax

FINAL SALE
-30%
-50%
-60%
-70%

Figure 6.2.1: Sale prices are often described as percent discounts. (credit: "Close-up of a discount sign" by Ivan Radic/Flickr, CC BY 2.0)

Learning Objectives

1. Calculate discounts.
2. Solve application problems involving discounts.
3. Calculate markups.
4. Solve application problems involving markups.
5. Compute sales tax.
6. Solve application problems involving sales tax.

Many people first encounter percentages during a retail transaction such as a percent discount (SALE! 25% off!!), or through sales tax ("Wait, I thought this was $1.99?"), a report that something has increased by some percentage of the previous value (NOW! 20% more!!). These are examples of percent decreases and percent increases. In this section, we discuss decrease, increase, and then the case of sales tax.

Calculating Discounts

Retailers frequently hold sales to help move merchandise. The sale price is almost always expressed as some amount off the original price. These are discounts, a reduction in the price of something. The price after the discount is sometimes referred to as the reduced price or the sale price.

When a reduction is a percent discount, it is an application of percent, which was introduced in Understanding Percent. The formula used was $\text{part} =$. In a discount application, the discount plays the role of the part, the percent discount is the percentage, and the original price plays the role of the total.

FORMULA

The formula for a discount based on a percentage is $\text{discount} = \text{percent discount} \times \text{original}$, with the percent discount expressed as a decimal. The price of the item after the discount is $\text{sale price} = \text{original price} -$.

These are often combined into the following formula

$$\begin{aligned} \text{sale price} &= \text{original price} - \text{percent discount} \times \text{original price} \\ &= \text{original price} \times (1 - \text{percent discount}) \end{aligned}$$

When the original price and the percent discount are known, the discount and the sale price can be directly computed.

Example $6.2.1$: Calculating Discount for a Percent Discount

Calculate the discount for the given price and discount percentage. Then calculate the sale price.

1. Original price = $75.80; percent discount is 25%
2. Original price = $168.90; percent discount is 30%

Answer

1. Substituting the values into the formula $\text{discount} = \text{percent discount} \times \text{original}$, we find that the discount is $\text{discount} = 0.25 \times 75.80 = 18.95$. The discount is $18.95.

   The sale price of the item is then $\text{sale price} = \text{original price} - \text{discount} = 75.80 - 18.95 = 56.85$, or $56.85.
2. Substituting the values into the formula $\text{discount} = \text{percent discount} \times \text{original}$, we find that the discount is $\text{discount} = 0.30 \times 168.90 = 50.67$. The discount is $50.67.

   The sale price of the item is then $\text{sale price} = \text{original price} - \text{discount} = 168.90 - 50.67 = 118.23$, or $118.23.

Your Turn $6.2.1$

Calculate the discount for the given original price and discount percentage. Then calculate the sale price.

Original price = $1,550.00; percent discount is 32%

Original price = $27.50; percent discount is 10%

Sometimes the original price and the sale price of an item is known. From this, the percent discount can be computed using the formula $\text{discount} = \text{percent discount} \times \text{original}$, by solving for the percent discount.

Example $6.2.2$: Calculating the Percent Discount from the Original and Sale Prices

Determine the percent discount based on the given original and sale prices.

1. Original price = $1,200.00; sale price = $900.00
2. Original price = $36.70; sale price = $29.52

Answer

1. **Step 1.** Find the discount. Using the original price and the sale price, we can find the discount with the formula $\text{sale price} = \text{original price} -$. Substituting and calculating, we find the discount to be $900.00 = 1,200.00 - \text{discount}$. Solving for the discount gives $300.00.

   Step 2. Find the percent discount. Substituting the discount of $300.00 and the original price of $1,200.00, into the formula $\text{discount} = \text{percent discount} \times \text{original}$, we can find the percent discount.

$$\begin{aligned} 300.00 &= \text{percent discount} \times 1{,}200.00 \\ \frac{300.00}{1,200.00} &= \text{percent discount} \\ 0.25 &= \text{percent discount} \end{aligned}$$

Converting to percent form, the percent discount is 25%.

2. **Step 1.** Find the discount. Using the original price and the sale price, we can find the discount with the formula $\text{sale price} = \text{original price} -$. Substituting and calculating, we find the discount to be $29.52 = 36.70 - \text{discount}$. Solving for the discount gives \$7.38.

   Step 2. Find the percent discount. Substituting the discount of \$7.38 and the original price of \$36.70, into the formula $\text{discount} = \text{percent discount} \times \text{original}$, we can find the percent discount.

$$\begin{aligned} 29.52 &= \text{percent discount} \times 36.70 \\ \frac{29.52}{36.70} &= \text{percent discount} \\ 0.2 &= \text{percent discount} \end{aligned}$$

   Converting to percent form, the percent discount is 20%.

Your Turn 6.2.2

Determine the percent discount based on the given original and sale prices.

Original price = \$250.00; sale price = \$162.50

Original price = \$19.50; sale price = \$17.16

Sometimes the sale price and the percent discount of an item are known. From this, the original price can be found. To avoid multiple steps, though, the formula that we will use is $\text{sale price} = \text{original price} \times (1 - \text{percent discount})$. The original price can be found by solving this equation for the original price.

Example 6.2.3: Calculating the Original Price from the Percent Discount and Sale Price

Determine the original price based on the percent discount and sale price.

1. Percent discount 10%; sale price = \$450.00
2. Percent discount 75%, sale price = \$90.00

Answer

1. Using the percent discount and the sale price, we can find the original price with the formula $\text{sale price} = \text{original price} \times (1 - \text{percent discount})$. Substituting and solving for the original price, we find

$$\begin{aligned} \text{sale price} &= \text{original price} \times (1 - \text{percent discount}) \\ 450.00 &= \text{original price} \times (1 - 0.10) \\ 450.00 &= \text{original price} \times (0.90) \\ 500.00 &= \text{original} \end{aligned}$$

   The original price of the item was \$500.00.

2. Using the percent discount and the sale price, we can find the original price with the formula $\text{saleprice} = \text{original price} \times (1 - \text{percent discount})$. Substituting and solving for the original price, we find

$$\begin{aligned} \text{sale price} &= \text{original price} \times (1 - \text{percent discount}) \\ 90.00 &= \text{original price} \times (1 - 0.75) \\ 90.00 &= \text{original price} \times (0.25) \\ 360.00 &= \text{original} \end{aligned}$$

   The original price of the item was \$360.00.

Your Turn 6.2.3

Determine the original price based on the percent discount and sale price.

Percent discount 15%; sale price = \$11.05

Percent discount 9%; sale price = \$200.20

Solve Application Problems Involving Discounts

In application problems, identify what is given and what is to be found, using the terms that have been learned, such as discount, original price, percent discount, and sale price. Once you have identified those, use the appropriate formula (or formulas) to find the solution(s).

Example 6.2.4: Determine Discount and New Price a Sale Rack Item

The sale rack at a clothing store is marked “All Items 30% off.” Ian finds a shirt that had an original price of \$80.00. What is the discount on the shirt? What is the sale price of the shirt?

Answer

We are asked to find the discount, and the sale price. We know the percent discount is 30%, or 0.30 in decimal form. The original price was \$80.

Substituting into the percent discount formula, we find that the discount is $\text{discount} = \text{percent discount} \times \text{original price} = 0.30 \times 80 =$.

The discount is \$24 on that shirt. The sale price is the original price minus the discount, so the sale price is \$80 – \$24 = \$56.

Your Turn 6.2.4

A bed originally priced at \$550, but is on sale, with a 60% discount. What is the discount on the bed? What is the sale price of the bed?

Example 6.2.5: Determine the Percent Discount of a Bus Pass

An annual pass on the city bus is priced at \$240. The student price, though, is \$168. What is the percent discount for students for the bus pass?

Answer

We know the original price of the item, \$240. We also know the sale price of the item, \$168. From this we know the discount is $\$240 - \$168 = \$72$. Substituting these values into the formula $\text{discount} = \text{percent discount} \times \text{original}$, we can find the percent discount.

$$\begin{aligned} \text{discount price} &= \text{percent discount} \times \text{original} \\ 72 &= \text{percent discount} \times 240 \\ \frac{72}{240} &= \frac{\text{percent discount} \times 240}{240} \\ 0.3 &= \text{percent} \end{aligned}$$

The student percent discount on the bus pass is 30%.

Your Turn 6.2.51

A pharmacy offers students at a nearby college a discount. Jerry purchases ibuprofen, which had an original price of $15.80. The cost to Jerry after the student discount was $13.43. What is the percent discount for students at the pharmacy?

Example 6.2.6: Finding the Original Price of a New Pair of Tires

Kendra’s car developed a flat, and the tire store told her that two tires had to be replaced. She got a 10% discount on the pair of tires, and the sale price came to $189.00. What was the original price of the tires?

Answer

Using the percent discount and the sale price, we can find the original price with the formula $\text{saleprice} = \text{original price} \times (1 - \text{percent discount})$. Substituting and solving for the original price, we find

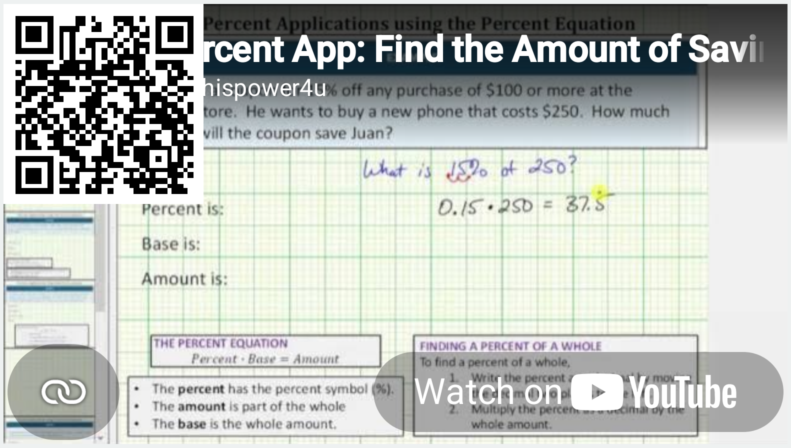
$$\begin{aligned} \text{sales price} &= \text{original price} \times (1 - \text{percent discount}) \\ 189.00 &= \text{original price} \times (1 - 0.10) \\ 189.00 &= \text{original price} \times (0.90) \\ 210.00 &= \text{original price} \end{aligned}$$

The original price of the two tires Kendra bought was $210.00.

Your Turn 6.2.6

Marisol needed to buy a new microwave. She got a 26% discount. The sale price Marisol paid was $43.66. What was the original price of the microwave?

Video


Percent Applications using the Percent Equation
rcent App: Find the Amount of Savi
hispower4u
% off any purchase of $100 or more at the
tore. He wants to buy a new phone that costs $250. How much
vill the coupon save Juan?
What is 15% of 250?
0.15 • 250 = 37.5
Percent is:
Base is:
Amount is:
THE PERCENT EQUATION
Percent · Base = Amount
The percent has the percent symbol (%).
The amount is part of the whole
The base is the whole amount.
FINDING A PERCENT OF A WHOLE
To find a percent of a whole,
Watch on YouTube

Computing Price Based on a Percent Off Coupon

WORK IT OUT

There are cases where retailers allow multiple discounts to be applied. However, it is rare that the discount percentages are added together. For example, if you have a 15% coupon and qualify for a 20% price reduction, the retailer typically does not add those two percentages together to determine the new price. The retailer instead applies one discount, then applies the second discount to the price obtained after the first discount was deducted.

Research the original prices of two different laptops offered by one retail outlet. Assume you will receive a student discount of 12% and your outlet of choice is having a 15% off sale on all laptops.

For each laptop:

1. List the original price and calculate the price after applying the student discount (12%) only.

2. Then find the price after applying the sale discount (15% off) to the price found in Step 1.
3. Determine the total saved on the laptop and what percent discount the total savings represents.
4. Now, apply the discounts in reverse order (first the sale discount, then the student discount).
5. Note anything interesting about your findings.

Calculate Markups

When retailers purchase goods to sell, they pay a certain price, called the **cost**. The retailer then charges more than that amount for the goods. This increase is called the **markup**. This selling price, or **retail price**, is what the retailer charges the consumer in order to pay their own costs and make a profit. Markup, then is very similar to discount, except we add the markup, while we subtract the discount.

FORMULA

The formula for a markup based on a percentage is $\text{markup} = \text{percent}$, with the percent markup expressed as a decimal. The price of the item after the markup is $\text{retail price} = \text{cost}+$.

These are often combined into the following formula

$$\text{retail price} = \text{cost} + \text{percent markup} \times \text{cost} = \text{cost} \times (1 + \text{percent markup})$$

It should be noted that the formulas used for a markup are very similar to those for a discount, with addition replacing the subtraction.

Example $6.2.7$: Determining the Retail Price Based on the Cost and the Percent Markup

Calculate the markup for the given cost and markup percentage. Then calculate the retail price.

1. Cost = \$62.00; percent markup is 15%
2. Cost = \$750.00; percent markup is 45%

Answer

1. Substituting the values into the formula $\text{markup} = \text{percent}$, we find that the markup is $\text{markup} = 0.15 \times 62.00 = 9.30$. The markup is \$9.30.
   The retail price of the item is then $\text{retail price} = \text{cost}+$, or \$62.00 + \$9.30 = \$71.30.
2. Substituting the values into the formula $\text{markup} = \text{percent}$, we find that the markup is $\text{markup} = 0.45 \times 750.00 = 337.50$. The markup is \$337.50.
   The retail price of the item is then $\text{retail price} = \text{cost}+$, or \$750.00 + \$337.50 = \$1,087.50.

Your Turn $6.2.7$

Calculate the markup for the given cost and markup percentage. Then calculate the retail price.

Cost = \$1,800.00; percent markup is 22%

Cost = \$10.50; percent markup is 10%

Sometimes the cost and the retail price of an item are known. From this, the percent markup can be computed using the formula $\text{markup} = \text{percent}$, by solving for the percent markup.

Example $6.2.8$: Calculating the Percent Markup from the Cost and Retail Price

Determine the percent markup based on the given cost and retail price. Round percentages to two decimal places.

1. Cost = \$90.00; retail price = \$103.50
2. Cost = \$5.20; retail price = \$9.90

Answer

1. **Step 1:** Using the cost and the retail price, we can find the markup with the formula $\text{retail price} = \text{cost} +$. Substituting and calculating, we find the markup to be $103.50 = 90.00 + \text{markup}$. Solving for the markup gives $13.50.
   Step 2: After substituting the markup, $13.50, and the original price, $90.00, into the formula $\text{markup} = \text{percent}$, we can find the percent markup.

   $$\begin{aligned} 13.50 &= \text{percent markup} \times 90.00 \\ \frac{13.50}{90.00} &= \text{percent markup} \\ 0.15 &= \text{percent markup} \end{aligned}$$

   Converting to percent form, the percent markup is 15%.
2. **Step 1:** Using the cost and the retail price, we can find the markup with the formula $\text{retail price} = \text{cost} +$. Substituting and calculating, we find the markup to be $9.90 = 5.20 + \text{markup}$. Solving for the markup gives $4.70.
   Step 2: After substituting the markup, $4.70, and the original price, $5.20, into the formula $\text{markup} = \text{percent}$, we can find the percent markup.

   $$\begin{aligned} 4.70 &= \text{percent markup} \times 5.20 \\ \frac{4.70}{5.20} &= \text{percent markup} \\ 0.9038 &= \text{percent markup} \end{aligned}$$

   Converting to percent form, the percent markup is 90.38%.

Your Turn 6.2.8

Determine the percent markup based on the given cost and retail price. Round percentages to two decimal places.

Cost = $120.00; retail price = $190.00

Cost = $0.38; retail price = $1.14

Sometimes the retail price and the percent markup of an item are known. From this, the cost can be found. To avoid multiple steps, though, the formula that we will use is $\text{retail price} = \text{cost} \times (1 + \text{percent markup})$. The cost can be found by solving this equation for the cost.

Example 6.2.9: Calculating the Cost from the Percent Markup and Retail Price

Determine the cost based on the percent markup and retail price.

1. Percent markup 20%; retail price = $10.62
2. Percent markup 125%; retail price = $26.55

Answer

1. Using the percent markup and the retail price, we can find the cost with the formula $\text{retail price} = \text{cost} \times (1 + \text{percent markup})$. Substituting and solving for the cost, we find

   $$\begin{aligned} \text{retail price} &= \text{cost} \times (1 + \text{percent markup}) \\ 10.62 &= \text{cost} \times (1 + 0.2) \\ 10.62 &= \text{cost} \times (1.2) \\ 8.85 &= \end{aligned}$$

   The cost of the item was $8.85.
2. Using the percent markup and the retail price, we can find the cost with the formula $\text{retailprice} = \text{cost} \times (1 + \text{percent markup})$. Substituting and solving for the original price, we find

$$\begin{aligned}\text{retail price} &= \text{cost} \times (1 + \text{percent markup})\\ 26.55 &= \text{cost} \times (1{+}2.25)\\ 26.55 &= \text{cost} \times (3.25)\\ 11.80 &= \end{aligned}$$

The cost of the item was $11.80.

Your Turn 6.2.9

Determine the cost based on the percent markup and retail price.

Percent markup 15%; retail price = $40.25

Percent markup 300%; retail price = $35.96

Solve Application Problems Involving Markups

As before when working with application problems, be sure to look for what is given and identify what you are to find. Once you have evaluated the problem, use the appropriate formula to find the solution(s). These application problems address markups.

Example 6.2.10: Determine Retail Price of a Power Bar

Janice works at a convenience store near campus. It sells protein bars at a 60% markup. If a bar costs the store $1.30, how much is the retail price at the convenience store?

Answer

We are asked to find the retail price. We know the percent markup is 60%. The cost of the bar was $1.30. Substituting into the percent markup formula, we find that the markup is $\text{markup} = \text{percent markup} \times \text{cost} = 0.60 \times 1.30 = 0.78$. The markup is $0.78 on that protein bar. The retail price is the cost plus the markup, so the retail price is $\text{retail price} = \text{cost} + \text{markup} = 1.30 + 0.78 = 2.08$. The retail price is $2.08.

Your Turn 6.2.10

A furniture outlet spends $360.00 to buy a bed. The store marks up the bed by 250%. What is the retail price of the bed?

Example 6.2.11: Determine the Percent Markup of a Phone

Javi began working at a phone outlet. In a recent shipment, he noticed that the cost of the phone to the store was $480.00. The phone sells for $840.00 in the store. What is the percent markup on the phone?

Answer

We know the cost of the phone, $480. We also know the retail price of the phone, $840.00. From this we know the markup is $\$840.00 - \$480.00 = \$360.00$. Substituting these values into the formula $\text{markup} = \text{percent}$, we can find the percent markup.

$$\begin{aligned}\text{markup} &= \text{percent markup} \times \text{cost}\\ 360 &= \text{percent markup} \times 480\\ \frac{360}{480} &= \frac{\text{percent markup} \times 480}{480}\\ 0.75 &= \text{percent markup}\end{aligned}$$

The markup on the phone is 75%.

Your Turn $6.2.11$

Maggie does some research into textbook costs. The *Sociology of the Family* text she finds sells for \$234.36 but costs the store only \$189.00. What is the percent markup on the sociology book?

Example $6.2.12$: Finding the Cost of a T-Shirt

Bob decided to order a t-shirt for his gaming friend online for \$29.50. He knows the markup on such t-shirts is 18%. What was the t-shirt's cost before the markup?

Answer

Using the percent markup and the retail price, \$29.50, we can find the cost with the formula $\text{retail price} = \text{original price} \times (1 + \text{percent markup})$. Substituting and solving for cost we find

$$\begin{aligned}\text{retail price} &= \text{cost} \times (1 + \text{percent markup})\\ 29.50 &= \text{cost} \times (1{+}0.18)\\ 29.50 &= \text{cost} \times (1.18)\\ 25.00 &= \text{cost}\end{aligned}$$

The cost of the t-shirt was \$25.00.

Your Turn $6.2.12$

Tina has opened a retail shop and purchased a unique hat for resale. Tina uses a 50% markup and sells the hat for \$57.00. How much did the hat cost Tina?

Compute Sales Tax

Sales tax is applied to the sale or lease of some goods and services in the United States but is not determined by the federal government. It is most often set, collected, and spent by individual states, counties, parishes, and municipalities. None of these sales tax revenues go to the federal government.

For example, North Carolina has a state sales tax of 4.75% while New Mexico has a state sales tax of 5%. Additionally, many counties in North Carolina charge an additional 2% sales tax, bringing the total sales tax for most (72 of the 100) counties in North Carolinians to 6.75%. However, in Durham, the county sales tax is 2.25% plus an additional 0.5% tax used to fund public transportation, bringing Durham County's sales tax to 7%. To find the sales tax in a particular place, then, add other locality sales taxes to the base state sales tax rate.

How much we pay in sales tax depends on where we are, and what we are buying.

To determine the amount of sales tax on taxable purchase, we need to find the product of the purchase price, or marked price, and the sales tax rate for that locality.

FORMULA

To calculate the amount of sales tax paid on the purchase price in a locality with sales tax given in decimal form, calculate $\text{sales tax} = \text{purchase price} \times \text{tax}$ The total price is then $\text{Total price} = \text{purchase price} + \text{purchase price} \times \text{tax rate} = \text{purchase price}\times(1 + \text{tax rate})$

Checkpoint

When the sales tax calculation results in a fraction of a penny, then normal rounding rules apply, round up for half a penny or more, but round down for less than half a penny.

You should notice that this the same as markup, except using a different term. Sales tax plays the role of markup, the purchase price plays the role of cost, and the tax rate plays the role of percent markup. This means all the strategies developed for markups apply to this situation, with the changes indicated.

✓ Example $6.2.13$: Sales Tax in Kankakee Illinois

The sales tax in Kankakee, Illinois, is 8.25%. Find the sales tax and total price of items based on the purchase price listed.

1. Purchase price = $428.99
2. Purchase price = $34.88

Answer

1. The sales tax is found using $\text{sales tax} = \text{purchase price} \times \text{tax}$. The purchase price is $428.99 and the tax rate is 8.25%. Substituting and calculating, the sales tax is $\text{sales tax} = \$428.99 \times 0.0825 = \$35.391675$. The sales tax needs to be rounded off. Since the third decimal place (fraction of a penny) is 1, we round down and the sales tax is $35.39. The total price is the sales tax plus the purchase price, so is $\$428.99 + \$35.88 = \$464.87$.
2. The sales tax on the item is found using $\text{sales tax} = \text{purchase price} \times \text{tax}$. The purchase price is $34.88 and the tax rate is 8.25%. Substituting and calculating, the sales tax is $\text{sales tax} = \$34.88 \times 0.0825 = \$2.8776$. The sales tax needs to be rounded off. Since the third decimal place (fraction of a penny) is 7, we round up and the sales tax is $2.88. The total price of the item is the sales tax plus the purchase price, so is $\$34.88 + \$2.88 = \$37.76$.

Your Turn $6.2.13$

The sales tax in Union County, Oregon, is 7%. Find the sales tax and total price of items based on the purchase price listed.

Purchase price = $1,499.00

Purchase price = $26.89

As before, the information available might be different than only the purchase price and the sales tax rate. In these cases, use either $\text{sales tax} = \text{purchase price} \times \text{tax}$ or $\text{Total price} = \text{purchase price} \times (1 + \text{tax rate})$ and solve for the indicated tax, price, or rate. These problems mirror those for percent markup.

Be aware, almost all sales tax rates are structured as full percentages, or half percent, or one-quarter percent, or three-quarter percent. This means the decimal value of the sales tax rate, written as a percent, will be either 0, as in 5.0%, 5 as in 7.5%, 25 as in 3.25%, or 75 as in 4.75%. When rounding for the sales tax percentage, be sure to use this guideline.

✓ Example $6.2.14$: Calculating the Sales Tax from the Purchase Price and the Total Price

Find the sales tax rate for the indicated purchase price and total price. Round using the guideline for sales tax percentages.

1. Purchase price = $329.50; total price = $354.21
2. Purchase Price = $13.77; total price = $14.39

Answer

1. **Step 1.** Find the sales tax paid. First, the amount of sales tax must be found. Subtracting the purchase price from the total price, the amount of sales tax is $24.71.
   Step 2. Find the sales tax rate. Using the purchase price, the sales tax, and the formula $\text{sales tax} = \text{purchase price} \times \text{tax}$, the sales tax rate can be found. Substituting and solving yields

$$\begin{aligned} \text{Sales Tax} &= \text{purchase price} \times \text{tax rate} \\ \$24.71 &= \$329.50 \times \text{tax rate} \\ \frac{\$24.71}{\$329.50} &= \text{tax rate} \\ 0.07499 &= \text{tax} \end{aligned}$$

   Keeping in mind the guideline for rounding sales tax rate, the sales tax rate is 7.5%.

2. **Step 1.** Find the sales tax paid. First, the amount of sales tax must be found. Subtracting the purchase price from the total price, the amount of sales tax is $0.62.
   Step 2. Find the sales tax rate. Using the purchase price, the sales tax, and the formula $\text{sales tax} = \text{purchase price} \times \text{tax}$, the sales tax rate can be found. Substituting and solving yields

$$\begin{aligned}\text{sales tax} &= \text{purchase price} \times \text{tax rate}\\ \$0.62 &= \$13.77 \times \text{tax rate}\\ \frac{\$0.62}{\$13.77} &= \text{tax rate}\\ 0.04503 &= \text{tax}\end{aligned}$$

   Keeping in mind the guideline for rounding sales tax rate, the sales tax rate is 4.5%.

Your Turn 6.2.14

Find the sales tax rate for the indicated purchase price and total price. Round using the guideline for sales tax percentages.

Purchase price = $83.90; total price = $88.30

Purchase price = $477.00; total price = $509.20

Example 6.2.15: Calculating the Purchase Price from the Sales Tax and Total Price

Find the purchase price for the indicated sales tax rate and total price.

1. Sales tax rate = 5.75%; total price = $36.56
2. Sales tax rate = 4.25%; total price = $97.17

Answer

1. When the sales tax rate and the total price are known, the formula $\text{total price} = \text{purchase price} \times (1 + \text{tax rate})$ can be used to find the purchase price. Substituting the tax rate and total price into the formula and solving, we find

$$\begin{aligned}\text{Total price} &= \text{purchase price} \times (1 + \text{tax rate})\\ \$36.56 &= \text{purchase price} \times (1 + 0.0575)\\ \frac{\$36.56}{1.0575} &= \text{purchase price}\\ \$34.57 &= \text{purchase}\end{aligned}$$

   The purchase price, the price before tax, was $34.57.

2. When the sales tax rate and the total price are known, the formula $\text{total price} = \text{purchase price} \times (1 + \text{tax rate})$ can be used to find the purchase price. Substituting the tax rate and total price into the formula and solving, we find

$$\begin{aligned}\text{Total price} &= \text{purchase price} \times (1 + \text{tax rate})\\ \$97.17 &= \text{purchase price} \times (1 + 0.0425)\\ \frac{\$97.17}{1.0425} &= \text{purchase price}\\ \$93.21 &= \text{purchase}\end{aligned}$$

   The purchase price, the price before tax, was $93.21.

Your Turn 6.2.15

Find the purchase price for the indicated sales tax rate and total price.

Sales tax rate = 8.25%; total price = $157.81

Sales tax rate = 6.75%; total price = $522.01

Solve Application Problems Involving Sales Tax

Solving problems involving sales tax follows the same ideas and steps as solving problems for markups. But here we will use the following formula:

$$\text{total price} = \text{purchase price} + \text{sales tax}$$

We can also use the formula:

$$\text{total price} = \text{purchase price} \times (1 + \text{sales tax rate})$$

.

This can be seen in the following examples.

✓ Example 6.2.16: Compute Sales Tax for Denver, Colorado

The sales tax rate in Denver Colorado is 8.81%. Keven buys a TV in Denver, and the purchase price (before taxes) is $499.00. How much will Keven pay in sales tax and what will be the total amount he spends when he buys the TV?

Answer

The sales tax rate in Denver is 8.81%. To find the sales tax Keven will pay, find 8.81% of the purchase price. In decimal form, that sales tax rate is 0.0881. Using the formula and substituting 499.00 for purchase price, we find that Keven will pay $\text{purchase price} \times \text{tax rate} = \$499 \times 0.0881 = \$43.96$ in sales tax for the TV.

The total price that Keven will pay is the purchase price plus the sales tax, or $\$499.00 + \$43.96 = \$542.96$.

Your Turn 6.2.16

Daryl decides to buy a new scooter in St. Louis, Missouri, where the sales tax is 9.68%. The scooter he chooses has a purchase price of $1,149. How much will Daryl pay in sales tax and what is the total price he spends on the scooter?

✓ Example 6.2.17: Compute Sales Tax for Austin, Texas

Jillian visits Austin, Texas, and purchases a new set of weights for her home. She spends, including sales tax, $467.64. The sales tax rate in Austin Texas is 8.25%. How much of the total price is sales tax?

Answer

The sales tax paid for this purchase is the difference in the total price and the purchase price. We know the total price is $467.64. We also know the sales tax rate, which is 8.25%. In decimal form, this is 0.0825. Using these values and the formula $\text{total price} = \text{purchase price} \times (1 + \text{tax rate})$ to find the purchase price.

$$\begin{aligned} \text{total price} &= \text{purchase price} \times (1 + \text{tax rate}) \\ \$467.64 &= \text{purchase price} \times (1 + 0.0825) \\ \$467.64 &= \text{purchase price} \times (1.0825) \\ \$432 &= \text{purchase price} \end{aligned}$$

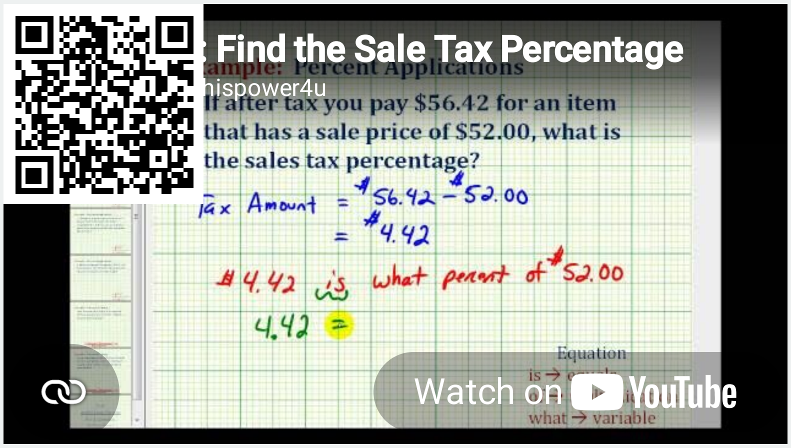
Knowing both the total price and the now the purchase price, we can find the difference, which is the sales tax.

The total price was $467.64. The purchase price was $432. The difference of the total price and the purchase price, or the sales tax, is then $467.64 − $432.00, which is $35.64. Jillian pays $35.64 in sales tax.

Your Turn 6.2.17

Elizabeth decides to buy new running shoes in her hometown of Springfield, Illinois, where the sales tax rate is 6.25%. If her total bill comes to $153, how much of the total price is sales tax?

Video


Find the Sale Tax Percentage
hispower4u
Example: Percent Applications

If after tax you pay $56.42 for an item that has a sale price of $52.00, what is the sales tax percentage?

Tax Amount = $56.42 − $52.00
= $4.42
$4.42 is what percent of $52.00
4.42 =
Equation
is → equals
what → variable
Watch on YouTube

Finding Sales Tax Percentage

Who Knew?

West Virginia was the first state to impose a sales tax. This happened on May 3, 1921.

Look up your locality on this website that lists standard state-level sales tax rates and compare the sales tax structure in your state to two nearby states (for the lower 48) and for any two states (Alaska and Hawaii).

Check Your Understanding

1. What is a discount?
2. What is a markup?
3. An item has a retail price of $45.00. What is the sale price after a 32% discount?
4. A retailer buys an item for $311.00. What is the retail price if their markup is 60%?
5. Does sales tax have the same formula as markup?
6. If the sales tax is 6.8%, what is the total price for an item that has a purchase price of $39.95?

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6.2.0: Exercises

For the following exercises, use the given values to find the indicated value. Round percent results to 2 decimal places. Round money results to the penny (2 decimal places).

? Exercise 6.2.0.1

Retail price = \$399.00, percent discount = 30%, find the sale price.

? Exercise 6.2.0.2

Retail Price = \$75.00, percent discount = 65%, find the sale price.

? Exercise 6.2.0.3

Retail price = \$125.00, sale price = \$90.00, find the percent discount.

? Exercise 6.2.0.4

Retail price = \$47.00, sale price = \$41.50, find the percent discount.

? Exercise 6.2.0.5

Sale price = \$145.70, percent discount = 20%, find the retail price.

? Exercise 6.2.0.6

Sale price = \$1,208.43, percent discount = 13%, find the retail price.

? Exercise 6.2.0.7

Retail price = \$26,790.00, percent discount = 8%, find the sale price.

? Exercise 6.2.0.8

Sale price = \$314.06, percent discount = 33%, find the retail price.

? Exercise 6.2.0.9

Retail price = \$145.50, sale price = \$117.90, find the percent discount.

? Exercise 6.2.0.10

Retail price = \$28.90, percent discount = 18%, find the sale price.

? Exercise 6.2.0.11

Sale price = \$17.59, percent discount = 12%, find the retail price.

? Exercise 6.2.0.12

Retail price = \$57.50, sale price = \$46.00, find the percent discount.

? Exercise 6.2.0.13

Cost = \$130.00, percent markup = 34%, find the retail price.

? Exercise 6.2.0.14

Cost = \$2.27, percent markup = 42%, find the retail price.

? Exercise 6.2.0.15

Cost = \$68.45, retail price = \$109.90, find the percent markup.

? Exercise 6.2.0.16

Cost = \$466.16, retail price = \$699.00, find the percent markup.

? Exercise 6.2.0.17

Retail price = \$98.99, percent markup = 25%, find the cost.

? Exercise 6.2.0.18

Retail price = \$799.00, percent markup = 55%, find the cost.

In the following exercises, find the sales tax and total paid.

? Exercise 6.2.0.19

Retail price = \$17.99; sales tax = 7.5%

? Exercise 6.2.0.20

Retail price = \$799.00; sales tax = 8.5%

? Exercise 6.2.0.21

Retail price = \$176.83; sales tax = 6.25%

? Exercise 6.2.0.22

Retail price = \$223.93; sales tax = 4.5%

In the following exercises, find the sales tax rate.

? Exercise 6.2.0.23

Purchase price = \$257.45; total price = \$273.54

? Exercise 6.2.0.24

Purchase price = \$14.99; total price = \$15.74

? Exercise 6.2.0.25

Purchase price = \$26.83; total price = \$28.84

? Exercise 6.2.0.26

Purchase price = \$2,399.90; total price = \$2,609.89

In the following exercises, find the purchase price.

? Exercise 6.2.0.27

Sales tax rate = 4.75%; total price = \$50.15

? Exercise 6.2.0.28

Sales tax rate = 8%; total price = \$1,069.20

? Exercise 6.2.0.29

Sales tax rate = 9.5%; total price = \$51.45

? Exercise 6.2.0.30

Sales tax rate = 5.75%; total price = \$3,065.69

? Exercise 6.2.0.31

Harris has a coupon for 20% off for any purchase. She finds a new tennis racket for \$278.00. How much is the price after the coupon is applied?

? Exercise 6.2.0.32

After the employee discount, Mariam will pay \$46.55. What is her employee discount rate if the retail price was \$53.50? Round to nearest full percent.

? Exercise 6.2.0.33

Resa purchased a new game for her cousin. After sales tax, she paid \$41.13. Find the sales tax rate she paid if the purchase price of the game was \$38.99.

? Exercise 6.2.0.34

Larissa opens a new secondhand bookstore. She buys a book for \$2.75. What is her percent markup if the sells the book for \$8.50. Round to the nearest percent.

? Exercise 6.2.0.35

Doug opens a used auto parts store. He pays \$30 for a car door. How much will he charge if his percent markup is 60%?

? Exercise 6.2.0.36

Gaia and Seth live in Osceola County in Florida, where the sales tax rate is 7.5%. They purchased some new camping gear. The price before taxes came to $784.62. How much do they pay after the sales tax is applied?

? Exercise 6.2.0.37

Theresa decides to purchase a new phone, which has a retail price of $799.00. Her discount is 20% through a friends and family plan. The sales tax in her county is 6.75%. How much will she pay after the discount? How much will she pay after the tax is applied?

? Exercise 6.2.0.38

Sakari manages a retail outlet. They receive a shipment of shirts. She sees on the shipping list that each shirt cost 24.50 to the store. The store marks up the shirts by 45%. The county in which she lives charges sales tax of 6.5%. What is the retail price of one of the shirts? After sales tax, how much will a customer pay for the shirt?

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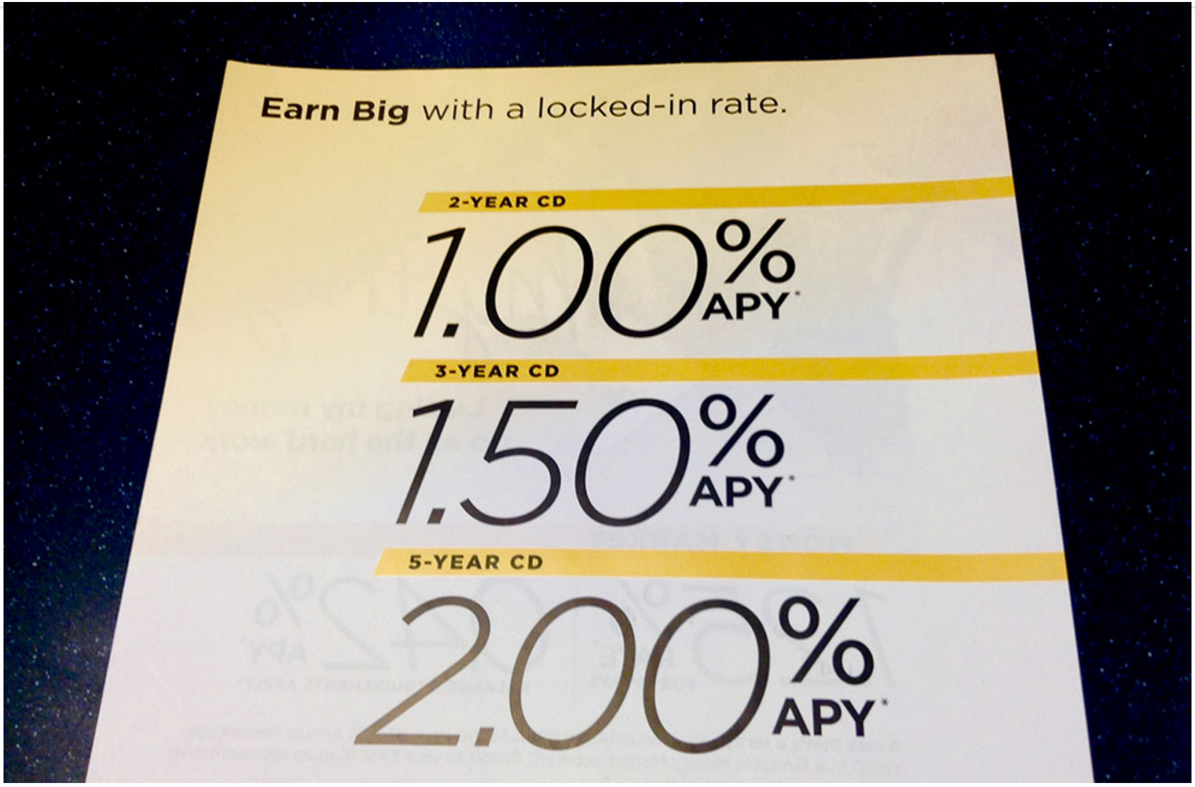
6.3: Simple Interest



Earn Big with a locked-in rate.

2-YEAR CD

1.00% APY*

3-YEAR CD

1.50% APY*

5-YEAR CD

2.00% APY*

Figure 6.3.1: Interest is how savings earns money. (credit: “Interest Rates” by Mike Mozart/Flickr, CC BY 2.0)

Learning Objectives

1. Compute simple interest.
2. Understand and compute future value.
3. Compute simple interest loans with partial payments.
4. Understand and compute present value.

There is truth in the phrase “You need to have money to make money.” In essence, if you have money to lend, you can lend it at a cost to a borrower and make money on that transaction.

When money is borrowed, the person borrowing the money (borrower) typically has to pay the person or entity that lent the money (the lender) more than the amount of money that was borrowed. This extra money is the interest that is to be paid. Interest is sometimes referred to as the cost to borrow, the cost of the loan, or the finance cost.

This idea also applies when someone deposits money in a bank account or some other form of investment. That person is essentially lending the money to the bank or company. The money earned by the depositor is also called interest. The interest is typically based on the amount borrowed, or the principal.

The pairing of borrower and lender can take various forms. The borrower may be a consumer using a credit card or taking out a loan from a bank, the lender. Companies also borrow from lending banks. Someone who invests in a company’s stock is the lender in this case; the company is essentially the borrower.

In this section, we examine the basic building block of interest paid on loans and borrowed credit and also the returns on investments like bank accounts, simple interest.

Compute Simple Interest

Let’s get some terminology understood. Interest to be paid by a borrower is often expressed as an annual percentage rate, which is the percent of the principal that is paid as interest for each year the money is borrowed. This means that the more that is borrowed, the more that must be paid back. Sometimes, the interest to be paid back is simple interest, which means that the interest is calculated on the amount borrowed only.

The length of time until the loan must be paid off is the **term** of the loan. The date when the loan must be paid off is when the loan is **due**. The day that the loan is issued is the origination date. We'll put this terminology to use in the following examples. Note that in this section we will use letters, called variables, to represent the different parts of the formulas we'll be using. This will help keep our formulas and calculations manageable.

Simple Interest Loans with Integer Year Terms

Calculating simple interest is similar to the percent calculations we made in Understanding Percent and Discounts, Markups, and Sales Tax, but must be multiplied by the term of the loan (in years, if dealing with an annual percentage rate).

FORMULA

The simple interest, I, to be paid on a loan with annual interest rate r for a number of years (term of the loan) t, with principal P, is found using I =P×r×tI=P×r×t, where the decimal form of the interest rate, r, is used. The total repaid, then is $T = P + I$ or, more directly, $T = P$+P×r×tT=P+P×r×t. This total is often referred to as the loan **payoff amount**, or more simply just the payoff.

When the annual interest rate, the principal, and the number of years that the money is borrowed is known, the interest to be paid can be found and from there the total to be repaid can be calculated.

Checkpoint

Be aware, interest paid to a lender is almost uniformly rounded up to the next cent.

Example ✓ 6.3.1: Simple Interest on Loans with Integer Year Terms

Calculate the simple interest to be paid on a loan with the given principal, annual percentage rate, and number of years. Then, calculate the loan payoff amount.

1. Principal P = $4,000, annual interest rate r = 5.5%, and number of years t = 4
2. Principal P = $14,800, annual interest rate r = 7.9%, and number of years t = 7

Answer

1. Substitute the principal P = $4,000, the decimal form of the annual interest rate r = 0.055, and number of years t = 4 into the formula for simple interest, and calculate.
   $I = P \times r \times t = 4{,}000 \times 0.055 \times 4 =$880I=P×r×t=4,000×0.055×4=880.
   The simple interest, or cost of the loan, to be paid on the loan is $880.
   The loan payoff amount, or the total to be repaid, is $T = P + I = 4{,}000 + 880 = 4{,}$880T=P+I=4,000+880=4,880, or $4,880.00.
2. Substitute the principal P = $14,800, the decimal form of the annual interest rate r = 0.079, and number of years t = 7 into the formula for simple interest, and calculate.
   $I = P \times r \times t = 14{,}800 \times 0.079 \times 7 = 8{,}184.4$.
   The simple interest, or cost to borrow, to be paid on the loan is $8,184.40.
   The loan payoff amount, or the total to be repaid, is $T = P + I = 14{,}800 + 8{,}184.4 = 22{,}984.4$, or $22,984.40.

Your Turn 6.3.1

Calculate the simple interest to be paid on a loan with the given principal, annual percentage rate, and number of years. Then calculate the loan payoff, or total to be repaid.

Principal P = $6,700, annual interest rate r = 11.99%, and number of years t = 3

Principal P = $25,800, annual interest rate r = 6.9%, and number of years t = 5

Example 6.3.2: Simple Interest Equipment Loan

Riley runs an auto repair shop, and needs to purchase a new brake lathe, which costs $11,995. She takes out a two-year, simple interest loan at an annual interest rate of 14.9%. How much interest will she pay and how much total will she repay on the loan?

Answer

Step 1. Determine the variables, or parts of the formula. The principal P is the cost of the brake lathe, so P = \$11,995. The interest rate Riley pays is 14.9%, or r = 0.149 in decimal form. The length of the loan is two years, so t = 2. We are first asked to find I, the interest Riley will pay.

Step 2. Substitute the known variables into the formula for simple interest I =P×r×tI=P×r×t and solve for I.

From Step 1 we have $I = P \times r \times t = 11995 \times 0.149 \times 2 = 3{,}574.51$.

This tells us that the simple interest, or cost to borrow, to be paid on the loan is $3,574.51.

Step 3. Use the formula $T = P + I$ to determine the total amount Riley will repay, T.

The total to be repaid is $T = P + I = 11{,}995 + 3{,}574.51 = 15{,}569.51$, or \$15,569.51.

Your Turn 6.3.2

Beth is the owner of a small retail store in downtown St. Louis. The windows in the storefront need replacing, so she needs to take out a $9,500 loan to get the repairs done. The rate she secures is 9.25% and the term of the loan is one year. How much interest will she pay and how much total will she repay on the loan?

Simple Interest Loans with Other Lengths of Terms

In the previous example and Your Turn exercise, the loans were paid back in one payment after an integer number of years. However, there are also loans lasting a length of time not equal to an integer number of years (like 1, 2, or 3 years or more), but in a number of months (like 4 months, 18 months, and so on). What model would apply to these situations?

When the loan is paid back after a term that is not an integer number of years but is instead a number of months, the term of the loan, or time, t, is expressed as a fraction of the year. So for a 2-month loan, the time, in years, is 2/12 = 1/6. For a 5-month loan, the time in years is 5/12. For an 18-month term, the term in years is 18/12 = 1.5.

Example 6.3.3: Loan to Purchase Equipment

Abeje needs a loan to purchase equipment for the gym she is going to open. She visits the bank and secures a 4-month loan of $20,000. Her annual percentage rate is 6.75%. How much interest will Abeje pay and what is her loan payoff amount?

Answer

Abeje's loan is for \$20,000, so her principal is P = 20,000. The interest rate Abeje will pay is 6.75%, or r = 0.0675 in decimal form. The length of the loan is 4 months, so t =412t=412. Substituting these in the formula for simple interest, we find her interest to be $I = P \times r \times t = 20{,}000 \times 0.0675 \times 4/12$ =450I=P×r×t=20,000×0.0675×4/12=450

The simple interest, or cost to borrow, to be paid on the loan is $450.00.

The payoff is $T = P + I = 20{,}000 + 450 = 20{,}$450T=P+I=20,000+450=20,450, or \$20,450.00.

Your Turn 6.3.3

Samuel needs to borrow $8,400 to pay for repairs to his small manufacturing facility. He manages to get a simple interest loan at 17.33%, to be paid after 6 months. How much interest will Samuel pay and what is Samuel's loan payoff amount?

Those examples dealt in months. However, some loans are for days only (45 days, 60 days, 120 days). In such cases, we find the daily interest rate. The fraction we will use for the daily interest rate is the interest rate (as a decimal) divided by 365. This may be referred to as Actual/365. In order to find the term of the loan, divide the number of days in the term of the loan by 365.

FORMULA

To determine the interest, I, on a loan with term t expressed in days, with principal of P, and interest rate in decimal form of r, calculate I =P×r365×tI=P×r365×t. Here, semanticsr365r365 represents the daily interest rate.

Alternately, the above formula is equivalent to I =P×r×t365I=P×r×t365, where the interest rate remains an annual rate, but the time is expressed as a fraction of the year.

Who Knew?

It seems reasonable to use 365 as the number of days in the year, since there are 365 days in most years. However, sometimes, banks have used (and continue to use) 360 as the number of days in a year. They may also treat all months as if they have 30 days. These differences lead to (sometimes small) differences in how much interest is paid. Since the number of days is in the denominator, a smaller denominator (360) will result in larger numbers (interest) that is 365 is used for the denominator. See this page from ACRE for a comparison.

Example 6.3.4: Loan for Moving Costs

David plans to move his family from Raleigh, North Carolina to Tempe, Arizona. His company will reimburse (pay after the move) David for the move. David does research and determines that movers will cost $5,600 to move his family's belongings to Tempe. He takes out a simple interest, 45-day loan at 11.75% interest to pay this cost. How much interest will be paid on this 45-day loan, and what is David's loan payoff amount?

Answer

This loan is in terms of days, so we will use the formula I =P×r365×tI=P×r365×t, where t is the number of days and r is the annual interest rate.

The principal for the loan is the moving cost, or P = 5,600. The annual interest rate that David will pay is 11.75%, which in decimal is 0.1175. The length of time for the loan is 45 days, so t = 45.

Substituting these values into the formula and calculating, we find that the interest to be paid is $I = 5{,}600 \times \frac{0.1175}{365} \times 45 = 81.13$, or $81.13 (remember, interest is almost always rounded up to the next cent).

The payoff for the loan is $5,681.13.

Your Turn 6.3.4

Heather runs a silk screen t-shirt shop, and seeks a short-term loan to pay for new inventory (paints, blades, shirts). They secure a $3,700 simple interest loan for 60 days, at an annual rate of 18.99%. How much will borrowing the money cost Heather, and what is her loan payoff amount?

Understand and Compute Future Value

Money can be invested for a specific amount of time and earn simple interest while invested. The terminology and calculations are the same as we've already seen. However, instead of the total to be paid back, the investor is interested in the total value of the investment after the interest is added. This is called the **future value** of the investment.

FORMULA

The future value, semanticsFVFV, of an investment that yields simple interest is $FV = P + I = P$+P×r×tFV=P+I=P+P×r×t, where P is the principal (amount invested at the start), r is the annual interest rate in decimal form, and t is the length of time the money is invested. The time t will be an integer if the term of the deposit is an integer number of years, will be number of months/12 if the term is in months, will be actual/365 if the deposit is for a number of days.

Example 6.3.5: Simple Interest on a Deposit

In the following, determine how much interest was earned on the investment and the future value of the investment, if the investment yields simple interest.

1. Principal is $1,000, annual interest rate is 2.01%, and time is 5 years
2. Principal is $5,000, annual interest rate is 1.85%, and time is 30 years
3. Principal is $10,000, annual interest rate is 1.25%, and time is 18 months
4. Principal is $7,000, annual interest rate is 3.26%, and time is 100 days

Answer

1. The principal is P = $1,000, the annual interest rate, in decimal form, is 0.0201, and the term is 5 years, or t = 5. Since the term is an integer number of years, the interest earned on the investment is $I = P \times r \times t = 1{,}000 \times 0.0201 \times 5 = 100.5$, or the interest earned was $100.50.
   To find the future value, we use the formula $FV = P$+IFV=P+I. Substituting the values and calculating, we find the future value of the investment to be $FV = P + I = 1{,}000 + 100.5 = 1100.5$. The future value of the investment at the end of 5 years is $1,100.50.
   Notice that the future value could have been calculated directly with $FV = P$+P×r×tFV=P+P×r×t
2. The principal is P = $5,000, the annual interest rate, in decimal form, is 0.0185, and the term is 30 years, or t = 30. Since the term is an integer number of years, the interest earned on the investment is $I = P \times r \times t = 5{,}000 \times 0.0185 \times 30 = 2$,775I=P×r×t=5,000×0.0185×30=2,775, or the interest earned was $2,775.00. To find the future value, we use the formula $FV = P$+IFV=P+I. Substituting the values and calculating, we find the future value of the investment to be $FV = P + I = 5{,}000 + 2{,}775 = 7$,775FV=P+I=5,000+2,775=7,775. The future value of the investment at the end of 30 years is $7,775.00.
3. The principal is P = $10,000, the annual interest rate, in decimal form, is 0.0125, and the term is 18 months. Since the term is in months, we have to write the months in terms of years. For 18 months, we use 18/12 as t. The interest earned on the investment is $I = P \times r \times t = 10{,}000 \times 0.0125 \times \frac{18}{12} = 187.5$, or the interest earned was $187.50. To find the future value, we use the formula $FV = P$+IFV=P+I. Substituting the values and calculating, we find the future value of the investment to be $FV = P + I = 10{,}000 + 187.5 = 10{,}187.5$. The future value of the investment at the end of 18 months is $10,187.50.
4. Principal is $7,000, annual interest rate is 3.26%, and time is 100 days. The principal is P = $7,000, the annual interest rate, in decimal form, is 0.0326, and the term is 100 days. Since the term is in days, we have to write the time using actual/365, or t = 100/365. The interest earned on the investment is $I = P \times r \times t = 7{,}000 \times 0.0326 \times \frac{100}{365} = 62.52$, or the interest earned was $62.52. To find the future value, we use the formula $FV = P$+IFV=P+I. Substituting the values and calculating, we find the future value of the investment to be $FV = P + I = 7{,}000 + 62.52 = 7{,}062.52$. The future value of the investment at the end of 100 days is $7,062.52.

Your Turn 6.3.5

In the following, determine how much interest was earned on the investment and the future value of the investment if the investment yields simple interest.

Principal is $4,500, annual interest rate is 1.88%, and time is 3 years

Principal is $2,000, annual interest rate is 2.03%, and time is 10 years

Principal is $120,000, annual interest rate is 3.1%, and time is 100 days

Principal is $4,680, annual interest rate is 1.55%, and time is 42 months

You may have noticed that for these problems, the future value was rounded down. When the future value is paid, the amount is typically rounded down.

A certificate of deposit (CD) is a savings account that holds a single deposit (the principal) for a fixed term at a fixed interest rate. Once the term of the CD is over, the CD may be redeemed (cashed in or withdrawn) and the owner of the CD receives the original principal plus the interest earned. The deposit often cannot be withdrawn until the term is up; if it can be withdrawn early, there is often a penalty imposed to do so.

✓ Example 6.3.6: Certificate of Deposit

Jonas deposits $2,500 in a CD bearing 3.25% simple interest for a term of 3 years. When he redeems his CD at the end of the 3 years, how much will he receive?

Answer

This is a future value example. We know that P = $2,500 is the amount deposited. The annual simple interest rate in decimal form is r = 0.0325. The term of the investment is t = 3 years.

Substituting those values into the future value formula, we have $FV = P + P \times r \times t = 2{,}500 + 2{,}500 \times 0.0325 \times 3 = 2{,}500 + 243.75 = 2{,}743.75$.

When the CD is redeemed, Jonas will receive $2,743.75.

Your Turn 6.3.6

Mia deposits $4,900 in a CD bearing 3.95%. The CD term is 7 years. When she redeems the CD, how much will Mia receive?

WORK IT OUT

The reason CD (certificate of deposit) rates look so small is because they are extremely safe investments. Though overall interest rates for CDs change over time and individual returns vary with the terms of the CD, investors are offered predictable interest income for their investments.

To investigate this yourself, search online to determine the strengths and weaknesses of CDs (investopedia.com offers good, basic information on investing). Then, online, identify five national banks and two local banks who offer CDs.

- Track the interest rates for the CDs at various terms (1 year, 3 years, 5 years) for each of the banks you found that offer CDs.
- Calculate the amount of interest earned for a $10,000 deposit for each CD at each of the terms.
- Compare the results from the various banks, CDs, and terms and decide which is the best investment. You may want to consider both the length of time that the money is locked up, and the return.

Paying Simple Interest Loans with Partial Payments

In every example above, there was one payment for the loan, or one withdrawal for the investment. However, for many loans (house, car, in-ground swimming pool), the loan will be paid back in two or more payments. Such a payment is called a **partial payment**, because they only pay off part of the loan.

When a partial payment is made, some of the payment pays for the principal, but the rest of the payment pays for interest on the principal. When making the first partial payment, the interest is calculated on the principal for the time between the origination date of the loan and the date of the payment. If another partial payment is made, the interest is calculated based on the remaining principal and the time between the previous partial payment and the current partial payment date.

✓ Example 6.3.7: Interest Paid in a Partial Payment on a Loan

1. A simple interest loan for $6,500 is taken out at 12.6% annual percentage rate. A partial payment is made 45 days into the loan period. How much of the partial payment will be for interest?
2. A simple interest loan for $13,700 is taken out at 6.55% annual interest rate. A partial payment is to be made after 60 days. How much of the partial payment will be for interest?

Answer

1. To find the interest paid in this partial payment, we calculate the interest on the principal for the time between the origination of the loan and the payment day, or 45 days.
   The principal is \$6,500. The annual interest rate, in decimal form, is 0.126.
   The interest paid for 45 days is found by substituting the values for principal P, rate r, and time t into the formula I =P×r365×tI=P×r365×t.
   Calculating, we have I =P×r365×t=6,500×0.126365×45=100.9726I=P×r365×t=6,500×0.126365×45=100.9726.
   Rounding up, the portion of the partial payment that will be paid for interest is \$100.98.
2. To find the interest paid in this partial payment, we calculate the interest on the principal for the time between the origination of the loan and the payment day, or 60 days.
   The principal is \$13,700. The annual interest rate, in decimal form, is 0.0655.
   The interest paid for those 60 days is found by substituting those values into the formula I =P×r365×tI=P×r365×t.
   Calculating, we have I =P×r365×t=13,700×0.0655365×60=147.5096I=P×r365×t=13,700×0.0655365×60=147.5096.
   Rounding up, the portion of the partial payment that will be paid for interest is \$147.51.

Your Turn 6.3.7

A simple interest loan for \$50,000 is taken out at 5.15% annual percentage rate. A partial payment is made 120 days into the loan period. How much of the partial payment will be for interest?

A simple interest loan for \$8,500 is taken out at 9.9% annual interest rate. A partial payment is to be made after 75 days. How much of the partial payment will be for interest?

Remaining Balance

The previous examples demonstrated how to determine the interest paid in a partial payment. Using this, we can determine the remaining balance after a partial payment.

Step 1: determine the amount of the payment, P, that is applied to interest, I.

Step 2: subtract the amount paid in interest from the payment, $(P - I)$. This is the amount applied to the balance.

Step 3: subtract the amount applied to the balance (the value obtained in Step 2) from the balance of the loan, $B - (P - I)$. This is the remaining balance after the partial payment.

Example 6.3.8: Determining the Remaining Balance on a Loan After a Partial Payment

1. A simple interest loan for \$45,500 is taken out at 11.8% annual percentage rate. A partial payment of \$20,000 is made 50 days into the loan period. After this payment, what will the remaining balance of the loan be?
2. A simple interest loan for \$150,000 is taken out at 5.85% annual percentage rate. A partial payment of \$50,000 is made 70 days into the loan period. After this payment, what will the remaining balance of the loan be?

Answer

1. The principal is \$45,500, which will be treated as the balance, B, of the loan. The annual simple interest rate, in decimal form, is 0.118. The time is t = 50 days.
   Step 1: Determine the amount of the partial payment that is applied to interest. To find this, substitute the values above into the formula I =P×r365×tI=P×r365×t and calculate. Calculating, the amount of the payment that is applied to interest is $I = 45{,}500 \times \frac{0.118}{365} \times 50 = 735.4795$. Rounding up, we have I = \$735.48.
   Step 2: The amount of the payment that is to be applied to the balance of the loan is partial payment minus the amount of the partial payment that is applied to the interest. The payment is \$2,000. The amount that is applied to the balance is $P - I = \$20{,}000 - \$735.48 = \$19{,}264.52$.
   Step 3: The remaining balance is found by subtracting the amount applied to the balance from the previous balance, or $B - (P - I) = \$45{,}500 - \$19{,}264.52 = \$26{,}235.48$.
   The remining balance after the partial payment is \$26,235.48.
2. The principal is \$150,000, which will be treated as the balance, B, of the loan. The annual simple interest rate, in decimal form, is 0.0585. The time is t = 70 days.
   Step 1: Determine the amount of the partial payment that is applied to interest. To find this, substitute the values above

into the formula I =P×r365×tI=P×r365×t and calculate. Calculating, the amount of the payment that is applied to interest is $I = 150{,}000 \times \frac{0.0585}{365} \times 70 = 1{,}682.8767$. Rounding up, we have I = \$1,682.88.

Step 2: The amount of the payment that is to be applied to the balance of the loan is partial payment minus the amount of the partial payment that is applied to the interest. The payment is \$50,000. The amount that is applied to the balance is $P - I = \$50{,}000 - \$1{,}682.88 = \$48{,}317.12$.

Step 3: The remaining balance is found by subtracting the amount applied to the balance from the previous balance, or $B - (P - I) = \$150{,}000 - \$48{,}317.12 = \$101{,}682.88$.

The remining balance after the partial payment is \$101,682.88.

Your Turn 6.3.8

A simple interest loan for \$1,400 is taken out at 12.5% annual percentage rate. A partial payment of \$700 is made 20 days into the loan period. After this payment, what will the remaining balance of the loan be?

A simple interest loan for \$23,000 is taken out at 7.25% annual percentage rate. A partial payment of \$10,000 is made 40 days into the loan period. After this payment, what will the remaining balance of the loan be?

Loan Payoff

Finally, we will determine the amount to be paid at the end of the loan. To do so, we apply the formula for the loan payoff to the remaining balance. However, the length of time for that remaining balance is the time between the partial payment and the day the loan is paid off.

Step 1: Determine the remaining balance after the partial payment.

Step 2: Calculate the number of days between the partial payment and the date the loan is paid off. This will be the time t in the payment formula.

Step 3: Calculate the amount to be paid at the end of the loan, or the payoff amount, using $\text{Payoff} = P$+P×r365×tPayoff=P+P×r365×t, where P is the remaining balance and t is the time found in Step 2.

Example 6.3.9: Finding Loan Pay Off After a Partial Payment

Laura takes out an \$18,400 loan for 120 days at 17.9% simple interest. She makes a partial payment of \$7,500 after 45 days. What is her payoff amount at the end of the loan?

Answer

The initial balance, or principal, of her loan is \$18,400. The interest rate in decimal form is 0.179. Her partial payment of \$7,500 is made after 45 days. Using these values, we can determine how much of the partial payment is applied to the balance. From there, we can determine her final loan payoff after 120 days.

Step 1: Determine the remaining balance after the partial payment. Using the partial payment process outlined in the previous example, we first find that the amount of the partial payment that is applied to the balance. Their interest paid in the partial payment is I =P×r365×t=18,400×0.179365×45=406.0603I=P×r365×t=18,400×0.179365×45=406.0603, or \$406.07 (remember to round up!). Using this and that the loan amount was for \$18,400, the remaining balance on the loan after the partial payment is $B - (P - I) = \$18{,}400 - (\$7{,}500 - \$406.07) = \$11{,}306.07$.

Step 2: The number of days between the partial payment and the date that the loan is to be paid off is 120 – 45 = 75. This means that the time between the partial payment and the final payment is 75 days.

Step 3: To calculate the payoff amount, use $\text{payoff} = P$+P×r365×tpayoff=P+P×r365×t, with P = \$11,306.07 (the remaining balance), t = 75 (from Step 2) and r = 0.179. The payoff amount, then, is $\text{payoff} = 11{,}306.07 + 11{,}306.07 \times \frac{0.179}{365} \times 75 = 11{,}721.9165$. Rounding up, the payoff amount is \$11,721.92.

Your Turn 6.3.9

Paola takes out a 75-day loan for $3,500.00. Her interest rate is 11.2%. If she makes a partial payment of $1,250.00 after 30 days, what will her payoff be at the end of the loan?

Repeated Partial Payments

Car loans and mortgages (loans for homes) are paid off through repeated partial payments, most often monthly payments. Since car loans are often 3 to 6 years, and mortgages 15 to 30 years, calculating each individual monthly payment one at a time is time consuming and tedious. Even a 3-year loan would involve applying the above steps 36 times! Fortunately, there is a formula for determining the amount of each partial payment for monthly payments on a simple interest loan.

FORMULA

The amount of monthly payments, A, for a loan with principal P, monthly simple interest rate r (in decimal form), for t number of months is found using the formula $A = P \times \frac{r \times (1+r)^t}{(1+r)^t - 1}$. The monthly interest rate is the annual rate divided by 12. The number of months is the number of years times 12.

Example 6.3.10: Calculating Car Payments

Desiree buys a new car, by taking a loan out from her credit union. The balance of her loan is $27,845.00. The annual interest rate that Desiree will pay is 7.3%. She plans to pay this off over 4 years. How much will Desiree’s monthly payment be?

Answer

To use the formula for monthly payments, we need the principal, the interest rate, and the number of years. The principal is $27,845. The annual rate, in decimal form, is 0.073. Dividing 0.073 by 12 gives the monthly interest rate $0.073/12 = 0.00608\bar{3}$. She takes the loan out for 4 years, which is $t = 12 \times 4 =$48t=12×4=48 months. Substituting these values into the formula, $A = P \times \frac{r \times (1+r)^t}{(1+r)^t - 1}$, we calculate:

$$\begin{aligned} A &= 27{,}845 \times \frac{0.00608\bar{3} \times (1 + 0.00608\bar{3})^{48}}{(1 + 0.00608\bar{3})^{48} - 1} \\ &= 27{,}845 \times \frac{0.00608\bar{3} \times (1.00608\bar{3})^{48}}{(1.00608\bar{3})^{48} - 1} \\ &= 27{,}845 \times \frac{0.00608\bar{3} \times 1.337918996}{1.337918996 - 1} \\ &= 27{,}845 \times \frac{0.008139007}{0.337918996} \\ &= 27{,}845 \times 0.024085675 \\ &= 670.6656 \end{aligned}$$

Using the formula and rounding up to the next cent, we see that Desiree’s monthly payment will be $670.67.

Your Turn 6.3.10

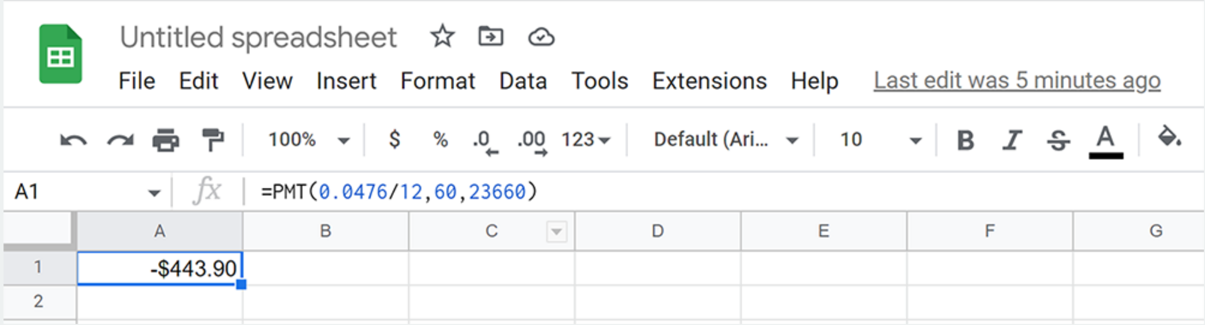
Russell buys a new car, by taking a loan out from the dealership. After all of the discussions are over, he finances (gets a loan for) $23,660. The annual interest rate that Russell will pay is 4.76%. He plans to pay this off over 5 years. How much will Russell’s monthly payment be?

Tech Check

The calculation of payments is long, and involves many steps. However, most spreadsheet programs, including Google Sheets, have a payment function. In Google Sheets, that function is PMT. To find the payment for an installment loan (like for a car), you need to enter the interest rate per period, the number of payments, and the loan amount. From Your Turn 6.36, the rate was 0.0476/12, the number of payments was 60, and the loan amount was $23,660. In Google Sheets, select any cell and enter the following:

=PMT(0.0476/12,60,23660)

And click the enter key. Immediately, in the cell you selected, the payment of $443.90 appears, though with a negative sign. The negative sign indicates it is a payment out of an account. Since we want to know the payment amount, we ignore the negative sign. The result, with the formula in the formula bar, is shown in Figure 6.5.



=PMT(0.0476/12,60,23660)

A1: -$443.90

Figure 6.5: Google Sheets payment function

In general, to use the PMT function in Google Sheets, enter

=PMT(r/12,t*12,P)

where r is the annual interest rate, t is the number of years, and P is the principal of the loan.

Understand and Compute Present Value for Simple Interest Investments

When finding the future value of an investment, we know how much is deposited, but we have no idea how much that money will be worth in the future. If we set a goal for the future, it would be useful to know how much to deposit now so an account reaches the goal. The amount that needs to be deposited now to hit a goal in the future is called the **present value**.

FORMULA

The present value, semanticsPVPV, of money deposited at an annual, simple interest rate of r (in decimal form) for time t (in years) with a specified future value of semanticsFVFV, is calculated with the formula $PV = \frac{FV}{(1+rt)}$.

Note: Present value, in this calculation, is always rounded up. Otherwise, future value may fall short of the target future value.

Understanding what this tells you is important. When you find the present value, that is how much you need to invest now to reach the goal semanticsFVFV, under the conditions (time and rate) at which the money will be invested.

Example 6.3.11

Compute the present value of the investment described. Interpret the result.

1. semanticsFVFV = $10,000, t = 15 years, annual simple interest rate of 5.5%
2. semanticsFVFV = $150,000, t = 20 years, annual simple interest rate of 6.25%
3. semanticsFVFV = $250,000, t = 486 months, annual simple interest rate of 4.75%

Answer

1. The future value is semanticsFVFV = $10,000. The time of the investment is in years, so t = 15. The annual, simple interest rate is 5.5%, which in decimal form is 0.055. We substitute those values into the formula and calculate. $PV = \frac{FV}{(1+rt)} = \frac{10{,}000}{(1+(0.055)\times(15))} = \frac{10{,}000}{(1+0.825)} = \frac{10{,}000}{(1.825)} = 5{,}479.452$. Rounding up, we see that the present value of $10,000 invested at a simple annual interest rate of 5.5% for 15 years is $5,479.46. This means that $5,479.46 needs to be invested so that, after 15 years at 5.5% interest, the investment will be worth $10,000.
2. The future value is semanticsFVFV = $150,000. The time of the investment is in years, so t = 20. The annual, simple interest rate is 6.25%, which in decimal form is 0.0625. We substitute those values into the formula and calculate. $PV = \frac{FV}{(1+rt)} = \frac{150{,}000}{(1+(0.0625)\times(20))} = \frac{150{,}000}{(1+1.25)} = \frac{150{,}000}{(2.25)} = 66{,}666.\bar{6}$. Rounding up, we see that the present value of $150,000 invested at a simple annual interest rate of 6.25% for 20 years is $66,666.67. This means that $66,666.67 needs to be invested so that, after 20 years at 6.25% interest, the investment will be worth $150,000.

3. The future value is semanticsFVFV = $250,000. The time of the investment is 486 months. This needs to be converted to years. To do so, divide the number of months by 12, giving $years = \frac{486}{12} = 40.5$, so t = 40.5 years. The annual, simple interest rate is 4.75%, which in decimal form is 0.0475. We substitute those values into the formula and calculate. $PV = \frac{FV}{(1+rt)} = \frac{250{,}000}{(1+(0.0475)\times(40.5))} = \frac{150{,}000}{(1+1.92375)} = \frac{150{,}000}{(2.92375)} = 129{,}954.5159$. Rounding up, we see that the present value of $250,000 invested at a simple annual interest rate of 4.75% for 486 months is $129,954.52. This means that $129,954.52 needs to be invested so that, after 486 months at 4.75% interest, the investment will be worth $150,000.

Your Turn 6.3.11

Compute the present value of the investment described. Interpret the result.

FV = $25,000, t = 10 years, annual simple interest rate of 7.5%

FV = $320,000, t = 35 years, annual simple interest rate of 6.5%

FV = $90,000, t = 270 months, annual simple interest rate of 3.75%

Example 6.3.12: Present Value of a CD

Beatriz will invest some money in a CD that yields 3.99% simple interest when invested for 30 years. How much must Beatriz invest so that after those 30 years, her CD is worth $300,000?

Answer

Beatriz needs to know how much to deposit now so that her CD is worth $300,000 after 30 years. This means she needs to know the present value of that $300,000. The time is 30 years and the annual simple interest rate, in decimal form, is 0.0399. Using that information and the formula for present value, we calculate the present value of that $300,000. $PV = \frac{FV}{(1+rt)} = \frac{300{,}000}{(1+(0.0399)\times(30))} = \frac{300{,}000}{(1+1.197)} = \frac{300{,}000}{(2.197)} = 136{,}549.8407$. Rounding up, Beatriz needs to invest $136,549.85 so that she has $300,000 in 30 years.

Your Turn 6.3.12

Kentaro will invest some money in a CD that yields 2.75% simple interest when invested for 5 years. How much must Kentaro invest so that after those 5 years his CD is worth $12,000?

Check Your Understanding

1. What is interest?
2. What is the principal in a loan?
3. Calculate the simple interest to be paid for a 6-year loan with principal $1,500.00 and annual interest rate of 12.99%
4. A simple interest loan for $24,200 is taken out at 10.55% annual percentage rate. A partial payment of $13,000 is made 25 days into the loan period. After this payment, what will the remaining balance of the loan be?
5. Find the monthly payment for a $9,800.00 loan at a 13.8% interest for 4 years.
6. Find the present value of an investment with future value $30,000 with a simple interest rate of 3.75% invested for 10 years.

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6.3.0: Exercises

? Exercise 6.3.0.1

If \$1,500.00 is invested in an account bearing 3.5% interest, what is the principal?

? Exercise 6.3.0.2

If \$1,500 is invested in an account bearing 3.5% interest, what is the interest rate?

? Exercise 6.3.0.3

What is simple interest?

? Exercise 6.3.0.4

What is the present value of an investment?

? Exercise 6.3.0.5

What is the future value of an investment?

? Exercise 6.3.0.6

What is a partial payment on a loan?

In the following exercises, calculate the simple interest and payoff for the loan with the given principal, simple interest rate, and time.

? Exercise 6.3.0.7

Principal P = \$5,000, annual interest rate r = 6.5%, and number of years t = 6

? Exercise 6.3.0.8

Principal P = \$3,500, annual interest rate r = 12%, and number of years t = 7

? Exercise 6.3.0.9

Principal P = \$7,800, annual interest rate r = 11.5%, and number of years t = 10

? Exercise 6.3.0.10

Principal P = \$62,500, annual interest rate r = 4.88%, and number of years t = 4

? Exercise 6.3.0.11

Principal P = \$4,600, annual interest rate r = 9.9%, for 18 months

? Exercise 6.3.0.12

Principal P = \$19,000, annual interest rate r = 16.9%, for 14 months

? Exercise $6.3.0.13$

Principal P = \$8,500, annual interest rate r = 10.66%, for 6 months

? Exercise $6.3.0.14$

Principal P = \$17,600, annual interest rate r = 17.9%, for 20 months

? Exercise $6.3.0.15$

Principal P = \$4,000, annual interest rate r = 8.5%, for 130 days

? Exercise $6.3.0.16$

Principal P = \$9,900, annual interest rate r = 15.9%, for 90 days

? Exercise $6.3.0.17$

Principal P = \$600, annual interest rate r = 16.8%, for 25 days

? Exercise $6.3.0.18$

Principal P = \$890, annual interest rate r = 9.75%, for 200 days

In the following exercises, find the future value of the investment with the given principal, simple interest rate, and time.

? Exercise $6.3.0.19$

Principal is \$5,300, annual interest rate is 2.07%, and time is 18 years.

? Exercise $6.3.0.20$

Principal is \$14,700, annual interest rate is 3.11%, and time is 10 years.

? Exercise $6.3.0.21$

Principal is \$5,600, annual interest rate is 2.55%, for 30 months.

? Exercise $6.3.0.22$

Principal is \$10,000, annual interest rate is 1.99%, for 15 months.

? Exercise $6.3.0.23$

Principal is \$2,000, annual interest rate is 3.22%, for 100 days.

? Exercise $6.3.0.24$

Principal is \$900, annual interest rate is 3.75%, for 175 days.

In the following exercises, determine the amount applied to principal for the indicated partial payment on the loan with the given principal, interest rate, and time when the partial payment was made.

? Exercise 6.3.0.25

A simple interest loan for $2,700 is taken out at 11.6% annual percentage rate. A partial payment of $1,500 is made 28 days into the loan period.

? Exercise 6.3.0.26

A simple interest loan for $900 is taken out at 18.9% annual percentage rate. A partial payment of $400 is made 30 days into the loan period.

? Exercise 6.3.0.27

A simple interest loan for $13,500 is taken out at 14.8% annual percentage rate. A partial payment of $8,000 is made 75 days into the loan period.

? Exercise 6.3.0.28

A simple interest loan for $9,900 is taken out at 9.875% annual percentage rate. A partial payment of $4,000 is made 65 days into the loan period.

In the following exercises, determine the remaining principal for the indicated partial payment on the loan with the given principal, interest rate, and time when the partial payment was made.

? Exercise 6.3.0.29

A simple interest loan for $2,700 is taken out at 11.6% annual percentage rate. A partial payment of $1,500 is made 28 days into the loan period.

? Exercise 6.3.0.30

A simple interest loan for $900 is taken out at 18.9% annual percentage rate. A partial payment of $400 is made 30 days into the loan period

? Exercise 6.3.0.31

A simple interest loan for $13,500 is taken out at 14.8% annual percentage rate. A partial payment of $8,000 is made 75 days into the loan period.

? Exercise 6.3.0.32

A simple interest loan for $9,900 is taken out at 9.875% annual percentage rate. A partial payment of $4,000 is made 65 days into the loan period.

In the following exercises, find the payoff value of the loan with the given principal, annual simple interest rate, term, partial payment, and time at which the partial payment was made.

? Exercise 6.3.0.33

Principal = $1,500, rate = 6.99%, term is 5 years, partial payment of $900 made 2 years into the loan.

Exercise 6.3.0.34

Principal = \$21,500, rate = 7.44%, term is 10 years, partial payment of 15,000 made after 6 years.

Exercise 6.3.0.35

Principal = \$6,800, rate = 11.9%, term is 200 days, partial payment of \$4,000 made after 100 days.

Exercise 6.3.0.36

Principal = \$800, rate = 13.99%, term is 150 days, partial payment of \$525 made after 50 days.

In the following exercises, find the monthly payment for a loan with the given principal, annual simple interest rate and number of years.

Exercise 6.3.0.37

Principal = \$4,500, rate = 8.75%, years = 3

Exercise 6.3.0.38

Principal = \$2,700, rate = 15.9%, years = 5

Exercise 6.3.0.39

Principal = \$13,980, rate = 10.5%, years = 4

Exercise 6.3.0.40

Principal = \$8,750, rate = 9.9%, years = 10

In the following exercises, find the present value for the given future value, FV, annual simple interest rate r, and number of years t.

Exercise 6.3.0.41

FV = \$25,000, t = 15 years, annual simple interest rate of 6.5%

Exercise 6.3.0.42

FV = \$12,000, t = 10 years, annual simple interest rate of 4.5%

Exercise 6.3.0.43

FV = \$15,000, t = 16 years, annual simple interest rate of 3.5%

Exercise 6.3.0.44

FV = \$100,000, t = 30 years, annual simple interest rate of 5.5%

Exercise 6.3.0.45

Rita takes out a simple interest loan for \$4,000 for 5 years. Her interest rate is 7.88%. How much will Rita pay when the loan is due?

? Exercise 6.3.0.46

Humberto runs a private computer networking company, and needs a loan of $31,500 for new equipment. He shops around for the lowest interest rate he can find. He finds a rate of 8.9% interest for a 10-year term. How much will Humberto's payoff be at the end of the 10 years?

? Exercise 6.3.0.47

Jaye needs a short-term loan of $3,500. They find a 75-day loan that charges 14.9% interest. What is Jaye's payoff?

? Exercise 6.3.0.48

Theethat's car needs new struts, which cost $1,189.50 installed, but he doesn't have the money to do so. He asks the repair shop if they offer any sort of financing. It offers him a short-term loan at 18.9% interest for 60 days. What is Theethat's payoff for the struts?

? Exercise 6.3.0.49

Michelle opens a gaming shop in her small town. She takes out an $8,500 loan to get started. The loan is at 9.5% interest and has a term of 5 years. Michelle decides to make a partial payment of $4,700 after 3 years. What will Michelle pay when the loan is due?

? Exercise 6.3.0.50

A small retailer borrows $3,750 for a repair. The loan has a term of 100 days at 13.55% interest. If the retailer pays a partial payment of $2,000 after 30 days, what will the loan payoff be when the loan is due?

? Exercise 6.3.0.51

Sharon invests $2,500 in a CD for her granddaughter. The CD has a term of 5 years and has a simple interest rate of 3.11%. After that 5-year period, how much will the CD be worth?

? Exercise 6.3.0.52

Jen and Fred have a baby, and deposit $1,500 in a savings account bearing 1.76% simple interest. How much will the account be worth in 18 years?

? Exercise 6.3.0.53

Yasmin decides to buy a used car. Her credit union offers 7.9% interest for 5-year loans on used cars. The cost of the car, including taxes and fees, is $11,209.50. How much will Yasmin's monthly payment be?

? Exercise 6.3.0.54

Cleo runs her own silk-screening company. She needs new silk-screening printing machines, and finds two that will cost her, in total, $5,489.00. She takes out a 3-year loan at 8.9% interest. What will her monthly payments be for the loan?

? Exercise 6.3.0.55

Kylie wants to invest some money in an account that yields 4.66% simple interest. Her goal is to have $20,000 in 15 years. How much should Kylie invest to reach that goal?

? Exercise $6.3.0.56$

Ishraq wants to deposit money in an account that yields 3.5% simple interest for 10 years, to help with a down payment for a home. Her goal is to have $25,000 for the down payment. How much does Ishraq need to deposit to reach that goal?

In the following exercises, use the **Cost of Financing**. The difference between the total paid for a loan, along with all other charges paid to obtain the loan, and the original principal of the loan is the cost of financing. It measures how much more you paid for an item than the original price. In order to find the cost of financing, find the total paid over the life of the loan. Add to that any fees paid for the loan. Then subtract the principal.

$\text{Cost of Financing} = \text{Total of payments} + \text{fees} - \text{principal.}$

? Exercise $6.3.0.57$

Yasmin decides to buy a used car. Her credit union offers 7.9% interest for 5-year loans on used cars. The cost of the car, including taxes and fees, is $11,209.50. How much did she pay the credit union over the 5 years? What was the cost of financing for Yasmin?

? Exercise $6.3.0.58$

Cleo runs her own silk-screening company. She needs new silk-screening printing machines, and finds two that will cost her, in total, $5,489.00. She takes out a 3-year loan at 8.9% interest. What was the cost of financing for Cleo?

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6.4: Compound Interest

Figure 6.4.1 The impact of compound interest (credit: "English Money" by Images Money/Flickr, CC BY 2.0)

Learning Objectives

1. Compute compound interest.
2. Determine the difference in interest between simple and compound calculations.
3. Understand and compute future value.
4. Compute present value.
5. Compute and interpret effective annual yield.

For a very long time in certain parts of the world, interest was not charged due to religious dictates. Once this restriction was relaxed, loans that earned interest became possible. Initially, such loans had short terms, so only simple interest was applied to the loan. However, when loans began to stretch out for years, it was natural to add the interest at the end of each year, and add the interest to the principal of the loan. After another year, the interest was calculated on the initial principal plus the interest from year 1, or, the interest earned interest. Each year, more interest was added to the money owed, and that interest continued to earn interest.

Since the amount in the account grows each year, more money earns interest, increasing the account faster. This growth follows a geometric series (Geometric Sequences). It is this feature that gives compound interest its power. This module covers the mathematics of compound interest.

Understand and Compute Compound Interest

As we saw in Simple Interest, an account that pays simple interest only pays based on the original principal and the term of the loan. Accounts offering compound interest pay interest at regular intervals. After each interval, the interest is added to the original principal. Later, interest is calculated on the original principal plus the interest that has been added previously.

After each period, the interest on the account is computed, then added to the account. Then, after the next period, when interest is computed, it is computed based on the original principal AND the interest that was added in the previous periods.

The following example illustrates how compounded interest works.

✓ Example ✓ 6.4.1: Interest Compounded Annually

Abena invests \$1,000 in a CD (certificate of deposit) earning 4% compounded annually. How much will Abena's CD be worth after 3 years?

Answer

Since the interest is compounded annually, the interest will be computed at the end of each year and added to the CD's value. The interest at the end of the following year will be based on the value found form the previous year.

Step 1: After the first year, the interest in Abena's CD is computed using the interest formula I =P×r×tI=P×r×t. The principal is P = 1,000, the rate, as a decimal, is 0.04, and the time is one year, so t = 1. Using that, the interest earned in the first year is $I = P \times r \times t = 1{,}000 \times 0.04 \times 1$ =40I=P×r×t=1,000×0.04×1=40, so the interest earned in the first year was $40.00. This is added to the value of the CD, making the CD worth $\$1{,}000 + \$40 = \$1{,}040$.

Step 2: At the end of the second year, interest is again computed, but is computed based on the CD's new value, $1,040. Using this new value and the interest formula (r and t are still 0.04 and 1, respectively), we see that the CD earned $I = P \times r \times t = 1{,}040 \times 0.04 \times 1 = 41.6$, or $41.60. This is added to the value of the CD, making the CD now worth $\$1{,}040.00 + \$41.60 = \$1{,}081.60$.

Step 3: At the end of the third year, interest is again computed, but is computed based on Abena's CD's new value, $1,081.60. Using this value and the interest formula (r and t are still 0.04 and 1, respectively), we see that the CD earned $I = P \times r \times t = 1{,}081.60 \times 0.04 \times 1 = 43.264$, or $43.26 (remember to round down). This is added to the value of the CD, making the CD now worth $\$1{,}081.60 + \$43.26 = \$1{,}124.86$.

After 3 years, Abena's CD is worth $1,124.86.

Your Turn 6.4.1

Oksana deposits $5,000 in a CD that earns 3% compounded annually. How much is the CD worth after 4 years?

Determine the Difference in Interest Between Simple and Compound Calculations

It is natural to ask, does compound interest make much of a difference? To find out, we revisit Abena's CD.

✓ Example 6.4.2: Comparing Simple to Compound Interest on a 3-Year CD

Abena invested $1,000 in a CD that earned 4% compounded annually, and the CD was worth $1,124.86 after 3 years. Had Abena invested in a CD with simple interest, how much would the CD have been worth after 3 years? How much more did Abena earn using compound interest?

Answer

Had Abena invested $1,000 in a 4% simple interest CD for 3 years, her CD would have been worth $P + P \times r \times t = 1{,}000 + 1{,}000 \times 0.04 \times 3 = 1{,}$120P+P×r×t=1,000+1,000×0.04×3=1,120, or $1,120.00. With interest compounded annually, Abena's CD was worth $1,124.86. The difference between compound and simple interest is $\$1{,}124.86 - \$1{,}120.00 = \$4.86$. So compound interest earned Abena $4.86 more than the simple interest did.

Your Turn 6.4.2

Oksana deposits $5,000 in a CD that earned 3% compounded annually and was worth $5,627.54 after 4 years. Had Oksana invested in a CD with simple interest, how much would the CD have been worth after 4 years? How much more did Oksana earn using compound interest?

Video

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Compound Interest

Understand and Compute Future Value

Imagine investing for 30 years and compounding the interest every month. Using the method above, there would be 360 periods to calculate interest for. This is not a reasonable approach. Fortunately, there is a formula for finding the future value of an investment that earns compound interest.

FORMULA

The future value of an investment, A, when the principal P is invested at an annual interest rate of r (in decimal form), compounded n times per year, for t years, is found using the formula $A = P$msupntA=P(1+rn)nt. This is also referred to as the future value of the investment.

Checkpoint

Note, sometimes the formula is presented with the total number of periods, n, and the interest rate per period, r. In that case the formula becomes $A = P(1+r)^n$.

Example $6.4.3$: Computing Future Value for Compound Interest

In the following, compute the future value of the investment with the given conditions.

1. Principal is \$5,000, annual interest rate is 3.8%, compounded monthly, for 5 years.
2. Principal is \$18,500, annual interest rate is 6.25%, compounded quarterly, for 17 years.

Answer

1. The principal is P = \$5,000, interest rate, in decimal form, r = 0.038, compounded monthly so n = 12, and for t = 5 years. Substituting these values into the formula, we find

$$\begin{aligned} A &= P\left(1+\tfrac{r}{n}\right)^{nt} = 5,000\left(1+\tfrac{0.038}{12}\right)^{12\,\times\,5} \\ &= 5,000\left(1+0.0031\bar{6}\right)^{60} \\ &= 5,000\left(1.0031\bar{6}\right)^{40} \\ &= 5,000 \times 1.20888663572 \\ &= 6,044.4332 \end{aligned}$$

   The future value of the investment is \$6,044.43.

2. The principal is P = \$18,500, interest rate, in decimal form, r = 0.0625, compounded quarterly so n = 4, and for t = 17 years. Substituting these values into the formula, we

$$\begin{aligned} A = P(1+\tfrac{r}{n})^{nt} &= 18,500(1+\tfrac{0.0625}{4})^{4 \times 17} \\ &= 18,500(1+0.015625)^{68} \\ &= 18,500(1.015625)^{68} \\ &= 18,500 \times 2.86992151999 \\ &= 53,093.5481 \end{aligned}$$

The future value of the investment is $53,093.54.

Your Turn 6.4.3

In the following, compute the future value of the investment with the given conditions.

Principal is $7,600, annual interest rate is 4.1%, compounded monthly, for 10 years.

Principal is $13,250, annual interest rate is 2.79%, compounded quarterly, for 25 years.

✓ Example 6.4.4: Interest Compounded Quarterly

Cody invests $7,500 in an account that earns 4.5% interest compounded quarterly (4 times per year). Determine the value of Cody's investment after 10 years.

Answer

Cody's initial investment is $7,500, so P = $7,500. The annual interest rate is 4.5%, which is 0.045 in decimal form. Compounding quarterly means there are four periods in a year, so n = 4. He invests the money for 10 years. Substituting those values into the formula, we calculate

$$\begin{aligned} A = P(1+\tfrac{r}{n})^{nt} &= 7,500(1+\tfrac{0.045}{4})^{4 \times 10} \\ &= 7,500(1+0.01125)^{40} \\ &= 7,500(1.01125)^{40} \\ &= 7,500 \times 1.564376865391 \\ &= 11,732.8265 \end{aligned}$$

After 10 years, Cody's initial investment of $7,500 is worth $11,732.82.

Your Turn 6.4.4

Maggie invests $3,000 in an account that earns 5.1% interest compounded monthly. How much is the account worth after 13 years?

✓ Example 6.4.5: Interest Compounded Daily

Kathy invests $10,000 in an account that yields 5.6% compounded daily. How much money will be in her account after 20 years?

Answer

Kathy's initial investment is $10,000, so P = $10,000. The annual interest rate is 5.6%, which is 0.056 in decimal form. Compounding daily means there are 364 periods in a year, so n = 365. She invests the money for 20 years, so t = 20. Substituting those values into the formula, we calculate

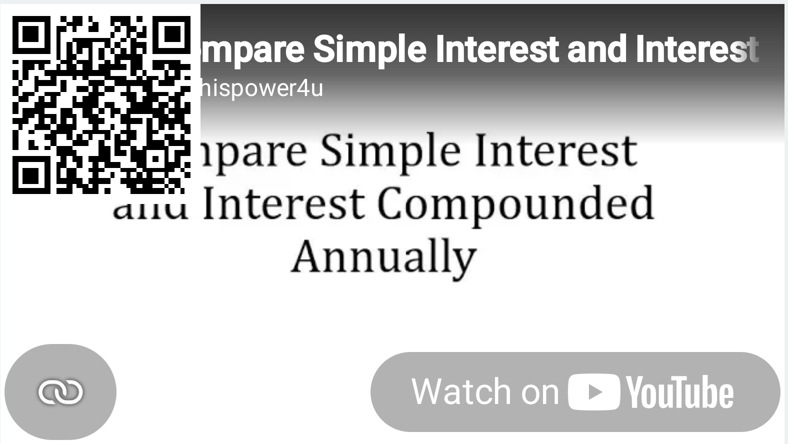
$$\begin{aligned} A = P(1+\tfrac{r}{n})^{nt} &= 10{,}000(1+\tfrac{0.056}{365})^{365 \times 20} \\ &= 10{,}000(1+0.000153424657534)^{7300} \\ &= 10{,}000(1.000153424657534)^{7300} \\ &= 10{,}000 \times 3.06459091598 \\ &= 30{,}645.909 \end{aligned}$$

After 20 years, Kathy’s initial investment of $10,000 is worth $30,645.90.

Your Turn 6.4.5

Jacob invests $3,000 in a CD that yields 3.4% compounded daily for 5 years. How much is his CD worth after 5 years?

Video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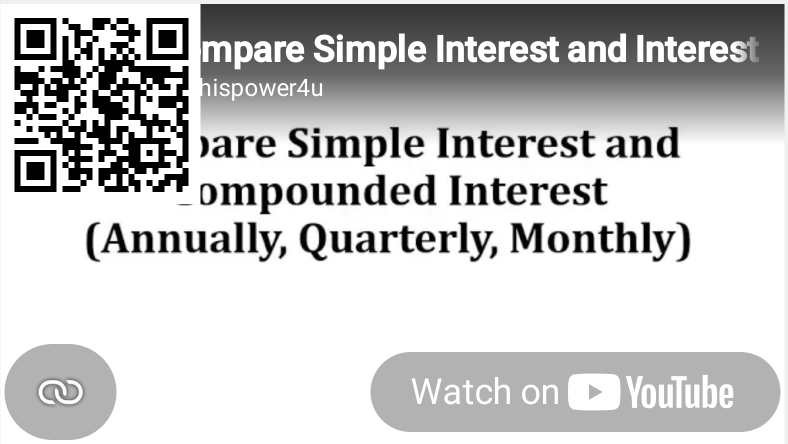
Watch on YouTube

- Compare Simple Interest to Interest Compounded Annually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Watch on YouTube

- Compare Simple Interest and Compound Interest for Different Number of Periods Per Year

WORK IT OUT

To truly grasp how compound interest works over a long period of time, create a table comparing simple interest to compound interest, with different numbers of periods per year, for many years would be useful. In this situation, the principal is $10,000, and the annual interest rate is 6%.

1. Create a table with five columns. Label the first column YEARS, the second column SIMPLE INTEREST, the third column COMPOUND ANNUALLY, the fourth column COMPOUND MONTHLY and the last column COMPOUND DAILY, as shown below.

YEARS	SIMPLE INTEREST	COMPOUND ANNUALLY	COMPOUND MONTHLY	COMPOUND DAILY

- In the years column, enter 1, 2, 3, 5, 10, 20, and 30 for the rows.
- Calculate the account value for each column and each year.
- Compare the results from each of the values you find. How do the number of periods per year (compoundings per year) impact the account value? How does the number of years impact the account value?
- Redo the chart, with an interest rate you choose and a principal you choose. Are the patterns identified earlier still present?

Understand and Compute Present Value

When investing, there is often a goal to reach, such as “after 20 years, I’d like the account to be worth $100,000.” The question to be answered in this case is “How much money must be invested now to reach the goal?” As with simple interest, this is referred to as the present value.

FORMULA

The money invested in an account bearing an annual interest rate of r (in decimal form), compounded n times per year for t years, is called the present value, semanticsPVPV, of the account (or of the money) and found using the formula $PV = \frac{A}{\text{msup}}$n×tPV=A(1+rn)n×t, where A is the value of the account at the investment’s end. Always round this value up to the nearest penny.

Example 6.4.6: Computing Present Value

Find the present value of the accounts under the following conditions.

1. A = $250,000, invested at 6.75 interest, compounded monthly, for 30 years.
2. A = $500,000, invested at 7.1% interest, compounded quarterly, for 40 years.

Answer

1. To reach a final account value of A = $250,000, invested at 6.75% interest, in decimal form r = 0.0675 (decimal form!), compounded monthly, so n = 12, for 30 years, substitute those values into the formula for present value. Calculating, we find the present value of the $250,000.

$$\begin{aligned} PV &= \frac{A}{\left(1+\frac{r}{n}\right)^{n\times t}} = \frac{250{,}000}{\left(1+\frac{0.0675}{12}\right)^{12\times 30}} \\ &= \frac{250{,}000}{(1+0.005625)^{360}} \\ &= \frac{250{,}000}{(1.005625)^{360}} \\ &= \frac{250{,}000}{7.5332454772} \\ &= 33{,}186.2277 \end{aligned}$$

   In order for this account to reach $250,000 after 30 years, $33,186.23 needs to be invested.

2. To reach a final account value of A = $500,000, invested at 7.1% interest, in decimal form r = 0.071, compounded quarterly, so n = 4, for 40 years, substitute those values into the formula for present value. Calculating, we find the present value of the $500,000.

$$\begin{aligned} PV &= \frac{A}{\left(1+\frac{r}{n}\right)^{n\times t}} = \frac{500{,}000}{\left(1+\frac{0.071}{4}\right)^{4\times 40}} \\ &= \frac{500{,}000}{(1+0.01775)^{160}} \\ &= \frac{500{,}000}{(1.01775)^{160}} \\ &= \frac{500{,}000}{16.6946672846} \\ &= 29,949.6834 \end{aligned}$$

   In order for this account to reach $500,000 after 40 years, $29,949.69 needs to be invested.

Your Turn 6.4.6

Find the present value of the accounts under the following conditions.

A = \$1,000,000, invested at 5.75% interest, compounded monthly, for 40 years.

A = \$175,000, invested at 3.8% interest, compounded quarterly, for 20 years.

Example 6.4.7: Investment Goal with Compound Interest

Pilar plans early for retirement, believing she will need \$1,500,000 to live comfortably after the age of 67. How much will she need to deposit at age 23 in an account bearing 6.35% annual interest compounded monthly?

Answer

Knowing how much to deposit at age 23 to reach a certain value later is a present value question. The target value for Pilar is \$1,500,000. The interest rate is 6.35%, which in decimal form is 0.0635. Compounded monthly means n = 12. She's 23 and will leave the money in the account until the age of 67, which is 44 years, making t = 44. Using this information and substituting in the formula for present value, we calculate

$$\begin{aligned} PV &= \frac{A}{\left(1+\frac{r}{n}\right)^{n \times t}} = \frac{1{,}500{,}000}{\left(1+\frac{0.0635}{12}\right)^{12 \times 44}} \\ &= \frac{1{,}500{,}000}{\left(1+0.005291\bar{6}\right)^{528}} \\ &= \frac{1{,}500{,}000}{\left(1.005291\bar{6}\right)^{528}} \\ &= \frac{1{,}500{,}000}{16.226302189} \\ &= 92{,}442.5037 \end{aligned}$$

Pilar will need to invest \$92,442,51 in this account to have \$1,500,000 at age 67.

Your Turn \(\PageIndex{7\)

Hajun turns 30 this year and begins to think about retirement. He calculates that he will need \$1,200,000 to retire comfortably. He finds a fund to invest in that yields 7.23% and is compounded monthly. How much will Hajun need to invest in the fund when he turns 30 so that he can reach his goal when he retires at age 65?

Compute and Interpret Effective Annual Yield

As we've seen, quarterly compounding pays interest 4 times a year or every 3 months; monthly compounding pays 12 times a year; daily compounding pays interest every day, and so on. Effective annual yield allows direct comparisons between simple interest and compound interest by converting compound interest to its equivalent simple interest rate. We can even directly compare different compound interest situations. This gives information that can be used to identify the best investment from a yield perspective.

Using a formula, we can interpret compound interest as simple interest. The effective annual yield formula stems from the compound interest formula and is based on an investment of \$1 for 1 year.

Effective annual yield is $Y = \left(1+\frac{r}{n}\right)^{n} - 1$ where Y = effective annual yield, r = interest rate in decimal form, and n = number of times the interest is compounded in a year. Y is interpreted as the equivalent annual simple interest rate.

Example 6.4.8: Determine and Interpret Effective Annual Yield for 6% Compounded Quarterly

Suppose you have an investment paying a rate of 6% compounded quarterly. Determine and interpret that effective annual yield of the investment.

Answer

Here, n = 4 (quarterly) and r = 0.06 (decimal form). Substituting into the formula we find that the effective annual yield is

$$\begin{aligned} Y &= \left(1+\frac{0.06}{4}\right)^4-1 \\ &= (1.015)^4-1 \\ &= 1.06136-1 \\ &= 0.0614 \\ &= 6.14\% \end{aligned}$$

Therefore, a rate of 6% compounded quarterly is equivalent to a simple interest rate of 6.14%.

Your Turn 6.4.8

Calculate and interpret the effective annual yield for an investment that pays at a 7% interest compounded quarterly.

Example 6.4.9: Determine and Interpret Effective Annual Yield for 5% Compounded Daily

Calculate and interpret the effective annual yield on a deposit earning interest at a rate of 5% compounded daily.

Answer

In this case, the rate is r = 0.05 and n = 365 (daily). Using the formula $Y=\left(1+\frac{r}{n}\right)^n-1$, we have

$$\begin{aligned} Y &= \left(1+\frac{0.05}{365}\right)^{365}-1 \\ &= (1.0001369863)^{365}-1 \\ &= 1.051267-1 \\ &= 0.0513 \end{aligned}$$

This tells us that an account earning 5% compounded daily is equivalent to earning 5.13% as simple interest.

Your Turn 6.4.9

Calculate and interpret the effective annual yield on a deposit earning 2.5% compounded daily.

Example 6.4.10: Choosing a Bank

Minh has a choice of banks in which he will open a savings account. He will deposit $3,200 and he wants to get the best interest he can. The banks advertise as follows:

Bank	Interest Rate
ABC Bank	2.08% compounded monthly
123 Bank	2.09% compounded annually
XYZ Bank	2.05% compounded daily

Which bank offers the best interest?

Answer

To compare these directly, Minh could change each interest rate to its effective annual yield, which would allow direct comparison between the rates. Computing the effective annual yield for all three choices gives:

ABC Bank: $Y=\left(1+\frac{0.0208}{12}\right)^{12}-1=0.0210=2.10\%$

123 Bank: $Y=\left(1+\frac{0.0209}{1}\right)^{1}-1=0.0209=2.09\%$

XYZ Bank: $Y=\left(1+\frac{0.0205}{365}\right)^{365}-1=0.0207=2.07\%$

ABC Bank has the highest effective annual yield, so Minh should choose ABC bank.

Your Turn 6.4.10

Isabella decides to deposit $5,500 in a CD but needs to choose between banks that offer CDs. She identifies four banks and finds out the terms of their CDs. Her findings are in the table below.

Bank	Interest Rate
Smith Bank	3.08% compounded quarterly
Park Bank	3.11% compounded annually
Town Bank	3.09% compounded daily
Community Bank	3.10% compounded monthly

Which bank has the best yield?

Check Your Understanding

1. What is compound interest?
2. Which yields more money, simple interest or compound interest?
3. Find the future value after 15 years of $8,560.00 deposited in an account bearing 4.05% interest compounded monthly.
4. $10,000 is deposited in an account bearing 5.6% interest for 5 years. Find the difference between the future value when the interest is simple interest and when the interest is compounded quarterly.
5. Find the present value of $75,000 after 28 years if money is invested in an account bearing 3.25% interest compounded monthly.
6. What can be done to compare accounts if the rates and number of compound periods per year are different?
7. Find the effective annual yield of an account with 4.89% interest compounded quarterly.

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6.4.0: Exercises

? Exercise 6.4.0.1

What is the difference between simple interest and compound interest?

? Exercise 6.4.0.2

What is a direct way to compare accounts with different interest rates and number of compounding periods?

? Exercise 6.4.0.3

Which type of account grows in value faster, one with simple interest or one with compound interest?

How many periods are there if interest is compounded?

? Exercise 6.4.0.4

Daily

? Exercise 6.4.0.5

Weekly

? Exercise 6.4.0.6

Monthly

? Exercise 6.4.0.7

Quarterly

? Exercise 6.4.0.8

Semi-annually

In the following exercises, compute the future value of the investment with the given conditions.

? Exercise 6.4.0.9

Principal = $15,000, annual interest rate = 4.25%, compounded annually, for 5 years

? Exercise 6.4.0.10

Principal = $27,500, annual interest rate = 3.75%, compounded annually, for 10 years

? Exercise 6.4.0.11

Principal = $13,800, annual interest rate = 2.55%, compounded quarterly, for 18 years

? Exercise 6.4.0.12

Principal = \$150,000, annual interest rate = 2.95%, compounded quarterly, for 30 years

? Exercise 6.4.0.13

Principal = \$3,500, annual interest rate = 2.9%, compounded monthly, for 7 years

? Exercise 6.4.0.14

Principal = \$1,500, annual interest rate = 3.23%, compounded monthly, for 30 years

? Exercise 6.4.0.15

Principal = \$16,000, annual interest rate = 3.64%, compounded daily, for 13 years

? Exercise 6.4.0.16

Principal = \$9,450, annual interest rate = 3.99%, compounded daily, for 25 years

In the following exercises, compute the present value of the accounts with the given conditions.

? Exercise 6.4.0.17

Future value = \$250,000, annual interest rate = 3.45%, compounded annually, for 25 years

? Exercise 6.4.0.18

Future value = \$300,000, annual interest rate = 3.99%, compounded annually, for 15 years

? Exercise 6.4.0.19

Future value = \$1,500,000, annual interest rate = 4.81%, compounded quarterly, for 35 years

? Exercise 6.4.0.20

Future value = \$750,000, annual interest rate = 3.95%, compounded quarterly, for 10 years

? Exercise 6.4.0.21

Future value = \$600,000, annual interest rate = 3.79%, compounded monthly, for 17 years

? Exercise 6.4.0.22

Future value = \$800,000, annual interest rate = 4.23%, compounded monthly, for 35 years

? Exercise 6.4.0.23

Future value = \$890,000, annual interest rate = 2.77%, compounded daily, for 25 years

? Exercise 6.4.0.24

Future value = $345,000, annual interest rate = 2.99%, compounded daily, for 19 years

In the following exercises, compute the effective annual yield for accounts with the given interest rate and number of compounding periods. Round to three decimal places.

? Exercise 6.4.0.25

Annual interest rate = 2.75%, compounded monthly

? Exercise 6.4.0.26

Annual interest rate = 3.44%, compounded monthly

? Exercise 6.4.0.27

Annual interest rate = 5.18%, compounded quarterly

? Exercise 6.4.0.28

Annual interest rate = 2.56%, compounded quarterly

? Exercise 6.4.0.29

Annual interest rate = 4.11%, compounded daily

? Exercise 6.4.0.30

Annual interest rate = 6.5%, compounded daily

The following exercises explore what happens when a person deposits money in an account earning compound interest.

? Exercise 6.4.0.31

Find the present value of $500,000 in an account that earns 3.85% compounded quarterly for the indicated number of years.

a. 40 years
b. 35 years
c. 30 years
d. 25 years
e. 20 years
f. 15 years

? Exercise 6.4.0.32

Find the present value of $1,000,000 in an account that earns 6.15% compounded monthly for the indicated number of years.

a. 40 years
b. 35 years
c. 30 years
d. 25 years
e. 20 years
f. 15 years

? Exercise 6.4.0.33

In the following exercises, the number of years can reflect delaying depositing money. 40 years would be depositing money at the start of a 40-year career. 35 years would be waiting 5 years before depositing the money. Thirty years would be waiting 10 years before depositing the money, and so on. What do you notice happens if you delay depositing money?

? Exercise 6.4.0.34

For each 5-year gap for exercise 32, compute the difference between the present values. Do these differences remain the same for each of the 5-year gaps, or do they differ? How do they differ? What conclusion can you draw?

? Exercise 6.4.0.35

Daria invests $2,500 in a CD that yields 3.5% compounded quarterly for 5 years. How much is the CD worth after those 5 years?

? Exercise 6.4.0.36

Maurice deposits $4,200 in a CD that yields 3.8% compounded annually for 3 years. How much is the CD worth after those 3 years?

? Exercise 6.4.0.37

Georgita is shopping for an account to invest her money in. She wants the account to grow to $400,000 in 30 years. She finds an account that earns 4.75% compounded monthly. How much does she need to deposit to reach her goal?

? Exercise 6.4.0.38

Zak wants to create a nest egg for himself. He wants the account to be valued at $600,000 in 25 years. He finds an account that earns 4.05% interest compounded quarterly. How much does Zak need to deposit in the account to reach his goal of $600,000?

? Exercise 6.4.0.39

Eli wants to compare two accounts for their money. They find one account that earns 4.26% interest compounded monthly. They find another account that earns 4.31% interest compounded quarterly. Which account will grow to Eli's goal the fastest?

? Exercise 6.4.0.40

Heath is planning to retire in 40 years. He'd like his account to be worth $250,000 when he does retire. He wants to deposit money now. How much does he need to deposit in an account yielding 5.71% interest compounded semi-annually to reach his goal?

? Exercise 6.4.0.41

Jo and Kim want to set aside some money for a down payment on a new car. They have 6 years to let the money grow. If they want to make a $15,000 down payment on the car, how much should they deposit now in an account that earns 4.36% interest compounded monthly?

? Exercise 6.4.0.42

A newspaper's business section runs an article about savings at various banks in the city. They find six that offer accounts that offer compound interest.
Bank A offers 3.76% compounded daily.

Bank B offers 3.85% compounded annually.
Bank C offers 3.77% compounded weekly.
Bank D offers 3.74% compounded daily.
Bank E offers 3.81% compounded semi-annually.
To earn the most interest on a deposit, which bank should a person choose?

? Exercise 6.4.0.43

Paola reads the newspaper article from exercise 32. She really wants to know how different they are in terms of dollars, not effective annual yield. She decides to compute the future value for accounts at each bank based on a principal of $100,000 that are allowed to grow for 20 years. What is the difference in the future values of the account with the highest effective annual yield, and the account with the second highest effective annual yield?

? Exercise 6.4.0.44

Paola reads the newspaper article from exercise 32. She really wants to know how different they are in terms of dollars, not effective annual yield. She decides to compute the future value for accounts at each bank based on a principal of $100,000 that are allowed to grow for 20 years. What is the difference in the future values of the account with the highest effective annual yield, and the account with the lowest effective annual yield?

? Exercise 6.4.0.45

Jesse and Lila need to decide if they want to deposit money this year. If they do, they can deposit $17,400 and allow the money to grow for 35 years. However, they could wait 12 years before making the deposit. At that time, they'd be able to collect $31,700 but the money would only grow for 23 years. Their account earns 4.63% interest compounded monthly. Which plan will result in the most money, depositing $17,400 now or depositing $31,700 in 12 years?

? Exercise 6.4.0.46

Veronica and Jose are debating if they should deposit $15,000 now in an account or if they should wait 10 years and deposit $25,000. If they deposit money now, the money will grow for 35 years. If they wait 10 years, it will grow for 25 years. Their account earns 5.25% interest compounded weekly. Which plan will result in the most money, depositing $15,000 now or depositing $25,000 in 10 years?

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6.5: Making a Personal Budget

Figure 6.7: Calculating a budget is important to your financial health. (credit: “Budget planning concept on white desk” by Marco Verch Professional Photographer/Flickr, CC BY 2.0)

Learning Objectives

1. Create a personal budget with the categories of expenses and income.
2. Apply general guidelines for a budget.

“That doesn’t fit in the budget.”

“We didn’t budget for that.”

“We need to figure out our budget and stick to it.”

A budget is an outline of how money and resources should be spent. Companies have them, individuals have them, your college has one. But do you have one?

Creating a realistic budget is an important step in careful stewardship of your financial health. Designing your budget will help understand the financial priorities you have, and the constraints on your life choices. You want to have enough income to pay not only for the necessities, but also for things that represent your wants, like trips or dinner out. You also may want to save money for large purchases or retirement. You do not want to just get by, and you do not want the problems associated with overdue balances, rising debt, and possibly losing something you have worked hard to obtain.

While creating a budget may seem intimidating at first, coming up with your basic budget outline is the hardest part. Over time, you will adjust not only the numbers, but the categories.

Creating a Budget

You should view creating a **budget** as a financial tool that will help you achieve your long-term goals. A budget is an estimation of income and expenses over some period of time. You will be able to track your progress, which will help you to prepare for the future by making smart investment decisions.

There are several budget-creating tools available, such as the apps Good budget and Mint, and Google Sheets. Getting started, though, begins well before you find an app. The following are steps that can be used to create your monthly budget.

1. **Track your income and expenses** Review your income and expenses for the past 6 months to a year. This will give you an idea of your current habits.
2. **Set your income baseline** Determine all the sources of income you will have. This income may from paychecks, investments, or freelance work. It even includes child support and gifts. Be sure to use income after taxes. This allows you to determine your maximum expenditures per month.

Checkpoint

For income that is not steady, such as gig work or freelance work, use the previous 6 to 12 months of income to find an average income from that gig or freelance work. Use this average in the budgeting process.

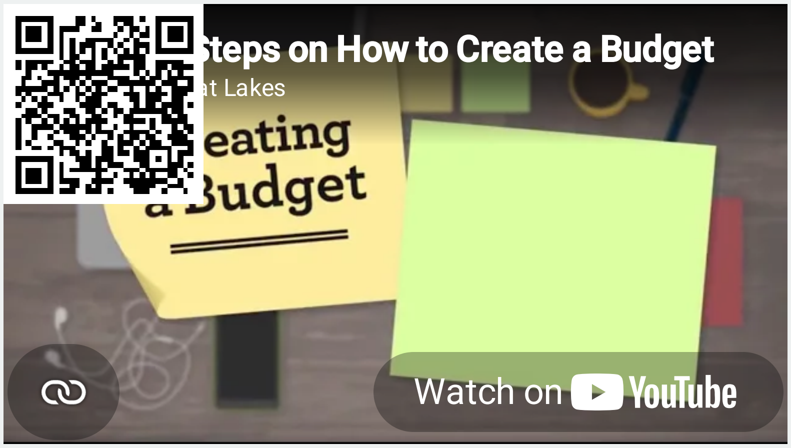
3. **Determine your expenses** Review your bills from the past 6 months. You should include mortgage payments or rent, insurance, car payments, utilities, groceries, transportation expenses, personal care, entertainment, and savings. Using your credit card statements and bank statements will help you determine these amounts. Be aware that some of the expenses will not change over time. These are referred to as fixed expenses, like rent, car payments, insurance, internet service, and the like. Other expenses may vary widely from month to month and are appropriately called variable expenses, and include such expenses as gasoline, groceries.

Checkpoint

Some expenses are yearly, such as insurance or property taxes. Other expanses may be quarterly (four times per year) or semiannual (twice per year). To budget for such bills by month, divide the bill total by the number of months the bill covers.

4. **Categorize your expenses** These categories may be housing, transportation, or food, for broad categories, or may get more specific, where you categorize car payments, car insurance, and gasoline separately. The categories are your choices. Be sure to account for the cost of maintaining a vehicle or home. The more specific you are, the better you'll understand your spending needs and habits.
5. **Total your monthly income and monthly expenses and compare** These values should be compared. If your expenses are higher than your income, then adjustments have to be made. Decisions of what to do with any extra income is part of the planning process also.
6. **Make plans for unplanned expenses** Ask anyone, an unexpected car repair can ruin a carefully crafted budget. Have a plan for how you can be ready for these random expenses. This often means creating a cushion in your budget.
7. **Use your budget to make decisions and adjust for any changes** Your budget is a changeable document. Add to it when you wish, refer to it when special purchases are to be made. Keeping your budget up to date helps accommodate changes in income and expenses.

Video



Creating a Budget

In this section, we will focus on income and expenses. One of the easiest ways to manage a budget is to create a table, with one column containing income sources, another with income values, a third with expense categories, and a last containing expenses. An example is shown in Table 6.1.

Table 6.1 Table with Budget

Income Source	Amount	Expense	Amount
Full-time job	$3,565	Rent	$975
Uber	$185	Car Payment	$355
TOTAL	$3,750	Student Loan	$418
		Electric	$76
		Food	$400
		Gasoline	$250
		Car Insurance	$165
		Clothing	$100
		Entertainment	$100
		TOTAL	$2,839

Who Knew?: Gross Pay and Take Home Pay

If you’ve ever had a paycheck, you know that taxes are taken out of your pay before you get your check. This amount of money varies from state to state, and sometimes even city to city. For a person making $50,000 per year gross salary in Salt Lake City, Utah, take home pay is about 75.6% of gross salary. In Detroit, Michigan, take home pay is about 74.5% of gross salary. Lakeland, Florida, take home pay is about 80.5% of gross salary. These also change based on how much a person earns! Before choosing a place to live, it makes sense to determine how much deductions from pay will impact your income.

Example 6.5.1: Creating a Budget

Heather has graduated college and currently works as a nurse for a rural medical group. Her net monthly income from that job is $3,765.40. She also works part-time on the weekends, earning another $672.00 per month. Her monthly expenses are rent at $1,050, car payments at $489, student loan payments at $728, car insurance at $139, utilities at $130, clothing at $150, entertainment (going out with friends, Netflix, Amazon Prime, movies) at $300, credit card debt at $200, food at $360, and gasoline at $275. Create her budget in a table, compare the total income to total expenses, and determine how much excess income per month she has or how much she falls short by each month.

Answer

Step 1: To begin, we create the table with appropriate headings.

Income Source	Amount	Expense	Amount

Step 2: Her income categories are her nursing job, with $3,765.40 per month, and her part-time job, with $672.00 per month. Entering these into the table, we have the following.

Income Source	Amount	Expense	Amount

Income Source	Amount	Expense	Amount
Nursing	$3,765.40		
Part-time	$672.00		

Step 3: Her monthly expenses are listed above. Entering the categories and the amount for each of those expenses, the table is now

Income Source	Amount	Expense	Amount
Nursing	$3,765.40	Rent	$1,050
Part-time	$672.00	Car Payment	$489
		Student Loan	$728
		Car Insurance	$139
		Utilities	$130
		Clothing	$150
		Entertainment	$300
		Credit Card	$200
		Food	$400
		Gasoline	$250

Step 4: Totaling the income and expenses, we see that her total income is $4,437.40 per month, and her total expenses are $3,836 per month. Comparing these, we see that Heather has $601.40 in excess income per month. This provides a cushion in her budget.

Your Turn 6.5.1

Mateo works as a union electrician in a suburban area. Monthly, his take home pay is $3,375. He sometimes does small side jobs for family or friends, and averages about $300 per month from these little jobs. His monthly expenses are his mortgage at $986.78, truck payments at $589.00, truck insurance at $312, utilities at $167, clothing at $150, entertainment at $400, credit card debt at $325, food at $470, and gasoline at $375. Create Mateo’s budget in a table, compare the total income to total expenses, and determine how much excess income per month he has or how much he falls short by each month.

✓ Example 6.5.2: Creating a Budget

Carol is working in a dental lab, creating dentures and bridges. Monthly her take home pay is $2,816 (based on $22 per hour minus payroll taxes). She also receives $320 per month in child support for her one daughter. Her monthly expenses are rent at $700, car payments at $229, student loan payments at $250, car insurance at $119, health insurance at $225, utilities at $80, clothing at $75, entertainment at $200, food at $275, and gasoline at $275. Create Carol’s budget in a table, compare the total income to total expenses, and determine how much excess income per month she has or how much she falls short by each month.

Answer

Step 1: To begin, we create the table with appropriate headings.

Income Source	Amount	Expense	Amount

Income Source	Amount	Expense	Amount

Step 2: Her income categories are from work, \$2,816, and child support, \$320, per month. Entering these into the table, we have the following.

Income Source	Amount	Expense	Amount
Job	\$2,816.00		
Child support	\$320.00		

Step 3: Her monthly expenses are listed above. Entering the categories and the amount for each of those expenses, the table is now

Income Source	Amount	Expense	Amount
Job	\$2,816.00	Rent	\$700
Child support	\$320.00	Car Payment	\$229
		Student Loan	\$250
		Car Insurance	\$119
		Utilities	\$80
		Health insurance	\$225
		Clothing	\$75
		Entertainment	\$200
		Food	\$275
		Gasoline	\$275

Step 4: Totaling the income and expenses, we see that her total income is \$3,136.00 per month, and her total expenses are \$2,428.00 per month. Comparing these, we see that Carol has \$708.00 in excess income per month. This is the cushion in her budget.

Your Turn 6.5.2

Maddy works as a mechanical engineer, making \$6,093.75 monthly after payroll taxes. Her monthly expenses are her mortgage at \$1,452.89, car payments at \$627.38, car insurance at \$179.00, health insurance at \$265.00, utilities at \$320, clothing at \$150, entertainment at \$400, credit card debt at \$450, food at \$370, and gasoline at \$175. Create Maddy's budget in a table, compare the total income to total expenses, and determine how much excess income per month she has or how much she falls short by each month.

Using the budget process, we can make decisions on adding expenses to the budget. To do so, check the cushion of the budget to see if there is room in the budget for the new expense.

✓ Example 6.5.3: Adding to an Existing Budget

In the example above, Carol had excess income of \$708.00. She looks up the cost of before-school care for her daughter. She finds that, monthly, the cost would be \$252.00 per month. Is this an affordable program for Carol? Add this expense to her budget table.

Answer

She can afford this, as the cost for the before school program is $252.00 and she had extra income of $708.00. Adding this to her budget, her budget table is now

Income Source	Amount	Expense	Amount
Job	$2,816.00	Rent	$700
Child support	$320.00	Car Payment	$229
		Student Loan	$250
		Car Insurance	$119
		Utilities	$80
		Health insurance	$225
		Clothing	$75
		Entertainment	$200
		Food	$275
		Gasoline	$275
		Before-school care	$252

Now, she has $456.00 in excess income per month.

Your Turn 6.5.3

Recall Heather’s budget from Example 6.49. She decides she wants to buy her own home, which would increase her expenses. Instead of $1,050.00 in rent, she would pay $1,240.00 for her mortgage. Her utilities costs would increase to $295.00 per month. Add these to Heather’s budget to determine if the changes are affordable.

The 50-30-20 Budget Philosophy

It isn’t clear, obvious, or easy to decide how much of your income to allocate to various categories of expenses. Many people pay their bills and then consider all the leftover money to be spending money. However, when developing your own budget, you may want to follow the **50-30-20 budget philosophy**, which provides a basic guideline for how your income could be allocated. Fifty percent of your budget is allotted to your needs, 30% of your budget is allotted to pay for your wants, and 20% of your budget is allotted for savings and debt service (paying off your debts).

Knowing what expenses are necessary and what expenses are wants is important, since wants and needs are often confused. The following are **necessary expenses** that represent basic living requirements and debt services. This list isn’t complete: mortgage/rent, utilities, car, car insurance, health care, groceries, gasoline, child care (for working parents), and minimum debt payments. The 50-30-20 budget philosophy suggests that 50%, or half, your income go to these necessities.

Wants, though, are things you could live without but still wish to have, such as Amazon Prime, restaurant dinners, coffee from Starbucks, vacation trips, and hobby costs. Even a gym membership or that new laptop are wants. Creating the room to afford these wants is important to our mental health. Not budgeting for things we want will negatively impact our quality of life.

The remaining 20% should be set aside, either in retirement funds, stocks, other investments, an emergency fund (recommendations are that an emergency fund have 3 months of income), and perhaps extra spent to pay down debt. This 20% is very useful for addressing those unexpected costs, such as repairs or replacement of items that no longer work. Without budgeting this cushion, any expense that is a surprise can cause us to miss necessary payments.

The list of necessary expenses was not complete. There are other expenses that could be included.

Necessary Expenses and Expenses that are Wants

For some people, an expense will be necessary while the same expense for someone else will be a want. A good example of this is internet service. Many people consider internet service as a need, especially those who work from home or who are not able to leave their homes. One could also call internet service a need if they have children in school. For others, internet service is a want. If a person’s job doesn’t require them to be online, if they are not in school, if they do not have kids, then internet service can be dropped. There are public options for internet service. One could even use their phone as a hot spot.

Cars often fall into the category of need, but could also fall into the want category, depending on where and how you live. Bikes, public transportation, and walking are all options that could replace a car. This would then remove the cost of gasoline and car insurance.

Another consideration when deciding if an item on your budget is a need or a want is about your choices and priorities. A car is a need for many. But the need for a car is not the same as the need for a specific car. If you choose to buy a car with payments that exceed your budgeted amount for the car, then that car is a want. The amount you exceed the budget now belongs in the want category.

The same can be said for housing. If you want an apartment that costs $1,250 per month, but your budget only allows for an apartment that costs $900, then $350 of the rent is a want.

The point of that is to carefully consider if an expense is a need as opposed to a want.

When your expenses exceed your income, you may want to change how you budget your income to line up with these guidelines. This may mean cutting back, finding less-expensive living arrangements, finding a less-expensive (and more fuel-efficient) car, or sacrificing some specialty groceries. Using these guidelines keeps your financial life manageable.

Better still, they can guide you as you begin your life after graduation.

In the example above, after Carol added before school care for her daughter to the budget, her budget was as shown below. Evaluate Carol’s budget using the 50-30-20 budget philosophy.

Income Source Amount Expense Amount Job $2,816.00 Rent $700 Child support $320.00 Car Payment $229 Student Loan $250 Car Insurance $119 Utilities $80 Health insurance $225 Clothing $75 Entertainment $200 Food $275 Gasoline $275 Before-school care $252

Answer

Carol’s total income is $3,136.00. Applying the 50-30-20 budget philosophy to this income requires the calculation of each of those percentages.

For the necessities, Carol should budget 50% of her income, or $0.5 \times \$3{,}136.00 = \$1{,}568.00$.

For her wants, she should budget 30% of her income, or $0.3 \times \$3{,}136.00 = \$940.80$.

For savings and extra debt service, she should budget 20% of her income, or $0.2 \times \$3{,}136.00 = \$627.20$.

In her budget, her necessities include all expenses except for entertainment. These expenses total $2,480, which exceeds the suggested budget amount of $1568.00. To follow the guidelines, Carol would have to cut back on these necessities.

For her wants, she spends $200.00 on entertainment, which is well below the suggested budget amount of $940.80. If she modifies how much she spends on needs, she may be able to increase the spending on her wants.

Her excess income is $456.00, which is below what she should be saving and using to pay down extra debt. If she does adjust how much she spends on needs, she could increase the amount for savings.

Your Turn 6.5.4

Recall Heather’s budget from Example 6.49, before she thought of moving. That budget is below. Evaluate Heather’s budget using the 50-30-20 budget philosophy.

Income Source	Amount	Expense	Amount

Income Source	Amount	Expense	Amount
Nursing	$3,765.40	Rent	$1,050
Part-time	$672.00	Car Payment	$489
		Student Loan	$728
		Car Insurance	$139
		Utilities	$130
		Clothing	$150
		Entertainment	$300
		Credit Card	$200
		Food	$400
		Gasoline	$250

Carmen is about to graduate and has been offered a job at a bank as a data scientist. She estimates her monthly take home pay to be $5,662.50. Apply the 50-30-20 philosophy to that monthly income. How should Carmen use this information?

Answer

Step 1. To apply the 50-30-20 budget philosophy to Carmen's income, she needs to calculate 50%, 30%, and 20% of her income. Fifty percent of her income is $0.5 \times \$5{,}662.50 = \$2{,}831.25$. Thirty percent of her income is $0.3 \times \$5{,}662.50 = \$1{,}698.75$. Twenty percent of her income is $0.2 \times \$5{,}662.50 = \$1{,}132.50$.

Step 2. She would then budget $2,831.25 for her needs, $1,698.75 for her wants, and $1,132.50 for savings and debt service.

Step 3. When choosing where to live, what to eat, and what to drive, she should make choices that keep those costs, combined with her debt service costs, gasoline, and utilities, below $2,831.25. This means she will have to make decisions about what her priorities are.

Step 4. She should then figure out what she wants to do with her money, and stay within the limits, that is, keep those costs below $1,698.75.

Step 5. Finally, she can begin building her savings with the remaining $1,132.50.

Your Turn 6.5.5

Elijah has finished an apprenticeship and is about to start his first job as an HVAC (heating, ventilation, and air conditioning) tech. He estimates that his net monthly income will be $3,263.44. Apply the 50-30-20 budget philosophy to his income to set guidelines for Elijah's budget. How should Elijah use this information?

Steve is thinking of moving out of his family's home. He currently works at a full-time job making $18 per hour, which will give him, approximately, a net annual income of $29,180 (working 40 hours per week for 52 weeks per year). He has student debt that he pays off at $218.00 per month, and already owns a car that he pays $162.00 per month for.

1. Apply the 50-30-20 budget philosophy to Steve's income.
2. If he follows the budget, how much does he have, after paying his car payment and student loan, to spend on necessities.
3. If he follows the budget, how much will he set aside for wants? For savings?
4. Discuss the affordability of moving out, based on Steve's budget.

Answer

Before the 50-30-20 philosophy can be applied, Steve's monthly income needs to be determined. This is found by dividing his annual income by 12. This gives $\$29{,}180/12=\$2{,}431.67$. This will be used for his monthly budget.

1. **To apply the 50-30-20 philosophy to Steve's income, find 50%, 30%, and 20% of his monthly income.**
   Needs (50%): 50% of his income is $0.50\times\$2{,}431.67=\$1{,}215.83$.
   Wants (30%): $0.30\times\$2{,}431.67=\$729.50$
   Savings (20%): $0.20\times\$2{,}431.67=\$486.33$
2. **The total for Steve's needs is $1,215.83. From this, he already pays $218.00 for his student loans, and $162.00 for his car payment. Together that is $380.00. Subtracting from the amount he should budget for his needs, he can spend $835.83 on other needs.**
3. **Steve budgeted $729.50 for wants, and $486.33 for savings and other debt servicing.**
4. **Steve will have other needs to pay for, including rent, utilities, food, heath care, gasoline, and car insurance. It is difficult to imagine Steve being able to afford to move out, unless he reallocates money that he would want to save, or use for entertainment and other wants, or takes on another job. Even if Steve uses all the money that the 50-30-20 budget sets aside for savings, he still only has $1,322.16 to spend on those necessities. It does not appear he can afford to move out.**

Your Turn 6.5.5

Fran wants to take a new job but will have to move to an area with a higher cost of living. With her current income, she can use the 50-30-20 budget philosophy. The new job will have a net pay of $43,700 annually. She will still have to pay her car payment of $295.00, her student loans that cost $264.00 per month, and her outstanding credit card debt, on which she pays $200 per month.

Apply the 50-30-20 budget philosophy to Fran's new income.

If she follows the budget, how much does she have, after paying her credit card debt, car payment and student loan, to spend on necessities.

If she follows the budget, how much will she set aside for wants? For savings?

Discuss the affordability of changing jobs and moving, based on Fran's budget.

Video

dgeting and the 50:30:20 rule | Bud
n Academy
50:30:20 rule
Khan Academy
Watch on YouTube

50-30–20 Budget Philosophy

Check Your Understanding

1. What is a budget?
2. What are necessary expenses?
3. David gathers his paystubs and bills from the past 6 months. His income, after taxes, is $3,450 per month. His rent, utilities included, is $925. His car payments are $178.54 per month, his car insurance is $129.49 per month, his credit cards cost him

$117.00 per month, he spends $195 per month on gas, his food costs are $290 per month. He also spends $21.99 on Amazon prime, $49.99 on his internet bill, and $400 per month going out. Create David’s monthly budget, including totals, based on that information.

4. Using David’s Budget from Exercise 28, how much income does he have per month after accounting for his expenses?
5. Apply the 50-30-20 budget philosophy to David’s budget.
6. Evaluate David’s budget with respect to the 50-30-20 budget philosophy.

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6.5.0: Exercises

In the following exercises, categorize each expense as a necessary expense or an expense that is a want.

? Exercise 6.5.0.1

Rent

? Exercise 6.5.0.2

Dinner at a restaurant.

? Exercise 6.5.0.3

Car payment

? Exercise 6.5.0.4

New game system

? Exercise 6.5.0.5

Gym membership

? Exercise 6.5.0.6

Electric bill

? Exercise 6.5.0.7

Heating bill

? Exercise 6.5.0.8

Phone bill

? Exercise 6.5.0.9

Netflix

? Exercise 6.5.0.10

Student Loan Payment

? Exercise 6.5.0.11

Explain how a necessary expense for one person could be a want expense for another person.

? Exercise 6.5.0.12

Explain how a necessary expense may be partly a necessary expense and partially a want expense.

In the following exercises, create the budget, including totals and how much the income exceeds or falls short of the expenses, based on the information given.

? Exercise 6.5.0.13

Per month: paychecks = $3,680, consulting = $900, Mortgage = $1,198.00, Utilities = $376, Cell phone = $67.50, Car payments = $627.85, Car insurance = $183.50, Student loans = $833, Food = $450, Gasoline = $275, Internet = $69, Dining out = $250, Credit cards = $375, entertainment = $300

? Exercise 6.5.0.14

Per month: paychecks = $2,750, child support = $500, Mortgage = $945.50, Utilities = $195, Cell phone = $37.50, Car payments = $298.23, Car insurance = $163.50, Student loans = $438, Food = $250, Gasoline = $175, Internet = $49, Netflix = $15, After school care = $711, Credit cards = $150, entertainment = $150

? Exercise 6.5.0.15

Per month: paychecks = $4,385, Rent = $1095, Utilities = $165, Cell phone = $67.50, Car payments = $467.35, Car insurance = $243.75, Student loans = $1,150, Food = $325, Gasoline = $260, Internet = $99, Netflix = $15, Amazon = $23, Gym membership = $49, entertainment = $650

? Exercise 6.5.0.16

Per month: paychecks = $3,460, Gig job = $173, Rent = $895, Utilities = $165, Car payments = $195.80, Car insurance = $123.30, Food = $265, Gasoline = $185, Internet = $39, Hulu = $15, Amazon = $23, Credit cards $97.60, Entertainment = $600

In the following exercises, determine the amount of money that should be allocated to each of the three categories of the 50-30-20 budget philosophy guidelines.

? Exercise 6.5.0.17

Referring to Exercise 13: Monthly income = $4,580.00

? Exercise 6.5.0.18

Referring to Exercise 14: Monthly income = $3,250.00

? Exercise 6.5.0.19

Referring to Exercise 15: Monthly income = $4,385.00

? Exercise 6.5.0.20

Referring to Exercise 16: Monthly income = $3,633.00

In the following exercises, evaluate the given budget with respect to the 50-30-20 budget philosophy guidelines.

For the following exercises, Kiera and Logan sit down to make their budget. Kiera works full time as a mental health counselor and sells kids toys on her own. Logan works as a branch manager at a local bank and works part-time at the nearby bar. They collect their financial document to work out their budget. Kiera's paychecks from her job as a mental health counselor, after taxes and per month, total $3,021. Logan's paychecks from the bank, after taxes and per month, total $3,827. Kiera's income from toy sales for the last 3 months were $140, $87, and $475. Logan's take-home pay from the bartending job for the last 3 months were $540, $310, and $449.

? Exercise 6.5.0.25

Determine how much income Kiera and Logan have per month.

For the following exercises, Kiera and Logan gather their bills from the last 6 months. Their fixed expenses, with costs, are rent for $1,350, Kiera's car payment for $275, Logan's car payment of $380, student loans (they each have students loans) for $934, car insurance for $289, internet service for $39, Netflix for $15, Amazon Prime for $24, gym membership for $99, and cell phones for $250. The variable cost expenses, and their average costs for the last 6 months, are utilities for $370, gasoline for $500, food for $475, clothing for $225, and miscellaneous entertainment expenses for $535. They always pay off their credit card bill and carry no balance.

? Exercise 6.5.0.27

Create their budget, using the income from Exercise 25.

? Exercise 6.5.0.28

Categorize each expense as a need or a want. Find the total for each, along with remaining income.

? Exercise 6.5.0.30

Determine if Kiera and Logan can afford to buy a new computer, which would cost $330 per month for the next 6 months.

In the following exercises, the Federal Paycheck Calculator was used to estimate monthly take-home pay. The annual salary, before taxes and deductions, is provided. Then, the monthly take-home pay after taxes and deductions is given (which means the monthly take-home pay is not just the annual salary divided by 12!). In each case, apply the 50-30-20 budget philosophy to the monthly take-home income. Note: These are based on living in Indianapolis, Indianapolis, unmarried and with no dependents.

? Exercise 6.5.0.31

Annual salary: $30,000. Monthly take home: $1,938

? Exercise 6.5.0.32

Annual salary: $40,000.00. Monthly take home: $2,564

? Exercise 6.5.0.33

Annual salary: $50,000. Monthly take home: $3,144

? Exercise 6.5.0.34

Annual salary: $70,000. Monthly take home: $4,229

? Exercise 6.5.0.35

Annual salary: $100,000. Monthly take home: $5,840

Exercise 6.5.0.36

Annual salary: 150,000. Monthly take home: $8,506

In the following exercises, the Federal Paycheck Calculator was used to estimate monthly take-home pay. The hourly pay, before taxes and deductions, is provided. Then, the monthly take-home pay after taxes and deductions is given (which means the monthly take-home pay is not just the hourly pay times 174 hours!). In each case, apply the 50-30-20 budget philosophy to the monthly take-home income. Note: These are based on living in Tempe, Arizona, unmarried and with no dependents.

Exercise 6.5.0.37

Hourly pay: $12.15 (minimum wage in Tempe, Arizona as of September 2022). Monthly take home: $1,698

Exercise 6.5.0.38

Hourly pay: $15.00. Monthly take home: $2,083

Exercise 6.5.0.39

Hourly pay: $17.50. Monthly take home: $2,421

Exercise 6.5.0.40

Hourly pay: $19.75. Monthly take home: $2,725

Exercise 6.5.0.41

Hourly pay: $25.00. Monthly take home: $3,369

Exercise 6.5.0.42

Hourly pay: $35.00. Monthly take home: $4,547

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6.6: Methods of Savings

Figure 6.6.1: Money wisely invested grows over time. (credit: “Stack of Cash” by Janak Raja/Flickr, Public Domain Mark 1.0)

Learning Objectives

1. Distinguish various basic forms of savings plans.
2. Compute return on investment for basic forms of savings plans.
3. Compute payment to reach a financial goal.

The stock market crash of 1929 led to the Great Depression, a decade-long global downturn in productivity and employment. A state of shock swept through the United States; the damage to people’s lives was immeasurable. Americans no longer trusted established financial institutions. By October 1931, the banking industry’s biggest challenge was restoring confidence to the American public. In the next 10 years, the federal government would impose strict regulations and guidelines on the financial industry. The Emergency Banking Act of 1933 created the Federal Deposit Insurance Corporation (FDIC), which insures bank deposits. The new federal guidelines helped ease suspicions among the general public about the banking industry. Gradually, things returned to normal, and today we have more investment instruments, many insured through the FDIC, than ever before.

In this section, we will first look at the different types of savings accounts and proceed to discuss the various types of investments. There is some overlap, but we will try to differentiate among these financial instruments. Saving money should be a goal of every adult, but it can also be a difficult goal to attain.

Distinguish Various Basic Forms of Savings Plans

There are at least three types of savings accounts. Traditional savings accounts, certificates of deposit (CDs), and money market accounts are three main savings account vehicles.

Savings Account

A savings account is probably the most well-known type of investment, and for many people it is their first experience with a bank. A savings account is a deposit account, held at a bank or other financial institution, which bears some interest on the deposited money. Savings accounts are intended as a place to save money for emergencies or to achieve short-term goals. They typically pay a low interest rate, but there is virtually no risk involved, and they are insured by the FDIC for up to $250,000.

Savings accounts have some strengths. They are highly flexible. Generally, there are no limitations on the number of withdrawals allowed and no limit on how much you can deposit. It is not unusual, however, that a savings account will have a minimum balance in order for the bank to pay maintenance costs. If your account should dip below the minimum, there are usually fees attached.

Who Knew?

Many banks are covered by FDIC insurance. The FDIC is the Federal Deposit Insurance Corporation and is an independent agency created by the U.S. Congress. One of its purposes is to provide insurance for deposits in banks, including savings accounts. Be aware, not all banks are FDIC insured. The FDIC insures up to $250,000 for a savings account, so you do not want your balance to exceed that federally insured limit.

Having your savings account at the same bank as your checking account does offer a real advantage. For example, if your checking account is approaching its lower limit, you can transfer funds from your savings account and avoid any bank fees. Similarly, if you have an excess of funds in your checking account, you can transfer funds to your savings account and earn some interest. Checking accounts rarely pay interest.

People in Mathematics: J.P. Morgan

J.P. Morgan was a wealthy banker around the turn of the 20th century. His business interests included railroads and the steel industry. However, it was in 1907 that a financial crisis, caused by poor banking decisions and followed by such great distrust in the banking system that a frenzy of withdrawals from banks occurred, that J.P. Morgan and other wealthy bankers lent from their own funds to help stabilize and save the system.

There are some weaknesses to savings accounts. Primarily, it is because savings accounts earn very low interest rates. This means they are not the best way to grow your money. Experts, though, recommend keeping a savings account balance to cover 3 to 6 months of living expenses in case you should lose your job, have a sudden medical expense, or other emergency.

Around tax time, you will receive a 1099-INT form stating the amount of interest earned on your savings, which is the amount that must be reported when you file your tax return. A 1099 form is a tax form that reports earnings that do not come from your employer, including interest earned on savings accounts. These 1099 forms have the suffix INT to indicate that the income is interest income.

Savings accounts earn interest, and those earnings can be found using the interest formulas from previous sections. The final value of these accounts is sometimes called the future value of the account.

Example 6.6.1: Single Deposit in a Savings Account

Violet deposits $4,520.00 in a savings account bearing 1.45% interest compounded annually. If she does not add to or withdraw any of that money, how much will be in the account after 3 years?

Answer

To find the compound interest, use the formula from Compound Interest, $A = P$msupntA=P(1+rn)nt, where A represents the amount in the account after t years, with initial deposit (or principal) of P, at an annual interest rate, in decimal form, of r, compounded n times per year. Violet has a principal of $4,520.00, which will earn an interest of $r = 0.0145$, compounded yearly (so $n = 1$), for $t = 3$ years. Substituting and calculating, we find that Violet’s account will be worth

$$\begin{aligned} A &= P\left(1+\tfrac{r}{n}\right)^{nt} \\ &= \$4{,}520.00\left(1+\tfrac{0.0145}{1}\right)^{1\times 3} \\ &= \$4{,}520.00(1.0145)^{3} \\ &= \$4{,}719.48 \end{aligned}$$

Or, Violet will have $4,719.48 after 3 years.

Your Turn 6.6.1

Brian deposits \$5,600 in a savings account that yields 1.23% interest compounded annually. If he leaves that deposit in the account and adds nothing new to the account, what will the account be worth in 5 years?

Who Knew?

Banks have not always offered interest on savings accounts. An 1836 publication from Indiana noted that banks in other states allow small interest on deposits. It specifically says that in these other states, these deposits are what business transactions are based upon. And that giving interest would encourage deposits, and thus increase the business that banks can do.

Journal of the House of Representatives of the Sate of Indiana

Certificates of Deposit, or CDs

We discussed **certificates of deposit** (CDs) in earlier sections. CDs differ from savings accounts in a few ways. First, the investment lasts for a fixed period of time, agreed to when the money is invested in the CD. These time periods often range from 6 months to 5 years. Money from the CD cannot be withdrawn (without penalty) until the investment period is up. Also, money cannot be added to an existing CD.

Certificates of deposit have features similar to savings accounts. They are insured by the FDIC. They are entirely safe. They do, though, offer a better interest rate. The trade-off is that once the money is invested in a CD, that money is unavailable until the investment period ends.

Example 6.6.2: 5-Year CD

Silvio deposits \$10,000 in a CD that yields 2.17% compounded semiannually for 5 years. How much is the CD worth after 5 years?

Answer

This also uses the compound interest formula from Compound Interest, $A = P$msupntA=P(1+rn)nt, Substituting the values P = \$10,000, r = 0.0217, n = 2 (semiannually means twice per year), and t = 5, we find the account will be worth

$$\begin{aligned} A &= P\left(1+\frac{r}{n}\right)^{nt} \\ &= \$10,000.00\left(1+\frac{0.0217}{2}\right)^{2\times 5} \\ &= \$10{,}000.00(1.01085)^{10} \\ &= \$11,239.53 \end{aligned}$$

The CD will be worth \$11,219.53 after 5 years.

Your Turn 6.6.2

Denise deposits \$3,500 in a CD bearing 2.23% interest compounded quarterly for 3 years. How much will Denise's CD be worth after those 3 years?

Money Market Account

A money market account is similar to a savings account, except the number of transactions (withdrawals and transfers) is generally limited to six each month. Money market accounts typically have a minimum balance that must be maintained. If the balance in the account drops below the minimum, there is likely to be a penalty. Money market accounts offer the flexibility of checks and ATM cards. Finally, the interest rate on a money market account is typically higher than the interest rate on a savings account.

✓ Example 6.6.3: Single Deposit to a Money Market Account

Marietta opens a money market account, and deposits $2,500.00 in the account. It bears 1.76% interest compounded monthly. If she makes no other transactions on the account, how much will be in the account after 4 years?

Answer

This, once again, uses the compound interest formula from Compound Interest: $A = P$msupntA=P(1+rn)nt, Substituting the values P = $25,000, r = 0.0176, n = 12, and t = 4, we find the account will be worth

$$\begin{aligned} A &= P\left(1+\frac{r}{n}\right)^{nt} \\ &= \$2{,}500.00\left(1+\frac{0.0176}{12}\right)^{12\times 4} \\ &= \$2{,}500.00(1.0014\bar{6})^{48} \\ &= \$2{,}682.20 \end{aligned}$$

The money market account will be worth $2,682.20 after 4 years.

Your Turn 6.6.3

Chuck opens a money market account, and deposits $8,500.00 in the account. It bears 1.83% interest compounded quarterly. If he leaves makes no other transactions on the account, how much will be in the account after 3 years?

Return on Investment

If we want to compare the profitability of different investments, like savings accounts versus other investment tools, we need a measure that evens the playing field. Such a measure is return on investment.

FORMULA

The return on investment, often denoted ROI, is the percent difference between the initial investment, P, and the final value of the investment, semanticsFVFV, or $\text{ROI} =$ mfracPPROI=FV-PP, expressed as a percentage.

Checkpoint

The length of time of the investment is not considered in ROI.

✓ Example 6.6.4: Calculating Return on Investment

1. Determine the return on investment for the 5-year CD from Example 6.56. Round the percentage to two decimal places.
2. Determine the return on investment for the money market account from Example 6.57. Round the percentage to two decimal places.

Answer

1. The initial deposit in the CD was $10,000, so P = $10,000. The value at the end of 5 years was $11,239.53. so semanticsFVFV = $11,239.53. Substituting and computing we find the return on investment.

$$\begin{aligned} \text{ROI} &= \frac{FV-P}{P} \\ &= \frac{\$11{,}239.53-\$10{,}000}{\$10{,}000} \\ &= \frac{\$1{,}239.53}{\$10,000} \\ &= 0.123953 \end{aligned}$$

The ROI is 12.40%.

2. The initial deposit in the money market was \$2,500, so P = \$2,500. The value at the end of 4 years was \$2,682.20. so semanticsFVFV = \$2,682.20. Substituting and computing we find the return on investment.

$$\begin{aligned}\text{ROI} &= \frac{FV - P}{P}\\ &= \frac{\$2{,}682.20 - \$2{,}500}{\$2{,}500}\\ &= \frac{\$182.20}{\$2{,}500}\\ &= 0.07288\end{aligned}$$

The ROI is 7.29%.

Your Turn 6.6.4

The amount of \$13,000 is invested in a savings account. After 10 years the account has \$15,250.00. Find the return on investment for this account.

The amount of \$6,500 is deposited in a money market account. After 7 years, the account has \$7,358.00. Find the return on investment for this account.

Video

e Return On Investment (ROI) in On
Minute Economics
$1,000
— $ 400
$600
Watch on YouTube
OneMinuteEconomics.com

Return on Investment, ROI

Annuities as Savings

In Compound Interest, we talked about the future value of a single deposit. In reality, people often open accounts that allow them to add deposits, or *payments*, to the account at regular intervals. This agrees with the 50-30-20 budget philosophy, where some income is saved every month. When a deposit is made at the end of each compounding period, such a savings account is called an ordinary annuity.

The formula for the future value of an ordinary annuity is FV =pmt×(1+r/n)n×t-1r/nFV=pmt×(1+r/n)n×t-1r/n, where semanticsFVFV is the future value of the annuity, semanticspmtpmt is the payment, r is the annual interest rate (in decimal form), n is the number of compounding periods per year, and t is the number of years.

Checkpoint

It is important to note that the number of deposits per year and the number of periods per year are the same.

Checkpoint

Another form of annuity if the annuity due, which has deposits at the start of each compounding period. This other annuity type has different formulas and is not addressed in this text.

Example 6.6.5: Future Value of an Ordinary Annuity

Jill has an account that bears 3.75% interest compounded monthly. She decides to deposit $250.00 each month, at the end of the compounding period, into this account. What is the future value of this account, after 8 years?

Answer

These are regular payments into an account bearing compound interest. She is depositing them at the end of each compounding period. This makes this an ordinary annuity. Substituting the values semanticspmtpmt = 250, r = 0.0375, n = 12, and t = 8 into the formula, we find the future value of the account.

$$\begin{aligned} FV &= pmt \times \frac{(1+r/n)^{n \times t}-1}{r/n} \\ &= 250 \times \frac{(1+0.0375/12)^{12 \times 8}-1}{0.0375/12} \\ &= 250 \times \frac{(1.003125)^{96}-1}{0.003125} \\ &= 250 \times \frac{1.34922752406-1}{0.003125} \\ &= 250 \times \frac{0.34922752406}{0.003125} \\ &= 250 \times 111.752807699 \\ &= 27,938.202 \end{aligned}$$

The account, after 8 years, will contain $27,938.20.

Your Turn 6.6.5

Kelly invests $525 every third month, at the end of the compounding period, into an account bearing 3.89% interest compounded quarterly. How much will be in the account after 15 years?

Who Knew?: Setting Savings Account Interest Rates

There are a number of factors that contribute to the amount a bank gives for savings accounts. The interest rate reflects how much the bank values deposits. It also reflects the money that the bank will earn when they lend out money. Finally, interest rates are impacted by the Federal Reserve Bank. When the Fed raises interest rates, so do banks.

People in Mathematics: The Federal Reserve Chairperson

The Federal Reserve Board monitors the risks in the financial system to help ensure a healthy economy for individuals, companies, and communities. The Board oversees the 12 regional reserve banks. The Chairperson of the Federal Reserve Board testifies to Congress twice per year, meets with the secretary of the Treasury, chairs the Federal Open Market Committee, and is the face of federal monetary policy. Currently, the Fed Chair is Jerome Powell, who has served since 2018.

Example 6.6.6: Saving for College

When Yusef was born, Rita and George began to save for Yusef’s college years by investing $2,500 each year in a savings account bearing 3.4% interest compounded annually. How much will they have saved after 18 years?

Answer

To find the future value of the account, we use the ordinary annuity formula FV =pmt×(1+r/n)n×t-1r/nFV=pmt×(1+r/n)n×t-1r/n. The payment is \$2,500, rate is 0.034, the number of compounding periods is 1, and the number of years is 18. Substituting these values and computing, we have

$$\begin{aligned} FV &= pmt \times \frac{(1+r/n)^{n \times t}-1}{r/n} \\ &= \$2{,}500 \times \frac{(1+0.034/1)^{1 \times 18}-1}{0.034/1} \\ &= \$2{,}500 \times \frac{(1.034)^{18}-1}{0.034} \\ &= \$2{,}500 \times \frac{1.82544897331-1}{0.034} \\ &= \$2{,}500 \times 24.2779109798 \\ &= \$60{,}694.77 \end{aligned}$$

After saving for 18 years, Rita and George will have \$60,694.77 for Yusef’s college.

Your Turn 6.6.6

Bemnet saves \$280 per month in a savings account bearing 3.11% interest compounded monthly. After 20 years, how much does Bemnet have in the account?

Tech Check

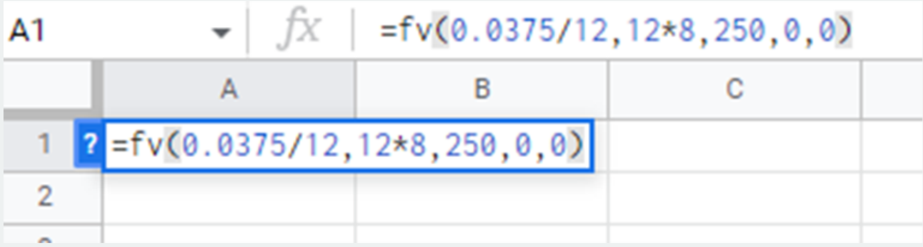
Google Sheets offers a function to calculate the future value of an ordinary annuity. To get Google Sheets to calculate the future value, you use the following:

=fv(rate,number_of_periods, payment, present_value, end_or_beginning).

To explain, the rate is the rate per compounding period. From our formula, that is r/n. Also, the number of periods must be entered. From our formula, that is $n \times t$. The payment is the amount deposited each period. Present value is 0 if we begin with no money and rely only on the payments to be made. However, if some money is available to put in the account before the payments start, that amount, an initial deposit, would be the value of semanticsPVPV. Finally, for an ordinary annuity, enter 0 for end or beginning. Using the values for Jill, the payment amount is \$250, r = 0.0375, n = 12, t = 8, and that there is no initial deposit, semanticsPVPV = 0, the Google Sheets formula is

=fv(0.0375/12,12*8,250,0,0).

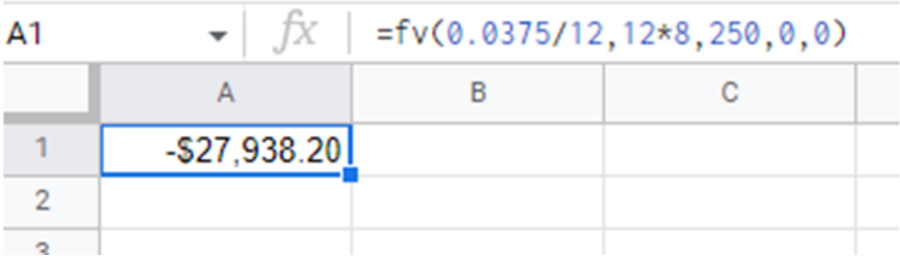
Figure 6.9 shows the formula in Google Sheets.



```
=fv(0.0375/12,12*8,250,0,0)
```

Figure 6.9: Google Sheets formula

Hitting the enter key shows the payment value (Figure 6.10).

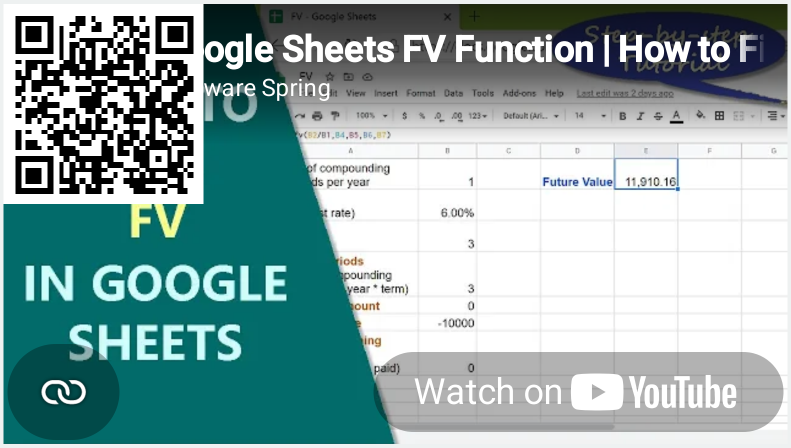


	A	B	C
1	-$27,938.20		
2			

Figure 6.10: Payment value

Notice that the future value is negative, since it is a payment leaving an account.

Video



Future Value Using Google Sheets

Compute Payment to Reach a Financial Goal

The formula used to get the future value of an ordinary annuity is useful, finding out what the final amount in the account will be. However, that isn't how planning works. To plan, we need to know how much to put into the ordinary annuity each compounding period in order to reach a goal. Fortunately, that formula exists.

FORMULA

The formula for the amount that needs to be deposited per period, semanticspmtpmt, of an ordinary annuity to reach a specified goal, semanticsFVFV, is $pmt =$ mfracFV×(r/n)(1+r/n)n×t-1pmt=FV×(r/n)(1+r/n)n×t-1, where r is the annual interest rate (in decimal form), n is the number of periods per year, and t is the number of years.

With this formula, it is possible to plan the amount to be saved.

✓ Example $6.6.7$: Saving for a Car

Yaroslava wants to save in order to buy a car, in 3 years, without taking out a loan. She determines that she'll need $35,500 for the purchase. If she deposits money into an ordinary annuity that yields 4.25% interest compounded monthly, how much will she need to deposit each month?

Answer

Yaroslava has a goal and needs to know the payments to make to reach the goal. Her goal is semanticsFVFV = $35,500, with an interest rate r = 0.0425, compounded per month so n = 12, and for 3 years, making t = 3. Substituting into the formula, Yaroslava finds the necessary payment.

$$\begin{aligned} pmt &= \frac{FV \times (r/n)}{(1+r/n)^{n \times t} - 1} \\ &= \frac{35{,}500 \times (0.0425/12)}{(1+0.0425/12)^{12 \times 3} - 1} \\ &= \frac{35{,}500 \times (0.003541\bar{6})}{(1.003541\bar{6})^{36} - 1} \\ &= \frac{125.7291\bar{6}}{0.13572901696} \\ &= 926.325 \end{aligned}$$

To reach her goal, Yaroslava would need to deposit \$926.33 in her account each month.

Checkpoint

This has been rounded up, so that the deposits don't fall short of the goal. However, some round off using the standard rounding rules: if the last digit is 1, 2, 3, or 4, the number is rounded down; if the last digit is 5, 6, 7, 8, or 9 the number is rounded up.

Your Turn 6.6.7

Chione decides to put new siding on her house. She finds that it will cost about \$27,800. She decides to begin saving for the purchase so that she doesn't take on debt to side the house. How much would Chione need to deposit each quarter in an ordinary annuity that yields 5.16% compounded quarterly for 5 years?

Tech Check

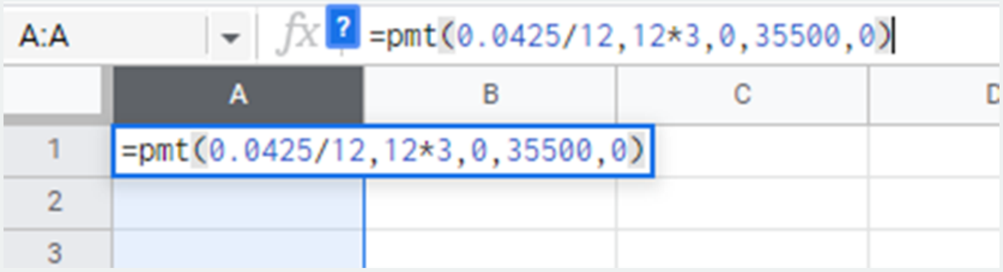
Google Sheets offers a function to calculate the payment necessary to reach a goal using ordinary annuities. To get Google Sheets to calculate the payment, you use the following:

=pmt(rate,number_of_periods, present_value, future_value, end_or_beginning).

To explain, the rate is per compounding period. From our formula, that is r/n. Also, the number of periods must be entered. From our formula, that is $n \times t$. The present value is the amount of money that the account begins with. If we begin with no money and rely only on the payments to be made, then this number is 0. However, if some money is available to put in the account before the payments start, that amount, an initial deposit, would be the value of semanticsPVPV. Next, enter the future value, semanticsFVFV. Finally, for an ordinary annuity, enter 0 for end or beginning. Using the values for Yaroslava, r = 0.0425, n = 12, t = 3, and that there is no initial deposit, semanticsPVPV = 0, the Google Sheets formula is

=pmt(0.0425/12,12*3,0,35500,0).

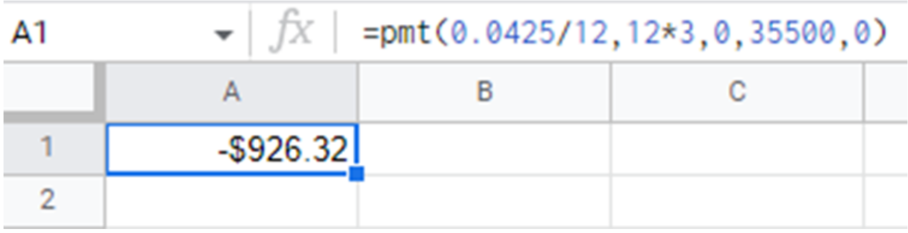
Figure 6.11 shows the formula in Google Sheets.



=pmt(0.0425/12,12*3,0,35500,0)

Figure 6.11: Google Sheets formula

Hitting the enter key shows the payment value (Figure 6.12).



`=pmt(0.0425/12,12*3,0,35500,0)`

-$926.32

Figure 6.12: Payment value

Notice that the payment is negative, since it is a payment leaving an account. Additionally, the payment is $926.32. We rounded that up, but Google Sheets rounded off.

Check Your Understanding

1. What is a savings account?
2. How does a CD differ from a savings account?
3. Which is more flexible, a CD or a money market account? Why?
4. If $7,500 is deposited in a 4-year CD earning 3.28% interest compounded monthly, how much is in the account after 4 years?
5. If the initial deposit in an account is $10,000 and the account is worth $12,560 after 7 years, what is the return on investment?
6. Find the future value of an account if $450.00 per quarter is invested in a savings account bearing 3.5% interest compounded quarterly for 10 years.
7. How much must be deposited per quarter in an account bearing 2.98% interest compounded quarterly if the account is to be worth $300,000 after 25 years?

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6.6.0: Exercises

? Exercise 6.6.0.1

Which account has the greatest flexibility, savings, certificate of deposit, or money market?

? Exercise 6.6.0.2

Why are interest rates on savings accounts, CDs, and money market accounts low?

? Exercise 6.6.0.3

Which of savings accounts, certificates of deposit, and money market accounts, allow for transactions?

? Exercise 6.6.0.4

How does number of years impact the return on investment?

In the following exercises, find the future value of the account based on the information given.

? Exercise 6.6.0.5

The amount of $3,000 deposited in a CD bearing 2.6% compounded semi-annually for 3 years.

? Exercise 6.6.0.6

The amount of $1,500 deposited in a money market account bearing 3.11% interest compounded monthly for 10 years.

? Exercise 6.6.0.7

The amount of $8,450 deposited in a savings account bearing 1.75% interest compounded monthly for 2 years.

? Exercise 6.6.0.8

The amount of $10,500 deposited in a savings account bearing 1.35% interest compounded quarterly for 20 years.

? Exercise 6.6.0.9

The amount of $24,800 deposited in a money market account bearing 2.53% interest compounded semi-annually for 13 years.

? Exercise 6.6.0.10

The amount of $16,400 deposited in a CD bearing 2.55% interest compounded quarterly for 18 years.

In the following exercises, find the return on investment based on the specified exercise. Round to two decimal places.

? Exercise 6.6.0.11

Account from Exercise 5.

? Exercise 6.6.0.12

Account from Exercise 6.

? Exercise 6.6.0.13

Account from Exercise 7.

? Exercise 6.6.0.14

Account from Exercise 8.

In the following exercises, find the future value of the ordinary annuities based on the payment, interest rate, compounding periods and length of time given.

? Exercise 6.6.0.15

The amount of \$150 deposited monthly in an account bearing 4.22% interest compounded monthly for 20 years.

? Exercise 6.6.0.16

The amount of \$500 deposited semi-annually in an account bearing 3.62% interest compounded semi-annually for 30 years.

? Exercise 6.6.0.17

The amount of \$250 deposited quarterly in an account bearing 3.61% interest compounded quarterly for 25 years.

? Exercise 6.6.0.18

The amount of \$250 deposited monthly in an account bearing 3.09% interest compounded monthly for 40 years.

? Exercise 6.6.0.19

The amount of \$1,500 deposited annually in an account bearing 3.34% interest compounded annually for 10 years.

? Exercise 6.6.0.20

The amount of \$1400 deposited semi-annually in an account bearing 2.78% interest compounded semi-annually for 30 years.

In the following exercises, find the payment per period necessary to reach a specified future value based on the given interest rate, compounding periods per year, and number of years. Recall, the number of payments per year and the number of compounding periods per year are the same.

? Exercise 6.6.0.21

Future value of \$1,000,000 from an account bearing 3.94% interest compounded monthly for 40 years.

? Exercise 6.6.0.22

Future value of \$500,000 from an account bearing 2.11% interest compounded quarterly for 30 years.

Exercise 6.6.0.23

Future value of $750,000 from an account bearing 3.27% interest compounded monthly for 25 years.

Exercise 6.6.0.24

Future value of $300,000 from an account bearing 3.59% interest compounded semiannually for 35 years.

Exercise 6.6.0.25

Future value of $1,000,000 from an account bearing 3.62% interest compounded annually for 25 years.

Exercise 6.6.0.26

Future value of $600,000 from an account bearing 4.02% interest compounded quarterly for 30 years.

Exercise 6.6.0.27

Dina deposits $3,000 in a 5-year CD that bears 3.25% interest compounded quarterly. What is the CD worth after those 5 years?

Exercise 6.6.0.28

Timothy deposits $1,200 in a savings account that bears 1.85% interest compounded monthly. If Timothy does not deposit or withdraw money from the account how much is in Timothy's account after 3 years?

Exercise 6.6.0.29

Leslie deposits $13,000 in a money market account that bears 2.55% interest compounded semi-annually. If Leslie does not withdraw or deposit money into the account, how much is in Leslie's account after 6 years?

Exercise 6.6.0.30

Jennifer deposits $8,500 in a 3-year CD bearing 2.71% interest compounded annually. How much is Jennifer's CD worth after those 3 years?

Exercise 6.6.0.31

Yasmin has analyzed her budget and decides to deposit $425 per month in an account bearing 3.99% interest compounded monthly. How much will be in the account after 20 years? After 30 years? After 40 years?

Exercise 6.6.0.33

Ashliegh wants to save for an early retirement. She thinks she needs $1,250,000 to retire at the age of 55, which is 30 years from now. How much must she deposit per month in an account bearing 3.48% interest compounded monthly to reach her goal?

Exercise 6.6.0.34

Colin plans out the next 38 years of his life. In order to retire in 38 years (age 65) with $1,450,000, how much should he deposit quarterly in an account bearing 4.21% interest compounded quarterly to reach his goal?

In the following exercises, different savings strategies will be compared.

? Exercise 6.6.0.36

Sam decides instead to delay investing in the account until her 35th birthday. How much will be in the account at age 65 (30 years)?

? Exercise 6.6.0.37

Sam decides to deposit the $300 per month until she turns 35 years old (11 years). She will then stop investing the $300 monthly, and just allow the money to earn interest until her 65th birthday (30 more years). How much will be in her account on her 65th birthday? Hint: First, compute the FV of the deposits. Then use that FV as the principal for a single deposit into an account bearing 3.75% interest compounded monthly.

? Exercise 6.6.0.38

Compare the results of the three investment strategies.

In the following exercises, different savings strategies will be compared.

? Exercise 6.6.0.40

Dahlia decides instead to delay investing in the account until her 34th birthday. How much will be in the account on her 68th birthday (34 years)?

? Exercise 6.6.0.41

Dahlia decides to deposit the $250 per month until her 34th birthday (11 years). She will then stop investing the $250 monthly, and just allow the money to earn interest until her 68th birthday (34 more years). How much will be in her account on her 65th birthday? Hint: First compute the FV of the deposits. Then use that FV as the principal for a single deposit into an account bearing 6.2% interest compounded monthly.

? Exercise 6.6.0.42

Compare the results of the three investment strategies.

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6.7: Investments

Learning Objectives

1. Distinguish between basic forms of investments including stocks, bonds, and mutual funds.
2. Understand what bonds are and how bond investments work.
3. Understand how stocks are purchased and gain or lose value.
4. Read and derive information from a stock table.
5. Define a mutual fund and how to invest.
6. Compute return on investment for basic forms of investments.
7. Compute future value of investments.
8. Compute payment to reach a financial goal.
9. Identify and distinguish between retirement savings accounts.

You can save your money in a safe or a vault (or worse, under the mattress!), but that money does not grow. It would be hard to save enough for retirement that way. What can be done to increase the value of the money you already have?

The answer is to invest it. Use the money that you have to earn more money back. For instance, as we saw in Methods of Savings, you can save it in a bank. Or, to reach loftier goals, invest in something more likely to grow, such as stocks.

Figure 6.7.1: Stocks are bought and sold to improve investment values. (credit: modification of work "FT ringing the Closing Bell at the NYSE" by Financial Times/Flickr, CC BY 2.0)

A great example of this is Apple stock. Anyone who bought stock in Apple Inc. (formerly Apple Computer, Inc.) in 1997 and held onto the shares earned a lot of money. To be more specific, $100 worth of Apple shares bought in 1980, when it was first sold to the public, was valued at $67,564 in 2019, or 676 times more! Perhaps you have heard a story like that, of an investment opportunity taken that paid off, or the story of an investment opportunity missed. But such stories are the exceptions.

In this section, we’ll investigate bonds, stocks, and mutual funds and their comparative strengths and weaknesses. We close the section with a discussion of retirement savings accounts.

Distinguish Between Basic Forms of Investments

Bonds, stocks, and mutual funds tend to offer higher returns, but to varying degrees, come with higher risks. Stocks and mutual funds also vary in how much they earn. Their predicted rates of return on investment are not guaranteed, but educated guesses based on market trends and historical performance.

We will use the methods and formulas we learned earlier to evaluate these forms of investment.

Bonds

Bonds are issued from big companies and from governments. Selling bonds is an alternative to an institution taking a loan from a bank. The funds from the selling of bonds are often used for large projects, like funding the building of a new highway or hospital.

Bonds are considered a conservative investment. They are bought for what is known as the issue price. The interest is fixed (does not change) at the time of purchase and is based on the issue price of the bond. The interest rate is often referred to as the coupon rate; the interest paid is often called the coupon yield. The interest paid is often higher than savings accounts and the risk is exceptionally low. The bond is for a fixed length of time. The end of this time is the maturity date of the bond.

There are several types of bonds:

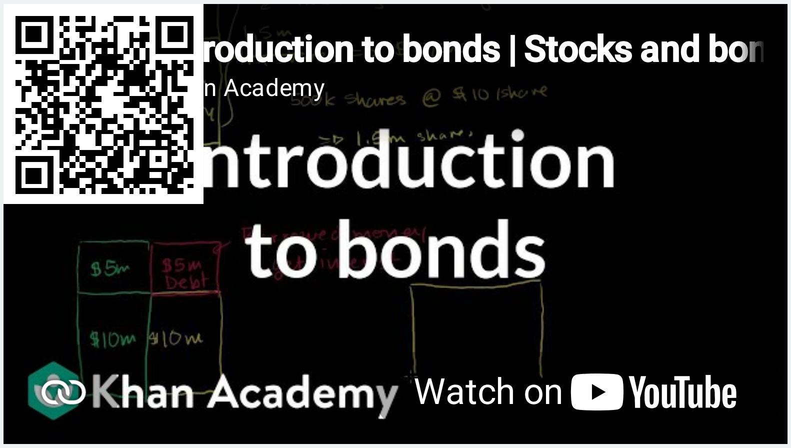
- **Treasury bonds** are issued by the federal government.
- **Municipal bonds** are issued by state and local governments.
- **Corporate bonds** are issued by major corporations.

There are other types of bonds available, but they are beyond the scope of this section.

Who Knew?: Trading Bonds

Bonds are often part of larger investment portfolios. These bonds may be traded. However, the interest paid is based on the price when the bond was bought (the issue price). These bonds can be bought and sold for more or less money than the issue price. If the bond is bought for more than the issue price, the interest is still paid on the issue price, not on the purchase price when the trade was made. This means the actual return on the bond decreases. If the bond is bought for less than the issue price, the return on the bond goes up.

Video



Bonds

Example 6.7.1: Bond Investment

Muriel purchases a $3,000 bond with a maturity of 4 years at a fixed coupon rate of 5.5% paid annually. How much is Muriel paid each year, and how much does she receive on the maturity date?

Answer

The coupon rate is 5.5%. 5.5% of her bond value is $0.055 \times \$3{,}000 = \$165$. After year 1, Muriel receives $165. She receives $165 after years 2 and 3 also. In year 4, when the bond matures, Muriel receives $3,165, or the interest and the initial investment, or principal.

Your Turn 6.7.1

Maureen invests $5,000 in a bond with a maturity date in 5 years at a fixed coupon rate of 4.75%. How much is Maureen paid each year and how much does she receive on the maturity date?

Stocks

Stocks are part ownership in a company. They come in units called shares. The performance and earnings of stocks is not guaranteed, which makes them riskier than any other investment discussed earlier. However, they can offer higher return on investment than the other investments. Their value grows in two ways. They offer dividends, which is a portion of the profit made by the company. And the price per share can increase based on how others see that value of the company changing. If the value of the company drops, or the company folds, the money invested in the stock also drops.

Most stock transactions are executed through a broker. Brokers' commissions can be a percentage of value of the trades made or a flat fee. There are full-service brokers who charge higher commission rates, but they also offer financial advice and perform the research that you may not have the time or the expertise to do on your own. A discount broker only executes the stock transactions, buying or selling, so they charge lower rates than full-service brokers. There are also brokers that offer commission-free trading.

An important thing to remember is that stocks might provide a very large return on investment, but the trade-off is the risk associated with owning stocks.

Who Knew?: Chapter 11 Bankruptcy and Stocks

In the fall of 2022, the parent company of Regal Theaters, named Cineworld, filed for Chapter 11 bankruptcy. According to news articles, the bankruptcy was necessitated due to its heavy debt load. Generally, a company can file for Chapter 11 bankruptcy to allow them time to reorganize and restructure debts. When this happens, the company, after the Chapter 11 process is over, offers new stock. This makes the previous stock worthless. However, the company may allow an exchange of old stock for a discounted amount of the new stock. This in effect reduces (maybe vastly) the wealth held by those who owned the original stock.

Example 6.7.2: Buying Stock in Company ABC

Haniah buys stock in the ABC company, investing a total of $13,000. She expects the stock to grow, through stock price increase and reinvestment of dividends, by 12.3% per year and compounded annually. If she leaves that money invested, how much will the stocks be worth in 20 years?

Answer

Calculating this is a compound interest calculation, if Haniah's assumption about the stock's performance is correct. If so, then the principal is $13,000, the rate is 0.123, the number of compounding periods per year is 1, and the time is 20 years. Substituting into the compound interest formula from Methods of Savings, and computing, we have $A = P(1 + \frac{r}{n})^{nt} = \$13{,}000(1 + \frac{0.123}{1})^{1 \times 20} = \$13{,}000 \times 10.1764223996 = \$132{,}293.49$. After 20 years, her stock is now worth $132,293.49.

Your Turn 6.7.2

Rixie deposits $23,000 in the stock of DEF company. She assumes the stock value to grow though stock price increase and reinvestment of dividends by 13.8% compounded annually. How much will her stock be worth in 12 years?

Who Knew?: Risk and Volkswagen

The question of risk hovers over every investment. How risky can it get? Volkswagen seems to be a rather safe investment. But in 2015, Volkswagen's stock tumbled 30% over a few days when it was revealed that the company had installed software that altered the emission performance of some of their diesel engines. Volkswagen's hope was that lower emissions would bolster US sales of some of their diesel models. This was a drastic drop, and many investors lost a lot of money. However, the stock has come back since then. This was mild compared to the 65% drop in the Martha Stewart Living Omnimedia stocks.

People in Mathematics: Warren Buffett

Warren Buffett is an investment legend. He began his career as an investment salesman in the 1950s. He formed Buffett associates in 1956. In 1965, he was in control of Berkshire Hathaway, which began as a merger between two textile companies. In his role there, he began to invest in a variety of companies. It is now a conglomerate holding company, and fully owns GEICO, Duracell, Diary Queen, and other large companies.

His investment philosophy involves finding stocks and bonds from companies that have high intrinsic worth compared to their stock or bond prices. This means he focuses not on the supply and demand side of stock investing, but instead on the company's worth in total. Using this philosophy, he has become one of the world's most successful investors.

Reading Stock Tables

Information about particular stocks is contained in **stock tables**. This information includes how much the stock is selling for, and its high and low values form the past year (52 weeks). In a newspaper, the stock table may look like this:

Table 6.2 excerpt from a stock table, 2008

52-Week High Low		Stock	SYM	Div	Yld %	P/E	Vol 100s	High	Low	Close	Net Chg
41.66	18.90	McDonald's	MCD	.72	2.9	12	7588	25.73	23.87	25.42	+0.31
22.60	13.20	Monsanto	MON	.52	2.4	55	15474	21.86	21.48	21.64	-0.29
17.05	8.30	Motorola	MOT	.16	1.7	dd	16149	10.57	8.88	10.43	+0.14
31.75	22.99	Mueller	MLI	-	-	16	1564	29.32	27.03	27.11	-0.02

The symbols and abbreviations are defined here:

52-week High	52-week Low	The highest and lowest price of the stock over the past 52 weeks
Stock	SYM	The name of the company and the symbol used for trading
	Annual DIV	The current annual dividend per share
	Yld %	Percent yield is = mfracdividendshare price×100=annual dividendshare price×100
	P/E	Price to earnings ratio, share price divided by earnings per share over past year (dd indicates loss)
	Vol 100s	The number of shares traded yesterday in 100s
High	Low	The highest and lowest prices at which stocks traded yesterday
	Close	The price at which the stock traded at the close of the market yesterday
	Net Chg	Net change; change in price from market close 2 days ago to yesterday's close

The formulas for yield and price to earnings is a good way to measure how much the stock returns per share. Their values are calculated in the stock table, but deserve attention here.

FORMULA

The price to earnings ratio of a stock, P/E, is $\mathrm{P/E}=$ mfracPriceDividendP/E=Share PriceDividend. The percent yield for a stock, Yld%, is Yld%=Annual DividendShare Price×100%Yld%=Annual DividendShare Price×100%.

It should be noted that the price of a stock increases and decreases every moment, and so these value change as the share price changes.

Example 6.7.3: Computing Percent Yield

1. Find the percent yield for a stock with a price of $30.69 and an annual dividend of $1.48.
2. Find the percent yield for a stock with a price of $62.25 and an annual dividend of $1.76.

Answer

1. Substituting the values for price, $30.69, and annual dividend, $1.48, we find the percent yield for the stock to be Yld%=Annual DividendShare Price×100%=$1.48$30.69×100%=4.82%Yld%=Annual DividendShare Price×100%=$1.48$30.69×100%=4.82%
2. Substituting the values for price, $62.25, and annual dividend, $1.76, we find the percent yield for the stock to be $\mathrm{Yld}\%=$ mfracDividendShare Price×100%=$1.76$62.25×100%=2.83%Yld%=Annual DividendShare Price×100%=$1.76$62.25×100%=2.83%

Your Turn 6.7.3

Find the percent yield for a stock with a price of $37.40 and an annual dividend of $1.60.

Find the percent yield for a stock with a price of $73.22 and an annual dividend of $2.41.

The stock table information is now, and has been, available online, from websites such as cnn.com/markets, markets.businessinsider.com/stocks, and marketwatch.com. The same information is available from these sites as from the newspaper listings, but are often accessed one stock at a time. Figure 6.14 shows the stock table for Lowe's on September 7, 2022.

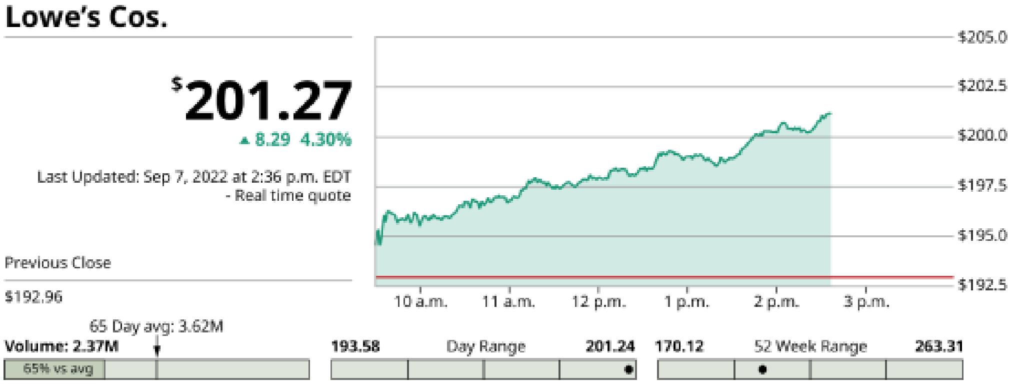
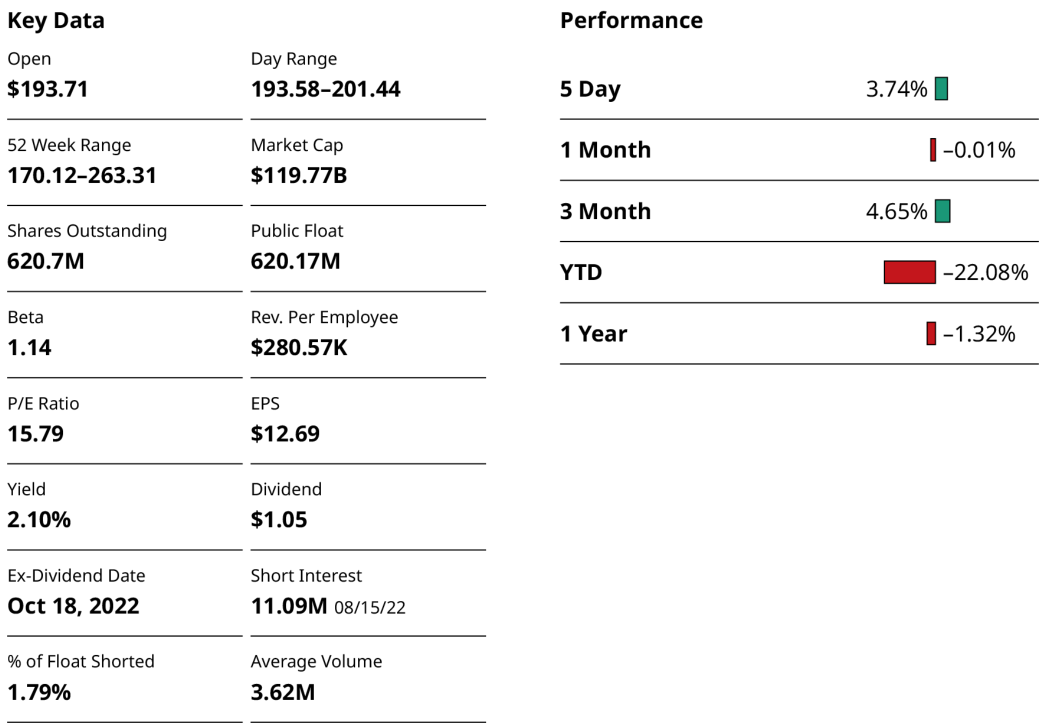


Figure 6.14: Key data for Lowe's stock 9/7/2022 (data source: marketwatch.com)

Other key data is further down on the website, and is shown in Figure 6.15, below.



Key Data

Open	**$193.71**	Day Range	**193.58–201.44**
52 Week Range	**170.12–263.31**	Market Cap	**$119.77B**
Shares Outstanding	**620.7M**	Public Float	**620.17M**
Beta	**1.14**	Rev. Per Employee	**$280.57K**
P/E Ratio	**15.79**	EPS	**$12.69**
Yield	**2.10%**	Dividend	**$1.05**
Ex-Dividend Date	**Oct 18, 2022**	Short Interest	**11.09M** 08/15/22
% of Float Shorted	**1.79%**	Average Volume	**3.62M**

Performance

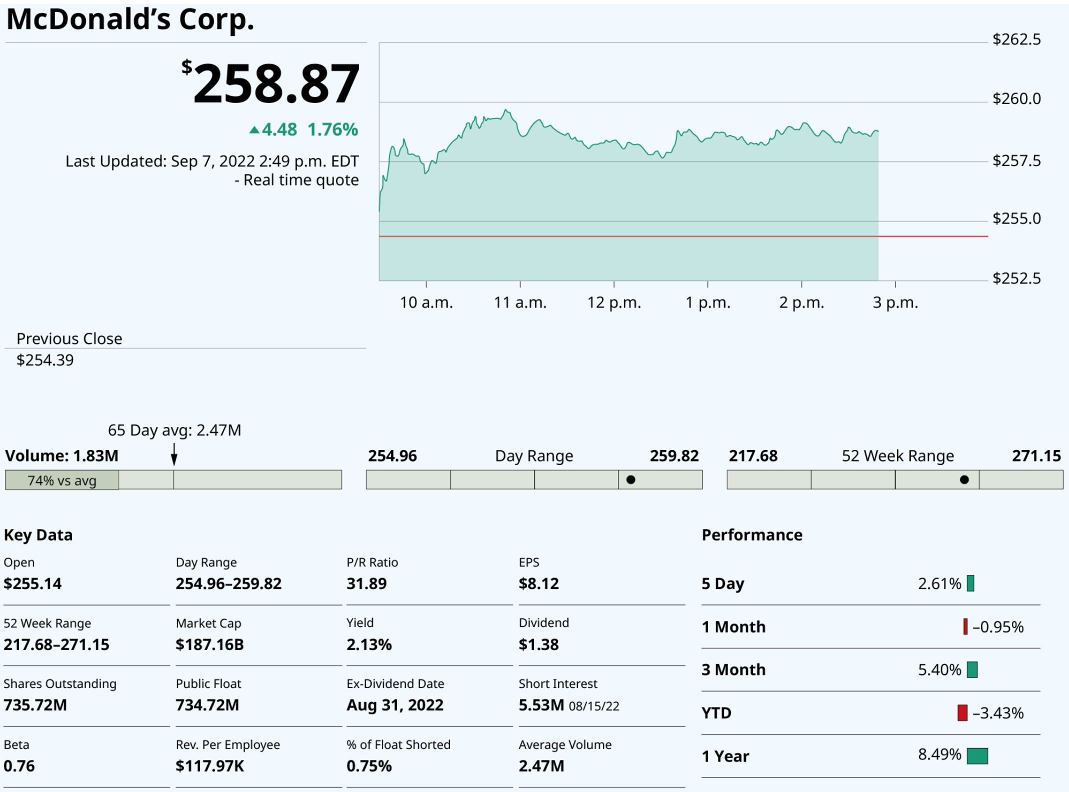
5 Day	3.74% 🟩
1 Month	🟥 −0.01%
3 Month	4.65% 🟩
YTD	🟥 −22.08%
1 Year	🟥 −1.32%

Figure 6.15: Key data for Lowe's stock 9/7/2022 (data source: marketwatch.com)

Notice that the 52-week high and low are now shown as the 52-week range. However, you get additional information, including the stock performance over the past 5 days, past month, past 3 months, the year to date (YTD), and over the past year. You can also read the number of shares outstanding, the expected date for the dividend (EX-DIVIDEND DATE), and importantly for the P/E ratio, the earning per share (EPS).

✓ Example $6.7.4$: Reading an Online Stock Table

Consider the stock table (Figure 6.16), and answer the questions based on the table.



McDonald's Corp.

$258.87

▲4.48 1.76%

Last Updated: Sep 7, 2022 2:49 p.m. EDT
- Real time quote

Previous Close
$254.39

65 Day avg: 2.47M

Volume: 1.83M
74% vs avg

254.96 Day Range **259.82**

217.68 52 Week Range **271.15**

Key Data

Open **$255.14**	Day Range **254.96–259.82**	P/R Ratio **31.89**	EPS **$8.12**
52 Week Range **217.68–271.15**	Market Cap **$187.16B**	Yield **2.13%**	Dividend **$1.38**
Shares Outstanding **735.72M**	Public Float **734.72M**	Ex-Dividend Date **Aug 31, 2022**	Short Interest **5.53M** 08/15/22
Beta **0.76**	Rev. Per Employee **$117.97K**	% of Float Shorted **0.75%**	Average Volume **2.47M**

Performance

5 Day	2.61%
1 Month	−0.95%
3 Month	5.40%
YTD	−3.43%
1 Year	8.49%

Figure ✓ 6.7.2 Key data for McDonald's stock 9/7/2022 (data source: marketwatch.com)

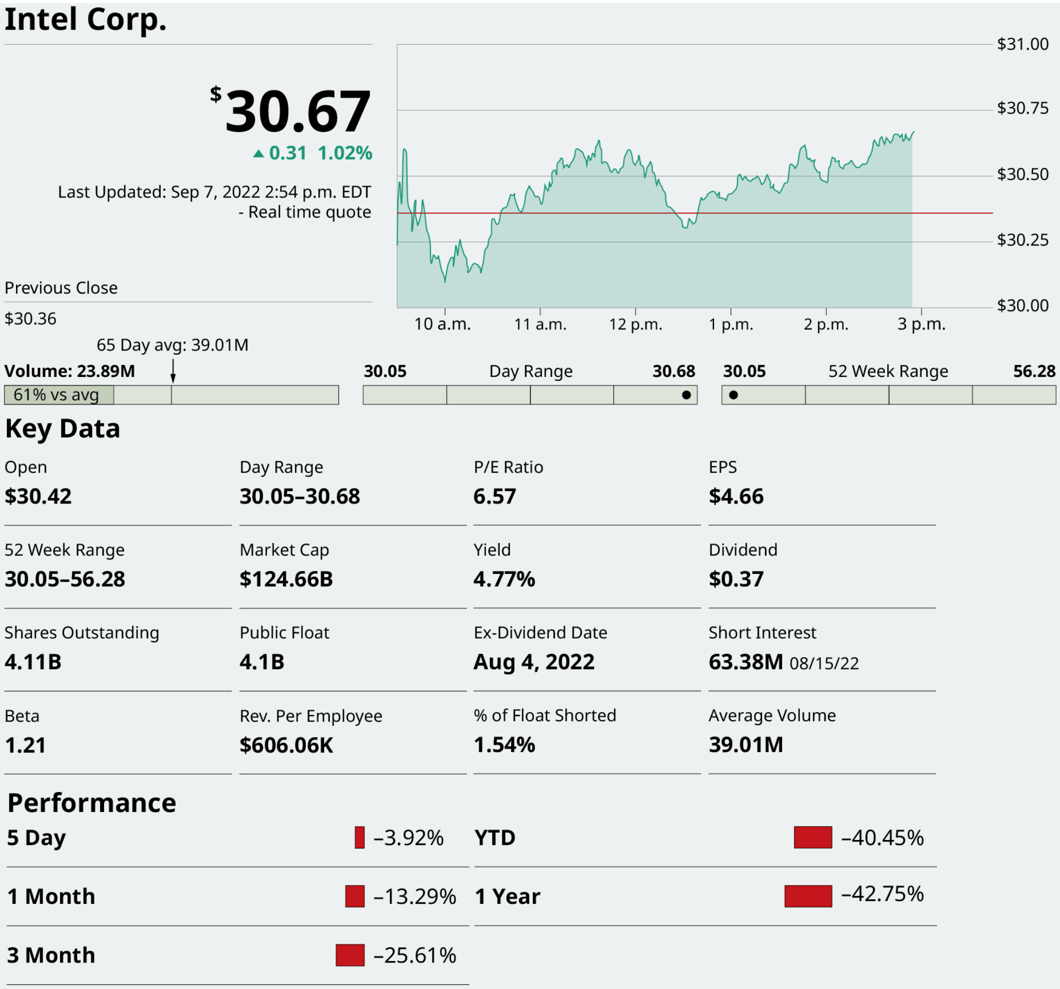
1. What is the current price for McDonald's Corp on this date?
2. What is the 52-wk high? 52-wk low?
3. When is the dividend expected?
4. What is its yield?
5. What is the earnings per share?

Answer

1. Looking at the table, the current price of a share is $258.87.
2. The high was $271.15, and the low was $217.68.
3. August 31, 2022
4. 2.13%
5. The EPS value is $8.12.

Your Turn 6.7.4

Consider the stock table below, and answer the questions based on the table.



Intel Corp.

$30.67

▲ 0.31 1.02%

Last Updated: Sep 7, 2022 2:54 p.m. EDT
- Real time quote

Previous Close

$30.36

65 Day avg: 39.01M

Volume: 23.89M

61% vs avg

30.05 Day Range **30.68**

30.05 52 Week Range **56.28**

Key Data

Open	Day Range	P/E Ratio	EPS
$30.42	**30.05–30.68**	**6.57**	**$4.66**
52 Week Range	Market Cap	Yield	Dividend
30.05–56.28	**$124.66B**	**4.77%**	**$0.37**
Shares Outstanding	Public Float	Ex-Dividend Date	Short Interest
4.11B	**4.1B**	**Aug 4, 2022**	**63.38M** 08/15/22
Beta	Rev. Per Employee	% of Float Shorted	Average Volume
1.21	**$606.06K**	**1.54%**	**39.01M**

Performance

5 Day	🟥 –3.92%	**YTD**	🟥 –40.45%
1 Month	🟥 –13.29%	**1 Year**	🟥 –42.75%
3 Month	🟥 –25.61%		

Figure 6.7.3: Key data for Intel stock 9/7/2022 (data source: marketwatch.com)

What is the current price for Intel Corp on this date?

What is the 52-wk high? 52-wk low?

When is the dividend expected?

What is its yield?

What is the earnings per share?

As mentioned, stocks earn money in two ways, through dividends and increase in share price.

✓ Example 6.7.5: Dividends Paid

Darma owns 150 shares of stock in the GDW company. This quarter, GDW is paying $0.87 per share in dividends. How much will Darma earn in dividends this quarter?

Answer

Each share pays $0.87, so Darma earns $150 \times \$0.87 = \$130.50$.

Your Turn 6.7.5

Yulia owns 300 shares of stock in YYZ company. It pays $1.12 per share this quarter. How much did Yulia earn this quarter on stock in YYZ?

Example 6.7.6: Stock Price Increases

Vincent buys 100 stocks in the REM company for $21.87 per share. One year later, he sells those 100 shares for $29.15 per share.

1. How much money did Vincent make?
2. What was his return on investment for that one year?

Answer

1. Vincent spent $21.87 per share to buy the stock. The total he spent on the stock was $\$21.87 \times 100 = \$2{,}187.00$. When he sold the stock, the price was $29.15, so he received $\$29.15 \times 100 = \$2{,}915.00$. He made $\$2{,}915.00 - \$2{,}187.00 = \$728.00$.
2. His return on investment was mfracEarningsOriginal Price=$728$2,187=33.29%EarningsOriginal Price=$728$2,187=33.29%.

Your Turn 6.7.6

Ginny buys 200 shares of stock in UUK company for $9.76 per share. At the end of the year, she sells those stocks for $10.02 per share.

1. How much money did Ginny make?
2. What was her return on investment for that one year?

Video

w to Read a Stock Summary - Learn
rtan Trading
HOW TO READ A STOCK SUMMARY
Watch on YouTube

Reading Stock Summary Online

Mutual Funds

A **mutual fund** is a collection of investments that are all bundled together. When you buy shares of a mutual fund, your money is pooled with the assets of other investors. This pooled money is invested in stocks, bonds, money market instruments, and other assets. Mutual funds are typically operated by professional money managers who allocate the fund's assets and attempt to produce capital gains or income for the fund's investors.

A key benefit of mutual funds is that they allow small or individual investors to invest in professionally managed portfolios of equities, bonds, and other securities. This means each shareholder participates proportionally in the gains or losses of the fund. The performance of a mutual fund is usually stated as how much the mutual fund's total value has increased or decreased. Since there are many different investments inside the mutual fund, the risk is reduced significantly, compared to direct ownership of stocks. Even so, mutual funds historically perform well and can earn more than 10% annually.

The investments that make up a mutual fund are structured and maintained to match stated investment objectives, which are specified in its **prospectus**. A prospectus is a pamphlet or brochure that provides information about the mutual fund. Before buying shares of a mutual fund, consult its prospectus, consider its goals and strategies to see if they match your goals and values and also research any associated fees.

✓ Example 6.7.7: Investing in a Mutual Fund

Kaitlyn has analyzed her \$12,862.50 quarterly budget using the 50-30-20 budget philosophy, and sees she should be saving or paying down debt with \$2,572.50 per quarter. She decides to invest \$1,300 quarterly a mutual fund that reports an average return of 11.62% over the 18-year life of the mutual fund. Assuming that this interest rate continues, and is compounded quarterly, how much will her mutual fund account be worth after 5 years?

Answer

Kaitlyn's plan is an ordinary annuity, and so the future value of her account can be found using the formula FV =pmt×(1+r/n)n×t-1r/nFV=pmt×(1+r/n)n×t-1r/n, with a payment of \$1,300, a rate of 0.1162, number of compounding periods 4, after 5 years. Substituting these values into the formula and calculating, we find

$$\begin{aligned} FV &= pmt \times \frac{(1+r/n)^{n \times t} - 1}{r/n} \\ &= \$1{,}300 \times \frac{(1.02905)^{20} - 1}{0.02905} \\ &= \$1{,}300 \times \frac{0.773084935178}{0.032905} \\ &= \$1{,}300 \times 26.6122189784 \\ &= \$34{,}595.88 \end{aligned}$$

Kaitlyn's mutual fund will be worth \$34,595.88 after 5 years.

Your Turn 6.7.7

Aidan decides to invest \$3,200 annually in a mutual fund. He expects the fund to have a 10.8% interest rate compounded annually. How much will Aidan's mutual fund account have after 15 years?

✓ Example 6.7.8: Investing in a Mutual Fund to Reach a Goal

Kaitlyn wants to retire with \$1,500,000 in her mutual fund account. She will invest for 35 years. The mutual fund reports an average return of 11.62% over the 18-year-long life of the mutual fund. Assuming that this interest rate continues, and is compounded quarterly, how much will she need to pay annually into her mutual fund to reach her goal?

Answer

Kaitlyn's plan is an ordinary annuity, and so the payment to reach her goal can be found using the formula pmt=FV×(r/n)(1+r/n)n×t-1pmt=FV×(r/n)(1+r/n)n×t-1, with a semanticsFVFV, or goal, of \$1,500,000, a rate of 0.1162, for 35 years. Substituting these values into the formula and calculating, we find

$$\begin{aligned} pmt &= \frac{FV \times (r/n)}{(1+r/n)^{n \times t} - 1} \\ &= \frac{\$1{,}500{,}000 \times (0.1162/1)}{(1+0.1162/1)^{1 \times 35} - 1} \\ &= \frac{\$174{,}300}{26.0558103113} \\ &= \$6{,}689.49 \end{aligned}$$

Kaitlyn needs to invest \$6,689.49 per year (or \$557.46 per month) into the mutual fund to reach \$1,500,000 in 35 years.

Your Turn 6.7.8

How much does Aidan need to invest annually in his mutual fund to reach a goal of \$1,000,000 in 40 years. He expects the fund to have a 10.8% interest rate compounded annually.

Return on Investment

As in Methods of Savings, the formula for return on investment is $\mathrm{ROI} =$ mfracPPROI=FV-PP. As indicated before, this formula does not take into account how long the investment took to reach its current value. It depends only on the initial value, P, and the value at the end of the investment, semanticsFVFV.

Example $6.7.9$: Return on Investment for a Bond

Recall Example 6.62, in which Muriel purchased a \$3,000 bond with a maturity of 4 years at a fixed coupon rate of 5.5% paid annually. What was Muriel’s return on investment?

Answer

Each year, Muriel received \$165. She received this money four times, so earned a total of \$660. This represents semanticsFVFV – *P*, or just the earnings. Using that we find that the ROI is $\text{ROI} = \frac{660}{3000} = 0.22$, or 22%.

Your Turn $6.7.9$

Maureen invests \$5,000 in a bond with a maturity date in 5 years at a fixed coupon rate of 4.75%. What is Maureen’s return on investment?

As mentioned, the ROI does not address the length of time of the investment. A good way to do that is to equate the ROI to an account bearing interest that is compounded annually.

The annual return is the average annual rate, or the annual percentage yield (APY) that would result in the same amount were the interest paid once a year.

FORMULA

The formula for annual return is $\text{annual return} =$ msupFVP)(1t)-1annual return=(FVP)(1t)-1, where t = the number of years, semantics FVFV = new value, and *P* = starting principal.

We apply this to the previous example.

Example $6.7.10$: Annual Return on Investment for a Bond

Recall Example 6.70, in which Muriel purchased a \$3,000 bond with a maturity of 4 years at a fixed coupon rate of 5.5% paid annually. What was Muriel’s annual return on investment? Interpret this as compound interest.

Answer

Muriel earned a total of \$660. This represents semanticsFVFV – *P*, or just the earnings. The starting principal was \$3,000. The value at the end of 4 years was \$3,000 + \$660 = \$3,660. The time of the investment was 4 years. Using that we find that the annual return is $\text{annual return} =$ msupFVP)(1t)-1=(\$3,660\$3,000)(14)-1=1.2214-1=1.050969=0.050969annual return=(FVP)(1t)-1=(\$3,660\$3,000)(14)-1=1.2214-1=1.050969=0.050969, or 5.10%. The 5.5% bond earned the equivalent of 5.10% compounded annually.

Your Turn $6.7.10$

Maureen invests \$5,000 in a bond with a maturity date in 5 years at a fixed coupon rate of 4.75%. What is Maureen’s annual return on investment?

In Example 6.71 and Your Turn, the annual return was lower than the interest rate of the investment. This is because the interest from a bond is simple interest, but annual yield equates to compounded annually.

Example $6.7.11$: Return on Investment for Stock in Company ABC

Haniah buys stock in the ABC company, investing a total of \$13,000. After 20 years, the stock is worth \$132,293.49, including reinvestment of dividends.

1. What is Haniah’s return on investment?
2. What is Haniah’s annual return?

Answer

1. To calculate Hanniah's return on investment, substitute $13,000 for P and $132,293.49 for semanticsFVFV in the formula ROI $=$ mfracPPROI=FV-PP and calculate. Doing so we find Haniah's return on investment to be $\text{ROI} = \frac{FV-P}{P} = \frac{\$132{,}293.49-\$13{,}000}{\$13{,}000} = 9.176422$, or 917.64%
2. To calculate Haniah's annual return, substitute $13,000 for P and $132,293.49 for semanticsFVFV in the formula $\text{annual return} =$ msupFVP)(1t)-1annual return=(FVP)(1t)-1 and calculate. Doing so we find her annual return to be $\text{annual return} =$ msupFVP)(1t)-1=($132,293.49$13,000)(120)-1=0.1230annual return=(FVP)(1t)-1=($132,293.49$13,000)(120)-1=0.1230, or 12.3%

Your Turn 6.7.11

Rixie deposits $23,000 in the stock of DEF company. After 12 years, her stock is worth $108,501.30.

What is Rixie's return on investment?

What is Rixie's annual return?

You should see that the annual return is equal to the annual compounded interest that was assumed for the stocks.

Compute Payment to Reach a Financial Goal

As in Methods of Savings, determining the payment necessary to reach a financial goal uses the payment formula for an ordinary annuity, pmt=FV×(r/n)(1+r/n)n×t-1pmt=FV×(r/n)(1+r/n)n×t-1. If dealing with mutual funds or stocks, an assumed annual interest rate, compounded, will be used. This value is often determined through research and informed speculation.

Example 6.7.12

Richard is saving for new siding for his home. He and his partner believe they will need $37,500 in 10 years to pay for the siding. How much should they invest yearly in a mutual fund they believe will have an annual interest rate of 12%, compounded annually, in order to reach their goal?

Answer

The necessary annual payment is found using the function pmt=FV×(r/n)(1+r/n)n×t-1pmt=FV×(r/n)(1+r/n)n×t-1 with semanticsFVFV = 37,500, r = 0.12, and n = 1. Substituting and calculating, we find the annual payment should be

$$\begin{aligned} pmt &= \frac{FV \times (r/n)}{(1+r/n)^{n \times t} - 1} \\ &= \frac{\$37{,}500 \times (0.12/1)}{(1+0.12/1)^{1 \times 10} - 1} \\ &= \frac{\$4{,}500}{(1.12)^{10} - 1} \\ &= \frac{\$4{,}500}{2.10584820834} \\ &= \$2{,}136.91 \end{aligned}$$

Your Turn 6.7.12

Pete and Erin want to save for their child's college. They think they will need $90,000 in 14 years. How much should they invest annually in a mutual fund they believe will yield 9.5% compounded annually?

Retirement Savings Plans

We close this section by investigating the three main forms of retirement savings accounts: traditional individual retirement accounts (IRAs), Roth IRAs, and 401(k) accounts. Each has distinct characteristics that are suited to different investors' needs.

Individual Retirement Accounts

A **traditional IRA** lets you contribute up to an amount set by the government, which may change from year to year. For example, the maximum contribution for 2022 is $6,000; $7,000 over age 50. Anyone is eligible to contribute to a traditional IRA, regardless of your income level. Your money grows tax-deferred, but withdrawals after age 59½ are taxed at current rates. Traditional IRAs also allow you to use the contribution itself as a deduction on a current year tax return.

Roth IRAs allow contributions at the same levels as traditional IRAs, with a maximum $6,000 for 2022; $7,000 over age 50. However, to be eligible to make contributions, your earned income must be below a certain level. A Roth IRA allows after-tax contributions. In other words, the contribution itself is not tax-deductible, as it is with the traditional IRA. However, your money grows tax-free. If you make no withdrawals until you are age 59½, there are no penalties. IRAs pay a modest interest rate.

In either case, IRA deposits have to be from earned income, which in effect means if your earned income is over $6,000 ($7,000) then you can deposit the maximum.

✓ Example 6.7.13: Comparing Roth IRAs to Traditional IRAs

Which type of IRA, Roth or traditional, has an income limit for its use?

Answer

Roth IRAs require income to be below a certain limit.

Your Turn 6.7.13

Which type of IRA, Roth or traditional, allow deposits before tax, but have earnings that are taxed after the age of 59 ½?

Who Knew?

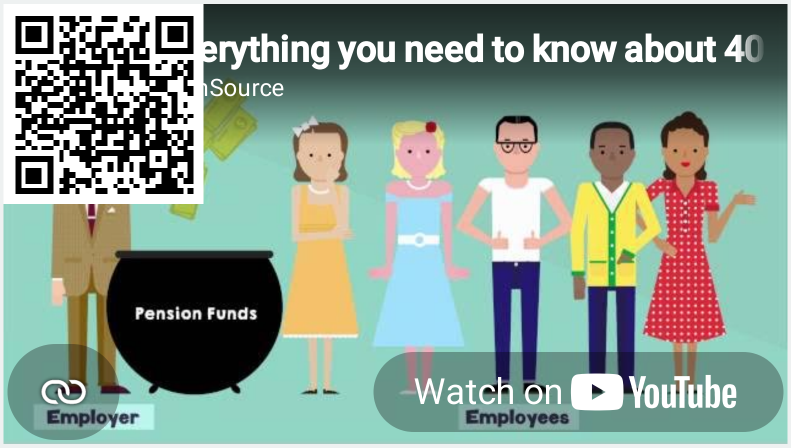
In 2022, the maximum that can be added to a Roth IRA was $6,000 for those under 50 years of age. For those over 50 years of age, the maximum that can be added to a Roth IRA is $7,000. However, to qualify for a Roth IRA in fall of 2022, a single person’s modified adjusted gross income (MAGI) must be below $129,000. Then, if a single person’s income is between $129,000 and $144,000, the maximum contribution is reduced from the limit for incomes below $129,000. For a married couples filing a joint tax return those values are $204,000 to $214,000.

401(k) Accounts

Your employer may offer a retirement account to you. These are often in the form of a **401(k)** account. There are traditional and Roth 401(k) accounts, which differ in how they are taxed, much as with other IRAs. In the traditional 401(k) plans, the money is deposited before tax is assessed, which means you do not pay taxes on this money. However, that means when money is withdrawn, it is taxed. These accounts are similar to mutual funds, in that the money is invested in a wide range of assets, spreading the risk.

One of the perks some employers offer is to match some amount of your contributions to the 401(k) plan. For instance, they may match your deposits up to 5% of your income. This is an instant 100% return on the money that was matched.

Video



401(k) Accounts

Example 6.7.14: Matching 401(k) Deposit

Alice signs up for her employer-based 401(k). The employer matches any 401(k) contribution up to 6% of the employee salary. Alice's annual salary is $51,600.

1. What is the most money that Alice can deposit that will be fully matched by the company?
2. How much total will be deposited into Alice's account if she deposits the full 6%?
3. How much return does Alice earn if she deposits exactly 6% in her 401(k)?

Answer

1. The employer will match up to 6% of any employee's salary. 6% of Alice's salary is $0.06 \times \$51{,}600 = \$3{,}096$. So Alice can deposit up to $3,096 and receive that amount in matching funds in her account.
2. Alice's contribution plus the company's contribution is $\$3{,}096 + \$3{,}096 = \$6{,}192$, which is the total that is deposited into Alice's account.
3. She earns a 100% return on the day she deposits her $3,096.

Your Turn 6.7.14

Jameis signs up for his employer-based 401(k). The employer matches any 401(k) contribution up to 7.5% of the employee salary. Jameis' annual salary is $72,800.

What is the most money that Jamie can deposit that will be fully matched by the company?

How much total will be deposited into Jameis' account if he deposits the full 7.5%?

How much return does Jameis earn if he deposits exactly 7.5% in her 401(k)?

401(k) plans with matching funds provide great value, as their rates of return are high compared to savings accounts, and are less risky that stocks since such funds invest across many investment vehicles. The next example demonstrates the power of constant deposits into a 401(k) plan that has some employer match.

Example 6.7.15: Constant Deposits into a 401(k) Plan

DeJean begins depositing $300 per month from his paycheck each month in his employer-based 401(k) account. The employer matches this deposit as it falls below their matching threshold. DeJean expects the return to average 10% per year, compounded annually.

1. How much will DeJean's account be worth if he keeps making those payments for 30 years?
2. What will his account be worth without the matching funds?

Answer

1. This is a form of an ordinary annuity, so the formula $FV =$pmt×(1+r/n)n×t-1r/nFV=pmt×(1+r/n)n×t-1r/n will be used. The company matches DeJean's full deposit, so each month $600 will be deposited. He is assuming the money will compound annually, so the amount deposited each year is needed as the value of pmt. For the year, he will deposit $12 \times \$600 = \$7{,}200$. The rate is 0.1, the number of compounding periods is 1, and the number of years is 30. Substituting and calculating, the value of DeJean's account after 30 years will be

   $$FV =$$

   pmt×(1+r/n)n×t-1r/n=$7,200×(1+0.1/1)1×30-10.1/1=$7,200×(1.1)30-10.1=$7,200×164.494022689=$1,184,356.96FV=pmt×(1+r/n)n×t-1r/n=$7,200×(1+0.1/1)1×30-10.1/1=$7,200×(1.1)30-10.1=$7,200×164.494022689=$1,184,356.96
2. This is a form of an ordinary annuity, so the formula $FV =$pmt×(1+r/n)n×t-1r/nFV=pmt×(1+r/n)n×t-1r/n will be used but the deposit is now only $300 per month without the matching funds. For the year, he will deposit $12 \times \$300 = \$3{,}600$. The rate is 0.1, the number of compounding periods is 1, and the number of years is 30. Substituting and calculating, the

value of DeJean’s account after 30 years will be

$$FV =$$

pmt×(1+r/n)n×t-1r/n=$3,600×(1+0.1/1)1×30-10.1/1=$3,600×(1.1)30-10.1=$3,600×164.494022689=$592,178.48FV=pmt×(1+r/n)n×t-1r/n=$3,600×(1+0.1/1)1×30-10.1/1=$3,600×(1.1)30-10.1=$3,600×164.494022689=$592,178.48

Your Turn 6.7.15

Crystal begins depositing $450 per month from her paycheck each month in her employer-based 401(k) account. The employer matches $350 of each deposit. Crystal expects the return to average 9% per year, compounded annually.

How much will Crystal’s account be worth if she keeps making those payments for 25 years?

What will her account be worth without the matching funds?

Check Your Understanding

1. Which investment has the highest risk?
2. What is different about who can use a Roth IRA versus a traditional IRA??
3. How do mutual funds reduce risk?
4. How much in earnings does a 10-year bond with issue price $5,000 that pays 4% interest annually is generated?
5. If an individual makes $115,000 annually, is the person eligible for a Roth IRA?
6. An employer 401(k) matches up to 4% of income by employees. Merisol’s annual salary is $87,500. If Merisol wants to deposit $4,400 annually in the 401(k), how much will be matched?
7. A person owns 300 shares of stock. It pays $.38 per share this quarter. How much does the share owner earn from the stock that quarter?
8. David purchased stocks for $3,500. The stocks paid dividends by reinvesting them in the stock. After 3 years, he sold the socks for $4,650. What was David’s annual return on that investment?
9. How much must be deposited annually in a mutual fund that is expected to bear 12.5% interest compounded annually if the account is to be worth $750,000 after 25 years?

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6.7.0: Exercises

Exercise 6.7.0.1

What is the maturity date for a bond?

Exercise 6.7.0.2

What is the issue price for a bond?

Exercise 6.7.0.3

Stock investments increase in value in what two ways?

Exercise 6.7.0.4

Which is the least risky of stocks, bonds, mutual funds, CDs?

Exercise 6.7.0.5

Which type of individual retirement account allows pre-tax deposits?

Exercise 6.7.0.6

What are the limits on contributions to individual retirement accounts in 2022?

Exercise 6.7.0.7

Of bonds, stocks, mutual funds, CDs, and money market accounts, which do not allow for withdrawal until a certain time period has passed?

Exercise 6.7.0.8

Which type of IRA allows the account to grow tax free, provided no withdrawals are made until after the age of $59\frac{1}{2}$?

Exercise 6.7.0.9

Which of bonds, mutual funds, and CDs are professionally managed?

Exercise 6.7.0.10

Why do mutual funds and IRAs have relatively low risk?

For the bonds with the given properties, find a. the amount paid each year and b. the total amount earned with the bond.

Exercise 6.7.0.11

Issue price of $10,000 pays 3.5% annually, matures in 5 years.

Exercise 6.7.0.12

Issue price of $3,400 pays 2.75% annually, matures in 10 years.

? Exercise 6.7.0.13

Issue price of \$1,000 pays 2.8% annually, matures in 5 years.

? Exercise 6.7.0.14

Issue price of \$5,000 pays 3.75% annually, matures in 15 years.

In the following exercises, find: a. the return on investment and b. the annual return for the bond described. Round to two decimal places.

? Exercise 6.7.0.15

Issue price of \$10,000 pays 3.5% annually, matures in 5 years.

? Exercise 6.7.0.16

Issue price of \$3,400 pays 2.75% annually, matures in 10 years.

? Exercise 6.7.0.17

Issue price of \$1,000 pays 2.8% annually, matures in 5 years.

? Exercise 6.7.0.18

Issue price of \$5,000 pays 3.75% annually, matures in 15 years.

? Exercise 6.7.0.19

40 shares of stock are owned. The dividend per share is \$0.38 for a quarter. How much was earned in dividends on this stock this quarter?

? Exercise 6.7.0.20

150 shares of stock are owned. The dividend per share is \$0.78 for a quarter. How much was earned in dividends on this stock this quarter?

? Exercise 6.7.0.21

100 shares of stock are owned. The dividend per share is \$0.18 for a quarter. How much was earned in dividends on this stock this quarter?

? Exercise 6.7.0.22

250 shares of stock are owned. The dividend per share is \$0.41 for a quarter. How much was earned in dividends on this stock this quarter?

For the following exercises, 70 shares of stock were purchased for \$31.50 per share 5 years ago. Over the 5 years, the total of all dividends earned from this stock was \$6.34 per share. The stock is sold for \$34.83.

Exercise 6.7.0.23

How much was earned with dividends and share price increase combined?

Exercise 6.7.0.24

What was the return on investment for these stocks?

Exercise 6.7.0.25

What was the annual return for these stocks?

For the following exercises, 10 shares of stock were purchased for $18.91 per share 3 years ago. Over the 3 years, the total of all dividends earned from this stock was $3.18 per share. The stock is sold for $22.01.

Exercise 6.7.0.26

How much was earned with dividends and share price increase combined?

Exercise 6.7.0.27

What was the return on investment for these stocks?

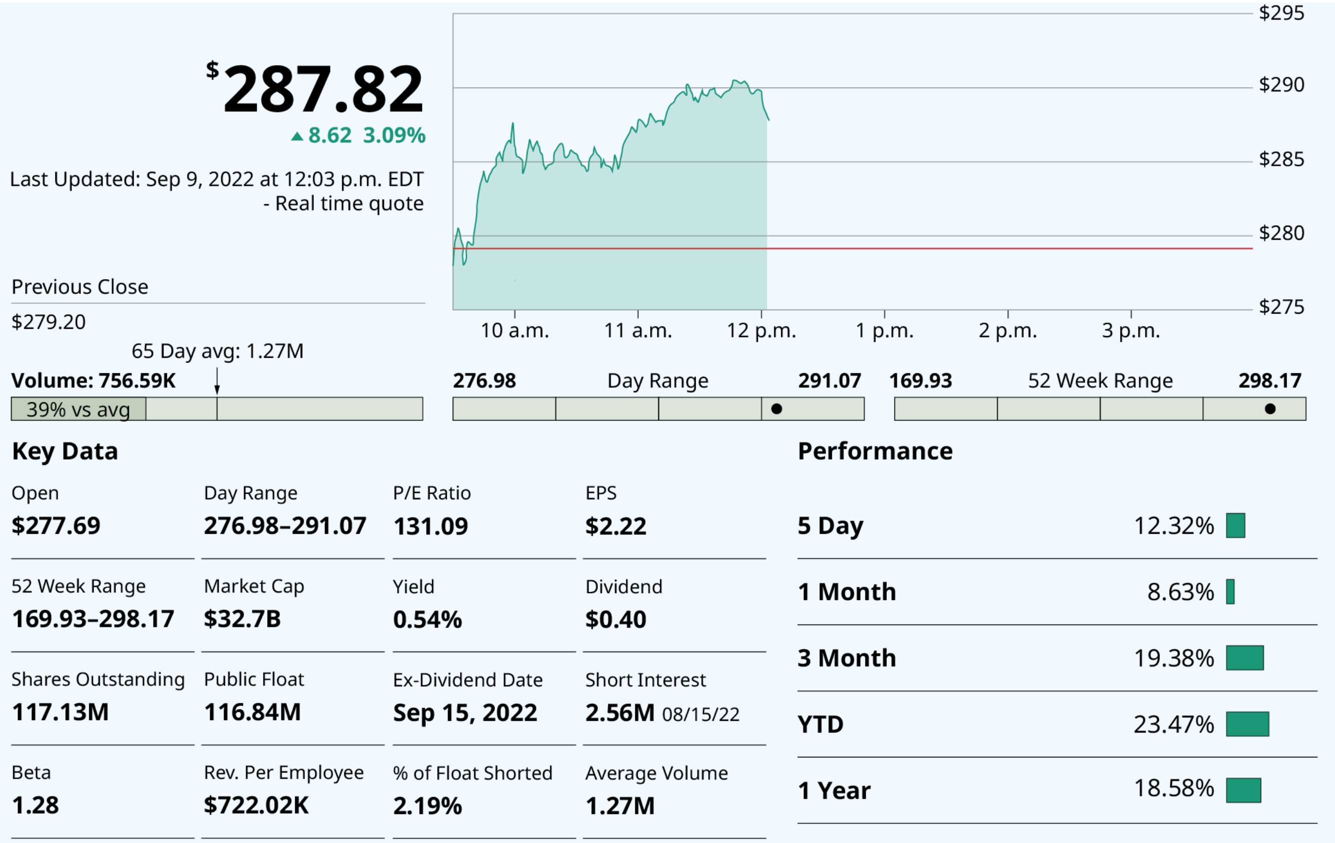
Exercise 6.7.0.28

What was the annual return for these stocks?

For the following exercises, use the given stock table to answer the following:

a. What was the 52-week low?
b. What was the dividend?
c. What is its year-to-date performance?
d. What is its yield?

Exercise 6.7.0.29



$287.82

▲8.62 3.09%

Last Updated: Sep 9, 2022 at 12:03 p.m. EDT - Real time quote

Previous Close

$279.20

65 Day avg: 1.27M

Volume: 756.59K

39% vs avg

276.98 Day Range **291.07**

169.93 52 Week Range **298.17**

Key Data

Open **$277.69**	Day Range **276.98–291.07**	P/E Ratio **131.09**	EPS **$2.22**
52 Week Range **169.93–298.17**	Market Cap **$32.7B**	Yield **0.54%**	Dividend **$0.40**
Shares Outstanding **117.13M**	Public Float **116.84M**	Ex-Dividend Date **Sep 15, 2022**	Short Interest **2.56M** 08/15/22
Beta **1.28**	Rev. Per Employee **$722.02K**	% of Float Shorted **2.19%**	Average Volume **1.27M**

Performance

5 Day	12.32%
1 Month	8.63%
3 Month	19.38%
YTD	23.47%
1 Year	18.58%

Stock table (data source: marketwatch.com)

Stock table (data source: marketwatch.com)

In the following exercises, find the future value of the mutual fund or IRA with the given annual deposit, the duration of the investment, and the assumed annual, compounded, percentage rate.

? Exercise 6.7.0.31

IRA, annual deposit = $6,000, 25 years, assumed percentage rate of 11.3%

? Exercise 6.7.0.32

IRA, annual deposit = $4,800, 15 years, assumed percentage rate of 9.7%

? Exercise 6.7.0.33

Mutual fund, annual deposit = $7,500, 35 years, assumed percentage rate of 10%

? Exercise 6.7.0.34

Mutual find, annual deposit = $12,000, 40 years, assumed percentage rate of 9%

In the following exercises, find the annual deposit necessary, into an IRA or mutual fund, to reach the stated financial goal, given the assumed annual, compounded, interest rate, and the duration of the deposits. Convert that annual deposit to a monthly amount.

Exercise 6.7.0.35

Mutual fund, goal = $1,000,000, 25 years, assumed percentage rate of 11.3%

Exercise 6.7.0.36

Mutual fund, goal = $100,000, 15 years, assumed percentage rate of 9.7%

Exercise 6.7.0.37

IRA, goal = $750,000, 35 years, assumed percentage rate of 10%

Exercise 6.7.0.38

IRA, goal = $1,750,000, 40 years, assumed percentage rate of 9%

Exercise 6.7.0.39

Francis's employer matches 401(k) contributions up to 7% of annual salary. She makes $98,500. What is the maximum amount that the company will match for Francis?

Exercise 6.7.0.40

Miles' employer matches 401(k) contributions up to 4% of annual salary. He makes $38,500. What is the maximum amount that the company will match for Miles?

Exercise 6.7.0.41

Ila's employer matches 401(k) contributions, up to 4% of salary. She makes $49,000. She wants to deposit $3,000 of her own earnings per year ($250 per month). How much, including her employer's matching funds, will be deposited into Ila's account each year?

Exercise 6.7.0.42

Georgia works at a company that matches 401(k) contributions up to 5.5% of salary. Georgia wants to deposit $6,000 of her own earnings per year ($500 per month) in her 401(k). If she makes $65,000 annually, how much, including employer contributions, will be deposited in Georgia's account annually?

In the following exercises, Cheryl's company offers a 401(k) account to all employees. The company will match employee contributions up to 6% of the employee's salary. She earns $81,000 per year. Cheryl decides to deposit, or contribute, $10,000 annually in the 401(k). She expects a return of 9.5% per year.

Exercise 6.7.0.43

How much will the company match?

Exercise 6.7.0.44

What is her total contribution per year?

? Exercise 6.7.0.45

If she deposits into the account for 25 years, how much will her 401(k) be worth?

In the following exercises, Lavanya's company offers a 401(k) account to all employees. Her annual salary is $58,000. The company will match employee contributions up to 7% of the employee's salary. Lavanya decides to deposit $3,200 annually in the 401(k). She expects a return of 8.5% per year.

? Exercise 6.7.0.46

How much will the company match?

? Exercise 6.7.0.47

What is her total contribution per year?

? Exercise 6.7.0.48

If she deposits into the account for 20 years, how much will her 401(k) be worth?

In the following exercises, Ruslana wants to use her 401(k) to save $1,400,000 when she retires in 36 years. She assumes the plan will yield 11% compounded annually. Her company will match contributions up to 5% of annual salary. Her salary is $48,000.

? Exercise 6.7.0.49

How much total will need to be added to her account each year to reach her goal?

? Exercise 6.7.0.50

What is 5% of Ruslana's salary?

? Exercise 6.7.0.51

The company will match up to 5% of Ruslana's salary, which means half the payment (up to 5% of salary) will be contributed by the company. How much is half the necessary payment?

? Exercise 6.7.0.52

Does the answer to Exercise 51 exceed the result from Exercise 50?

? Exercise 6.7.0.53

If the answer to Exercise 52 is no, then the company contributes half the deposit in the 401(k). How much will the company contribute annually to the 401(k) if this is the case?

? Exercise 6.7.0.54

If the answer to Exercise 52 is yes, then the company contributes only the 5% match of Ruslana's salary. In this case, how much does the company contribute?

? Exercise 6.7.0.55

How much will Ruslana need to contribute per year, after the employer contribution?

? Exercise 6.7.0.56

Divide the answer to Exercise 55 by 12 to find the monthly contribution Ruslana will make.

In the following exercises, Remy wants to use his 401(k) to save $1,750,000 when he retires in 30 years. He assumes the plan will yield 10.4% compounded annually. His company will match contributions up to 6% of annual salary. His salary is $78,000.

? Exercise 6.7.0.57

How much total will need to be added to his account each year to reach his goal?

? Exercise 6.7.0.58

What is 6% of Remy’s salary?

? Exercise 6.7.0.59

The company will match up to 6% of Remy’s salary, which means half the payment (up to 6% of salary) will be contributed by the company. How much is half the necessary payment?

? Exercise 6.7.0.60

Does the answer to Exercise 59 exceed the result from Exercise 58?

? Exercise 6.7.0.61

If the answer to Exercise 60 is no, then the company contributes half the deposit in the 401(k). How much will the company contribute annually to the 401(k) if this is the case?

? Exercise 6.7.0.62

If the answer to Exercise 60 is yes, then the company contributes only the 6% of Remy’s salary. In this case, how much does the company contribute?

? Exercise 6.7.0.63

How much will Remy need to contribute per year, after the employer contribution?

? Exercise 6.7.0.64

Divide the answer to Exercise 63 by 12 to find the monthly contribution Remy will make.

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6.8: The Basics of Loans

Figure $6.8.1$: Loans are contracts that allow people to buy now but require them to pay more. (credit: "Closing" by Tim Pierce/Flickr, CC BY 2.0)

Learning Objectives

1. Describe various reasons for loans.
2. Describe the terminology associated with loans.
3. Understand how credit scoring works.
4. Calculate the payment necessary to pay off a loan.
5. Read an amortization table.
6. Determine the cost to finance for a loan.

New car envy is real. Some people look at a new car and feel that they too should have a new car. The search begins. They find the model they want, in the color they want, with the features they want, and then they look at the price. That's often the point where the new car fever breaks and the reality of borrowing money to purchase the car enters the picture. This borrowing takes the form of a loan.

In this section, we look at the basics of loans, including terminology, credit scores, payments, and the cost of borrowing money.

Reasons for Loans

Even if you want a new car because you need one, or if you need a new computer since your current one no longer runs as fast or smoothly as you would like, or you need a new chimney because the one on your house is crumbling, it's likely you do not have that cost in cash. Those are very large purchases. How do you buy that if you don 't have the cash?
You borrow the money.

And for helping you with your purchase, the company or bank charges you interest.

Loans are taken out to pay for goods or services when a person does not have the cash to pay for the goods or services. We are most familiar with loans for the big purchases in our lives, such as cars, homes, and a college education. Loans are also taken out to pay for repairs, smaller purchases, and home goods like furniture and computers.

Loans can come from a bank, or from the company selling the goods or providing the service. The borrower agrees to pay back more than the amount borrowed. So there is a cost to borrowing that should be considered when deciding on a purchase bought with credit or a borrowed money.
Even using a credit card is a form of a loan.

Essentially, a loan can be obtained for just about any purchase, large or small, that has a cost beyond a person’s cash on hand.

The Terminology of Loans

There are many words and acronyms that get used in relation to loans. A few are below.

APR is the annual percentage rate. It is the annual interest paid on the money that was borrowed. The principal is the total amount of the loan, or that has been financed. A fixed interest rate loan has an interest rate that does not change during the life of the loan. A variable interest rate loan has an interest rate that may change during the life of the loan. The **term** of the loan is how long the borrower has to pay the loan back. An installment loan is a loan with a fixed period, and the borrower pays a fixed amount per period until the loan is paid off. The periods are almost uniformly monthly. Loan amortization is the process used to calculate how much of each payment will be applied to principal and how much is applied to interest. Revolving credit, also known as open-end credit, is how most credit cards work but is also a kind of loan account. (We will learn about credit cards in Credit Cards) You can use up to some specified value, called the limit, any way you want, and as long as you pay the issuer of the credit according to their terms, you can keep borrowing from this account.

These and other terminologies can be researched further at Forbes.

Who Knew?: Credit Scores

Not everyone pays the same interest for the same loan. One person might get an APR of 2.9% while another pays 6.9%. These rates are based on your credit score.

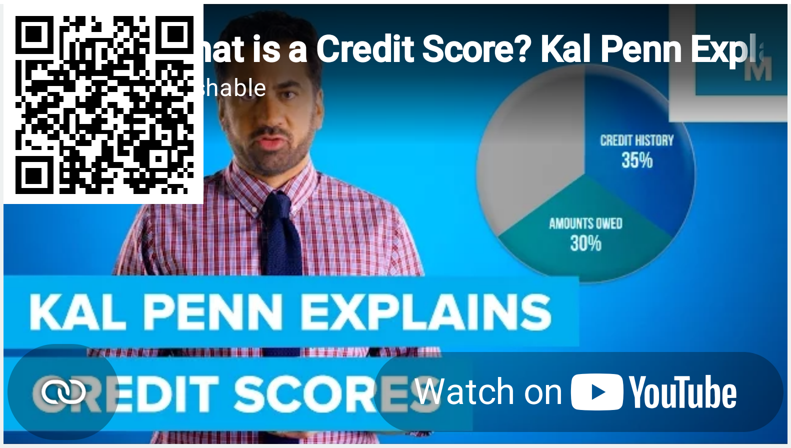
Data about you and your credit is collected by three credit bureaus—Experian, Equifax, and TransUnion. They calculate your score using one of two main models: FICO and Vantage Score. The score they develop is based on the following categories:

- **Payment History:** Making your payments on time and not missing payments is by far the most important factor. All three credit types—revolving, installment, and open—contribute to this factor.
- **Credit Utilization or Amount Owed:** How much do you owe on your credit card accounts? This category is concerned with the ratio of how much you owe on revolving credit accounts relative to your available credit, also known as your credit utilization ratio.
  This is the only category that depends solely on your revolving credit accounts.
- **Length of Credit History:** This is the average age of your credit history, including the age of the oldest and newest accounts. All three types of credit accounts play a role in this category.
- **Credit Mix:** This number represents the different types of credit accounts you have, such as credit cards, car loan, mortgages, and whether you are successful managing both revolving and installment accounts.
- **New Credit:** Have you recently opened a new account or applied for new credit? Lenders want to know how much new credit you are taking on. So, if you are planning to buy a car and make another large purchase with a credit card, you may want to space these purchases out.

If you have done well in these categories, your credit score will be high, and you will qualify for lower interest rates because you are not perceived as being a risky investment. However, if you do poorly in these categories, your score will be low and you will pay higher interest rates since you present a greater risk.

Check out this nerdwallet article on credit scores to learn more!

Video



Credit Scores Explained

Who Knew?: Where Do Interest Rates Come From?

There are many factors that impact your interest rate beyond your credit score. Banks have the authority to set their own rates, so competition between banks impacts interest rates. Bank A doesn't want to charge interest rates that are too high, since borrowers will find banks with better rates. Banks also don't want to charge too little interest.

The too little interest is more involved than the too high. The bank needs to make a profit on its loans. Deposits at the bank are used by the bank to generate loans. The bank has to pay those depositors interest. The bank must charge more for loans they give than they pay to people with deposits in the bank.

Banks also borrow money from each other. These loans have an interest rate, and once more, the bank making a loan must make a profit. More directly, they must charge more for loans they give than they pay for loans they take. In the United States, banks may also borrow from the Federal Reserve, which also charges an interest rate, which is also called the discount rate.

This is where a bank's prime rate comes from. A bank's prime rate is the interest rate it will give to its very best customers, which means most customers will pay more than the bank's prime rate. To confuse the issue, there is also the *Wall Street Journal*'s prime rate. It is the average of the prime rates charged by individual banks. The *Wall Street Journal* surveys several banks to generate this value.

Banks also increase the interest rate charged to customers based on both the credit risk presented by the customer, and the risk associate with what the loan will be used for.

Calculating Loan Payments

Loan payments are made up of two components. One component is the interest that accrued during the payment period. The other component is part of the principal. This should remind you of partial payments from Simple Interest.

Over the course of the loan, the amount of principal remaining to be paid decreases. The interest you pay in a month is based on the remaining principal, just as in the partial payments of Simple Interest.

FORMULA

The amount of interest, I, to be paid for one period of a loan with remaining principal P is $I =$, where r is the interest rate in decimal form and n is he number of payments in a year (most often $n = 12$). Since the interest is for the one period, the time is 1 and does not impact the calculation. Note, interest paid to lenders is always rounded up to the next penny.

✓ Example ✓ 6.8.1: Interest for a Monthly Payment of a Loan

Find the interest to be paid for the period on loans with the following remaining principal and given annual interest rate. Each period is a month.

1. Remaining principal is $13,450, interest rate is 6.75%

2. Remaining principal is $8,460, interest rate is 5.99%

Answer

1. Substituting $13,450 for the remaining principal P, 0.0675 for r, and n = 12 since the period is a month into the formula, we find that the interest to be paid this period is $I = P \times \frac{r}{n} = \$13,450 \times \frac{0.0675}{12} = \$75.66$.
2. Substituting $8,460 for the remaining principal P, 0.0599 for r, and n = 12 since the period is a month into the formula, we find that the interest to be paid this period is $I = P \times \frac{r}{n} = \$8,450 \times \frac{0.0599}{12} = \$42.18$.

Your Turn 6.8.1

Find the interest to be paid for the period on loans with the following remaining principal and given annual interest rate. Each period is a month.

Remaining principal is $56,945, interest rate is 7.5%

Remaining principal is $25,850, interest rate is 2.9%

The payment of the loan has to be such that the principal of the loan is paid off with the last payment. In any period, the amount of interest is defined by the formula above, but changes from period to period since the principal is decreasing with each payment. The trick is knowing how much principal should be paid each payment so that the loan is paid off at the stated time. Fortunately, that is found using the following formula.

FORMULA

The payment, pmt, per period to pay down a loan with beginning principal P is $pmt = \frac{P \times (r/n) \times (1+r/n)^{n \times t}}{(1+r/n)^{n \times t} -}$, where r is the annual interest rate in decimal form, t is the number of years of the loan, and n is the number of payments per year (typically, loans are paid monthly making n = 12).

Note, payment to lenders is always rounded up to the next penny.

Checkpoint

Often, the formula takes the form $pmt = \frac{P \times (r) \times (1+r)^{n}}{(1+r)^{n} -}$, *where* r *is the interest rate per period (annual rate divided by the number of periods per year), and* n *is the total number of payments to be made.*

✓ Example 6.8.2: Calculating the Payment for a Loan

In the following, calculate the payment necessary to pay off the loan with the given details. The payments are monthly.

1. A car loan taken out for $28,500 at an annual interest rate of 3.99% for 5 years.
2. A home loan taken out for $136,700 and an annual interest rate of 5.75% for 15 years.

Answer

1. The loan is for $28,500, which is the principal. The rate is 3.99%, so r = 0.0399. The term of the loan is 5 years, so t =5. Monthly payments means n = 12. Substituting these values for P, r, n, and t into the formula $pmt = \frac{P \times (r/n) \times (1+r/n)^{n \times t}}{(1+r/n)^{n \times t} -}$ and calculating, we find the payment for the loan.

$$\begin{aligned} pmt &= \frac{P \times (r/n) \times (1+r/n)^{n \times t}}{(1+r/n)^{n \times t} - 1} \\ &= \frac{\$28{,}500 \times (0.0399/12) \times (1+0.399/12)^{12 \times 5}}{(1+0.0399/12)^{12 \times 5} - 1} \\ &= \frac{\$28{,}500 \times (0.003325) \times (1.003325)^{60}}{(1.003325)^{60} - 1} \\ &= \frac{\$115.6470437}{0.220388273} \\ &= \$524.75 \end{aligned}$$

The monthly payment needed is $524.75.

2. The loan is for $136,000, which is the principal P. The rate is 5.75% so r = 0.0575. The term of the loan is 15 years, so n = 15. Monthly payments mean n =12. Substituting these values for P, r, n, and t into the formula $pmt = \frac{P \times (r/n) \times (1+r/n)^{n \times t}}{(1+r/n)^{n \times t} -}$ and calculating, we find the payment for the loan.

$$\begin{aligned} pmt &= \frac{P \times (r/n) \times (1+r/n)^{n \times t}}{(1+r/n)^{n \times t} - 1} \\ &= \frac{\$136{,}700 \times (0.0575/12) \times (1+0.0575/12)^{12 \times 15}}{(1+0.0575/12)^{12 \times 15} - 1} \\ &= \frac{\$28{,}500 \times (0.0047917) \times (1.0047917)^{180}}{(1.003325)^{180} - 1} \\ &= \frac{\$1{,}548.600986}{1.364201118} \\ &= \$1{,}135.18 \end{aligned}$$

The monthly payment needed is $1,135.18.

Your Turn 6.8.2

In the following, calculate the payment necessary to pay off the loan with the given details. The payments are monthly.

A home improvement loan taken out for $17,950 at an annual interest rate of 7.5% for 10 years.

A solar panel installation loan taken out for $33,760 and an annual interest rate of 4.3% for 20 years.

Tech Check: Using Google Sheets to Calculate Loan Payments

Google Sheets has a formula to calculate the monthly payment necessary to pay off a loan with a specified interest rate and term. The formula is the PMT formula. It uses the principal P, interest rate r, and t years. Follow these steps

Open a Google worksheet and click on any cell.

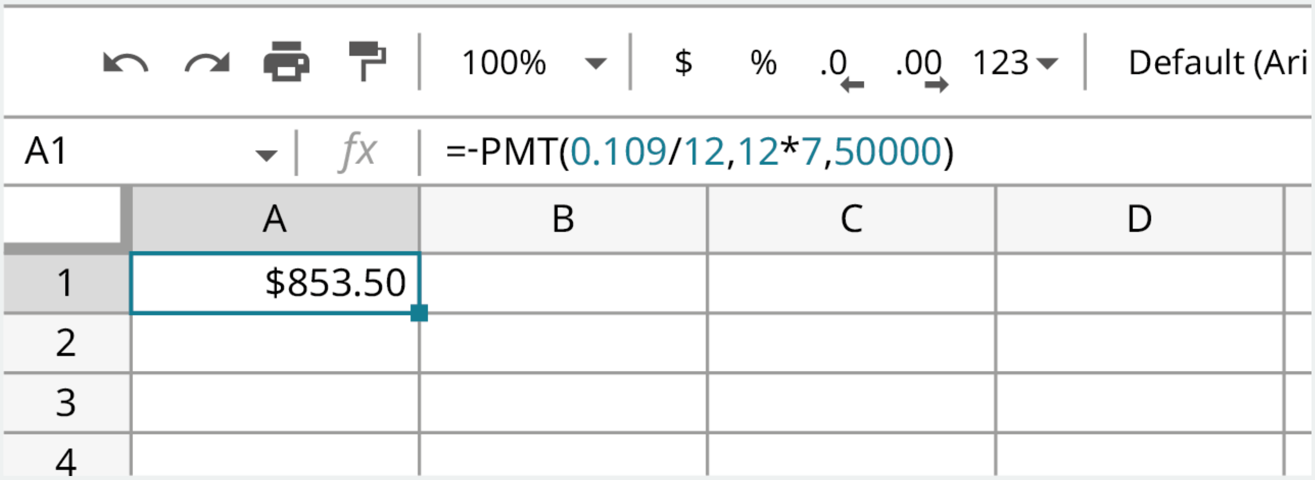
Type =-PMT(r/12,12*t,P).

Hit the enter key.

The cell displays the payment.

Please note the negative sign. Since Google Sheets is a spreadsheet program, is sees the payment as funds leaving the account, and so they are, by default, negative. The negative sign in front of the formula makes the result positive.

For example, for a $50,000 loan at 10.9% interest for 7 years, you would type =-PMT(0.109/12,12*7,50000). The formula and the result are shown in Figure 6.19.



=-PMT(0.109/12,12*7,50000)

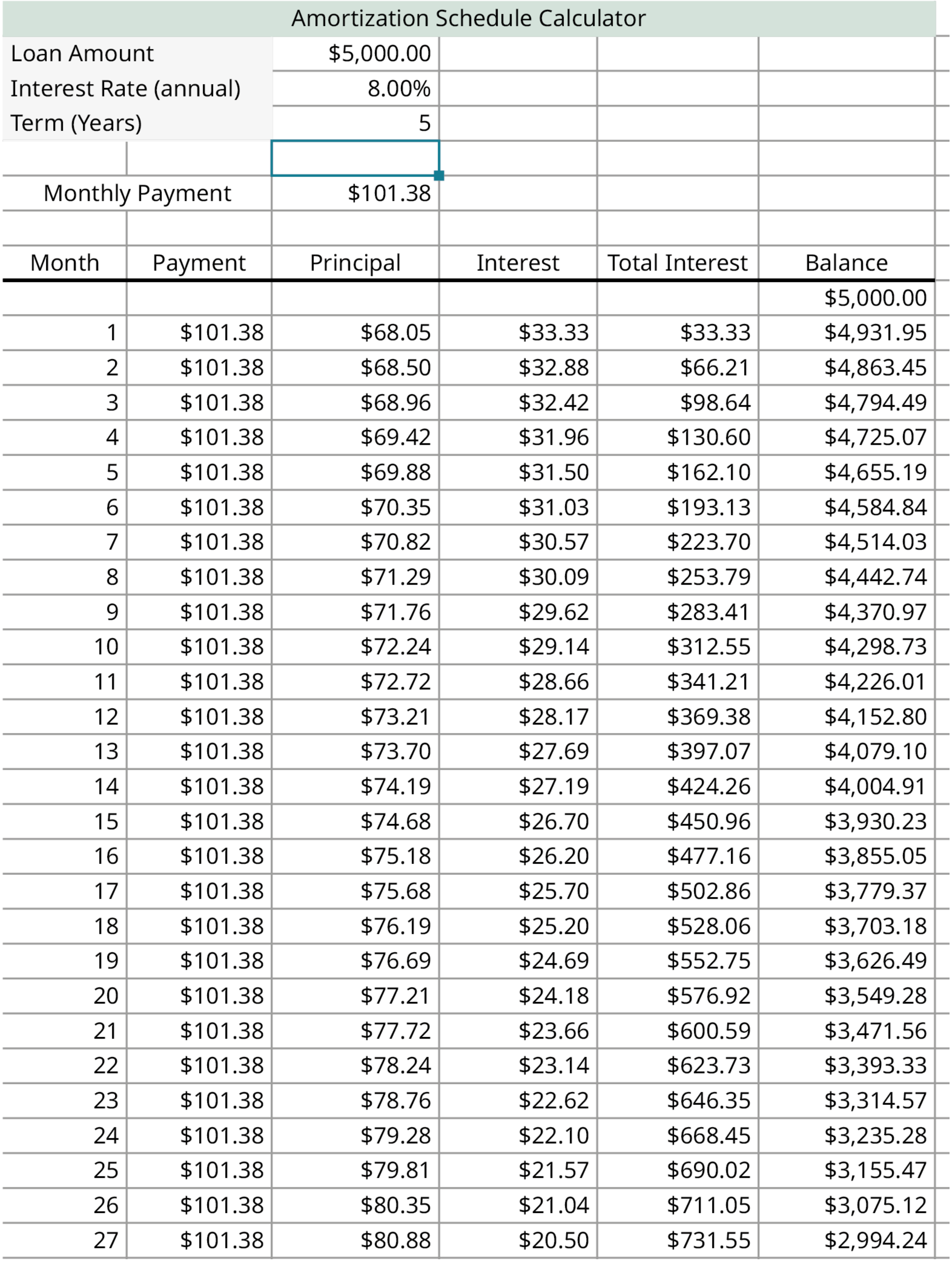
$853.50

Figure 6.8.2: Google Sheets formula

Alternatively, you can also use an online calculator to find monthly payments for a loan, such as the one at Caluculator.net

Reading Amortization Tables

An **amortization table** or amortization schedule is a table that provides the details of the periodic payments for a loan where the payments are applied to both the principal and the interest. The principal of the loan is paid down over the life of the loan. Typically, the payments each period are equal. Importantly, one of the columns will show how much of each payment is used for interest, another column shows how much is applied to the outstanding principal, and another column shows the remaining principal or balance Figure 6.20.



Amortization Schedule Calculator

Loan Amount	$5,000.00
Interest Rate (annual)	8.00%
Term (Years)	5
Monthly Payment	$101.38

Month	Payment	Principal	Interest	Total Interest	Balance
					$5,000.00
1	$101.38	$68.05	$33.33	$33.33	$4,931.95
2	$101.38	$68.50	$32.88	$66.21	$4,863.45
3	$101.38	$68.96	$32.42	$98.64	$4,794.49
4	$101.38	$69.42	$31.96	$130.60	$4,725.07
5	$101.38	$69.88	$31.50	$162.10	$4,655.19
6	$101.38	$70.35	$31.03	$193.13	$4,584.84
7	$101.38	$70.82	$30.57	$223.70	$4,514.03
8	$101.38	$71.29	$30.09	$253.79	$4,442.74
9	$101.38	$71.76	$29.62	$283.41	$4,370.97
10	$101.38	$72.24	$29.14	$312.55	$4,298.73
11	$101.38	$72.72	$28.66	$341.21	$4,226.01
12	$101.38	$73.21	$28.17	$369.38	$4,152.80
13	$101.38	$73.70	$27.69	$397.07	$4,079.10
14	$101.38	$74.19	$27.19	$424.26	$4,004.91
15	$101.38	$74.68	$26.70	$450.96	$3,930.23
16	$101.38	$75.18	$26.20	$477.16	$3,855.05
17	$101.38	$75.68	$25.70	$502.86	$3,779.37
18	$101.38	$76.19	$25.20	$528.06	$3,703.18
19	$101.38	$76.69	$24.69	$552.75	$3,626.49
20	$101.38	$77.21	$24.18	$576.92	$3,549.28
21	$101.38	$77.72	$23.66	$600.59	$3,471.56
22	$101.38	$78.24	$23.14	$623.73	$3,393.33
23	$101.38	$78.76	$22.62	$646.35	$3,314.57
24	$101.38	$79.28	$22.10	$668.45	$3,235.28
25	$101.38	$79.81	$21.57	$690.02	$3,155.47
26	$101.38	$80.35	$21.04	$711.05	$3,075.12
27	$101.38	$80.88	$20.50	$731.55	$2,994.24

Figure 6.8.3: Amortization table

Example 6.8.3: Reading from an Amortization Table

Using the partial amortization table (Figure 6.23), answer the following questions.

Amortization Schedule Calculator					
Loan Amount		$10,000.00			
Interest Rate (annual)		4.75%			
Term (Years)		20			
Monthly Payment		$64.62			
Month	Payment	Principal	Interest	Total Interest	Balance
					$10,000.00
1	$64.62	$25.04	$39.58	$39.58	$9,974.96
2	$64.62	$25.14	$39.48	$79.07	$9,949.82
3	$64.62	$25.24	$39.38	$118.45	$9,924.59
4	$64.62	$25.34	$39.28	$157.74	$9,899.25
5	$64.62	$25.44	$39.18	$196.92	$9,873.81
6	$64.62	$25.54	$39.08	$236.01	$9,848.27
7	$64.62	$25.64	$38.98	$274.99	$9,822.63
8	$64.62	$25.74	$38.88	$313.87	$9,796.89
9	$64.62	$25.84	$38.78	$352.65	$9,771.05
10	$64.62	$25.95	$38.68	$391.33	$9,745.10
11	$64.62	$26.05	$38.57	$429.90	$9,719.05
12	$64.62	$26.15	$38.47	$468.37	$9,692.90
13	$64.62	$26.25	$38.37	$506.74	$9,666.65
14	$64.62	$26.36	$38.26	$545.00	$9,640.29
15	$64.62	$26.46	$38.16	$583.16	$9,613.83
16	$64.62	$26.57	$38.05	$621.22	$9,587.26
17	$64.62	$26.67	$37.95	$659.17	$9,560.59
18	$64.62	$26.78	$37.84	$697.01	$9,533.81
19	$64.62	$26.88	$37.74	$734.75	$9,506.92
20	$64.62	$26.99	$37.63	$772.38	$9,479.93
21	$64.62	$27.10	$37.52	$809.91	$9,452.84
22	$64.62	$27.20	$37.42	$847.32	$9,425.63
23	$64.62	$27.31	$37.31	$884.63	$9,398.32
24	$64.62	$27.42	$37.20	$921.83	$9,370.90
25	$64.62	$27.53	$37.09	$958.93	$9,343.37
26	$64.62	$27.64	$36.98	$995.91	$9,315.73
27	$64.62	$27.75	$36.87	$1,032.79	$9,287.98

Figure 6. ✓ 6.8.4 Amortization table

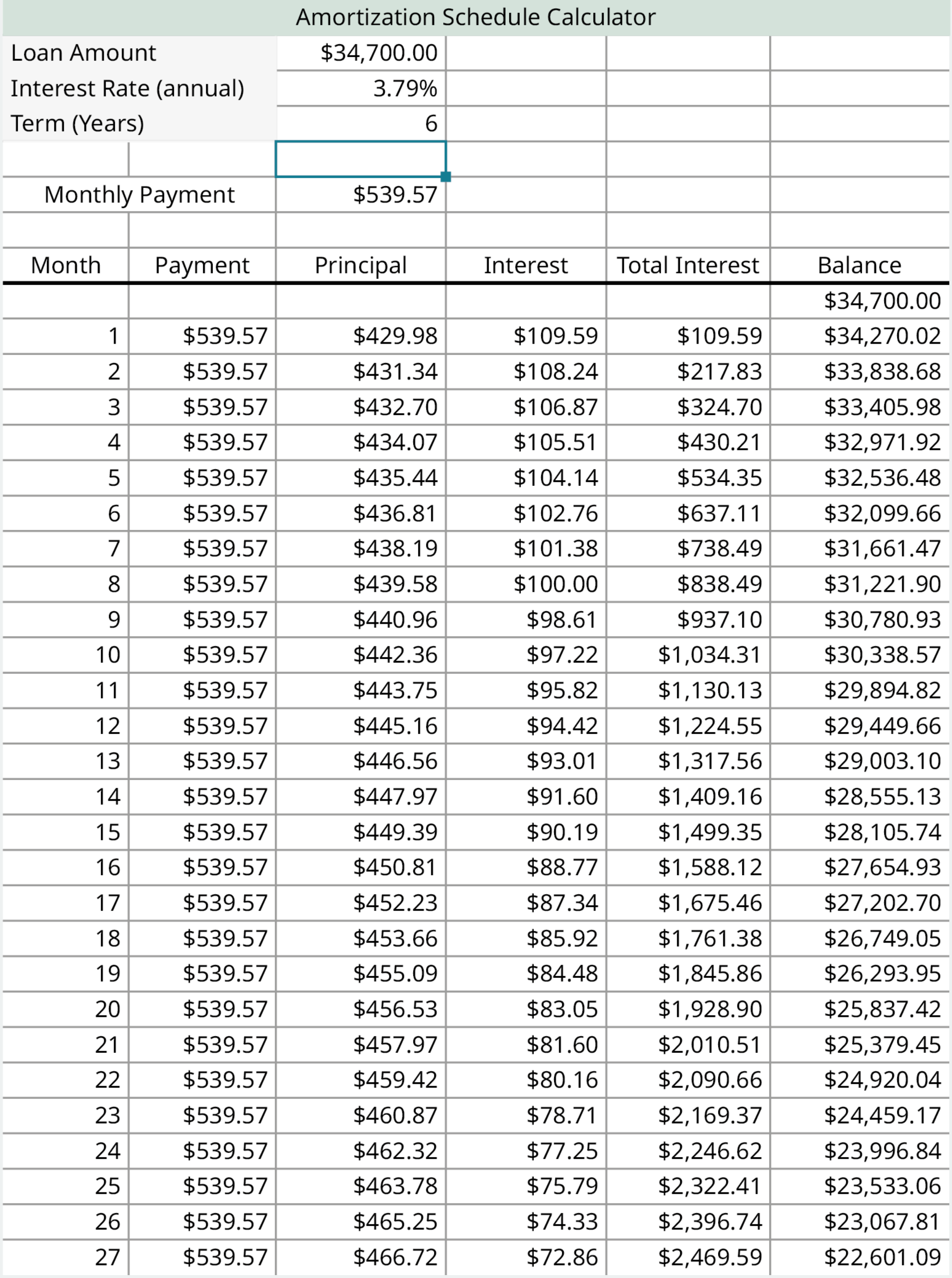
1. What is the loan amount (principal), the interest rate, and the term of the loan?
2. How much is the monthly payment?
3. How much remaining balance is there after the payment in month 15?
4. How much was the interest in payment 10?
5. What is the total of the interest paid after payment 18?
6. What happens to the amount paid in interest each month?

Answer

1. Reading the values at the top of the table, we see the principal is $10,000, the interest rate is 4.75%, and has a term of 20 years.
2. The monthly payment is listed below the term of the loan, and is $64.62.
3. $9,613.83
4. $38.68
5. $697.01
6. The amount paid to interest decreases each month.

Your Turn 6.8.3

Using the partial amortization table below, answer the following questions.

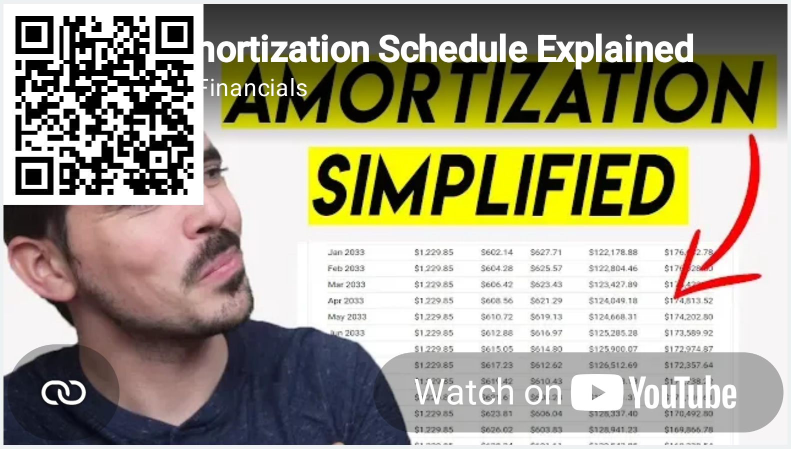


Amortization Schedule Calculator					
Loan Amount		$34,700.00			
Interest Rate (annual)		3.79%			
Term (Years)		6			
Monthly Payment		$539.57			
Month	Payment	Principal	Interest	Total Interest	Balance
					$34,700.00
1	$539.57	$429.98	$109.59	$109.59	$34,270.02
2	$539.57	$431.34	$108.24	$217.83	$33,838.68
3	$539.57	$432.70	$106.87	$324.70	$33,405.98
4	$539.57	$434.07	$105.51	$430.21	$32,971.92
5	$539.57	$435.44	$104.14	$534.35	$32,536.48
6	$539.57	$436.81	$102.76	$637.11	$32,099.66
7	$539.57	$438.19	$101.38	$738.49	$31,661.47
8	$539.57	$439.58	$100.00	$838.49	$31,221.90
9	$539.57	$440.96	$98.61	$937.10	$30,780.93
10	$539.57	$442.36	$97.22	$1,034.31	$30,338.57
11	$539.57	$443.75	$95.82	$1,130.13	$29,894.82
12	$539.57	$445.16	$94.42	$1,224.55	$29,449.66
13	$539.57	$446.56	$93.01	$1,317.56	$29,003.10
14	$539.57	$447.97	$91.60	$1,409.16	$28,555.13
15	$539.57	$449.39	$90.19	$1,499.35	$28,105.74
16	$539.57	$450.81	$88.77	$1,588.12	$27,654.93
17	$539.57	$452.23	$87.34	$1,675.46	$27,202.70
18	$539.57	$453.66	$85.92	$1,761.38	$26,749.05
19	$539.57	$455.09	$84.48	$1,845.86	$26,293.95
20	$539.57	$456.53	$83.05	$1,928.90	$25,837.42
21	$539.57	$457.97	$81.60	$2,010.51	$25,379.45
22	$539.57	$459.42	$80.16	$2,090.66	$24,920.04
23	$539.57	$460.87	$78.71	$2,169.37	$24,459.17
24	$539.57	$462.32	$77.25	$2,246.62	$23,996.84
25	$539.57	$463.78	$75.79	$2,322.41	$23,533.06
26	$539.57	$465.25	$74.33	$2,396.74	$23,067.81
27	$539.57	$466.72	$72.86	$2,469.59	$22,601.09

Figure 6.8.5: Amortization table

1. What is the loan amount (principal), the interest rate, and the term of the loan?
2. How much is the monthly payment?
3. How much remaining balance is there after the payment in month 20?
4. How much was the interest in payment 5?
5. What is the total of the interest paid after payment 24?
6. What happens to the amount paid to the principal each month?

Video



Reading an Amortization Table

Cost of Finance

There are often costs associated with a loan beyond the interest being paid. The **cost of finance** of a loan is the sum of all costs, fees, interest, and other charges paid over the life of the loan.

Example $6.8.4$: Cost of Financing a Personal Loan

Irena signed for a loan of $15,000 at 6.33% for 5 years. When she took out the loan, Irena paid a $750 origination fee. Over the course of the loan, she pays $2,537.96 in interest. What was her cost to finance the loan?

Answer

> The cost of finance is the sum is all interest and any fees paid for the loan. The fees paid were $750.00 and the interest was $2,537.96. Her cost of finance for this loan was $3,287.96.

Your Turn $6.8.4$

Samantha takes out a $28,000 loan for 6 years at 5.5% interest. When she took out the loan, she paid $4,937.18 in interest, an origination fee of $1,400 and an application fee of $100. What was her cost of finance for this loan?

Check Your Understanding

Give three reasons for taking out a loan.

What is an installment loan?

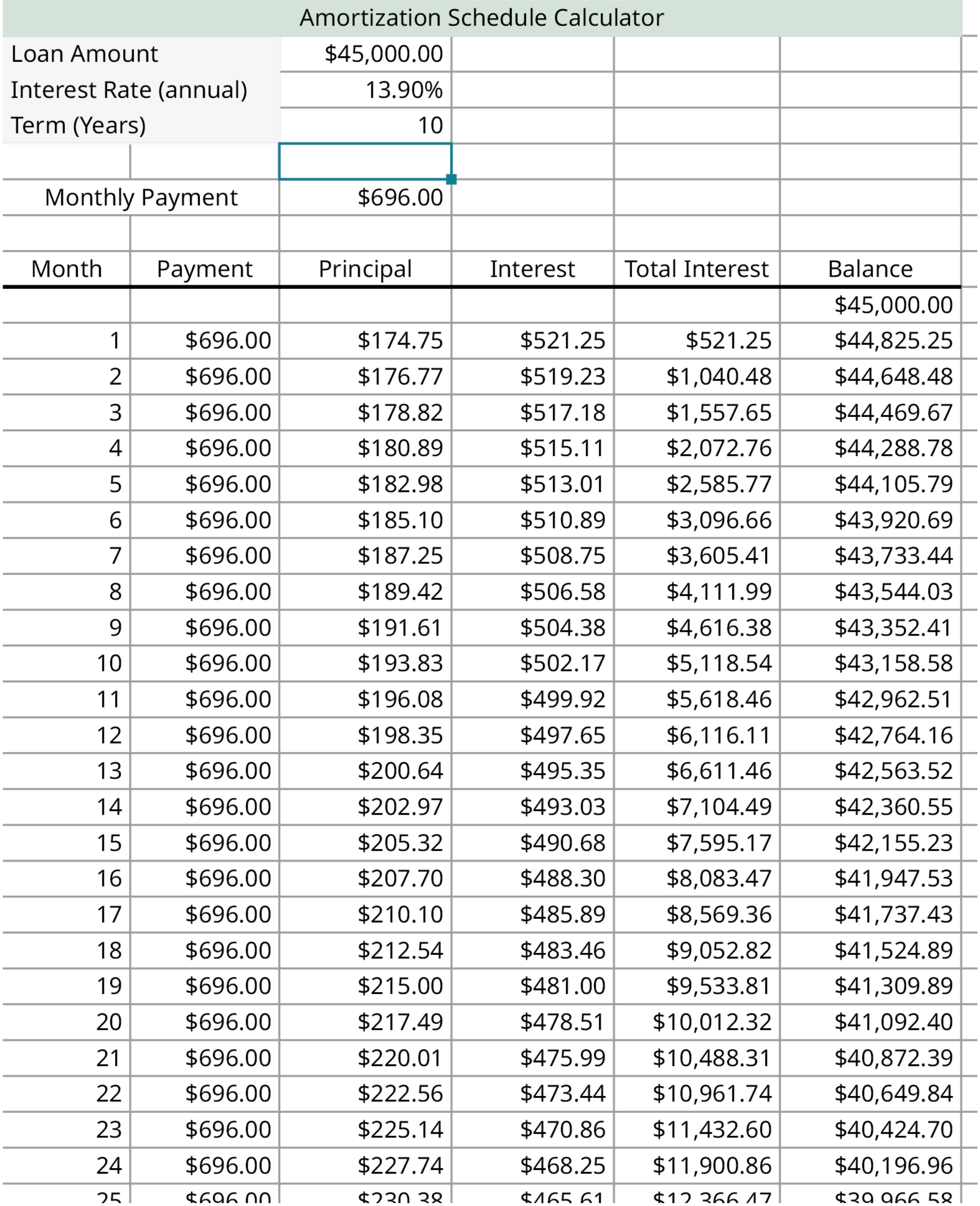
In an installment loan what happens to the amount of the payment that is applied to interest in later payments compared to earlier payments?

An installment loan is paid monthly. The remaining principal is $5,498. If the interest rate is 12.9%, how much interest is paid that month?

A business loan is taken out for $250,000 for 15 years at 6.9% interest. What are the payments for the loan?

Consider the amortization table below. What is the balance after the eleventh payment?

The total interest paid on a loan is $38,519.50. The origination fee was $300 while the processing fee was $250. An additional fee was charged to the borrowers, amounting to another $475. How much was the cost of financing for this loan?



Amortization Schedule Calculator					
Loan Amount		$45,000.00			
Interest Rate (annual)		13.90%			
Term (Years)		10			
Monthly Payment		$696.00			
Month	Payment	Principal	Interest	Total Interest	Balance
					$45,000.00
1	$696.00	$174.75	$521.25	$521.25	$44,825.25
2	$696.00	$176.77	$519.23	$1,040.48	$44,648.48
3	$696.00	$178.82	$517.18	$1,557.65	$44,469.67
4	$696.00	$180.89	$515.11	$2,072.76	$44,288.78
5	$696.00	$182.98	$513.01	$2,585.77	$44,105.79
6	$696.00	$185.10	$510.89	$3,096.66	$43,920.69
7	$696.00	$187.25	$508.75	$3,605.41	$43,733.44
8	$696.00	$189.42	$506.58	$4,111.99	$43,544.03
9	$696.00	$191.61	$504.38	$4,616.38	$43,352.41
10	$696.00	$193.83	$502.17	$5,118.54	$43,158.58
11	$696.00	$196.08	$499.92	$5,618.46	$42,962.51
12	$696.00	$198.35	$497.65	$6,116.11	$42,764.16
13	$696.00	$200.64	$495.35	$6,611.46	$42,563.52
14	$696.00	$202.97	$493.03	$7,104.49	$42,360.55
15	$696.00	$205.32	$490.68	$7,595.17	$42,155.23
16	$696.00	$207.70	$488.30	$8,083.47	$41,947.53
17	$696.00	$210.10	$485.89	$8,569.36	$41,737.43
18	$696.00	$212.54	$483.46	$9,052.82	$41,524.89
19	$696.00	$215.00	$481.00	$9,533.81	$41,309.89
20	$696.00	$217.49	$478.51	$10,012.32	$41,092.40
21	$696.00	$220.01	$475.99	$10,488.31	$40,872.39
22	$696.00	$222.56	$473.44	$10,961.74	$40,649.84
23	$696.00	$225.14	$470.86	$11,432.60	$40,424.70
24	$696.00	$227.74	$468.25	$11,900.86	$40,196.96
25	$696.00	$230.38	$465.61	$12,366.47	$39,966.58

26	$696.00	$233.05	$462.95	$12,829.42	$39,733.53
27	$696.00	$235.75	$460.25	$13,289.67	$39,497.78

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6.8.0: Exercises

? Exercise 6.8.0.1

How do fixed interest rates and variable interest rates differ?

? Exercise 6.8.0.2

What is revolving credit?

? Exercise 6.8.0.3

What is loan amortization?

? Exercise 6.8.0.4

What are the two components of a loan payment?

? Exercise 6.8.0.5

Name three details that are presented on an amortization schedule.

For the following exercises, calculate the interest due for the monthly installment payment given the remaining principal and interest rate.

? Exercise 6.8.0.6

Remaining principal = $21,872.99, interest rate = 13.9%

? Exercise 6.8.0.7

Remaining principal = $2,845.43, interest rate = 4.99%

? Exercise 6.8.0.8

Remaining principal = $78,913.76, interest rate = 2.9%

? Exercise 6.8.0.9

Remaining principal = $6,445.22, interest rate = 5.65%

In the following exercises, calculate the monthly payment for the loan.

? Exercise 6.8.0.10

Principal = $8,600, annual interest rate = 6.75%, term is 5 years

? Exercise 6.8.0.11

Principal = $19,400, annual interest rate = 2.25%, term is 6 years

Exercise 6.8.0.12

Principal = $11,870, annual interest rate = 3.59%, term is 3 years

Exercise 6.8.0.13

Principal = $41,900, annual interest rate = 8.99%, term is 15 years

Exercise 6.8.0.14

Principal = $26,150, annual interest rate = 11.1%, term is 7 years

Exercise 6.8.0.15

Principal = $46,350, annual interest rate = 2.9%, term is 6 years

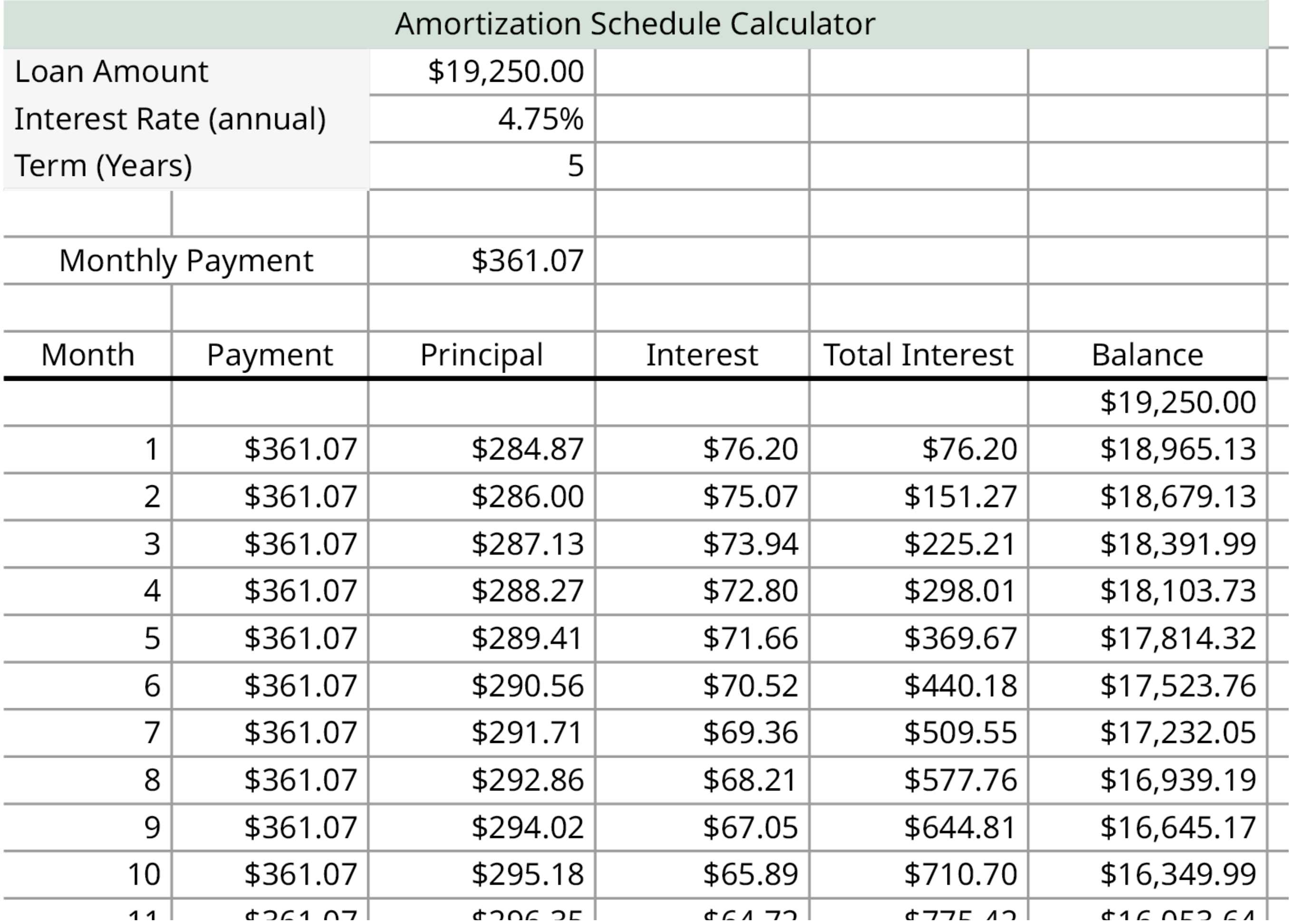
Exercise 6.8.0.16

Principal = $175,800, annual interest rate = 4.73%, term is 25 years

Exercise 6.8.0.17

Principal = $225,000, annual interest rate = 5.06%, term is 30 years

In the following exercises, use the amortization table below to answer the questions.



Amortization Schedule Calculator					
Loan Amount		$19,250.00			
Interest Rate (annual)		4.75%			
Term (Years)		5			
Monthly Payment		$361.07			
Month	Payment	Principal	Interest	Total Interest	Balance
					$19,250.00
1	$361.07	$284.87	$76.20	$76.20	$18,965.13
2	$361.07	$286.00	$75.07	$151.27	$18,679.13
3	$361.07	$287.13	$73.94	$225.21	$18,391.99
4	$361.07	$288.27	$72.80	$298.01	$18,103.73
5	$361.07	$289.41	$71.66	$369.67	$17,814.32
6	$361.07	$290.56	$70.52	$440.18	$17,523.76
7	$361.07	$291.71	$69.36	$509.55	$17,232.05
8	$361.07	$292.86	$68.21	$577.76	$16,939.19
9	$361.07	$294.02	$67.05	$644.81	$16,645.17
10	$361.07	$295.18	$65.89	$710.70	$16,349.99
11	$361.07	$296.35	$64.72	$775.42	$16,053.64

11	$361.07	$296.35	$64.72	$775.42	$16,053.64
12	$361.07	$297.52	$63.55	$838.96	$15,756.11
13	$361.07	$298.70	$62.37	$901.33	$15,457.41
14	$361.07	$299.88	$61.19	$962.51	$15,157.53
15	$361.07	$301.07	$60.00	$1,022.51	$14,856.46
16	$361.07	$302.26	$58.81	$1,081.32	$14,554.19
17	$361.07	$303.46	$57.61	$1,138.93	$14,250.73
18	$361.07	$304.66	$56.41	$1,195.34	$13,946.07
19	$361.07	$305.87	$55.20	$1,250.54	$13,640.20
20	$361.07	$307.08	$53.99	$1,304.54	$13,333.12
21	$361.07	$308.29	$52.78	$1,357.31	$13,024.83
22	$361.07	$309.51	$51.56	$1,408.87	$12,715.32
23	$361.07	$310.74	$50.33	$1,459.20	$12,404.58
24	$361.07	$311.97	$49.10	$1,508.30	$12,092.61
25	$361.07	$313.20	$47.87	$1,556.17	$11,779.40
26	$361.07	$314.44	$46.63	$1,602.80	$11,464.96
27	$361.07	$315.69	$45.38	$1,648.18	$11,149.27
28	$361.07	$316.94	$44.13	$1,692.31	$10,832.33

? Exercise 6.8.0.18

What was the loan amount, or starting principal?

? Exercise 6.8.0.19

What was the interest rate?

? Exercise 6.8.0.20

What is the term of the loan?

? Exercise 6.8.0.21

What are the monthly payments?

? Exercise 6.8.0.22

What is the remaining principal after payment 17?

? Exercise 6.8.0.23

How much of payment 27 was for interest?

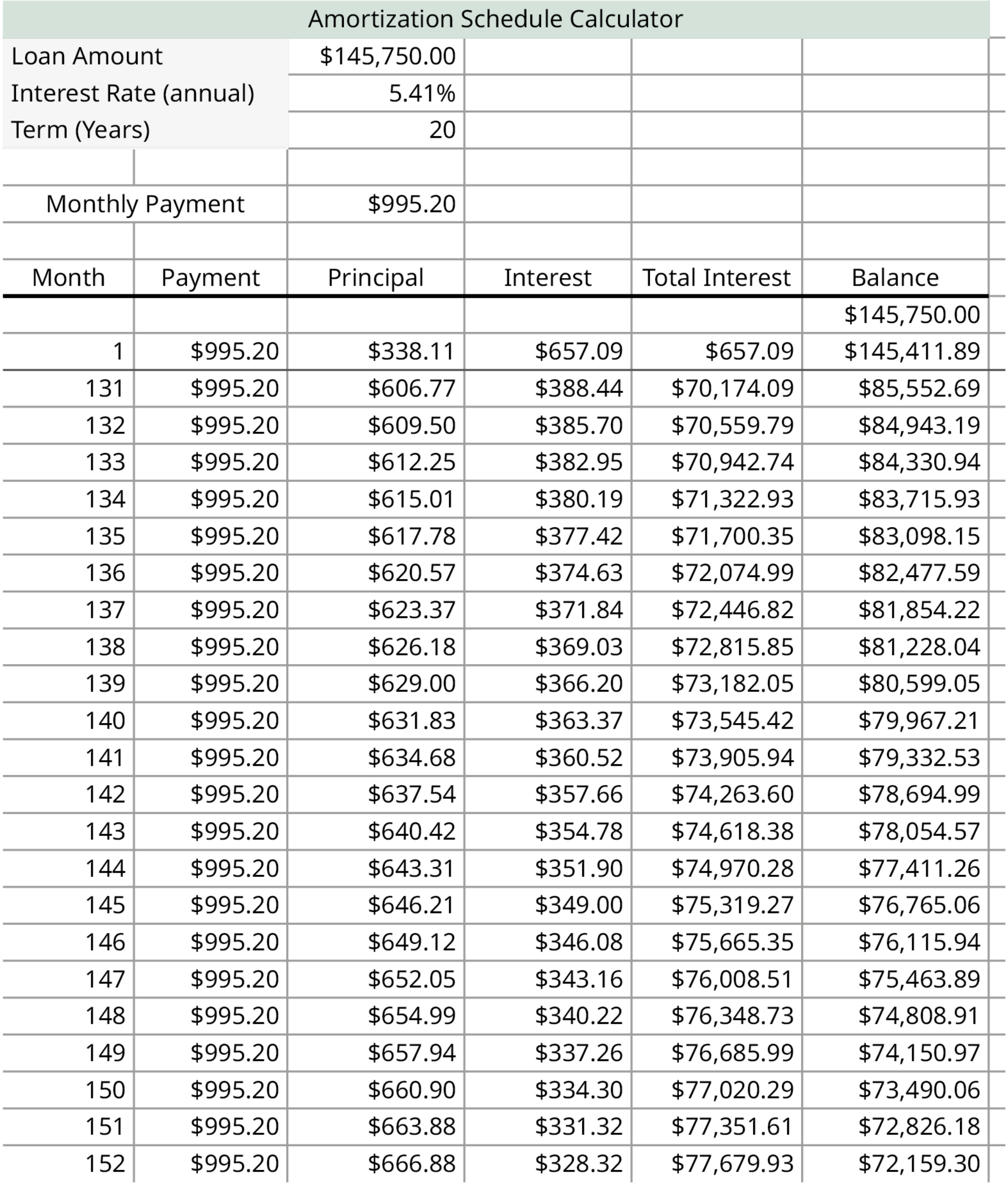
Exercise 6.8.0.24

How much of payment 19 was for principal?

Exercise 6.8.0.25

How much total interest was paid after payment 22?

In the following exercises, use the amortization table for payments 131–152 of a mortgage below to answer the questions.



Amortization Schedule Calculator

Loan Amount	$145,750.00
Interest Rate (annual)	5.41%
Term (Years)	20
Monthly Payment	$995.20

Month	Payment	Principal	Interest	Total Interest	Balance
					$145,750.00
1	$995.20	$338.11	$657.09	$657.09	$145,411.89
131	$995.20	$606.77	$388.44	$70,174.09	$85,552.69
132	$995.20	$609.50	$385.70	$70,559.79	$84,943.19
133	$995.20	$612.25	$382.95	$70,942.74	$84,330.94
134	$995.20	$615.01	$380.19	$71,322.93	$83,715.93
135	$995.20	$617.78	$377.42	$71,700.35	$83,098.15
136	$995.20	$620.57	$374.63	$72,074.99	$82,477.59
137	$995.20	$623.37	$371.84	$72,446.82	$81,854.22
138	$995.20	$626.18	$369.03	$72,815.85	$81,228.04
139	$995.20	$629.00	$366.20	$73,182.05	$80,599.05
140	$995.20	$631.83	$363.37	$73,545.42	$79,967.21
141	$995.20	$634.68	$360.52	$73,905.94	$79,332.53
142	$995.20	$637.54	$357.66	$74,263.60	$78,694.99
143	$995.20	$640.42	$354.78	$74,618.38	$78,054.57
144	$995.20	$643.31	$351.90	$74,970.28	$77,411.26
145	$995.20	$646.21	$349.00	$75,319.27	$76,765.06
146	$995.20	$649.12	$346.08	$75,665.35	$76,115.94
147	$995.20	$652.05	$343.16	$76,008.51	$75,463.89
148	$995.20	$654.99	$340.22	$76,348.73	$74,808.91
149	$995.20	$657.94	$337.26	$76,685.99	$74,150.97
150	$995.20	$660.90	$334.30	$77,020.29	$73,490.06
151	$995.20	$663.88	$331.32	$77,351.61	$72,826.18
152	$995.20	$666.88	$328.32	$77,679.93	$72,159.30

Exercise 6.8.0.26

What was the loan amount, or starting principal?

Exercise 6.8.0.27

What was the interest rate?

Exercise 6.8.0.28

What is the term of the loan?

Exercise 6.8.0.29

What are the monthly payments?

Exercise 6.8.0.30

What is the remaining principal after payment 140?

Exercise 6.8.0.31

How much of the principal has been paid off after payment 151?

Exercise 6.8.0.32

How much of payment 138 was for principal?

Exercise 6.8.0.33

How much total interest was paid after payment 135?

In the following exercises find the cost to finance for the loan.

Exercise 6.8.0.34

Total interest paid is $94,598.36, origination fee was $450, processing fee was $300, commission fee was $1,457.50.

Exercise 6.8.0.35

Total interest paid was $3,209.34, origination fee was $100, processing fee was $200.

Exercise 6.8.0.36

Total interest paid was $8,295.50, fees were $875.

Exercise 6.8.0.37

Total interest paid was $56,114.90, origination fee was $1,230, and filing fee was $250.

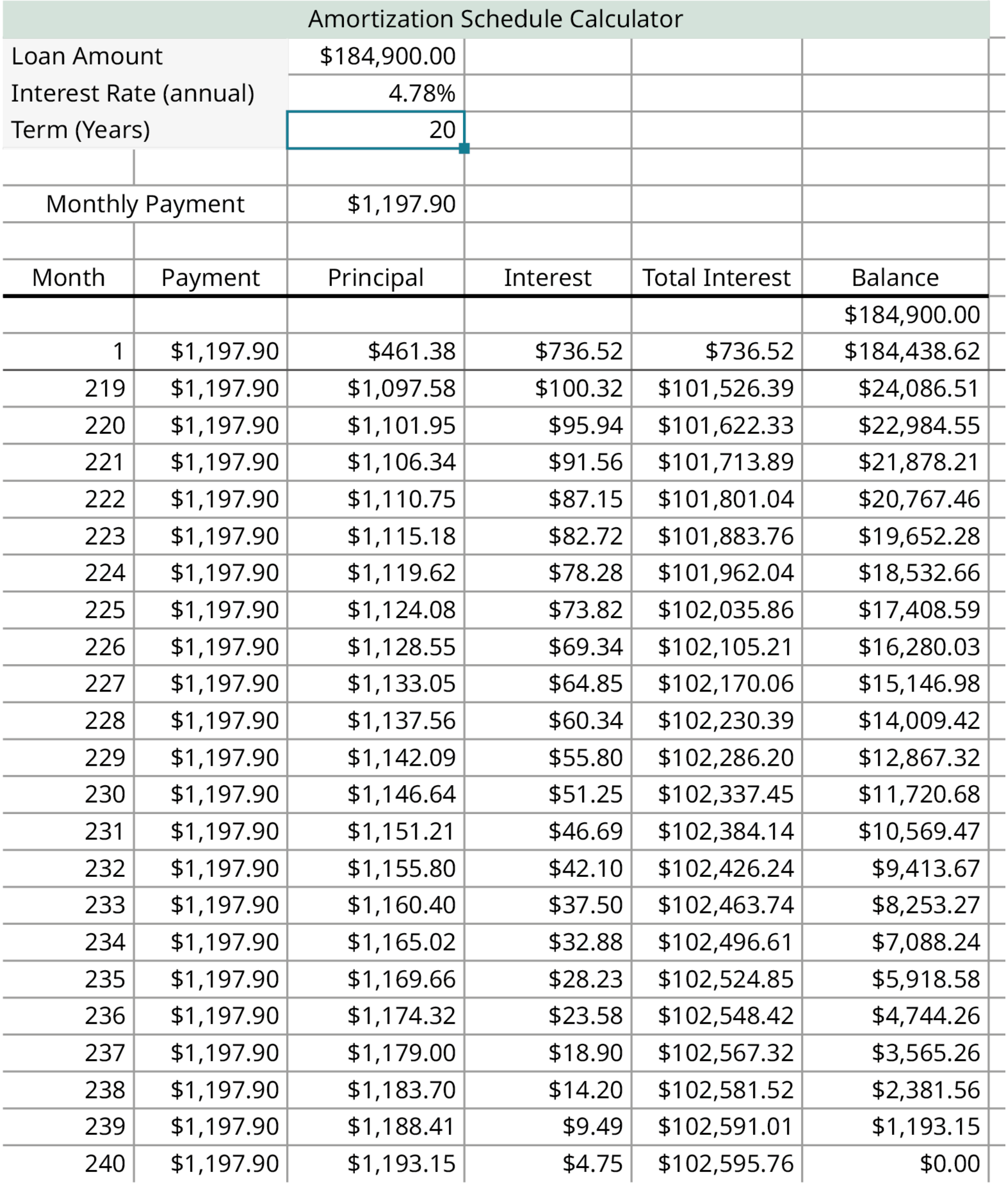
Exercise 6.8.0.38

Kylie takes out a $16,780 loan from her credit union for a new car. The loan's term is 4 years with an interest rate of 6.77%. What are Kylie's monthly payments?

Exercise 6.8.0.39

Crissy and Jonesy take out a $13,200 loan for repairs to their roof. The loan is for 10 years at 7.15% interest. What are their payments for this loan?

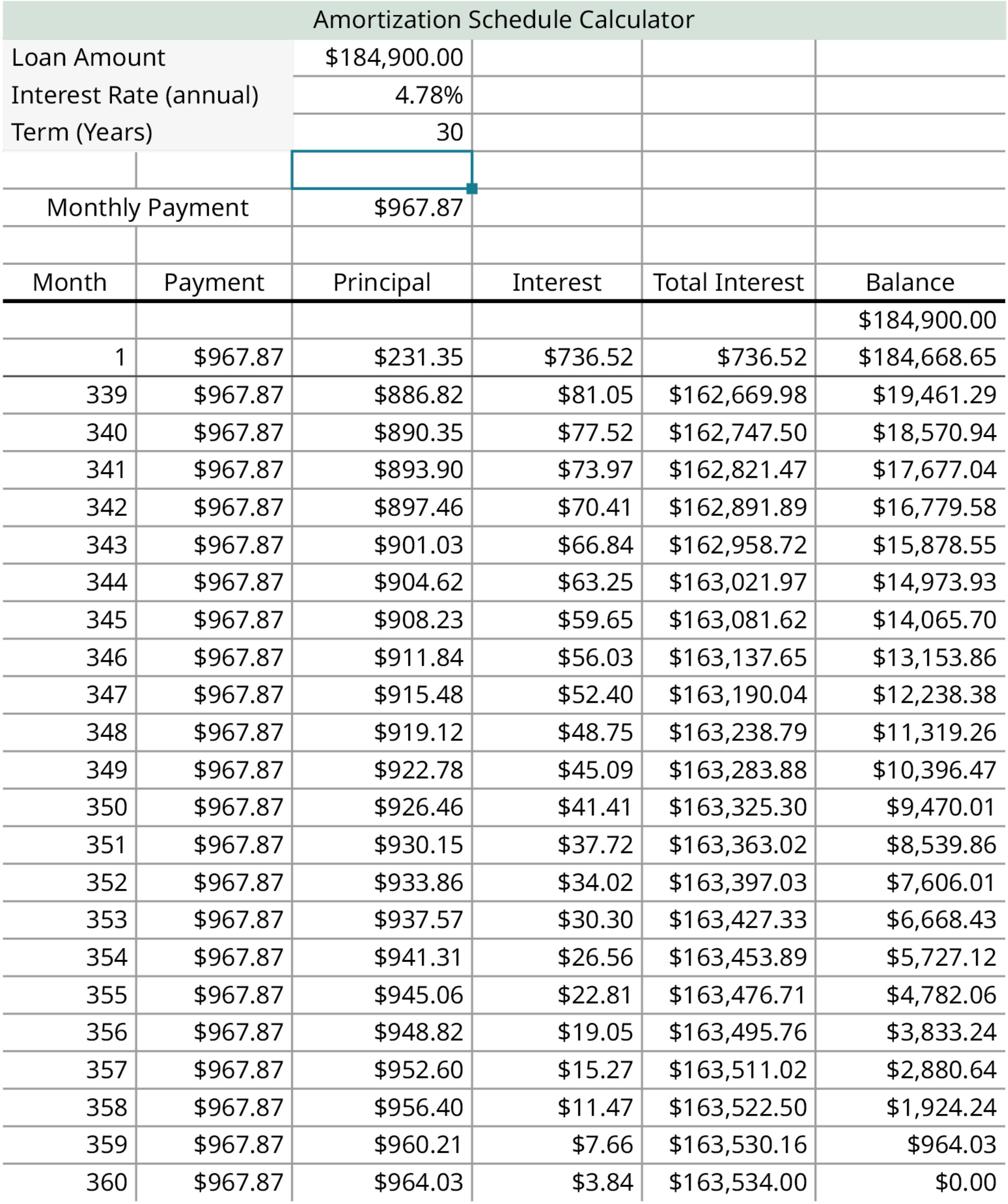
Below are two amortization tables. They are for loans with the same principal and interest rate, but different terms, the first a 20-year term and the second a 30-year term. The tables show the last payments of each loan. Use those two tables for the following exercises.



Amortization Schedule Calculator

Loan Amount	$184,900.00
Interest Rate (annual)	4.78%
Term (Years)	20
Monthly Payment	$1,197.90

Month	Payment	Principal	Interest	Total Interest	Balance
					$184,900.00
1	$1,197.90	$461.38	$736.52	$736.52	$184,438.62
219	$1,197.90	$1,097.58	$100.32	$101,526.39	$24,086.51
220	$1,197.90	$1,101.95	$95.94	$101,622.33	$22,984.55
221	$1,197.90	$1,106.34	$91.56	$101,713.89	$21,878.21
222	$1,197.90	$1,110.75	$87.15	$101,801.04	$20,767.46
223	$1,197.90	$1,115.18	$82.72	$101,883.76	$19,652.28
224	$1,197.90	$1,119.62	$78.28	$101,962.04	$18,532.66
225	$1,197.90	$1,124.08	$73.82	$102,035.86	$17,408.59
226	$1,197.90	$1,128.55	$69.34	$102,105.21	$16,280.03
227	$1,197.90	$1,133.05	$64.85	$102,170.06	$15,146.98
228	$1,197.90	$1,137.56	$60.34	$102,230.39	$14,009.42
229	$1,197.90	$1,142.09	$55.80	$102,286.20	$12,867.32
230	$1,197.90	$1,146.64	$51.25	$102,337.45	$11,720.68
231	$1,197.90	$1,151.21	$46.69	$102,384.14	$10,569.47
232	$1,197.90	$1,155.80	$42.10	$102,426.24	$9,413.67
233	$1,197.90	$1,160.40	$37.50	$102,463.74	$8,253.27
234	$1,197.90	$1,165.02	$32.88	$102,496.61	$7,088.24
235	$1,197.90	$1,169.66	$28.23	$102,524.85	$5,918.58
236	$1,197.90	$1,174.32	$23.58	$102,548.42	$4,744.26
237	$1,197.90	$1,179.00	$18.90	$102,567.32	$3,565.26
238	$1,197.90	$1,183.70	$14.20	$102,581.52	$2,381.56
239	$1,197.90	$1,188.41	$9.49	$102,591.01	$1,193.15
240	$1,197.90	$1,193.15	$4.75	$102,595.76	$0.00



Amortization Schedule Calculator					
Loan Amount		$184,900.00			
Interest Rate (annual)		4.78%			
Term (Years)		30			
Monthly Payment		$967.87			
Month	Payment	Principal	Interest	Total Interest	Balance
					$184,900.00
1	$967.87	$231.35	$736.52	$736.52	$184,668.65
339	$967.87	$886.82	$81.05	$162,669.98	$19,461.29
340	$967.87	$890.35	$77.52	$162,747.50	$18,570.94
341	$967.87	$893.90	$73.97	$162,821.47	$17,677.04
342	$967.87	$897.46	$70.41	$162,891.89	$16,779.58
343	$967.87	$901.03	$66.84	$162,958.72	$15,878.55
344	$967.87	$904.62	$63.25	$163,021.97	$14,973.93
345	$967.87	$908.23	$59.65	$163,081.62	$14,065.70
346	$967.87	$911.84	$56.03	$163,137.65	$13,153.86
347	$967.87	$915.48	$52.40	$163,190.04	$12,238.38
348	$967.87	$919.12	$48.75	$163,238.79	$11,319.26
349	$967.87	$922.78	$45.09	$163,283.88	$10,396.47
350	$967.87	$926.46	$41.41	$163,325.30	$9,470.01
351	$967.87	$930.15	$37.72	$163,363.02	$8,539.86
352	$967.87	$933.86	$34.02	$163,397.03	$7,606.01
353	$967.87	$937.57	$30.30	$163,427.33	$6,668.43
354	$967.87	$941.31	$26.56	$163,453.89	$5,727.12
355	$967.87	$945.06	$22.81	$163,476.71	$4,782.06
356	$967.87	$948.82	$19.05	$163,495.76	$3,833.24
357	$967.87	$952.60	$15.27	$163,511.02	$2,880.64
358	$967.87	$956.40	$11.47	$163,522.50	$1,924.24
359	$967.87	$960.21	$7.66	$163,530.16	$964.03
360	$967.87	$964.03	$3.84	$163,534.00	$0.00

Exercise 6.8.0.40

How much total interest was paid for the 20-year loan? The 30-year loan? Calculate the difference.

Exercise 6.8.0.41

What was the payment for the 20-year loan? The 30-year loan? Calculate the difference.

Exercise 6.8.0.42

Compare the two loans. Why would the 30-year be preferable over the 20-year loan? Why would the 20-year loan be preferable to the 30-year loan?

In the following exercises, compare payments per month and interest rates.

Exercise 6.8.0.43

A loan for $20,000 is taken out for 5 years. Find the payment is the interest rate is:

a. 2.9%
b. 3.9%
c. 4.9%
d. 5.9%
e. Did the payments increase by the same amount for each 1% jump in interest rate? Describe the pattern.

Exercise 6.8.0.44

A loan for $50,000 is taken out for 7 years. Find the payment is the interest rate is:

a. 3.5%
b. 4.5%
c. 5.5%
d. 6.5%
e. Did the payments increase by the same amount for each 1% jump in interest rate?

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6.9: Understanding Student Loans

Figure 6.9.1: College is often paid for through loans. (credit: "Sidewalk Scene in front of Science Building" by IMCBerea College Wikimedia Commons, CC- BY 2.0)

Learning Objectives

1. Describe how to obtain a student loan.
2. Distinguish between federal and private student loans and state distinctions.
3. Understand the limits on student loans.
4. Summarize the standard prepayment plan.
5. Understand student loan consolidation.
6. Summarize and describe benefits or drawbacks of other repayment plans.
7. Summarize possible courses of action if a student loan defaults.

Obtaining a Student Loan

All college students are eligible to apply for a loan regardless of their financial situation or credit rating. Federal student loans do not require a co-signer or a credit check. Most students do not have a credit history when they begin college, and the federal government is aware of this. However, private loans will generally require a co-signer as well as a credit check. The co-signer will assume responsibility for paying off the loan if the student cannot make the payments.

The first step in applying for student loans is to fill out the FAFSA (Free Application for Student Aid). FAFSA determines financial need and what type of loan the student is qualified to obtain. For students who are still dependents on their parent's taxes, the parents also fill out the FAFSA, as their wealth and income impacts what the dependent student is eligible for. Students who cannot demonstrate financial need will also be helped by applying with FAFSA, as it will help guide them to the type of loan most appropriate. The FAFSA must be submitted each year.

Checkpoint

The FAFSA deadline is the spring of the student's next academic year. The deadline is often in March. Do not allow this deadline to pass.

As soon as an offer letter from the college is received, the student should start the application process. The college will determine the loan amount needed. Also, there are limits on the amount a student can borrow. There are both yearly limits and aggregate limits. See the table later in this section that outlines the loan limits per school year and in the aggregate.

If the student receives a direct subsidized loan, there is a limit on the eligibility period. The time limit on eligibility depends on the college program into which the student enrolls. The school publishes how long a program is expected to take. The eligibility period is 150% of that published time. For example, if a student is enrolled in a 4-year program, such as a bachelor's degree program, their eligibility period is 6 years, as 1.50(4) = 6. Therefore, the student may receive direct subsidized loans for a period of 6 years.

Types and Features of Student Loans

Once a tuition statement is received, and all the non-loan awards are analyzed that are applicable to the costs of college (such as scholarships and grants), there still may be quite of bit of an expense to attend college. This difference between what college will cost (including tuition, room and board, books, computers) and the non-loan awards received is the **college funding gap.**

✓ Example ✓ 6.9.1: College Funding Gap

Ishraq receives her award and tuition letter from the college she wants to attend. Her tuition, fees, books, and room and board all come to $24,845 for the year. Her non-loan awards include an instant scholarship from the school for $7,500, a scholarship she earned for enrolling in a STEM program for $3,750, and a $1,000 scholarship from her church. What is Ishraq's college funding gap?

Answer

Her awards total to $12,250. Her cost to attend is $24,845. Her college funding gap is then $\$24{,}845 - \$12{,}250 = \$12{,}595$. She will need to find $12,595 in funding.

Your Turn 6.9.1

Yuan-Teng receives his award and tuition letter from the college he wants to attend. His tuition, fees, books, and room and board all come to $34,845 for the year. His non-loan awards include an instant scholarship from the school for $17,500, three scholarships he earned from sources he found online, which total $5,600, and a $2,000 scholarship from his employer. What is Yuan-Teng's college funding gap?

There are several loan types, which basically break down into four broad categories: subsidized loans, unsubsidized loans, PLUS loans, and private loans. These loans are meant to fill the college funding gap.

Federal subsidized loans are backed by the U.S. Department of Education. These loans are intended for undergraduate students who can demonstrate financial need. Subsidized federal loans, including Stafford loans, defer payments until the student has graduated. During the deferment, the government pays the interest while the student is enrolled at least half-time. These loans are generally made directly to students. However, there are restrictions on how the money can be used. It can only be used for tuition, room and board, computers, books, fees, and college-related expenses. Interest rates are not based on the financial markets but determined by Congress. Federal loans are backed by the Department of Education.

Federal unsubsidized loans, including unsubsidized Stafford loans, are available for undergraduate and graduate students who cannot demonstrate financial need. If the student meets the program requirements, they are automatically approved. The student is not required to pay these loans during their time in college (enrolled at least half-time). However, the interest rate is generally higher and there is no deferment period, as with subsidized loans. Interest begins accruing as soon as the money is disbursed.

Checkpoint

The immediate accrual of interest means the balance of the loan grows as the student attends school. A loan that was for $10,000 can grow past $13,000 over five years of college. Some advisors tell students to pay the interest portion of the loan while it is deferred to prevent this growth of debt.

Parent Loans for Undergraduate Students (PLUS) are federal loans made directly to parents. They are available even if parents are not deemed financially needy. A credit check is performed and approval is not automatic. The limit to what parents can borrow from a PLUS loan each year is still the college funding gap, but the aggregate of the PLUS loans does not have a limit. This means the PLUS loan can cover whatever is left in the funding gap once all other aid and loans are applied. Payments do not begin until the student is out of school, but interest begins to accrue the moment funds are disbursed. Because the parents take out the loan, the parents are responsible for paying back the loan.

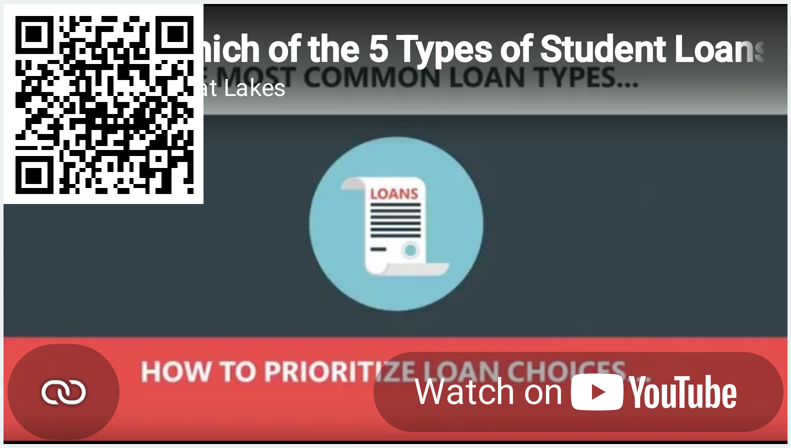
Private student loans are backed by a bank or credit institution and require a credit check, and interest rates are variable. As private loans are not subsidized by the government, no one pays the interest but the borrower. The student does not have to start repaying the loan until after graduation, but interest starts to accrue immediately. This loan has fewer repayment options, more fees and penalties, and the loan cannot be discharged through bankruptcy. Many students need a co-signer to acquire a private loan. Like PLUS loans, private student loans can cover whatever is left in the funding gap once all other aid and loans are applied

Student loans, in general, have a term of 10 years, that is, the loans are paid back over 10 years. This can vary, but 10 years is the standard.

Who Knew?: School-Channel Loans and Direct-to-Consumer Loans

Private loans can fall into one of two categories: school-channel loans and direct to consumer loans. **School-channel loans** are disbursed directly to the school. The school verifies the loan does not exceed the cost to attend school. **Direct–to-consumer loans** do not have the verification process. Those proceeds are sent directly to the borrower. They are processed more quickly, but often have higher interest rates.

Video


ich of the 5 Types of Student Loans
MOST COMMON LOAN TYPES...
at Lakes
LOANS
HOW TO PRIORITIZE LOAN CHOICES
Watch on YouTube

Types of Student Loans

Limits on Student Loans

As mentioned earlier, there are limits to how much a student can borrow, per year and in total. The following table shows a general breakdown of the amounts the federal government and private lenders will lend. Amounts are based on level of need and whether the student is a dependent or an independent student. Independent students include those who are at least 24 years old, married, a professional, a graduate student, a veteran, a member of the armed forces, an emancipated minor, or an orphan. The amounts shown are as of this writing in 2022.

Year	Dependent Students Maximum Amounts	Independent Students Maximum Amounts

Year	Dependent Students Maximum Amounts	Independent Students Maximum Amounts
First-Year Undergraduate	$5,500 but no more than $3,500 may be in subsidized loans	$9,500 but no more than $3,500 may be in subsidized loans
Second-Year Undergraduate	$6,500 but no more than $4,500 may be subsidized loans	$10,500 but no more than $4,500 in subsidized loans
Third Year and Additional Years	$7,500 but no more than $5,500 may be in subsidized loans	$12,500 but no more than $5,500 may be in subsidized loans
Graduate and Professional	Not applicable	$20,500 in unsubsidized loans
Limits	$31,000 but no more than $23,000 in subsidized loans	$57,500 for undergraduates but no more than $23,000 in subsidized. $138,500 for graduate or professional but no more than $65,500 may be subsidized loans.

Check out this Edvisors page on the limits of student borrowing to learn more!

✓ Example 6.9.2: Loan for Year 5 of College

Efraim is a dependent undergraduate student enrolled in a biology program. He's about to attend for the fifth year. In year 1 he took out $5,000 in federal subsidized and unsubsidized loans, in year 2 he took out $6,400 in federal subsidized and unsubsidized loans, in years 3 and 4, he took out the maximum federal subsidized and unsubsidized loans amounts. He needs federal subsidized and unsubsidized loans for his fifth year of school. How much can he obtain in federal subsidized and unsubsidized student loans?

Answer

The sum of his previous loans is $\$5{,}000+\$6{,}400+\$7{,}500+\$7{,}500=\$26{,}400$. The limit for federal subsidized and unsubsidized loans is $31,000, so in year 5 he can get student loans in the amount of $\$31{,}000-\$26{,}400=\$4{,}600$.

Your Turn 6.9.2

Tiana in an independent college student entering the sixth and final year of her engineering program. In the previous years, she took out student loans of $5,500, $8,470, $10,000, $7,890, and $11,900. She needs a loan to finish her final year. How much in federal subsidized and unsubsidized student loans can she obtain?

Putting this all together, we have a way to determine the student loans needed for a student to attend college.

- First, determine the funding gap. If the student or family can cover the gap, then no loans are necessary.
- Second, determine how much in federal subsidized and unsubsidized student loans can be taken out. If the total federal loans available is more than the funding gap, no other loans are needed.
- Third, if the federal subsidized and unsubsidized loans do not cover the gap, PLUS and private student loans can be taken out to cover the remainder of the gap.

At each step, if the student and family can cover some or all of the gap, they can do so without taking out a loan.

✓ Example 6.9.3: College Funding Gap and PLUS and Private Student Loans

Olivia receives her award and tuition letter from the college she wants to attend. Her tuition, fees, books, and room and board all come to $44,845 for her second year. Her non-loan awards include an instant scholarship from the school for $13,500, a scholarship she earned for enrolling in an engineering program for $5,750, and a $2,000 scholarship from her parent's workplace. For her first year, what is Olivia's college funding gap? How much can Olivia borrow in federal subsidized and unsubsidized student loans? Once Olivia takes out her maximum subsidized and unsubsidized federal student loans, how much will have to be paid for using PLUS and private student loans?

Answer

Her awards total to $21,250. Her cost to attend is $44,845. Her college funding gap is then $\$44{,}845 - \$21{,}250 = \$23{,}595$. The maximum in federal student loans that Olivia can borrow is $6,500 in year 2. The remaining funding gap is $\$23{,}595 - \$6{,}500 = \$17{,}095$. Private student loans, PLUS loans, or other sources must be used to cover this gap.

Your Turn 6.9.3

Makenzy receives her award and tuition letter from the college she wants to attend. Her tuition, fees, books, and room and board all come to $39,200 for her third year. Her non-loan awards include an instant scholarship from the school for $19,500, three scholarships she earned from sources she found online which total $3,850, and a $5,000 scholarship from her employer. What is Makenzy's college funding gap? How much can Makenzy borrow in federal subsidized and unsubsidized student loans? Once she takes out her maximum subsidized and unsubsidized federal student loans, how much will Makenzy and her family have to pay using PLUS and private student loans?

Student Loan Interest Rates

Student loans are first and foremost loans. Students will pay them back and will pay interest. In the fall of 2022, the federal student loan interest rate was 4.99%. Private student loans rates ranged between 3.22% and 13.95%. Finding the lowest interest rate you can helps with the payments, and especially helps if the loan is not federally subsidized. Remember, if the loan is not federally subsidized, the student is on the hook for the interest that is accumulating with the loan.

Interest Accrual

The interest on student loans begins as soon as the loan is disbursed (paid to the borrower). When the loan is federally subsidized, the government pays that interest for the student. This means the loan for a subsidized loan of $3,000 is still a loan for $3,000 when the student graduates. However, if the loan is not federally subsidized, the student is responsible for the interest that accrues on the loan. The $3,000 loan from year 1 of college is now a loan for more due to that added interest. The interest on that loan grew while the student was in college. The formula for growth of the loan's balance is the same as compound interest formula from Compound Interest, $A = P$msup.

Example 6.9.4

Denise takes out unsubsidized student loan, in August, in her first year of college for $2,000. She manages an interest rate of 8%. She graduates after her fifth year of college, in May. She does not pay the interest on the loan during her time in college. What is the balance of her first year loan in May of her graduation year?

Answer

The principal of the loan is $2,000. Her interest rate is 8%. Since student loans are typically paid monthly, there are 12 periods per year. Since the time she has had the loan is not in years, we will use the number of months for the value of nt in the formula. She has had the loan for 4 years and 9 months, meaning 57 period have passed. Substituting those values into the formula and calculating, we find her balance in May of her graduating year is $A = P(1+\frac{r}{n})^{nt} = \$2{,}000(1+\frac{0.08}{12})^{57} = \$2{,}000(1.0\bar{6})^{57} = \$2{,}920.89$.

Your Turn 6.9.4

Priya takes out unsubsidized student loan, in August, in her first year of college for $2,000. She manages an interest rate of 7%. She graduates after her sixth year of college, in May. She does not pay the interest on the loan during her time in college. What is the balance of her first year loan in May of her graduation year?

The Standard Repayment Plan

There are various repayment plans available. The one most likely to apply to a student loan is the **standard repayment plan,** which is available to everyone. Borrowers pay a fixed amount monthly so the loan is paid in full within 10 years. Consolidated loans, discussed later in this section, also qualify for the standard repayment plan, and may allow the payoff period to range from 10 to 30 years. Direct subsidized and unsubsidized loans, PLUS loans, and federal Stafford loans are eligible.

Since these are loans, they are paid back with interest. As with most installment loans, their payments are due monthly. The formula for paying back these loans is the same as the formula used for paying loans in The Basics of Loans:

$$pmt = \frac{P \times (r/n) \times (1 + r/n)^{n \times t}}{(1 + r/n)^{n \times t} -}$$

.

Using that formula, we can calculate how much the payment is for a student loan. Remember that all loan payments are rounded up to the next penny.

✓ Example 6.9.5: Standard Repayment Plan

Find the payment for the following student loans using the standard repayment plan:

1. Loan is $3,500, interest is 4.99%
2. Loan is for $6,200, interest is 6.75%

Answer

1. The principal is P = $3,500 and the rate is r = 0.0499. Since this is the standard repayment plan, there are n = 12 payments per year for 10 years. Substituting those values into $pmt = \frac{P \times (r/n) \times (1 + r/n)^{n \times t}}{(1 + r/n)^{n \times t} -}$ and calculating gives a monthly payment of

$$\begin{aligned} pmt &= \frac{P \times (r/n) \times (1 + r/n)^{n \times t}}{(1 + r/n)^{n \times t} - 1} \\ &= \frac{\$3{,}500 \times (0.0499/12) \times (1 + 0.0499/12)^{12 \times 10}}{(1 + 0.0499/12)^{12 \times 10} - 1} \\ &= \frac{\$3{,}500 \times (0.004158\bar{3}) \times (1.004158\bar{3})^{120}}{(1.004158\bar{3})^{120} - 1} \\ &= \frac{\$23.9469911276}{0.645370131872} \\ &= \$37.11 \end{aligned}$$

2. The principal is P = $6,200 and the rate is r = 0.0675. Since this is the standard repayment plan, there are n = 12 payments per year for 10 years. Substituting those values into $pmt = \frac{P \times (r/n) \times (1 + r/n)^{n \times t}}{(1 + r/n)^{n \times t} -}$ and calculating gives a monthly payment of

$$\begin{aligned} pmt &= \frac{P \times (r/n) \times (1 + r/n)^{n \times t}}{(1 + r/n)^{n \times t} - 1} \\ &= \frac{\$6{,}200 \times (0.0675/12) \times (1 + 0.0675/12)^{12 \times 10}}{(1 + 0.0675/12)^{12 \times 10} - 1} \\ &= \frac{\$6{,}200 \times (0.005625) \times (1.005625)^{120}}{(1.005625)^{120} - 1} \\ &= \frac{68.3662233426}{0.960321816275} \\ &= \$71.20 \end{aligned}$$

Your Turn \(\PageIndex{5}

Find the payment for the following student loans using the standard repayment plan:

Loan is $7,800, interest is 4.99%

Loan is for $11,450, interest is 7.75%

✓ Example 6.9.6: Standard Repayment Plan for an Unsubsidized Loan

Erson has a balance of $8,132.55 when he starts paying off the 8.6% unsubsidized student loan he took out in his third year. How much are his payments if the term for his loan is the standard 10 years?

Answer

Using the payment formula, $pmt = \frac{P \times (r/n) \times (1 + r/n)^{n \times t}}{(1 + r/n)^{n \times t} -}$, with P = \$8,132.55, r = 0.086, t = 10 and n = 12, we calculate that his monthly payment will be

$$\begin{aligned} pmt &= \frac{P \times (r/n) \times (1 + r/n)^{n \times t}}{(1 + r/n)^{n \times t} - 1} \\ &= \frac{\$8{,}132.55 \times (0.086/12) \times (1 + 0.086/12)^{12 \times 10}}{(1 + 0.086/12)^{12 \times 10} - 1} \\ &= \frac{\$8{,}132.55 \times (0.0071\bar{6}) \times (1.0071\bar{6})^{120}}{(1.0071\bar{6})^{120} - 1} \\ &= \frac{137.310962384}{1.35592393158} \\ &= \$101.27 \end{aligned}$$

Your Turn 6.9.6

Kathleen has an unsubsidized student loan from graduate school. She begins to pay it back in the September after she earned her doctorate. The balance at that point was \$22,666.21. The interest rate on the loan is 6.7%. How much are her payments if she pays it off in the standard of 10 years?

Student Loan Consolidation

When a student graduates, they may have multiple different student loans. Keeping track of them and paying them off separately can be a burden. Instead, these loans can be consolidated into a single loan. If they are federal loans the combination is called **federal consolidation.** Combining private loans is often referred to as **refinancing.** Refinancing, or **private consolidation,** can be used to combine both private and federal student loans. Be aware that consolidated federal loans may still be subject to the rules and protections that govern subsidized loans. Refinancing loans, private or federal, are no longer subject to those rules and guidelines. Check out this Experian article about consolidation and refinancing for more deatil.

In consolidation of federal direct student loans, the interest rate is the weighted average of the interest rates on the subsidized loans. This means the interest rate remains the same. However, if the term is extended, then the student will pay back more over time than if they did not extend the loan term.

In refinancing, it is possible to obtain a lower interest rate on the student loans, which may lower how much is paid per month and lower the total paid back over time. These monthly payments are calculated using the same formula as for any other loan payment, $pmt = \frac{P \times (r/n) \times (1 + r/n)^{n \times t}}{(1 + r/n)^{n \times t} -}$. The term of the refinanced loan may also be changed, which would also impact the payment per month.

In either case, consolidating or refinancing, the monthly financial burden on the student can decrease. However, if the term is extended, the total amount repaid may increase.

✓ Example 6.9.7: Federal Loan Consolidation and Interest Rates

Ernest has four federal student loans that he wants to consolidate. He combines them into one loan. What is the maximum Ernest can reduce the interest rate by?

Answer

Consolidating subsidized loans has no impact on the interest rate of the loans, so the maximum that the interest rate can be reduced is 0%.

Your Turn 6.9.7

Ryann has five federal student loans that he wishes to consolidate. He wants to extend the term to 15 years. How does this impact Ryann's interest rate? The total Ryann pays back?

✓ Example 6.9.8: Payments for Consolidated Student Loans

Brianna consolidates her student loans, some federal and some private, into a single refinanced student loan with a principal of $27,800. The interest rate that Brianna received was 8.375%. If Brianna's new term is 15 years, how much are her payments per month?

Answer

The principal is P = $27,800 and the rate is r = 0.08375. Since the payments are monthly, n =12. The loan term is for 15 years, so t = 15. years. Substituting those values into $pmt = \frac{P \times (r/n) \times (1+r/n)^{n \times t}}{(1+r/n)^{n \times t} -}$ and calculating gives a monthly payment of

$$\begin{aligned} pmt &= \frac{P \times (r/n) \times (1+r/n)^{n \times t}}{(1+r/n)^{n \times t} - 1} \\ &= \frac{\$27{,}800 \times (0.08375/12) \times (1+0.08375/12)^{12 \times 15}}{(1+0.08375/12)^{12 \times 15} - 1} \\ &= \frac{\$6{,}200 \times (0.0069791\bar{6}) \times (1.0069791\bar{6})^{180}}{(1.0069791\bar{6})^{180} - 1} \\ &= \frac{\$678.477992464}{2.49693370968} \\ &= \$271.73 \end{aligned}$$

Your Turn 6.9.8

Brian consolidates his federal student loans into one loan at 4.99%. The consolidated balance is $20,340, and he extends the term to 12 years. What are Brian's monthly payments for the consolidated loan?

Other Repayment Plans

There are various other repayment plans available to students. Plans other than the standard repayment plan typically require the student to meet certain criteria. The following plans are independent of student income, but may make early payments easier.

- **Graduated repayment plans** are plans where the amount of payments gradually increases so that the loan is paid off in 10 years, or within 10 to 30 years for consolidated loans. Payments start off small and increase approximately every 2 years. Almost all loan types are eligible, including direct subsidized and unsubsidized loans, Stafford loans, PLUS loans, and consolidated loans.
- **Extended repayment plans** are available to the direct loan borrower if the outstanding direct loans are over $30,000. The payments, fixed or graduated, are designed so that the loans are satisfied within 25 years. Eligible loans include both direct subsidized or unsubsidized loans, Stafford loans, PLUS loans, and consolidated loans.

If student earnings are such that the standard, graduated, or extended repayment plans are unaffordable, then one can make payments that are based on their **discretionary income.** Discretionary income is federally defined to be the difference between (adjusted) gross income and 150% of the poverty guideline for location and family size. This discretionary income then depends on where one lives (contiguous United States or Hawaii or Alaska) and how many dependents one has. If married, a spouse's income will be included in the adjusted gross income. Understanding discretionary income is necessary to understand how income driven payments plans work.

✓ Example 6.9.9: Discretionary Income

1. The poverty guideline for a single person living in Arkansas, is $12,000. If Harriet is a single person in Arkansas with a (adjusted) gross income of $23,500, what is her discretionary income?
2. For California, the poverty guideline for a person with four people in the household is $27,750. If such a person has a (adjusted) gross income of $48,600, what is their discretionary income?

Answer

1. The poverty guideline for a single person in Arkansas is $12,000. 150% of that guideline value is $1.5 \times \$12{,}000 = \$18{,}000$. The gross income that Harriet makes over that $18,000 is her discretionary income. That gross income is $23,500, so her discretionary income is $\$23{,}500 - \$18{,}000 = \$5{,}500$.

2. The poverty guideline for a household of four in California is $27,750. 150% of that guideline value is $1.5 \times \$27{,}750 = \$41{,}625$. The gross income that Harriet makes over that $41,625 is her discretionary income. That gross income is $48,600, so her discretionary income is $\$48{,}600 - \$41{,}625 = \$6{,}975$.

Your Turn 6.9.9

The poverty guideline for two-person household in Hawaii is $21,060. If Jamie lives in a two-person household and has a gross income of $36,900, what is his discretionary income?

The following plans all depend on discretionary income.

- **Pay As You Earn (PAYE) repayment plans** have monthly payments that are 10% of discretionary income based on a student's updated income and family size. If a borrower files a joint tax return, their spouse's income and debt may also be considered. Eligible loans include direct subsidized and unsubsidized loans, PLUS loans made to students, and some consolidated loans. These loans are forgiven (student does not pay the remaining balance) after 20 years of monthly payments if they were direct federal student loans.
- **Revised Pay As You Earn (REPAYE) repayment plans** have payment amounts that are based on income and family size and calculated as 10% of discretionary income. Eligible loans include direct subsidized and unsubsidized loans, direct PLUS loans, and some consolidated loans. These loans are forgiven, that is, the student does not pay more, after 20 or 25 years, provided they were direct federal student loans.
- **Income-Based Repayment (IBR) plans** sound like a few of the others and there are similarities in all of them. The payments are either 10% or 15% of discretionary income, but this plan is meant for those with a relatively high debt. Every year, income and family size must be updated, and payments are calculated based on those figures. Eligible loans include direct subsidized and unsubsidized loans, Stafford loans, and PLUS loans made to students, but not PLUS loans made to parents.
- **Income-Contingent Repayment (ICR) plans** have payments that are either 20% of discretionary income or whatever would be paid if a student were on a fixed payment plan for more than 12 years, whichever is less. Eligible loans include direct subsidized and unsubsidized, PLUS loans to students, and consolidated loans.

There are many similarities among these repayment plans, and it is easy to misunderstand the nuances of each. Therefore, be careful entering into any type of repayment contract until you fully understand all the details and repercussions of the plan you choose. For more detail, see this nerdwallet article "Income-Driven Repayment: Is It Right for You?" to learn more!

✓ Example 6.9.10: Payments for a REPAYE Program

Warren qualifies for a REPAYE payment plan. His gross income is $32,700. He is single and live in Montana, so the federal poverty guideline for Warren is $12,000.

1. What is Warren's discretionary income?
2. Under the REPAYE plan, he pays 10% of his discretionary income, but monthly. How much are Warren's REPAYE payments?

Answer

1. The poverty guideline for Warren is $12,000. 150% of that guideline value is $1.5 * \$12{,}000 = \$18{,}000$. The gross income that Warren makes over that $18,000 is his discretionary income. That gross income is $32,700, so his discretionary income is $\$32{,}700 - \$18{,}000 = \$14{,}700$.
2. 10% of Warren's discretionary income is $0.1 \times \$14{,}700 = \$1{,}470$. He pays monthly, so his monthly payments are $1,470 divided by 12, or $122.50 per month.

Your Turn 6.9.10

Lauren, a Missouri resident, qualifies for a REPAYE payment plan. Her gross income is $39,200. She has one child, so there are two people in her household. The federal poverty guideline for Lauren is $18,310.

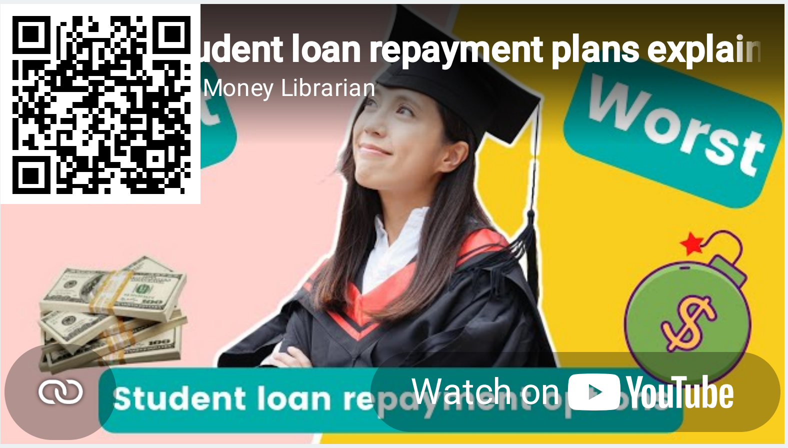
What is Lauren's discretionary income?

Under the REPAYE plan, she pays 10% of her discretionary income, but monthly. How much are Lauren’s REPAYE payments?

Using income to determine payments initially seems excellent. However, if there is no forgiveness at the end of the loan, then the income driven payment plans can cause problems. For one, your payment may not be sufficient to cover the interest rate of your loans. In that case, your loan balance actually *increases* as you make your payments. Eventually, you are paying for not only your original loan balance, but interest that has been growing and compounding over time. Also, if the loan term is extended, you may pay more, perhaps a lot more, money over time. You may find yourself in the position of paying these loans for decades.

With those possible drawbacks, great care must be taken to avoid large problems down the line.

Video



Repayment Plans

Student Loan Default and Consequences

The first day a payment is late, the account becomes **delinquent**. After 90 days, this delinquency is reported to the credit bureaus, and goes into **default**. This is serious, as now a credit score is affected, meaning that it will be harder to buy a car, a home, get a credit card, or a cell phone. Even renting an apartment may be a task not easily overcome. The default rate for students who do not complete their degree is three times higher than for students who do.

Further, defaulting on a student loan may mean that the borrower loses eligibility for repayment plans, as the balance and any unpaid interest may become due immediately, and any tax refunds may be withheld and applied to the loan, and wages may be garnished. One should immediately contact your loan servicer and try to make other arrangements for repayment if this situation becomes apparent, as different repayment plans are available, if actions are taken quickly.

There are several options that may be open to avoid defaulting. One is called **rehabilitation,** or is the process in which a borrower may bring a student loan out of default by adhering to specified repayment requirements, and the other is consolidation. Certain criteria must be met to enter these programs.

Both of these options are detailed, including the criteria required for eligibility, on the studentaid.gov loan management page.

Professionals advise hiring an attorney if one of these paths is chosen.

Check Your Understanding

1. When is the FAFSA typically due?
2. If the college has a program that it advertises will take 6 years, how long is the eligibility period?
3. What is a student’s funding gap?
4. What is a main difference between subsidized federal loans and unsubsidized loans?
5. What is the limit on the total in federal subsidized and unsubsidized loans that can be borrowed for an undergraduate degree?
6. If a student takes out a private student loan for $9,800 at 5.99% interest but does not begin paying it back for 66 months, how much is then owed on the loan?
7. How long is the standard repayment plan for student loans?
8. How much of a student’s discretionary income is paid for student loans under the REPAYE payment plan?

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6.9.0: Exercises

? Exercise 6.9.0.1

What form must be filled out before applying for a student loan?

? Exercise 6.9.0.2

What two issues limit the amount of a federal subsidized or unsubsidized student loan a student can take out?

? Exercise 6.9.0.3

What is the limit to the number of years a student can take out a student loan?

? Exercise 6.9.0.4

What types of student loans have deferred payments and do not accrue interest?

? Exercise 6.9.0.5

Which type of student loan is made directly to parents?

? Exercise 6.9.0.6

Which types of student loans require a credit check?

? Exercise 6.9.0.7

What is the limit on the total amount borrowed in federal subsidized and unsubsidized student loans for an independent undergraduate student?

? Exercise 6.9.0.8

What is the limit on subsidized student loans in the third year of an undergraduate program if the student is a dependent student?

? Exercise 6.9.0.9

What is the maximum allowed to be borrowed on a PLUS loan?

? Exercise 6.9.0.10

What is the maximum allowed to be borrowed on a private student loan?

? Exercise 6.9.0.11

What is the current interest rate on federal subsidized loans?

? Exercise 6.9.0.12

What feature does a graduated payment plan have?

? Exercise 6.9.0.13

If a student is struggling with paying back their student loans, what can the student do?

? Exercise 6.9.0.14

When a student consolidates federal subsidized and unsubsidized loans, does the interest rate change? Does the term of the loan change?

? Exercise 6.9.0.15

If a person with federal subsidized and unsubsidized loans enters a pay as you earn program (PAYE), how long will the person pay on their student loans?

? Exercise 6.9.0.16

How long is it between the time a person is delinquent on a student loan and being in default on the student loan?

? Exercise 6.9.0.17

What is discretionary income?

? Exercise 6.9.0.18

What are the two remedies if a borrower default on their student loans?

? Exercise 6.9.0.19

Toni's total cost to attend college including all fees, books, and room and board is $18,450. If Toni receives $5,500 in an instant scholarship from the college and has $2,175 in scholarships, what is Toni's funding gap?

? Exercise 6.9.0.20

Arthur's total cost to attend college including all fees, books, and room and board is $31,500. If Arthur receives $13,500 in an instant scholarship from the college and has $3,475 in private scholarships, what is Arthur's funding gap?

? Exercise 6.9.0.21

Amelia is an independent student in her fourth year of an urban studies undergraduate degree. What is the maximum she can borrow in federal subsidized and unsubsidized student loans that year?

? Exercise 6.9.0.22

Yaroslav is a dependent student in his second year of an undergraduate psychology program. What is the maximum that he can borrow in federal subsidized and unsubsidized student loans that year?

? Exercise 6.9.0.23

Toni's total cost to attend her third year of undergraduate school, including all fees, books, and room and board, is $18,450. Toni receives $5,500 in an instant scholarship from the college and has $2,175 in scholarships. She is a dependent student. After receiving the maximum federal student loans (subsidized and unsubsidized), and without her parents taking out a PLUS loan, how much will she need to borrow in private student loans to cover the funding gap?

? Exercise 6.9.0.24

Arthur’s total cost to attend his second year of his undergraduate program in data science, including all fees, books, and room and board, is $31,500. Arthur receives $13,500 in an instant scholarship from the college and has $3,475 in private scholarships. He is a dependent student. After receiving the maximum federal student loans (subsidized and unsubsidized), and without his parents taking out a PLUS loan, how much will he need to borrow in private student loans to cover the funding gap?

? Exercise 6.9.0.25

Nantumbwe takes out a private student loan for $7,950 in her second year of her undergraduate health administration program. She begins paying her loan after 39 months. If her interest rate was 8.75%, how much is her balance when she begins to pay off her loan?

? Exercise 6.9.0.26

Cameron takes out a private student loan for $11,385 in their third year of their undergraduate communications program. They begin paying their loan after 27 months. If Cameron’s interest rate was 9.125%, how much is their balance when they begin to pay off the loan?

? Exercise 6.9.0.27

Farzana refinances her loans for a reduced interest rate before she begins to pay them off. The total of her refinanced loan is $39,447.50. The new interest rate is 6.99%. How much are Farzana’s payments if she has a 10-year term on this loan?

? Exercise 6.9.0.28

Carly refinances her loans for a reduced interest rate before she begins to pay them off. The total of her refinanced loan is $61,332.00. The new interest rate is 7.45%. How much are Carly’s payments if she has a 10-year term on this loan?

? Exercise 6.9.0.29

Andrew consolidates his federal student loans to make paying them easier. They total $27,582.00. The consolidated loan is at 4.99% interest. How much are Andrew’s payments if his new term is 15 years?

? Exercise 6.9.0.30

Manuel consolidates his federal student loans to make paying them easier. They total $19,470.00. The consolidated loan is at 4.99% interest. How much are Manuel’s payments if his term is 10 years?

? Exercise 6.9.0.31

Steven has four members in his Georgia household and makes $53,400 annually. The federal poverty guideline for him is $27,750. What is Steven’s discretionary income?

? Exercise 6.9.0.32

Akash has three members in his New Jersey household and makes $38,120 annually. The federal poverty guideline for him is $23,030. What is Akash’s discretionary income?

❓ Exercise 6.9.0.33

Kateryna has two members in her Iowa household and earns \$42,500 annually. The federal poverty guideline for her is \$18,310. She qualifies for a REPAYE repayment plan. What is the maximum monthly payment Kateryna would pay?

❓ Exercise 6.9.0.34

Keirstin lives by herself in her New Mexico home and earns \$28,530 annually. The federal poverty guideline for her is \$13,590. She qualifies for a REPAYE repayment plan. What is the maximum monthly payment Keirstin would pay?

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6.10: Credit Cards

Figure 6.10.1: Credit cards are both a convenience and a danger. (credit: "Credit Cards" by Sean MacEntee/Flickr, CC BY 2.0)

> **Learning Objectives**
>
> 1. Apply for a credit card armed with basic knowledge.
> 2. Distinguish between three basic types of credit cards.
> 3. Compare and contrast the benefits and drawbacks of credit cards.
> 4. Read and understand the basic parts of a credit card statement.
> 5. Compute interest, balance due, and minimum payment due for a credit card.

It can be difficult to get along these days without at least one credit card. Most hotels and rental car agencies require that a credit card is used. There are even a number of retailers and restaurants that no longer accept cash. They make online purchasing easier. And nothing contributes more to a good credit rating than a solid history of making credit card payments on time.

Being granted a credits card is a privilege. Used unwisely that privilege can become a curse and the privilege may be withdrawn. In this section, we will talk about the different types of credit cards and their advantages and disadvantages. The more knowledge a cardholder has about the credit card industry, the better able credit accounts can be managed, and that knowledge may cause major adjustments to a cardholder’s lifestyle.

All credit cards are not equal, but they all represent consumers borrowing money, usually from a bank, to pay for needs and “wants.” As such, they are a type of loan, and your repayment may include interest. (You might want to review Section 6.8, which discusses loans and repayment plans.)

There are many institutions and credit cards to choose from. Use caution as you shop around for a credit card that suits you. Your top concern is likely the interest rates on purchases and cash advances. But be careful to also read the small print regarding charges for late payments, and other fees such as an **annual fee,** where the credit card charges you (the cardholder) a fee each year for the privilege of using the cards. Many cards charge no such fee, but there are many that charge modest to heavy fees. Make sure to understand rules for **reward programs**, where the credit card issuer grants benefits based on one’s spending. Finally, once one applies for and is granted a credit card, pay attention to the credit limit the bank offers. Once a company is owed that much money, use of the card for purchases should be curtailed until some of the debt is paid off.

> **Checkpoint**
>
> *The interest rate will not matter if the balance is paid every month. When the balance is paid every month, there is NO INTEREST charged.*

Types of Credit Cards

There are basically three types of credit cards: bank-issued credit cards, store-issued credit cards, and travel/entertainment credit cards. We will look at all three and explain the good and the bad qualities of each.

Bank-Issued Credit Cards

Perhaps the most widely used credit card type is the **bank-issued credit card**, like Visa or MasterCard (and even American Express and Discover cards). These types of cards are an example of revolving credit, meaning that additional credit is extended before the previous balance is paid—but only up to the assigned credit limit. Bank-issued cards are considered the most convenient, as they can be used to purchase anything, including apparel, furniture, groceries, fuel for automobiles, meals, hotel bills, and so on, just as if paying with cash. The interest rates on bank-issued credit cards are usually lower than those for other credit cards we'll discuss, and the credit limits are generally higher. Currently, bank-issued cards have an average 20.09% APR.

Store-Issued Credit Cards

Store-issued credit cards are issued by retailers. One can hardly walk into a store these days without being offered a discount on purchases if one applies for the store credit card. These cards can only be used in that store or family of stores that issues the card. However, if a store credit card is associated with Visa, MasterCard, or American Express, then the card might be used the same way that the bank-issued cards are used. This is called cobranding. The logo of the bank-issued card will be present on the store card. Many stores offer both types. Like other credit cards, they may come with an annual fee.

Store credit cards usually charge higher interest rates than bank-issued cards. Currently, store credit cards have an APR (annual percentage rate) of 24.15%. Any rewards offered by store credit cards are usually limited to purchases made in their own store, and it typically takes longer to accumulate enough rewards or points to redeem them, whereas cobranded credit cards offer opportunities to earn rewards on all purchases, regardless of whether purchases are made in the issuing store or not.

Store credit cards usually offer lower credit limits, at least in the beginning. After being proved to be a responsible credit card owner, credit limits can be raised. Nevertheless, store credit cards are a good choice for those new to the credit card industry. If on-time payments are consistently made, it is an excellent way to get started building a credit history.

Travel/Entertainment Cards, or Charge Cards

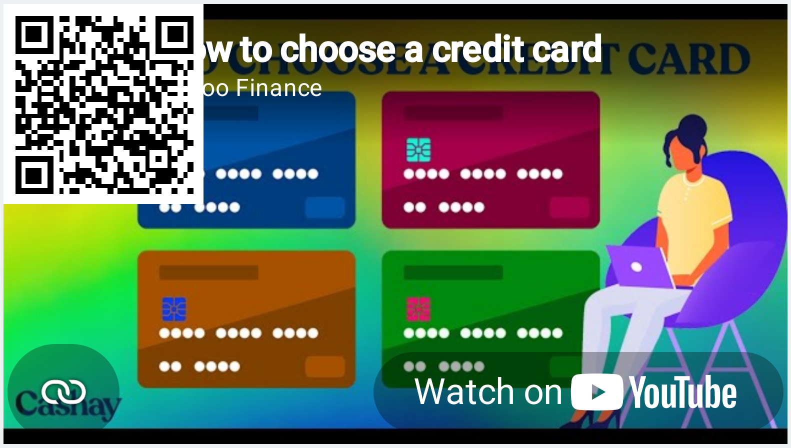
This is the third type of credit card. The **travel and entertainment cards,** also known as **charge cards,** first and foremost offer very high limits or unlimited credit, but they must be paid in full every month. They generally charge high annual fees and impose expensive penalties should a payment be late. On the other hand, they typically have longer grace periods and offer many and various kinds of rewards.

Check out this nerdwallet article about the differences between a charge card and a credit card.

Checkpoint

An interest rate will greatly depend on credit score. Responsible use of credit cards will increase a credit score. See the WHO KNEW? from The Basics of Loans.

Video



Choosing a Credit Card

Who Knew?: Top Travel and Entertainment Cards

At one time, Diner's Club was the premier entertainment card. To be accepted into the Diner's Club and be rewarded with a charge card meant one was special. Today, an American Express card is held with the same reverence as Diner's Club was in the past. It was a privilege to own one of these cards. There is a lot of competition going on to supplant Diner's Club, as shown by the Chase Sapphire Preferred Card's travel rewards and benefits.

Another thing worth mentioning here and something that appears to be an unusual offering is part of the American Express Gold and higher-level cards. Cardholders can actually open a savings account, buy a CD, or apply for a personal or business loan from American Express. The company boasts a higher interest rate than what is paid on traditional savings accounts and CDs.

Example 6.10.1: Comparing Credit Cards

1. Which type of credit card is paid off every month so has no interest to be paid, but comes with high fees?
2. Which type of credit cards are the most widely accepted?
3. Which type of credit cards are the most limited?

Answer

1. Charge cards are to be paid off completely each month.
2. Bank-issued credit cards are the most flexible to use, because they are not limited to which retailers or service providers accept them.
3. Store issues credit cards are the most limited, since they only work in that family of stores.

Your Turn 6.10.1

Which type of credit card typically have the highest interest rates?

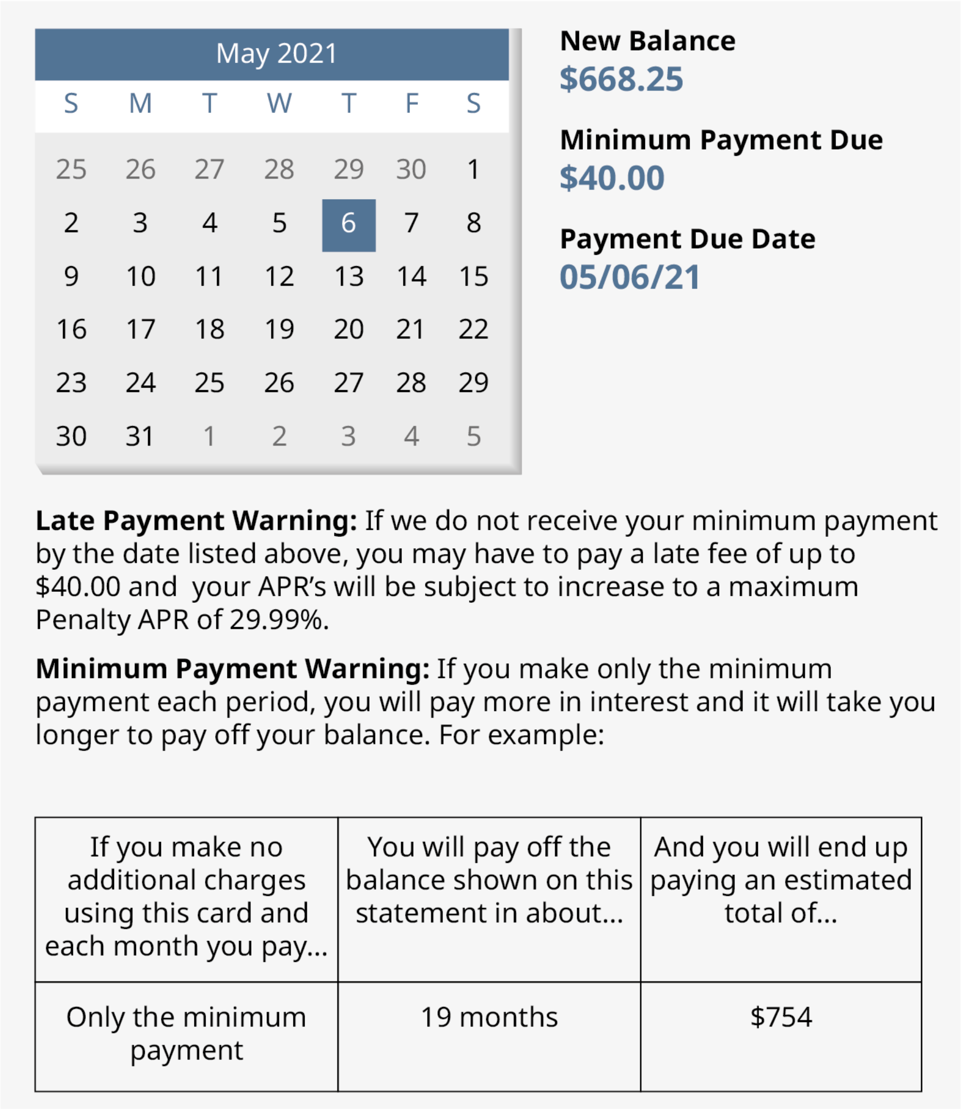
What should be considered other than interest rate when selecting a card?

Which type of cards typically have high annual fees?

Credit Card Statements

Cardholders usually receive monthly statements and have 21 days to pay the minimum amount due. The statements itemize and summarize activity on the credit card for that statement's billing period. The billing period for a credit card is generally a month long, but typically does not start and end on the first and last days of the month. The statement will include the current balance, interest rate, the minimum payment due, and the due date. Be aware, different companies produce statements that are laid out differently. The information will be clearly labeled though.

The due date is a top concern. Missing a due date is one of the worst things a cardholder can do financially, and this is by far the biggest downfall of owning a credit card. Not only is the cardholder subject to late fees, but when a payment is late more than once there is a high probability that the cardholder will be negatively reported to the credit bureaus, which can quickly erode a credit score. Figure 6.25 shows an excerpt from an actual statement from a Chase Bank Visa card, based on the current $668.25 balance.



May 2021

S	M	T	W	T	F	S
25	26	27	28	29	30	1
2	3	4	5	6	7	8
9	10	11	12	13	14	15
16	17	18	19	20	21	22
23	24	25	26	27	28	29
30	31	1	2	3	4	5

New Balance
$668.25

Minimum Payment Due
$40.00

Payment Due Date
05/06/21

Late Payment Warning: If we do not receive your minimum payment by the date listed above, you may have to pay a late fee of up to $40.00 and your APR's will be subject to increase to a maximum Penalty APR of 29.99%.

Minimum Payment Warning: If you make only the minimum payment each period, you will pay more in interest and it will take you longer to pay off your balance. For example:

If you make no additional charges using this card and each month you pay...	You will pay off the balance shown on this statement in about...	And you will end up paying an estimated total of...
Only the minimum payment	19 months	$754

Figure 6.10.2: Credit card statement

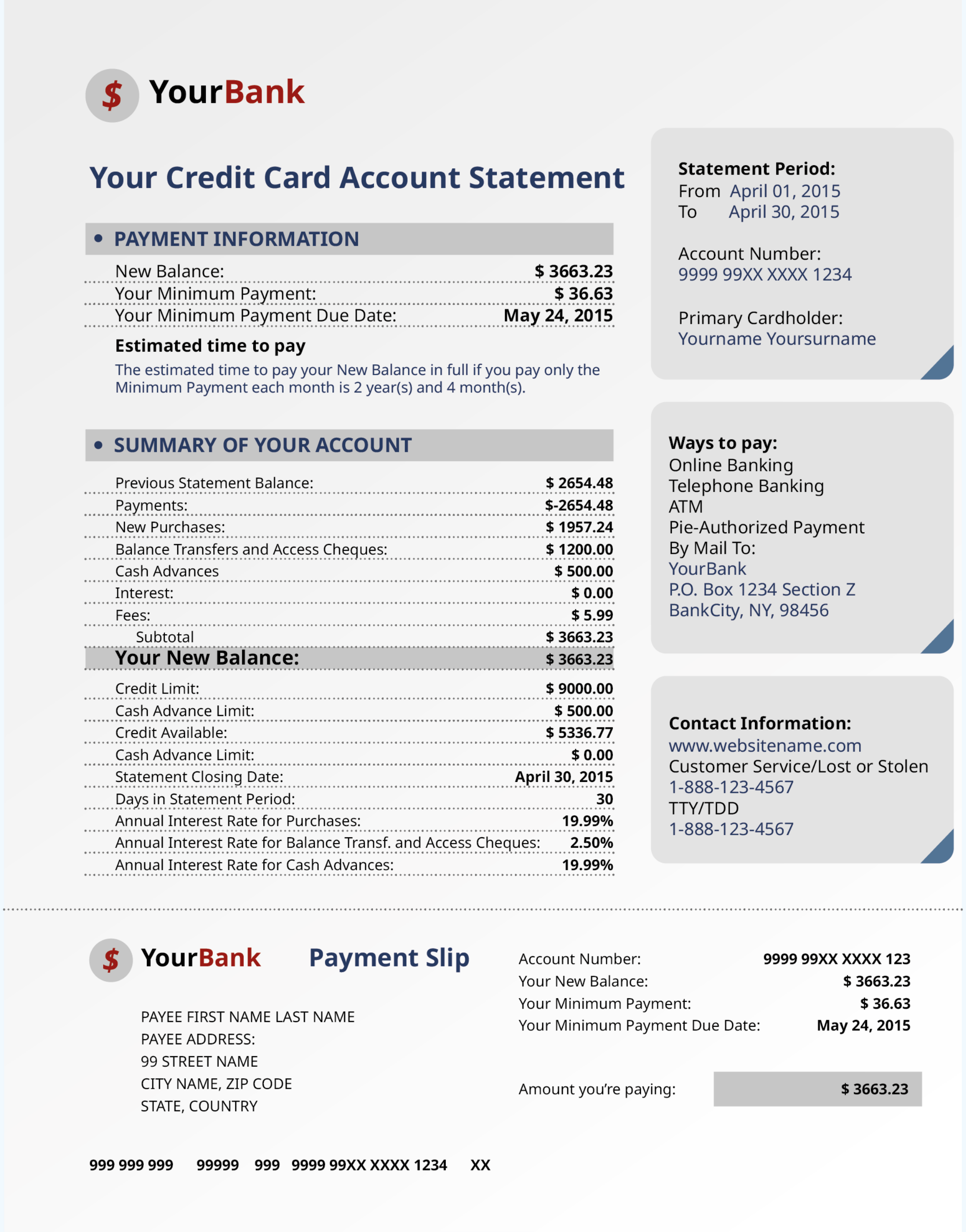
Specifically pay attention to the late payment penalty and minimum payment warning statements. stating that if no other purchases are made and you continue making only the minimum payment, it will take 19 months to pay off the balance and you will pay $754.00. You can't say you were not warned.

It is critical that you examine your statement every month because it is always a possibility that your account may have been compromised. If you should notice fraudulent charges on your statement, notifying the credit card company is often enough to have those charges researched by the company and removed. The card with the fraudulent charges will be canceled and a new card with a new account number will be sent to you.

✓ Example 6.10.2: Reading a Credit Card Statement

On the credit card statement Figure 6.26, identify

1. The balance due
2. The minimum required payment
3. The length of time it takes to pay off the balance by paying the minimum payments and without charging more to the card
4. The interest rate for purchases



YourBank

Your Credit Card Account Statement

• PAYMENT INFORMATION

New Balance:	**$ 3663.23**
Your Minimum Payment:	**$ 36.63**
Your Minimum Payment Due Date:	**May 24, 2015**

Estimated time to pay

The estimated time to pay your New Balance in full if you pay only the Minimum Payment each month is 2 year(s) and 4 month(s).

• SUMMARY OF YOUR ACCOUNT

Previous Statement Balance:	**$ 2654.48**
Payments:	**$-2654.48**
New Purchases:	**$ 1957.24**
Balance Transfers and Access Cheques:	**$ 1200.00**
Cash Advances	**$ 500.00**
Interest:	**$ 0.00**
Fees:	**$ 5.99**
Subtotal	**$ 3663.23**
Your New Balance:	**$ 3663.23**
Credit Limit:	**$ 9000.00**
Cash Advance Limit:	**$ 500.00**
Credit Available:	**$ 5336.77**
Cash Advance Limit:	**$ 0.00**
Statement Closing Date:	**April 30, 2015**
Days in Statement Period:	**30**
Annual Interest Rate for Purchases:	**19.99%**
Annual Interest Rate for Balance Transf. and Access Cheques:	**2.50%**
Annual Interest Rate for Cash Advances:	**19.99%**

Statement Period:
From April 01, 2015
To April 30, 2015

Account Number:
9999 99XX XXXX 1234

Primary Cardholder:
Yourname Yoursurname

Ways to pay:
Online Banking
Telephone Banking
ATM
Pie-Authorized Payment
By Mail To:
YourBank
P.O. Box 1234 Section Z
BankCity, NY, 98456

Contact Information:
www.websitename.com
Customer Service/Lost or Stolen
1-888-123-4567
TTY/TDD
1-888-123-4567

YourBank **Payment Slip**

PAYEE FIRST NAME LAST NAME
PAYEE ADDRESS:
99 STREET NAME
CITY NAME, ZIP CODE
STATE, COUNTRY

Account Number:	**9999 99XX XXXX 123**
Your New Balance:	**$ 3663.23**
Your Minimum Payment:	**$ 36.63**
Your Minimum Payment Due Date:	**May 24, 2015**
Amount you're paying:	**$ 3663.23**

999 999 999 99999 999 9999 99XX XXXX 1234 XX

Figure 6.10.3: Credit card account statement

Answer

1. The balance due is under the payment information heading and is $3,663.23.
2. The minimum payment due is also under the payment information heading, and is $36.63.
3. The time to pay off the balance using only minimum payments is below the payment information, and says it takes 2 years and 4 months to pay off the balance.
4. The interest rate for purchases is toward the bottom of the statement. It is 19.99%.

Your Turn 6.10.2

Referring to the statement above, answer the following:

What is the statement period?

What is the credit limit?

How much in fees were charged?

Video

w To Read Your Credit Card Statem
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ERT
ading
a Credit Card
Statement!
Watch on YouTube

Reading Credit Card Statements

Compute Interest, Balance Due, and Minimum Payment Due for a Credit Card

Computing all of these values depends on understanding and computing the average daily balance on a credit card. Once that is known, the interest, balance due, and minimum payment can be found.

Above all else, if you pay off the entire balance each month, interest is not charged.

Average Daily Balance

Most credit card companies compute interest using the **average daily balance** method.

To find the average daily balance on your credit card, determine the balance on the card each day of the billing period (often that month), and take the average. One process to find that average daily balance follows these steps:

1. Start with a list of transactions with their dates and amounts.
2. For each day that had transactions, find the total of the transactions for the day. Expenditures are treated as positive values, payments are treated as negative values.
3. Create a table containing each day with a different balance. The balance is the previous balance plus the day’s total transactions.
4. Add a column for the number of days those balances until the balance changed.
5. Add a column that contains the balances multiplied by the number of days until the balance changed.
6. Find the sum of that last column.
7. Divide the sum by the number of days in the billing period (often the number of days in the month). This is the average daily balance.

Example 6.10.3: Computing Average Daily Balance

The billing cycle goes from May 1 to May 31. The balance at the start of the billing cycle is \$450.21. The list of transactions on the card is below.

Date	Activity	Amount
1-May	Billing Date Balance	\$450.21

Date	Activity	Amount
10-May	Payment	$120.00
15-May	Groceries	$83.43
26-May	Auto Parts	$45.12
26-May	Restaurant	$85.34
30-May	Shoes	$98.23

Find the average daily balance for the credit card during the month of May.

Answer

To find the average daily balance, we use the following steps.

1. Start with a list of transactions with their dates and amounts.
   This list is provided.
2. For each day that had transactions, find the total of the transactions for the day.
   The only day with more than one transaction was May 26. The sum of those transactions is $130.46. Treating the payment as a negative value, the daily transaction amounts are

Date	Amount
1-May	$450.21
10-May	-$120.00
15-May	$83.43
26-May	$130.46
30-May	$98.23

3. Create a table containing each day with a different balance.
   The new table with dates that had different balances is below.

Date	Balance
1-May	$450.21
10-May	$330.21
15-May	$413.64
26-May	$544.10
30-May	$642.33

4. Now, add a column for the number of days those balances until the balance changed. The days until the balance changes is found by finding the difference in the dates. For instance, from May 15 to May 26 was 11. Adding that column to the table we have

Date	Balance	Days Until Balance Changes
1-May	$450.21	9
10-May	$330.21	5
15-May	$413.64	11
26-May	$544.10	4

Date	Balance	Days Until Balance Changes
30-May	$642.33	2

The last entry was 2 since there are 31 days in May.

5. Add a column that contains the balances multiplied by the number of days until the balance changed. We create the column and multiply the values.

Date	Balance	Days Until Balance Changes	Balance Times Days
1-May	$450.21	9	$4,051.89
10-May	$330.21	5	$1,651.05
15-May	$413.64	11	$4,550.04
26-May	$544.10	4	$2,176.40
30-May	$642.33	2	$1,284.66

6. Find the sum of that last column. Adding that last column we have a sum of $13,714.04.
7. There are 31 days in May, so divide the sum by 31, which gives an average of $442.39, which is the average daily balance.

Your Turn 6.10.3

The billing cycle goes from June 1 to June 30. The previous month's balance is $563.80. The transactions are in the table below.

Date	**Activity**	**Amount**
1-Jun	Balance	$563.80
2-Jun	Gasoline	$47.50
2-Jun	Groceries	$63.42
15-Jun	Movie	$38.75
15-Jun	Payment	$250.00
27-Jun	Pharmacy	$31.21
28-Jun	Auto fuel	$48.00

Find the average daily balance for this credit card.

Calculating the Interest for a Credit Card

The interest charged for a credit card is based on the daily interest rate of the card, the number of days in the billing cycle, and the average daily balance on the card.

FORMULA

The interest charge, I, for a credit card during a billing cycle is $I =$, where ADB is the average daily balance, r is the annual percentage rate, and d is the number of days in the billing cycle. As before, interest is rounded up to the next penny.

✓ Example 6.10.4: Calculating Interest for a Credit Card Billing Cycle

Compute the interest charged for the credit card based on the given average daily balance (ABD), annual interest rate, and number of days in the billing cycle.

1. ADB = $2,765.00, annual interest rate 13.99%, billing cycle of 30 days
2. ADB = $789.30, annual interest rate 17.99%, billing cycle of 31 days
3. ADB = $1,037.85, annual interest rate 11.99%, billing cycle of 28 days

Answer

1. Substituting $2,765.00 for ADB, 0.1399 for r and 30 for d and calculating, we find the interest charge to be $I = \frac{\text{ADB} \times r \times d}{365} = \frac{\$2{,}765.00 \times 0.1399 \times 30}{365} = \$31.80$.
2. Substituting $789.30 for ADB, 0.1799 for r and 31 for d and calculating, we find the interest charge to be $I = \frac{\text{ADB} \times r \times d}{365} = \frac{\$789.30 \times 0.1799 \times 31}{365} = \$12.06$.
3. Substituting $1,037.85 for ADB, 0.1199 for r and 28 for d and calculating, we find the interest charge to be $I = \frac{\text{ADB} \times r \times d}{365} = \frac{\$1{,}037.85 \times 0.1199 \times 28}{365} = \$9.55$.

Your Turn 6.10.4

Compute the interest charged for the credit card based on the given average daily balance (ABD), annual interest rate, and number of days in the billing cycle.

ADB = $2,135.00, annual interest rate 12.9%, billing cycle of 30 days

ADB = $1,589.63, annual interest rate 9.99%, billing cycle of 31 days

ADB = $6,803.41, annual interest rate 14.9%, billing cycle of 28 days

Who Knew?: Credit Cards Charge Stores Fees

The interest you pay is not the only way a credit card company generates revenue. It also charges fees to the retailers, online stores, and service providers that allow the consumer, you, to use your credit card to pay them. These are called processing fees. Currently they typically range from 2.87% to 4.35% of each transaction. That means if you use your credit card at a store and spend $100.00, the store will have to pay the credit card company somewhere between $2.87 and $4.35.

One type of processing fee is the interchange fee. Mastercard charges the vendor 1.35% of the sale, plus an additional percentage up to 3.25%, and a fixed $001 fee for each transaction.

Added to that is an assessment fee. This fee is 0.14% for Visa cards.

Calculating the Balance of a Credit Card

The **balance,** or sometimes balance due, on a credit card is the previous balance, plus all expenses, minus all payments and credits, plus the interest on the card. As stated before, if the card was paid off, there is no interest to be paid.

✓ Example 6.10.5: Calculating the Balance of a Credit Card

Find the balance on the credit card with the given interest charge and balance before interest was charged. The cards were not paid off previously.

1. Balance before interest is $708.50, interest charge is $8.15
2. Balance before interest is $1,395.10, interest charge is $21.32

Answer

1. Adding the balance before interest to the interest charge, we find the balance to be $716.65.
2. Adding the balance before interest to the interest charge, we find the balance to be $1,416.42.

Your Turn 6.10.5

Find the balance on the credit card with the given interest charge and balance before interest was charged. The cards were not paid off previously.

Balance before interest is $560.00, interest charge is $6.44

Balance before interest is $3,218.00, interest charge is $49.17

The next example puts all those steps together.

✓ Example 6.10.6: Find Balance Due from Transactions and Interest Rate

Kaylen's credit card charges 16.9% annual interest. His current billing period is from November 1 to November 30. The balance on November 1 was $1,845.23. Use Kaylen's following transactions to determine his balance due at the end of the billing cycle.

Date	Activity	Amount
1-Nov	Billing Date Balance	$1,845.23
3-Nov	Groceries	$78.50
4-Nov	Tablet	$159.00
4-Nov	Online Game Purchase	$39.99
4-Nov	Restaurant	$47.10
10-Nov	Payment	$300.00
13-Nov	Gasoline	$58.75
13-Nov	Clothing	$135.00
18-Nov	Gift	$30.00
18-Nov	Restaurant	$21.75
28-Nov	Gasoline	$43.79

Answer

The first step is to find Kaylen's average daily balance. To find the average daily balance, we use the following steps.

1. Start with a list of transactions with their dates and amounts.
   This list is provided.
2. For each day that had transactions, find the total of the transactions for the day. The days with more than one transaction were Nov. 4, Nov. 13, and Nov. 18. Treating the payment on November 10 as a negative value, the daily transaction amounts are

Date	Amount
1-Nov	$1,845.23
3-Nov	$78.50
4-Nov	$246.09
10-Nov	-$300.00
13-Nov	$193.75
18-Nov	$51.75
28-Nov	$43.79

3. Create a table containing each day with a different balance. The new table with dates that had different balances is below.

Date	Balance
1-Nov	$1,845.23
3-Nov	$1,923.73
4-Nov	$2,169.82
10-Nov	$1,869.82
13-Nov	$2,063.57
18-Nov	$2,115.32
28-Nov	$2,159.11

4. Now, add a column for the number of days those balances until the balance changed. The days until the balance changes is found by finding the difference in the dates. For instance, from May 15 to May 26 was 11. Adding that column to the table we have
   The last entry was 3 since there are 30 days in November (the 28th, 29th, and 30th).

Date	Balance	Days Until Balance Changes
1-Nov	$1,845.23	2
3-Nov	$1,923.73	1
4-Nov	$2,169.82	6
10-Nov	$1,869.82	3
13-Nov	$2,063.57	5
18-Nov	$2,115.32	10
28-Nov	$2,159.11	3

5. Add a column that contains the balances multiplied by the number of days until the balance changed. We create the column and multiply the values.

Date	Balance	Days Until Balance Changes	Balance Times Days
1-Nov	$1,845.23	2	$3,690.46
3-Nov	$1,923.73	1	$1,923.73
4-Nov	$2,169.82	6	$13,018.92
10-Nov	$1,869.82	3	$5,609.46
13-Nov	$2,063.57	5	$10,317.85
18-Nov	$2,115.32	10	$21,153.20
28-Nov	$2,159.11	3	$6,477.33

6. Find the sum of that last column. Adding that last column we have a sum of $62,190.95.
7. There are 30 days in November, so divide the sum by 30, which gives an average of $2,073.03, which is the average daily balance.

With the average daily balance, we can determine the interest that is charged for November. Substituting ADB = $2,073.03, $r = 0.169$, and $d = 30$ into the formula $I =$ and calculating, we find the interest to be

$I = \frac{\text{ADB} \times r \times d}{365} = \frac{\$2,073.03 \times 0.169 \times 30}{365} = \$28.80.$

This interest is added to the final balance from the table in step 3, $2,159.11, which yields a balance due of $2,101.83.

Your Turn 6.10.6

Angel's credit card charges 16.9% annual interest. His current billing period is from August 1 to August 31. The balance on August 1 was $982.45. Use Angel's following transactions to determine his balance due at the end of the billing cycle.

Date	Activity	Amount
1-Aug	Billing Date Balance	$982.45
5-Aug	Food	$125.31
13-Aug	Payment	$500.00
14-Aug	Gasoline	$51.65
14-Aug	Pizza	$36.99
14-Aug	Shoes	$89.45
19-Aug	Electric bill	$178.34
21-Aug	Internet	$36.99
21-Aug	Food	$93.45
30-Aug	Gasoline	$43.18

Minimum Payment Due

The **minimum payment** due is the smallest required amount to be paid on a credit card to avoid late fees and penalties, such as an increased interest rate. The calculations for this may differ from card to card. They also depend in the balance of the credit card. General guidelines for minimum payment due are:

- For larger balances (usually over $1,000), the minimum payment will be some percentage of the balance due.
- For moderate balances (between $25 and $1,000), the minimum would be a specified dollar amount. $25 seems to be a common value.
- If the balance is small (under $25 for instance), then the minimum payment is the balance.

Those are just guidelines. Individual cards may vary in these values.

Minimum payments should only be paid if money is short in a given month. The length of time to pay off a credit card using minimum payments is quite long, and results in paying a lot of interest. It is strongly discouraged.

Example 6.10.7: Calculate the Minimum Payment Due

The FYA credit card company has the following minimum payment policy. For balances over $1,000, the minimum payment is 2.5% of the balance due plus fees, but not interest. For balances between $500.00 and $999.99, the minimum payment is $50.00. For balances $499.99 and under, the minimum payment is $25.00 or the balance due, whichever is smaller.

In the following, calculate the minimum payment due given the credit card minimum payment policy, the balance due and fees charged.

1. Balance due is $1,309.00, no fees
2. Balance due is $265.50, $35 in fees
3. Balance due is $784.90, no fees

Answer

1. The balance is over $1,000, so the minimum payment is 2.5% of the balance due plus fees. 2.5% of the balance due is $0.025 \times \$1{,}309.00 = \$32.73$. Since there are no fees, the minimum payment due is $32.73.
2. The balance is under $499.99, so the minimum payment due is $25.00.
3. The balance is between $500.00 and $999.00, so the minimum payment due is $50.00.

Your Turn 6.10.7

The CLH credit card company has the following minimum payment policy. For balances over $1,000, the minimum payment is 1% of the balance due plus interest and fees. For balances between $25.00 and $999.99, the minimum payment is $25.00 plus any fees. For balances under $25.00, the minimum payment is the balance due plus any fees.
In the following, calculate the minimum payment due given the credit card minimum payment policy, the balance due and fees charged.

Balance due is $2,308.00, billing cycle interest is $24.39, no fees

Balance due is $265.50, $59.00 in fees

Balance due is $19.90, no fees

Check out this nerdwallet article about minimum payments for more!

Check Your Understanding

1. What are the three main types of credit cards?
2. Which type of credit card typically has the highest interest rate?
3. Name three important pieces of information that are included in a credit card statement.
4. A credit card has an average daily balance of $1,428.50 during a 31-day billing cycle. Find the interest if the interest rate is 15.9%.
5. The billing cycle goes from June 15 to July 14, or 30 days. The balance at the start of the billing cycle is $3,825.50. The list of transactions on the card is below. Find the average daily balance.

Date	Activity	Amount
15-Jun	Billing Date Balance	$3,825.50
20-Jun	Food	$125.31
20-Jun	Clothing	$345.00
29-Jun	Payment	$750.00
1-Jul	Restaurant	$94.80
6-Jul	Gasoline	$49.75
6-Jul	Home Repair Supplies	$683.94
6-Jul	Internet	$49.99
10-Jul	Cell Phone	$85.00
10-Jul	Food	$175.24

6. Find the balance due on a credit card statement if ADB = $487.65, the annual interest rate is 16.8%, the billing cycle was 30 days, and the balance at the end of the billing cycle, before interest is added, was $689.47.

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6.10.0: Exercises

? Exercise 6.10.0.1

Name three main criteria to choose a credit card by.

? Exercise 6.10.0.2

Which type of credit card is most convenient to use?

? Exercise 6.10.0.3

Which type of credit card comes with no preset spending limit?

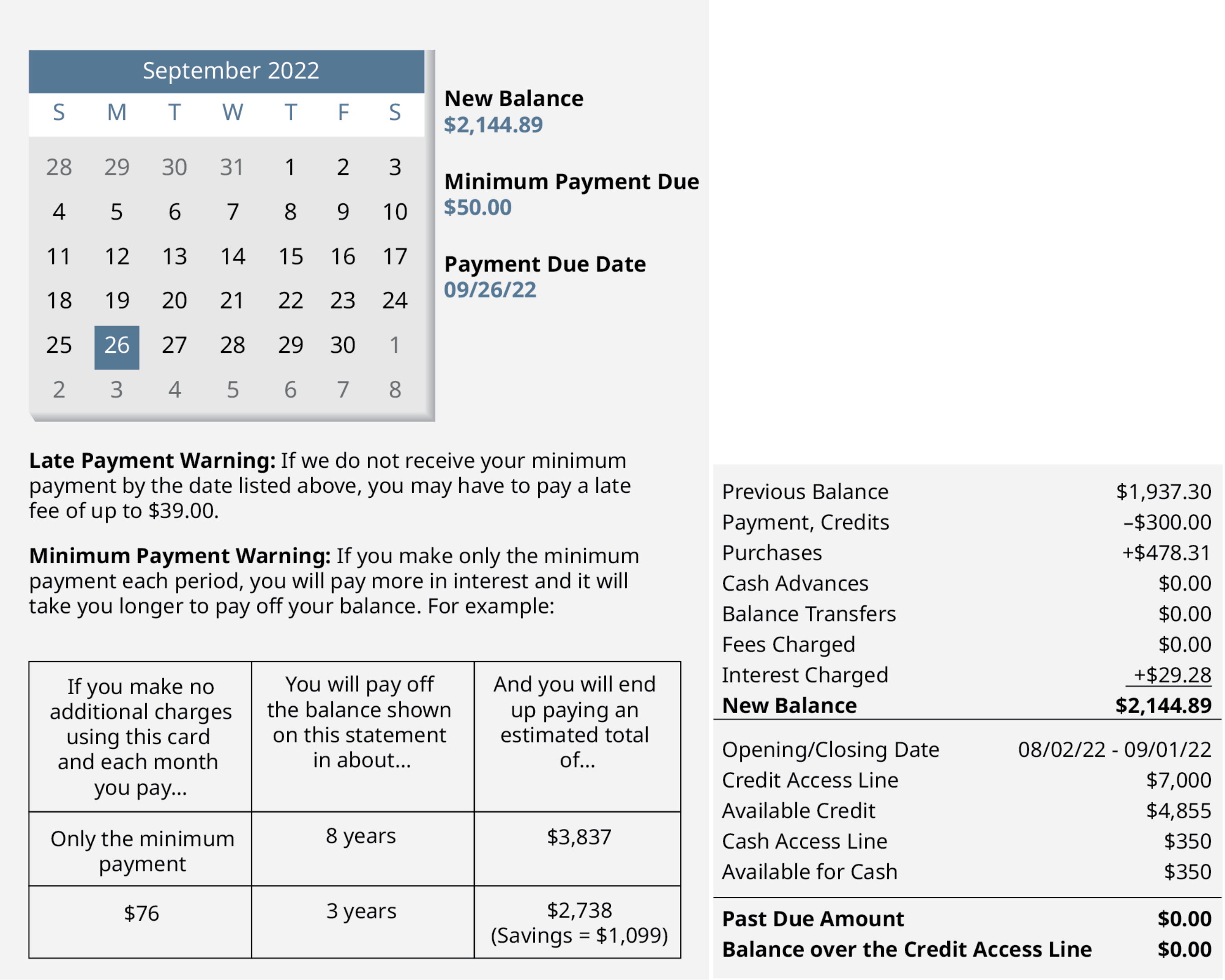
? Exercise 6.10.0.4

How much interest does a credit card holder pay if they pay off the balance every month?

? Exercise 6.10.0.5

Which type of credit card comes with the highest annual fees?

Use the following credit card statement for the following exercises.



September 2022						
S	M	T	W	T	F	S
28	29	30	31	1	2	3
4	5	6	7	8	9	10
11	12	13	14	15	16	17
18	19	20	21	22	23	24
25	26	27	28	29	30	1
2	3	4	5	6	7	8

New Balance
$2,144.89

Minimum Payment Due
$50.00

Payment Due Date
09/26/22

Late Payment Warning: If we do not receive your minimum payment by the date listed above, you may have to pay a late fee of up to $39.00.

Minimum Payment Warning: If you make only the minimum payment each period, you will pay more in interest and it will take you longer to pay off your balance. For example:

If you make no additional charges using this card and each month you pay…	You will pay off the balance shown on this statement in about…	And you will end up paying an estimated total of…
Only the minimum payment	8 years	$3,837
$76	3 years	$2,738 (Savings = $1,099)

Previous Balance	$1,937.30
Payment, Credits	–$300.00
Purchases	+$478.31
Cash Advances	$0.00
Balance Transfers	$0.00
Fees Charged	$0.00
Interest Charged	+$29.28
New Balance	**$2,144.89**
Opening/Closing Date	08/02/22 - 09/01/22
Credit Access Line	$7,000
Available Credit	$4,855
Cash Access Line	$350
Available for Cash	$350
Past Due Amount	**$0.00**
Balance over the Credit Access Line	**$0.00**

? Exercise 6.10.0.6

What is the minimum payment due?

? Exercise 6.10.0.7

How long will it take to pay the balance if the minimum only is paid and no new purchases are made?

? Exercise 6.10.0.8

How much interest was charged in this billing cycle?

? Exercise 6.10.0.9

What is the balance on the credit card?

Exercise 6.10.0.10

What is the credit limit on this card?

Exercise 6.10.0.11

What is the late payment fee for this card?

Exercise 6.10.0.12

When is the payment due?

Exercise 6.10.0.13

The billing cycle for a credit card goes from April 1 to April 30. The balance at the start of the billing cycle is $1,598.00. The list of transactions on the card is below. Find the average daily balance for the billing cycle.

Date	Activity	Amount
1-Apr	Billing Date Balance	$1,598.00
9-Apr	Gasoline	$51.24
9-Apr	Food	$105.56
9-Apr	Payment	$675.00
13-Apr	Camping Trip	$229.75
21-Apr	Gasoline	$38.45
22-Apr	Gifts	$148.88
22-Apr	Food	$49.75
30-Apr	Gym Payment	$74.99

Exercise 6.10.0.14

The billing cycle for a credit card goes from September 1 to September 30. The balance at the start of the billing cycle is $384.25. The list of transactions on the card is below. Find the average daily balance for the billing cycle.

Date	Activity	Amount
1-Sep	Billing Date Balance	$384.25
2-Sep	Food	$94.54
5-Sep	Gasoline	$25.65
5-Sep	Internet	$39.99
6-Sep	Payment	$380.00
9-Sep	Insurance	$174.52
16-Sep	Food	$83.54
16-Sep	Day Care	$350.00
20-Sep	Tires	$2,337.56
21-Sep	Child Clothing	$27.65

Date	Activity	Amount
21-Sep	Gasoline	$31.00
28-Sep	Television	$299.95

Exercise 6.10.0.15

The billing cycle for a credit card goes from October 10 to November 9. The balance at the start of the billing cycle is $930.50. The list of transactions on the card is below. Find the average daily balance for the billing cycle.

Date	Activity	Amount
10-Oct	Billing Date Balance	$930.50
11-Oct	Clothing	$350.00
14-Oct	Computer	$865.84
17-Oct	Food	$106.51
21-Oct	Payment	$700.00
21-Oct	Restaurant	$134.52
21-Oct	Hotel	$387.56
30-Oct	Hockey Game	$76.47
5-Nov	Memorabilia	$150.00
5-Nov	Restaurant	$94.45
6-Nov	Gasoline	$49.19

Exercise 6.10.0.16

The billing cycle for a credit card goes from February 15 to March 16 during a non-leap year. The balance at the start of the billing cycle is $292.82. The list of transactions on the card is below. Find the average daily balance for the billing cycle.

Date	Activity	Amount
15-Feb	Billing Date Balance	$292.82
21-Feb	Food	$64.57
22-Feb	Gasoline	$31.50
22-Feb	Food	$71.94
28-Feb	Insurance	$133.25
28-Feb	Payment	$100.00
3-Mar	Gasoline	$26.61
12-Mar	School Trip	$300.00

In the following exercises, compute the interest charged for the credit card based on the given average daily balance (ABD), annual interest rate, and number of days in the billing cycle.

Exercise 6.10.0.17

ADB = $350.00, annual interest rate 14.9%, billing cycle of 30 days.

Exercise 6.10.0.18

ADB = $4,312.00, annual interest rate 9.99%, billing cycle of 31 days.

Exercise 6.10.0.19

ADB = $563.38, annual interest rate 17.9%, billing cycle of 30 days.

Exercise 6.10.0.20

ADB = $1,043.53, annual interest rate 11.9%, billing cycle of 31 days.

In the following exercises, find the balance on the credit card with the given interest charge and balance before interest was charged. The cards were not paid off previously.

Exercise 6.10.0.21

Balance before interest is $1630.00, interest charge is $16.48.

Exercise 6.10.0.22

Balance before interest is $621.00, interest charge is $7.81.

Exercise 6.10.0.23

Balance before interest is $1,380.00, interest charge is $15.35

Exercise 6.10.0.24

Balance before interest is $2,774.00, interest charge is $44.05.

Exercise 6.10.0.25

Alsaggr’s credit card charges 11.9% annual interest. His current billing period is from April 1 to April 30. The balance on April 1 was $1,598.00. Use Alsaggr’s following transactions to determine his balance due at the end of the billing cycle.

Date	Activity	Amount
1-Apr	Billing Date Balance	$1,598.00
9-Apr	Gasoline	$51.24
9-Apr	Food	$105.56
9-Apr	Payment	$675.00
13-Apr	Camping Trip	$229.75
21-Apr	Gasoline	$38.45
22-Apr	Gifts	$148.88
22-Apr	Food	$49.75
30-Apr	Gym Payment	$74.99

? Exercise 6.10.0.26

Marisa’s credit card charges 8.9% annual interest. Her current billing period is from September 1 to September 30. The balance on September 1 was $384.25. Use Marisa’s following transactions to determine her balance due at the end of the billing cycle.

Date	Activity	Amount
1-Sep	Billing Date Balance	$384.25
2-Sep	Food	$94.54
5-Sep	Gasoline	$25.65
5-Sep	Internet	$39.99
9-Sep	Insurance	$174.52
16-Sep	Food	$83.54
16-Sep	Day Care	$350.00
20-Sep	Tires	$2,337.56
21-Sep	Child Clothing	$27.65
21-Sep	Gasoline	$31.00
28-Sep	Television	$299.95

? Exercise 6.10.0.27

Haley’s credit card charges 18.9% annual interest. Her current billing period is from October 10 to November 9. The balance on October 10 was $930.50. Use Haley’s following transactions to determine her balance due at the end of the billing cycle.

Date	Activity	Amount
10-Oct	Billing Date Balance	$930.50
11-Oct	Clothing	$350.00
14-Oct	Computer	$865.84
17-Oct	Food	$106.51
21-Oct	Payment	$700.00
21-Oct	Restaurant	$134.52
21-Oct	Hotel	$387.56
30-Oct	Hockey Game	$76.47
5-Nov	Memorabilia	$150.00
5-Nov	Restaurant	$94.45
6-Nov	Gasoline	$49.19

? Exercise 6.10.0.28

Pavly’s credit card charges 10.9% annual interest. His current billing period is from February 15 to March 16 in a non-leap year. The balance on February 15 was $292.82. Use Pavly’s following transactions to determine his balance due at the end of the billing cycle.

Date	Activity	Amount
15-Feb	Billing Date Balance	$292.82
21-Feb	Food	$64.57
22-Feb	Gasoline	$31.50
22-Feb	Food	$71.94
28-Feb	Insurance	$133.25
28-Feb	Payment	$100.00
3-Mar	Gasoline	$26.61
12-Mar	School Trip	$300.00

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6.11: Buying or Leasing a Car

Figure 6.11.1: The choice to lease or buy is based on cost and other concerns. (credit: “Car Insurance” by Pictures of Money/Flickr, CC BY 2.0)

Learning Objectives

1. Evaluate basics of car purchasing.
2. Compute purchase payments and identify the related interest cost.
3. Evaluate the basics of leasing a car.
4. Identify and contrast the pros and cons of purchasing versus leasing a car.
5. Investigate the types of car insurance.
6. Solve application problems involving owning and maintaining a car.

There are people who don’t need a car and won’t purchase one. But for many people, whether or not to have a car is not a question. Having a car is a basic necessity for these people.

Obtaining a car can be daunting. The models, the features, the additional costs, and finding funding are all steps that need to be taken. One of the big decisions is whether to buy the car or to lease the car. This section will address some of the issues associated with each option.

The Basics of Car Purchasing

The biggest questions you will answer before purchasing a car are, what do you want and what do you need?

Does it have to be new? Does it have to be a make and model you are familiar with? Does it have to have assisted driving? What other details are important to you? For a new vehicle, every feature beyond standard features comes with additional cost, which leads to the question that constrains all of your decisions about a car. How much can you afford to spend on a car?

What you can afford must include insurance costs (discussed later in this section) and maintenance and upkeep. Once you have this in mind, you can search for a car that matches, as closely as possible, what you want and can afford. Most, if not all, dealers have websites that you can search through to identify the car you want. If new cars are not affordable, used cars cost less but come with the wear and tear of use.

The sticker price of the car, called the manufacturer’s suggested retail price (MSRP), or the negotiated price you arrive at, isn’t the end of the cost to buying a car. There are many fees that accompany the purchase of the car, and perhaps even sales tax. These include but aren’t necessarily limited to the following:

- the title and registration fee, which includes registering your car with the state, getting the license plate, and assigning the title of the car to the lender. This cannot be avoided.
- a destination fee, which covers the cost of delivering the vehicle to the dealer
- a documentation fee, sometimes referred to a processing fee of handling fee, is the cost of all the paperwork the dealer did to get you the car
- a dealer preparation fee, which is for washing the car and other preparation of that sort. You should try to negotiate that out of the cost of the dealer tries to charge for that
- extended warranties and maintenance plans, which help cover some of the costs of caring for the car.
- Sales tax.

You could pay for these immediately, but they are often added to the financing of the car, meaning they become part of the principal of your loan.

Example 6.11.1: Total Cost to Purchase a Car

Nichole negotiates with her car dealership so that the price is $21,800. She needs to pay the 6.75% sales tax on the car. Other fees are $31.00 for title and registration, $1,000 in destination fees, and a documentation fee of $175. What is the total cost of Nichole’s car?

Answer

We add the car’s sales cost, sales tax and all other fees to arrive at this value. The sales tax is 6.75% on the price she negotiated, so is $\$21{,}800 \times 0.0675 = \$1{,}471.50$. Adding these up, we have a total cost of $\$21{,}800 + \$1{,}471.50 + \$31.00 + \$1{,}000 + \$175 = \$24{,}477.50$.

Your Turn 6.11.1

Luther negotiates for the price of his car, reaching agreement at $28,975. He needs to pay 8% in sales tax, 2.1% in ownership tax, a $950 destination fee, processing fees totaling $370, and registration fee of $617. What is the total cost for Luther’s purchase?

One way to bring down payments on a car is to provide a down payment or a trade in. This is money applied to the purchase price before financing happens. Be warned, the sales tax applies to the full purchase price! If you reduce the amount financed, the payments respond by going down. This often becomes part of the negotiating process.

Example 6.11.2: Total Cost to Purchase a Car with Down Payment

Sophia negotiates a $19,800 price for her new car. The sales tax is 9.5% in her area, and the dealership charges her $300 in documentation fees. Her title, plates, and registration come to $321.50. The dealership adds to this a destination fee of $1,100. If she places a down payment of $5,000 on the car, what is the total she will finance for the car?

Answer

The price was $19,800. The sales tax of 9.5% is based on this number. The sales tax comes to $\$19{,}800 \times 0.095 = \$1{,}881$. Adding all the fees to the price and the sales tax brings the total cost of the car to $\$19{,}800 + \$1{,}881 + \$300 + \$321.50 + \$1{,}100 = \$23{,}402.50$. Her down payment is applied to this number, so the $5,000 is subtracted from $23,402.50. The subtraction yields the amount to be financed, which is $18,402.50.

Your Turn 6.11.2

Carlos buys a car with a negotiated price of $36,250. The sales tax in his region is 6.5%. The dealer charges a $1,200 destination fee and a $450 documentation fee. He must pay for title, registration, and license plates, which come to $21.50. If he has a $7,500 down payment, how much will he have to finance?

When purchasing a car, the total cost to obtain the car is not the only factor in your monthly price. You will also pay an interest rate for the loan you obtain. The interest rate you will get is dependent on your credit score (see The Basics of Loans). But you can choose from different lenders. The dealership will likely offer to finance your car loan. Frequently, dealerships offer special financing with very low rates. This is to help move inventory, and may indicate their desire to make sales. This might make negotiating easier. Even if the dealership offers financing, check with your bank or credit union to determine the interest rates they are offering. To reduce your payments, choose the lowest rate you can find.

Purchase Payments and Interest

Whether or not you buy a new car or a used car, if you finance the purchase, you are taking out a loan. The interest rates available for used cards are frequently higher than those for new cars. These loan payments work exactly the same way as other loans do as far as payments are concerned. The payment function comes from The Basics of Loans. The difference between financing a new car or a used car is that financing a new car typically comes with a lower interest rate and a longer term that financing a used car.

FORMULA

The payment, pmt, per period to pay off a loan with beginning principal P is $pmt = \frac{P \times (r/n) \times (1 + r/n)^{n \times t}}{(1 + r/n)^{n \times t} -}$, where r is the annual interest rate in decimal form, t is the term in years, and n is the number of payments per year (typically, loans are paid monthly making n = 12).

Note, payment to lenders is always rounded up to the next penny.

Checkpoint

Often, the formula takes the form $pmt = \frac{P \times (r) \times (1 + r)^n}{(1 + r)^n -}$*, where* r *is the interest rate per period (annual rate divided by the number of periods per year), and* n *is the total number of payments to be made.*

Example 6.11.3: New Car Payments

In the following, calculate the monthly payment using the given total to be financed, the interest rate, and the term of the car loan.

1. Total to be financed is \$31,885, interest rate is 2.9%, for 5 years.
2. Total to be financed is \$22,778, interest rate is 4.5%, for 6 years.

Answer

1. The amount to be financed is the principal, P, which is \$31,885. The rate r is 0.029, and the term is t = 5 years. These are monthly payments, so n = 12. Substituting and calculating, we find the monthly payment to be

$$\begin{aligned} pmt &= \frac{P \times (r/n) \times (1 + r/n)^{n \times t}}{(1 + r/n)^{n \times t} - 1} \\ &= \frac{\$31{,}885 \times (0.029/12) \times (1 + 0.029/12)^{12 \times 5}}{(1 + 0.029/12)^{12 \times 5} - 1} \\ &= \frac{\$31{,}885 \times (0.00241\bar{6}) \times (1.00241\bar{6})^{60}}{(1.00241\bar{6})^{60} - 1} \\ &= \frac{\$31{,}885 \times (0.00241\bar{6}) \times (1.15583736592)}{1.15583736592 - 1} \\ &= \frac{\$89.0635298302}{0.15583736592} \\ &= \$571.52 \end{aligned}$$

2. The amount to be financed is the principal, P, which is \$22,778. The rate r is 0.045, and the term is t = 6 years. These are monthly payments, so n = 12. Substituting and calculating, we find the monthly payment to be

$$\begin{aligned} pmt &= \frac{P \times (r/n) \times (1+r/n)^{n \times t}}{(1+r/n)^{n \times t} - 1} \\ &= \frac{\$22{,}778 \times (0.045/12) \times (1+0.045/12)^{12 \times 6}}{(1+0.045/12)^{12 \times 6} - 1} \\ &= \frac{\$22{,}778 \times (0.00375) \times (1.00375)^{72}}{(1.00375)^{72} - 1} \\ &= \frac{\$22{,}778 \times (0.00375) \times (1.3093031051)}{1.3093031051 - 1} \\ &= \frac{\$111.837397673}{0.3093031051} \\ &= \$361.58 \end{aligned}$$

Your Turn 6.11.3

In the following, calculate the monthly payment using the given total to be financed, the interest rate, and the term of the car loan.

Total to be financed is $18,325, interest rate is 6.75%, for 4 years.

Total to be financed is $41,633, interest rate is 3.9%, for 6 years.

Example 6.11.4: Used Car Payments

Calculate the monthly payment for the used car if the total to be financed is $16,990, the interest rate is 7.5%, and the loan term is 3 years.

Answer

The amount to be financed is the principal, P, which is $16,990. The rate r is 0.075, and the term is t = 3 years. These are monthly payments, so n = 12. Substituting and calculating, we find the monthly payment to be

$$\begin{aligned} pmt &= \frac{P \times (r/n) \times (1+r/n)^{n \times t}}{(1+r/n)^{n \times t} - 1} \\ &= \frac{\$16{,}990 \times (0.075/12) \times (1+0.075/12)^{12 \times 3}}{(1+0.04775/12)^{12 \times 3} - 1} \\ &= \frac{\$16{,}990 \times (0.00625) \times (1.00625)^{36}}{(1.00625)^{36} - 1} \\ &= \frac{\$16990 \times (0.00375) \times (1.25144613551)}{1.25144613551 - 1} \\ &= \frac{\$132.887936515}{0.25144613551} \\ &= \$528.50 \end{aligned}$$

Your Turn 6.11.4

Calculate the monthly payment for a car loan that has $21,845 being financed at an interest rate of 6.3% for 4 years.

The Basics of Leasing a Car

Leasing a car is an alternative to purchasing a car. It is still a loan, and acts like one in many respects. They typically last either 24 months or 36 months, though other terms are available. Leases also come with mileage limits, frequently 10,000, 12,000, or 15,000 miles per year. When the lease is over, the car is returned to the dealer. At that time there may be fees that have to be paid, such as for damage to the car or for extra miles driven over the limit.

There are two components to lease costs. One is the monthly payment for the lease. The other is the fees for leasing, These often are paid before the lease is complete. These include:

- a down payment, which is your initial payment that is applied to the price of the car. It reduces the amount you finance, much the same as when you purchase a car. It is recommended that this be negotiated away.
- the acquisition fee, sometimes called the bank fee. This is the money charge for the company to set up the lease. It is essentially a paperwork fee. It is not likely that this can be negotiated.

- a security deposit, which might be required. It is about the same as 1 month's payment for the lease. The deposit is returned to you if the car is in good shape at the end. This can be negotiated away.
- disposition fees, which cover the cost the company will incur when they take your car back and are typically between $200 and $450.
- the title, registration, and license fees, just as with the purchase of a car.
- sales tax, which will likely be applied. The sales tax only covers the depreciated portion of the car (more on depreciation later) in many states. Since this depends on the state in which the car is leased, you should determine the sales tax rules for where you lease the car.

As you can imagine, this can come to a fairly high dollar amount.

✓ Example 6.11.5: Cost to Obtain a Lease

Donna wants to lease a Subaru Outback in Eden, New York. Find the total cost of obtaining her lease if there is no down payment, $175.00 in acquisition fees, a security deposit of $300.00, $350.00 in disposition fees, $102.50 in title and registration fees, and sales tax of $3,536.05.

Answer

Adding these values together we find the total cost is $4,463.55.

Your Turn 6.11.5

RJ wants to lease a Tahoe in Salt Lake City, Utah. Find the total cost of obtaining his lease if he has a $5,000.00 down payment, $225 in acquisition fees, a security deposit of $750.00, $500.00 in disposition fees, and sales tax of $4,092.

People in Mathematics: Zollie Frank

Zollie Frank and Armund Shoen founded one of the original leasing companies, Four Wheels, in 1939. Their company leased automobiles to corporations. They began by leasing 5 cars to the Petrolager pharmaceutical company in year one. This saved Petrolager money and provided a steady cash flow to the Four Wheels business. In year two, the number of cars leased to Petrolager was 75. Their new idea was to lease cars directly to companies for one year. Previously, such companies might pay for mileage, gas, and a partial down payment. Sadly, the salesmen who were being so helped often left the company before the car was paid for, and so the company lost the down payment money.

The lease was for $45 per month per car for one year.

You have some obligations when you lease a car. You must keep the car in good condition, cleaned, maintained, and free of anything more than minor damage. If the car is in poor condition when the car is returned, you will be responsible for the cost to bring the car to an acceptable condition. You are also expected to keep the mileage under its limit. If you go over, you will pay 10 to 25 cents per mile over.

Lease Payments

Lease payments are similar to regular loan payments, but have some other details. Calculating a lease payment involves knowing the following values:

- **The price of the car.** This is the cost you would pay for the car after applying all discounts, incentives, and negotiations.
- **Residual Value.** This is the manufacturer's estimate of the car's value after a set period of time. The residual value is expressed as a percentage of the manufacturer's suggested retail price (MSRP).
- **Months.** This is the length of the lease. Most leases are either 24- or 36-month leases, but other terms are available.
- **Monthly Depreciation**. The monthly depreciation is the difference between the price of the car and the residual value, divided by the number of months of the lease, and represents the monthly loss of value of the car while it's being used.
- **Money Factor (MF).** This is the interest rate, but expressed in a different way for a lease. Converting from the money factor to the annual percentage rate (APR) is done by multiplying the MF by 2400. Naturally, converting an APR to a MF is done by dividing the APR by 2400.

FORMULA

The monthly depreciation for a car, MD, is $\text{MD} =$ mfrac, P is the price paid for the car, R is the residual value of the car, and n is the number of months of the lease.

The annual percentage rate for a lease is $\text{APR} =$, where MF is the money factor of the lease. The MF for a lease is $\text{MF} = \text{APR}/2,$.

Example 6.11.6: Monthly Depreciation of a Car

1. The purchase price of a car is \$25,000. Its residual price is \$14,500. What is its monthly depreciation for a 36-month lease?
2. The purchase price of a car is \$30,000. Its residual price is \$18,600. What is its monthly depreciation for a 24-month lease?

Answer

1. The monthly depreciation formula is $\text{MD} =$ mfrac, Substituting \$25,000 for P, \$14,500 for R, and 36 for n, we find MD to be $\text{MD} = \frac{P-R}{n} = \frac{\$25{,}000 - \$14{,}500}{36} = \frac{\$10{,}500}{36} = \$291.67$.
2. The monthly depreciation formula is $\text{MD} =$ mfrac, Substituting \$30,000 for P, \$18,600 for R, and 24 for n, we find MD to be $\text{MD} = \frac{P-R}{n} = \frac{\$30{,}000 - \$18{,}600}{24} = \frac{\$11{,}400}{24} = \$475.00$.

Your Turn 6.11.6

The purchase price of a car is \$27,500. Its residual price is \$17,875.00. What is its monthly depreciation for a 24-month lease?

Example 6.11.7: Converting Between APR and MF

1. Find the annual percentage rate if the money factor is 0.00001875.
2. Find the money factor if the APR is 6.25%.

Answer

1. The APR is the money factor times 2400, so $\text{APR} = 2400 \times \text{MF} = 2400 \times 0.00001875 = 0.045$. Expressed as a percentage, the APR is 4.5%.
2. The MF is the APR divided by 2400, so $\text{MF} = \text{APR}/2400 = 0.0625/2400 = 0.000026041\bar{6}$.

Your Turn 6.11.7

Find the annual percentage rate if the money factor is $0.00001208\bar{3}$.

Find the money factor if the APR is 7.9%.

Once the values above are found, the payment for the lease can be calculated.

FORMULA

The payment, PMT, for a lease is $PMT = \frac{(P-R)}{n} + (P+R)$, where P is the price paid for the car, R is the residual value of the car, n is the number of months of the lease, and MF is the money factor for the lease.

Example 6.11.8: Calculating Car Lease Payments

Calculate the lease payments for car with the following price, residual price, length of lease, and money factor or APR.

1. Price is \$28,344, residual price is \$18,140.16, 24-month lease, money factor is 0.000025.
2. Price is \$22,500, residual price is \$13,050, 36-month lease, APR is 7.5%.

Answer

1. Substituting the values P = \$28,344, R = \$18,140.16, n = 24 and MF = 0.000025 into the formula and calculating, the monthly lease payment is

$$\begin{aligned} PMT &= \frac{(P-R)}{n} + (P+R) \times \text{MF} \\ &= \frac{(\$28{,}344 - \$18{,}140.16)}{24} + (\$28{,}344 + \$18{,}140.16) \times 0.000025 \\ &= \frac{\$10{,}203.84}{24} + (\$46{,}484.16) \times 0.000025 \\ &= \$426.33 \end{aligned}$$

2. Given the APR, we find the MF which is $\text{MF} = \text{APR}/2400 = 0.075/2400 = 0.00003125$. Substituting the values P = \$28,344, R = \$18,140.16, n = 24 and the MF into the formula and calculating, the monthly lease payment is

$$\begin{aligned} PMT &= \frac{(P-R)}{n} + (P+R) \times \text{MF} \\ &= \frac{(\$22{,}500 - \$13{,}050)}{36} + (\$22{,}500 + \$13{,}050) \times 0.00003125 \\ &= \frac{\$9{,}450}{24} + (\$35{,}550) \times 0.000025 \\ &= \$263.62 \end{aligned}$$

Your Turn 6.11.8

Calculate the lease payments for car with the following price, residual price, length of lease, and money factor or APR.

Price is \$38,750, residual price is \$18,140.16, 36-month lease, money factor is 0.000035.

Price is \$45,600, residual price is \$21,312.50, 24-month lease, APR is 11.7%.

Comparing Purchasing and Leasing

When deciding to buy or lease a car, the differences between the two options should be carefully evaluated. The following is a list of points of comparison between the two.

- The payments for a lease are likely less than the payment for purchasing.
- When leasing, you get a new car after the lease term is over, typically 24 or 36 months. Buying the car means the same car is driven until it is re-sold and a new one bought. Essentially, leasing a car is equivalent to renting a car.
- The leased car is new, so all warrantees are in force and you drive the car during its best years. When the car is purchased, it may be kept past its warrantees and may be driven until it is quite old.
- Each time you lease a new car, all the fees and taxes must be paid again. When buying a car, these fees are only paid once.
- Leasing contracts carry restrictions on the mileage you can drive per year, and going over incurs more cost at the end of the lease. Buying the car means no such mileage limits.
- When leasing, you are obligated to keep the vehicle in good condition and maintained according to the dealer’s schedule. Some dealerships will even pay for oil changes over the life of the lease. When the car is purchased, the upkeep schedule is the choice of the owner.
- When a car is purchased and kept for long enough, the warranty expires and the owner is responsible for all maintenance items and repairs. The warrantee for a car won’t expire during the lease term.
- When a new car is purchased and the loan is paid off the car is still owned by the buyer and may be traded in when a new car is to be purchased. When leasing, the car is returned to the dealer when the lease term is over.

When deciding between the two, you are choosing between these features. If you aren’t willing to drive an older car or deal with the upkeep that accompanies an older car, you may want to lease. This means you will need to pay those beginning costs each time the lease is up. If you want to own the car after the payments are over, then you may want to buy a car. This means you are paying for all the upkeep after the warrantees expire, but you have no limits on mileage and own the car at the end. It really depends on your preferences.

✓ Example 6.11.9: Lease or Buy

In the following, determine if a lease or purchase of a car is better.

1. Joyce is concerned with large repairs and does not want to deal with them.

2. Maurice prefers to drive newer cars.

Answer

1. Since Joyce does not want to deal with repairs, so leasing would be a better choice. This way, the warranty covers most of the big repairs that could need to be done.
2. Since Maurice likes to drive new cars, leasing is a better option, since he will lease a new car every 2 to 3 years.

Your Turn 6.11.9

Christopher does a lot of driving, averaging over 20,000 miles per year.

John eventually wants to not make car payments.

Car Insurance

Car insurance is meant to cover costs associated with accidents involving cars. Most states (all except New Hampshire and Virginia) require some insurance. Without insurance, the state may not let you get a license for your car or register your car. Your state’s requirements can be hard to follow. Fortunately, insurance companies and brokers will make sure your insurance is sufficient for your state and will warn you if you try to not meet the requirements. Of course, they may offer more than what is sufficient, so it is your responsibility to determine how much coverage you want, as long as the minimum insurance requirements are met. The cost of insurance should be accounted for when evaluating the affordability of buying or leasing a car.

Whether your car is leased or owned, you do need insurance. This contributes to the cost of having the vehicle. Leasing or owning makes no difference to the insurance company you choose, because they are insuring you based on what you are driving, your driving record, and other information about you including where you live and your age. These insurance policies have many components that address different costs that can come from auto accidents. This may make details confusing, and you may not realize what you are paying for until you must use it. Here is a brief outline of the different components of auto insurance, many of which are required by the state that issues your driver’s license.

- **Liability insurance** is mandatory coverage in most states. Liability insurance covers property damage and injuries to others should you be found legally responsible for an accident. You are required to have the minimum amount of coverage, as determined by your state, in both areas.
- **Collision insurance** is insurance covering the damage caused to your car if involved in an accident with another vehicle.
- **Comprehensive insurance** is an extra level of coverage if involved in an accident with another vehicle and covers other things like theft, vandalism, fire, or weather events as outlined in your policy. There is a deductible assigned to each type of insurance, an amount that you pay out of pocket before your comprehensive coverage takes effect. Comprehensive insurance is often required if you lease or finance the purchase of a vehicle.
- **Uninsured or underinsured motorist insurance:** If you are hit by an uninsured or underinsured motorist, this insurance will help pay medical bills and damage to your car.
- **Medical payments insurance** is mandatory in some states and helps pay for medical costs associated with an accident, regardless of who is at fault.
- **Personal injury protection insurance** is coverage for certain medical bills and other expenses due to a car accident. Other covered expenses may include loss of income or childcare, depending on your policy.
- **Gap insurance** is designed to cover the gap between what is owed on the car and what the car is worth in the event your car is a total loss.
- **Rental reimbursement insurance** is coverage for a rental car while your car is under repair resulting from an accident.

You can also purchase other special insurance policies, such as classic car insurance, new car replacement insurance, and sound system replacement insurance, to name a few. It is important that you determine exactly what you need, as insurance policies can be expensive and vary according to your age, driving history, and where you live.

✓ Example 6.11.10: Types of Insurance

1. Which component of insurance pays if you are in an accident with a motorist without insurance?
2. Which component of insurance pays for the remaining principal owed on your car in the case of a total loss?

Answer

1. Uninsured motorist insurance covers accidents with those who have no insurance.
2. Gap insurance will cover the gap between what is owed on the car and what it is worth if an accident results in a total loss.

Your Turn 6.11.10

Which component of insurance covers medical bills in the event of an accident?

Which component of insurance pays for costs to others if you have an accident and are found legally responsible for those damages?

✓ Example 6.11.11: Monthly Cost of Owning a Car

If your car payment is $287.50 per month and your car insurance is $930 every 6 months, what is the cost of the car per month when accounting for the insurance?

Answer

The cost of the car including insurance is the monthly payment, $287.50, plus the monthly cost of the insurance. The insurance cost per month is $\$930/6 = \$155$ since the insurance cost is for every 6 months. Adding those the cost with insurance is $442.50.

Your Turn 6.11.11

If your car payment is $410.86 per month and your car insurance is $2,190 per year, what is the cost of the car per month when accounting for the insurance?

Maintaining a Car

Cars are not a buy it and forget it item. They require upkeep, which adds to the cost of owning the car. Tires, brakes, and wipers need replacing. Oli changes, inspections, so many things other than gasoline. Below is a list of some maintenance requirements for cars, along with cost and roughly how often they should happen.

Maintenance	Frequency	Cost Range
New Tires	Every 5 years	$25–$300 per tire
Oil Change	Every 3,000–6,000 miles	$35–$75
Wipers	Every 6–12 months	$20–$40
Inspection	Annual	$10–$50
Brake pads	10,000–20,000 miles	$200–$300
Air Filter	15,000–30,000 miles	$35–$80

When designing a budget, these expected costs should be accounted for. Extra money per month should be saved in addition to this budget category, to handle unanticipated, and perhaps very costly, repairs.

✓ Example 6.11.12

Estella needs to budget for her car maintenance. She expects to buy new tires each 4 years, which will cost her $480 to replace them all. Oil changes near her cost $49.99, and she believes she will get one every 4 months. Her inspection costs $15 per year. Wipers for her car cost $95 for all three and she anticipates changing them every year. She drives less than 30,000 miles per year, so she plans to replace the air filter once per year. The air filter for her car costs $57.50. How much should she budget per month to cover these costs?

Answer

Her yearly costs are the wipers, inspection, and tires, which total $167.50. Tires will be bought every 5 years, so per year she should budget $96. Her oil changes, which will happen three times per year, cost $49.99 each, so she'll spend $149.97 for the year on oil changes. Adding these up, her yearly budget should include $413.47 for maintenance. Dividing by 12 gives the monthly budget for maintenance, which is $34.46 (rounded up to the next penny).

Your Turn 6.11.12

Natalie needs to budget for her car maintenance. She expects to buy new tires each 5 years, which will cost her $390 to replace them all. Oil changes near her cost $59.99, and she believes she will get one every 3 months. Her inspection costs $25 per year. Wipers for her car cost $115 for all three and she anticipates changing them every year. She drives less than 30,000 miles per year, so she plans to replace the air filter once per year. The air filter for her car costs $46.25. How much should she budget per month to cover these costs?

Check Your Understanding

1. What is destination fee?
2. What is a title and registration fee?
3. Calculate the monthly payment if the total to be financed is $34,570, the interest rate is 3.5%, for 5 years.
4. What might be different for a used car loan and a new car loan?
5. What fees are similar between leasing a car and buying a car?
6. If the cost of a car is $30,000, and the residual value of the car after 3 years is $18,000, what is the monthly depreciation for the car?
7. If the money factor for a lease is 0.000015625, what is the annual interest rate?
8. For people who do not mind driving an older car or maintaining a car, which is preferable, a lease or purchasing?
9. What does collision insurance cover?

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6.11.0: Exercises

? Exercise 6.11.0.1

What are the two main questions to answer before buying a car?

? Exercise 6.11.0.2

What is a documentation fee?

? Exercise 6.11.0.3

What is a dealer preparation fee?

? Exercise 6.11.0.4

What is a down payment?

? Exercise 6.11.0.5

How long do leases typically last?

? Exercise 6.11.0.6

Can you drive a leased car any mileage?

? Exercise 6.11.0.7

What are three fees associated with leasing a car?

? Exercise 6.11.0.8

What is the monthly depreciation for a leased car?

? Exercise 6.11.0.9

How are the money factor and the annual percentage rate related?

? Exercise 6.11.0.10

Name two advantages to leasing over buying a car.

? Exercise 6.11.0.11

Name two advantages to buying a car over leasing a car.

? Exercise 6.11.0.12

What does liability insurance cover?

Exercise 6.11.0.13

What does collision insurance cover?

Exercise 6.11.0.14

Many expenses associated with a car can be anticipated. Name three maintenance expenses that can be anticipated.

In the following exercises, find the total cost to purchase the car.

Exercise 6.11.0.15

Alexia negotiates a purchase price of $17,850 for her new car. The sales tax in her area is 6.5%. Her license, plates, and registration come to $285.00. The dealership charges her a $600 destination fee and a $150 processing fee. How much will she finance in total for the car?

Exercise 6.11.0.16

Stephanie negotiates a purchase price of $25,670 for her new car. The sales tax in her area is 8.0%. Her license, plates, and registration come to $389.00. The dealership charges her a $700 destination fee and a $345 processing fee. How much will she finance in total for the car?

Exercise 6.11.0.17

Matthew negotiates a purchase price of $35,100 for her new car. The sales tax in his area is 7.25%. His license, plates, and registration come to $325.00. The dealership charges him a $900 destination fee and a $125 processing fee. How much will he finance in total for the car?

Exercise 6.11.0.18

Madisyn negotiates a purchase price of $45,800 for her new car. The sales tax in her area is 7.25%. Her license, plates, and registration come to $199.00. The dealership charges her a $1,000 destination fee and a $275 processing fee. How much will she finance in total for the car?

In the following exercises, calculate the car payment based on the total financed and the interest rate.

Exercise 6.11.0.19

Total to be financed is $36,775, interest rate is 2.75%, for 6 years.

Exercise 6.11.0.20

Total to be financed is $29,350, interest rate is 3.9%, for 5 years.

Exercise 6.11.0.21

Total to be financed is $27,180, interest rate is 1.99%, for 7 years.

Exercise 6.11.0.22

Total to be financed is $15,489, interest rate is 6.75%, for 4 years.

In the following exercises, find the total cost to obtain the lease.

Exercise 6.11.0.23

A $2,000.00 down payment, $120 in acquisition fees, a security deposit of $350.00, $200.00 in disposition fees, and sales tax of $2,860.

Exercise 6.11.0.24

No down payment, $260 in acquisition fees, a security deposit of $450.00, $400.00 in disposition fees, and sales tax of $3,155.00.

Exercise 6.11.0.25

A $4,000 down payment, $360 in acquisition fees, a security deposit of $900.00, $1,000.00 in disposition fees, and sales tax of $4,275.

Exercise 6.11.0.26

A $7,000.00 down payment, $225 in acquisition fees, a security deposit of $800.00, $675.00 in disposition fees, and sales tax of $3,673.

In the following exercises, find the monthly depreciation of the car.

Exercise 6.11.0.27

The purchase price of a car is $34,000. Its residual price is $23,500. What is its monthly depreciation for a 24-month lease?

Exercise 6.11.0.28

The purchase price of a car is $23,500. Its residual price is $11,750. What is its monthly depreciation for a 36-month lease?

In the following exercises find the APR based on the MF.

Exercise 6.11.0.29

MF = 0.00004125

Exercise 6.11.0.30

MF = $0.000046458\overline{3}$

In the following exercises find the MF based on the APR.

Exercise 6.11.0.31

8.75%

Exercise 6.11.0.32

6.25%

In the following exercises, find the lease payment based on the given information.

? Exercise 6.11.0.33

Price is $41,700, residual price is $27,105, 24-month lease, money factor is 0.000025.

? Exercise 6.11.0.34

Price is $22,165, residual price is $12,855.70, 24-month lease, money factor is 0.0000275.

? Exercise 6.11.0.35

Price is $30,650, residual price is $16,857.50, 36-month lease, APR is 8.25%.

? Exercise 6.11.0.36

Price is $24,800, residual price is $14,384, 36-month lease, APR is 5.85%.

? Exercise 6.11.0.37

Sara needs to budget for her car maintenance. She expects to buy new tires each 3 years, which will cost her $540 to replace them all. Oil changes near her cost $39.99, and she believes she will get one every 4 months. Her inspection costs $20 per year. Wipers for her car cost $115 for all three and she anticipates changing them every year. She drives less than 30,000 miles per year, so she plans to replace the air filter once per year. The air filter for her car costs $36.25. She plans to get new brake pads every year, which cost her $267. How much should she budget per month to cover these costs?

? Exercise 6.11.0.38

Mwibeleca needs to budget for his car maintenance. He expects to buy new tires each 3 years, which will cost her $560 to replace them all. Oil changes near his cost $69.99, and he believes he will get one every 4 months. His inspection costs $25 per year. Brake pads will cost $215 each year. Wipers for his car cost $143.50 for all three and he anticipates changing them every year. He drives less than 30,000 miles per year, so he plans to replace the air filter once per year. The air filter for his car costs $62.88. How much should Mwibeleca budget per month to cover these costs?

? Exercise 6.11.0.39

John will either lease or buy a car. The total cost to purchase the car is $35,830, and he would finance the car for 5 years at 2.99%. If he leases, he would pay $3,287.50 for the lease, and then his payments would be based on a price of $32,750, a residual price of $20,305, 36 months, with a money factor of 0.00001725. Compare the payments to purchase the car to the payments of the lease plus the lease cost divided by 36.

? Exercise 6.11.0.40

Zachary will either lease or buy a car. The total cost to purchase the car is $22,945, and he would finance the car for 6 years at 1.99%. If he leases, he would pay $2,387.75 for the lease, and then his payments would be based on a price of $21,350, a residual price of $12,390.30, 36 months, with a money factor of 0.000018375. Compare the payments to purchase the car to the payments of the lease plus the lease cost divided by 36.

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6.12: Renting and Homeownership

New Development
FOR SALE

Figure 6.12.1: Purchasing a home is a big investment, while renting is a lower cost alternative. (credit: "New construction, new development house for sale" by jongorey/houseandhammer.com, CC BY 2.0)

Learning Objectives

1. Evaluate advantages and disadvantages of renting.
2. Evaluate advantages and disadvantages of homeownership.
3. Calculate the monthly payment for a mortgage and related interest cost.
4. Read and interpret an amortization schedule.
5. Solve application problems involving affordability of a mortgage.

After renting an apartment for 10 years, you realize that it may be time to purchase a home. Your job is stable, and you could use more space. It is time to investigate becoming a homeowner. What are the things that you must consider, and what is the financial benefit of owning as opposed to renting? This section is about the advantages, disadvantages, and costs of homeownership as opposed to renting.

Advantages and Disadvantages of Renting

When renting, you will likely sign a lease, which is a contract between a renter and a landlord. A landlord is the person or company that owns property that is rented. The lease will detail your responsibilities, restrictions on activities, deposits, fees, maintenance, repairs, and rent during the term of the lease. It also defines what your landlord can, and cannot, do with the property while you occupy the property.

Like leasing a car, there are advantages to renting but also some disadvantages. Some advantages are:

- Lower cost.
- Short-term commitment.
- Little to no maintenance cost. The landlord pays for or performs most maintenance.
- You need not stay at end of lease. Once the lease term is over (the lease is up), you are not obligated to stay.
- If renting in an apartment complex, there may be a pool, gym, or community room for renters to use.

Of course, there are disadvantage too:

- No tax incentives.
- Housing cost is not fixed. When the lease is up, the rent can change.
- No equity. When you are done living in a rental, you have built no value.
- Restrictions on occupants. There may be a limit on how many can live in the apartment.
- Restrictions on decorating. The property is not yours, so any decorating or improvements need landlord permission.
- Limits on pets. Permission for pets, and their number and type, will be set forth in the lease.

- May not be able to remain when lease term is over. The landlord can, at the end of your lease, invite you to leave.
- The building may be sold, and the new landlord may institute changes to the lease when the previous lease expires.

Renting has fees to be paid at the start of the lease. Typically, when you rent, you will pay first and last months' rent and a security deposit. A security deposit is a sum of money that the landlord holds until the renter leaves the rental property. The deposit will cover repairs for damage to the apartment during the renter's stay but may be returned if the apartment is in good condition. If your landlord runs a credit check on you, the landlord may charge you for that.

Advantages of Buying a Home

The advantages to buying a home mirror the disadvantages of renting, and the disadvantages of home ownership mirror the advantages of renting.

Some advantages to buying a home are:

- There are tax incentives. The interest you pay for your mortgage (more on that later) is deductible on your federal income tax.
- There are no restrictions on pets or occupants, unless laws in your area specify limits for homes.
- You can redecorate any way you wish, limited only by the laws in your area.
- Once your mortgage is set with a fixed-interest rate, your housing cost is fixed.
- Your home grows equity, that is, the difference between what you owe and what the house is worth grows. You can use the equity to secure loans, and you recover the equity (and more if you're fortunate) when you sell the house.
- As long as you pay your mortgage and maintain the home to the standards of your community, you can stay as long as you wish.

Some disadvantages to home ownership are:

- The cost is higher than renting. Mortgages and associated costs are typically higher than rent for a similar living space.
- The owner is responsible for upkeep, maintenance, and repairs. These can be extremely costly.
- The owner cannot walk away from the property. It can be sold, but simply leaving the property, especially if not paid off yet, has serious consequences.

The big question of affordability looms large over the decision to rent or buy. Renting, strictly from an affordability viewpoint, comes with much less initial outlay and smaller commitment. If you do not have sufficient income to regularly save for possibly expensive repairs, or your credit isn't quite as good as it needs to be, then renting may be the best choice. Of course, even if you can afford to buy a home, you may choose to rent based on the comparative advantages.

Video

nting versus Buying a home | Housi
n Academy
enting versus
buying a home
Khan Academy
Watch on YouTube

Rent or Buy

Buying a home really involves two buyers. You and the mortgage company. The mortgage company has interest in the home, as they are providing the funds for the home. They want to protect their investment, and many fees are about the bank as much as the buyer. They fund a mortgage based on the value they assign the property. Not you. This means they will want some certainty that the home is sound, and you are a good investment.

Who Knew?: Closing Costs

When a home is bought, there are many costs that need to be paid at the time of purchase, which are lumped under the term closing costs. At the start of 2022, the average closing costs for a single-family home exceeded $6,800. These costs include:

- The appraisal fee, which is what is paid to someone to establish the home’s worth. The value of the home to the bank may differ from what the home is listed for, or what an app tells you the home is worth. It may run approximately $350.
- The home inspection fee. The inspection should reveal any problems with the house that will need to be fixed either before or after you obtain the home.
- The title search. The is a records search to insure there are no issues with who actually owns the property. It can cost about 0.5% to 1% of the amount you are financing.
- Prepaid taxes. You will need to pay about 6 months of taxes at the time of purchase.
- The credit report fee. This is a fee for checking your credit. You might pay $25 or more for this.
- The origination fee. This is the price the mortgage company charges you to cover the costs of creating the mortgage. This could be 0.5% to 1% (or more) of the amount you are borrowing.
- The application fee. This is just a processing fee and could come to several hundred dollars.
- The underwriting fee. This covers the cost of verifying your financial qualifications. It could be a flat fee, or some small percentage of the amount financed. Such as 0.5% or 1%.
- Attorney fees. If you use an attorney, you will have to pay the attorney.
- State of local fees. This may include a filing fee charged by the county or municipality in which you reside.

That’s a long list, and it is not even complete. When buying, be prepared to see these costs. It can be surprising. But in the end, you will have equity in the home, which means when you sell your home, you will get some of your money back.

Video

mple Conventional Closing Costs -
heedah Hill
OSING COSTS
FULLY EXPLAINED!
Watch on YouTube

Closing Costs

In the end, you must weigh your options and carefully consider your priorities in choosing to rent or buy a home.

Mortgages

Some people will purchase a home or condo with cash, but the majority of people will apply for a mortgage. A mortgage is a long-term loan and the property itself is the security. The bank decides the minimum down payment (with your input), the payment schedule, the duration of the loan, whether the loan can be assumed by another party, and the penalty for late payments. The title of the home belongs to the bank.

Since a mortgage is a loan, everything about loans from The Basics of Loans holds true, including the formula for the payments.

Monthly Mortgage Payments

The formula to calculate your monthly payments of principal and interest uses APR as the annual interest rate.

FORMULA

The payment, pmt, per month to pay down a mortgage with beginning principal P is $pmt = \frac{P \times (r/12) \times (1 + r/12)^{12 \times t}}{(1 + r/12)^{12 \times t} -}$, where r is the annual interest rate in decimal form and t is the number of years of the payment.
Note, payment to lenders is always rounded up to the next penny.

To find the total amount of your payments over the life of the loan, multiply your monthly payments by the number of payments.

Example 6.12.1: 30-Year Mortgage at 4.8% Interest

Evan buys a house. His 30-year mortgage comes to \$132,650 with 4.8% interest. Find Evan's monthly payments.

Answer

Using the information above, $P = \$132{,}650$, $r = 0.048$ and $t = 30$. Substituting those values into the formula $pmt = \frac{P \times (r/12) \times (1 + r/12)^{12 \times t}}{(1 + r/12)^{12 \times t} -}$ and calculating, we find the payment is

$$\begin{aligned} pmt &= \frac{P \times (r/12) \times (1 + r/12)^{12 \times t}}{(1 + r/12)^{12 \times t} - 1} \\ &= \frac{\$132{,}650 \times (0.048/12) \times (1 + 0.048/12)^{12 \times 30}}{(1 + 0.048/12)^{12 \times 30} - 1} \\ &= \frac{\$132{,}650 \times (0.004) \times (1.004)^{360}}{(1.004)^{360} - 1} \\ &= \frac{\$2{,}233.07781448}{3.20858992551} \\ &= \$695.97 \end{aligned}$$

His mortgage payment is \$695.97.

Your Turn 6.12.1

Paulo buys a house. His 20-year mortgage comes to \$153,899 with 4.21% interest. Find Evan's monthly payments.

To find the total amount of your payments over the life of the loan, multiply your monthly payments by the number of payments. This can be useful information, but not too many people reach the end of their mortgage. They tend to move before the mortgage is paid off.

FORMULA

The total paid, T, on an t year mortgage with monthly payments pmt is $T =$.

Example 6.12.2: 30-Year Mortgage at 5.35% Interest

Cassandra buys a house. Her 30-year mortgage comes to \$99,596 with 5.35% interest. If Cassandra pays off the mortgage over those 30 years, how much will she have paid in total?

Answer

To find the total paid over the life of the mortgage, use the formula $T =$. To calculate this, the payment must be found. Using the information above, $P = \$99{,}596$, $r = 0.0535$ and $t = 30$. Substituting those values into the formula $pmt = \frac{P \times (r/12) \times (1 + r/12)^{12 \times t}}{(1 + r/12)^{12 \times t} -}$ and calculating, we find the payment is

$$\begin{aligned} pmt &= \frac{P \times (r/12) \times (1 + r/12)^{12 \times t}}{(1 + r/12)^{12 \times t} - 1} \\ &= \frac{\$99{,}596 \times (0.0535/12) \times (1 + 0.0535/12)^{12 \times 30}}{(1 + 0.048/12)^{12 \times 30} - 1} \\ &= \frac{\$99.596 \times (0.004458\bar{3}) \times (1.004458\bar{3})^{360}}{(1.004458\bar{3})^{360} - 1} \\ &= \frac{\$2{,}202.45911222}{3.96013414693} \\ &= \$556.16 \end{aligned}$$

Using the mortgage payment of $556.16 and t = 30 years in the formula $T =$, the total that Cassandra will pay for the mortgage is $200,217.60.

Your Turn 6.12.2

Arthur buys a house. Their 15-year mortgage comes to $225,879 with 4.91% interest. If Arthur pays off the mortgage over those 15 years, how much will they have paid in total?

With the principal of the mortgage and how much total is paid over the life of the mortgage, the cost of financing can be found by subtracting the principal of the mortgage from the total paid over the life of the mortgage.

FORMULA

The cost of financing a mortgage, CoF, is $\text{CoF} = T-$ where P is the mortgage's starting principal and T is the total paid over the life of the mortgage.

Example 6.12.3: 30-Year Mortgage at 5.35% Interest

Cassandra buys a house. Her 30-year mortgage comes to $99,596 with 5.35% interest. What was Cassandra's cost of financing?

Answer

In Example 6.111, we found that the total Cassandra will pay for the $99,569 mortgage is $200,217.60. Subtracting those we find the cost of financing $\text{CoF} = T - P = \$200{,}217.60 - \$99{,}596 = \$100{,}621.60$.

Your Turn 6.12.3

Arthur buys a house. Their 15-year mortgage comes to $225,879 with 4.91% interest. What was Arthur's cost of financing?

Who Knew?: Private Mortgage Insurance (PMI)

When you purchase a home, you will have to pay a down payment. This means you have money tied to the property, which lenders believe makes you less likely to walk away from a property. The amount of the down payment will be decided between you and the mortgage company. However, if your down payment is less than 20% of the property value, you will be required to pay private mortgage insurance (PMI). This is insurance you pay for so that the mortgage company is protected if *you* default on the loan. It often comes to between 0.5% and 2.25% of the original loan amount. It increases your monthly payment. Once you reach 20% of the loan value, you can request that the PMI be dropped. Even if you do not request cancelling the PMI, it will eventually and automatically be dropped.

For more, see this article on ways to get rid of PMI.

Reading and Interpreting Amortization Tables

Amortization tables were addressed in The Basics of Loans. They are most frequently encountered when analyzing mortgages.

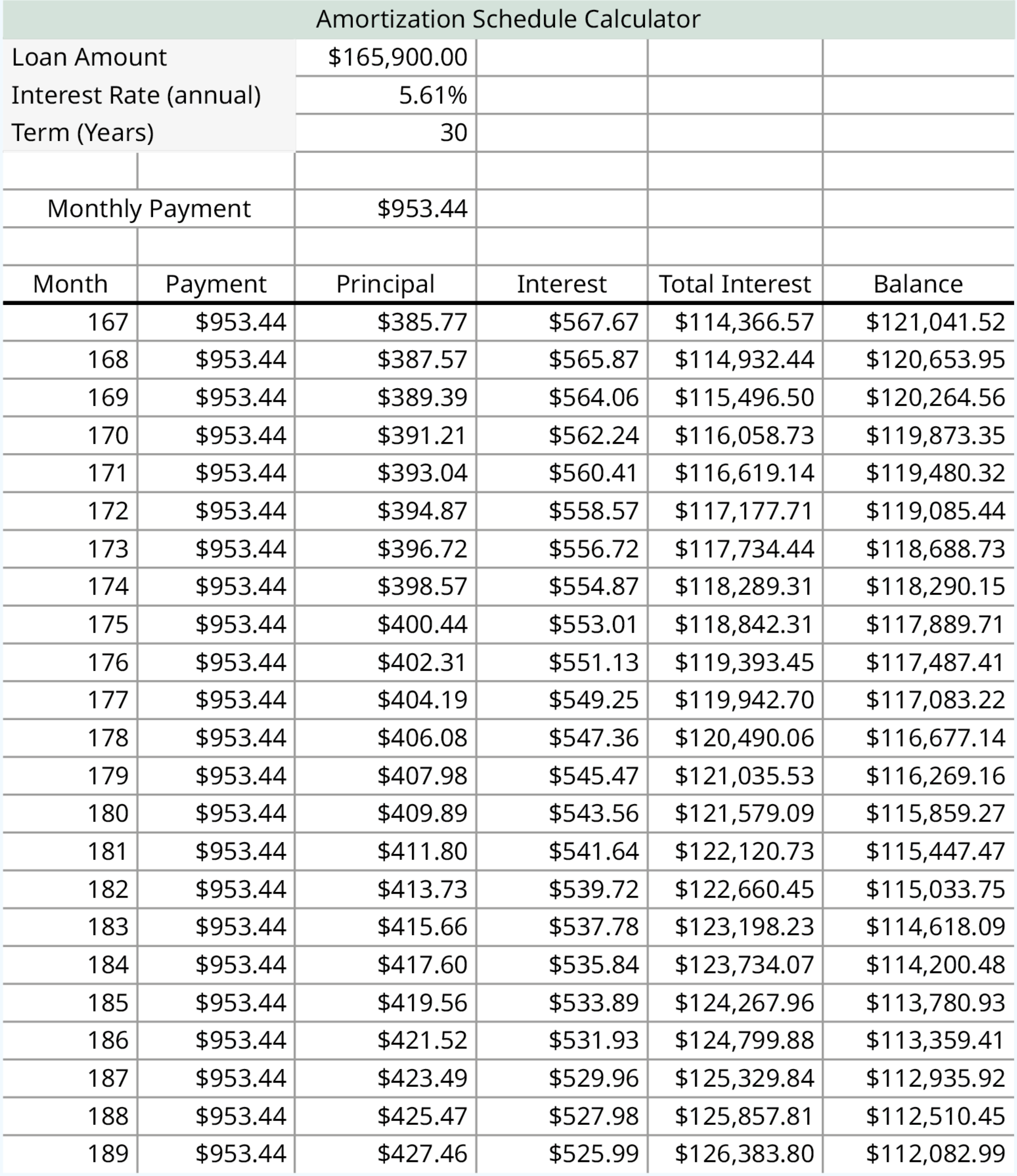
The amortization table for a 30-year mortgage is quite long, containing 360 rows. A full table will not be reproduced here. We can, though, read information from a portion of an amortization table.

Example 6.12.4: Amortization Table for a 30-Year, $165,900 Mortgage

Figure ✓ 6.12.2 shows a portion of an amortization table for a 30-year, $165,900 mortgage. Use that table to answer the following questions.

1. What is the interest rate?
2. How much are the payments?

3. How much of payment 175 goes to principal?
4. How much of payment 180 goes to interest?
5. What’s the remaining balance on the mortgage after payment 170?



Amortization Schedule Calculator					
Loan Amount	$165,900.00				
Interest Rate (annual)	5.61%				
Term (Years)	30				
Monthly Payment	$953.44				
Month	Payment	Principal	Interest	Total Interest	Balance
167	$953.44	$385.77	$567.67	$114,366.57	$121,041.52
168	$953.44	$387.57	$565.87	$114,932.44	$120,653.95
169	$953.44	$389.39	$564.06	$115,496.50	$120,264.56
170	$953.44	$391.21	$562.24	$116,058.73	$119,873.35
171	$953.44	$393.04	$560.41	$116,619.14	$119,480.32
172	$953.44	$394.87	$558.57	$117,177.71	$119,085.44
173	$953.44	$396.72	$556.72	$117,734.44	$118,688.73
174	$953.44	$398.57	$554.87	$118,289.31	$118,290.15
175	$953.44	$400.44	$553.01	$118,842.31	$117,889.71
176	$953.44	$402.31	$551.13	$119,393.45	$117,487.41
177	$953.44	$404.19	$549.25	$119,942.70	$117,083.22
178	$953.44	$406.08	$547.36	$120,490.06	$116,677.14
179	$953.44	$407.98	$545.47	$121,035.53	$116,269.16
180	$953.44	$409.89	$543.56	$121,579.09	$115,859.27
181	$953.44	$411.80	$541.64	$122,120.73	$115,447.47
182	$953.44	$413.73	$539.72	$122,660.45	$115,033.75
183	$953.44	$415.66	$537.78	$123,198.23	$114,618.09
184	$953.44	$417.60	$535.84	$123,734.07	$114,200.48
185	$953.44	$419.56	$533.89	$124,267.96	$113,780.93
186	$953.44	$421.52	$531.93	$124,799.88	$113,359.41
187	$953.44	$423.49	$529.96	$125,329.84	$112,935.92
188	$953.44	$425.47	$527.98	$125,857.81	$112,510.45
189	$953.44	$427.46	$525.99	$126,383.80	$112,082.99

Figure ✓ 6.12.2 Amortization table

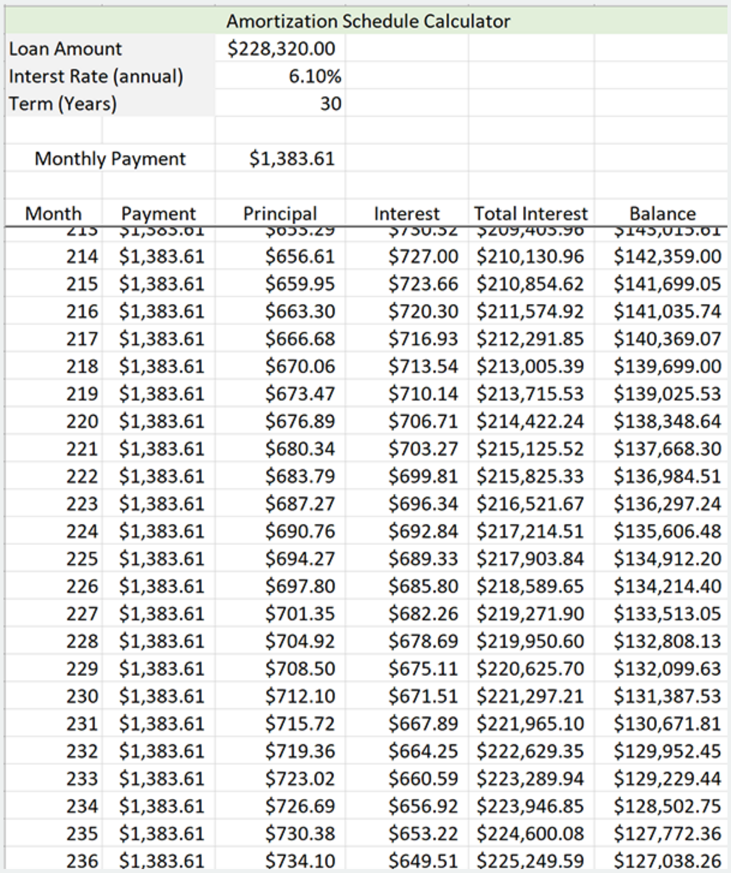
Answer

1. Reading at the top of the table, we see the interest rate is 5.61%.

2. Reading from the top of the table or from the column labeled Payment, we see the payments are $953.44 per month.
3. In the row for payment 175, we see that the amount that goes to principal is $400.44.
4. In the row for payment 180, we see that the amount that goes to interest is $543.56.
5. In the row for payment 170, we see the remaining balance is $119,873.35.

Your Turn \(\PageIndex{4}\)

Below is a portion of an amortization table for a 30-year, $228,320 mortgage. Use that table to answer the following questions.



Amortization Schedule Calculator

Loan Amount: $228,320.00
Interst Rate (annual): 6.10%
Term (Years): 30

Monthly Payment: $1,383.61

Month	Payment	Principal	Interest	Total Interest	Balance
213	$1,383.61	$653.29	$730.32	$209,403.96	$143,015.61
214	$1,383.61	$656.61	$727.00	$210,130.96	$142,359.00
215	$1,383.61	$659.95	$723.66	$210,854.62	$141,699.05
216	$1,383.61	$663.30	$720.30	$211,574.92	$141,035.74
217	$1,383.61	$666.68	$716.93	$212,291.85	$140,369.07
218	$1,383.61	$670.06	$713.54	$213,005.39	$139,699.00
219	$1,383.61	$673.47	$710.14	$213,715.53	$139,025.53
220	$1,383.61	$676.89	$706.71	$214,422.24	$138,348.64
221	$1,383.61	$680.34	$703.27	$215,125.52	$137,668.30
222	$1,383.61	$683.79	$699.81	$215,825.33	$136,984.51
223	$1,383.61	$687.27	$696.34	$216,521.67	$136,297.24
224	$1,383.61	$690.76	$692.84	$217,214.51	$135,606.48
225	$1,383.61	$694.27	$689.33	$217,903.84	$134,912.20
226	$1,383.61	$697.80	$685.80	$218,589.65	$134,214.40
227	$1,383.61	$701.35	$682.26	$219,271.90	$133,513.05
228	$1,383.61	$704.92	$678.69	$219,950.60	$132,808.13
229	$1,383.61	$708.50	$675.11	$220,625.70	$132,099.63
230	$1,383.61	$712.10	$671.51	$221,297.21	$131,387.53
231	$1,383.61	$715.72	$667.89	$221,965.10	$130,671.81
232	$1,383.61	$719.36	$664.25	$222,629.35	$129,952.45
233	$1,383.61	$723.02	$660.59	$223,289.94	$129,229.44
234	$1,383.61	$726.69	$656.92	$223,946.85	$128,502.75
235	$1,383.61	$730.38	$653.22	$224,600.08	$127,772.36
236	$1,383.61	$734.10	$649.51	$225,249.59	$127,038.26

Figure $6.12.3$: Amortization table

What is the interest rate?

How much are the payments?

How much of payment 235 goes to principal?

How much of payment 215 goes to interest?

What’s the remaining balance on the mortgage after payment 227?

At what payment does the amount that is applied to mortgage finally exceed the amount applied to interest?

Escrow Payments

The last few examples have looked at mortgage payments, which cover the principal and interest. However, when you take out a mortgage, the payment is sometimes much higher than that. This is because your mortgage company also has you pay into an **escrow account**, which is a savings account maintained by the mortgage company.

Your insurance payments will be set by your insurer and the mortgage company will pay them on time for you from your escrow account. Your property taxes are set by where you live and are typically a percentage of your property’s **assessed value**. The assessed value is the estimation of the value of your home and does not necessary reflect the purchase or resale value of the home. Your property taxes will also be paid on time by the mortgage company from your escrow account.

For example, in Kalamazoo, Michigan, the effective tax rate for property is 1.69% of the assessed value of the home. These escrow payments, which cover bills for the home, can increase the monthly payments for your home well beyond the basic principal and interest payment.

Example 6.12.5: Adding Escrow Payments to Mortgage Payments

Jenna decides to purchase a home, with mortgage of \$108,450 at 6% interest for 30 years. The assessed value of her home is \$75,600. Her property taxes come to 5.7% of her assessed value. Jenna also has to pay her home insurance every 6 months, which is \$744 per six months. How much, including escrow, will Jenna pay per month?

Answer

Using the payment function to find her mortgage payments, $pmt = \frac{P \times (r/12) \times (1 + r/12)^{12 \times t}}{(1 + r/12)^{12 \times t} -}$, with P = \$108,405, r = 0.06, and t = 30, her payments are

$$\begin{aligned} pmt &= \frac{P \times (r/12) \times (1 + r/12)^{12 \times t}}{(1 + r/12)^{12 \times t} - 1} \\ &= \frac{\$108{,}405 \times (0.06/12) \times (1 + 0.06/12)^{12 \times 30}}{(1 + 0.06/12)^{12 \times 30} - 1} \\ &= \$649.95 \end{aligned}$$

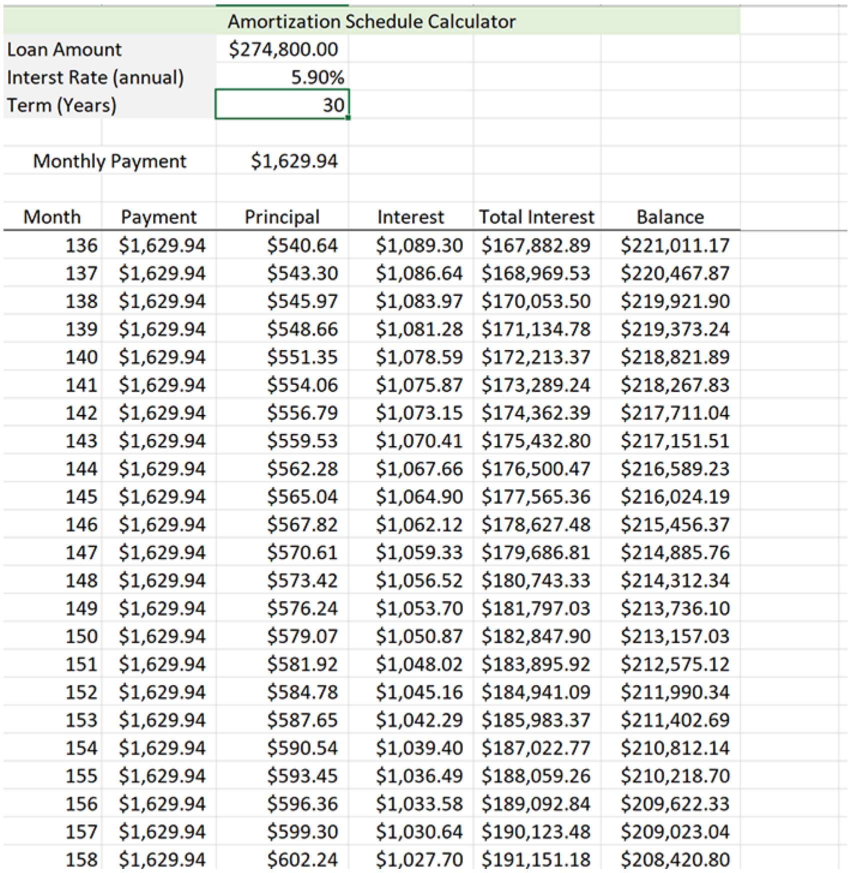
Jenna also pays into escrow 1/12 of her property taxes per month. Her property taxes are 5.7% of the assessed value of \$75,600, which comes to $0.057 \times \$75{,}600 = \$4309.20$. This is an annual tax, so she pays 1/12 of that each month, or \$359.10. Jenna's home insurance is \$744 per 6 months, so each month she pays \$124.00 for insurance. Adding these together, her monthly payment is $\$649.95 + \$359.10 + \$124.00 = \$1{,}133.05$. This is quite a bit more than the \$649.95 for the principal and interest.

Your Turn 6.12.5

Destiny decides to purchase a home, with mortgage of \$159,195.50 at 5.75% interest for 30 years. The assessed value of her home is \$100,000. Her property taxes come to 5.42% of her assessed value. Destiny also has to pay her home insurance every 6 months, and that comes to \$843 per 6 months. How much, including escrow, will Destiny pay per month?

Check Your Understanding

1. Does renting or buying have tax advantages?
2. Which has more restrictions, renting or buying?
3. Which has housing cost that does not change?
4. What is the name given to a loan for a home?
5. What are the monthly payments for a 30-year mortgage of \$108,993 with 6.14% interest.
6. What is the cost of financing for a 30-year mortgage of \$108,993 with 6.14% interest if the mortgage is paid off?
7. Consider the following amortization table. What is the amount of payment 141 that goes to principal?



Amortization Schedule Calculator

Loan Amount	$274,800.00
Interst Rate (annual)	5.90%
Term (Years)	30
Monthly Payment	$1,629.94

Month	Payment	Principal	Interest	Total Interest	Balance
136	$1,629.94	$540.64	$1,089.30	$167,882.89	$221,011.17
137	$1,629.94	$543.30	$1,086.64	$168,969.53	$220,467.87
138	$1,629.94	$545.97	$1,083.97	$170,053.50	$219,921.90
139	$1,629.94	$548.66	$1,081.28	$171,134.78	$219,373.24
140	$1,629.94	$551.35	$1,078.59	$172,213.37	$218,821.89
141	$1,629.94	$554.06	$1,075.87	$173,289.24	$218,267.83
142	$1,629.94	$556.79	$1,073.15	$174,362.39	$217,711.04
143	$1,629.94	$559.53	$1,070.41	$175,432.80	$217,151.51
144	$1,629.94	$562.28	$1,067.66	$176,500.47	$216,589.23
145	$1,629.94	$565.04	$1,064.90	$177,565.36	$216,024.19
146	$1,629.94	$567.82	$1,062.12	$178,627.48	$215,456.37
147	$1,629.94	$570.61	$1,059.33	$179,686.81	$214,885.76
148	$1,629.94	$573.42	$1,056.52	$180,743.33	$214,312.34
149	$1,629.94	$576.24	$1,053.70	$181,797.03	$213,736.10
150	$1,629.94	$579.07	$1,050.87	$182,847.90	$213,157.03
151	$1,629.94	$581.92	$1,048.02	$183,895.92	$212,575.12
152	$1,629.94	$584.78	$1,045.16	$184,941.09	$211,990.34
153	$1,629.94	$587.65	$1,042.29	$185,983.37	$211,402.69
154	$1,629.94	$590.54	$1,039.40	$187,022.77	$210,812.14
155	$1,629.94	$593.45	$1,036.49	$188,059.26	$210,218.70
156	$1,629.94	$596.36	$1,033.58	$189,092.84	$209,622.33
157	$1,629.94	$599.30	$1,030.64	$190,123.48	$209,023.04
158	$1,629.94	$602.24	$1,027.70	$191,151.18	$208,420.80

8. What is the name of the account that the mortgage company holds your taxes and insurance in?

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6.12.0: Exercises

In the following exercises, indicate if the advantage listed is for renting or buying a home.

? Exercise 6.12.0.1

Short-term commitment.

? Exercise 6.12.0.2

Tax advantage.

? Exercise 6.12.0.3

Freedom to remodel.

? Exercise 6.12.0.4

Builds equity.

? Exercise 6.12.0.5

Cost is lower.

? Exercise 6.12.0.6

You do not pay for repairs.

? Exercise 6.12.0.7

No pet restrictions

? Exercise 6.12.0.8

More flexibility to move.

? Exercise 6.12.0.9

Housing cost is fixed.

? Exercise 6.12.0.10

May have other amenities.

In the following exercises, find the mortgage payment for the given loan amount, interest rate, and term.

? Exercise 6.12.0.11

Loan amount is $78,560, interest rate is 5.87%, 30-year mortgage.

? Exercise 6.12.0.12

Loan amount is $125,800, interest rate is 6.5%, 30-year mortgage.

? Exercise 6.12.0.13

Loan amount is $96,400, interest rate is 4.9%, 15-year mortgage.

? Exercise 6.12.0.14

Loan amount is $267,450, interest rate is 5.25%, 20-year mortgage.

In the following exercises, find the total paid on the mortgage if it is fully paid through the term.

? Exercise 6.12.0.15

Loan amount is $78,560, interest rate is 5.87%, 30-year mortgage.

? Exercise 6.12.0.16

Loan amount is $125,800, interest rate is 6.5%, 30-year mortgage.

? Exercise 6.12.0.17

Loan amount is $96,400, interest rate is 4.9%, 15-year mortgage.

? Exercise 6.12.0.18

Loan amount is $267,450, interest rate is 5.25%, 20-year mortgage.

In the following exercises, find the cost of financing for the mortgages if they are fully paid.

? Exercise 6.12.0.19

Loan amount is $78,560, interest rate is 5.87%, 30-year mortgage.

? Exercise 6.12.0.20

Loan amount is $125,800, interest rate is 6.5%, 30-year mortgage.

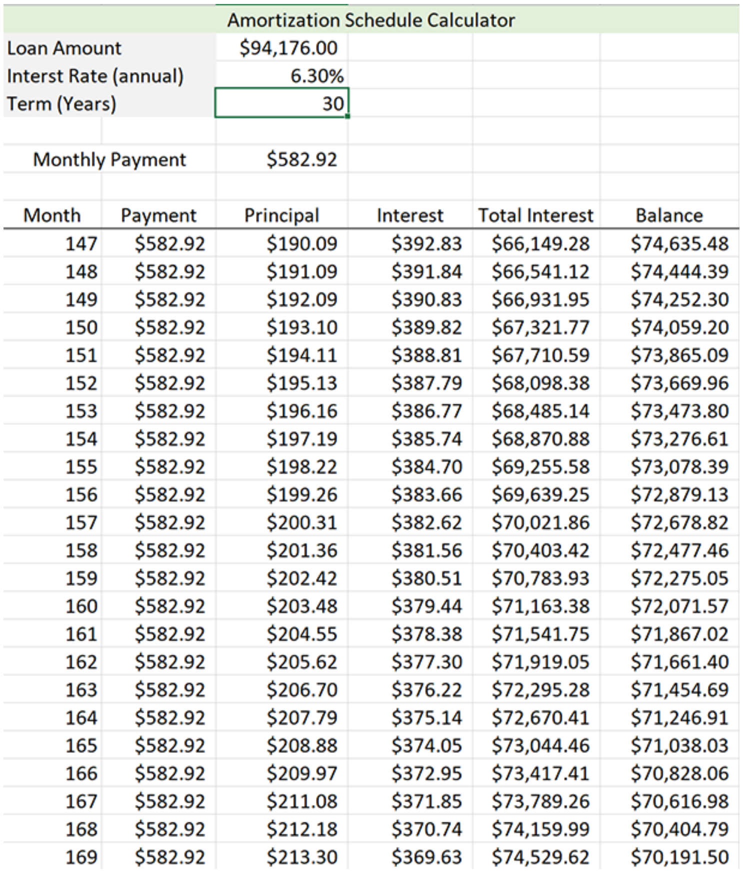
? Exercise 6.12.0.21

Loan amount is $96,400, interest rate is 4.9%, 15-year mortgage.

? Exercise 6.12.0.22

Loan amount is $267,450, interest rate is 5.25%, 20-year mortgage.

In the following exercises, use the amortization table to answer the question.



	Amortization Schedule Calculator				
Loan Amount	$94,176.00				
Interst Rate (annual)	6.30%				
Term (Years)	30				
Monthly Payment	$582.92				
Month	Payment	Principal	Interest	Total Interest	Balance
147	$582.92	$190.09	$392.83	$66,149.28	$74,635.48
148	$582.92	$191.09	$391.84	$66,541.12	$74,444.39
149	$582.92	$192.09	$390.83	$66,931.95	$74,252.30
150	$582.92	$193.10	$389.82	$67,321.77	$74,059.20
151	$582.92	$194.11	$388.81	$67,710.59	$73,865.09
152	$582.92	$195.13	$387.79	$68,098.38	$73,669.96
153	$582.92	$196.16	$386.77	$68,485.14	$73,473.80
154	$582.92	$197.19	$385.74	$68,870.88	$73,276.61
155	$582.92	$198.22	$384.70	$69,255.58	$73,078.39
156	$582.92	$199.26	$383.66	$69,639.25	$72,879.13
157	$582.92	$200.31	$382.62	$70,021.86	$72,678.82
158	$582.92	$201.36	$381.56	$70,403.42	$72,477.46
159	$582.92	$202.42	$380.51	$70,783.93	$72,275.05
160	$582.92	$203.48	$379.44	$71,163.38	$72,071.57
161	$582.92	$204.55	$378.38	$71,541.75	$71,867.02
162	$582.92	$205.62	$377.30	$71,919.05	$71,661.40
163	$582.92	$206.70	$376.22	$72,295.28	$71,454.69
164	$582.92	$207.79	$375.14	$72,670.41	$71,246.91
165	$582.92	$208.88	$374.05	$73,044.46	$71,038.03
166	$582.92	$209.97	$372.95	$73,417.41	$70,828.06
167	$582.92	$211.08	$371.85	$73,789.26	$70,616.98
168	$582.92	$212.18	$370.74	$74,159.99	$70,404.79
169	$582.92	$213.30	$369.63	$74,529.62	$70,191.50

Exercise 6.12.0.23

What is the term of the mortgage?

Exercise 6.12.0.24

How much of payment 165 applies to interest?

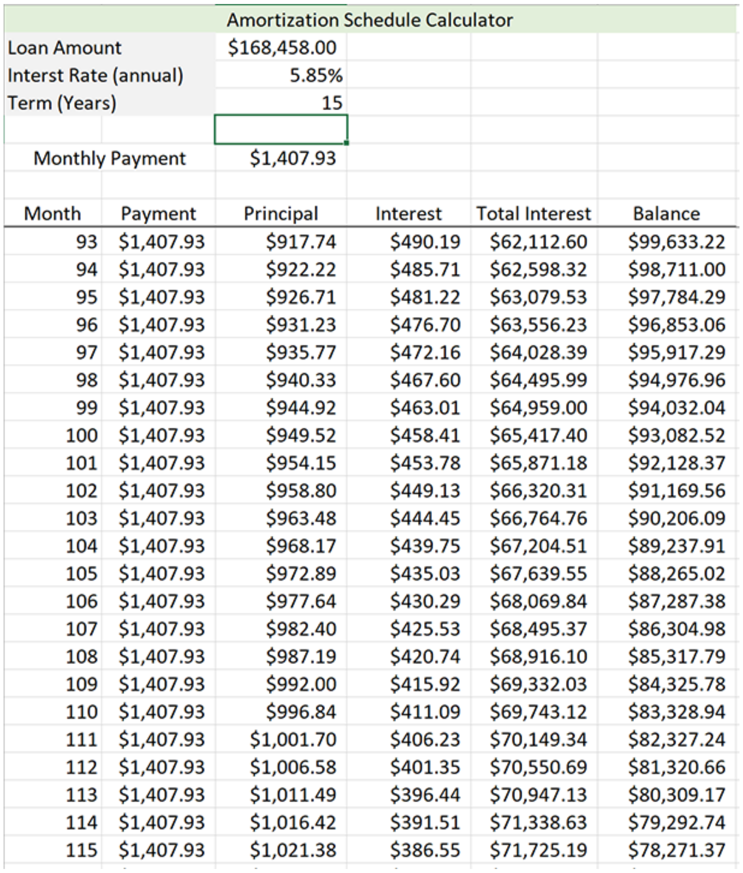
Exercise 6.12.0.25

What is the remaining balance after payment 155?

Exercise 6.12.0.26

How much total interest was paid after payment 149?

In the following exercises, use the amortization schedule to answer the question.



Amortization Schedule Calculator

Loan Amount	$168,458.00
Interst Rate (annual)	5.85%
Term (Years)	15
Monthly Payment	$1,407.93

Month	Payment	Principal	Interest	Total Interest	Balance
93	$1,407.93	$917.74	$490.19	$62,112.60	$99,633.22
94	$1,407.93	$922.22	$485.71	$62,598.32	$98,711.00
95	$1,407.93	$926.71	$481.22	$63,079.53	$97,784.29
96	$1,407.93	$931.23	$476.70	$63,556.23	$96,853.06
97	$1,407.93	$935.77	$472.16	$64,028.39	$95,917.29
98	$1,407.93	$940.33	$467.60	$64,495.99	$94,976.96
99	$1,407.93	$944.92	$463.01	$64,959.00	$94,032.04
100	$1,407.93	$949.52	$458.41	$65,417.40	$93,082.52
101	$1,407.93	$954.15	$453.78	$65,871.18	$92,128.37
102	$1,407.93	$958.80	$449.13	$66,320.31	$91,169.56
103	$1,407.93	$963.48	$444.45	$66,764.76	$90,206.09
104	$1,407.93	$968.17	$439.75	$67,204.51	$89,237.91
105	$1,407.93	$972.89	$435.03	$67,639.55	$88,265.02
106	$1,407.93	$977.64	$430.29	$68,069.84	$87,287.38
107	$1,407.93	$982.40	$425.53	$68,495.37	$86,304.98
108	$1,407.93	$987.19	$420.74	$68,916.10	$85,317.79
109	$1,407.93	$992.00	$415.92	$69,332.03	$84,325.78
110	$1,407.93	$996.84	$411.09	$69,743.12	$83,328.94
111	$1,407.93	$1,001.70	$406.23	$70,149.34	$82,327.24
112	$1,407.93	$1,006.58	$401.35	$70,550.69	$81,320.66
113	$1,407.93	$1,011.49	$396.44	$70,947.13	$80,309.17
114	$1,407.93	$1,016.42	$391.51	$71,338.63	$79,292.74
115	$1,407.93	$1,021.38	$386.55	$71,725.19	$78,271.37

? Exercise 6.12.0.27

What is the interest rate for the mortgage?

? Exercise 6.12.0.28

How much of payment 110 applies to principal?

? Exercise 6.12.0.29

What is the remaining balance after payment 94?

? Exercise 6.12.0.30

How much total interest was paid after payment 111?

In the following exercises, find the total monthly payment including both the mortgage payment and the escrow payment.

? Exercise 6.12.0.31

Mortgage of $87,690 at 6.2% interest for 30 years. Assessed value of the home is $75,600. Property taxes come to 5.65% of assessed value. Home insurance of $815 paid every 6 months.

? Exercise 6.12.0.32

Mortgage of $143,900 at 5.05% interest for 30 years. Assessed value of the home is $90,150. Property taxes come to 5.88% of assessed value. Home insurance of $924 paid every 6 months.

? Exercise 6.12.0.33

Mortgage of $65,175 at 6.48% interest for 30 years. Assessed value of the home is $62,800. Property taxes come to 6.75% of assessed value. Home insurance of $558 paid every 6 months.

? Exercise 6.12.0.34

Mortgage of $245,950 at 5.35% interest for 30 years. Assessed value of the home is $156,500. Property taxes come to 6.41% of assessed value. Home insurance of $972 paid every 6 months.

For the following exercises, read the following: **Fifteen-year mortgage compared to 30-year mortgage.** Mortgage interest rates are often higher for 30-year mortgages than 15-year mortgages. However, the payments for 15-year mortgages are considerably higher. The following exercises explore the difference between a 15- and 30-year mortgage for a mortgage of $100,000.

? Exercise 6.12.0.35

The 15-year mortgage interest rate is 5.65%.

a. Find the payment.
b. Determine the total that would be paid if the mortgage was completed.
c. Find the cost of financing for this mortgage.

? Exercise 6.12.0.36

The 30-year mortgage rate is 6.4%.

a. Find the payment.
b. Determine the total that would be paid if the mortgage was completed.
c. Find the cost of financing for this mortgage.

? Exercise 6.12.0.37

How different are the payments, the total paid, and the cost to finance?

? Exercise 6.12.0.38

Summarize the answer in the previous question.

For the following exercises, read the following: **Fifteen-year mortgage compared to 30-year mortgage.** A 15-year mortgage comes with advantages, the biggest being the home is paid off much sooner, and equity is built much more quickly. Mortgage interest rates are often higher for 30-year mortgages than 15-year mortgages. However, the payments for 15-year mortgages are considerably higher. The following exercises explore the difference between a 15- and 30-year mortgage for a mortgage of $200,000.

? Exercise 6.12.0.39

The 15-year mortgage interest rate is 5.6%.

a. Find the payment.
b. Determine the total that would be paid if the mortgage was completed.
c. Find the cost of financing for this mortgage.

? Exercise 6.12.0.40

The 30-year mortgage rate is 6.25%.

a. Find the payment.
b. Determine the total that would be paid if the mortgage was completed.
c. Find the cost of financing for this mortgage.

? Exercise 6.12.0.41

How different are the payments, the total paid, and the cost to finance?

? Exercise 6.12.0.42

Summarize the answer in the previous question.

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6.13: Income Tax

Figure 6.13.1: Federal income tax is a concern for most US citizens. (credit: "1040 US tax form" by Marco Verch Professional Photographer/Flickr, CC BY 2.0)

Learning Objectives

1. Determine gross, adjusted gross, and taxable income.
2. Apply exemptions, deductions, and credits to basic income tax calculations.
3. Compute FICA tax.
4. Solve tax application problems for working students.

Before the start of the American Civil War in 1861, most of the country's revenue came from tariffs on trade and excise taxes. However, this fell far short of the high cost of the war. Because of this, the federal government enacted the nation's first income tax with the Revenue Act of 1861, which created the Internal Revenue Service as we know it today.

No one likes paying income tax, but it is a reality of life. In this section, we will learn about Form 1040, the U.S. Individual Income Tax Return, and ways to prepare for tax time.

The U.S. tax code may change from year to year. Because of this, this section includes examples of how taxes, deductions, and exemptions might be computed. The types of income, deductions, and exemptions that are used in the examples are used in the current tax code.

Gross, Adjusted Gross, and Taxable Income

Your income drives how much you pay in taxes. The more you earn, the more you are likely to pay. But your income alone is not the full story. When you add all the money you earned from your job, freelance work, interest from savings, and other sources, you have your gross income. If you are an employee, your income from your job will be reported on a W-2, which is sent to you by your employer. Income from freelance work will be reported on a 1099-MISC form, and is sent by the company that paid you. Income from interest is reported on a 1099-INT form and comes from the entity that paid the interest.

Before you determine how much you owe in taxes, you will make certain adjustments to that gross income. You will deduct, or subtract, some of income from the gross income. That's your adjusted gross income, or AGI. That is still not what you are taxed on.

Next, you need to apply exemptions to your income. These are pieces of income that the government does not tax. After that is done, you reach your taxable income. We will look at each of these parts of the taxable income.

Who Knew?: Gifts and Winning

Money given as a gift my be taxed if the gift amount is high enough. If you win $50,000 in the lottery, that money is taxed as income. If you give a family member a large cash gift, that gift will be subject to a tax provided that the gift exceeds the federally set limits.

You will notice that your paycheck already has taxes taken out of it. Your employer will withhold some of your income, sending it directly to the federal, state, and local governments. It is an estimate of how much you will owe in income tax. In the end, it reduces how much you will pay when your taxes are due. If they withhold too much income, you will receive the extra they withheld in the form of a refund.

✓ Example ✓ 6.13.1: Computing Gross Income

Roger is preparing to do his taxes. He worked two jobs, with reported income of $27,500 and $13,200. His two CDs yielded $327 together. He also won a split club raffle for $2,000. What was Roger's gross earnings?

Answer

This is the sum of his wages, winnings and interest earned. Add these Roger's gross income was $43,027.

Your Turn 6.13.1

Chloe prepares for her taxes by collecting her W-2 and her 1099-MISC documents. The wages from her regular job were $41,780. She did some freelance work and earned $5,500 from that job. What is Chloe's gross income?

Your adjusted gross income (AGI) is computed before your taxes are determined. It begins with the gross income, and then subtracts from that income any deduction. **Deductions** are expenditures on your part that the government won't tax. These deductions include money deposited into tax-deferred investments, and mortgage interest that you paid, charitable contributions if you made any, medical bills over a threshold, medical insurance under certain circumstances, and property taxes. If you add all these up, and they are all legal deductions, the sum is subtracted from your gross income, leaving the AGI.

✓ Example 6.13.2: Compute Adjusted Gross Income

Chandra's gross income is $58,400. She deposited $3,000 into her tax-deferred retirement account for the year. Her mortgage interest paid was $1,250 for the year, and her property taxes came to $4,200. What is her AGI?

Answer

To get Chandra 's AGI, total her deductions and subtract that total from her gross income. The total of her deductions is $\$3{,}000 + \$1{,}250 + \$4{,}200 = \$8{,}450$. She subtracts that total from her gross income, making her AGI $\$58{,}400 - \$8{,}450 = \$57{,}950$.

Your Turn 6.13.2

Isabelle's gross income is $93,450. She paid $7,840 in mortgage interest and $3,810 in property taxes. She also donated $1,500 to her favorite charity. What is Isabelle's AGI?

Remember that your AGI is not your **taxable income**. Exemptions need to be subtracted from the AGI to reach your taxable income. Exemptions are income that the government does not tax. Some examples of exempt income are disbursements from health savings accounts for qualified medical expenses, bond interest, some IRA distributions, and gifts given that are under $16,000. Note that exemptions are different from deductions: exemptions are excused incomes, whereas deductions are excused expenditures.

✓ Example 6.13.3: Taxable Income

Yelizaveta's AGI is $75,490. However, she gave a $2,000 gift to her mother. What is her taxable income?

Answer

Taxable income is AGI minus any exemptions. Gifts below $16,000 are exempted, so her taxable income is $\$75{,}490-\$2{,}000=\$75{,}490$.

Your Turn 6.13.3

Ammie has an AGI of $43,100. However, $3,400 was a disbursement from her health savings account. What is her taxable income?

Tax Credits

Another piece of the tax puzzle is tax credits. This is money subtracted from the tax you owe.

Tax credits are very different from deductions or exemptions. Deductions and exemptions are taken away from your gross income before the tax you owe is calculated. A tax credit, is subtracted, dollar for dollar, from your tax bill. Once the tax you owe is calculated, subtract the any tax credits from that calculated tax.

Some of the tax credits are refundable. This means that if subtracting them from your tax results in a negative number, you receive a tax refund. For more details, see this article about tax credits.

The federal government has placed income limits and restrictions and on those eligible to receive tax credits because their value is so high. Here is a partial list of tax credits that you might qualify for:

- **Earned income credit** is a refundable tax credit for low- to moderate-income workers and ranges from $560 to $6,935 depending on dependents and income. This is refundable
- **American opportunity credit** is a credit taken by parents who have children enrolled in college at least half time and pursuing a degree. This credit is worth $2,500 per student for the first 4 years of undergraduate school, subject to income limits. This is a refundable tax credit.
- **Lifetime learning credit** is a credit is equivalent to 20% of educational expenses, up to $2,000 per year, subject to income limits. There is no cap to how many years you can apply for this credit.
- **Child tax credit** is worth $2,000 per child under the age of 17 if that child lives at home at least half the year, subject to income limits. This is a refundable tax credit.
- **Child and dependent care tax credit** was designed to help pay for child care while the parent works. The amount of the credit is dependent on your income. However, the maximum amount that can be received is, in 2022, $4,000 for one eligible person, or $8,000 for two or more qualifying people. A dependent qualifies if they are a child under 13 years old, a spouse who is unable to care for themselves, or some other qualifying person. This is a refundable tax credit.
- **Premium tax credit** was created by the Affordable Care Act, and it is one that is received by many people throughout the year. In essence it is a health insurance premium subsidy. The amount of the credit is based on your income and the price of health insurance in your area. This is a refundable tax credit.

✓ Example 6.13.4: Apply a Tax Credit

Kaitlyn has calculated the tax she owes to be $5,200. However, she receives an earned income tax credit of $1,715. How much does Kaitlyn owe after applying the earned income tax credit?

Answer

The amount of taxes Katilyn owes is her calculated tax of $5,200 minus the credit she receives. The amount she owes in taxes is $\$5{,}200-\$1{,}715=\$3{,}485$.

Your Turn 6.13.4

Antonio's taxes owed based on his income are $3,950. However, he qualifies for the earned income tax credit of $1,925 and a child tax credit of $2,000. How much does Antonio owe after applying the tax credits?

Example 6.13.5: Tax Credits and a Refund

Chanajah calculated their tax owed, which came to $4,300. They have an earned income tax credit of $2,190, a child tax credit of $2,000, and a child and dependent care tax credit of $4,000. How much tax does Chanajah owe, or how much will they get in a refund?

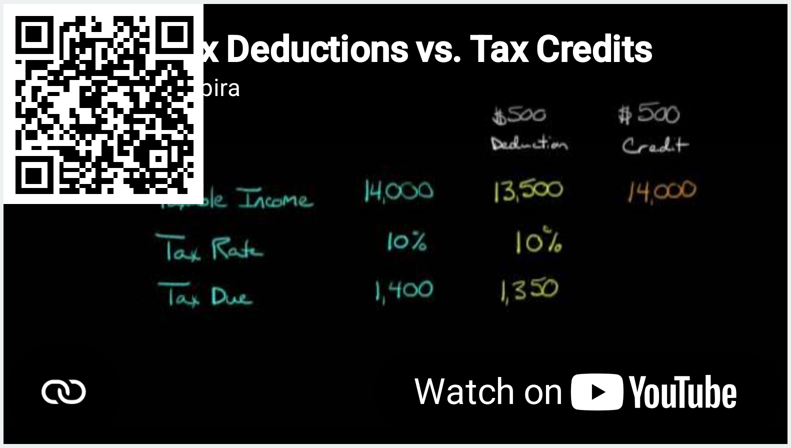
Answer

Adding Chanajah's tax credits together, we find their total to be $8,190. That is more than the tax they owe, which was $4,300. Each of those credits is refundable, which means they will receive a refund. Subtracting the credits from the tax owed yields $\$4{,}300-\$8{,}190=-\$3{,}890$. This is negative, so represents a refund of $3,890.

Your Turn 6.13.5

Ismail owed $2,350 in taxes. He has a child tax credit of $2,000 and an earned income tax credit of $1,640. After applying the tax credit, how much does Ismail owe or how much does he get in a refund?

Video



Deductions Versus Credits

Computing FICA Taxes

FICA stands for the Federal Insurance Contributions Act of 1935. FICA taxes are used solely to fund Social Security and Medicare and are separate from federal income tax. It amounts to 7.65% of your gross pay, which is withheld from your paycheck automatically. Your employer is required to match the 7.65% amount. Of the 7.65%, 6.2% goes to Social Security (SSI), and 1.45% goes to Medicare.

As of 2022, SSI tax only applies to the first $147,000 of earnings. Any gross income above that is not taxed for social security. This limit changes every year.

Medicare tax, on the other hand, applies to the entirety of your gross income.

Example 6.13.6: Computed FICA Taxes

McKenzi earned $2,700 in gross income, before taxes, in a given 2-week period. How much does she owe in FICA taxes, and how much of that is for SSI?

Answer

The FICA tax is 7.65% of her gross earnings. 7.65% of her $2,700 is $206.55. Also, the SSI is 6.2% of her income, or $167.40

Your Turn 6.13.6

Arianne earned $3,200 over a 2-week pay period. How much FICA tax does she pay, and how much of that is for SSI?

Example \(\PageIndex{7}\): Social Security Tax with Higher Income

Renard earned $195,000 in wages for the year. How much in SSI taxes does Renard owe for the year?

Answer

Since Renard earned more than the taxable limit of $147,000 dollars, he only pays the 6.2% SSI tax on $147,000. This comes to $9,114.

Your Turn 6.13.7

Andy earned $169,450 in wages this year. How much in SSI taxes does Andy owe?

Calculating Your Income Tax

Your income tax bill and your income tax rate are based on your taxable income. The tax system in the United States is progressive, meaning that the tax rates are marginal so the higher your taxable income the higher the tax rate you will pay. Taxable income is broken into brackets, or ranges of income. Each bracket has a different tax rate. The tax brackets and rates for single filers as of 2022 are given below:

Table https://www.irs.gov/newsroom/irs-pro...-tax-year-2022)

Bracket	Lower Income Limit	Upper Income Limit	Tax Rate
1	0	$10,275	10%
2	$10,276	$41,775	12%
3	$41,776	$89,075	22%
4	$89,076	$170,050	24%
5	$170,051	$215,950	32%
6	$215,951	$539,900	35%
7	$539,901		37%

So if your taxable income is $76,500 and you are filing as a single filer, your tax bill will be 22% of that $76,500, right?

Wrong. Your income is split among those brackets and the money in each bracket is taxed at that bracket's tax rate. Seems confusing. Here is a list of steps to follow to find the tax owed.

Step 1: Find the bracket for the income.

Step 2: For each bracket below the income bracket, the tax from that bracket is:

Step 2a: Find the difference between the upper limit of that bracket and upper limit of the next lower bracket. If this is bracket 1, use 0 as the upper limit of the previous bracket.

Step 2b: The tax from that bracket is the bracket tax rate applied to the difference from Step 2a.

Step 3: For the bracket that the income belongs to, find the income minus the lower limit for the bracket.

Step 4: The tax for the bracket of the income is tax rate for that bracket applied to the difference found in Step 3.

Step 5: Add these various tax values to get the total income tax.

There are various tax brackets, and the rates may change in any given year. The income limits may also change. For all examples going forward, we will use the single filer tax brackets, even if those brackets are not appropriate (e.g., married or head of household filers).

✓ Example 6.13.8: Income Tax on Taxable Income

Faith has a taxable income of $103,650. How much income tax does Faith owe?

Answer

Step 1: Faith's income belongs to the 4th tax bracket, $89,076 to $170,050. So the taxes from the first three brackets follow steps 2a and 2b.

For bracket 1:

Step 2a: The upper limit is $10,275, the upper limit of the previous bracket was 0, so the difference is $10,275.

Step 2b: For the first bracket, she owes 10% of $10,275, or $1,027.50.

For bracket 2:

Step 2a: The upper limit is $41,775, the upper limit from the previous bracket was $10,275, so the difference is $31,500

Step 2b: For the second bracket, she owes 12% of $31,500, or $3,780.

For bracket 3:

Step 2a: The upper limit is $89,075, the upper limit from the previous bracket is $41,175, so the difference was $47,300

Step 2b: For the third bracket, she owes 22% of $47,300, or $10,406.

Step 3: For bracket 4, her income is $103,650, the upper limit of the previous bracket is $89,075. The difference of those is $\$103{,}650 - \$89{,}075 = \$14{,}575$.

Step 4: The tax he owes for that bracket is 24% of the difference, which is $3,498.

Step 5: The total in taxes that Emmanuel owes is the sum of the taxes found above, or $\$1{,}027.5 + \$3{,}780 + \$10{,}406 + \$3{,}498 = \$18{,}711.50$

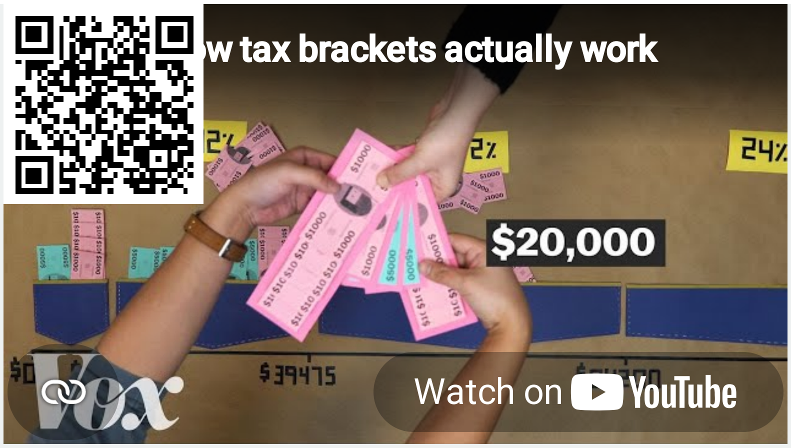
Your Turn 6.13.8

Jenna's taxable income is $73,500. Find how much income tax she owes.

Checkpoint

Remember that your employer will estimate how much tax you will owe and withholds it from paychecks. This means you may have already paid some, if not all and more, of the income tax you owe.

Video



Tax Brackets

Example 6.13.9: Finding Income Tax Owed

Emmanuel is preparing his taxes. His W-2 from work shows gross income for the year of $95,250. He also has a 1099-MISC for some freelance art work he did, amounting to $7,500. Emmanuel also deposited $4,500 into a tax-deferred retirement plan. He paid $7,920 in mortgage interest for the year and $3,740 in property taxes. He also qualified for $4,000 in child tax credits. Based on this information, how much tax does Emmanuel owe or how much does he get in a refund?

Answer

We need to know Emmanuel's taxable income, based on gross income, deductions, and exemptions.

His gross income is the amount from his W-2 and his 1099-MISC. Adding these gives a gross income of $\$95{,}250+\$7{,}500=\$102{,}750$.

Subtracting his deductions will yield his AGI. His deductions are $7,920 for mortgage interest, $3,740 for property taxes, and $4,500 deposited into his retirement account. Adding these, his total deductions are $16,160. Subtracting the deductions from the gross income, we find his AGI to be $86,590.

Emmanuel seems to have no exemptions, so his AGI and his taxable income are the same.

We find the taxes Emmanuel owes using the process outlined above.

Step 1: His income belongs to the third tax bracket, $41,776 to $89,0875. So the taxes from the first two brackets follow steps 2a and 2b.

For bracket 1:

Step 2a: The upper limit is $10,275, the upper limit of the previous bracket was 0, so the difference is $10,275.

Step 2b: For the first bracket, he owes 10% of $10,275, or $1,027.50.

For bracket 2:

Step 2a: The upper limit is $41,775, the upper limit from the previous bracket was $10,275, so the difference is $31,500

Step 2b: For the second bracket, he owes 12% of $31,500, or $3,780.

Step 3: For bracket 3, his income is $86,590, the upper limit of the previous bracket is $41,775. The difference of those is $\$86{,}590-\$41{,}775=\$44{,}815$.

Step 4: The tax he owes for that bracket is 22% of the difference, which is $9,859.30.

Step 5: The total in taxes that Emmanuel owes is the sum of the taxes found above, or $\$1027.5+\$3780+\$9{,}859.30=\$14{,}666.80$.

Once his taxes are computed, he subtracts his tax credits. His only tax credit is $4,000. The total he owes in taxes is $\$14{,}666.80-\$4{,}000=\$10{,}666.80$.

Your Turn 6.13.9

Jacob is preparing his taxes. His W-2 from work shows gross income for the year of $39,885. He also has a 1099-INT for a savings account, amounting to $378. Jacob also deposited $2,500 into a tax-deferred retirement plan. He also qualified for $2,000 in child tax credits and $4,000 in child and dependent care tax credit. Based on this information, how much tax does Jacob owe or how much does he receive in a refund?

Check Your Understanding

1. Gross income consists of what?
2. Taxable income is found after what is done to gross income?
3. If the gross income is $50,000 and there are three deductions of $2,000, $3,000, and $8,000, what is the adjusted gross income?
4. When is a tax credit applied?
5. Do you pay 6.2% for SSI tax for all your earnings?
6. For a taxable income of $41,800, how much income tax is owed?
7. If $5,600 is owed in taxes but there are $4,000 in tax credits, how much income tax is owed?

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6.13.0: Exercises

Exercise 6.13.0.1

What is gross income?

Exercise 6.13.0.2

How is adjusted gross income derived from gross income?

Exercise 6.13.0.3

Is there a difference between AGI and taxable income? If so, what is the difference?

Exercise 6.13.0.4

How are deductions different from tax credits?

Exercise 6.13.0.5

What are the two components of the FICA tax?

Exercise 6.13.0.6

What taxes do employers match?

Exercise 6.13.0.7

If you are in the 32% tax bracket, is your tax computed on your full income?

Exercise 6.13.0.8

What does it mean for a tax credit to be refundable?

In the following exercises, compute the gross income.

In the following exercises find the adjusted gross income, AGI, based on gross income and deductions.

Exercise 6.13.0.13

Gross income was \$65,700, \$2,280 deposited in a tax-deferred IRA, mortgage interest was \$4,715, property taxes were \$3,065.

Exercise 6.13.0.14

Gross income was \$183,200, \$5,000 deposited in a tax-deferred IRA, \$7,300 in charitable contributions, \$8,350 in mortgage interest, and \$6,900 in property taxes.

Exercise 6.13.0.15

Gross income was \$31,200, \$1,500 deposited in a tax-deferred IRA account.

Exercise 6.13.0.16

Gross income was \$41,500, \$1,250 deposited in a tax-deferred IRA account, \$4,210 in mortgage interest, \$2,980 in property tax.

In the following exercises, find the taxable income based on AGI and exemptions.

Exercise 6.13.0.17

AGI of \$34,560, \$2,500 disbursement from a health savings account for a qualifying medical expense.

Exercise 6.13.0.18

AGI of \$56,750, gift given to a family member of \$8,000, \$550 in bond interest.

Exercise 6.13.0.19

AGI of \$120,940, gift given to a family member of \$15,000, bond interest of \$4,500.

Exercise 6.13.0.20

AGI of \$28,450, gift given to a family member of \$2,000.

In the following exercises, find the taxes owed or refund received based on income tax bill and tax credits.

Exercise 6.13.0.21

Tax bill of \$5,300, child tax credit of \$4,000, earned income tax credit of \$1,630.

Exercise 6.13.0.22

Tax bill is \$17,300, child tax credit of \$4,000, child and dependent care tax credit \$8,000, lifetime learning credit of \$1,000.

Exercise 6.13.0.23

Tax bill of \$5,205, child tax credit of \$6,000, earned income tax credit of \$2,450.

Exercise 6.13.0.24

Tax bill of \$11,300, child tax credit of \$2,000, earned income tax credit of \$650, child and dependent care credit of \$4,000.

Exercise 6.13.0.25

Tax bill of \$13,750, child tax credit of \$2,000, earned income tax credit of \$780, child and dependent care credit of \$2,000.

In the following exercises, determine the SSI tax and the total FICA tax for the given incomes.

Exercise 6.13.0.26

Earning for pay period were $3,500.

Exercise 6.13.0.27

Earnings for pay period were $1,400.

Exercise 6.13.0.28

Earnings for pay period were $3,150.

Exercise 6.13.0.29

Earnings for the year were $135,000.

Exercise 6.13.0.30

Earnings for the year were $203,400.

In the following exercises, find the income tax owed using the taxable income and the tax table below, for a person filing single.

Bracket	Lower Income Limit	Upper Income Limit	Tax Rate
1	0	$10,275	10%
2	$10,276	$41,775	12%
3	$41,776	$89,075	22%
4	$89,076	$170,050	24%
5	$170,051	$215,950	32%
6	$215,951	$539,900	35%
7	$539,901		37%

Exercise 6.13.0.31

Taxable income of $36,250.

Exercise 6.13.0.32

Taxable income of $63,500.

Exercise 6.13.0.33

Taxable income of $209,450.

Exercise 6.13.0.34

Taxable income of $92,250.

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SECTION OVERVIEW

6.14: Chapter Summary

6.14.0: Key Terms

6.14.1: Key Concepts

6.14.2: Videos

6.14.3: Formula Review

6.14.4: Projects

6.14.5: Chapter Review

6.14.6: Chapter Test

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6.14.0: Key Terms

6.1 Understanding Percent

- Percent
- Fractional form
- Decimal form
- Total
- Base
- Percent of the total
- Part
- Amount

6.2 Discounts, Markups, and Sales Tax

- Discount
- Cost
- Markup
- Retail price

6.3 Simple Interest

- Interest
- Principal
- Annual percentage rate
- Simple interest
- Term
- Due
- Origination date
- Payoff amount
- Future value
- Partial payment
- Present value

6.4 Compound Interest

- Compound interest
- Effective annual yield

6.5 Making a Personal Budget

- Budget
- Necessary expenses
- Fixed expenses
- Variable expenses
- 50-30-20 budget philosophy

6.6 Methods of Savings

- Savings account
- 1099 form
- Certificate of deposit
- Money market account
- Return on investment
- Ordinary annuity

6.7 Investments

- Bonds
- Maturity date
- Stocks
- Dividend
- Mutual fund
- Prospectus
- Issue price
- Shares
- Stock table
- Individual retirement account
- Roth IRA
- 401(k)

6.8 The Basics of Loans

- Fixed interest rate
- Variable interest rate
- Installment loan
- Loan amortization
- Revolving credit
- Amortization table
- Cost of finance

6.9 Understanding Student Loans

- FAFSA
- College funding gap
- Subsidized loan
- Unsubsidized loan
- Parent loan for undergraduate students
- Private student loan
- School-channel loan
- Direct-to-consumer loan
- Standard repayment plan
- Federal consolidation
- Refinancing
- Private consolidation
- Graduated repayment plan
- Extended repayment plan
- Discretionary income
- Pay as you earn (PAYE) repayment plan
- Revised pay as you earn (REPAYE) repayment plan
- Income-based (IBR) repayment plan
- Income-contingent (ICR) repayment plan
- Delinquent
- Default
- Rehabilitation

6.10 Credit Cards

- Reward program
- Annual fee
- Credit limit
- Bank-issued credit card

- Store-issued credit card
- Travel and entertainment cards
- Charge cards
- Billing period
- Balance
- Minimum payment
- Average daily balance

6.11 Buying or Leasing a Car

- Title and registration fees
- Destination fee
- Documentation fee
- Dealer preparation fee
- Extended warranty
- Down payment
- Acquisition fee
- Security deposit
- Disposition fees
- Liability insurance
- Collision insurance
- Comprehensive insurance
- Uninsured or underinsured motorist insurance
- Medical payment insurance
- Personal injury insurance
- Gap insurance
- Rental reimbursement insurance

6.12 Renting and Homeownership

- Lease
- Landlord
- Mortgage
- Escrow account
- Assessed value

6.13 Income Tax

- Gross income
- Adjusted gross income
- Exemption
- Taxable income
- Deduction
- Tax credit
- Earned income credit
- American opportunity credit
- Lifetime learning credit
- Child tax credit
- Child and dependent care tax credit
- Premium tax credit

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6.14.1: Key Concepts

6.1 Understanding Percent

- Know what a percent is as a fraction, a decimal, and as a part of the whole.
- Use the percent equation to find any of the three values that are related by the equation.
- Apply the percent equation in applications.

6.2 Discounts, Markups, and Sales Tax

- Discounts are markdowns from an original price.
- Mark-ups are increases to the price paid by a retailer to cover their costs.
- be able to calculate the markup based on a percentage of the cost
- Sales taxes vary from state to state and often county to county.
- Retail prices, sales prices and percent discounts can be calculated if the other two values are known.
- Original costs, retail prices, and percent markup can be calculated if the other two values are known.
- In calculations, sales tax acts like a markup.

6.3 Simple Interest

- Interest is money that is paid by a borrower for the privilege of borrowing the money.
- Simple interest is computed by substituting the principal, interest rate, and number of years into the formula $I =$
- The payoff for a loan is the amount of principal remaining on a loan plus the interest that accumulated on the loan since the last payment.
- The future value of an investment yielding simple interest is the original principal plus the interest earned on the investment.
- When making a partial payment, some of the payment pays off all the accumulated interest, while the remainder of the payment is applied to the principal of the loan.
- Finding the present value of an investment is used to determine how much should be invested now in order to achieve a specific goal.

6.4 Compound Interest

- Compound interest means that the interest earned during one period will earn interest in later periods. Essentially, the amount of the principal grows from period to period.
- The important values in computing compound interest are the interest rate, the principal, the length of time the investment, and the number of times the investment is compounded.
- Compound interest has minimal impact early, but later has a very large impact.
- You can determine how much to invest today in order to reach a goal for some time later.
- Compound interest can be translated into an effective annual yield, which allows for comparison between investment options.

6.5 Making a Personal Budget

- A budget is a set of guidelines for how to allocate your income.
- Budgeting helps to plan for many of life’s expenses
- Budgets are used to compare income to expenses. When expenses exceed income, changes have to be made.
- Budgets can help evaluate the affordability of life changes.
- One guideline for setting a budget is the 50-30-20 budget philosophy. The guidelines suggest that 50% of income is allocated to necessary expenses, 30% to expenses that wants, and 20% to savings and other debt reduction.

6.6 Methods of Savings

- There are three main types of savings accounts, saving accounts, certificates of deposit (CD), and money market accounts.
- Savings account are very risk free, and so yield low interest rates.
- The differences in the three types of savings accounts relate to their convenience.
- Savings account typically have a lower interest rate that money market accounts, which typically have lower interest rates than CDs.
- Ordinary annuities more accurately reflect how we save, in that money is deposited repeatedly over time.

- Spreadsheet software, such as Google Sheets, have built in functions that can be used to quickly calculate both the future value of an ordinary annuity account, but also the payment necessary to reach a goal using an ordinary annuity.

6.7 Investments

- There are many different investments with different returns and risks.
- Bonds are loans form the purchaser to the entity selling the bond.
- Bonds have some tax benefits, low to no risk, and a low return.
- Stocks represent part ownership in a company. As such, stock holders share in the profits, and losses, of the company.
- Information, including price, P/E, yearly highs and lows, and dividend amount can be found in online stock tables available on many websites.
- Mutual funds represent collections of professionally administered investment vehicles. Have shares in a mutual fund has lower risk than ownership of stocks.
- Retirement accounts employ some of the same strategies as mutual funds, in that they spread the risk and are professionally managed.
- IRAs and Roth IRAs differ on when taxes are paid on the money, and who can use them. Roth IRAs have income limits while traditional IRAs do not.

6.8 The Basics of Loans

- There are many reasons for a loan, but primarily it is taken out for a large expense when cash is not available.
- Each payment for an installment loan consists of an interest portion and a principal portion.
- There is a formula to calculate the payment necessary to pay off a loan in installments.
- Amortization schedules, or tables, show how each payment is applied to principal and interest. It also includes other details such as remaining balance and total interest paid.
- Loans often have other fees associated with them such as origination fees or application fees. The total of the interest paid and the fees is the cost of finance.

6.9 Understanding Student Loans

- The FAFSA must be filled out each year that a student wishes to borrow for.
- A student’s funding gap determines how much they need in loans to pay for college.
- Federal subsidized student loans defer payments until after graduation and interest does not accrue on these loans.
- Unsubsidized student loans defer payment until after graduation but interest begins accruing as soon as the loan finds are disbursed.
- There are both yearly and aggregate limits for student loans to prevent over-borrowing, among other reasons.
- Federal direct loans have a low interest rate set by the government, but other student loans have varying rates of interest set by the banks.
- The standard repayment plan lasts 10 years and is made up of monthly payments.
- Consolidating or refinancing student loans merges many student loans into one loan.
- If only federal loans are consolidated, the interest rate is the same as the individual loans, currently set at 4.99%.
- If other loans are refinanced together, the interest rate may be lower with the new loan.
- Other repayment plans are available. Such a plan may have payment that start small and grow as the loan is paid off, or it may have a longer term, or may be based on the discretionary income of the student.
- Being delinquent on a student loan is a precursor to being in default. Making payments in a timely fashion allows the student to avoid this situation.

6.10 Credit Cards

- Credit cards can be a flexible way to pay for almost anything, but can become a financial hazard if used unwisely.
- When deciding which credit card to apply for, evaluate the interest rate, fees (annual and late), reward programs and credit limit. Be sure they meet your criteria.
- Paying off the balance of your credit card every month will control your spending and will never result in paying interest.
- Credit card statements hold all important information about your credit card, including payment, balances, charges and billing cycle dates.

- Although the minimum payment is attractive precisely because it is so small, paying only the minimum results is a long payoff term and higher interest costs.

6.11 Buying or Leasing a Car

- There are many factors to consider when choosing to buy or lease a car.
- The cost of the car is increased by a number of fees and sales tax.
- There are advantages to buying a car and advantages to leasing a car. The decision between the two depends on the preference of the buyer.
- Insurance covers costs associate with accidents. It is made up of various components.
- The costs of owning a car, including insurance and maintenance, should be a part of the budgeting process.
- Budgeting for unexpected repairs can ease the stress of encountering large repair bill.

6.12 Renting and Homeownership

- There are many points of comparison between renting and buying a house.
- Before deciding to buy a house, you should carefully consider all the responsibilities that come with home ownership.
- Renting comes with more restrictions on the renter, but with fewer costs and is easier to move from.
- Owning a house has more costs but has more freedom, plus the owner creates equity.
- Mortgages are loans, and payments are calculated in the same way as any other loan.
- Amortization tables help a homeowner understand the mortgage and how the payments are applied to the principal and interest.
- In addition to paying the amount financed for a mortgage, the monthly payment will include an escrow payment, which covers insurance and taxes.

6.13 Income Tax

- Federal income tax is based on income after certain adjustments.
- Gross income is income from all sources, including gifts and winnings.
- Before taxes are calculated, the taxable income is found by subtracting deductions and exemptions from gross income.
- Income tax is progressive, increasing in rate as income increases.
- Being in the 32% tax bracket means some of your income is taxed at 10%, some at 12%, some at 22%, some at 24%, and the rest at 32%.
- Income in each tax bracket is taxed at that bracket's rate, which means in 2022 the first $10,275 earned is taxed at 10% only.
- Tax credits are subtracted from the taxes that are owed.
- Some tax credits are refundable, which means they can make the amount you owe negative, which results in a refund.

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6.14.2: Videos

6.1 Understanding Percent

- Finding Percent of a Total
- Finding the Total from the Percent and the Part
- Finding the Percent When the Total and the Part Are Known

6.2 Discounts, Markups, and Sales Tax

- Computing Price Based on a Percent Off Coupon
- Finding Sales Tax Percentage

6.4 Compound Interest

- Compound Interest
- Compare Simple Interest to Interest Compounded Annually
- Compare Simple Interest and Compound Interest for Different Number of Periods Per Year

6.5 Making a Personal Budget

- Creating a Budget
- 50-30–20 Budget Philosophy

6.6 Methods of Savings

- Return on Investment, ROI
- Future Value Using Google Sheets

6.7 Investments

- Bonds
- Reading Stock Summary Online
- Mutual Funds
- 401(k) Accounts

6.8 The Basics of Loans

- Credit Scores Explained
- Reading an Amortization Table

6.9 Understanding Student Loans

- Types of Student Loans
- Repayment Plans

6.10 Credit Cards

- Choosing a Credit Card
- Reading Credit Card Statements

6.12 Renting and Homeownership

- Rent or Buy
- Closing Costs

6.13 Income Tax

- Deductions Versus Credits
- Tax Brackets

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6.14.3: Formula Review

Formula Review

6.1 Understanding Percent

part $=$ percent x total

6.2 Discounts, Markups, and Sales Tax

discount $=$ percent discount $\times$ original price

sale price $=$ original price - discount

sale price $=$ original price - percent discount $\times$ original price $=$ original price $\times(1-$ percent discount $)$

markup $=$ percent markup $\times$ cost

retail price $=$ cost + markup

retail price $=$ cost + percent markup $\times$ cost $=$ cost $\times(1+$ percent markup $)$

sales tax $=$ purchase price $\times$ tax rate

Total price $=$ purchase price + purchase price $\times$ tax rate $=$ purchase price $\times(1+$ tax rate $)$

6.3 Simple Interest

$$I=P\times r\times t$$
$$T=P+I$$
$$T=P+P\times r\times t$$
$$I=P\times\frac{r}{365}\times t$$
$$FV=P+I=P+P\times r\times t$$
$$A=P\times\frac{r\times(1+r)^{t}}{(1+r)^{t}-1}$$
$$PV=\frac{FV}{(1+rt)}$$

6.4 Compound Interest

$$A=P\left(1+\frac{r}{n}\right)^{nt}$$
$$PV=\frac{A}{\left(1+\frac{r}{n}\right)^{n\times t}}$$
$$Y=\left(1+\frac{r}{n}\right)^{n}-1$$

6.6 Methods of Savings

$$A=P\left(1+\frac{r}{n}\right)^{nt}$$
$$\mathrm{ROI}=\frac{FV-P}{P}$$
$$FV=pmt\times\frac{(1+r/n)^{n\times t}-1}{r/n}$$
$$pmt=\frac{FV\times(r/n)}{(1+r/n)^{n\times t}-1}$$

6.7 Investments

$$\text{annual return } = \left(\frac{FV}{P}\right)^{\left(\frac{1}{t}\right)} - 1$$

$$\text{P/E} = \frac{\text{Share Price}}{\text{Dividend}}$$

$$\text{Yld\%} = \frac{\text{Annual Dividend}}{\text{Share Price}} \times 100\%$$

6.8 The Basics of Loans

$$I = P \times \frac{r}{n}$$

$$pmt = \frac{P \times (r/n) \times (1 + r/n)^{n\times t}}{(1 + r/n)^{n\times t} - 1}$$

6.9 Understanding Student Loans

funding gap = total cost - all aid

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$pmt = \frac{P\times(r/n)\times(1+r/n)^{n\times t}}{(1+r/n)^{n\times t}-1}$$

discretionary income $=$ gross income $-1.5\times$ poverty guideline

6.10 Credit Cards

$$I = \frac{\text{ADB} \times r \times d}{365}$$

6.11 Buying or Leasing a Car

$$pmt = \frac{P \times (r/n) \times (1 + r/n)^{n\times t}}{(1 + r/n)^{n\times t} - 1}$$

$$\text{MD} = \frac{P - R}{n}$$

$$\text{APR} = 2400 \times \text{MF}$$

$$\text{MF} = \text{APR}/2,400$$

$$PMT = \frac{(P - R)}{n} + (P + R) \times \text{MF}$$

6.12 Renting and Home Ownership

$$pmt = \frac{P \times (r/12) \times (1 + r/12)^{12\times t}}{(1 + r/12)^{12\times t} - 1}$$

$$T = pmt \times 12 \times t$$

$$\text{CoF} = T - P$$

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6.14.4: Projects

Creating Your Future Budget

In this project, you will create a budget based on a job you are likely to have after you graduate.

1. Go online and research the average starting salary for the profession you are studying for. Use at least two sources. Be sure to record the web address from your search.
2. Approximate your monthly take-home pay. You may use the SmartAsset website to estimate this.
3. Use the 50-30-20 budget philosophy to determine how much you should budget for needs, wants and savings, or extra debt reduction.
4. Create a list of likely expenses. This list must include rent/mortgage, utilities, food, and school loan repayment. You may also want to include car payments, gasoline, and other items.
5. Categorize each expense as need, want, or savings.
6. Using the amounts found in step 3, decide how much to allocate to each of your expenses. It may help to quickly research how much rent is where you want to live.
7. Discuss the choices you had to make, and why you prioritized some expenses over others.

Interest Rate and Time: What Is the Relationship?

The interplay between interest rate and time for an annuity is not easily seen. How the amount that must be deposited per compounding period, pmt, changes based on the time and interest rate would be useful to understand. In this project, you will explore this relationship. We will use a fixed future value of FV = \$1,000,000 and a fixed number of periods per year, 12 (monthly compounding). With those, we'll find various annuity payments that must be made to reach the goal.

The annual interest rate for the investment is in the top row. The number of years for the investment is in the left column. In each cell (or box), find the monthly payment necessary to reach the goal of \$1,000,000.

		Annual Interest Rate					
		1.5%	2.0%	3.0%	5.0%	7.5%	10.0%
Number of Years	10						
	15						
	20						
	30						
	40						
	45						

Describe how the interest rates and number of years impact the payment necessary to reach the goal of \$1,000,000.

Finding a Home

In this project you will identify a home you like, and then estimate the costs associated with that home.

1. Find a home in your region that you would like to buy using an online search of listings in your area. Zillow is a good place to begin.
2. Find the asking price for this home. Assume you would pay that price.
3. Find an estimation for closing costs in your area. Assume you finance those costs also.
4. Estimate the taxes to be paid on the home per year. It is likely that the online listing of the home has an estimate for the taxes for the house.
5. Use Google to determine the average homeowner's insurance cost in your region.
6. Use the internet to determine the average interest rate for a 30-year mortgage.

7. Find how much you would pay per month, based on the answers to the previous questions, including the escrow payments for taxes and insurance.
8. Assume you will pay $50 per $100,000 borrowed in PMI. Add this to the monthly payment.

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6.14.5: Chapter Review

Understanding Percent

? Exercise 6.14.5.1

Convert 0.45 to percent form.

? Exercise 6.14.5.2

What is 70% of 200?

? Exercise 6.14.5.3

Rewrite the fraction $\frac{2}{4}0100$ as a percentage.

? Exercise 6.14.5.4

20 is what percent of 500?

Discounts, Markups, and Sales Tax

? Exercise 6.14.5.5

An item with a retail price of $250.00 is on sale for 30% off. What is the sale price of the item? Round to the nearest penny.

? Exercise 6.14.5.6

The sales price of an item is $38.50 after a 40% discount. What was the retail price of the item? Round to the nearest penny.

? Exercise 6.14.5.7

A retailer buys an item for $1,000.00 and has a 65% markup. What is the retail price of the item? Round to the nearest penny.

? Exercise 6.14.5.8

The sale price of an item is $46.00 and the retail price is $65.00. What is the percent discount? Round to two decimal places.

? Exercise 6.14.5.9

The retail price of an item is $345.38 and sales tax in the region is 8.25%. How much is the sales tax?

Simple Interest

? Exercise 6.14.5.10

$3,000 is invested in a 5-year CD earning 2.25% interest. How much is the CD worth in 5 years?

? Exercise 6.14.5.11

A $4,000.00 simple interest loan is taken out for 3 years at 12.5%. How much is owed when the loan comes due?

? Exercise $6.14.5.12$

A \$10,000 loan with annual simple interest of 14.9% is taken out for 90 days. How much is due in 90 days?

? Exercise $6.14.5.13$

A simple interest loan for \$20,000 is taken out at 13.9% annual interest rate. A partial payment of \$13,500.00 is made 30 days into the loan period. After this payment, what will the remaining balance be?

? Exercise $6.14.5.14$

Find the present value of an investment with future value of \$15,000 if the investment earns 3.55% simple interest for 10 years?

Compound Interest

? Exercise $6.14.5.15$

Find the future value after 20 years of \$15,000 deposited in an account bearing 4.26% interest compounded quarterly.

? Exercise $6.14.5.16$

What is the effective annual yield for an account bearing 3.21% interest compounded quarterly? Round to two decimal places if necessary.

? Exercise $6.14.5.17$

Find the present value of \$100,000 in an account bearing 5.25% interest compounded daily after 30 years.

Making a Personal Budget

? Exercise $6.14.5.18$

In budgeting, what is the difference between an expense that is a need and an expense that is a want?

? Exercise $6.14.5.19$

Apply the 50-30-20 budget philosophy to a monthly income of \$6,000.00.

? Exercise $6.14.5.20$

Create the budget for a person with the following income and bills, and determine how much the income exceeds or falls short of the expenses:
Job = \$5,250, Side job = \$550, rent = \$1,150.00, utilities = \$150.00, internet = \$39.99, student loans = \$375.00, food = \$550.00, bus and subway = \$112.00, credit cards = \$200.00, entertainment = \$400.00, clothing = \$150.00.

? Exercise $6.14.5.21$

In question 20, the budget for a person was created. Can the person afford to buy a car if the monthly payments will be \$365.50, monthly car insurance will be \$114.75, and gasoline will be \$200.00?

Methods of Savings

? Exercise 6.14.5.22

Why is a CD less flexible than a money market account?

? Exercise 6.14.5.23

Which is likely to have the highest interest rate, a savings account, CD, or money market account?

? Exercise 6.14.5.24

$3,000 is deposited in a money market account bearing 3.88% interest compounded monthly. How much is the account worth in 15 years?

? Exercise 6.14.5.25

Refer to question 24. What is the return on investment for the money market account? Round to two decimal places, if necessary.

? Exercise 6.14.5.26

$300 is deposited monthly in an ordinary annuity that bears 6.5% interest compounded monthly. How much is in the annuity after 30 years?

? Exercise 6.14.5.27

How much must be deposited quarterly in an account bearing 4.89% interest compounded quarterly for 30 years so the account has a future value of $500,000?

Investments

? Exercise 6.14.5.28

Why is investing in a stock riskier than in a mutual fund?

? Exercise 6.14.5.29

Which investment offers high rates of return with reduced risk?

? Exercise 6.14.5.30

Which investment offer a fixed rate of return over time?

? Exercise 6.14.5.31

Which is the riskiest investment?

? Exercise 6.14.5.32

How much is earned on a 10-year bond with issue price of $1,000 that pays 3.9% annually?

? Exercise 6.14.5.33

Henri’s employer matches up to 6% of salary for IRA contributions. Henri’s annual income is $52,500. If Henri wants to deposit $3,000 annually in his IRA, how much in matching funds does the employer deposit in the IRA?

? Exercise 6.14.5.34

How much must be deposited quarterly in a mutual fund that is expected to earn 11.2% interest compounded quarterly if the account is to be worth $1,000,000 after 38 years?

? Exercise 6.14.5.35

Lorraine purchases stocks for $5,000. The stocks paid dividends by reinvesting the dividends in stock. She sold the stock for $13,750 after 6 years. What was Lorraine’s annual return on that investment? Round to two decimal places if necessary.

? Exercise 6.14.5.36

What are two ways stocks earn money?

The Basics of Loans

? Exercise 6.14.5.37

What is an installment loan?

? Exercise 6.14.5.38

An installment loan with monthly payments has an outstanding balance of $3,560. If the annual interest rate is 9%, how much interest will be paid that month?

? Exercise 6.14.5.39

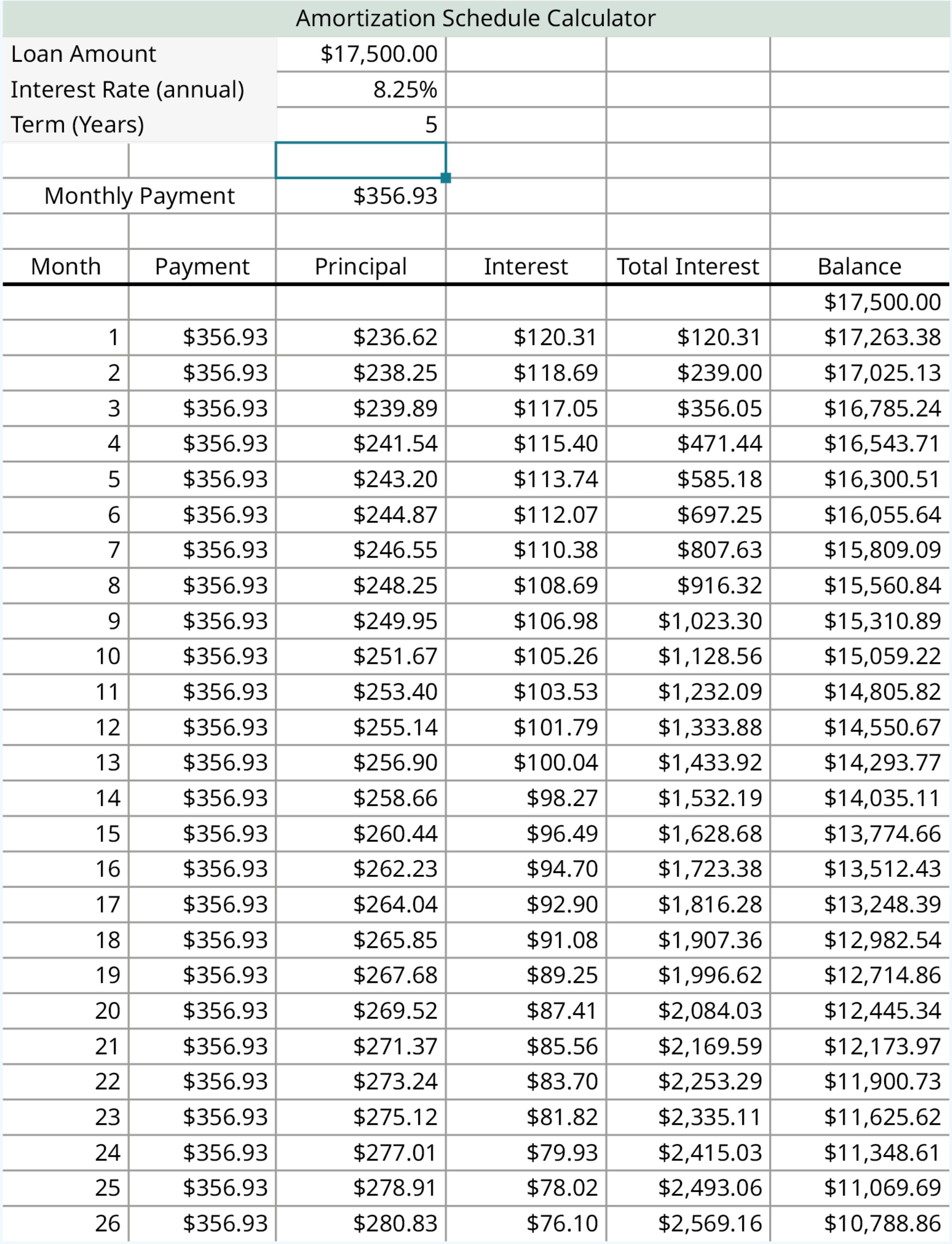
A loan for $42,000 is taken out for 6 years at 7.5% interest. What are the payments for that loan?

? Exercise 6.14.5.40

The total interest paid on a loan was $4,500. The loan had an origination fee of $400.00, a $125.00 processing fee, and a filing fee of $250.00. How much was the cost of financing for that loan?

? Exercise 6.14.5.41

Using the amortization table below, what is the remaining balance after payment 25?



Amortization Schedule Calculator

Loan Amount		$17,500.00			
Interest Rate (annual)		8.25%			
Term (Years)		5			
Monthly Payment		$356.93			

Month	Payment	Principal	Interest	Total Interest	Balance
					$17,500.00
1	$356.93	$236.62	$120.31	$120.31	$17,263.38
2	$356.93	$238.25	$118.69	$239.00	$17,025.13
3	$356.93	$239.89	$117.05	$356.05	$16,785.24
4	$356.93	$241.54	$115.40	$471.44	$16,543.71
5	$356.93	$243.20	$113.74	$585.18	$16,300.51
6	$356.93	$244.87	$112.07	$697.25	$16,055.64
7	$356.93	$246.55	$110.38	$807.63	$15,809.09
8	$356.93	$248.25	$108.69	$916.32	$15,560.84
9	$356.93	$249.95	$106.98	$1,023.30	$15,310.89
10	$356.93	$251.67	$105.26	$1,128.56	$15,059.22
11	$356.93	$253.40	$103.53	$1,232.09	$14,805.82
12	$356.93	$255.14	$101.79	$1,333.88	$14,550.67
13	$356.93	$256.90	$100.04	$1,433.92	$14,293.77
14	$356.93	$258.66	$98.27	$1,532.19	$14,035.11
15	$356.93	$260.44	$96.49	$1,628.68	$13,774.66
16	$356.93	$262.23	$94.70	$1,723.38	$13,512.43
17	$356.93	$264.04	$92.90	$1,816.28	$13,248.39
18	$356.93	$265.85	$91.08	$1,907.36	$12,982.54
19	$356.93	$267.68	$89.25	$1,996.62	$12,714.86
20	$356.93	$269.52	$87.41	$2,084.03	$12,445.34
21	$356.93	$271.37	$85.56	$2,169.59	$12,173.97
22	$356.93	$273.24	$83.70	$2,253.29	$11,900.73
23	$356.93	$275.12	$81.82	$2,335.11	$11,625.62
24	$356.93	$277.01	$79.93	$2,415.03	$11,348.61
25	$356.93	$278.91	$78.02	$2,493.06	$11,069.69
26	$356.93	$280.83	$76.10	$2,569.16	$10,788.86

Understanding Student Loans

Exercise 6.14.5.42

If a college program takes 5 years, how long is the student’s eligibility period for student loans?

Exercise 6.14.5.43

What is the funding gap for a student with cost of college equal to $35,750 and non-loan financial aid of $24,150?

Exercise 6.14.5.44

What is the maximum amount of federal subsidized and unsubsidized loans for a dependent student in their third year of an undergraduate data science program?

Exercise 6.14.5.45

A single person has gross income of $28,500. Their poverty guideline is $12,000. What is their discretionary income?

Credit Cards

Exercise 6.14.5.46

The billing cycle for a credit card goes from March 15 to April 14. The balance at the start of the billing cycle is $450.00. The list of transactions on the card is below. Find the average daily balance for the billing cycle.

Date	Activity	Amount
15-Mar	Billing Date Balance	$450.00
21-Mar	Movie	$50.00
28-Mar	Gasoline	$65.00
31-Mar	Snacks	$15.50
1-Apr	Dinner	$63.60
1-Apr	Payment	$300.00
1-Apr	Gasoline	$48.90
9-Apr	Plane Tickets	$288.50

Exercise 6.14.5.47

The average daily balance for Greg’s last credit card statement was $1,403.50. The card charges 15.9% interest. If the billing cycle for that statement was 30 days, how much interest is Greg charged?

Exercise 6.14.5.48

The billing cycle for a credit card goes from March 15 to April 14. The balance at the start of the billing cycle is $450.00. The list of transactions on the card is below. The interest rate is 15.9%. What is the balance due at the end of the billing cycle?

Date	Activity	Amount
15-Mar	Billing Date Balance	$450.00
21-Mar	Movie	$50.00

Date	Activity	Amount
28-Mar	Gasoline	\$65.00
31-Mar	Snacks	\$15.50
1-Apr	Dinner	\$63.60
1-Apr	Payment	\$300.00
1-Apr	Gasoline	\$48.90
9-Apr	Plane Tickets	\$288.50

Buying or Leasing a Car

? Exercise 6.14.5.49

Name three common fees when leasing a car.

? Exercise 6.14.5.50

Name three common fees when buying a car.

? Exercise 6.14.5.51

If the cost of a car is \$35,500, and the residual value of the car after 3 years is \$27,800, what is the monthly depreciation for the car?

? Exercise 6.14.5.52

What is the annual interest rate if the money factor is 0.000045?

? Exercise 6.14.5.53

What is the payment for a lease on a car if the price of the car is \$29,900, the residual price of the car is \$16,500, the lease is for 24 months, and the APR is 8.9%?

Renting and Homeownership

? Exercise 6.14.5.54

Name two advantages of renting a residence.

? Exercise 6.14.5.55

Name two advantages of buying a home.

? Exercise 6.14.5.56

What is PMI?

Exercise 6.14.5.57

What are the monthly payments for a 25-year mortgage for $165,000 at 4.8% interest, not including escrow?

Exercise 6.14.5.58

What are the monthly payments, including escrow, for a 25-year mortgage for $165,000 at 4.8% interest, if the property taxes are $5,200 annually, the homeowner's insurance is $825 every 6 months, and the PMI is $45 per month?

Income Tax

Exercise 6.14.5.59

What is the maximum gift that a person can receive that is an exemption on federal income taxes?

Exercise 6.14.5.60

Find the gross income if a person earns $51,500 at their full-time job, they earn $5,300 for contracted services, and $5,000 in gambling winnings.

Exercise 6.14.5.61

Find adjusted gross income for a person with a gross income of $67,850, if they had a $2,000 gift, they paid $1,000 in mortgage interest, $4,250 in property taxes, and they contributed $750 to a qualifying charity.

Exercise 6.14.5.62

How much SSI is paid by a person with an annual income of $183,500?

Exercise 6.14.5.63

How much in FICA taxes are paid from a person's paycheck with gross income of $3,450?

Exercise 6.14.5.64

How much in federal income tax does a person filing single pay if their taxable income is $205,240? Use the tax table below.

Bracket	Lower Income Limit	Upper Income Limit	Tax Rate
1	0	$10,275	10%
2	$10,276	$41,775	12%
3	$41,776	$89,075	22%
4	$89,076	$170,050	24%
5	$170,051	$215,950	32%
6	$215,951	$539,900	35%
7	$539,901		37%

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6.14.6: Chapter Test

? Exercise 6.14.6.1

45 is what percent of 180?

? Exercise 6.14.6.2

Find 47% of 31. Round to two decimal places.

? Exercise 6.14.6.3

What is the sale price of a $90 shirt if there is a 25% discount for the shirt?

? Exercise 6.14.6.4

A small boutique buys purses for $27.50 per purse. What is the boutique's retail price for the purses if they add a 90% markup?

? Exercise 6.14.6.5

What was the discount on a computer is the retail price was $1,200 and the sale price is $950? Round to two decimal places if necessary.

? Exercise 6.14.6.6

A 15% simple interest loan of $2,000 is taken out for 75 days. How much is owed when the loan is due?

? Exercise 6.14.6.7

What is the future value of a $15,000 investment made at 4.65% interest compounded yearly for 10 years?

? Exercise 6.14.6.8

Apply the 50-30-20 budget philosophy to a monthly income of $3,920.

? Exercise 6.14.6.9

$2,500 is deposited in an account that bears 5.25% interest compounded quarterly for 15 years. What is the return on investment for that investment? Round to two decimal places if necessary.

? Exercise 6.14.6.10

How much per month must be paid into an ordinary annuity so its future value is $150,000 after 10 years if the annuity earns 3.85% interest compounded monthly?

? Exercise 6.14.6.11

Bethanie's employer matches up to 7% of annual salary for deposits into the company sponsored IRA. Bethanie earns $62,400 annually. She deposits $8,000 annually inter her IRA. How much does the company deposit in Bethanie's IRA?

Exercise 6.14.6.12

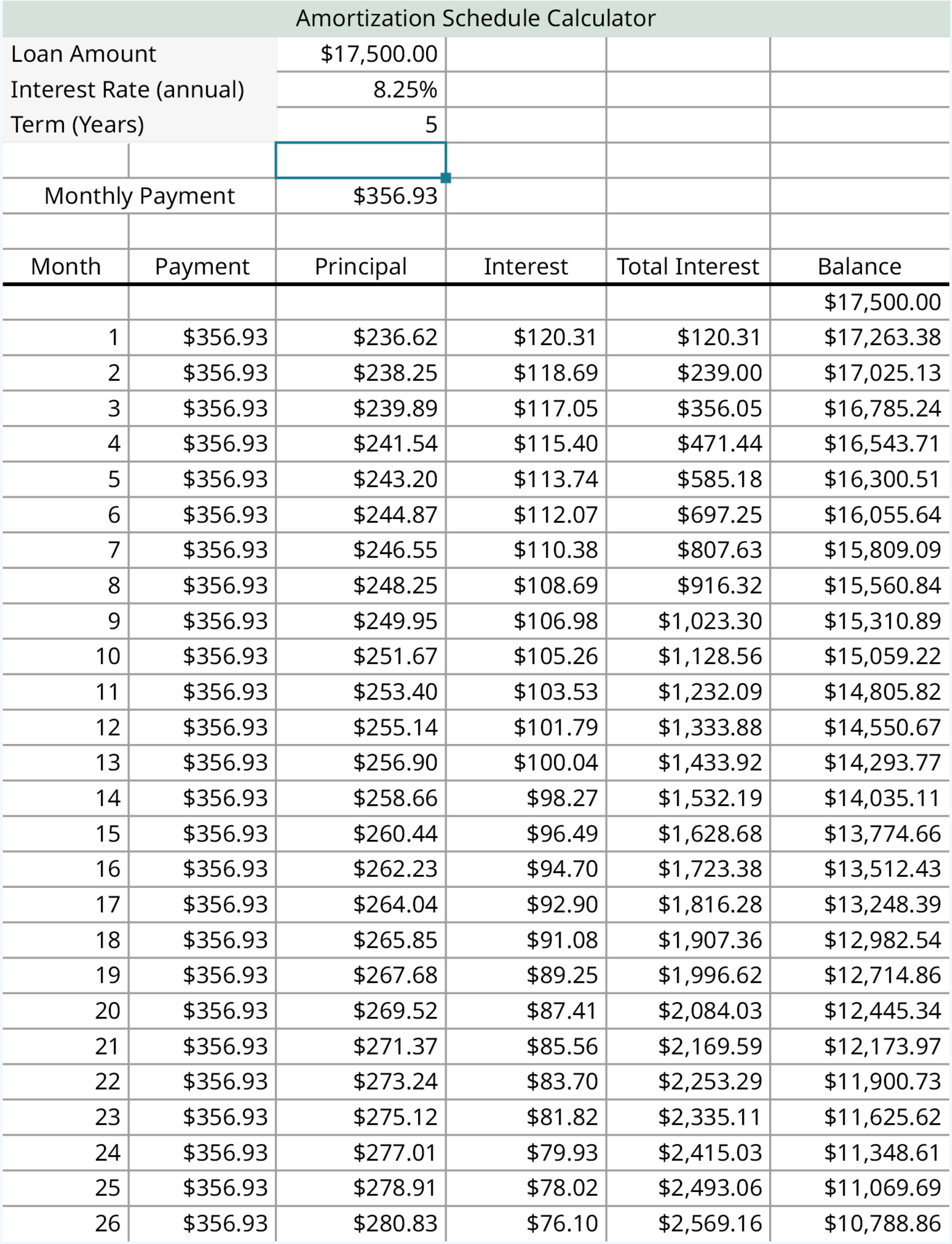
If \$360 is deposited monthly in a mutual fund that is expected to earn 9.35% compounded monthly, how much will the mutual fund be worth after 23 years?

Exercise 6.14.6.13

A \$14,250, 4-year loan is taken out at 9.15% interest. What are the monthly payments for the loan?

Exercise 6.14.6.14

Using the amortization table below, how much total interest has been paid after payment 20?



Amortization Schedule Calculator					
Loan Amount		$17,500.00			
Interest Rate (annual)		8.25%			
Term (Years)		5			
Monthly Payment		$356.93			
Month	Payment	Principal	Interest	Total Interest	Balance
					$17,500.00
1	$356.93	$236.62	$120.31	$120.31	$17,263.38
2	$356.93	$238.25	$118.69	$239.00	$17,025.13
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6	$356.93	$244.87	$112.07	$697.25	$16,055.64
7	$356.93	$246.55	$110.38	$807.63	$15,809.09
8	$356.93	$248.25	$108.69	$916.32	$15,560.84
9	$356.93	$249.95	$106.98	$1,023.30	$15,310.89
10	$356.93	$251.67	$105.26	$1,128.56	$15,059.22
11	$356.93	$253.40	$103.53	$1,232.09	$14,805.82
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14	$356.93	$258.66	$98.27	$1,532.19	$14,035.11
15	$356.93	$260.44	$96.49	$1,628.68	$13,774.66
16	$356.93	$262.23	$94.70	$1,723.38	$13,512.43
17	$356.93	$264.04	$92.90	$1,816.28	$13,248.39
18	$356.93	$265.85	$91.08	$1,907.36	$12,982.54
19	$356.93	$267.68	$89.25	$1,996.62	$12,714.86
20	$356.93	$269.52	$87.41	$2,084.03	$12,445.34
21	$356.93	$271.37	$85.56	$2,169.59	$12,173.97
22	$356.93	$273.24	$83.70	$2,253.29	$11,900.73
23	$356.93	$275.12	$81.82	$2,335.11	$11,625.62
24	$356.93	$277.01	$79.93	$2,415.03	$11,348.61
25	$356.93	$278.91	$78.02	$2,493.06	$11,069.69
26	$356.93	$280.83	$76.10	$2,569.16	$10,788.86

Exercise 6.14.6.15

What is the maximum amount of federal subsidized and unsubsidized loans per year for an independent student in a graduate program?

Exercise 6.14.6.16

Braden takes out a private student loan for $6,250 in his second year of his undergraduate IT program. He begins paying the loan after 39 months. If Braden's interest rate was 8%, how much is his balance when he begins to pay off the loan?

Exercise 6.14.6.17

The billing cycle for Therese's credit card is July 7 to August 6 (31 days). The interest rate on the card is 17.9%. The balance on her card at the start of the billing cycle is $925.00. The table below has her transactions for the billing cycle. What is her balance due at the end of the cycle?

Date	Activity	Amount
7-Jul	Billing Date Balance	$925.00
10-Jul	Child Care	$850.00
10-Jul	Gasoline	$51.00
12-Jul	Food	$135.50
18-Jul	Payment	$750.00
18-Jul	Pizza	$35.00
18-Jul	Gift	$25.00
21-Jul	Food	$121.75
30-Jul	Gasoline	$47.50
1-Aug	Department store	$173.00
4-Aug	Food	$98.00

Exercise 6.14.6.18

The cost of a car is $45,750. The residual price after 2 years is $28,822.50. What is the monthly depreciation for the car?

Exercise 6.14.6.19

Find the payment for a 3-year lease on a car if the price of the car is $30,000, the residual price of the car is $20,000, and the APR is 8.5%.

Exercise 6.14.6.20

What are the monthly payments, excluding escrow, for a 20-year mortgage for $131,500 at 5.15% interest?

Exercise 6.14.6.21

What are the monthly payments, including escrow, for a 20-year mortgage for $131,500 at 5.15% interest, with property taxes of $3,170 per year and homeowner's insurance of $715 every 6 months?

? Exercise 6.14.6.22

If a person earns $2700 gross on a paycheck, how much is taken out in FICA taxes?

? Exercise 6.14.6.23

Find the taxable income for a person who has $81,200 in earnings, $1,250 reported on a 1099-INT for a savings account, $5,400 reported on a 1099-MISC for a contract job they did, paid $5,250 in mortgage interest, paid $4,700 in property taxes, and has $4,200 in bond interest.

? Exercise 6.14.6.24

Find the federal income tax owed by a person filing single who has a taxable income of $114,750. Use the table below to determine the taxes.

Bracket	Lower Income Limit	Upper Income Limit	Tax Rate
1	0	$10,275	10%
2	$10,276	$41,775	12%
3	$41,776	$89,075	22%
4	$89,076	$170,050	24%
5	$170,051	$215,950	32%
6	$215,951	$539,900	35%
7	$539,901		37%

? Exercise 6.14.6.25

If a person owes $20,000 in federal income tax, but qualifies for $4,000 in child tax credit and $8,000 for child and dependent care credit, how much federal tax does the person actually pay?

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CHAPTER OVERVIEW

7: Probability

7.0: Introduction
7.1: The Multiplication Rule for Counting
  7.1.0: Exercises
7.2: Permutations
  7.2.0: Exercises
7.3: Combinations
  7.3.0: Exercises
7.4: Tree Diagrams, Tables, and Outcomes
  7.4.0: Exercises
7.5: Basic Concepts of Probability
  7.5.0: Exercises
7.6: Probability with Permutations and Combinations
  7.6.0: Exercises
7.7: What Are the Odds?
  7.7.0: Exercises
7.8: The Addition Rule for Probability
  7.8.0: Exercises
7.9: Conditional Probability and the Multiplication Rule
  7.9.0: Exercises
7.10: The Binomial Distribution
  7.10.0: Exercises
7.11: Expected Value
  7.11.0: Exercises
7.12: Chapter Summary
  7.12.0: Key Terms
  7.12.1: Key Concepts
  7.12.2: Formula Review
  7.12.3: Projects
  7.12.4: Chapter Review
  7.12.5: Chapter Test

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7.0: Introduction

Casinos are big business; according to the American Gaming Association, commercial casinos in the United States brought in over \$43 billion in revenue in 2019. Casinos must walk a fine line in order to be profitable. Their customers must lose more money than they win, on average, in order to stay in business. But if the chances of a single customer winning more money than they lose is too small, people will stop coming in the door to play the games.

Figure 7.0.1: Roulette is a game whose outcomes are based entirely on the concept of probability. (credit: modification of work "Roulette wheel" by Håkan Dahlström/Flickr, CC BY 2.0)

In this chapter, we'll study the techniques a casino must use to determine how likely it is that a customer will win a particular game, and then how the casino decides how much money a winner will rake in so that the customers are happy, but the casino also turns a profit in the long run. In order to figure out those likelihoods, we have to be able to somehow consider every possible outcome of these games. For example, in a game that involves players receiving 5 cards from a deck of 52, there are 2,598,960 possibilities for each player. We'll start off this chapter by learning how to count those possible outcomes.

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7.1: The Multiplication Rule for Counting

Figure 7.1.1 : The Multiplication Rule for Counting allows us to compute more complicated probabilities, like drawing two aces from a deck. (credit: “Pair of Aces – Poker” by Poker Photos/Flickr, CC BY 2.0)

Learning Objectives

Apply the Multiplication Rule for Counting to solve problems.

One of the first bits of mathematical knowledge children learn is how to count objects by pointing to them in turn and saying: “one, two, three, …” That’s a useful skill, but when the number of things that we need to count grows large, that method becomes onerous (or, for *very* large numbers, impossible for humans to accomplish in a typical human lifespan). So, mathematicians have developed short cuts to counting big numbers. These techniques fall under the mathematical discipline of combinatorics, which is devoted to counting.

Multiplication as a Combinatorial Short Cut

One of the first combinatorial short cuts to counting students learn in school has to do with areas of rectangles. If we have a set of objects to be counted that can be physically arranged into a rectangular shape, then we can use multiplication to do the counting for us. Consider this set of objects (Figure 7.1.2):

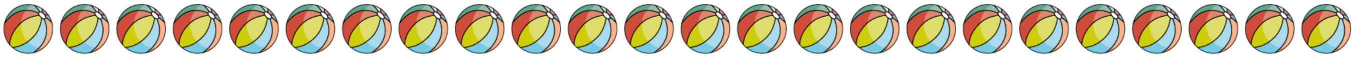

Figure 7.1.2

Certainly we can count them by pointing and running through the numbers, but it’s more efficient to group them (Figure 7.1.3).

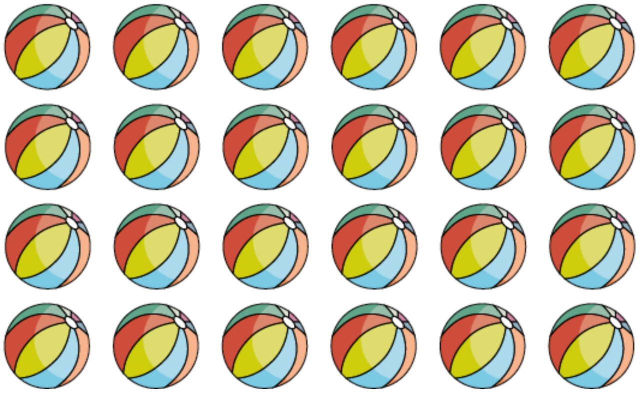

Figure $7.1.3$

If we group the balls by 4s, we see that we have 6 groups (or, we can see this arrangement as 4 groups of 6 balls). Since multiplication is repeated addition (i.e., $6 \times 4 = 4 + 4 + 4 + 4 + 4 + 4$), we can use this grouping to quickly see that there are 24 balls.

Let's generalize this idea a little bit. Let's say that we're visiting a bakery that offers customized cupcakes. For the cake, we have three choices: vanilla, chocolate, and strawberry. Each cupcake can be topped with one of four types of frosting: vanilla, chocolate, lemon, and strawberry. How many different cupcake combinations are possible? We can think of laying out all the possibilities in a grid, with cake choices defining the rows and frosting choices defining the columns (Figure $7.1.4$).

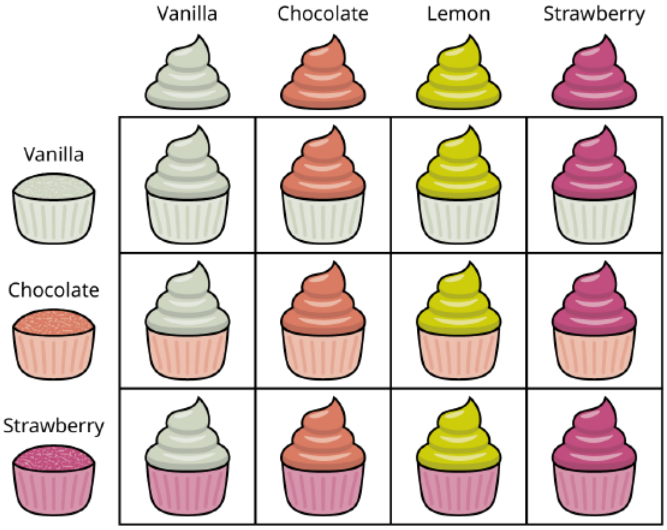

Figure $7.1.4$

Since there are 3 rows (cakes) and 4 columns (frostings), we have $3 \times 4 = 12$ possible combinations. This is the reasoning behind the Multiplication Rule for Counting, which is also known as the Fundamental Counting Principle. This rule says that if there are n ways to accomplish one task and m ways to accomplish a second task, then there are $n \times m$ ways to accomplish both tasks. We can tack on additional tasks by multiplying the number of ways to accomplish those tasks to our previous product.

✓ Example ✓ 7.1.1: Using the Multiplication Rule for Counting

Every card in a standard deck of cards has two identifying characteristics: a suit (clubs, diamonds, hearts, or spades; these are indicated by these symbols, respectively: ♣, ♢, ♡, ♠) and a rank (ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, jack, queen, and king; the letters A, J, Q, and K are used to represent the words). Each possible pair of suit and rank appears exactly once in the deck. How many cards are in the standard deck?

Answer

Since there are 4 suits and 13 ranks, the number of cards must be $4 \times 13 =$

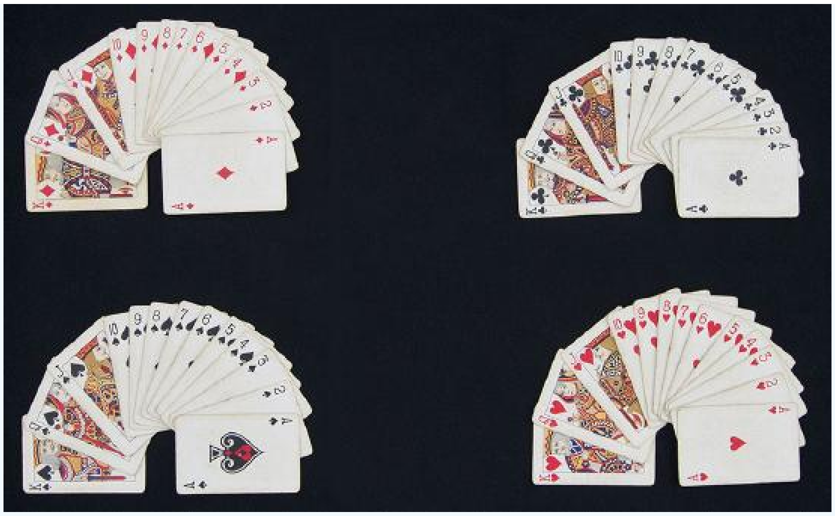

Figure 7.1.5 Standard Deck of Cards, Sorted by Rank and Suit (credit: "Playing Cards, USS Arkansas" by Naval History & Heritage Command/Flickr, CC BY 2.0)

Your Turn 7.1.1

Joe’s Pizza Shack offers pizzas with 4 different types of crust and a choice of 15 toppings. How many different one-topping pizzas can be made at Joe’s?

✓ Example 7.1.2: Using the Multiplication Rule for Counting for 4 Groups

The University Combinatorics Club has 31 members: 8 seniors, 7 juniors, 5 sophomores, and 11 first-years. How many possible 4-person committees can be formed by selecting 1 member from each class?

Answer

Since we have 8 choices for the senior, 7 choices for the junior, 5 for the sophomore, and 11 for the first-year, there are $8 \times 7 \times 5 \times 11 = 3{,}080$ different ways to fill out the committee.

Your Turn 7.1.2

The menu for Joe’s Pizza Shack offers pizzas with 4 different types of crust and a choice of 15 toppings. Suppose that Joe’s also offers a choice of 3 sauces and 2 cheese blends. How many different one-topping pizzas can be made at Joe’s now?

✓ Example 7.1.3: Using the Multiplication Rule for Counting for More Groups

The standard license plates for vehicles in a certain state consist of 6 characters: 3 letters followed by 3 digits. There are 26 letters in the alphabet and 10 digits (0 through 9) to choose from. How many license plates can be made using this format?

Answer

Since there are 26 different letters and 10 different digits, the total number of possible license plates is $26 \times 26 \times 26 \times 10 \times 10 \times 10 = 17,576,000$.

Your Turn 7.1.3

At a certain college, ID cards are issued to all students, faculty, and staff. These cards have unique ID codes for each person: a letter to indicate the person’s status (S for students, F for faculty, and E for staff), followed by 5 digits and finally 3 letters (these letters can be anything). How many different ID codes can be created using this scheme?

Check Your Understanding

1. A website that lets you build custom belts has 18 different buckles and 30 different straps. How many different belts can be made using those materials?
2. A chain of chicken restaurants offers a combo that includes your choice of 3 or 5 chicken strips, along with your choice of side dish. If there are 7 side dishes, how many different ways are there to build this combo meal?
3. When you flip a coin, there are 2 possible outcomes: heads and tails. Let's say you flip a coin 10 times, and after each you write down the result of the flip (H for heads, T for tails). How many different results (strings of 10 characters, where each is either an H or a T) are possible?
4. A T-shirt company allows shoppers to customize their shirts in several ways. There are 5 sizes, 8 shirt colors, 4 designs, and 5 design colors. How many different shirts can be made?
5. Josephine is trying to build her class schedule for next semester. Because of her work schedule, she has only 4 class periods that can work for her, and she must take 4 classes. If there are 15 classes that she could take during the first period, 18 during the second, 12 during the third, and 8 during the fourth, how many different schedules could Josephine build?

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7.1.0: Exercises

An ice-cream parlor sells 26 different flavors of ice cream. A basic sundae has one scoop of any flavor of ice cream, your choice of one of 3 sauces, and any one of 8 different toppings.

? Exercise $7.1.0.1$

How many different basic sundaes are possible?

? Exercise $7.1.0.2$

The ice-cream parlor also sells a medium sundae. The options are the same except it starts with 2 scoops of ice cream, which can be the same flavor or different flavors. How many different medium sundaes are there?

? Exercise $7.1.0.3$

The ice-cream parlor also sells a large sundae. The choice of a large sundae allows you to choose any 3 scoops of ice cream, any 2 sauces (they can be the same, or you can choose 2 different ones), and any 3 toppings (that might be 3 servings of the same topping, or 2 servings of one topping and a single serving of another, or 3 different toppings). How many different large sundaes are possible?

? Exercise $7.1.0.4$

A company that builds custom computers offers 4 hard drive sizes, 4 memory sizes, 3 graphics cards, and 3 display options. How many computer configurations do they offer, if customers choose one of each customization?

? Exercise $7.1.0.5$

A video game allows users to customize their avatars. There are 12 hair styles that users may choose from, as well as 5 hair colors, 8 skin tones, 24 shirts, 12 pants, and 8 shoes. How many different avatars are possible?

? Exercise $7.1.0.6$

A small company has 3 divisions: Sales, Research and Development, and Manufacturing. One person from each division will be chosen to create an advisory board for the management group. If there are 8 people in Sales, 15 in Research and Development, and 48 in Manufacturing, how many different compositions of the advisory board are possible?

? Exercise $7.1.0.7$

A multiple-choice quiz has 5 questions, each of which has 4 possible answers. How many different ways are there to respond to this quiz?

? Exercise $7.1.0.8$

The teacher decides to make the quiz from above a little harder by offering 5 responses on each of the 5 questions. How many ways are there to respond to this quiz?

? Exercise $7.1.0.9$

In the United States, radio and television broadcast stations are assigned unique identifiers known as call signs. Call signs consist of 4 letters. The first is either K or W (generally speaking, stations with a K call sign are west of the Mississippi River and stations with a W call sign are east of the river, though there are several exceptions to this rule); the remaining 3 letters can be anything. How many possible call signs are there under this system?

? Exercise 7.1.0.10

Little sister has asked big brother to play a new game she’s invented. It uses a modified deck of cards with 3 suits and only the numbered cards (those with rank 2 through 10). How many cards are in her deck?

? Exercise 7.1.0.11

The board game *Mastermind* has 2 players. One of them is designated the codemaker who creates a code that consists of a series of 4 colors (indicated in the game with 4 colored pegs), which may contain repeats. The other player, who is the codebreaker, tries to guess the code. If there are 6 colors that the codemaker can use to make the code, how many possible codes can be made?

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7.2: Permutations

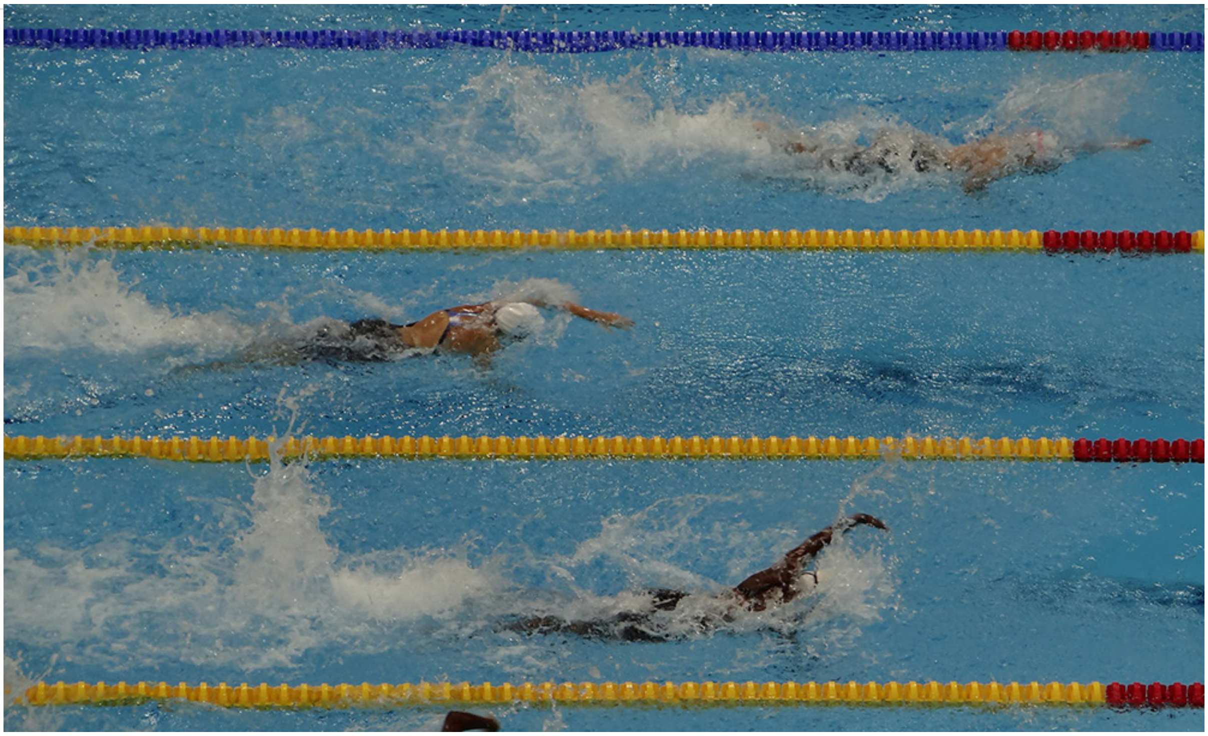

Figure 7.2.1: We can use permutations to calculate the number of different orders of finish in an Olympic swimming heat. (credit: "London 2012 Olympics Park Stratford London" by Gary Bembridge/Flickr, CC BY 2.0)

Learning Objectives

1. Use the Multiplication Rule for Counting to determine the number of permutations.
2. Compute expressions containing factorials.
3. Compute permutations.
4. Apply permutations to solve problems.

Swimming events are some of the most popular events at the summer Olympic Games. In the finals of each event, 8 swimmers compete at the same time, making for some exciting finishes. How many different orders of finish are possible in these events? In this section, we'll extend the Multiplication Rule for Counting to help answer questions like this one, which relate to permutations. A permutation is an ordered list of objects taken from a given population. The length of the list is given, and the list cannot contain any repeated items.

Applying the Multiplication Rule for Counting to Permutations

In the case of the swimming finals, one possible permutation of length 3 would be the list of medal winners (first, second, and third place finishers). A permutation of length 8 would be the full order of finish (first place through eighth place). Let's use the Multiplication Rule for Counting to figure out how many of each of these permutations there are.

Example

7.2.1: Using the Multiplication Rule for Counting to Find the Number of Permutations

The final heat of Olympic swimming events features 8 swimmers (or teams of swimmers).

1. How many different podium placements (first place, second place, and third place) are possible?
2. How many different complete orders of finish (first place through eighth place) are possible?

Answer

1. Let’s start with the first place finisher. How many options are there? Since 8 swimmers are competing, there are 8 possibilities. Once that first swimmer completes the race, there are 7 swimmers left competing for second place. After the second finisher is decided, there are 6 swimmers remaining who could possibly finish in third place. Thus, there are 8 possibilities for first place, 7 for second place, and 6 for third place. The Multiplication Rule for Counting then tells us there are $8 \times 7 \times 6 = 336$ different ways the winners’ podium can be filled out.
2. To look at the complete order of finish, we can continue the pattern we can see in part 1 of this example: There are 5 possibilities for fourth place, 4 for fifth place, 3 for sixth place, 2 for seventh place, and then just 1 swimmer is left to finish in eighth place. Using the Multiplication Rule for Counting, we see that there are $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40,320$ possible orders of finish.

Your Turn 7.2.1

You have a hand of 5 cards (that happen to create what’s called a royal flush in the game of poker): 10♠, J♠, Q♠, K♠, and A♠. Into how many different orders can you put those cards?

Factorials

The pattern we see in Example 7.4 occurs commonly enough that we have a name for it: factorial.
For any positive whole number n, we define the factorial of n (denoted $n!$ and read "n factorial") to be the product of every whole number less than or equal to n. We also define 0! to be equal to one. We will use factorials in a couple of different contexts, so let's get some practice doing computations with them.

Example 7.2.2: Computing Factorials

Compute the following:

1. 4!
2. $\frac{8!}{6!}$
3. $\frac{9!}{3!4!}$

Answer

1. $4! = 4 \times 3 \times 2 \times 1 = 24$
2. There are two ways to approach this calculation. The first way is to compute the factorials first, then divide:

$$\frac{8!}{6!} = \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{6 \times 5 \times 4 \times 3 \times 2 \times 1} = \frac{40{,}320}{720} = 56$$

   However, there is an easier way! You may notice in the second step that there are several terms that can be canceled; that’s always the case whenever we divide factorials. In this case, notice that we can rewrite the numerator like this:

$$8! = 8 \times 7 \times (6 \times 5 \times 4 \times 3 \times 2 \times 1) = 8 \times 7 \times 6!$$

   With that in mind, we can proceed this way by canceling out the 6!:

$$\frac{8!}{6!} = \frac{8 \times 7 \times \mathbf{6!}}{\mathbf{6!}} = 8 \times 7 = 56$$

   That’s much easier!

3. Let’s approach this one using our canceling technique. When we see two factorials in either the numerator or denominator, we should focus on the larger one first. So:

$$\frac{9!}{3!4!} = \frac{9 \times 8 \times 7 \times 6 \times 5 \times \mathbf{4!}}{3!\mathbf{4!}} = \frac{9 \times 8 \times 7 \times \mathbf{6} \times 5}{\mathbf{3} \times \mathbf{2} \times \mathbf{1}} = 9 \times 8 \times 7 \times 5 = 2{,}520$$

Your Turn 7.2.2

Compute the following:

$6!$

$\frac{12!}{10!}$

$\frac{8!}{4!4!}$

Permutations

As we've seen, factorials can pop up when we're computing permutations. In fact, there is a formula that we can use to make that connection explicit. Let's define some notation first. If we have a collection of n objects and we wish to create an ordered list of r of the objects (where $1 \le r \le n$), we'll call the number of those permutations msub (read "the number of permutations of n objects taken r at a time"). We formalize the formula we'll use to compute permutations below.

FORMULA

${}_nP_r = \frac{n!}{(n-r)!}$

If you wondered why we defined $0! = 1$ earlier, it was to make formulas like this one work; if we have n objects and want to order all of them (so, we want the number of permutations of n objects taken n at a time), we get msubmsub$P_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n!$. Next, we'll get some practice computing these permutations.

Example 7.2.3: Computing Permutations

Find the following numbers:

1. The number of permutations of 12 objects taken 3 at a time
2. The number of permutations of 8 objects taken 5 at a time
3. The number of permutations of 32 objects taken 2 at a time

Answer

1. $$\text{msubmsub}P_3 = \frac{12!}{(12-3)!} = \frac{12!}{9!} = \frac{12 \times 11 \times 10 \times \mathbf{9!}}{\mathbf{9!}} = 12 \times 11 \times 10 = 1,$$

2. $$\text{msubmsub}P_5 = \frac{8!}{(8-5)!} = \frac{8 \times 7 \times 6 \times 5 \times 4 \times \mathbf{3!}}{\mathbf{3!}} = 8 \times 7 \times 6 \times 5 \times 4 = 6,$$

3. $$\text{msubmsub}P_2 = \frac{32!}{(32-2)!} = \frac{32 \times 31 \times \mathbf{30!}}{\mathbf{30!}} = 32 \times 31 =$$

Your Turn 7.2.3

Find the following numbers:

The number of permutations of 6 objects taken 2 at a time

The number of permutations of 14 objects taken 4 at a time

The number of permutations of 19 objects taken 3 at a time

Example 7.2.4: Applying Permutations

1. A high school graduating class has 312 students. The top student is declared valedictorian, and the second-best is named salutatorian. How many possible outcomes are there for the valedictorian and salutatorian?
2. In the card game blackjack, the dealer's hand of 2 cards is dealt with 1 card faceup and 1 card facedown. If the game is being played with a single deck of (52) cards, how many possible hands could the dealer get?

3. The University Combinatorics Club has 3 officers: president, vice president, and treasurer. If there are 18 members of the club, how many ways are there to fill the officer positions?

Answer

1. This is the number of permutations of 312 students taken 2 at a time, and msubmsub$P_2 = 97,$.
2. We want the number of permutations of 52 cards taken 2 at a time, and msubmsub$P_2 = 2,$.
3. Here we're looking for the number of permutations of 18 members taken 3 at a time, and msubmsub$P_3 = 4,$.

Your Turn 7.2.4

One of the big draws at this year's state fair is the pig race. There are 15 entrants, and prizes are given to the top three finishers. How many different combinations of top-three finishes could there be?

Who Knew?: Very Big Permutations

Permutations involving relatively small sets of objects can get very big, very quickly. A standard deck contains 52 cards. So, the number of different ways to shuffle the cards—in other words, the number of permutations of 52 objects taken 52 at a time—is $52! \approx 8 \times 10^{67}$ (written out, that's an 8 followed by 67 zeroes). The estimated age of the universe is *only* about 4×10^{17} seconds. So, if a very bored all-powerful being started shuffling cards at the instant the universe began, it would have to have averaged at least $\frac{8 \times 10^{67}}{4 \times 10^{17}} \approx 2 \times 10^{50}$ shuffles *per second since the beginning of time* to have covered every possible arrangement of a deck of cards. That means the next time you pick up a deck of cards and give it a good shuffle, it's almost certain that the particular arrangement you created has never been created before and likely never will be created again.

Check Your Understanding

1. Compute 5!.
2. Compute $\frac{10!}{7!3!}$.
3. Compute ${}_{12}P_3$.
4. Compute ${}_{8}P_4$.
5. The standard American edition of the board game *Monopoly* has a deck of 15 orange Chance cards. In how many different ways could the first 4 Chance cards drawn in a game appear?

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7.2.0: Exercises

For the following exercises, give a whole number that's equal to the given expression.

Exercise 7.2.0.1

$3!$

Exercise 7.2.0.2

$9!$

Exercise 7.2.0.3

$\frac{7!}{2!2!3!}$

Exercise 7.2.0.4

$\frac{8!}{5!2!}$

Exercise 7.2.0.5

$\frac{21!}{18!2!}$

Exercise 7.2.0.6

$\frac{28!}{26!2!}$

Exercise 7.2.0.7

$\frac{34!}{30!3!}$

Exercise 7.2.0.8

$\frac{17!}{12!5!}$

Exercise 7.2.0.9

${}_4P_3$

Exercise 7.2.0.10

${}_7P_5$

Exercise 7.2.0.11

${}_{12}P_{10}$

Exercise 7.2.0.12

${}_{14}P_{10}$

Exercise 7.2.0.13

${}_{10}P_{8}$

Exercise 7.2.0.14

${}_{15}P_{11}$

The following exercises are about the card game euchre, which uses a partial standard deck of cards: It only has the cards with ranks 9, 10, J, Q, K, and A for a total of 24 cards. Some variations of the game use the 8s or the 7s and 8s, but we'll stick with the 24-card version.

Exercise 7.2.0.15

A euchre hand contains 5 cards. How many ways are there to receive a 5-card hand (where the order in which the cards are received matters, i.e., 9♡, J♡, K♣, 9♠, 10♠ is different from 9♠ J♡, 9♡, K♣, 10♠?

Exercise 7.2.0.16

After all 4 players get their hands, the remaining 4 cards are placed facedown in the center of the table. How many arrangements of 4 cards are there from this deck?

Exercise 7.2.0.17

Euchre is played with partners. How many ways are there for 2 partners to receive 5-card hands (where the order in which the cards are received matters)?

Exercise 7.2.0.18

How many different arrangements of the full euchre deck are possible (i.e., how many different shuffles are there)?

The following exercises involve a horse race with 13 entrants.

Exercise 7.2.0.19

How many possible complete orders of finish are there?

Exercise 7.2.0.20

An exacta bet is one where the player tries to predict the top two finishers in order. How many possible exacta bets are there for this race?

Exercise 7.2.0.21

A trifecta bet is one where the player tries to predict the top three finishers in order. How many possible trifecta bets are there for this race?

Exercise 7.2.0.22

A superfecta bet is one where the player tries to predict the top four finishers in order. How many possible superfecta bets are there for this race?

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7.3: Combinations

Figure 7.3.1: Combinations help us count things like the number of possible card hands, when the order in which the cards were drawn doesn't matter. (credit: "IMG_3177" by Zanaca/Flickr, CC BY 2.0)

Learning Objectives

1. Distinguish between permutation and combination uses.
2. Compute combinations.
3. Apply combinations to solve applications.

In Permutations, we studied permutations, which we use to count the number of ways to generate an ordered list of a given length from a group of objects. An important property of permutations is that the order of the list matters: The results of a race and the selection of club officers are examples of lists where the order is important. In other situations, the order is not important. For example, in most card games where a player receives a hand of cards, the order in which the cards are received is irrelevant; in fact, players often rearrange the cards in a way that helps them keep the cards organized.

Combinations: When Order Doesn't Matter

In situations in which the order of a list of objects doesn't matter, the lists are no longer permutations. Instead, we call them combinations.

✓ Example 7.3.1: Distinguishing Between Permutations and Combinations

For each of the following situations, decide whether the chosen subset is a permutation or a combination.

1. A social club selects 3 members to form a committee. Each of the members has an equal share of responsibility.
2. You are prompted to reset your email password; you select a password consisting of 10 characters without repeats.
3. At a dog show, the judge must choose first-, second-, and third-place finishers from a group of 16 dogs.
4. At a restaurant, the special of the day comes with the customer's choice of 3 sides taken from a list of 6 possibilities.

Answer

1. Since there is no distinction among the responsibilities of the 3 committee members, the order isn't important. So, this is a combination.
2. The order of the characters in a password matter, so this is a permutation.
3. The order of finish matters in a dog show, so this is a permutation.
4. A plate with mashed potatoes, peas, and broccoli is functionally the same as a plate with peas, broccoli, and mashed potatoes, so this is a combination.

Your Turn 7.3.1

Decide whether the following represent permutations or combinations:

On Halloween, you give each kid who comes to your door 3 pieces of candy, taken randomly from a candy dish.

Your class is going on a field trip, but there are too many people for one vehicle. Your instructor chooses half the class to take the first vehicle.

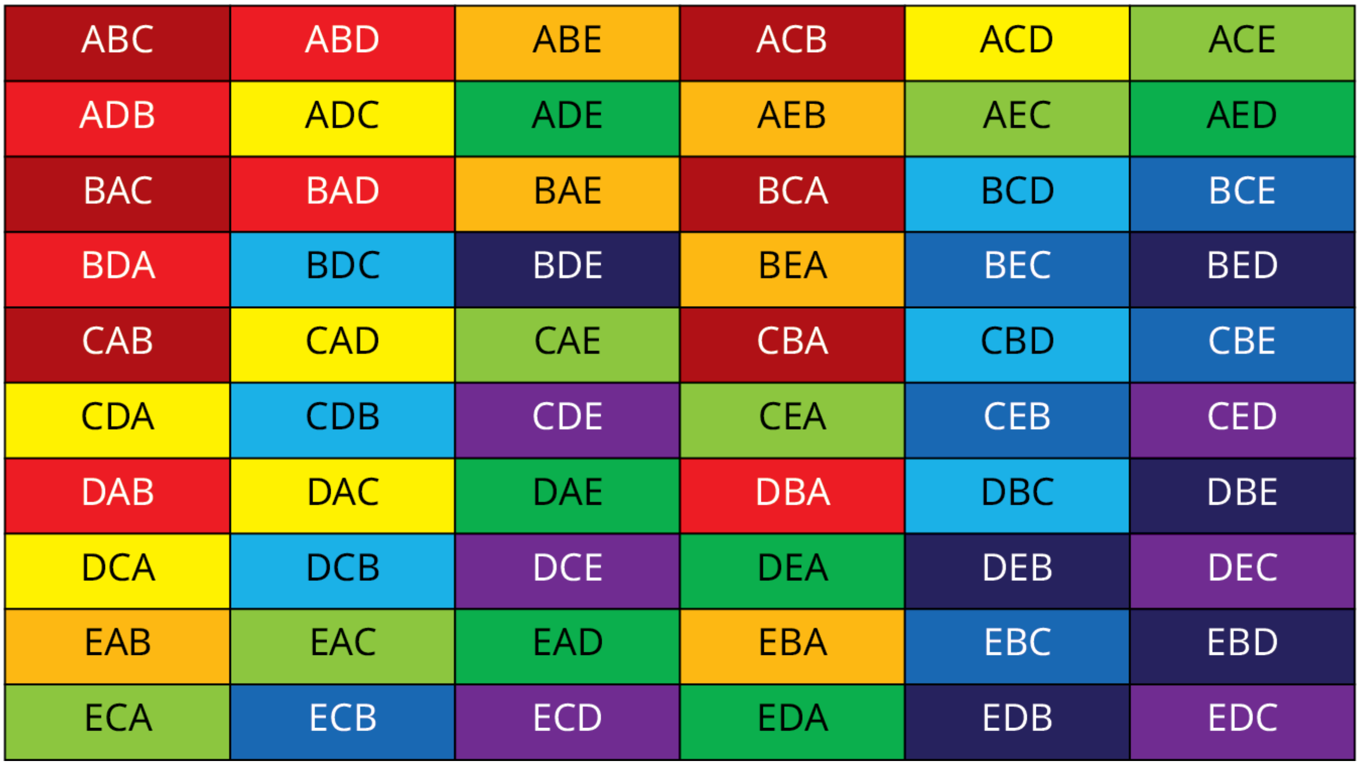
\(\frac{12!}{{10!}\) \(\frac{8!}{{4!4!}\)

Counting Combinations

Permutations and combinations are certainly related, because they both involve choosing a subset of a large group. Let’s explore that connection, so that we can figure out how to use what we know about permutations to help us count combinations. We’ll take a basic example. How many ways can we select 3 letters from the group A, B, C, D, and E? If order matters, that number is msubmsub$P_3 =$. That’s small enough that we can list them all out in the table below.

ABC	ABD	ABE	ACB	ACD	ACE
ADB	ADC	ADE	AEB	AEC	AED
BAC	BAD	BAE	BCA	BCD	BCE
BDA	BDC	BDE	BEA	BEC	BED
CAB	CAD	CAE	CBA	CBD	CBE
CDA	CDB	CDE	CEA	CEB	CED
DCA	DAC	DAE	DBA	DBC	DBE
DCA	DCB	DCE	DEA	DEB	DEC
EAB	EAC	EAD	EBA	EBC	EBD
ECA	ECB	ECD	EDA	EDB	EDC

Now, let’s look back at that list and color-code it so that groupings of the same 3 letters get the same color, as shown in Figure 7.9:



ABC	ABD	ABE	ACB	ACD	ACE
ADB	ADC	ADE	AEB	AEC	AED
BAC	BAD	BAE	BCA	BCD	BCE
BDA	BDC	BDE	BEA	BEC	BED
CAB	CAD	CAE	CBA	CBD	CBE
CDA	CDB	CDE	CEA	CEB	CED
DAB	DAC	DAE	DBA	DBC	DBE
DCA	DCB	DCE	DEA	DEB	DEC
EAB	EAC	EAD	EBA	EBC	EBD
ECA	ECB	ECD	EDA	EDB	EDC

Figure 7.3.2

After color-coding, we see that the 60 cells can be seen as 10 groups (colors) of 6. That's no coincidence! We've already seen how to compute the number of permutations using the formula To compute the number of combinations, let's count them another way using the Multiplication Rule for Counting. We'll do this in two steps:

Step 1: Choose 3 letters (paying no attention to order).

Step 2: Put those letters in order.

The number of ways to choose 3 letters from this group of 5 (A, B, C, D, E) is the number of combinations we're looking for; let's call that number msub (read "the number of combinations of 5 objects taken 3 at a time"). We can see from our chart that this is ten (the number of colors used). We can generalize our findings this way: remember that the number of permutations of n things taken r at a time is msubmsub$P_r = \frac{n!}{(n-r)!}$. That number is *also* equal to msubmsub$C_r \times r!$, and so it must be the case that $\frac{n!}{(n-r)!}$msubmsub$C_r \times r!$. Dividing both sides of that equation by $r!$ gives us the formula below.

FORMULA

msubmsub$_nC_r = \frac{n!}{r!(n-r)!}$

nCr=n!r!(n−r)!

Example 7.3.2: Using the Combination Formula

Compute the following:

1. $_8C_3$
2. $_{12}C_5$
3. $_{15}C_9$

Answer

1. $_8C_3 = \frac{8!}{3!(8-3)!} = \frac{8\times7\times6\times\mathbf{5!}}{\mathbf{3}\times\mathbf{2}\times\mathbf{1}\times\mathbf{5!}} = 8\times7 = 56$
2. $_{12}C_5 = \frac{12!}{5!(12-5)!} = \frac{12\times11\times10\times9\times8\times7!}{5\times4\times3\times2\times1\times7!} = 11\times9\times8 = 792$
3. $_{15}C_9 = \frac{15!}{9!(15-9)!} = \frac{\mathbf{15}\times14\times13\times\mathbf{12}\times11\times10\times\mathbf{9!}}{\mathbf{9!}\times\mathbf{6}\times\mathbf{5}\times4\times\mathbf{3}\times\mathbf{2}\times1} = \frac{14\times13\times11\times10}{4} = 5,005$

Your Turn 7.3.2

Compute the following:

$_6C_4$

$_{10}C_8$

$_{14}C_5$

Example 7.3.3: Applying the Combination Formula

1. In the card game Texas Hold'em (a variation of poker), players are dealt 2 cards from a standard deck to form their hands. How many different hands are possible?
2. The board game *Clue* uses a deck of 21 cards. If 3 people are playing, each person gets 6 cards for their hand. How many different 6-card *Clue* hands are possible?
3. Palmetto Cash 5 is a game offered by the South Carolina Education Lottery. Players choose 5 numbers from the whole numbers between 1 and 38 (inclusive); the player wins the jackpot of $100,000 if the randomizer selects those numbers in any order. How many different sets of winning numbers are possible?

Answer

1. A standard deck has 52 cards, and a hand has 2 cards. Since the order doesn't matter, we use the formula for counting combinations:

$$\text{msubmsub}C_2 = \frac{52!}{2!(52-2)!} = \frac{52 \times 51 \times \mathbf{50!}}{2 \times 1 \times \mathbf{50!}} = \frac{52 \times 51}{2} = 1{,}326.$$

2. Again, the order doesn't matter, so the number of combinations is:

$$\text{msubmsub}C_6 = \frac{21!}{6!(21-6)!} = \frac{21 \times \mathbf{20} \times 19 \times \mathbf{18} \times 17 \times 16 \times \mathbf{15!}}{\mathbf{6} \times \mathbf{5} \times \mathbf{4} \times \mathbf{3} \times 2 \times 1 \times \mathbf{15!}} = \frac{21 \times 19 \times 17 \times 16}{2} = 54{,}264.$$

3. There are 38 numbers to choose from, and we must pick 5. Since order doesn't matter, the number of combinations is:

$$\text{msubmsub}C_5 = \frac{38!}{5!(38-5)!} = \frac{38 \times 37 \times 36 \times 35 \times 34 \times \mathbf{33!}}{5 \times 4 \times 3 \times 2 \times 1 \times \mathbf{33!}} = 501{,}492.$$

Your Turn 7.3.3

At a charity event with 58 people in attendance, 3 raffle winners are chosen. All receive the same prize, so order doesn't matter. How many different groups of 3 winners can be chosen?

A sorority with 42 members needs to choose a committee with 4 members, each with equal responsibility. How many committees are possible?

Checkpoint

The notation and nomenclature used for the number of combinations is not standard across all sources. You'll sometimes see $\binom{n}{r}$ *instead of* msub *. Sometimes you'll hear that expression read as "*n *choose* r*" as shorthand for "the number of combinations of* n *objects taken* r *at a time."*

People in Mathematics: Early Eastern Mathematicians

Although combinations weren't really studied in Europe until around the 13th century, mathematicians of the Middle and Far East had already been working on them for hundreds of years. The Indian mathematician known as Pingala had described them by the second century BCE; Varāhamihira (fl. sixth century) and Halayudha (fl. 10th century) extended Pingala's work. In the ninth century, a Jain mathematician named Mahāvīra gave the formula for combinations that we use today.

In 10th-century Baghdad, a mathematician named Al-Karaji also knew formulas for combinations; though his work is now lost, it was known to (and repeated by) Persian mathematician Omar Khayyam, whose work survives. Khayyam is probably best remembered as a poet, with his *Rubaiyat* being his most famous work.

Meanwhile, in 11th-century China, Jia Xian also was working with combinations, as was his 13th-century successor Yang Hui.

It is not known whether the discoveries of any of these men were known in the other regions, or if the Indians, Persians, and Chinese all came to their discoveries independently. We do know that mathematical knowledge and sometimes texts did get passed along trade routes, so it can't be ruled out.

✓ Example 7.3.4: Combining Combinations with the Multiplication Rule for Counting

The student government at a university consists of 10 seniors, 8 juniors, 6 sophomores, and 4 first-years.

1. How many ways are there to choose a committee of 8 people from this group?
2. How many ways are to choose a committee of 8 people if the committee must consist of 2 people from each class?

Answer

1. There are 28 people to choose from, and we need 8. So, the number of possible committees is msubmsub$C_8 = 3{,}108,.$
2. Break the selection of the committee members down into a 4-step process: Choose the seniors, then choose the juniors, then the sophomores, and then the first-years, as shown in the table below:

Class	Number of Ways to Choose Committee Representatives
senior	msubmsub$C_2 =$
junior	msubmsub$C_2 =$
sophomore	msubmsub$C_2 =$
first-year	msubmsub$C_2 =$

The Multiplication Rule for Counting tells us that we can get the total number of ways to complete this task by multiplying together the number of ways to do each of the four subtasks. So, there are $45 \times 28 \times 15 \times 6 = 113{,}400$ possible committees with these restrictions.

Your Turn 7.3.4

How many ways are there to choose a hand of 6 cards from a standard deck with the constraint that 3 are ♠, 2 are ♡, and 1 is ♣?

Check Your Understanding

1. Suppose you want to count the number of ways that you can arrange the apps on the home screen on your phone. Should you use permutations or combinations?
2. Your little brother is packing up for a family vacation, but there's only room for 3 of his toys. If you want to know how many possible groups of toys he can bring, should you use permutations or combinations?
3. Compute ${}_{12}C_{10}$.
4. Compute ${}_{16}C_3$.
5. You're planning a road trip with some friends. Though you have 6 friends you'd consider bringing along, you only have room for 3 other people in the car. How many different possibilities are there for your road trip squad?
6. You're packing for a trip, for which you need 3 shirts and 3 skirts. If you have 8 shirts and 5 skirts that would work for the trip, how many different ways are there to pack for the trip?

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7.3.0: Exercises

For the following exercises, decide whether the situation describes a permutation or a combination.

? Exercise 7.3.0.1

You're packing for vacation, and you need to pick 5 shirts.

? Exercise 7.3.0.2

You and your friends are about to play a game, and you need to decide who will have the first turn, second turn, and so on.

? Exercise 7.3.0.3

You are watching your favorite reality show, and you want to know how many possibilities there are for the order of finish for the top three.

? Exercise 7.3.0.4

You are going to be working in groups of 4 with your classmates, and you want to know how many possibilities there are for the composition of your group.

For the following exercises, express your answers as whole numbers.

? Exercise 7.3.0.5

${}_5C_3$

? Exercise 7.3.0.6

${}_8C_2$

? Exercise 7.3.0.7

${}_8C_6$

? Exercise 7.3.0.8

${}_{12}C_3$

? Exercise 7.3.0.9

${}_{12}C_5$

? Exercise 7.3.0.10

${}_{14}C_3$

? Exercise 7.3.0.11

${}_{14}C_{10}$

? Exercise 7.3.0.12

$_{15}C_5$

? Exercise 7.3.0.13

$_{15}C_{13}$

? Exercise 7.3.0.14

$_{18}C_3$

? Exercise 7.3.0.15

$_{18}C_6$

? Exercise 7.3.0.16

$_{20}C_4$

? Exercise 7.3.0.17

In most variations of the card game poker, a hand consists of 5 cards, where the order doesn't matter. How many different poker hands are there?

? Exercise 7.3.0.18

A professor starts each class by choosing 3 students to present solutions to homework problems to the class. If there are 41 students in the class, in how many different ways can the professor make those selections?

? Exercise 7.3.0.19

An election for at-large members of a school board has 7 candidates; 3 will be elected. How many different ways can those 3 seats be filled?

? Exercise 7.3.0.20

There are 20 contestants on a reality TV show; at the end of the first episode, 10 are eliminated. How many different groups of eliminated contestants are possible?

? Exercise 7.3.0.21

At a horse race, bettors can place a bet called an exacta box. For this bet, the player chooses 2 horses; if those horses finish first and second (in either order), the player wins. In a race with 12 horses in the field, how many possible exacta box bets are there?

The following exercises are about the card game euchre, which uses a partial standard deck of cards: it only has the cards with ranks 9, 10, J, Q, K, and A (for a total of 24 cards). Some variations of the game use the 8s or the 7s and 8s, but we'll stick with the 24-card version.

Exercise 7.3.0.22

A euchre hand contains 5 cards. How many ways are there to receive a 5-card hand (where the order in which the cards are received doesn't matter, i.e., 9♡, J♡, K♣, 9♠, 10♠ is the same as 9♠ J♡, 9♡, K♣, 10♠)?

Exercise 7.3.0.23

After all 4 players get their hands, the remaining 4 cards are placed face down in the center of the table. How many different groups of 4 cards are there from this deck?

Exercise 7.3.0.24

Euchre is played with partners. How many ways are there for 2 partners to receive 5-card hands (where, as above, the order doesn't matter)? Hint: After the first person gets their cards, there are $52-5=47$ cards left for the second person.

You and 5 of your friends are at an amusement park, and are about to ride a roller coaster. The cars have room for 6 people arranged in 3 rows of 2, so you and your friends will perfectly fill one car.

Exercise 7.3.0.25

How many ways are there to choose the 2 people in the front row?

Exercise 7.3.0.26

Assuming the front row has been selected, how many ways are there to choose the 2 people in the middle row?

Exercise 7.3.0.27

Assuming the first 2 rows have been selected, how many ways are there to choose the 2 people in the back row?

Exercise 7.3.0.28

Using the Multiplication Rule for Counting and your answers to the earlier parts of this exercise, how many ways are there for your friends to sort yourselves into rows to board the roller coaster?

The University Combinatorics Club has 18 members. Four of them will be selected to form a committee.

Exercise 7.3.0.29

How many different committees of 4 are possible, assuming all of the duties are shared equally?

Exercise 7.3.0.30

Instead of sharing responsibility equally, one person will be chosen to be the committee chair. How many different committees are possible? Count these by selecting a chair first, then selecting the remaining 3 members of the committee from the remaining club members and use the Multiplication Rule for Counting. Show your work.

Exercise 7.3.0.31

Let's count the number of committees with chairs a different way: First, choose 4 people for the committee (as in the first question), then choose 1 of the 4 to be chair. Show your work. Do you get the same number?

Powerball® is a multistate lottery game, which costs $2 to play. Players fill out a ticket by choosing 5 numbers between 1 and 69 (these are the white balls) and then a single number between 1 and 26 (this is the Powerball).

Exercise 7.3.0.32

How many different ways are there to choose the white balls? Players who match these 5 numbers exactly (but not the Powerball) win $1 million.

Exercise 7.3.0.33

How many ways are there to choose the Powerball? Players who correctly pick the Powerball win $4.

Exercise 7.3.0.34

How many ways are there to play the game altogether? Players who match all 5 white balls and the Powerball win (or share) the grand prize. (The grand prize starts at $40 million; if no players win the grand prize, the value goes up for the next drawing. The highest value it has ever reached is $1.586 *billion*!) How many ways are there to fill out a single Powerball ticket?

Exercise 7.3.0.35

You are in charge of programming for a music festival. The festival has a main stage, a secondary stage, and several smaller stages. There are 40 bands confirmed for the festival. Five of those will play the main stage, and 8 will play the secondary stage. How many ways are there for you to allocate bands to these 2 stages?

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7.4: Tree Diagrams, Tables, and Outcomes

Figure 7.4.1: In genetics, the characteristics of an offspring organism depends on the characteristics of its parents. (credit: “Pea Plant” by Maria Keays/Flickr, CC BY 2.0))

Learning Objectives

1. Determine the sample space of single stage experiment.
2. Use tables to list possible outcomes of a multistage experiment.
3. Use tree diagrams to list possible outcomes of a multistage experiment.

In the 19th century, an Augustinian friar and scientist named Gregor Mendel used his observations of pea plants to set out his theory of genetic propagation. In his work, he looked at the offspring that resulted from breeding plants with different characteristics together. For applications like this, it is often insufficient to only know in how many ways a process might end; we need to be able to list all of the possibilities. As we’ve seen, the number of possible outcomes can be very large! Thus, it’s important to have a strategy that allows us to systematically list these possibilities to make sure we don’t leave any out. In this section, we’ll look at two of these strategies.

Single Stage Experiments

When we are talking about combinatorics or probability, the word “experiment” has a slightly different meaning than it does in the sciences. Experiments can range from very simple (“flip a coin”) to very complex (“count the number of uranium atoms that undergo nuclear fission in a sample of a given size over the course of an hour”). Experiments have unknown outcomes that generally rely on something random, so that if the experiment is repeated (or replicated) the outcome might be different. No matter what the experiment, though, analysis of the experiment typically begins with identifying its sample space.

The sample space of an experiment is the set of all of the possible outcomes of the experiment, so it’s often expressed as a set (i.e., as a list bound by braces; if the experiment is “randomly select a number between 1 and 4,” the sample space would be written $\{1,2,3,4\}$).

✓ Example 7.4.1: Finding the Sample Space of an Experiment

For each of the following experiments, identify the sample space.

1. Flip a coin (which has 2 faces, typically called “heads” and “tails”) and note which face is up.
2. Flip a coin 10 times and count the number of heads.
3. Roll a 6-sided die and note the number that is on top.
4. Roll two 6-sided dice and note the sum of the numbers on top.

Answer

1. If we use “H” to denote “heads is facing up” and “T” to denote “tails is facing up”, then the sample space is {H, T}.
2. It’s possible (though unlikely) that there will be no heads flipped; the outcome in that case would be “0.” It’s also possible (more likely, but still quite unlikely) that only one flip will result in heads. Any other whole number is possible, up to the maximum: We’re flipping the coin 10 times, so we can’t get any more than 10 heads. So, the sample space is {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}.
3. There are 6 numbers on the die: 1, 2, 3, 4, 5, and 6. So, the sample space for a single roll of the die is {1, 2, 3, 4, 5, 6}.
4. If we roll 2 dice, the smallest possible sum we could get is $1+1=2$ and the biggest is $6+6=12$. Every other whole number between those two is possible. So, the sample space is {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}.

Your Turn 7.4.1

Identify the sample space of each experiment.

You draw a card from a standard deck and note its suit.

You draw a card from a standard deck and note its rank.

You roll a 4-sided die and note the number on the bottom. (A 4-sided die is shaped like a pyramid, so when it comes to rest, there’s no single side facing up.)

You roll three 4-sided dice and note the sum of the numbers on the bottom.

Multistage Experiments

Some experiments have more complicated sample spaces because they occur in stages. These stages can occur in succession (like drawing cards one at a time) or simultaneously (rolling 2 dice). Sample spaces get more complicated as the complexity of the experiment increases, so it’s important to choose a systematic method for identifying all of the possible outcomes. The first method we’ll discuss is the table.

Using Tables to Find Sample Spaces

Tables are useful for finding the sample space for experiments that meet two criteria: (1) The experiment must have only two stages, and (2) the outcomes of each stage must have no effect on the outcomes of the other. When the stages do not affect each other, we say the stages are independent. Otherwise, the stages are dependent and so we can’t use tables; we’ll look at a method for analyzing dependent stages soon.

✓ Example 7.4.2: Determining Independence

Decide whether the two stages in these experiments are independent or dependent.

1. You flip a coin and note the result, and then flip the coin again and note the result.
2. You draw 2 cards from a standard deck (52 cards), one at a time.

Answer

1. No matter what happens on the first flip, the second flip has the same sample space: {H, T} (You’ll sometimes hear the phrase “The coin has no memory”). So, these stages are independent.

2. Let’s say that the first card you draw is A♠. The sample space for the second draw consists of all the cards *except* A♠ (since that card is no longer in the deck, you can’t draw it again). If instead that first card was 2♡, the sample space for the second draw is different: it’s every card except 2♡. Since the sample space for the second card changes based on the result of the first draw, these stages are dependent.

Your Turn 7.4.2

Decide whether the two stages in these experiments are dependent or independent.

You’re getting dressed to go to a party, and you plan to wear a blouse and a skirt. You choose the blouse first, then the skirt (assume that you’d be comfortable wearing any of your skirts with any of your blouses).

On further reflection, you realize that some of your skirts clash with some of your blouses. So, you choose the blouse first, and then choose a skirt that goes with your chosen blouse.

If you have a two-stage experiment with independent stages, a table is the most straightforward way to identify the sample space. To build a table, you list the outcomes of one stage of the experiment along the top of the table and the outcomes of the other stage down the side. The cells in the interior of the table are then filled using the outcomes associated with each cell’s row and column. Let’s look at an example.

Example 7.4.3: Using Tables to Identify Sample Spaces

Identify the sample spaces of these experiments using tables.

1. You roll two dice: one 4-sided and one 6-sided.
2. You’re in an ice-cream shop, and you’re going to get a single scoop of ice cream with a topping. The flavors of ice cream you’re considering are vanilla, chocolate, and rocky road; the toppings are fudge, whipped cream, and sprinkles.
3. The pea plants you’re breeding have two possible pod colors: green and yellow. These colors are decided by a particular gene, which comes in two types: “G” for green, and “g” for yellow (In genetics, capital letters usually denote *dominant* genes, while lower-case letters denote *recessive* genes). Each plant has two genes. If you breed a Gg pea plant with a gg plant, the offspring plant will get one gene from each parent. What are the possible outcomes?

Answer

1. **Step 1:** Make the outline of a table, with the results of the 4-sided roll on one side and the results of the 6-sided roll on the other. In practice, it doesn’t matter which you choose; for this example, we’ll put the 4-sided results on top (labeling the columns of the chart) and the 6-sided results on the side (labeling the rows of the chart) as shown in the following table:

		4-Sided Roll			
		1	2	3	4
6-Sided Roll	**1**				
	2				
	3				
	4				
	5				
	6				

Step 2: Fill in the results in each cell of the table below. Use the notation of an ordered pair, with the row label first.

		4-Sided Roll			
		1	2	3	4

		4-Sided Roll			
		1	2	3	4
6-Sided Roll	**1**	(1,1)	(1,2)	(1,3)	(1,4)
	2	(2,1)	(2,2)	(2,3)	(2,4)
	3	(3,1)	(3,2)	(3,3)	(3,4)
	4	(4,1)	(4,2)	(4,3)	(4,4)
	5	(5,1)	(5,2)	(5,3)	(5,4)
	6	(6,1)	(6,2)	(6,3)	(6,4)

So, for example, (3,2) represents the outcome where the 6-sided roll results in a 3, and the 4-sided roll gives us a 2. Thus, the sample space of the experiment is {(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,3), (4,4), (5,1), (5,2), (5,3), (5,4), (6,1), (6,2), (6,3), (6,4)}.

Step 1: Let's put the flavors on the rows and the toppings on the columns of the following table:

		Toppings		
		fudge	whipped cream	sprinkles
Flavors	**vanilla**			
	chocolate			
	rocky road			

Step 2: We can fill in the cells of the table below with the resulting combinations.

		Toppings		
		fudge	whipped cream	sprinkles
Flavors	**vanilla**	vanilla with fudge	vanilla with whipped cream	vanilla with sprinkles
	chocolate	chocolate with fudge	chocolate with whipped cream	chocolate with sprinkles
	rocky road	rocky road with fudge	rocky road with whipped cream	rocky road with sprinkles

So, the sample space is {vanilla with fudge, vanilla with whipped cream, vanilla with sprinkles, chocolate with fudge, chocolate with whipped cream, chocolate with sprinkles, rocky road with fudge, rocky road with whipped cream, rocky road with sprinkles}.

Step 1: We'll put the parents' genes (P1 and P2) as labels on the rows and columns of the following table:

		P2	
		g	g
P1	**G**		
	g		

Step 2: We'll fill in the offspring's gene composition, listing parent 1's gene first in the table below.

		P2	
		g	g
P1	**G**	Gg	Gg
	g	gg	gg

Thus, the sample space is {Gg, Gg, gg, gg}. (Diagrams like this, which allow us to identify the genotypes of offspring, are called *Punnett squares* in honor of Reginald Punnett (1875–1967), who first used them in the context of genetics.)

Your Turn 7.4.3

Use a table to identify the sample space of an experiment in which you flip a coin and roll a 6-sided die.

Using Tree Diagrams to Identify Sample Spaces

In experiments where there are more than two stages, or where the stages are dependent, a tree diagram is a helpful tool for systematically identifying the sample space. Tree diagrams are built by first drawing a single point (or *node*), then from that node we draw one branch (a short line segment) for each outcome of the first stage. Each branch gets its own node at the other end (which we typically label with the corresponding outcome for that branch); from each of these, we draw another branch for each outcome of the second stage, assuming that the outcome of the first stage matches the branch we were on. If there are other stages, we can continue from there by continuing to add branches and nodes. This sounds really complicated, but it's easier to understand through an example.

Example 7.4.4: Using a Tree Diagram to Identify a Sample Space

Use a tree diagram to find the sample spaces of each of the following experiments:

1. You flip a coin 3 times, noting the outcome of each flip in order.
2. You flip a coin. If the result is heads, you roll a 4-sided die. If it's tails, you roll a 6-sided die.
3. You are planning to go on a hike with a group of friends. There are 3 trails to consider: Abel Trail, Borel Trail, and Condorcet Trail. One of your friends, Jess, requires a wheelchair; if she joins you, the group couldn't handle the rocky Condorcet Trail.

Answer

1. **Step 1:** Let's start by placing our first node (Figure ✓ 7.4.2).

Figure ✓ 7.4.2

Step 2: We'll add two branches, one for each outcome of the first coin flip, and label them (Figure ✓ 7.4.3).

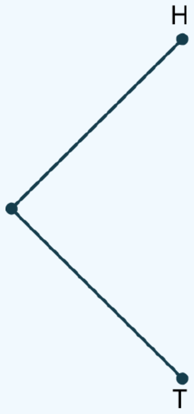

Figure ✓ 7.4.3

Step 3: We're ready for stage two of the experiment: another coin flip. At each node, we add in branches that represent those outcomes (Figure ✓ 7.4.4).

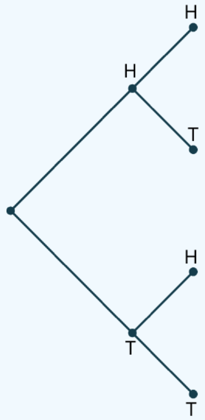

Figure ✓ 7.4.4

Finally, we can add another set of branches for the outcomes of the third stage (Figure ✓ 7.4.5).

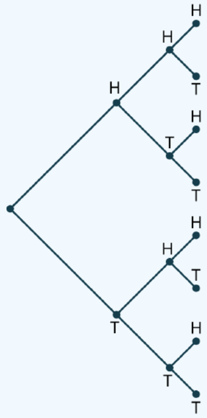

Figure ✓ 7.4.5

(These final nodes are called *leaves.)*

Step 4: We can write down the outcomes in the sample space by tracing the path out to each leaf, writing down the outcome at each node we pass through. For example, this leaf (Figure ✓ 7.4.6):

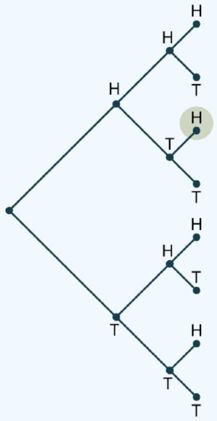

Figure ✓ 7.4.6

is reached via this path (Figure 7.16):

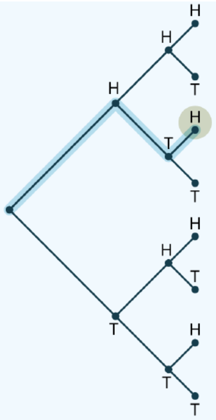
H
H
H
T
T
H
T
H
H
T
T
H
T
T

Figure ✓ 7.4.7

Step 5: We'll label that leaf as "HTH" (Figure ✓ 7.4.8), since the path passes through nodes labeled H, T, and H on its way out to our leaf.

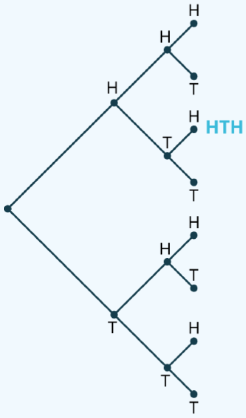
H
H
H
T
T
H
HTH
T
H
H
T
T
H
T
T

Figure ✓ 7.4.8

Step 6: We can label the remaining leaves using the same method (Figure ✓ 7.4.9).

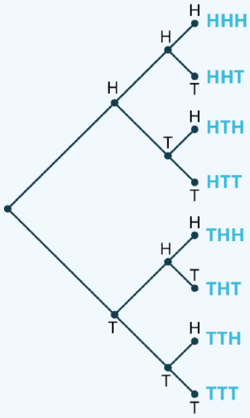
HHH
HHT
HTH
HTT
THH
THT
TTH
TTT

Figure ✓ 7.4.9

The sample space is the labels on the leaves: {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}.

Step 1: We'll start with our initial node (Figure ✓ 7.4.10}).

Figure ✓ 7.4.10

Step 2: We'll add in branches for the outcomes of the first stage (Figure ✓ 7.4.11), which is the coin flip.

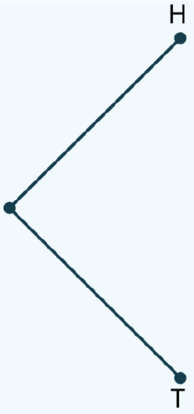

Figure ✓ 7.4.11

Step 3: The second stage of the experiment depends on the outcome of the first stage.

a. If the outcome of the first stage was H, then we roll a 4-sided die. So, *only* on the node for H, we'll add in the outcomes of a 4-sided die roll (Figure

✓ 7.4.12).

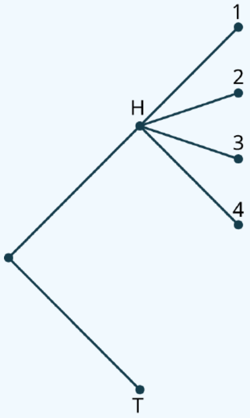

Figure ✓ 7.4.12

If the outcome of the first stage was T, then we roll a 6-sided die. So, we'll add those branches to the node for T (Figure ✓ 7.4.13).

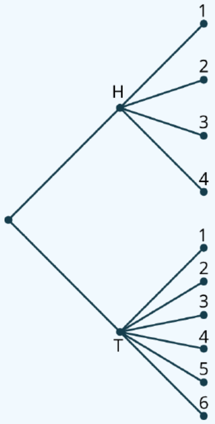

Figure ✓ 7.4.13

Step 4: We can label the leaves to get the sample space (Figure 7.4.14).

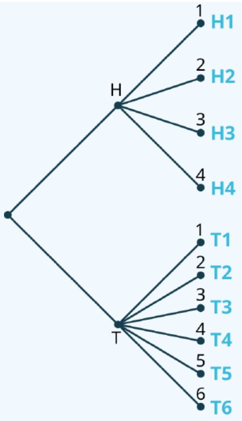
H
1
H1
2
H2
3
H3
4
H4
T
1
T1
2
T2
3
T3
4
T4
5
T5
6
T6

Figure ✓ 7.4.14

The sample space is: {H1, H2, H3, H4, T1, T2, T3, T4, T5, T6}.

c. Step 1: Let's label the trails A, B, and C for ease of labeling. Even though the trails are listed first in the exercise, we can't use the trail choice as our first stage: the trails available to us depend on whether Jess is able to join the trip. So, the first stage is whether Jess joins us (J) or not (N) (Figure ✓ 7.4.15).

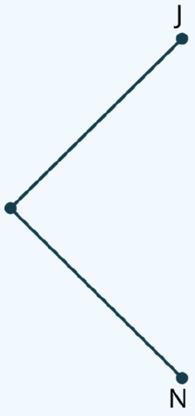

Figure ✓ 7.4.15

Step 2: We list the appropriate trails on each branch (Figure ✓ 7.4.16).

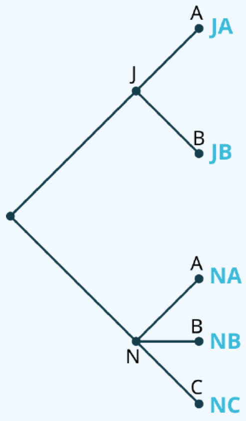
J
A
JA
B
JB
N
A
NA
B
NB
C
NC

Figure ✓ 7.4.16

So, the sample space is {JA, JB, NA, NB, NC}

Your Turn 7.4.4

You have a modified deck of cards containing only J♡, Q♡, and K♡. You draw 2 cards without replacing them (where order matters). Use a tree diagram to identify the sample space.

Check Your Understanding

1. You flip a coin 6 times and note the number of heads. What is the sample space of this experiment?
2. You are ordering a combo meal at a restaurant. The meal comes with either 8 or 12 chicken nuggets, and your choice of crinkle fries, curly fries, or onion rings. Create a table to help you identify the sample space containing your combo meal possibilities.
3. You need one more class to fill out your schedule for next semester. You want to take either History 101 (H), English 220 (E), or Sociology 112 (S). There are two professors teaching the history class: Anderson (A) and Burr (B); one professor teaching the English class: Carter (C); and three people teaching sociology: Johnson (J), Kirk (K), and Lambert (L). Create a tree diagram that helps you identify all your options.
4. Identify the sample space from Exercise 18.
5. Identify the sample space from Exercise 19.

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7.4.0: Exercises

In the following exercises, you are rolling a special 6-sided die that has both a colored letter and a colored number on each face. The faces are labeled with: a red 1 and a blue A, a red 1 and a green A, an orange 1 and a green B, an orange 2 and a red C, a purple 3 and a brown D, an orange 4 and a blue E. You will roll the die once and take note of something about the face that is showing. Identify the sample space:

? Exercise $7.4.0.1$

You note the number.

? Exercise $7.4.0.2$

You note the letter.

? Exercise $7.4.0.3$

You note the color of the number.

? Exercise $7.4.0.4$

You note the color of the letter.

? Exercise $7.4.0.5$

You note the number and letter.

? Exercise $7.4.0.6$

You note the number and its color.

? Exercise $7.4.0.7$

You note the letter and its color.

In Example 7.13, we used a table to identify the possible genetic result when two pea plants are bred; offspring plants get one of the two genes from each parent. Use tables to identify the sample space of offspring for the parents listed in the following exercises:

? Exercise $7.4.0.8$

GG and gg

? Exercise $7.4.0.9$

GG and Gg

? Exercise $7.4.0.10$

Gg and Gg

You’re visiting a pasta bar, where you have your choice of pasta types (cavatappi, ziti, or penne) and sauce (marinara, alfredo, or pesto).

Exercise 7.4.0.11

Make a table that shows the sample space for your choices.

Exercise 7.4.0.12

Write out the sample space for the pasta bar.

You have a new art print that you'd like to get framed. You have 3 good choices for the material of the frame: oak, maple, and cherry. You'll also choose a framing mat, for which there are 4 possible colors that will work with your print: plum, lilac, periwinkle, and violet.

Exercise 7.4.0.13

Make a table that shows the sample space for your choices.

Exercise 7.4.0.14

Write out the sample space for the framing choices.

You are shopping for a new laptop. The brand you're considering offers laptops with screen sizes 11", 13", 14", 15", and 17". It also offers five memory choices: 4GB, 8GB, 12GB, 16GB, and 32GB.

Exercise 7.4.0.15

Make a table that shows the sample space for your choices.

Exercise 7.4.0.16

Write out the sample space for the laptop choices.

For the following exercises, decide whether the described two-stage experiments have independent or dependent stages.

Exercise 7.4.0.17

Siobhan and Tristan are trying to decide where they will have dinner. Siobhan wants to go to Antoine's, Burger Hut, or the Chowder Palace. Tristan prefers Burger Hut, Chowder Palace, or Duck Duck Taco. They will flip a coin to decide who gets to choose, then that person selects a restaurant from their list.

Exercise 7.4.0.18

On the TV game show *The Price Is Right*, contestants play games to try to win prizes. One of these games is called "Let 'Em Roll." In this game, players roll five 6-sided dice. These dice each have 3 faces labeled with a car; the other 3 faces are labeled with prize money amounts (one each of $500, $1,000, and $1,500). Players get to roll all 5 dice, and then have the opportunity to win additional rolls. If all 5 dice show a car, the contestant wins a new car. If not, the contestant can use any additional rolls to reroll the dice with prize money showing to try to win the car (or, they can take the total amount of money showing on the dice and end the game). Josh is playing this game, and he has 2 rolls total (these are the two stages of the experiment; the number of cars showing on the dice will be the reported outcome of each stage).

Exercise 7.4.0.19

In the game Rock, Paper, Scissors, each of 2 players secretly chooses 1 of the 3 title objects. The players reveal their choices simultaneously using hand signals. If the players choose the same object, the game is a tie; if they choose different objects, the winner is determined by the following rules: rock beats scissors, scissors beats paper, paper beats rock. Jim and Eva are gearing up to play the game. The two stages of the experiment are Jim's and Eva's choices.

? Exercise 7.4.0.20

Paul is at the racetrack and is about to place a daily double bet, where he will try to predict the winner of 2 consecutive races. The stages of the experiment are the two races.

? Exercise 7.4.0.21

Mishka is looking for a song to play on a jukebox. The machine requires the user to choose an artist first, then choose a song from among that artist’s songs. The stages of the experiment are choosing the artist and choosing the song.

John is playing a game that involves flipping a coin and rolling dice. The coin flip happens first. If the outcome is heads, the player rolls a 4-sided die. If the outcome is tails, the player rolls a 6-sided die.

? Exercise 7.4.0.22

Create a tree diagram to display the possible outcomes of this game.

? Exercise 7.4.0.23

Give the sample space for the game.

In the casino game roulette, a wheel with colored and numbered pockets is spun. At the same time, a marble is spun in the opposite direction in such a way that after a minute or so the marble drops into one of the pockets. Players try to guess the color (red, black, or green) or the number that will appear.

? Exercise 7.4.0.24

Draw a tree diagram that shows the possible color outcomes for 3 spins of the wheel.

? Exercise 7.4.0.25

Give the sample space for the three-stage experiment.

? Exercise 7.4.0.26

Siobhan and Tristan are trying to decide where they will have dinner. Siobhan wants to go to Antoine’s, Burger Hut, or the Chowder Palace. Tristan prefers Burger Hut, Chowder Palace, or Duck Duck Taco. They will flip a coin to decide who gets to choose, then that person selects a restaurant from their list. Use an appropriate method to identify the sample space of possible outcomes, which include the person who chooses and the restaurant.

? Exercise 7.4.0.27

On the TV game show *The Price Is Right*, contestants play games to try to win prizes. One of these games is called “Let ‘Em Roll.” In this game, players roll five 6-sided dice. These dice each have 3 faces labeled with a car; the other 3 faces are labeled with prize money amounts (one each of \$500, \$1000, and \$1500). Players get to roll all 5 dice, then have the opportunity to win additional rolls. If all 5 dice show a car, the contestant wins a new car. If not, the contestant can use any additional rolls to reroll the dice with prize money showing to try to win the car (or, they can take the total amount of money showing on the dice and end the game). Kathleen is playing this game, and she has 3 rolls to try to win the car. On her first roll, 3 of the dice showed cars. Use an appropriate method to find the sample space of the number of cars showing after each of the following rolls. Give those outcomes as ordered pairs: (number of cars after the second roll, number of cars after the third roll).

Exercise $7.4.0.28$

In the game Rock, Paper, Scissors, each of 2 players chooses 1 of the 3 title objects. The players reveal their choices simultaneously using hand signals. Jim and Eva are gearing up to play the game. List the sample space of all the possible outcomes of the first round of their game as ordered pairs of the form (Jim’s choice, Eva’s choice).

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7.5: Basic Concepts of Probability

Figure $7.5.1$: When you roll two dice, some outcomes (like rolling a sum of seven) are more likely than others (rolling a sum of twelve). (credit: “Dice Isn’t Just A Game; It's A Way of Life” by Leah Love/Flickr, CC BY 2.0)

Learning Objectives

1. Define probability including impossible and certain events.
2. Calculate basic theoretical probabilities.
3. Calculate basic empirical probabilities.
4. Distinguish among theoretical, empirical, and subjective probability.
5. Calculate the probability of the complement of an event.

It all comes down to this. The game of *Monopoly* that started hours ago is in the home stretch. Your sister has the dice, and if she rolls a 4, 5, or 7 she’ll land on one of your best spaces and the game will be over. How likely is it that the game will end on the next turn? Is it more likely than not? How can we measure that likelihood? This section addresses this question by introducing a way to measure uncertainty.

Introducing Probability

Uncertainty is, almost by definition, a nebulous concept. In order to put enough constraints on it that we can mathematically study it, we will focus on uncertainty strictly in the context of experiments. Recall that experiments are processes whose outcomes are unknown; the sample space for the experiment is the collection of all those possible outcomes. When we want to talk about the likelihood of particular outcomes, we sometimes group outcomes together; for example, in the *Monopoly* example at the beginning of this section, we were interested in the roll of 2 dice that might fall as a 4, 5, or 7. A grouping of outcomes that we’re interested in is called an event. In other words, an event is a subset of the sample space of an experiment; it often consists of the outcomes of interest to the experimenter.

Once we have defined the event that interests us, we can try to assess the likelihood of that event. We do that by assigning a number to each event (E) called the probability of that event ($P(E)$). The probability of an event is a number between 0 and 1 (inclusive). If the probability of an event is 0, then the event is impossible. On the other hand, an event with probability 1 is certain to occur. In general, the higher the probability of an event, the more likely it is that the event will occur.

✓ Example ✓ 7.5.1: Determining Certain and Impossible Events

Consider an experiment that consists of rolling a single standard 6-sided die (with faces numbered 1-6). Decide if these probabilities are equal to zero, equal to one, or somewhere in between.

1. $P(\text{roll a } 4)$
2. $P(\text{roll a } 7)$
3. $P(\text{roll a positive number})$
4. $P\left(\text{rolla } \frac{1}{3}\right)$
5. $P(\text{roll an even number})$
6. $P(\text{roll a single-digit number})$

Answer

Let's start by identifying the sample space. For one roll of this die, the possible outcomes are $\{1, 2, 3, 4, 5{,}6\}$. We can use that to assess these probabilities:

1. We see that 4 is in the sample space, so it's possible that it will be the outcome. It's not certain to be the outcome, though. So, $0 < P(\text{roll a } 4) < 1$.
2. Notice that 7 is not in the sample space. So, $P(\text{roll a } 7) = 0$.
3. Every outcome in the sample space is a positive number, so this event is certain. Thus, $P(\text{roll a positive number}) = 1$.
4. Since $\frac{1}{3}$ is not in the sample space, $P\left(\text{rolla } \frac{1}{3}\right) = 0$.
5. Some outcomes in the sample space are even numbers (2, 4, and 6), but the others aren't. So, $0 < P(\text{roll an even number}) < 1$.
6. Every outcome in the sample space is a single-digit number, so $P(\text{roll a single-digit number}) = 1$.

Your Turn 7.5.1

Jorge is about to conduct an experiment that consists of flipping a coin 4 times and recording the number of heads (H). Decide if these probabilities are equal to zero, equal to one, or somewhere in between.

$P(\mathrm{H} < 5)$

$P(\mathrm{H} < 4)$

$P(\mathrm{H} \geq 5)$

Three Ways to Assign Probabilities

The probabilities of events that are certain or impossible are easy to assign; they're just 1 or 0, respectively. What do we do about those in-between cases, for events that might or might not occur? There are three methods to assign probabilities that we can choose from. We'll discuss them here, in order of reliability.

Method 1: Theoretical Probability

The theoretical method gives the most reliable results, but it cannot always be used. If the sample space of an experiment consists of equally likely outcomes, then the theoretical probability of an event is defined to be the ratio of the number of outcomes in the event to the number of outcomes in the sample space.

FORMULA

For an experiment whose sample space S consists of equally likely outcomes, the **theoretical probability** of the event E is the ratio

$$P(E) = \frac{n(E)}{n(S)},$$

where $n(E)$ and $n(S)$ denote the number of outcomes in the event and in the sample space, respectively.

✓ Example 7.5.2: Computing Theoretical Probabilities

Recall that a standard deck of cards consists of 52 unique cards which are labeled with a rank (the whole numbers from 2 to 10, plus J, Q, K, and A) and a suit (♣, ♢, ♡, or ♠). A standard deck is thoroughly shuffled, and you draw one card at random (so every card has an equal chance of being drawn). Find the theoretical probability of each of these events:

1. The card is $10\spadesuit$.
2. The card is a ♡.
3. The card is a king (K).

Answer

There are 52 cards in the deck, so the sample space for each of these experiments has 52 elements. That will be the denominator for each of our probabilities.

1. There is only one $10\spadesuit$ in the deck, so this event only has one outcome in it. Thus, $P(10\spadesuit) =$.
2. There are 13 ♡s in the deck, so $P(\heartsuit) = \frac{13}{52} =$.
3. There are 4 cards of each rank in the deck, so $P(\mathrm{K}) = \frac{4}{52} =$.

Your Turn 7.5.2

You are about to roll a fair (meaning that each face has an equal chance of landing up) 6-sided die, whose faces are labeled with the numbers 1 through 6. Find the theoretical probabilities of each outcome.

You roll a 4.

You roll a number greater than 2.

You roll an odd number.

Checkpoint

It is critical that you make sure that every outcome in a sample space is equally likely before you compute theoretical probabilities!

✓ Example 7.5.3: Using Tables to Find Theoretical Probabilities

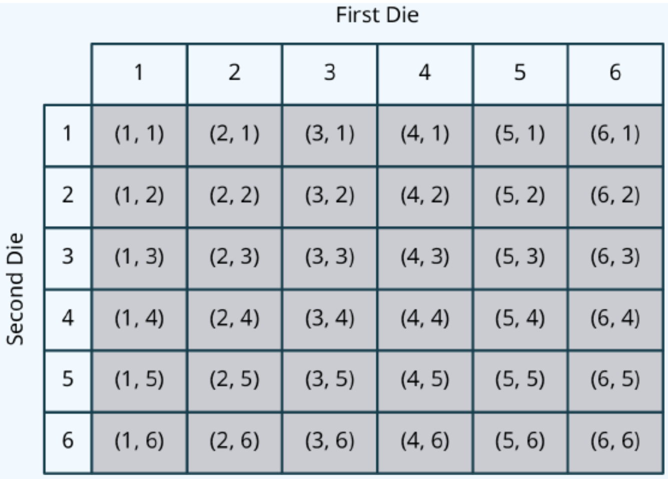
In the Basic Concepts of Probability, we were considering a *Monopoly* game where, if your sister rolled a sum of 4, 5, or 7 with 2 standard dice, you would win the game. What is the probability of this event? Use tables to determine your answer.

Now, each of the 36 ordered pairs in the table represent an equally likely outcome.

Answer

We should think of this experiment as occurring in two stages: (1) one die roll, then (2) another die roll. Even though these two stages will usually occur simultaneously in practice, since they're independent, it's okay to treat them separately.

Step 1: Since we have two independent stages, let's create a table (Figure ✓ 7.5.2), which is probably the most efficient method for determining the sample space.



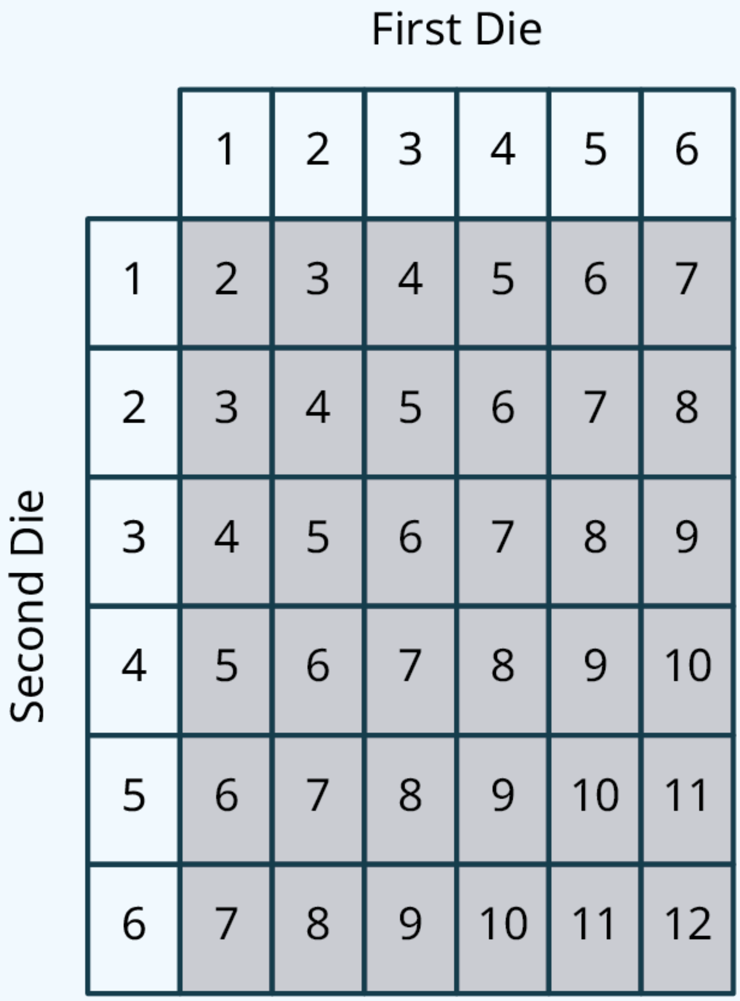
First Die

Second Die

	1	2	3	4	5	6
1	(1, 1)	(2, 1)	(3, 1)	(4, 1)	(5, 1)	(6, 1)
2	(1, 2)	(2, 2)	(3, 2)	(4, 2)	(5, 2)	(6, 2)
3	(1, 3)	(2, 3)	(3, 3)	(4, 3)	(5, 3)	(6, 3)
4	(1, 4)	(2, 4)	(3, 4)	(4, 4)	(5, 4)	(6, 4)
5	(1, 5)	(2, 5)	(3, 5)	(4, 5)	(5, 5)	(6, 5)
6	(1, 6)	(2, 6)	(3, 6)	(4, 6)	(5, 6)	(6, 6)

Figure ✓ 7.5.2

Step 2: To make our analysis easier, let’s replace each ordered pair with the sum (Figure ✓ 7.5.3).



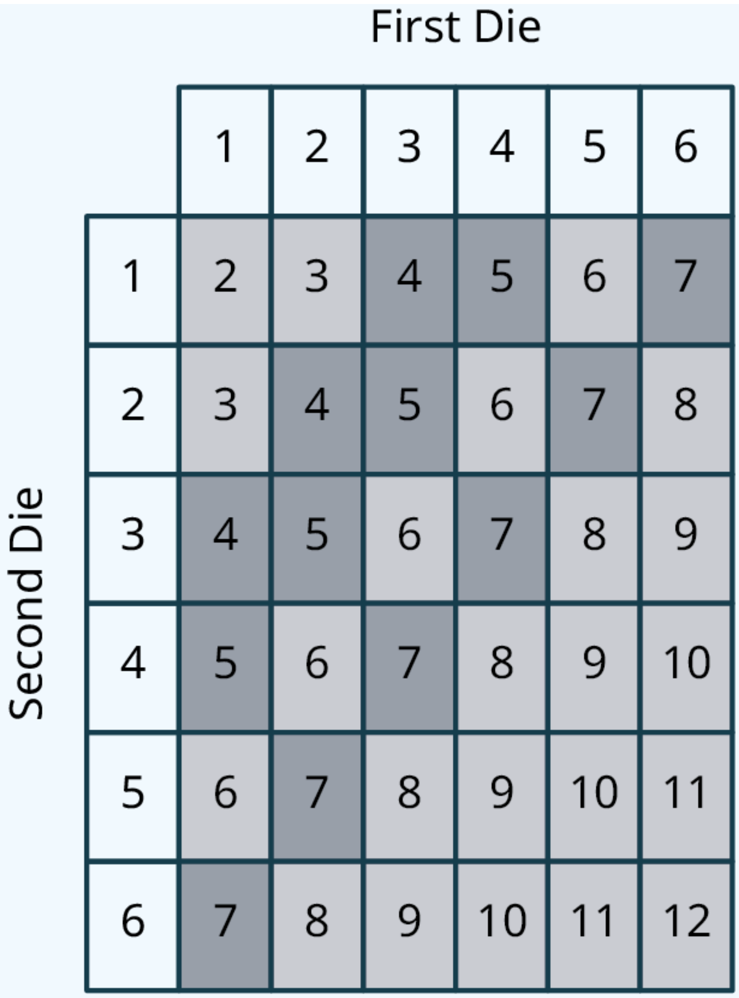
First Die

Second Die

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Figure ✓ 7.5.3

Step 3: Since the event we’re interested in is the one consisting of rolls of 4, 5, or 7. Let’s shade those in (Figure ✓ 7.5.4).



First Die

Second Die

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Figure 7.✔ $7.5.4$

Our event contains 13 outcomes, so the probability that your sister rolls a losing number is $\frac{13}{36}$.

Your Turn $7.5.3$

If you roll a pair of 4-sided dice with faces labeled with the numbers 1 through 4, what is the probability of rolling a sum of 6 or 7?

✔ Example $7.5.4$: Using Tree Diagrams to Compute Theoretical Probability

If you flip a fair coin 3 times, what is the probability of each event? Use a tree diagram to determine your answer

1. You flip exactly 2 heads.
2. You flip 2 consecutive heads at some point in the 3 flips.
3. All 3 flips show the same result.

Answer

Let’s build a tree to identify the sample space (Figure ✔ $7.5.5$).

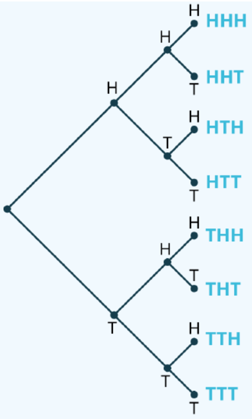


Figure ✓ 7.5.5

The sample space is {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}, which has 8 elements.

1. Flipping exactly 2 heads occurs three times (HHT, HTH, THH), so the probability is $\frac{3}{8}$.
2. Flipping 2 consecutive heads at some point in the experiment happens 3 times: HHH, HHT, THH. So, the probability is $\frac{3}{8}$.
3. There are 2 outcomes that all show the same result: HHH and TTT. So, the probability is $\frac{2}{8} = \frac{1}{4}$.

Your Turn 7.5.4

You have a modified deck of cards containing only 3♡, 4♡, and 5♡. You draw 2 two cards without replacing them (where order matters). What is the probability of each event?

The first card drawn is 3♡.

The first card drawn has a lower number than the second card.

One of the cards drawn is 4♡.

The first card drawn has a lower number than the second card.

One of the cards drawn is 4♡.

People in Mathematics: Gerolamo Cardano

The first known text that provided a systematic approach to probabilities was written in 1564 by Gerolamo Cardano (1501–1576). Cardano was a physician whose illegitimate birth closed many doors that would have otherwise been open to someone with a medical degree in 16th-century Italy. As a result, Cardano often turned to gambling to help ends meet. He was a remarkable mathematician, and he used his knowledge to gain an edge when playing at cards or dice. His 1564 work, titled *Liber de ludo aleae* (which translates as *Book on Games of Chance*), summarized everything he knew about probability. Of course, if that book fell into the hands of those he played against, his advantage would disappear. That’s why he never allowed it to be published in his lifetime (it was eventually published in 1663). Cardano made other contributions to mathematics; he was the first person to publish the third degree analogue of the Quadratic Formula (though he didn’t discover it himself), and he popularized the use of negative numbers.

Method 2: Empirical Probability

Theoretical probabilities are precise, but they can’t be found in every situation. If the outcomes in the sample space are not equally likely, then we’re out of luck. Suppose you’re watching a baseball game, and your favorite player is about to step up to the plate. What is the probability that he will get a hit?

In this case, the sample space is {hit, not a hit}. That doesn’t mean that the probability of a hit is $\frac{1}{2}$, since those outcomes aren’t equally likely. The theoretical method simply can’t be used in this situation. Instead, we might look at the player’s statistics up to this point in the season, and see that he has 122 hits in 531 opportunities. So, we might think that the probability of a hit in the next plate appearance would be about $\frac{122}{531} \approx 0.23$. When we use the outcomes of previous replications of an experiment to assign a probability to the next replication, we're defining an empirical probability. Empirical probability is assigned using the outcomes of previous replications of an experiment by finding the ratio of the number of times in the previous replications the event occurred to the total number of previous replications.

Empirical probabilities aren't exact, but when the number of previous replications is large, we expect them to be close. Also, if the previous runs of the experiment are not conducted under the exact set of circumstances as the one we're interested in, the empirical probability is less reliable. For instance, in the case of our favorite baseball player, we might try to get a better estimate of the probability of a hit by looking only at his history against left- or right-handed pitchers (depending on the handedness of the pitcher he's about to face).

Who Knew?: Probability and Statistics

One of the broad uses of statistics is called statistical inference, where statisticians use collected data to make a guess (or inference) about the population the data were collected from. Nearly every tool that statisticians use for inference is based on probability. Not only is the method we just described for finding empirical probabilities one type of statistical inference, but some more advanced techniques in the field will give us an idea of how close that empirical probability might be to the actual probability!

Example $7.5.5$: Finding Empirical Probabilities

Assign an empirical probability to the following events:

1. Jose is on the basketball court practicing his shots from the free throw line. He made 47 out of his last 80 attempts. What is the probability he makes his next shot?
2. Amy is about to begin her morning commute. Over her last 60 commutes, she arrived at work 12 times in under half an hour. What is the probability that she arrives at work in 30 minutes or less?
3. Felix is playing *Yahtzee* with his sister. Felix won 14 of the last 20 games he played against her. How likely is he to win this game?

Answer

1. Since Jose made 47 out of his last 80 attempts, assign this event an empirical probability of $\frac{47}{80} \approx 59\%$.
2. Amy completed the commute in under 30 minutes in 12 of the last 60 commutes, so we can estimate her probability of making it in under 30 minutes this time at $\frac{12}{60} = 20\%$.
3. Since Felix has won 14 of the last 20 games, assign a probability for a win this time of $\frac{14}{20} = 70\%$.

Your Turn $7.5.5$

Jessie is in charge of quality control at a factory manufacturing SUVs. Today, she's checking the placement of the taillight housing. Of the last thousand units off the line, 13 had faulty placement. What empirical probability might Jesse assign to the next vehicle coming off the line having bad placement?

WORK IT OUT: Buffon's Needle

A famous early question about probability (posed by Georges-Louis Leclerc, Comte de Buffon in the 18th century) had to do with the probability that a needle dropped on a floor finished with wooden slats would lay across one of the seams. If the distance between the slats is exactly the same length as the needle, then it can be shown using calculus that the probability that the needle crosses a seam is $\frac{2}{\pi}$. Using toothpicks or matchsticks (or other uniformly long and narrow objects), assign an empirical probability to this experiment by drawing parallel lines on a large sheet of paper where the distance between the lines is equal to the length of your dropping object, then repeatedly dropping the objects and noting whether the object touches one of the lines. Once you have your empirical probability, take its reciprocal and multiply by 2. Is the result close to π?

Method 3: Subjective Probability

In cases where theoretical probability can't be used and we don't have prior experience to inform an empirical probability, we're left with one option: using our instincts to guess at a subjective probability. A subjective probability is an assignment of a probability to an event using only one’s instincts.

Subjective probabilities are used in cases where an experiment can only be run once, or it hasn’t been run before. Because subjective probabilities may vary widely from person to person and they’re not based on any mathematical theory, we won’t give any examples. However, it’s important that we be able to identify a subjective probability when we see it; they will in general be far less accurate than empirical or theoretical probabilities.

✓ Example $7.5.6$: Distinguishing among Theoretical, Empirical, and Subjective Probabilities

Classify each of the following probabilities as theoretical, empirical, or subjective.

1. An eccentric billionaire is testing a brand new rocket system. He says there is a 15% chance of failure.
2. With 4 seconds to go in a close basketball playoff game, the home team need 3 points to tie up the game and send it to overtime. A TV commentator says that team captain should take the final 3-point shot, because he has a 38% chance of making it (greater than every other player on the team).
3. Felix is losing his *Yahtzee* game against his sister. He has one more chance to roll 2 dice; he’ll win the game if they both come up 4. The probability of this is about 2.8%.

Answer

1. This experiment has never been run before, so the given probability is subjective.
2. Presumably, the commentator has access to each player’s performance statistics over the entire season. So, the given probability is likely empirical.
3. Rolling 2 dice results in a sample space with equally likely outcomes. This probability is theoretical. (We’ll learn how to calculate that probability later in this chapter.)

Your Turn $7.5.7$

Classify each of the following probabilities as theoretical, empirical, or subjective.

You have entered a raffle with 500 entrants. Your probability of winning is 0.2%.

Your little brother takes the bus to school each morning. On the first day of school, you believe that the probability that the bus arrives between 7:15 AM and 7:30 AM is about 80%.

Your little brother takes the bus to school each morning. On the last day of school, you believe that the probability that the bus arrives between 7:15 AM and 7:30 AM is about 73%.

Who Knew?: Benford’s Law

In 1938, Frank Benford published a paper (“The law of anomalous numbers,” in *Proceedings of the American Philosophical Society*) with a surprising result about probabilities. If you have a list of numbers that spans at least a couple of orders of magnitude (meaning that if you divide the largest by the smallest, the result is at least 100), then the digits 1–9 are not equally likely to appear as the first digit of those numbers, as you might expect. Benford arrived at this conclusion using empirical probabilities; he found that 1 was about 6 times as likely to be the initial digit as 9 was!

New Probabilities from Old: Complements

One of the goals of the rest of this chapter is learning how to break down complicated probability calculations into easier probability calculations. We’ll look at the first of the tools we can use to accomplish this goal in this section; the rest will come later.

Given an event E, the **complement** of E (denoted E') is the collection of all of the outcomes that are *not* in E. (This is language that is taken from set theory, which you can learn more about elsewhere in this text.) Since every outcome in the sample space either is or is not in E, it follows that $n(E)+n(E')=n(S)$. So, if the outcomes in S are equally likely, we can compute theoretical probabilities $P(E)=\frac{n(E)}{n(S)}$ and $P(E')=\frac{n(E')}{n(S)}$. Then, adding these last two equations, we get

$$\begin{aligned} P(E)+P(E') &= \frac{n(E)}{n(S)}+\frac{n(E')}{n(S)} \\ &= \frac{n(E)+n(E')}{n(S)} \\ &= \frac{n(S)}{n(S)} \\ &= 1 \end{aligned}$$

Thus, if we subtract $P(E')$ from both sides, we can conclude that $P(E)=1-P(E')$. Though we performed this calculation under the assumption that the outcomes in S are all equally likely, the last equation is true in every situation.

FORMULA

$P(E)=1-P(E')$

How is this helpful? Sometimes it is easier to compute the probability that an event *won't* happen than it is to compute the probability that it *will*. To apply this principle, it's helpful to review some tricks for dealing with inequalities. If an event is defined in terms of an inequality, the complement will be defined in terms of the opposite inequality: Both the direction and the inclusivity will be reversed, as shown in the table below.

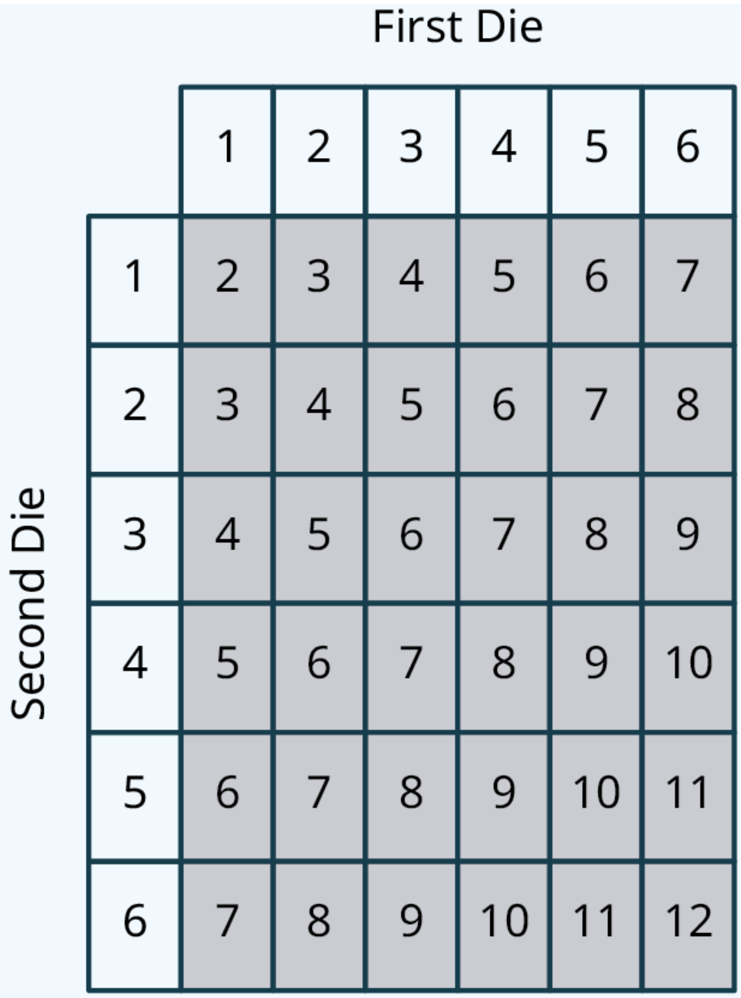
If E is defined with:	then E' is defined with:
$<$	$\geq$
$\leq$	$>$
$>$	$\leq$
$\geq$	$<$

✓ Example 7.5.8: Using the Formula for Complements to Compute Probabilities

1. If you roll a standard 6-sided die, what is the probability that the result will be a number greater than one?
2. If you roll two standard 6-sided dice, what is the probability that the sum will be 10 or less?
3. If you flip a fair coin 3 times, what is the probability that at least one flip will come up tails?

Answer

1. Here, the sample space is $\{1, 2, 3, 4, 5, 6\}$. It's easy enough to see that the probability in question is $\frac{5}{6}$, because there are 5 outcomes that fall into the event "roll a number greater than 1." Let's also apply our new formula to find that probability. Since E is defined using the inequality $\text{roll} >$, then E' is defined using $\text{roll} \leq$. Since there's only one outcome (1) in E', we have $P(E')=$. Thus, $P(E)=1-P(E')=$.
2. In Example 7.18, we found the following table of equally likely outcomes for rolling 2 dice (Figure ✓ 7.5.6):



	First Die					
Second Die	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Figure ✓ 7.5.6

Here, the event E is defined by the inequality $\text{sum} \leq$. Thus, E' is defined by $\text{sum} >$. There are three outcomes in E': two 11s and one 12. Thus, $P(E) = 1 - P(E') = 1 - \frac{3}{36} =$.

- In Example 7.15, we found the sample space for this experiment consisted of these equally likely outcomes: {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}. Our event E is defined by $\text{T} \geq 1$, so E' is defined by $\text{T} < 1$. The only outcome in E' is the first one on the list, where zero tails are flipped. So, $P(E) = 1 - P(E') = 1 - \frac{1}{8} =$.

Your Turn $7.5.8$

If you roll a pair of 4-sided dice with faces labeled 1 through 4, what is the probability that the sum of the resulting numbers will be greater than 3? Hint: You found this sample space in an earlier Your Turn.

Check Your Understanding

1. You have two coins: a nickel and a quarter. You flip them both. Find the probabilities of these events:
2. Both come up heads.
3. The quarter comes up heads.
4. You get one heads and one tails.
5. You get three tails.
6. Decide whether the given probabilities were most likely derived theoretically, empirically, or subjectively.
7. A poker player has a 16% chance of making a hand called a flush on the next card.
8. Your friend Jacob tells you that there’s a 20% chance he’ll get married in the next 5 years.

9. Ashley has a coin that they think might not be fair, so they flip it 100 times and note that the result was heads 58 times. So, Ashley says the probability of flipping heads on that coin is about 58%.
10. If you flip a fair coin 50 times, the probability of getting 20 or fewer heads is about 10.1% (a fact we'll learn how to verify later).
11. If E is the event "number of heads is 20 or fewer", describe the event E' using an inequality.
12. Find $P(E')$.

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7.5.0: Exercises

For the following exercises, we are considering two special 6-sided dice. Each face is labeled with a number and a letter: the first die has faces 1A, 1B, 2A, 2C, 4A, 4E; the second has faces 1A, 1A, 2A, 2B, 3A, 3C. Assume that each face has an equal probability of landing face up.

? Exercise 7.5.0.1

Use a table to identify the sample space of the experiment in which we roll both dice and note the sum of the two numbers that are showing.

? Exercise 7.5.0.2

What is the probability that we roll a sum less than 8?

? Exercise 7.5.0.3

What is the probability that we roll a sum larger than 8?

? Exercise 7.5.0.4

What is the probability that we roll a sum less than or equal to 2?

? Exercise 7.5.0.5

What is the probability that we roll a sum greater than 2?

? Exercise 7.5.0.6

What is the probability that we roll an even sum?

? Exercise 7.5.0.7

What is the probability that we roll an odd sum?

? Exercise 7.5.0.8

Use a table to identify the sample space of the experiment in which we roll both dice and note the two letters that are showing.

? Exercise 7.5.0.9

What is the probability that no As are showing?

? Exercise 7.5.0.10

What is the probability that at least one A is showing?

? Exercise 7.5.0.11

What is the probability that two As are showing?

Exercise 7.5.0.12

What is the probability that two of the same letter are showing?

Exercise 7.5.0.13

What is the probability that the letter on the first die comes alphabetically before the letter on the second die?

Exercise 7.5.0.14

What is the probability that two vowels are showing?

Exercise 7.5.0.15

What is the probability that two consonants are showing?

Exercise 7.5.0.16

What is the probability that one consonant and one vowel are showing?

In the following Exercises, decide whether the given probability was likely determined theoretically, empirically, or subjectively.

Exercise 7.5.0.17

Carolyn is breeding two pea plants, and notes that 8 of the 25 offspring plants have yellow peas. So, she concludes that there's a 32% chance that an offspring of these two plants will have yellow peas.

Exercise 7.5.0.18

At the beginning of the semester, Malik estimates there's a 30% chance that he'll earn As in all his classes.

Exercise 7.5.0.19

Abbie is deciding where she will attend college in the fall. Right now, she thinks there's an 80% chance she'll attend an in-state school.

Exercise 7.5.0.20

If you roll three standard 6-sided dice, the probability that the sum will be 8 is $\frac{2}{1}216$.

Exercise 7.5.0.21

According to the app he uses to play the game, Jason has won 18 of the last 100 games of solitaire he's played. So, the probability that he wins the next one is about 18%.

Exercise 7.5.0.22

Jim and Anne are both in a club with 10 members. If 3 people are chosen at random to form a committee, then the probability that both Jim and Anne are chosen is $\frac{1}{1}5$.

For the following exercises, use the following table of the top 15 players by number of plate appearances (PA) in the 2019 Major League Baseball season to assign empirical probabilities to the given events. A plate appearance is a batter's opportunity to try to get a hit. The other columns are runs scored (R), hits (H), doubles (2B), triples (3B), home runs (HR), walks (BB), and strike outs (SO).

Name	Team	PA	R	H	2B	3B	HR	BB	SO
Marcus Semien	OAK	747	123	187	43	7	33	87	102
Whit Merrifield	KCR	735	105	206	41	10	16	45	126
Ronald Acuna Jr.	ATL	715	127	175	22	2	41	76	188
Jonathan Villar	BAL	714	111	176	33	5	24	61	176
Mookie Betts	BOS	706	135	176	40	5	29	97	101
Rhys Hoskins	PHI	705	86	129	33	5	29	116	173
Jorge Polanco	MIN	704	107	186	40	7	22	60	116
Rafael Devers	BOS	702	129	201	54	4	32	48	119
Ozzie Albies	ATL	702	102	189	43	8	24	54	112
Eduardo Escobar	ARI	699	94	171	29	10	35	50	130
Xander Bogaerts	BOS	698	110	190	52	0	33	76	122
José Abreu	CHW	693	85	180	38	1	33	36	152
Pete Alonso	NYM	693	103	155	30	2	53	72	183
Freddie Freeman	ATL	692	113	176	34	2	38	87	127
Alex Bregman	HOU	690	122	164	37	2	41	119	83

Exercise $7.5.0.23$

Mookie Betts gets a home run in his next plate appearance.

Exercise $7.5.0.24$

Xander Bogaerts strikes out in his next plate appearance.

Exercise $7.5.0.25$

Jonathan Villar gets a hit in his next plate appearance.

? Exercise 7.5.0.26

Rhys Hoskins gets a walk in his next plate appearance.

? Exercise 7.5.0.27

José Abreu scores a run in his next plate appearance.

? Exercise 7.5.0.28

Eduardo Escobar hits a triple in his next plate appearance.

? Exercise 7.5.0.29

Whit Merrifield hits a double in his next plate appearance.

? Exercise 7.5.0.30

Ronald Acuna Jr. gets an extra-base hit (double, triple, or home run) in his next plate appearance.

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7.6: Probability with Permutations and Combinations

Figure 7.6.1: Bingo and many lottery games depend on selecting one or more numbers at random from a list; often this is done by drawing numbered balls from a bin. (credit: "Redundant Bingo Balls" by Greg Clarke/Flickr, CC BY 2.0)

Learning Objectives

1. Calculate probabilities with permutations.
2. Calculate probabilities with combinations.

In our earlier discussion of theoretical probabilities, the first step we took was to write out the sample space for the experiment in question. For many experiments, that method just isn't practical. For example, we might want to find the probability of drawing a particular 5-card poker hand. Since there are 52 cards in a deck and the order of cards doesn't matter, the sample space for this experiment has msubmsub$C_5 = 2{,}598,$ possible 5-card hands. Even if we had the patience and space to write them all out, sorting through the results to find the outcomes that fall in our event would be just as tedious.

Luckily, the formula for theoretical probabilities doesn't require us to know every outcome in the sample space; we just need to know how many outcomes there are. In this section, we'll apply the techniques we learned earlier in the chapter (The Multiplication Rule for Counting, permutations, and combinations) to compute probabilities.

Using Permutations to Compute Probabilities

Recall that we can use permutations to count how many ways there are to put a number of items from a list in order. If we're looking at an experiment whose sample space looks like an ordered list, then permutations can help us to find the right probabilities.

✓ Example ✓ 7.6.1: Using Permutations to Compute Probabilities

1. In horse racing, an exacta bet is one where the player tries to predict the top two finishers in particular race in order. If there are 9 horses in a race, and a player decided to make an exacta bet at random, what is the probability that they win?
2. You are in a club with 10 people, 3 of whom are close friends of yours. If the officers of this club are chosen at random, what is the probability that you are named president and one of your friends is named vice president?

3. A bag contains slips of paper with letters written on them as follows: A, A, B, B, B, C, C, D, D, D, D, E. If you draw 3 slips, what is the probability that the letters will spell out (in order) the word BAD?

Answer

1. Since order matters for this situation, we'll use permutations. How many different exacta bets can be made? Since there are 9 horses and we must select 2 in order, we know there are msubmsub$P_2 =$ possible outcomes. That's the size of our sample space, so it will go in the denominator of the probability. Since only one of those outcomes is a winner, the numerator of the probability is 1. So, the probability of randomly selecting the winning exacta bet is $\frac{1}{56}$.
2. There are 10 people in the club, and 2 will be chosen to be officers. Since the order matters, there are msubmsub$P_2 =$ different ways to select officers. Next, we must figure out how many outcomes are in our event. We'll use the Multiplication Rule for Counting to find that number. There is only 1 choice for president in our event, and there are 3 choices for vice president. So, there are $1 \times 3 = 3$ outcomes in the event. Thus, the probability that you will serve as president with one of your friends as vice president is $\frac{3}{90} = \frac{1}{30}$.
3. There are 12 slips of paper in the bag, and 3 will be drawn. So, there are msubmsub$P_3 =$ possible outcomes. Now, we'll compute the number of outcomes in our event. The first letter drawn must be a B, and there are 3 of those. Next must come an A (2 of those) and then a D (4 of those). Thus, there are $3 \times 2 \times 4 = 24$ outcomes in our event. So, the probability that the letters drawn spell out the word BAD is $\frac{24}{1320} = \frac{1}{55}$.

Your Turn 7.6.1

Another bag of letters contains C, C, C, C, D, D, I, I, I, T, T, T, Y, Y, Y, Y. What is the probability that 4 letters chosen at random will spell, in order, CITY?

Combinations to Computer Probabilities

If the sample space of our experiment is one in which order doesn't matter, then we can use combinations to find the number of outcomes in that sample space.

✓ Example 7.6.2: Using Combinations to Compute Probabilities

1. Palmetto Cash 5 is a game offered by the South Carolina Education Lottery. Players choose 5 numbers from the whole numbers between 1 and 38 (inclusive); the player wins the jackpot of $100,000 if the randomizer selects those numbers in any order. If you buy one ticket for this game, what is the probability that you win the top prize by choosing all 5 winning numbers?
2. There's a second prize in the Palmetto Cash 5 game that a player wins if 4 of the player's 5 numbers are among the 5 winning numbers. What's the probability of winning the second prize?
3. *Scrabble* is a word-building board game. Players make hands of 7 letters by selecting tiles with single letters printed on them blindly from a bag (2 tiles have nothing printed on them; these blanks can stand for any letter). Players use the letters in their hands to spell out words on the board. Initially, there are 100 tiles in the bag. Of those, 44 are (or could be) vowels (9 As, 12 Es, 9 Is, 8 Os, 4 Us, and 2 blanks; we'll treat Y as a consonant). What is the probability that your initial hand has no vowels?

Answer

1. There are 38 numbers to choose from, and the order of the 5 we pick doesn't matter. So, there are msubmsub$C_5 = 501,$ outcomes in the sample space. Only one outcome is in our winning event, so the probability of winning is $\frac{1}{501,492}$.
2. As in part 1 of this example,, there are 501,492 outcomes in the sample space. The tricky part here is figuring out how many outcomes are in our event. To qualify, the outcome must contain 4 of the 5 winning numbers, plus one losing number. There are msubmsub$C_4 =$ ways to choose the 4 winning numbers, and there are $38 - 5 = 33$ losing numbers. So, using the Multiplication Rule for Counting, there are $5 \times 33 = 165$ outcomes in our event. Thus, the probability of winning the second prize is $\frac{165}{501,492} = \frac{55}{167,164}$, which is about 0.00033.
3. The number of possible starting hands is msubmsub$C_7 = 16,007,560,$. There are $100 - 44 = 56$ consonants in the bag, so the number of all-consonant hands is msubmsub$C_7 = 231,917,$. Thus, the probability of drawing all consonants is $\frac{231,917,40}{16,007,560,800} = \frac{32,139}{2,425,388} \approx 0.0145$.

Your Turn 7.6.2

At a charity event with 58 people in attendance, 3 raffle winners are chosen. All receive the same prize, so order doesn't matter. You are attending with 4 of your friends. What is the probability that at least one of you or your friends wins a raffle prize? Hint: Find the probability that none of you wins, and use the formula for complements.

If you draw a hand of 5 cards from a standard deck, what is the probability that 2 cards are ♠ and 3 cards are ♡?

Check Your Understanding

1. For the following exercises, you are drawing *Scrabble* tiles without replacement from a bag containing the letters A, C, E, E, I, N, N, S, S, W.
2. What is the probability that you draw (in order) the letters W-I-N?
3. What is the probability that you draw (in order) the letters W-I-S-E?
4. What is the probability that you draw (in order) the letters S-E-E-N?
5. What is the probability that you draw (in any order) the letters W-I-N?
6. What is the probability that you draw (in any order) the letters W-I-S-E?
7. What is the probability that you draw (in any order) the letters S-E-E-N?

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7.6.0: Exercises

The following exercises deal with our version of the game blackjack. In this card game, players are dealt a hand of two cards from a standard deck. The dealer's cards are dealt with the second card face up, so the order matters; the other players' hands are dealt entirely face down, so order doesn't matter. The goal of the game is to build a hand whose point value is as close as possible to 21 without going over. The point values of each card are as follows: numbered cards are worth the number on the face (for example, 8♣ is worth 8 points); jacks, queens, and kings are each worth 10 points, and aces are worth either 1 or 11 points (the player can choose). Players whose hands are worth less than 21 points may ask to be dealt additional cards one at a time until they either go over 21 points or they choose to stop.

? Exercise 7.6.0.1

What is the probability that a player (not the dealer) is dealt an initial hand worth 21 points? This can only happen with an ace and a card worth 10 points (10, J, Q, or K).

? Exercise 7.6.0.2

What is the probability that the dealer is dealt an initial hand worth 21 points, with an ace showing?

? Exercise 7.6.0.3

What is the probability that a player is dealt 2 cards worth 10 points each?

? Exercise 7.6.0.4

What is the probability that a player is dealt an initial hand with an 8 and a 3?

? Exercise 7.6.0.5

What is the probability that a player is dealt an initial hand with two 8s?

? Exercise 7.6.0.6

What is the probability that a player is dealt 2 ♡?

? Exercise 7.6.0.7

In some versions of the game, a player wins automatically if they draw a hand of 5 cards that doesn't go over 21 points. One way this can happen is if they draw 5 cards, all of which are A, 2, 3, or 4. What is the probability of drawing 5 cards from that collection?

In horse racing, a trifecta bet is one where the player tries to predict the top three finishers in order. In the following exercises, find the probability of choosing a winning trifecta bet at random when the field contains the given number of horses.

? Exercise 7.6.0.8

6 horses

? Exercise 7.6.0.9

8 horses

? Exercise 7.6.0.10

10 horses

In the following exercises, you are about to draw *Scrabble* tiles from a bag without replacement; the bag contains the letters A, A, C, E, E, E, L, L, N, O, R, S, S, S, T, X.

? Exercise 7.6.0.11

What is the probability of drawing the letters E-A-R, in order?

? Exercise 7.6.0.12

What is the probability of drawing the letters E-A-R, in any order?

? Exercise 7.6.0.13

What is the probability of drawing the letters S-E-A-L, in order?

? Exercise 7.6.0.14

What is the probability of drawing the letters S-E-A-L, in any order?

? Exercise 7.6.0.15

What is the probability of drawing the letters L-A-S-S, in order?

? Exercise 7.6.0.16

What is the probability of drawing the letters L-A-S-S, in any order?

? Exercise 7.6.0.17

What is the probability of drawing 3 tiles that are all vowels?

? Exercise 7.6.0.18

What is the probability of drawing 3 tiles that are all consonants?

? Exercise 7.6.0.19

What is the probability of drawing 4 tiles in the pattern vowel-consonant-vowel-consonant, in order?

? Exercise 7.6.0.20

What is the probability of drawing 2 vowels and 2 consonants, in any order?

? Exercise 7.6.0.21

What is the probability of drawing at least 1 vowel when drawing four tiles? (Hint: use the Formula for Complements.)

Exercise 7.6.0.22

What is the probability of drawing at least 1 consonant when drawing four tiles?

The following exercises involve the board game *Clue*, which involves a deck of 21 cards: 6 suspects, 6 weapons, and 9 rooms. At the beginning of the game, 1 card of each of the 3 types is secretly removed from the deck (the object of the game is to identify those 3 cards). The remaining 18 cards are dealt out to the players. Assuming there are 3 players, each player gets 6 cards. Find the probabilities of a player being dealt hands with the given characteristics.

Exercise 7.6.0.23

All 6 cards are rooms.

Exercise 7.6.0.24

5 cards are suspects (the sixth can be anything).

Exercise 7.6.0.25

None of the cards are rooms.

Exercise 7.6.0.26

None of the cards are suspects.

Exercise 7.6.0.27

3 cards are suspects and 3 are weapons.

Exercise 7.6.0.28

There are 2 cards of each type.

Exercise 7.6.0.29

There are 3 rooms, 2 suspects, and 1 weapon.

Exercise 7.6.0.30

There are 4 rooms and 5 suspects.

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7.7: What Are the Odds?

Figure 7.7.1: Scratch-off lottery tickets, as well as many other games, represent the likelihood of winning using odds. (credit: “My Scratch-off Winnings” by Shoshanah/Flickr, CC BY 2.0)

Learning Objectives

1. Compute odds.
2. Determine odds from probabilities.
3. Determine probabilities from odds.

A particular lottery instant-win game has 2 million tickets available. Of those, 500,000 win a prize. If there are 500,000 winners, then it follows that there are 1,500,000 losing tickets. When we evaluate the risk associated with a game like this, it can be useful to compare the number of ways to win the game to the number of ways to lose. In the case of this game, we would compare the 500,000 wins to the 1,500,000 losses. In other words, there are 3 losing tickets for every winning ticket. Comparisons of this type are the focus of this section.

Computing Odds

The ratio of the number of equally likely outcomes in an event E to the number of equally likely outcomes *not* in the event E' is called the odds for (or odds in favor of) the event. The opposite ratio (the number of outcomes not in the event to the number in the event E' to the number in the event E is called the odds against the event.

Checkpoint

Both odds and probabilities are calculated as ratios. To avoid confusion, we will always use fractions, decimals, or percents for probabilities, and we’ll use colons to indicate odds. The rules for simplifying fractions apply to odds, too. Thus, the odds for winning a prize in the game described in the section opener are $500{,}000{:}1{,}500{,}000 = 1{:}3$ *and the odds against winning a prize are* $3{:}1$. *These would often be described in words as “the odds of winning are one to three in favor” or “the odds of winning are three to one against.”.*

Checkpoint

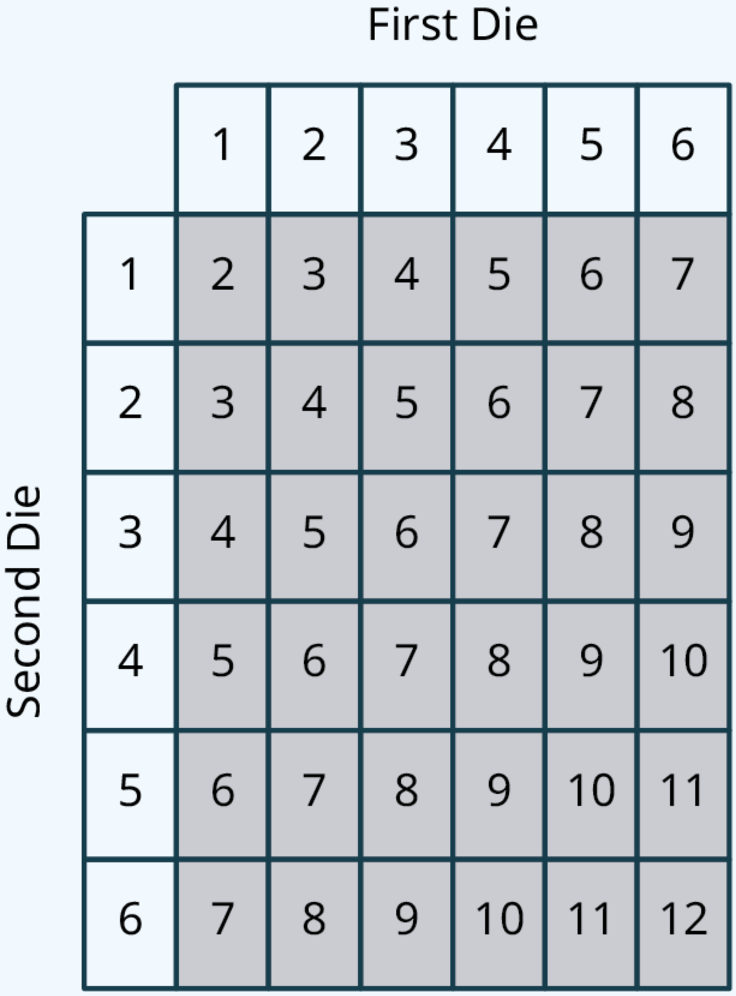
Notice that, while probabilities must always be between zero and one inclusive, odds can be any (non-negative) number, as we'll see in the next example.

Example ✓ 7.7.1: Computing Odds

1. If you roll a fair 6-sided die, what are the odds for rolling a 5 or higher?
2. If you roll two fair 6-sided dice, what are the odds against rolling a sum of 7?
3. If you draw a card at random from a standard deck, what are the odds for drawing a ♡?
4. If you draw 2 cards at random from a standard deck, what are the odds against them both being ♠?

Answer

1. The sample space for this experiment is $\{1, 2, 3, 4, 5, 6\}$. Two of those outcomes are in the event "roll a five or higher," while four are not. So, the odds for rolling a five or higher are $2{:}4 = 1{:}2$.
2. In Example 7.18, we found the sample space for this experiment using the following table (Figure ✓ 7.7.1):



First Die

Second Die	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Figure ✓ 7.7.2

There are 6 outcomes in the event "roll a sum of 7," and there are 30 outcomes not in the event. So, the odds against rolling a 7 are $30{:}6 = 5{:}1$.

- There are 13 ♡ in a standard deck, and $52 - 13 = 39$ others. So, the odds in favor of drawing a ♡ are $13 : 39 = 1{:}3$.
- There are msub13C2=7813C2=78 ways to draw 2 ♠, and msub52C2−78=1,24852C2−78=1,248 ways to draw 2 cards that are *not* both ♠. So, the odds against drawing 2 ♠ are $1{,}248{:}78 = 16{:}1$.

Your Turn 7.7.1

You roll a pair of 4-sided dice with faces labeled 1 through 4.

What are the odds for rolling a sum greater than 3?

What are the odds against both dice giving the same number?

Odds as a Ratio of Probabilities

We can also think of odds as a ratio of probabilities. Consider again the instant-win game from the section opener, with 500,000 winning tickets out of 2,000,000 total tickets. If a player buys one ticket, the probability of winning is $\frac{500{,}000}{2{,}000{,}000} = \frac{1}{4}$, and the probability of losing is $1 - \frac{1}{4} = \frac{3}{4}$. Notice that the ratio of the probability of winning to the probability of losing is $\frac{1}{4}{:}\frac{3}{4} = 1{:}3$, which matches the odds in favor of winning.

FORMULA

For an event E,

odds

forE=n(E):n(E′)=P(E):P(E′)=P(E):(1−P(E))odds againstE=n(E′):n(E)=P(E′):P(E)=(1−P(E)):P(E)odds
forE=n(E):n(E′)=P(E):P(E′)=P(E):(1−P(E))odds againstE=n(E′):n(E)=P(E′):P(E)=(1−P(E)):P(E)

We can use these formulas to convert probabilities to odds, and vice versa.

Example 7.7.2: Converting Probabilities to Odds

Given the following probabilities of an event, find the corresponding odds for and odds against that event.

1. $P(E)$ =35P(E)=35
2. $P(E) = 17\%$

Answer

1. Using the formula, we have:

   odds

   forE=P(E):(1−P(E))=35:(1−35)=35:25=3:2.odds forE=P(E):(1−P(E))=35:(1−35)=35:25=3:2.

   (Note that in the last step, we simplified by multiplying both terms in the ratio by 5.)
   Since the odds for E are $3{:}2$, the odds against E must be $2{:}3$.
2. Again, we'll use the formula:

   odds

   forE=P(E):(1−P(E))=0.17:(1−0.17)=0.17:0.83≈1:4.88.odds forE=P(E):(1−P(E))=0.17:(1−0.17)=0.17:0.83≈1:4.88.

   (In the last step, we simplified by dividing both terms in the ratio by 0.17.)
   It follows that the odds against E are approximately $4.88{:}1$.

Your Turn 7.7.2

If the probability of an event E is 80%, find the odds for and the odds against E.

Now, let's convert odds to probabilities. Let's say the odds for an event are A :Bformula above, we have $A{:}B = P(E){:}(1 - P(E))$. Converting to fractions and solving for $P(E)$, we get:

$$\begin{aligned}\tfrac{A}{B} &= \tfrac{P(E)}{1-P(E)}\\ A(1-P(E)) &= B \times P(E)\\ A - A\times P(E) &= B\times P(E)\\ A &= A\times P(E) + B\times P(E)\\ A &= (A+B)\times P(E)\\ \tfrac{A}{A+B} &= P(E).\end{aligned}$$

Let’s put this result in a formula we can use.

FORMULA

If the odds in favor of E are $A{:}B$, then

$P(E) = \frac{A}{A+B}$.

✓ Example ✓ 7.7.3: Converting Odds to Probabilities

Find $P(E)$ if E:

1. The odds of E are $2{:}1$ in favor
2. The odds of E are $6{:}1$ against

Answer

1. Using the formula we just found, we have $P(E) = \frac{2}{2+1}$ =23P(E)=22+1=23.
2. If the odds against are $6{:}1$, then the odds for are $1{:}6$. Thus, using the formula, $P(E) = \frac{1}{1+6}$ =17P(E)=11+6=17.

Your Turn 7.7.3

Find $P(E)$ if E:

The odds of E are $15{:}1$ against

The odds of E are $2.5{:}1$ in favor

Checkpoint

Some places, particularly state lottery websites, will use the words “odds” and “probability” interchangeably. Never assume that the word “odds” is being used correctly! Compute one of the odds/probabilities yourself to make sure you know how the word is being used!

Check Your Understanding

1. For the following exercises, you are rolling a 6-sided die with 3 orange faces, 2 green faces, and 1 blue face.
2. What are the odds in favor of rolling a green face?
3. What are the odds against rolling a blue face?
4. What are the odds in favor of rolling an orange face?
5. What are the odds in favor of an event with probability $\frac{3}{8}$?
6. What are the odds against an event with probability $\frac{2}{1}$3?
7. What is the probability of an event with odds $9{:}4$ against?
8. What is the probability of an event with odds $5{:}7$ in favor?

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7.7.0: Exercises

For the following exercises, find the probabilities of events with the given odds in favor.

? Exercise 7.7.0.1

9:4

? Exercise 7.7.0.2

2:3

? Exercise 7.7.0.3

2:3

? Exercise 7.7.0.4

5:4

? Exercise 7.7.0.5

1:50

? Exercise 7.7.0.6

7:5

? Exercise 7.7.0.7

1:7

? Exercise 7.7.0.8

10:9

For the following exercises, find the probabilities of events with the given odds against.

? Exercise 7.7.0.9

1:8

? Exercise 7.7.0.10

2:3

? Exercise 7.7.0.11

3:2

Exercise 7.7.0.12

$5:4$

Exercise 7.7.0.13

$1:50$

Exercise 7.7.0.14

$7:5$

Exercise 7.7.0.15

$1:7$

Exercise 7.7.0.16

$10:9$

In the following exercises, find the odds in favor of events with the given probabilities. Give your answer as a ratio of whole numbers. If neither of those two numbers is 1, also give an answer as a ratio involving both 1 and a number greater than or equal to 1 (for example, the odds $5:2$ and $3:8$ can be reduced to $2.5:1$ and $1:2.67$).

Exercise 7.7.0.17

$\frac{2}{7}$

Exercise 7.7.0.18

$\frac{1}{2}17$

Exercise 7.7.0.19

$\frac{8}{9}$

Exercise 7.7.0.20

$\frac{3}{8}$

Exercise 7.7.0.21

$\frac{9}{2}5$

Exercise 7.7.0.22

$\frac{6}{7}$

Exercise 7.7.0.23

$\frac{1}{0}13$

? Exercise 7.7.0.24

$\frac{8}{1}5$

In the following exercises, find the odds against events with the given probabilities. Give your answer as a ratio of whole numbers. If neither of those two numbers is 1, also give an answer as a ratio involving both 1 and a number greater than or equal to 1 (for example, the odds $5:2$ and $3:8$ can be reduced to $2.5:1$ and $1:2.67$).

? Exercise 7.7.0.25

$\frac{2}{7}$

? Exercise 7.7.0.26

$\frac{1}{2}17$

? Exercise 7.7.0.27

$\frac{8}{9}$

? Exercise 7.7.0.28

$\frac{3}{8}$

? Exercise 7.7.0.29

$\frac{9}{2}5$

? Exercise 7.7.0.30

$\frac{6}{7}$

? Exercise 7.7.0.31

$\frac{1}{0}13$

? Exercise 7.7.0.32

$\frac{8}{1}5$

In the following exercises, you are drawing from a deck containing only these 10 cards:
$A\heartsuit$, $A\spadesuit$, $A\clubsuit$, $A\diamondsuit$, $K\spadesuit$, $K\clubsuit$, $Q\heartsuit$, $Q\spadesuit$, $J\heartsuit$, $J\spadesuit$.

? Exercise 7.7.0.33

Let E be the event "draw an ace."

a. What is the probability of E?
b. What are the odds in favor of E?
c. What are the odds against E?

Exercise 7.7.0.34

Let F be the event “draw a $\heartsuit$”.

a. What is the probability of F?
b. What are the odds in favor of F?
c. What are the odds against F?

Exercise 7.7.0.35

Let T be the event “draw two ♠ (without replacement).”

a. What is the probability of T ?
b. What are the odds in favor of T?
c. What are the odds against T?

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7.8: The Addition Rule for Probability

Figure $7.8.1$: Students can be sorted using a variety of possible categories like class year, major, whether they are a varsity athlete, and so forth. (credit: “Multicultural Mashup Melds Languages, Cultures at COD 36” by COD Newsroom/Flickr, CC BY 2.0)

Learning Objectives

1. Identify mutually exclusive events.
2. Apply the Addition Rule to compute probability.
3. Use the Inclusion/Exclusion Principle to compute probability.

Up to this point, we have looked at the probabilities of *simple* events. Simple events are those with a single, simple characterization. Sometimes, though, we want to investigate more complicated situations. For example, if we are choosing a college student at random, we might want to find the probability that the chosen student is a varsity athlete *or* in a Greek organization. This is a compound event: there are two possible criteria that might be met. We might instead try to identify the probability that the chosen student is *both* a varsity athlete *and* in a Greek organization. In this section and the next, we'll cover probabilities of two types of compound events: those build using “or” and those built using “and.” We'll deal with the former first.

Mutual Exclusivity

Before we get to the key techniques of this section, we must first introduce some new terminology. Let's say you're drawing a card from a standard deck. We'll consider 3 events: H is the event “the card is a ♡,” T is the event “the card is a 10,” and S is the event “the card is a ♠.” If the card drawn is J♠, then H and T didn't occur, but S did. If the card drawn is instead 10♠, then H didn't occur, but both T and S did.

We can see from these examples that, if we are interested in several possible events, more than one of them can occur simultaneously (both T and S, for example). But, if you think about all the possible outcomes, you can see that H and S can *never* occur simultaneously; there are no cards in the deck that are both ♡ and ♠. Pairs of events that cannot both occur simultaneously are called mutually exclusive. Let's go through an example to help us better understand this concept.

Example $7.8.1$: Identifying Mutually Exclusive Events

Decide whether the following events are mutually exclusive. If they are not mutually exclusive, identify an outcome that would result in both events occurring.

1. You are about to roll a standard 6-sided die. E is the event “the die shows an even number” and F is the event “the die shows an odd number.”
2. You are about to roll a standard 6-sided die. E is the event “the die shows an even number” and S is the event “the die shows a number less than 4.”
3. You are about to flip a coin 4 times. J is the event “at least 2 heads are flipped” and K is the event “fewer than 3 tails are flipped.”

Answer

1. Let's look at the outcomes for each event: $E = \{2, 4, 6\}$ and $F = \{1, 3, 5\}$. There are no outcomes in common, so E and F are mutually exclusive.
2. Again, consider the outcomes in each event: $E = \{2, 4, 6\}$ and $S = \{1, 2, 3\}$. Since the outcome 2 belongs to both events, these are *not* mutually exclusive.
3. Suppose the results of the 4 flips are HTTH. Then at least 2 heads are flipped, and fewer than 3 tails are flipped. That means that both J and K occurred, and so these events are *not* mutually exclusive.

Your Turn 7.8.1

Suppose you're about to draw one card from a deck containing only these 10 cards: A♡, A♠, A♣, A♢, K♠, K♣, Q♡, Q♠, J♡, J♠. Decide whether these events are mutually exclusive:

E is the event "the card is an ace" and F is the event "the card is a king."

R is the event "the card is a ♡ " and E is the event "the card is an ace."

R is the event "the card is a ♡ " and F is the event "the card is a king."

The Addition Rule for Mutually Exclusive Events

If two events are mutually exclusive, then we can use addition to find the probability that one or the other event occurs.

FORMULA

If E and F are mutually exclusive events, then

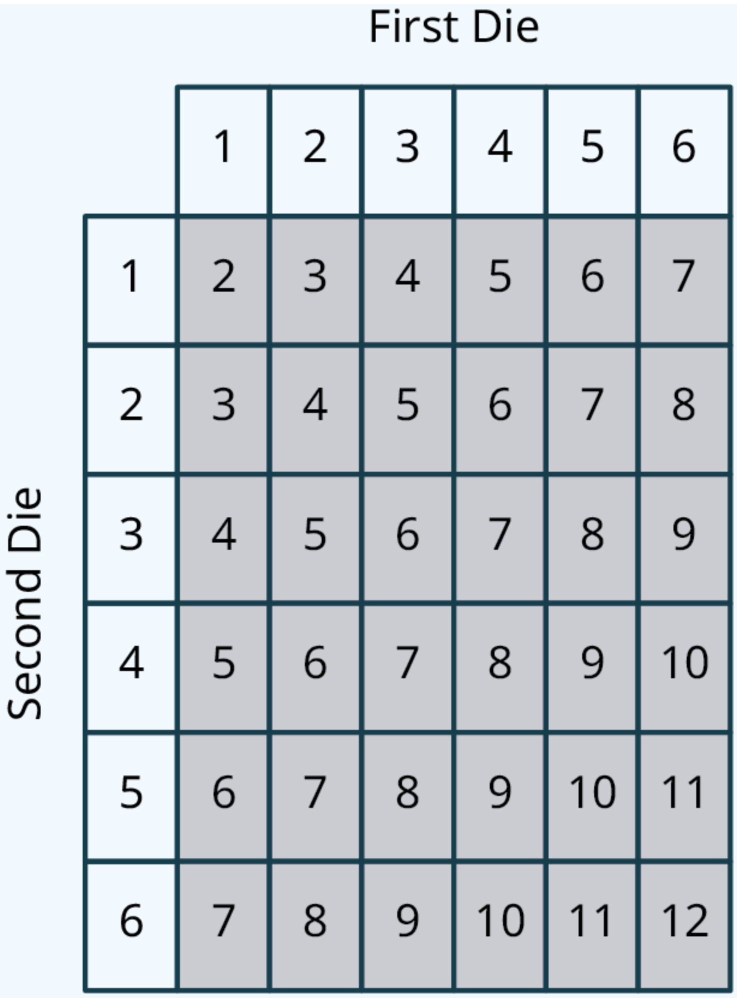
$P(E \text{ or } F) = P(E) + P(F)$

Why does this formula work? Let's consider a basic example. Suppose we're about to draw a *Scrabble* tile from a bag containing A, A, B, E, E, E, R, S, S, U. What is the probability of drawing an E or an S? Since 3 of the tiles are marked with E and 2 are marked with S, there are 5 tiles that satisfy the criteria. There are ten tiles in the bag, so the probability is $\frac{5}{10} = \frac{1}{2}$. Notice that the probability of drawing an E is $\frac{3}{10}$ and the probability of drawing an S is $\frac{2}{10}$; adding those together, we get $\frac{3}{10} + \frac{2}{10} = \frac{5}{10}$. Look at the numerators in the fractions involved in the sum: the 3 represents the number of E tiles and the 2 is the number of S tiles. This is why the Addition Rule works: The total number of outcomes in one event or the other is the sum of the numbers of outcomes in each of the individual events.

✓ Example 7.8.2: Using the Addition Rule

For each of the given pairs of events, decide if the Addition Rule applies. If it does, use the Addition Rule to find the probability that one or the other occurs.

1. You are rolling a standard 6-sided die. Event A is "roll an even number" and event B is "roll a 3."
2. You are drawing a card at random from a standard 52-card deck. Event R is "draw a ♡" and event S is "draw a king."
3. You are rolling a pair of standard 6-sided dice. Example 7.18 might help.



Second Die \ First Die	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Figure ✓ 7.8.2

Answer

1. Since 3 is not an even number, these events are mutually exclusive. So, we can use the Addition Rule: since $P(A) =$ and $P(B) =$, we get $P(A \text{ or } B) = \frac{3}{6} + \frac{1}{6} =$.
2. If the card drawn is $\mathrm{K}\heartsuit$, then both R and S occur. So, they aren't mutually exclusive, and the Addition Rule doesn't apply.
3. Since 10 is not odd, these events are mutually exclusive. Since $P(E) =$ and $P(F) =$, the Addition Rule gives us $P(E \text{ or } F) = \frac{18}{36} + \frac{3}{36} =$.

Your Turn \(\PageIndex{2}\)

Suppose you're about to draw one card from a deck containing only these 10 cards: $\mathrm{A}\heartsuit$, $\mathrm{A}\spadesuit$, $\mathrm{A}\clubsuit$, $\mathrm{AA}\diamondsuit$, $\mathrm{K}\spadesuit$, $\mathrm{K}\clubsuit$, $\mathrm{Q}\heartsuit$, $\mathrm{Q}\spadesuit$, $\mathrm{J}\heartsuit$, $\mathrm{J}\spadesuit$. If appropriate, use the Addition Rule to find the probability that one or the other of these events occurs:

E is the event "the card is an ace" and F is the event "the card is a king."

R is the event "the card is a $\heartsuit$ " and E is the event "the card is an ace."

R is the event "the card is a $\heartsuit$ " and F is the event "the card is a king."

Finding Probabilities When Events Aren't Mutually Exclusive

Let's return to the example we used to explore the Addition Rule: We're about to draw a *Scrabble* tile from a bag containing A, A, B, E, E, E, R, S, S, U. Consider these events: J is "draw a vowel" and K is "draw a letter that comes after L in the alphabet." Since there are 6 vowels, $P(J) =$. There are 4 tiles with letters that come after L alphabetically, so $P(K) =$. What is $P(J \text{ or } K)$? If we blindly apply the Addition Rule, we get $\frac{6}{10} + \frac{4}{10} = 1$, which would mean that the compound event J or K is certain. However, it's possible to draw a B, in which case neither J nor K happens. Where's the error?

The events are not mutually exclusive: the outcome U belongs to both events, and so the Addition Rule doesn't apply. However, there's a way to extend the Addition Rule to allow us to find this probability anyway; it's called the Inclusion/Exclusion Principle. In this example, if we just add the two probabilities together, the outcome U is included in the sum twice: It's one of the 6 outcomes represented in the numerator of $\frac{6}{10}$, and it's one of the 4 outcomes represented in the numerator of $\frac{4}{10}$. So, that particular outcome has been "double counted." Since it has been *included* twice, we can get a true accounting by *excluding* it once: $\frac{6}{10} + \frac{4}{10} - \frac{1}{10} = \frac{9}{10}$. We can generalize this idea to a formula that we can apply to find the probability of any compound event built using "or."

FORMULA

Inclusion/Exclusion Principle: If E and F are events that contain outcomes of a single experiment, then

$$P(E \text{ or } F) = P(E) + P(F) - P(E \text{ and } F)$$

.

It's worth noting that this formula is truly an extension of the Addition Rule. Remember that the Addition Rule requires that the events E and F are mutually exclusive. In that case, the compound event $(E \text{ and } F)$ is impossible, and so $P(E \text{ and } F) = 0$. So, in cases where the events in question are mutually exclusive, the Inclusion/Exclusion Principle reduces to the Addition Rule.

✓ Example 7.7.8.3: Using the Inclusion/Exclusion Principle

Suppose we have events E, F, and G, associated with these probabilities:

$$\begin{aligned} P(E) &= 0.45 \\ P(F) &= 0.6 \\ P(G) &= 0.55 \\ P(E \text{ and } F) &= 0.2 \\ P(E \text{ and } G) &= 0.2 \\ P(F \text{ and } G) &= 0.25 \end{aligned}$$

Compute the following:

1. $P(E \text{ or } F)$
2. $P(E \text{ or } G)$
3. $P(F \text{ or } G)$

Answer

1. Using the Inclusion/Exclusion Principle, we get:

$$\begin{aligned} P(E \text{ or } F) &= P(E) + P(F) - P(E \text{ and } F) \\ &= 0.45 + 0.6 - 0.2 \\ &= 0.85. \end{aligned}$$

2. Again, we'll apply the Inclusion/Exclusion Principle:

$$\begin{aligned} P(E \text{ or } G) &= P(E) + P(G) - P(E \text{ and } G) \\ &= 0.45 + 0.55 - 0.2 \\ &= 0.8. \end{aligned}$$

3. Applying the Inclusion/Exclusion Principle one more time:

$$\begin{aligned} P(F \text{ or } G) &= P(F) + P(G) - P(F \text{ and } G) \\ &= 0.6 + 0.55 - 0.25 \\ &= 0.9. \end{aligned}$$

Your Turn 7.8.3

You are about to roll a special 6-sided die that has both a colored letter and a colored number on each face. The faces are labeled with: a red 1 and a blue A, a red 1 and a green A, an orange 1 and a green B, an orange 2 and a red C, a purple 3 and a brown D, an orange 4 and a blue E. Find the probabilities of these events:

The number is orange or even.

The letter is green or an A.

The number is even or the letter is green.

Check Your Understanding

You are about to draw a card at random from a deck containing only these 10 cards: $A\heartsuit$, $A\spadesuit$, $A\clubsuit$, $A\diamondsuit$, $K\spadesuit$, $K\clubsuit$, $Q\heartsuit$, $Q\spadesuit$, $J\heartsuit$, $J\spadesuit$. Compute the following probabilities:

1. You draw an ace or a king.
2. You draw a $\spadesuit$ or a $\clubsuit$.
3. You draw an ace or a $\heartsuit$.
4. You draw a jack or a $\heartsuit$.
5. You draw a jack or a $\clubsuit$.
6. You draw a king or a $\diamondsuit$.

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7.8.0: Exercises

For the following exercises, we are considering a special 6-sided die, with faces that are labeled with a number and a letter: 1A, 1B, 2A, 2C, 4A, and 4E. You are about to roll this die once.

? Exercise 7.8.0.1

What is the probability of rolling a 1 or a 2?

? Exercise 7.8.0.2

What is the probability of rolling a 4 or a B?

? Exercise 7.8.0.3

What is the probability of rolling an even number or a consonant?

? Exercise 7.8.0.4

What is the probability of rolling a 2 or an E?

? Exercise 7.8.0.5

What is the probability of rolling an odd number or a vowel?

? Exercise 7.8.0.6

What is the probability of rolling an odd number or a consonant?

In the following exercises, you are drawing a single card from a standard 52-card deck.

? Exercise 7.8.0.7

What is the probability that you draw a ♡ or a ♠?

? Exercise 7.8.0.8

What is the probability that you draw a ♡ or a 5?

? Exercise 7.8.0.9

What is the probability that you draw a 2 or a 3?

? Exercise 7.8.0.10

What is the probability that you draw a card with an even number on it?

? Exercise 7.8.0.11

What is the probability that you draw a card with an even number on it or a ♣?

Exercise 7.8.0.12

What is the probability that you draw an ace or a king?

Exercise 7.8.0.13

What is the probability that you draw a face card (king, queen, or jack)?

Exercise 7.8.0.14

What is the probability that you draw a face card or a ♣?

For the following exercises, use the table provided here, which breaks down the enrollment at a certain liberal arts college by class year and area of study:

		Class Year				
		First-Year	Sophomore	Junior	Senior	Totals
Area Of Study	**Arts**	138	121	148	132	539
	Humanities	258	301	275	283	1117
	Social Science	142	151	130	132	555
	Natural Science/Mathematics	175	197	203	188	763
	Totals	713	770	756	735	2974

Exercise 7.8.0.15

What is the probability that a randomly selected student is a first-year or sophomore?

Exercise 7.8.0.16

What is the probability that a randomly selected student is a junior or an arts major?

Exercise 7.8.0.17

What is the probability that a randomly selected student is majoring in the social sciences or the natural sciences/mathematics?

Exercise 7.8.0.18

What is the probability that a randomly selected student is a social science major or a sophomore?

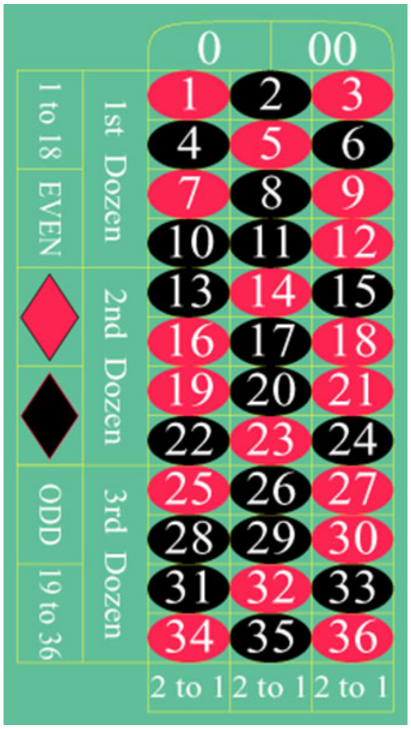
Exercise 7.8.0.19

What is the probability that a randomly selected student is a senior or is a humanities major?

Exercise 7.8.0.20

What is the probability that a randomly selected student is majoring in the arts or humanities?

The following exercises are about the casino game roulette. In this game, the dealer spins a marble around a wheel that contains 38 pockets that the marble can fall into. Each pocket has a number (each whole number from 0 to 36, along with a double zero) and a color (0 and 00 are both green; the other 36 numbers are evenly divided between black and red). Players make bets on which number (or groups of numbers) they think the marble will land on. The figure shows the layout of the numbers and colors, as well as some of the bets that can be made.


0
00
1 2 3
4 5 6
7 8 9
10 11 12
13 14 15
16 17 18
19 20 21
22 23 24
25 26 27
28 29 30
31 32 33
34 35 36
2 to 1
2 to 1
2 to 1
1st Dozen
2nd Dozen
3rd Dozen
1 to 18
EVEN
ODD
19 to 36

Roulette Table (credit: "American Roulette Table Layout" by Film8ker/Wikimedia Commons, Public Domain)

What is the probability of winning at least one of the following pairs of bets on a single spin of the wheel?

? Exercise 7.8.0.21

First dozen (wins if any of the numbers 1–12 come up) or second dozen (wins on 13–24)

? Exercise 7.8.0.22

Red (wins on any of the 18 red numbers) or black (wins on any of the 18 black numbers)

? Exercise 7.8.0.23

Even (wins on any even number 2–36; 0 and 00 both lose this bet) or red

? Exercise 7.8.0.24

Middle column (the numbers 2, 5, 8, 11, …, 35) or black

? Exercise 7.8.0.25

Middle column or red

? Exercise 7.8.0.26

Right column (the numbers 3, 6, 9, …, 36) or black

? Exercise 7.8.0.27

Right column or red

? Exercise 7.8.0.28

Odd or black

? Exercise 7.8.0.29

Even or black

? Exercise 7.8.0.30

The street bet (a bet on 3 numbers that make up a row on the table) on 1, 2, 3 or odd

? Exercise 7.8.0.31

The street bet on 1, 2, 3 or even

? Exercise 7.8.0.32

The corner bet (a bet on 4 numbers that form a square on the table) on 1, 2, 4, 5 or first dozen

? Exercise 7.8.0.33

The corner bet on 1, 2, 4, 5 or second dozen

? Exercise 7.8.0.34

The basket bet (which wins on 0, 00, 1, 2, 3) or red

? Exercise 7.8.0.35

The basket bet or black

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7.9: Conditional Probability and the Multiplication Rule

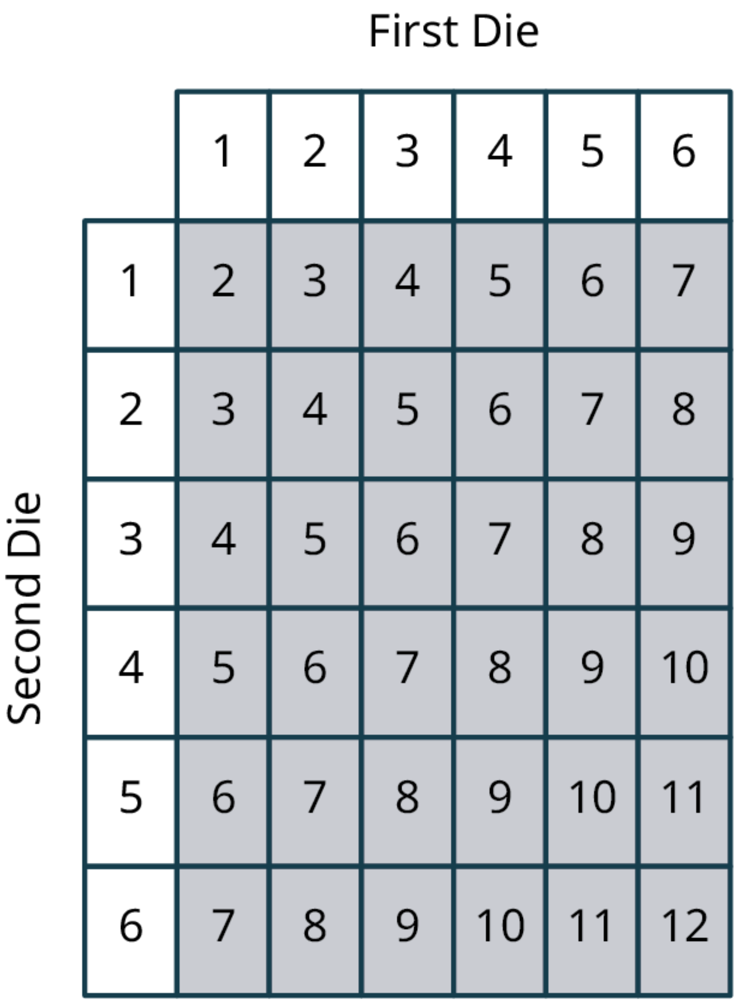
Figure 7.9.1: If you roll two dice by throwing them one at a time, the face showing on the first die will affect the possible outcomes for the sum of the two dice. (credit: “dice” by Ciarán Archer/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

1. Calculate conditional probabilities.
2. Apply the Multiplication Rule for Probability to compute probabilities.

Back in Example 7.9.2 , we constructed the following table (Figure 7.38) to help us find the probabilities associated with rolling two standard 6-sided dice:



First Die

Second Die

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Figure $7.9.2$

For example, 3 of these 36 equally likely outcomes correspond to rolling a sum of 10, so the probability of rolling a 10 is $\frac{3}{36} = \frac{1}{12}$. However, if you choose to roll the dice one at a time, the probability of rolling a 10 will change after the first die comes to rest. For example, if the first die shows a 5, then the probability of rolling a sum of 10 has jumped to $\frac{1}{6}$—the event will occur if the second die also shows a 5, which is 1 of 6 equally likely outcomes for the second die. If instead the first die shows a 3, then the probability of rolling a sum of 10 drops to 0—there are no outcomes for the second die that will give us a sum of 10.

Understanding how probabilities can shift as we learn new information is critical in the analysis of our second type of compound events: those built with “and.” This section will explain how to compute probabilities of those compound events.

Conditional Probabilities

When we analyze experiments with multiple stages, we often update the probabilities of the possible final outcomes or the later stages of the experiment based on the results of one or more of the initial stages. These updated probabilities are called conditional probabilities.

In other words, if O is a possible outcome of the first stage in a multistage experiment, then the probability of an event E conditional on O (denoted $P(E|O)$, read “the probability of E given O”) is the updated probability of E under the assumption that O occurred.

In the example that opened this section, we might consider rolling two dice as a multistage experiment: rolling one, then the other. If we define E to be the event “roll a sum of 10,” O to be the event “first die shows 5,” and Q to be the event “first die shows 3,” then we computed $P(E) = \frac{1}{12}$, $P(E|O) = \frac{1}{6}$, and $P(E|Q) = 0$.

Example ✓ 7.9.1: Computing Conditional Probabilities

1. April is playing a coin-flipping game with Ben. She will flip a coin 3 times. If the coin lands on heads more than tails, April wins; if it lands on tails more than heads, Ben wins. Let A be the event “April wins,” H be “first flip is heads,” and T be “first flip is tails.” Compute $P(A)$, $P(A|H)$, and $P(A|T)$.
2. You are about to draw 2 cards without replacement from a deck containing only these 10 cards: A♡, A♠, A♣, A♢, K♠, K♣, Q♡, Q♠, J♡, J♠. We’ll define the following events: F is “both cards are the same rank,” A is “first card is an ace,” and K is “first card is a king.” Compute $P(F|A)$ and $P(F|K)$.
3. Jim’s sock drawer contains 5 black socks and 3 blue socks. To avoid waking his partner, Jim doesn’t want to turn the lights on, so he puts on 2 socks at random. Let M be the event “Jim’s 2 socks match,” let K be the event “the sock on Jim’s left foot is black,” and let L be the event “the sock on Jim’s left foot is blue.” Compute $P(M)$, $P(M|K)$, and $P(M|L)$.

Answer

1. **Step 1.** The sample space is {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}. The event A consists of the first 4 of those outcomes: HHH, HHT, HTH, and THH. Thus, $P(A) = \frac{4}{8} =$.

   Step 2. Now, let’s compute $P(A|H)$. We are assuming the result of the first flip is heads. That leaves us with 4 possible outcomes: HHH, HHT, HTH, and HTT. Of those, April wins 3 (HHH, HHT, HTH) and loses one (HTT). So, $P(A|H) = \frac{3}{4}$.

   Step 3. If the result of the first flip is instead tails, the 4 possible outcomes are THH, THT, TTH, and TTT. Of those, April wins 1 (THH) and loses 3 (THT, TTH, TTT). So, $P(A|T) = \frac{1}{4}$.
2. **Step 1.** If the event A happens, then 1 of the 4 aces is drawn first; the remaining cards in the deck are 3 aces, 2 kings, 2 queens, and 2 jacks. In order for the event F to occur, the second card drawn has to be an ace. Since there are 3 aces among the remaining 9 cards, $P(F|A) = \frac{3}{9} = \frac{1}{3}$.

   Step 2. If the event K happens instead, then the first card drawn is a king. That leaves 4 aces, 1 king, 2 queens, and 2 jacks in the deck. Under the assumption that the first card is a king, the event F will occur only if the second card is also a king. Since only one of the remaining 9 cards is a king, we have $P(F|K) = \frac{1}{9}$.
3. **Step 1.** We can view the event M as a compound event using “or”: both socks are blue or both socks are black. Let’s compute the probability that both socks are blue using combinations. We’re choosing 2 socks from a group of 8; 3 of the 8 are blue. So, $P(\text{both socks blue}) = \text{mfrac} =$. Similarly, $P(\text{both socks black}) = \text{mfrac} =$. Therefore, since these events are mutually exclusive, we can use the Addition Rule: $P(M) = P(\text{both socks blue}) + P(\text{both socks black}) = \frac{3}{28} + \frac{10}{28} =$.

   Step 2. If the sock on Jim’s left foot is black (i.e., K occurred), then there are 4 remaining black socks of the 7 in the drawer. So, $P(M|K) = \frac{4}{7}$.

   Step 3. If the sock on Jim’s left foot is blue (L occurred), then there are 2 blue socks among the 7 remaining in the drawer. So, $P(M|L) = \frac{2}{7}$.

Your Turn 7.9.1

You are about to roll a special 6-sided die that has both a colored letter and a colored number on each face. The faces are labeled with: a red 1 and a blue A, a red 1 and a green A, an orange 1 and a green B, an orange 2 and a red C, a purple 3 and a brown D, an orange 4 and a blue E. Find the given conditional probabilities:

$P(\text{the letter is a vowel}|\text{the number is orange})$

$P\left(\text{the number is }1|\text{the number is red}\right)$

$P\left(\text{the number is red}|\text{the number is }1\right)$

Checkpoint

In Tree Diagrams, Tables, and Outcomes, we introduced the concept of dependence between stages of a multistage experiment. We stated at the time that two stages were dependent if the result of one stage affects the other stage. We explained that dependence in terms of the sample space, but sometimes that dependence can be a little more subtle; it’s more properly *understood in terms of conditional probabilities. Two stages of an experiment are dependent if $P(E|F) \neq P(E|F')$ for some outcome of the second stage E and outcome of the first stage F.*

Who Knew?: Protecting Bombers in World War II

In his book *How Not to Be Wrong*, Jordan Ellenberg recounts this anecdote: During World War II, the American military wanted to add additional armor plating to bomber aircraft, in order to reduce the chances that they get shot down. So, they collected data on planes after returning from missions. The data showed that the fuselage, wings, and fuel system had many more bullet holes (per unit area) than the engine compartments, so the military brass wanted to add additional armor to the parts of the plane that were hit most often. Luckily, before they added the armor to the planes, they asked for a second opinion. Abraham Wald, a Jewish mathematician who had fled the rising Nazi regime, pointed out that it was far more important that the armor plating be added to areas where there were *fewer* bullet holes. Why? The planes they were studying had already completed their missions, so the military was essentially looking at conditional probabilities: the probability of suffering a bullet strike, *given that* the plane made it back safely. More bullet holes in an area on the plane indicated that was a region that wasn't as important for the plane's survival!

Compound Events Using "And" and the Multiplication Rule

For multistage experiments, the outcomes of the experiment as a whole are often stated in terms of the outcomes of the individual stages. Commonly, those statements are joined with "and." For example, in the sock drawer example just above, one outcome might be "the left sock is black and the right sock is blue." As with "or" compound events, these probabilities can be computed with basic arithmetic.

FORMULA

Multiplication Rule for Probability: If E and F are events associated with the first and second stages of an experiment, then $P(E \text{ and } F) = P(E) \times P(F|E)$.

Checkpoint

In The Addition Rule for Probability, we considered probabilities of events connected with "and" in the statement of the Inclusion/Exclusion Principle. These two scenarios are different; in the statement of the Inclusion/Exclusion Principle, the events connected with "and" are both events associated with the ***same*** *single-stage experiment (or the same stage of a multistage experiment). In the Multiplication Rule, we're looking at events associated with* ***different*** *stages of a multistage experiment.*

✓ Example $7.9.2$: Using the Multiplication Rule for Probability

You are president of a club with 10 members: 4 seniors, 3 juniors, 2 sophomores, and 1 first-year. You need to choose 2 members to represent the club on 2 college committees. The first person selected will be on the Club Awards Committee and the second will be on the New Club Orientation Committee. The same person cannot be selected for both. You decide to select these representatives at random.

1. What is the probability that a senior is chosen for both positions?
2. What is the probability that a junior is chosen first and a sophomore is chosen second?
3. What is the probability that a sophomore is chosen first and a senior is chosen second?

Answer

1. We need the probability that a senior is chosen first and a senior is chosen second. These are two stages of a multistage experiment, so we'll apply the Multiplication Rule for Probability: $P(\text{senior chosen first and senior chosen second}) = P(\text{senior chosen first}) \times P(\text{senior chosen second}|\text{senior chosen first})$. Since there are 4 seniors among the 10 members, $P(\text{senior chosen first}) = \frac{4}{10} =$. Next, assuming a senior is chosen first, there are 3 seniors among the 9 remaining members. So, $P(\text{senior chosen second}|\text{senior chosen first}) = \frac{3}{9} = \frac{1}{3}$. Putting this all together, we get $P(\text{senior chosen first and senior chosen second}) = \frac{2}{5} \times \frac{1}{3} = \frac{2}{15}$.

2. There are 3 juniors among the 10 members, so $P(\text{junior chosen first}) = \frac{3}{10}$. Assuming a junior is chosen first, there are 2 sophomores among the remaining 9 members, so $P(\text{sophomore chosen second}|\text{junior chosen first}) = \frac{2}{9}$. Thus, using the Multiplication Rule for Probability, we have $P(\text{junior chosen first and sophomore chosen second}) = \frac{3}{10} \times \frac{2}{9} = \frac{1}{15}$.
3. The probability that a sophomore is chosen first is $\frac{2}{10} = \frac{1}{5}$, and the probability that a senior is chosen second given that a sophomore was chosen first is $\frac{4}{9}$. Thus, using the Multiplication Rule for Probability, we have:
$P(\text{sophomore chosen first and senior chosen second}) = \frac{1}{5} \times \frac{4}{9} =$.

Your Turn 7.9.2

You're drawing 2 cards in order from a deck containing only the cards A♡, A♣, K♠, 10♡, 9♡, 9♠, 9♢, and 6♣. Compute the following:

$P(\text{draw a } \heartsuit \text{ first and a } \spadesuit \text{ second})$

$P(\text{draw a 9 first and a 6 second})$

$P(\text{draw two hearts})$

WORK IT OUT: The Birthday Problem

One of the most famous problems in probability theory is the Birthday Problem, which has to do with shared birthdays in a large group. To make the analysis easier, we'll ignore leap days, and assume that the probability of being born on any given date is $\frac{1}{365}$. Now, if you have 366 people in a room, we're guaranteed to have at least one pair of people who share a single birthday. Imagine filling the room by first admitting someone born on January 1, then someone born on January 2, and so on… The 365th person admitted would be born on December 31. If you add one more person to the room, that person's birthday would *have* to match someone else's.

Let's look at the other end of the spectrum. If you choose two people at random, what is the probability that they share a birthday? As with many probability questions, this is best addressed by find out the probability that they do *not* share a birthday. The first person's birthday can be anything (probability 1), and the second person's birthday can be anything other than the first person's birthday (probability $\frac{364}{365}$). The probability that they have *different* birthdays is $1 \times \frac{364}{365} = \frac{364}{365}$. So, the probability that they share a birthday is $1 - \frac{364}{365} = \frac{1}{365}$.

What if we have three people? The probability that they all have different birthdays can be obtained by extending our previous calculation: The probability that two people have different birthdays is $\frac{364}{365}$, so if we add a third to the mix, the probability that they have a different birthday from the other two is $\frac{363}{365}$. So, the probability that all three have different birthdays is $\frac{364}{365} \times \frac{363}{365} \approx 0.9918$, and thus the probability that there's a shared birthday in the group is $1 - 0.9918 \approx 0.0082$.

The big question is this: How many people do we need in the room to have the probability of a shared birthday greater than $\frac{1}{2}$? Make a guess, then with a partner keep adding hypothetical people to the group and computing probabilities until you get there!

It is often useful to combine the rules we've seen so far with the techniques we used for finding sample spaces. In particular, trees can be helpful when we want to identify the probabilities of every possible outcome in a multistage experiment. The next example will illustrate this.

Example 7.9.3: Using Tree Diagrams to Help Find Probabilities

The board game *Clue* uses a deck of 21 cards: 6 suspects, 6 weapons, and 9 rooms. Suppose you are about to draw 2 cards from this deck. There are 6 possible outcomes for the draw: 2 suspects, 2 weapons, 2 rooms, 1 suspect and 1 weapon, 1 suspect and 1 room, or 1 weapon and 1 room. What are the probabilities for each of these outcomes?

Answer

Step 1: Let's start by building a tree diagram that illustrates both stages of this experiment. Let's use S, W, and R to indicate drawing a suspect, weapon, and room, respectively (Figure ✓ 7.9.3).

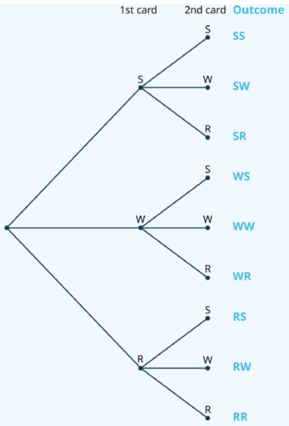


Figure ✓ 7.9.3

Step 2: We want to start computing probabilities, starting with the first stage. The probability that the first card is a suspect is $\frac{6}{21} = \frac{2}{7}$. The probability that the first card is a weapon is the same: $\frac{2}{7}$. Finally, the probability that the first card is a room is $\frac{9}{21} = \frac{3}{7}$.

Step 3: Let's incorporate those probabilities into our tree: label the edges going into each of the nodes representing the first-stage outcomes with the corresponding probabilities (Figure ✓ 7.9.4).

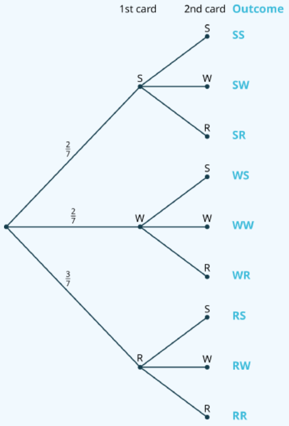


Figure ✓ 7.9.4

Note that the sum of the probabilities coming out of the initial node is 1; this should always be the case for the probabilities coming out of any node!

Step 4: Let's look at the case where the first card is a suspect. There are 3 edges emanating from that node (leading to the outcomes SS, SW, and SR). We'll label those edges with the appropriate *conditional* probabilities, under the assumption that the first card is a suspect. First, there are 5 remaining suspect cards among the 20 left in the deck, so $P(\text{second is suspect}|\text{first is suspect}) = \frac{5}{20} = \frac{1}{4}$. Using similar reasoning, we can compute $P(\text{second is weapon}|\text{first is suspect}) = \frac{6}{20} = \frac{3}{10}$ and $P(\text{second is room}|\text{first is suspect}) = \frac{9}{20}$.

Step 5: Checking our work, we see that the sum of these 3 probabilities is again equal to 1. Let's add those to our tree (Figure ✓ 7.9.5).

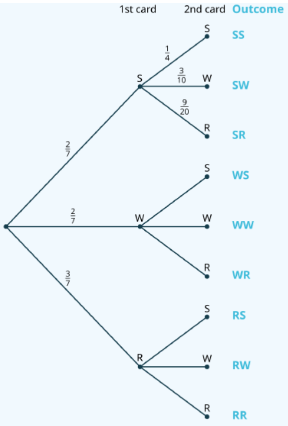

1st card
2nd card
Outcome
S
$\frac{1}{4}$
SS
S
$\frac{3}{10}$
W
SW
$\frac{9}{20}$
R
SR
$\frac{2}{7}$
S
WS
$\frac{2}{7}$
W
W
WW
$\frac{3}{7}$
R
WR
S
RS
R
W
RW
R
RR

Figure ✓ 7.9.5

Step 6: Let's continue filling in the conditional probabilities at the other nodes, always checking to make sure the sum of the probabilities coming out of any node is equal to 1 (Figure ✓ 7.9.6).

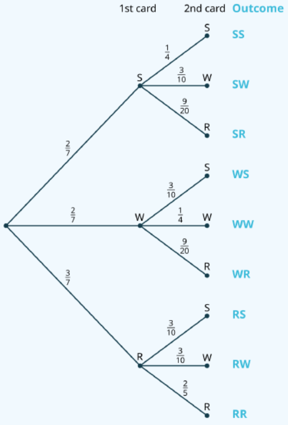

1st card
2nd card
Outcome
S
$\frac{1}{4}$
SS
S
$\frac{3}{10}$
W
SW
$\frac{9}{20}$
R
SR
$\frac{2}{7}$
S
$\frac{3}{10}$
WS
$\frac{2}{7}$
W
$\frac{1}{4}$
W
WW
$\frac{9}{20}$
$\frac{3}{7}$
R
WR
S
$\frac{3}{10}$
RS
R
$\frac{3}{10}$
W
RW
$\frac{2}{5}$
R
RR

Figure ✓ 7.9.6

Step 7: We can compute the probability of landing on any final node by multiplying the probabilities along the path we would take to get there. For example, the probability of drawing a suspect first and a weapon second (i.e., ending up on the node labeled "SW") is $\frac{2}{7} \times \frac{3}{10} = \frac{3}{35}$

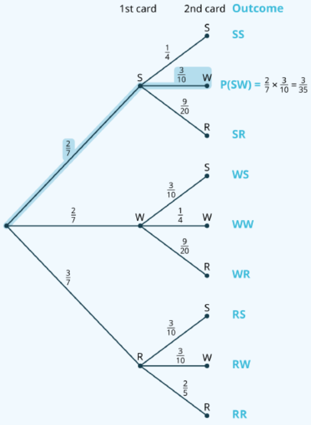

1st card
2nd card
Outcome
S
$\frac{1}{4}$
SS
S
$\frac{3}{10}$
W
$P(SW) = \frac{2}{7} \times \frac{3}{10} = \frac{3}{35}$
$\frac{9}{20}$
R
SR
$\frac{2}{7}$
S
$\frac{3}{10}$
WS
$\frac{2}{7}$
W
$\frac{1}{4}$
W
WW
$\frac{9}{20}$
$\frac{3}{7}$
R
WR
S
$\frac{3}{10}$
RS
R
$\frac{3}{10}$
W
RW
$\frac{2}{5}$
R
RR

Figure ✓ 7.9.7

Step 8: Let's fill in the rest of the probabilities (Figure ✓ 7.9.8).

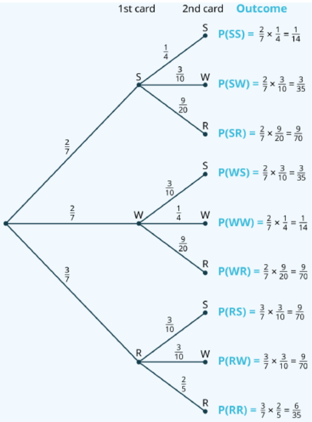

1st card
2nd card
Outcome
S
W
R
$\frac{2}{7}$
$\frac{2}{7}$
$\frac{3}{7}$
$\frac{1}{4}$
$\frac{3}{10}$
$\frac{9}{20}$
$\frac{3}{10}$
$\frac{1}{4}$
$\frac{9}{20}$
$\frac{3}{10}$
$\frac{3}{10}$
$\frac{2}{5}$
$P(SS) = \frac{2}{7} \times \frac{1}{4} = \frac{1}{14}$
$P(SW) = \frac{2}{7} \times \frac{3}{10} = \frac{3}{35}$
$P(SR) = \frac{2}{7} \times \frac{9}{20} = \frac{9}{70}$
$P(WS) = \frac{2}{7} \times \frac{3}{10} = \frac{3}{35}$
$P(WW) = \frac{2}{7} \times \frac{1}{4} = \frac{1}{14}$
$P(WR) = \frac{2}{7} \times \frac{9}{20} = \frac{9}{70}$
$P(RS) = \frac{3}{7} \times \frac{3}{10} = \frac{9}{70}$
$P(RW) = \frac{3}{7} \times \frac{3}{10} = \frac{9}{70}$
$P(RR) = \frac{3}{7} \times \frac{2}{5} = \frac{6}{35}$

Figure ✓ 7.9.8

Step 9: A helpful feature of tree diagrams is that the final outcomes are always mutually exclusive, so the Addition Rule can be directly applied. For example, the probability of drawing one suspect and one room (in any order) would be $P(SR)+P(RS)=\frac{9}{70}+\frac{9}{70}$ =935P(SR)+P(RS)=970+970=935. We can find the probabilities of the other outcomes in a similar fashion, as shown in the following table:

Outcome	Probability
2 suspects	$\frac{1}{14}$
2 weapons	$\frac{1}{14}$
2 rooms	$\frac{6}{35}$
1 suspect and 1 weapon	$\frac{3}{35}+\frac{3}{35}=\frac{6}{35}$
1 suspect and 1 room	$\frac{9}{70}+\frac{9}{70}=\frac{9}{35}$
1 weapon and 1 room	$\frac{9}{70}+\frac{9}{70}=\frac{9}{35}$

Checking once again, the sum of these 6 probabilities is 1, as expected.

Your Turn 7.9.31

You are about to perform the following two-stage experiment. First, you will flip a coin. If the result is heads, roll a standard 6-sided die. If the result of the coin flip is tails, roll a modified 6-sided die with faces labeled 1, 1, 1, 2, 2, 3. Use a tree diagram to find the probability of rolling each of the numbers from 1 to 6.

WORK IT OUT: The Monty Hall Problem

On the original version of the game show *Let's Make a Deal*, originally hosted by Monty Hall and now hosted by Wayne Brady, one contestant was chosen to play a game for the grand prize of the day (often a car). Here's how it worked: On the stage were three areas concealed by numbered curtains. The car was hidden behind one of the curtains; the other two curtains hid worthless prizes (called "Zonks" on the show). The contestant would guess which curtain concealed the car. To build tension, Monty would then reveal what was behind one of the *other* curtains, which was always one of the Zonks (Since Monty knew where the car was hidden, he always had at least one Zonk curtain that hadn't been chosen that he could reveal). Monty then turned to the contestant and asked: "Do you want to stick with your original choice, or do you want to switch your choice to the other curtain?" What should the contestant do? Does it matter?

With a partner or in a small group, simulate this game. You can do that with a small candy (the prize) hidden under one of three cups, or with three playing cards (just decide ahead of time which card represents the "Grand Prize"). One person plays the host, who knows where the prize is hidden. Another person plays the contestant and tries to guess where the prize is hidden. After the guess is made, the host should reveal a losing option that wasn't chosen by the contestant. The contestant then has the

> option to stick with the original choice or switch to the other, unrevealed option. Play about 20 rounds, taking turns in each role and making sure that both contestant strategies (stick or switch) are used equally often. After each round, make a note of whether the contestant chose "stick" or "switch" and whether the contestant won or lost. Find the empirical probability of winning under each strategy. Then, see if you can use tree diagrams to verify your findings.

Check Your Understanding

For the following exercises, you are rolling two 6-sided dice, each of which has 3 orange faces, 2 green faces, and 1 blue face.

1. What is the probability of rolling 2 orange faces?
2. What is the probability of rolling 2 green faces?
3. What is the probability of rolling 1 orange and 1 green face (in any order)?
4. For the following exercises, you are about to draw 2 cards at random (without replacement) from a deck containing only these 10 cards: $A\heartsuit$, $A\spadesuit$, $A\clubsuit$, $A\diamondsuit$, $K\spadesuit$, $K\clubsuit$, $Q\heartsuit$, $Q\spadesuit$, $J\heartsuit$, $J\spadesuit$.
5. What is the probability of drawing 2 aces?
6. What is the probability of drawing an ace first and a king second?
7. What is the probability of drawing a $\clubsuit$ and a $\spadesuit$ (in any order)?

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7.9.0: Exercises

For the following exercises, we are considering a special 6-sided die, with faces that are labeled with a number and a letter: 1A, 1B, 2A, 2C, 4A, and 4E. You are about to roll this die twice.

? Exercise 7.9.0.1

What is the probability of rolling two 1s?

? Exercise 7.9.0.2

What is the probability of rolling two vowels?

? Exercise 7.9.0.3

What is the probability of rolling an even number first and an odd number second?

? Exercise 7.9.0.4

What is the probability of rolling an even number and an odd number in any order?

? Exercise 7.9.0.5

What is the probability of rolling a consonant first and a 1 second?

? Exercise 7.9.0.6

What is the probability of rolling one number less than 3 and one number greater than 3, in any order?

In the following exercises, you are about to draw *Scrabble* tiles from a bag; the bag contains the letters A, A, C, E, E, E, L, L, N, O, R, S, S, S, T, X.

? Exercise 7.9.0.7

If you draw 1 tile at random, compute

a. $P(\text{tile shows A})$
b. $P(\text{tile shows A} \mid \text{tile shows a vowel})$

? Exercise 7.9.0.8

If you draw 1 tile at random, compute:

a. $P(\text{tile shows a vowel})$
b. $P(\text{tile shows a vowel} \mid \text{tile shows a letter that comes after M alphabetically})$

? Exercise 7.9.0.9

If you draw 2 tiles with replacement, compute $P(\text{both are vowels})$.

? Exercise 7.9.0.10

If you draw 2 tiles without replacement, compute $P(\text{both are vowels})$.

Exercise 7.9.0.11

If you draw 2 tiles with replacement, compute $P(\text{first is a vowel and second is a consonant})$.

Exercise 7.9.0.12

If you draw 2 tiles without replacement, compute $P(\text{first is a vowel and second is a consonant})$.

Exercise 7.9.0.13

If you draw 2 tiles with replacement, compute $P(\text{one is a vowel and one is a consonant})$.

Exercise 7.9.0.14

If you draw 2 tiles without replacement, compute $P(\text{one is a vowel and one is a consonant})$.

Exercise 7.9.0.15

If you draw 2 tiles with replacement, compute $P(\text{both are Es})$.

Exercise 7.9.0.16

If you draw 2 tiles without replacement, compute $P(\text{both are Es})$.

For the following exercises, use the table provided, which breaks down the enrollment at a certain liberal arts college by class year and area of study.

		Class Year				
		First-Year	Sophomore	Junior	Senior	Totals
Area Of Study	**Arts**	138	121	148	132	539
	Humanities	258	301	275	283	1117
	Social Science	142	151	130	132	555
	Natural Science/Mathematics	175	197	203	188	763
	Totals	713	770	756	735	2974

Exercise 7.9.0.17

Compute the probability that a randomly selected student is a sophomore, given that they are majoring in the arts.

Exercise 7.9.0.18

Compute the probability that a randomly selected student is majoring in the arts, given that they are a sophomore.

Exercise 7.9.0.19

If two seniors are chosen at random, compute the probability that both are social science majors. Give your answer as a decimal, rounded to 5 decimal places.

? Exercise 7.9.0.20

If two humanities majors are chosen at random, compute the probability that the first is a senior and the second is a junior. Give your answer as a decimal, rounded to 5 decimal places.

? Exercise 7.9.0.21

If two natural science/mathematics majors are chosen at random, compute the probability that one is a sophomore and one is a senior (in any order). Give your answer as a decimal, rounded to 5 decimal places.

? Exercise 7.9.0.22

If two students are chosen at random, compute the probability that one is an arts major and one is a social science major, in any order. Give your answer as a decimal, rounded to 5 decimal places.

In the following exercises deal with the game “Punch a Bunch,” which appears on the TV game show *The Price Is Right*. In this game, contestants have a chance to punch through up to 4 paper circles on a board; behind each circle is a card with a dollar amount printed on it. There are 50 of these circles; the dollar amounts are given in this table:

Dollar Amount	Frequency
$25,000	1
$10,000	2
$5,000	4
$2,500	8
$1,000	10
$500	10
$250	10
$100	5

Contestants are shown their selected dollar amounts one at a time, in the order selected. After each is revealed, the contestant is given the option of taking that amount of money or throwing it away in favor of the next amount. (You can watch the game being played in the video Playing “Punch a Bunch.”) Jeremy is playing “Punch a Bunch” and gets 2 punches.

? Exercise 7.9.0.23

What is the probability that both punches are worth less than $1,000?

? Exercise 7.9.0.24

What is the probability that both punches are worth more than $2,500?

? Exercise 7.9.0.25

What is the probability that the second punch is worth more than the first punch, given that the first punch was worth $250?

? Exercise 7.9.0.26

What is the probability that the second punch is worth more than the first punch, given that the first punch was worth $1,000?

? Exercise 7.9.0.27

What is the probability that the second punch is worth less than the first punch, given that the first punch was worth \$250?

? Exercise 7.9.0.28

What is the probability that the second punch is worth less than the first punch, given that the first punch was worth \$1,000?

? Exercise 7.9.0.29

What is the probability that both punches are worth \$100?

? Exercise 7.9.0.30

What is the probability that both punches are worth the same amount?

In the following exercises, we consider two baseball teams playing a best-of-three series (meaning the first team to win two games wins the series). Team A is a little bit better than Team B, so we expect Team A will win 55% of the time.

? Exercise 7.9.0.31

What is the probability that Team A wins the series given that Team B wins the first game?

? Exercise 7.9.0.32

What is the probability that Team B wins the series given that Team B wins the first game?

? Exercise 7.9.0.33

What is the probability that Team B wins the series given that Team A wins the first game?

? Exercise 7.9.0.34

What is the probability that Team A wins the series given that Team A wins the first game?

? Exercise 7.9.0.35

Build a tree diagram that shows all possible outcomes of the series. Label the edges with appropriate probabilities.

? Exercise 7.9.0.36

What is the probability that Team A wins the series?

? Exercise 7.9.0.37

If instead Team A has a 75% chance of winning each game, what is the probability that Team A wins the series?

? Exercise 7.9.0.38

If instead Team A has a 90% chance of winning each game, what is the probability that Team A wins the series?

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7.10: The Binomial Distribution

Figure 7.10.1: If one baseball team has a 65% chance of beating another in any single game, what’s the likelihood that they win a best-of-seven series? (credit: “baseball game” by Britt Reints/Flickr, CC BY 2.0)

Learning Objectives

1. Identify binomial experiments.
2. Use the binomial distribution to analyze binomial experiments.

It’s time for the World Series, which determines the champion for this season in Major League Baseball. The scrappy Los Angeles Angels are facing the powerhouse Cincinnati Reds. Computer models put the chances of the Reds winning any single game against the Angels at about 65%. The World Series, as its name implies, isn’t just one game, though: it’s what’s known as a “best-of-seven” contest: the teams play each other repeatedly until one team wins 4 games (which could take up to 7 games total, if each team wins three of the first 6 matchups). If the Reds truly have a 65% chance of winning a single game, then the probability that they win the series should be greater than 65%. Exactly how much bigger?

If you have the patience for it, you could use a tree diagram like we used in Example 7.33 to trace out all of the possible outcomes, find all the related probabilities, and add up the ones that result in the Reds winning the series. Such a tree diagram would have $2^7 = 128$ final nodes, though, so the calculations would be very tedious. Fortunately, we have tools at our disposal that allow us to find these probabilities very quickly. This section will introduce those tools and explain their use.

Binomial Experiments

The tools of this section apply to multistage experiments that satisfy some pretty specific criteria. Before we move on to the analysis, we need to introduce and explain those criteria so that we can recognize experiments that fall into this category. Experiments that satisfy each of these criteria are called binomial experiments. A binomial experiment is an experiment with a fixed number of repeated independent binomial trials, where each trial has the same probability of success.

Repeated Binomial Trials

The first criterion involves the structure of the stages. Each stage of the experiment should be a replication of every other stage; we call these replications trials. An example of this is flipping a coin 10 times; each of the ten flips is a trial, and they all occur under the same conditions as every other. Further, each trial must have only two possible outcomes. These two outcomes are typically labeled “success” and “failure,” even if there is not a positive or negative connotation associated with those outcomes. Experiments with more than two outcomes in their sample spaces are sometimes reconsidered in a way that forces just two outcomes; all we need to do is completely divide the sample space into two parts that we can label “success” and “failure.” For example, your grade on an exam might be recorded as A, B, C, D, or F, but we could instead think of the grades A, B, C, and D as “success” and a grade of F as “failure.” Trials with only two outcomes are called binomial trials (the word *binomial* derives from Latin and Greek roots that mean “two parts”).

Independent Trials

The next criterion that we'll be looking for is independence of trials. Back in Tree Diagrams, Tables, and Outcomes, we said that two stages of an experiment are independent if the outcome of one stage doesn't affect the other stage. Independence is necessary for the experiments we want to analyze in this section.

Fixed Number of Trials

Next, we require that the number of trials in the experiment be decided before the experiment begins. For example, we might say "flip a coin 10 times." The number of trials there is fixed at 10. However, if we say "flip a coin until you get 5 heads," then the number of trials could be as low as 5, but theoretically it could be 50 or a 100 (or more)! We can't apply the tools from this section in cases where the number of trials is indeterminate.

Constant Probability

The next criterion needed for binomial experiments is related to the independence of the trials. We must make sure that the probability of success in each trial is the same as the probability of success in every other trial.

✓ Example ✓ 7.10.1: Identifying Binomial Experiments

Decide whether each of the following is a binomial experiment. For those that aren't, identify which criterion or criteria are violated.

1. You roll a standard 6-sided die 10 times and write down the number that appears each time.
2. You roll a standard 6-sided die 10 times and write down whether the die shows a 6 or not.
3. You roll a standard 6-sided die until you get a 6.
4. You roll a standard 6-sided die 10 times. On the first roll, we define "success" as rolling a 4 or greater. After the first roll, we define "success" as rolling a number greater than the result of the previous roll.

Answer

1. Since we're noting 1 of 6 possible outcomes, the trials are not binomial. So, this isn't a binomial experiment.
2. We have 2 possible outcomes ("6" and "not 6"), the trials are independent, the probability of success is the same every time, and the number of trials is fixed. This is a binomial experiment.
3. Since the number of trials isn't fixed (we don't know if we'll get our first 6 after 1 roll or 20 rolls or somewhere in between), this isn't a binomial experiment.
4. Here, the probability of success might change with every roll (on the first roll, that probability is $\frac{1}{2}$; if the first roll is a 6, the probability of success on the next roll is zero). So, this is not a binomial experiment.

Your Turn 7.10.1

Decide whether the following experiments are binomial experiments:

1. Draw a card from a well-shuffled deck, note its suit, and replace it. Repeat this process 5 times.
2. Draw 5 cards from a well-shuffled deck and count the number of ♣.
3. Draw a card from a well-shuffled deck, note whether it is a ♣ or not, and replace it. Repeat this process 5 times.
4. Draw cards from a well-shuffled deck until you get a ♣.

The Binomial Formula

If we flip a coin 100 times, you might expect the number of heads to be around 50, but you probably wouldn't be surprised if the actual number of heads was 47 or 52. What is the probability that the number of heads is exactly 50? Or falls between 45 and 55? It seems unlikely that we would get more than 70 heads. Exactly how unlikely is that?

Each of these questions is a question about the number of successes in a binomial experiment (flip a coin 100 times, "success" is flipping heads). We could theoretically use the techniques we've seen in earlier sections to answer each of these, but the number of calculations we'd have to do is astronomical; just building the tree diagram that represents this situation is more than we could complete in a lifetime; it would have $2^{100} \approx 1.3 \times 10^{30}$ final nodes! To put that number in perspective, if we could draw 1,000 dots every second, and we started at the moment of the Big Bang, we'd currently be about 0.00000003% of the way to drawing out those final nodes. Luckily, there's a shortcut called the Binomial Formula that allows us to get around doing all those calculations!

FORMULA

Binomial Formula: Suppose we have a binomial experiment with n trials and the probability of success in each trial is p. Then:

$$P(\text{ number of successes is } a)={}_nC_a\times p^a\times(1-p)^{n-a}$$

We can use this formula to answer one of our questions about 100 coin flips. What is the probability of flipping exactly 50 heads? In this case, $n=100, p=\frac{1}{2}$, and $a=50$, so $P(\text{ flip 50 heads })={}_{100}C_{50}\times\left(\frac{1}{2}\right)^{50}\times\left(1-\frac{1}{2}\right)^{100-50}$. Unfortunately, many calculators will balk at this calculation; that first factor $\left({}_{100}C_{50}\right)$ is an enormous number, and the other two factors are very close to zero. Even if your calculator can handle numbers that large or small, the arithmetic can create serious errors in rounding off.

Tech Check

Luckily, spreadsheet programs have alternate methods for doing this calculation. In Google Sheets, we can use the BINOMDIST function to do this calculation for us. Open up a new sheet, click in any empty cell, and type "=BINOMDIST(50,100,0.5,FALSE)" followed by the Enter key. The cell will display the probability we seek; it's about 8%. Let's break down the syntax of that function in Google Sheets: enter "=BINOMDIST(a, n, p, FALSE)" to find the probability of a successes in n trials with probability of success p.

Example 7.10.2: Using the Binomial Formula

1. Find the probability of rolling a standard 6-sided die 4 times and getting exactly one 6 *without using technology*.
2. Find the probability of rolling a standard 6-sided die 60 times and getting exactly ten 6s using technology.
3. Find the probability of rolling a standard 6-sided die 60 times and getting exactly eight 6s using technology.

Answer

1. We'll apply the Binomial Formula, where $n=4, a=1$, and $p=\frac{1}{6}$:

$$\begin{aligned}P(\text{ rolling one } 6)&={}_4C_1\times\left(\frac{1}{6}\right)^1\times\left(\frac{5}{6}\right)^{4-1}\\&=\frac{4!}{1!(4-1)!}\times\frac{1}{6}\times\left(\frac{5}{6}\right)^3\\&=4\times\frac{1}{6}\times\frac{5^3}{6^3}\\&=\frac{4\times5^3}{6^4}\\&=\frac{500}{1,296}\end{aligned}$$

2. Here, $n=60, a=10$, and $p=\frac{1}{6}$. In Google Sheets, we'll enter "=BINOMDIST(10, 60, 1/6, FALSE)" to get our result: 0.137 .

3. This experiment is the same as in Exercise 2 of this example; we're simply changing the number of successes from 10 to 8 . Making that change in the formula in Google Sheets, we get the probability 0.116.

Your Turn 7.10.2

Compute the probabilities (rounded to 3 decimal places) of the following events related to rolling a standard 4-sided die (with faces labeled 1, 2, 3, and 4):

1. You roll the die 10 times and get exactly four 2s.
2. You roll the die 20 times and get exactly four 2s.
3. You roll the die 30 times and get exactly four 2s.

The Binomial Distribution

If we are interested in the probability of more than just a single outcome in a binomial experiment, it's helpful to think of the Binomial Formula as a function, whose input is the number of successes and whose output is the probability of observing that many successes. Generally, for a small number of trials, we'll give that function in table form, with a complete list of the possible outcomes in one column and the probability in the other.

For example, suppose Kristen is practicing her basketball free throws. Assume Kristen always makes 82% of those shots. If she attempts 5 free throws, then the Binomial Formula gives us these probabilities:

Shots Made	Probability
0	0.000189
1	0.004304
2	0.0392144
3	0.1786432
4	0.4069096
5	0.3707398

A table that lists all possible outcomes of an experiment along with the probabilities of those outcomes is an example of a probability density function (PDF). A PDF may also be a formula that you can use to find the probability of any outcome of an experiment.

Checkpoint

Because they refer to the same thing, some sources will refer to the Binomial Formula as the Binomial PDF.

If we want to know the probability of a range of outcomes, we could add up the corresponding probabilities. Going back to Kristen's free throws, we can find the probability that she makes 3 or fewer of her 5 attempts by adding up the probabilities associated with the corresponding outcomes (in this case: 0, 1, 2, or 3):

$$\begin{aligned} P(\text{makes 3 or fewer}) &= P(a=0)+P(a=1)+P(a=2)+P(a=3) \\ &= 0.000189+0.004304+0.0392144+0.1786432 \\ &= 0.2223506 \end{aligned}$$

The probability that the outcome of an experiment is less than or equal to a given number is called a *cumulative probability*. A table of the cumulative probabilities of all possible outcomes of an experiment is an example of a cumulative distribution function (CDF). A CDF may also be a formula that you can use to find those cumulative probabilities.

Checkpoint

Cumulative probabilities are always associated with events that are defined using $\leq$. If other inequalities are used to define the event, we must restate the definition so that it uses the correct inequality.

Here are the PDF and CDF for Kristen's free throws:

Shots Made	Probability	Cumulative
0	0.000189	0.000189
1	0.004304	0.004493
2	0.0392144	0.0437073
3	0.1786432	0.2223506
4	0.4069096	0.6292602

Shots Made	Probability	Cumulative
5	0.3707398	1

Tech Check

Google Sheets can also compute cumulative probabilities for us; all we need to do is change the “FALSE” in the formulas we used before to "TRUE."

Example 7.10.3: Using the Binomial CDF

Suppose we are about to flip a fair coin 50 times. Let H represent the number of heads that result from those flips. Use technology to find the following:

1. $P(H \leq 22)$
2. $P(H < 26)$
3. $P(H > 28)$
4. $P(H \geq 20)$
5. $P(20 < H < 25)$

Answer

1. The event here is defined by H ≤22H≤22, which is the inequality we need to have if we want to use the Binomial CDF. In Google Sheets, we’ll enter “=BINOMDIST(22, 50, 0.5, TRUE)” to get our answer: 0.2399.
2. This event uses the wrong inequality, so we need to do some preliminary work. If H <26H<26, that means H ≤25H≤25 (because H has to be a whole number). So, we’ll enter “=BINOMDIST(25, 50, 0.5, TRUE)” to find $P(H < 26) = P(H \leq 25) = 0.5561$.
3. The inequality associated with this event is pointing in the wrong direction. If E is the event H >28H>28, that means that E contains the outcomes {29, 30, 31, 32, 33, …}. Thus, E' must contain the outcomes {…, 25, 26, 27, 28}. In other words, E' is defined by H ≤28H≤28. Since it uses $\leq$, we can find $P(E')$ using “=BINOMDIST(28, 50, 0.5, TRUE)”: 0.8389 So, using the formula for probabilities of complements, we have
$$\begin{aligned} P(E) &= 1 - P(E') \\ &= 1 - 0.8389 \\ &= 0.1611. \end{aligned}$$
4. As in part 3, this inequality is pointing in the wrong direction. If F is the event H ≥20H≥20, then F contains the outcomes {20, 21, 22, 23, …}. That means F' contains the outcomes {…, 16, 17, 18, 19}, and so F' is defined by H ≤19H≤19. So, we can find $P(F')$ using “=BINOMDIST(19, 50, 0.5, TRUE)”: 0.0595. Finally, using the formula for probabilities of complements, we get:
$$\begin{aligned} P(F) &= 1 - P(F') \\ &= 1 - 0.0595 \\ &= 0.9405. \end{aligned}$$
5. If $20 < H < 25$, that means we are interested in the outcomes {21, 22, 23, 24}. This doesn’t look like any of the previous situations, but there is a way to find this probability using the Binomial CDF. We need to put everything in terms of “less than or equal to,” so we’ll first note that all of our outcomes are less than or equal to 24. But we don’t want to include values that are less than or equal to 20. So, we have three events: let I be the event defined by $20 < H < 25$ (note that we’re trying to find $P(I)$). Let J be defined by H ≤24H≤24, and let K be defined by H ≤20H≤20. Of these three events, J contains the most outcomes. If J occurs, then either K or I must have occurred. Moreover, K and I are mutually exclusive. Thus, $P(J) = P(K) + P(I)$, by the Addition Rule. Solving for the probability that we want, we get
$$\begin{aligned} P(I) &= P(J) - P(K) \\ &= P(H \leq 24) - P(H \leq 20) \\ &= 0.44386 - 0.10132 \\ &= 0.34254. \end{aligned}$$

Your Turn 7.10.3

You are about to roll a standard 6-sided die 20 times. Let S denote a success, which will be rolling a number greater than 4. Find the probabilities of the following events, rounded to 4 decimal places:

1. $S \leq 5$
2. $S < 8$
3. $S > 10$
4. $S \geq 7$
5. $5 < S < 8$

Finally, we can answer the question posed at the beginning of this section. Remember that the Reds are facing the Angels in the World Series, which is won by the team who is first to win 4 games. The Reds have a 65% chance to win any game against the Angels. So, what is the probability that the Reds win the World Series? At first glance, this is not a binomial experiment: The number of games played is not fixed, since the series ends as soon as one team wins 4 games. However, we can extend this situation to a binomial experiment: Let's assume that 7 games are always played in the World Series, and the winner is the team who wins more games. In a way, this is what happens in reality; it's as though the first team to lose 4 games (and thus cannot win more than the other team) forfeits the rest of their games. So, we can treat the actual World Series as a binomial experiment with seven trials. If W is the number of games won by the Reds, the probability that the Reds win the World Series is $P(W \geq 4)$. Using the techniques from the last example, we get $P(\text{Reds win the series}) = 0.8002$.

People in Mathematics: Abraham de Moivre

Abraham de Moivre was born in 1667 in France to a Protestant family. Though he was educated in Catholic schools, he remained true to his faith; in 1687, he fled with his brother to London to escape persecution under the reign of King Louis XIV. Once he arrived in England, he supported himself as a freelance math tutor while he conducted his own research. Among his interests was probability; in 1711, he published the first edition of *The Doctrine of Chances: A Method of Calculating the Probabilities of Events in Play*. This book was the second textbook on probability (after Cardano's *Liber de ludo aleae*). De Moivre discovered an important connection between the binomial distribution and the *normal distribution* (an important concept in statistics; we'll explore that distribution and its connection to the binomial distribution in Chapter 8). De Moivre also discovered some properties of a new probability distribution that later became known as the Poisson distribution.

Check Your Understanding

You are rolling a 6-sided die with 3 orange faces, 2 green faces, and 1 blue face.

1. If you roll the die 5 times and note the color showing on each roll, is this a binomial experiment?
2. If you roll the die 5 times and count the number times you roll a green face, is this a binomial experiment?
3. If you count how many times you roll the die until you get a blue face, is this a binomial experiment?

Suppose you're rolling the same colored 6-sided die 10 times. Let O, G, and B represent the number of times the die lands with an orange, green, and blue side up, respectively. Find these probabilities (round to 4 decimal places):

4. $P(O \leq 7)$

5. $P(G = 5)$

6. $P(1 \leq B \leq 4)$

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7.10.0: Exercises

In the following exercises, decide whether the described experiments are binomial experiments. For those that are not, explain why they aren't.

? Exercise 7.10.0.1

A golfer practices putts from 1 foot, 2 feet, 3 feet, 4 feet, and 5 feet; "success" is defined as making the putt.

? Exercise 7.10.0.2

A game designer rolls a pair of dice 100 times and counts the number of times the sum is at least 10.

? Exercise 7.10.0.3

A student who is completely unprepared for a multiple-choice pop quiz guesses on all 10 questions. There are 4 choices for each of the first 5 questions and 5 choices for each of the last 5 questions. "Success" is defined as answering the question correctly.

? Exercise 7.10.0.4

A baseball player is practicing pitching; he throws pitches until he gets 50 strikes.

? Exercise 7.10.0.5

A statistician stops 20 college students at random outside a dining hall and notes their class year.

? Exercise 7.10.0.6

An employee at a bowling alley watches each patron's first ball and counts how many are strikes over the course of his shift.

In the following exercises, you have an 8-sided die with 4 faces colored orange, 3 colored red, and 1 colored yellow. You're going to roll the die 100 times: Let Y be the number of times a yellow face is showing, R be the number of times a red face is showing, and O be the number of times an orange face is showing. Find the given probabilities, rounded to 4 decimal places:

? Exercise 7.10.0.7

$P(O=52)$

? Exercise 7.10.0.8

$P(R<40)$

? Exercise 7.10.0.9

$P(Y\geq 10)$

? Exercise 7.10.0.10

$P(O>55)$

Exercise 7.10.0.11

$P(R = 35)$

Exercise 7.10.0.12

$P(Y < 5)$

Exercise 7.10.0.13

$P(45 < O < 55)$

Exercise 7.10.0.14

$P(R \leq 30)$

Exercise 7.10.0.15

$P(Y \geq 20)$

Exercise 7.10.0.16

$P(O < 40)$

The following exercises are about series of games, where Team A faces Team B in a best-of series (like the World Series). Find the probability that Team A wins the series in each of the following scenarios:

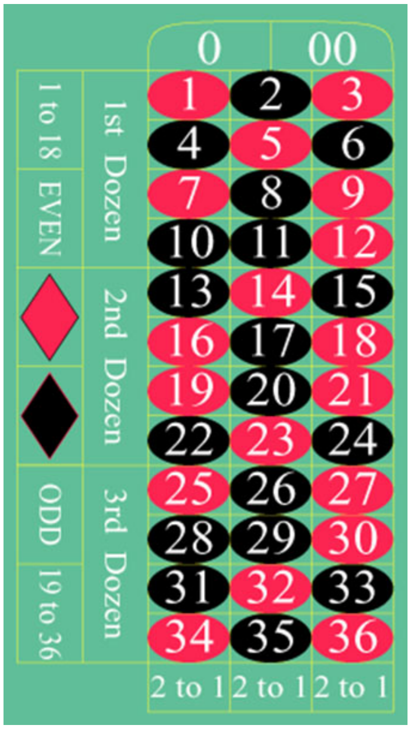
Exercise 7.10.0.27

Give the table of the PDF for flipping a fair coin 5 times and counting the heads. Do the calculations without using technology.

Exercise 7.10.0.28

Give the table of the CDF for flipping a fair coin 5 times and counting the heads. Do the calculations without using technology.

The following exercises are about the casino game roulette. In this game, the dealer spins a marble around a wheel that contains 38 pockets that the marble can fall into. Each pocket has a number (each whole number from 0 to 36, along with a “double zero”) and a color (0 and 00 are both green; the other 36 numbers are evenly divided between black and red). Players make bets on which number (or groups of numbers) they think the marble will land on. The figure shows the layout of the numbers and colors, as well as some of the bets that can be made.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30
31
32
33
34
35
36
2 to 1
2 to 1
2 to 1
1st Dozen
2nd Dozen
3rd Dozen
1 to 18
EVEN
ODD
19 to 36

Roulette Table (credit: "American Roulette Table Layout" by Film8ker/Wikimedia Commons, Public Domain)

? Exercise 7.10.0.29

If a player bets $1 on red and wins, the player gets $2 back (the original $1 bet plus $1 winnings). What is the probability that the player wins more than they lose if the player bets on red on 5 consecutive spins?

? Exercise 7.10.0.30

What is the probability that the player wins more than they lose if the player bets on red on 15 consecutive spins?

? Exercise 7.10.0.31

What is the probability that the player wins more than they lose if the player bets on red on 30 consecutive spins?

? Exercise 7.10.0.32

What is the probability that the player wins more than they lose if the player bets on red on 100 consecutive spins?

? Exercise 7.10.0.33

What is the probability that the player wins more than they lose if the player bets on red on 200 consecutive spins?

? Exercise 7.10.0.34

What is the probability that the player wins more than they lose if the player bets on red on 1,000 consecutive spins?

? Exercise 7.10.0.35

What is the probability that the player wins more than they lose if the player bets on red on 5,000 consecutive spins?

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7.11: Expected Value

Figure 7.11.1: The concept of expected value allows us to analyze games that involve randomness, like Roulette. (credit: “Roulette Table and Roulette Wheel in a Casino with People betting on numbers” by Marco Verch/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

- Calculate the expected value of an experiment.
- Interpret the expected value of an experiment.
- Use expected value to analyze applications.

The casino game roulette has dozens of different bets that can be made. These bets have different probabilities of winning but also have different payouts. In general, the lower the probability of winning a bet is, the more money a player wins for that bet. With so many options, is there one bet that’s “smarter” than the rest? What’s the best play to make at a roulette table? In this section, we’ll develop the tools we need to answer these questions.

Expected Value

Many experiments have numbers associated with their outcomes. Some are easy to define; if you roll 2 dice, the sum of the numbers showing is a good example. In some card games, cards have different point values associated with them; for example, in some forms of the game rummy, aces are worth 15 points; 10s, jacks, queens, and kings are worth 10; and all other cards are worth 5. The outcomes of casino and lottery games are all associated with an amount of money won or lost. These outcome values are used to find the expected value of an experiment: the mean of the values associated with the outcomes that we would observe over a large number of repetitions of the experiment. (See Conditional Probability and the Multiplication Rule for more on means.)

That definition is a little vague; How many is “a large number?” In practice, it depends on the experiment; the number has to be large enough that every outcome would be expected to appear at least a few times. For example, if we’re talking about rolling a standard 6-sided die and we note the number showing, a few dozen replications should be enough that the mean would be representative. Since the probability of each outcome is $\frac{1}{6}$, we would expect to see each outcome about 8 times over the course of 48 replications. However, if we’re talking about the Powerball lottery, where the probability of winning the jackpot is about $\frac{1}{292{,}000{,}000}$, we would need several *billion* replications to ensure that every outcome appears a few times. Luckily, we can find the theoretical expected value before we even run the experiment the first time.

FORMULA

Expected Value: If O represents an outcome of an experiment and $n(O)$ represents the value of that outcome, then the expected value of the experiment is:

$$\sum n(O) \times P(O)$$

where Σ semantics is the “sum,” meaning we add up the results of the formula that follows over all possible outcomes.

✓ Example 7.11.1: Finding Expected Values

Find the expected values of the following experiments.

1. Roll a standard 6-sided die and note the number showing.
2. Roll two standard 6-sided dice and note the sum of the numbers showing.
3. Draw a card from a well-shuffled standard deck of cards and note its rummy value (15 for aces; 10 for tens, jacks, queens, and kings; 5 for everything else).

Answer

1. **Step 1:** Let's start by writing out the PDF table for this experiment.

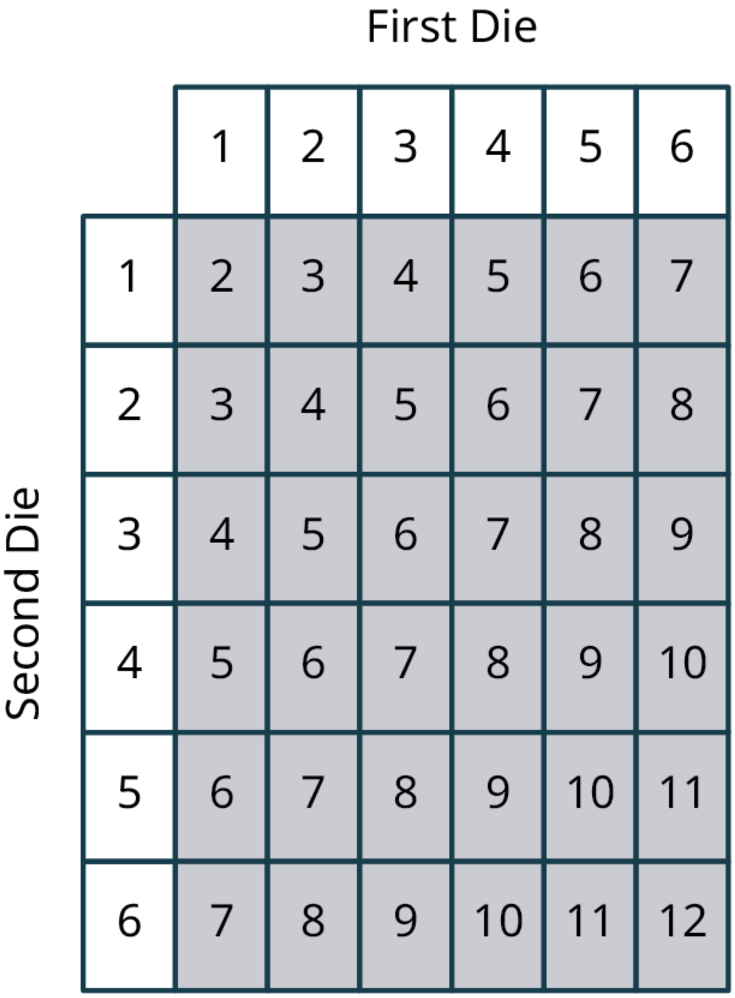
Value	Probability
1	$\frac{1}{6}$
2	$\frac{1}{6}$
3	$\frac{1}{6}$
4	$\frac{1}{6}$
5	$\frac{1}{6}$
6	$\frac{1}{6}$

Step 2: To find the expected value, we need to find $n(O) \times P(O)$ for each possible outcome in the table below.

Value	Probability	$n(O) \times P(O)$
1	$\frac{1}{6}$	$1 \times \frac{1}{6} = \frac{1}{6}$
2	$\frac{1}{6}$	$2 \times \frac{1}{6} = \frac{1}{3}$
3	$\frac{1}{6}$	$3 \times \frac{1}{6} = \frac{1}{2}$
4	$\frac{1}{6}$	$4 \times \frac{1}{6} = \frac{2}{3}$
5	$\frac{1}{6}$	$5 \times \frac{1}{6} = \frac{5}{6}$
6	$\frac{1}{6}$	$6 \times \frac{1}{6} = 1$

Step 3: We add all of the values in that last column: $\frac{1}{6} + \frac{1}{3} + \frac{1}{2} + \frac{2}{3} + \frac{5}{6} + 1 = \frac{7}{2} = 3.5$. So, the expected value of a single roll of a die is 3.5.

- Back in Example 7.18, we made this table of all of the equally likely outcomes (Figure 7.47):



	First Die 1	2	3	4	5	6
Second Die 1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Figure $7.11.2$

Step 1: Let's use Figure $7.11.2$ to create the PDF for this experiment, as shown in the following table:

Value	Probability
2	$\frac{1}{36}$
3	$\frac{1}{18}$
4	$\frac{1}{12}$
5	$\frac{1}{9}$
6	$\frac{5}{36}$
7	$\frac{1}{6}$
8	$\frac{5}{36}$
9	$\frac{1}{9}$
10	$\frac{1}{12}$
11	$\frac{1}{18}$
12	$\frac{1}{36}$

Step 2: To find the expected value, we need to find $n(O) \times P(O)$ for each possible outcome in the table below.

Value	Probability	$n(O) \times P(O)$

Value	Probability	$n(O) \times P(O)$
2	$\frac{1}{36}$	$\frac{1}{18}$
3	$\frac{1}{18}$	$\frac{1}{6}$
4	$\frac{1}{12}$	$\frac{1}{3}$
5	$\frac{1}{9}$	$\frac{5}{9}$
6	$\frac{5}{36}$	$\frac{5}{6}$
7	$\frac{1}{6}$	$\frac{7}{6}$
8	$\frac{5}{36}$	$\frac{10}{9}$
9	$\frac{1}{9}$	1
10	$\frac{1}{12}$	$\frac{5}{6}$
11	$\frac{1}{18}$	$\frac{11}{18}$
12	$\frac{1}{36}$	$\frac{1}{3}$

Step 3: We add all of the values in that last column: $\frac{1}{6}+\frac{1}{3}+\frac{1}{2}+\frac{2}{3}+\frac{5}{6}+1=\frac{7}{2}=3.5$. So, the expected value of a single roll of a die is 3.5 .

So, the expected value is 7.So, the expected value is 7.

Step 1: Let’s make a PDF table for this experiment. There are 3 events that we care about, so let’s use those events in the table below:

Event	Probability
{A}	$\frac{1}{13}$
{10, J, Q, K}	$\frac{4}{13}$
{2, 3, 4, 5, 6, 7, 8, 9}	$\frac{8}{13}$

Step 2: Let’s add a column to the following table for the values of each event:

Event	Probability	Value
{A}	$\frac{1}{13}$	15
{10, J, Q, K}	$\frac{4}{13}$	10
{2, 3, 4, 5, 6, 7, 8, 9}	$\frac{8}{13}$	5

Step 3: We’ll add the column for the product of the values and probabilities to the table below:

Event	Probability	Value	$n(O) \times P(O)$
{A}	$\frac{1}{13}$	15	$\frac{15}{13}$
{10, J, Q, K}	$\frac{4}{13}$	10	$\frac{40}{13}$
{2, 3, 4, 5, 6, 7, 8, 9}	$\frac{8}{13}$	5	$\frac{40}{13}$

Step 4: We’ll find the sum of the last column: $\frac{15}{13}+\frac{40}{13}+\frac{40}{13}=\frac{95}{13}\approx 7.3$. Thus, the expected *Rummy* value of a randomly selected card is about 7.3.

Your Turn 7.11.1

Find the expected value of the number showing when you roll a special 6-sided die with faces $\{1, 1, 2, 3, 5, 8\}$.

Find the expected value of the number of heads showing if you flip a coin 3 times.

You are about to play a game, where you flip a coin 3 times. If all 3 flips result in heads, you win \$20. If you get 2 heads, you win \$10. If you flip 1 or 0 heads, you win nothing. What is the expected value of your winnings?

Let's make note of some things we can learn from Example 7.37. First, as Exercises 1 and 3 demonstrate, the expected value of an experiment might not be a value that could come up in the experiment. Remember that the expected value is interpreted as a mean, and the mean of a collection of numbers doesn't have to actually be one of those numbers.

Second, looking at Exercise 1, the expected value (3.5) was just the mean of the numbers on the faces of the die: $\frac{1+2+3+4+5+6}{6} = 3.5$. This is no accident! If we break that fraction up using the addition in the numerator, we get $\frac{1}{6}+\frac{2}{6}+\frac{3}{6}+\frac{4}{6}+\frac{5}{6}+\frac{6}{6}$, which we can rewrite as $1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6}$. That's exactly the computation we did to find the expected value! In fact, expected values can always be treated as a special kind of mean called a **weighted mean**, where the weights are the probabilities associated with each value. When the probabilities are all equal, the weighted mean is just the regular mean.

Interpreting Expected Values

As we noted, the expected value of an experiment is the mean of the values we would observe if we repeated the experiment a large number of times. (This interpretation is due to an important theorem in the theory of probability called the Law of Large Numbers.) Let's use that to interpret the results of the previous example.

Example 7.11.2: Interpreting Expected Values

Interpret the expected values of the following experiments.

1. Roll a standard 6-sided die and note the number showing.
2. Roll 2 standard 6-sided dice and note the sum of the numbers showing.
3. Draw a card from a well-shuffled standard deck of cards and note its *Rummy* value (15 for aces; 10 for tens, jacks, queens, and kings; 5 for everything else).

Answer

1. If you roll a standard 6-sided die many times, the mean of the numbers you roll will be around 3.5.
2. If you roll a pair of standard 6-sided dice many times, the mean of the sums of the numbers you roll will be about 7.
3. If you draw a card from a well-shuffled deck many times, the mean of the *Rummy* values of the cards would be around 7.3.

Your Turn 7.11.2

Interpret the expected value of the number showing when you roll a special 6-sided die with faces $\{1, 1, 2, 3, 5, 8\}$.

Interpret the expected value of the number of heads showing if you flip a coin 3 times.

You are about to play a game where you flip a coin 3 times. If all 3 flips result in heads, you win \$20. If you get 2 heads, you win \$10. If you flip 1 or 0 heads, you win nothing. Interpret the expected value of your winnings.

Who Knew?: Pascal's Wager

The French scholar Blaise Pascal (1623–1662) was among the earliest mathematicians to study probabilities, and was the first to accurately describe and compute expected values. In his book *Pensées* (*Thoughts*), he turned the analysis of expected values to his belief in the Christian God. He said that there is no way for people to establish the probability that God exists, but since the "winnings" on a bet that God exists (and that you then lead your life accordingly) are essentially infinite, the expected value of taking that bet is always positive, no matter how unlikely it is that God exists.

Using Expected Value

Now that we know how to find and interpret expected values, we can turn our attention to using them. Suppose someone offers to play a game with you. If you roll a die and get a 6, you get $10. However, if you get a 5 or below, you lose $1. Is this a game you'd want to play? Let's look at the expected value: The probability of winning is $\frac{1}{6}$ and the probability of losing is $\frac{5}{6}$, so the expected value is $\$10 \times \frac{1}{6} + (-\$1) \times \frac{5}{6} = \frac{5}{6} \approx \$0.83$. That means, on average, you'll come out ahead by about 83 cents every time you play this game. It's a great deal! On the other hand, if the winnings for rolling a 6 drop to $3, the expected value becomes $\$3 \times \frac{1}{6} + (-\$1) \times \frac{5}{6} = -\frac{1}{3} \approx -\$0.33$, meaning you should expect to lose about 33 cents on average for every time you play. Playing that game is not a good idea! In general, this is how casinos and lottery corporations make money: Every game has a negative expected value for the player.

Who Knew?: Expected Values in Football

In the 21st century, data analytics tools have revolutionized the way sports are coached and played. One tool in particular is used in football at crucial moments in the game. When a team faces a fourth down (the last possession in a series of four possessions, a fairly common occurrence), the coach faces a decision: Run one play to try to gain a certain number of yards, or kick the ball away to the other team. Here's the interesting part of the decision: If the team "goes for it" and runs the play and they are successful, then they keep possession of the ball and can continue in their quest to score more points. If they are unsuccessful, then they lose possession of the ball, giving the other team an opportunity to score points. If, instead, the team punts, or kicks the ball away, then the other team gets possession of the ball, but in a worse position for them than if the original team goes for it and fails. To analyze this situation, data analysts have generated empirical probabilities for every fourth down situation, and computed the expected value (in terms of points) for each decision. Coaches frequently use those calculations when they decide which option to take!

People in Mathematics: Pierre de Fermat and Blaise Pascal

In 1654, a French writer and amateur mathematician named Antoine Gombaud (who called himself the Chevalier du Mére) reached out to his gambling buddy Blaise Pascal to answer a question that he'd read about called the "problem of points." The question goes like this: Suppose you're playing a game that is scored using points, and the first person to earn 5 points is the winner. The game is interrupted with the score 4 points to 2. If the winner stood to win $100, how should the prize money be divided between the players? Certainly the person who is 1 point away from victory should get more, but *how much* more?

We have developed tools in this section to answer this question. At its heart, it's a question about conditional probabilities and expected value. At the time that Pascal first started thinking about it, though, those ideas hadn't yet been invented. Pascal reached out to a colleague named Pierre de Fermat, and over the course of a couple of months, their correspondence with each other would eventually solve the problem. In the process, they first described conditional probabilities and expected values!

Apart from their work in probability, these men are famous for other work in mathematics (and, in Pascal's case, philosophy and physics). Fermat is remembered for his work in geometry and in number theory. After his death, the statement of what came to be called "Fermat's Last Theorem" was discovered scribbled in the margin of a book, with the note that Fermat had discovered a "marvelous proof that this margin is too small to contain." The theorem says that any equation of the form $a^n + b^n =$cnan+bn=cn has no positive integer solutions if $n \geq 3$. No proof of that theorem was discovered until 1994, when Andrew Wiles used computers and new branches of geometry to finally prove the theorem!

Pascal is remembered for the "arithmetical triangle" that is named for him (though he wasn't the first person to discover it; see the section on the binomial distribution for more), as well as work in geometry. In physics, Pascal worked on hydrodynamics and air pressure (the SI unit for pressure is named for him), and in philosophy, Pascal advocated for a mathematical approach to philosophical problems.

✓ Example $7.11.3$: Using Expected Values

In the casino game keno, a machine chooses at random 20 numbers between 1 and 80 (inclusive) without replacement. Players try to predict which numbers will be chosen. Players don't try to guess all 20, though; generally, they'll try to predict between 1 and 10 of the chosen numbers. The amount won depends on the number of guesses they made and the number of guesses that were correct.

1. At one casino, a player can try to guess just 1 number. If that number is among the 20 selected, the player wins \$2; otherwise, the player loses \$1. What is the expected value?
2. At the same casino, if a player makes 2 guesses and they're both correct, the player wins \$14; otherwise, the player loses \$1. What is the expected value?
3. Players can also make 3 guesses. If 2 of the 3 guesses are correct, the player wins \$1. If all 3 guesses are correct, the player wins \$42. Otherwise, the player loses \$1. What is the expected value?
4. Which of these games is the best for the player? Which is the best for the casino?

Answer

1. There are 20 winning numbers out of 80, so if we try to guess one of them, the probability of guessing correctly is $\frac{20}{80} = \frac{1}{4}$. The probability of losing is then $\frac{3}{4}$, and so the expected value is $\$2 \times \frac{1}{4} + (-\$1) \times \frac{3}{4} = -\$0.25$.
2. There are msub20C2=19020C2=190 winning choices out of msub80C2=3,16080C2=3,160 total ways to choose 2 numbers. So, the probability of winning is $\frac{190}{3{,}160}$ and the probability of losing is $\frac{3{,}160-190}{3{,}160} = \frac{2{,}970}{3{,}160}$. So, the expected value of the game is $\$14 \times \frac{190}{3{,}160} + (-\$1) \times \frac{2970}{3160} \approx -\$0.10$.
3. **Step 1:** Let's start with the big prize. There are msub20C3=1,14020C3=1,140 ways to correctly guess 3 winning numbers out of msub80C3=82,16080C3=82,160 ways to guess three numbers total. That means the probability of winning the big prize is $\frac{1{,}140}{82{,}160} \approx 0.01388$.

   Step 2: Let's find the probability of the second prize. The denominator is the same: 82,160. Let's figure out the numerator. To win the second prize, the player must pick 2 of the 20 winning numbers and one of the 60 losing numbers. The number of ways to do that can be found using the Multiplication Rule for Counting: there are msub20C2=19020C2=190 ways to pick 2 winning numbers and 60 ways to pick 1 losing number, so there are $190 \times 60 = 11{,}400$ ways to win the second prize. So, the probability of winning that second prize is $\frac{11{,}400}{82{,}160} \approx 0.13875$.

   Step 3: Since the overall probability of winning is $\frac{1{,}140}{82{,}160} + \frac{11{,}400}{82{,}160} = \frac{12{,}540}{82{,}160} \approx 0.15263$, the probability of losing must be $1 - 0.15263 = 0.84737$. So, the expected value is $\$42 \times 0.01388 + \$1 \times 0.13875 + (-\$1) \times 0.84737 \approx -\$0.13$.
4. The bet that's the best for the player is the one with the highest expected value for the player, which is guessing two numbers. The best one for the casino is the one with the lowest expected value for the player, which is guessing one number.

Your Turn 7.11.3

The casino game craps involves rolling 2 standard 6-sided dice. While the main game involves repeated rolls of the dice, players can also bet on the outcomes of single rolls. Find the expected values of the following three \$1 bets. Then decide which bet is the best for the player and which bet is the best for the casino.

Players can bet on the next roll of the dice being a sum of 7. Winners get \$4 and losers lose \$1.

Players can bet on the next roll of the dice being a sum of 12. Winners get \$30 and losers lose \$1

Players can bet that the next roll of the dice will be "any craps" (a sum of 2, 3, or 12). Winners get \$7 and losers lose \$1.

WORK IT OUT: Make Your Own Lottery

By yourself or with a partner, devise your own lottery scheme. Assume you would have access to one or more machines that choose numbers randomly. What will a lottery draw look like? How many numbers are players choosing from? How many will be drawn? Will they be drawn with replacement or without replacement? What conditions must be met for a player to win first or second (or more!) prize? Once you've decided that, decide the payoff structure for winners, and how much the game will cost to play. Try to make the game enticing enough that people will want to play it, but with enough negative expected value that the lottery will make money. Aim for the expected value to be about −0.25 times the cost of playing the game.

Check Your Understanding

You are about to roll a 20-sided die with faces labeled as follows: 5 faces have a 1, 6 faces have a 3, 4 faces have a 5, 3 faces have a 7, and 2 faces have a 9.

1. What is the expected value of the number showing on the die after it's rolled?

2. Interpret your answer.

You are about to play a game in which you draw 3 ping-pong balls without replacement from a barrel. The barrel contains 6 green balls and 4 red balls. If all 3 of your selections are green, you win $5. If 2 of the 3 are green, you win $1. If 2 or more of your selections are red, you lose $5.

3. What is the expected value of this game?

4. Interpret your answer.

5. Is it advantageous to you to play the game? How do you know?

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7.11.0: Exercises

You roll a standard 6-sided die and win points equal to the square of the number shown.

Exercise 7.11.0.1

What's the expected value of the number of points you win?

Exercise 7.11.0.2

Interpret your answer.

In the classic board game *The Game of Life*, players have the chance to play the market. A spinner with 10 equally likely spaces is spun to choose a random number. If the result is 3 or less, the player loses $25,000. If the result is 7 or more, the player wins $50,000. If the result is a 4, 5, or 6, the player doesn't win or lose anything.

Exercise 7.11.0.3

What is the expected value of playing the market?

Exercise 7.11.0.4

Interpret the answer.

The Game of Life players also occasionally have the opportunity to speculate. Players choose any 2 of the 10 numbers on the spinner and then give it a spin. If one of their numbers is chosen, they win $140,000; if not, they lose $10,000.

Exercise 7.11.0.5

What is the expected value of this speculation?

Exercise 7.11.0.6

Interpret your answer.

Exercise 7.11.0.7

Which is better for *The Game of Life* players: playing the market or speculating? How do you know?

A charitable organization is selling raffle tickets as a fundraiser. They intend to sell 5,000 tickets at $10 each. One ticket will be randomly selected to win the grand prize of a new car worth $35,000.

Exercise 7.11.0.8

What is the expected value of a single ticket?

Exercise 7.11.0.9

Interpret your answer.

Exercise 7.11.0.10

The organization is worried they won't be able to sell all the tickets, so they announce that, in addition to the grand prize, they will offer 10 second prizes of $500 in cash. What is the new expected value of a single ticket?

? Exercise 7.11.0.11

Interpret your answer.

In the following exercises involve randomly selecting golf balls from a bucket. The bucket contains 4 yellow balls (numbered 1-4) and 6 white balls (numbered 1-6).

? Exercise 7.11.0.12

If you draw a single ball, what is the expected number of yellow balls selected?

? Exercise 7.11.0.13

Suppose you draw 2 balls *with* replacement.

a. Give a PDF table for the possible outcomes for the number of yellow balls selected.
b. What is the expected number of yellow balls selected?

? Exercise 7.11.0.14

Suppose you draw 2 balls *without* replacement.

a. Give a PDF table for the possible outcomes for the number of yellow balls selected.
b. What is the expected number of yellow balls selected?

? Exercise 7.11.0.15

Suppose you draw 3 balls *with* replacement.

a. Give a PDF table for the possible outcomes for the number of yellow balls selected.
b. What is the expected number of yellow balls selected?

? Exercise 7.11.0.16

Suppose you draw 3 balls *without* replacement.

a. Give a PDF table for the possible outcomes for the number of yellow balls selected.
b. What is the expected number of yellow balls selected?

? Exercise 7.11.0.17

If you draw a single ball, what is the expected value of the number on the ball?

? Exercise 7.11.0.18

Suppose you draw 2 balls *with* replacement.

a. Give a PDF table for the possible outcomes for the sum of the numbers on the selected balls.
b. What is the expected sum of the numbers on the balls?

? Exercise 7.11.0.19

Suppose you draw 2 balls *without* replacement.

a. Give a PDF table for the possible outcomes for the sum of the numbers on the selected balls.
b. What is the expected sum of the numbers on the balls?

The following exercises deal with the game “Punch a Bunch,” which appears on the TV game show *The Price Is Right*. In this game, contestants have a chance to punch through up to 4 paper circles on a board; behind each circle is a card with a dollar amount printed on it. There are 50 of these circles; the dollar amounts are given in this table:

Dollar Amount	Frequency
\$25,000	1
\$10,000	2
\$5,000	4
\$2,500	8
\$1,000	10
\$500	10
\$250	10
\$100	5

Contestants are shown their selected dollar amounts one at a time, in the order selected. After each is revealed, the contestant is given the option of taking that amount of money or throwing it away in favor of the next amount. (You can watch the game being played in the video Playing "Punch a Bunch.") Anita is playing “Punch a Bunch” and gets 2 punches.

? Exercise 7.11.0.20

If Anita got \$500 on her first punch, what’s the expected value of her second punch?

? Exercise 7.11.0.21

If Anita got \$500 on her first punch, should she throw out her \$500 and take the results of her second punch? How do you know?

? Exercise 7.11.0.22

If Anita got \$1,000 on her first punch, what’s the expected value of her second punch?

? Exercise 7.11.0.23

If Anita got \$1,000 on her first punch, should she throw out her \$1,000 and take the results of her second punch? How do you know?

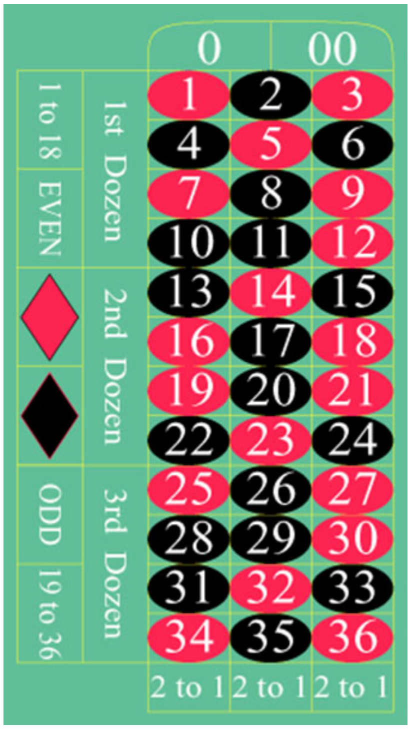
? Exercise 7.11.0.24

If Anita got \$2,500 on her first punch, what’s the expected value of her second punch?

? Exercise 7.11.0.25

If Anita got \$2,500 on her first punch, should she throw out her \$2,500 and take the results of her second punch? How do you know?

The following exercises are about the casino game roulette. In this game, the dealer spins a marble around a wheel that contains 38 pockets that the marble can fall into. Each pocket has a number (each whole number from 0 to 36, along with a “double zero”) and a color (0 and 00 are both green; the other 36 numbers are evenly divided between black and red). Players make bets on which number (or groups of numbers) they think the marble will land on. The figure shows the layout of the numbers and colors, as well as some of the bets that can be made.


0
00
1 2 3
4 5 6
7 8 9
10 11 12
13 14 15
16 17 18
19 20 21
22 23 24
25 26 27
28 29 30
31 32 33
34 35 36
2 to 1
2 to 1
2 to 1
1st Dozen
2nd Dozen
3rd Dozen
1 to 18
EVEN
ODD
19 to 36

Roulette Table (credit: "American Roulette Table Layout" by Film8ker/Wikimedia Commons, Public Domain)

? Exercise 7.11.0.26

If a player makes a $1 bet on a single number, they win $35 if that number comes up, but lose $1 if it doesn't. What is the expected value of this bet?

? Exercise 7.11.0.27

Interpret your answer to the previous question.

? Exercise 7.11.0.28

Suppose a player makes the $1 bet on a single number in 5 consecutive spins. What is the expected value of this series of bets? (Hint: use the Binomial Distribution.)

? Exercise 7.11.0.29

Interpret your answer to the previous question.

? Exercise 7.11.0.30

If a player makes a $10 bet on first dozen, they win $20 if one of the numbers 1–12 comes up but lose $10 otherwise. What is the expected value of this bet?

? Exercise 7.11.0.31

Interpret your answer to the previous question.

? Exercise 7.11.0.32

Suppose a player makes the $10 bet on first dozen in 4 consecutive spins. What is the expected value of that series of bets?

? Exercise 7.11.0.33

Interpret your answer to the previous question.

? Exercise 7.11.0.34

If a player makes a $10 basket bet, they win $60 if 0, 00, 1, 2, or 3 come up but lose $10 otherwise. What is the expected value of this bet?

? Exercise 7.11.0.35

Interpret your answer to previous question.

? Exercise 7.11.0.36

Which is better for the player: a $10 first dozen bet or a $10 basket bet? How do you know?

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SECTION OVERVIEW

7.12: Chapter Summary

7.12.0: Key Terms

7.12.1: Key Concepts

7.12.2: Formula Review

7.12.3: Projects

7.12.4: Chapter Review

7.12.5: Chapter Test

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7.12.0: Key Terms

7.1 The Multiplication Rule for Counting

- combinatorics
- Multiplication Rule for Counting (Fundamental Counting Principle)

7.2 Permutations

- permutation
- factorial

7.3 Combinations

- combination

7.4 Tree Diagrams, Tables, and Outcomes

- experiment
- replication
- sample space
- independent/dependent

7.5 Basic Concepts of Probability

- event
- probability
- theoretical probability
- empirical probability
- subjective probability

7.7 What Are the Odds?

- odds (for/against)

7.8 The Addition Rule for Probability

- mutually exclusive

7.9 Conditional Probability and the Multiplication Rule

- conditional probability

7.10 The Binomial Distribution

- binomial experiment
- probability density function (PDF)
- cumulative distribution function (CDF)

7.11 Expected Value

- expected value

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7.12.1: Key Concepts

7.1 The Multiplication Rule for Counting

- The Multiplication Rule for Counting is used to count large sets.

7.2 Permutations

- Using the Multiplication Rule for Counting to enumerate permutations.
- Simplifying and computing expressions involving factorials.
- Using factorials to count permutations.

7.3 Combinations

- Permutations are used to count subsets when order matters; combinations work when order doesn't matter.
- Combinations can also be computed using factorials.

7.4 Tree Diagrams, Tables, and Outcomes

- We identify the sample space of an experiment by identifying all of its possible outcomes.
- Tables can help us find a sample space by keeping the possible outcomes organized.
- Tree diagrams provide a visualization of the sample space of an experiment that involves multiple stages.

7.5 Basic Concepts of Probability

- The theoretical probability of an event is the ratio of the number of equally likely outcomes in the event to the number of equally likely outcomes in the sample space.
- Empirical probabilities are computed by repeating the experiment many times, and then dividing the number of replications that result in the event of interest by the total number of replications.
- Subjective probabilities are assigned based on subjective criteria, usually because the experiment can’t be repeated and the outcomes in the sample space are not equally likely.
- The probability of the complement of an event is found by subtracting the probability of the event from one.

7.6 Probability with Permutations and Combinations

- We use permutations and combinations to count the number of equally likely outcomes in an event and in a sample space, which allows us to compute theoretical probabilities.

7.7 What Are the Odds?

- Odds are computed as the ratio of the probability of an event to the probability of its compliment.

7.8 The Addition Rule for Probability

- The Addition Rule is used to find the probability that one event or another will occur when those events are mutually exclusive.
- The Inclusion/Exclusion Principle is used to find probabilities when events are not mutually exclusive.

7.9 Conditional Probability and the Multiplication Rule

- Conditional probabilities are computed under the assumption that the condition has already occurred.
- The Multiplication Rule for Probability is used to find the probability that two events occur in sequence.

7.10 The Binomial Distribution

- Binomial experiments result when we count the number of successful outcomes in a fixed number of repeated, independent trials with a constant probability of success.
- The binomial distribution is used to find probabilities associated with binomial experiments.
- Probability density functions (PDFs) describe the probabilities of individual outcomes in an experiment; cumulative distribution functions (CDFs) give the probabilities of ranges of outcomes.

7.11 Expected Value

- The expected value of an experiment is the sum of the products of the numerical outcomes of an experiment with their corresponding probabilities.
- The expected value of an experiment is the most likely value of the average of a large number of replications of the experiment.

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7.12.2: Formula Review

7.2 Permutations

- $${}_nP_r = \frac{n!}{(n-r)!}$$

7.3 Combinations

- The formula for counting combinations is:

$$\text{msubmsub}C_r = \frac{n!}{r!(n-r)!}$$

7.5 Basic Concepts of Probability

- For an experiment whose sample space S consists of equally likely outcomes, the *theoretical probability* of the event E is the ratio $P(E) = \frac{n(E)}{n(S)}$ where $n(E)$ and $n(S)$ denote the number of outcomes in the event and in the sample space, respectively.
- $P(E) = 1 - P(E')$

7.7 What Are the Odds?

- For an event E,

$$\begin{gathered}\text{oddsfor } E = n(E) : n(E') = P(E) : P(E') = P(E) : (1 - P(E)) \\ \text{oddsagainst } E = n(E') : n(E) = P(E') : P(E) = (1 - P(E)) : P(E)\end{gathered}$$

- If the odds in favor of E are $A : B$, then $P(E) = \frac{A}{A+B}$.

7.8 The Addition Rule for Probability

- If E and F are mutually exclusive events, then

$$P(E \text{ or } F) = P(E) + P(F)$$

.

- If E and F are events that contain outcomes of a single experiment, then

$$P(E \text{ or } F) = P(E) + P(F) - P(E \text{ and } F)$$

.

7.9 Conditional Probability and the Multiplication Rule

- If E and F are events associated with the first and second stages of an experiment, then $P(E \text{ and } F) = P(E) \times P(F|E)$.

7.10 The Binomial Distribution

- Suppose we have a binomial experiment with n trials and the probability of success in each trial is p. Then:

$$P(\text{number of successesis } a) = {}_nC_a \times p^a \times (1-p)^{n-a}$$

7.11 Expected Value

- where Σ stands for the sum, meaning we add up the results of the formula that follows over all possible outcomes.

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7.12.3: Projects

1. The Binomial Distribution is one of many examples of a *discrete probability distribution*. Other examples include the Geometric, Hypergeometric, Multinomial, Poisson, and Negative Binomial Distributions. Choose one of these distributions, and find out what makes it different from the Binomial Distribution. In what situations can it be applied? How is it used? Once you have an idea of how it's used, write a series of five questions like the ones in this chapter that can be answered with that distribution, and find the answers.
2. *Binomial* is a word that also comes up in algebra; the word describes polynomials with two terms. At first glance, there isn't much to indicate that these two uses of the word are related, but it turns out there is a connection. Explore the connection between the Binomial Distribution and the algebraic concept of binomial expansion, (the process of multiplying out expressions like $(x+y)^n$ for a positive whole number n). Search for a connection with the mathematical object known as Pascal's Triangle.
3. Hazard is a dice game that was mentioned in Chaucer's *Canterbury Tales*. It was a popular game of chance played in taverns and coffee houses well into the 18th century; its popularity at the time of the foundation of probability theory means that it was a common example in early texts on finding expected values and probabilities. Find the rules of the game, and get some practice playing it. Then, analyze the choices that the caster gets to make, and decide which is most advantageous, using the language of expected values.

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7.12.4: Chapter Review

The Multiplication Rule for Counting

Exercise $7.12.4.1$

You are booking a round trip flight for vacation. If there are 4 outbound flight options and 7 return flight options, how many different options do you have?

Exercise $7.12.4.2$

You are putting together a social committee for your club. You'd like broad representation, so you will choose one person from each class. If there are 8 seniors, 12 juniors, 10 sophomores, and 6 first-years, how many committees are possible?

Exercise $7.12.4.3$

The Big Breakfast Platter at Jimbo's Sausage Haus gives you your choice of 4 flavors of sausage, 5 preparations for eggs, 3 different potato options, and 4 different breads. If you choose one of each, how many different Big Breakfast Platters can be selected?

Exercise $7.12.4.4$

The multiple-choice quiz you're about to take has 10 questions with 4 choices for each. How many ways are there to fill out the quiz?

Permutations

Exercise $7.12.4.5$

Compute $\frac{8!}{2!3!3!}$.

Exercise $7.12.4.6$

Compute $\frac{12!}{8!3!}$.

Exercise $7.12.4.7$

Compute $\frac{211!}{210!}$

Exercise $7.12.4.8$

Compute ${}_5P_3$.

Exercise $7.12.4.9$

Compute ${}_{15}P_3$.

Exercise $7.12.4.10$

Compute ${}_{22}P_5$.

? Exercise 7.12.4.11

As you plan your day, you see that you have 6 tasks on your to-do list. You'll only have time for 5 of those. How many schedules are possible for you today?

? Exercise 7.12.4.12

As captain of your intramural softball team, you are responsible for setting the 10-person batting order for the team. If there are 12 people on the team, how many batting orders are possible?

Combinations

? Exercise 7.12.4.13

If you're trying to decide which 4 of your 12 friends to invite to your apartment for a dinner party, are you using permutations or combinations?

? Exercise 7.12.4.14

If you're trying to decide which of your guests sits where at your table, are you using permutations or combinations?

? Exercise 7.12.4.15

Compute ${}_7C_4$.

? Exercise 7.12.4.16

Compute ${}_{13}C_8$.

? Exercise 7.12.4.17

How many ways are there to draw a hand of 8 cards from a deck of 16 cards?

? Exercise 7.12.4.18

In a card game with 4 players and a deck of 12 cards, how many ways are there to deal out the four 3-card hands?

Tree Diagrams, Tables, and Outcomes

? Exercise 7.12.4.19

If you draw a card at random from a standard 52-card deck and note its suit, what is the sample space?

? Exercise 7.12.4.20

If you draw 2 cards at random from a standard 52-card deck and note the 2 suits (without paying attention to the order), what is the sample space?

? Exercise 7.12.4.21

If you draw 2 *Scrabble* tiles in order without replacement from a bag containing E, E, L, S, what is the sample space?

Exercise 7.12.4.22

If you draw 2 *Scrabble* tiles without replacement and ignoring order from a bag containing E, E, L, S, what is the sample space?

Exercise 7.12.4.23

If you draw 2 *Scrabble* tiles without replacement and ignoring order from a bag containing E, E, L, S, what is the sample space?

Exercise 7.12.4.24

If you draw 2 *Scrabble* tiles with replacement and ignoring order from a bag containing E, E, L, S, what is the sample space?

Basic Concepts of Probability

Exercise 7.12.4.25

If you read that the probability of flipping 10 heads in a row is $\frac{1}{1}024$, is that probability most likely theoretical, empirical, or subjective?

Exercise 7.12.4.26

If someone tells you that there is a 40% chance that a Democrat wins the U.S. Presidential election in 2132, is that probability most likely theoretical, empirical, or subjective?

Exercise 7.12.4.27

If your professor says that you have a 20% chance of getting an A in her class because 20% of her students historically have earned As, is that probability most likely theoretical, empirical, or subjective?

In the following exercises, you are about to roll a standard 12-sided die (with faces labeled 1–12).

Exercise 7.12.4.28

What is the probability of rolling a negative number?

Exercise 7.12.4.29

What is the probability of rolling a number less than 20?

Exercise 7.12.4.30

What is the probability of rolling an 11?

Exercise 7.12.4.31

What is the probability of rolling a number less than 7?

Exercise 7.12.4.32

What is the probability of *not* rolling an 11?

Exercise 7.12.4.33

What is the probability of rolling a multiple of 4?

Exercise 7.12.4.34

Over the last 30 years, it has rained 12 times on May 1. What empirical probability would you assign to the event "it rains next May 1"?

Probability with Permutations and Combinations

In the following exercises, you’re drawing cards from a special deck of cards containing $2\heartsuit$, $2\clubsuit$, $2\diamondsuit$, $2\spadesuit$, $3\heartsuit$, $3\clubsuit$, $3\spadesuit$, $4\heartsuit$, $4\clubsuit$, $5\heartsuit$.

Exercise 7.12.4.35

If you draw 4 cards without replacement, what is the probability of drawing a 2, 3, 4, and 5 in order?

Exercise 7.12.4.36

If you draw 4 cards without replacement, what is the probability of drawing a 2, 3, 4, and 5 in *any* order?

Exercise 7.12.4.37

If you draw 3 cards without replacement, what is the probability that you draw a $\heartsuit$, a $\clubsuit$, and a $\heartsuit$, in order?

Exercise 7.12.4.38

If you draw 3 cards without replacement, what is the probability that you draw 2 $\heartsuit$ and 1 $\clubsuit$, in any order?

What Are the Odds?

Exercise 7.12.4.39

If you roll a standard 20-sided die (with faces numbered 1–20), what are the odds against rolling a number less than 5?

Exercise 7.12.4.40

If you roll a standard 20-sided die (with faces numbered 1–20), what are the odds in favor of rolling greater than a 5?

Exercise 7.12.4.41

If $P(E) = \frac{4}{7}$, what are the odds in favor of E?

Exercise 7.12.4.42

If $P(E) = \frac{5}{1}7$, what are the odds against E?

The Addition Rule for Probability

In the following exercises, you’re drawing a single card from a special deck of cards containing $2\heartsuit$, $2\clubsuit$, $2\diamondsuit$, $2\spadesuit$, $3\heartsuit$, $3\clubsuit$, $3\spadesuit$, $4\heartsuit$, $4\clubsuit$, $5\heartsuit$.

? Exercise 7.12.4.43

What is the probability of drawing a 2 or a 3?

? Exercise 7.12.4.44

What is the probability of drawing a ♣ or a ♠?

? Exercise 7.12.4.45

What is the probability of drawing a 2 or a $\heartsuit$?

? Exercise 7.12.4.46

What is the probability of drawing an even number or a ♣?

Conditional Probability and the Multiplication Rule

In the following exercises, you're drawing from a special deck of cards containing $2\heartsuit$, $2\clubsuit$, $2\diamondsuit$, $2\spadesuit$, $3\heartsuit$, $3\clubsuit$, $3\spadesuit$, $4\heartsuit$, $4\clubsuit$, $5\heartsuit$.

? Exercise 7.12.4.47

If you draw a single card, what is:

a. $P(\text{draw a 2})$
b. $P(\text{draw a 2}|\text{draw a } \heartsuit)$
c. $P(\text{draw a 2}|\text{draw a } \spadesuit)$

? Exercise 7.12.4.48

If you draw a single card, what is:

a. $P(\text{draw a } \heartsuit)$
b. $P(\text{draw a } \heartsuit|\text{draw a 3})$
c. $P(\text{draw a } \heartsuit|\text{draw a 2})$

In the following exercises, you are playing the following game that involves rolling 2 dice, one at a time. First, you roll a standard 6-sided die. If the result is a 4 or less, your second roll uses a standard 4-sided die. If the result of the first roll is a 5 or 6, your second roll uses a standard 6-sided die. Find these probabilities:

? Exercise 7.12.4.49

$P(\text{first roll is a 3 and second roll is a 3})$

? Exercise 7.12.4.50

$P(\text{first roll is a 6 and second roll is a 6})$

? Exercise 7.12.4.51

$P(\text{second roll is a 6})$

Exercise 7.12.4.52

$P(\text{second roll is a } 1)$

The Binomial Distribution

In the following exercises, decide whether the described experiment is a binomial experiment. If it is, identify the number of trials and the probability of success in each trial. If it isn't, explain why it isn't.

Exercise 7.12.4.53

Draw 5 cards with replacement from a standard deck and count the number of ♠.

Exercise 7.12.4.54

Draw 5 cards without replacement from a standard deck and count the number of ♠.

Exercise 7.12.4.55

Draw cards from a standard deck and count how many cards are chosen before the first ♠ appears.

In the following exercises, you are about to roll a standard 20-sided die. Round answers to 4 decimal places.

Exercise 7.12.4.56

Suppose you are going to roll the die 4 times. Give a full PDF table for the number of times a number greater than 16 appears.

Exercise 7.12.4.57

If you roll the die 10 times, what is the probability that a number between 1 and 5 (inclusive) comes up exactly once?

Exercise 7.12.4.58

If you roll the die 40 times, what is the probability that 20 comes up fewer than 2 times?

Exercise 7.12.4.59

If you roll the die 40 times, what is the probability that 20 comes up 4 or more times?

Exercise 7.12.4.60

If you roll the die 100 times, what is the probability that the number of times the die lands on something less than or equal to 7 is between 30 and 35 (inclusive)?

Exercise 7.12.4.61

If you roll the die 100 times, what is the probability that the number of times the die lands on something less than or equal to 7 is exactly 36?

Exercise 7.12.4.62

If you roll the die 100 times, what is the probability that the die lands on 20 between 5 and 8 times, inclusive?

Expected Value

Exercise 7.12.4.63

You are playing a game where you roll a pair of standard 6-sided dice. You win \$32 if you get a sum of 12, and lose \$1 otherwise. What is the expected value of this game?

Exercise 7.12.4.64

Interpret your answer.

You are playing a game where you roll a standard 12-sided die 4 times. If you roll 12 four times, you win \$1,000. If you roll 12 three times, you win \$100. If you roll 12 twice, you win \$10. If you roll 12 one time, you don't win or lose anything. If you roll don't roll a single 12, you lose \$1.

Exercise 7.12.4.65

What is the expected value of this game?

Exercise 7.12.4.66

Interpret your answer.

Exercise 7.12.4.67

Which game would be better to play? Why?

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7.12.5: Chapter Test

Each of the following exercises involve drawing a *Scrabble* tile from a bag. These tiles are labeled with a letter and a point value, as follows: A(1), C(3), D(2), E(1), E(1), J(8), K(5), O(1), R(1), R(1).

? Exercise 7.12.5.1

How many ways are there to draw a vowel and then a consonant from the bag?

? Exercise 7.12.5.2

How many ways are there to draw a tile worth an even number of points and then a tile worth an odd number of points from the bag?

? Exercise 7.12.5.3

How many ways are there to draw 4 tiles from the bag without replacement, if order matters?

? Exercise 7.12.5.4

How many ways are there to draw 4 consonants from the bag without replacement, if order matter?

? Exercise 7.12.5.5

How many ways are there to draw 4 tiles from the bag with replacement, if order does not matter?

? Exercise 7.12.5.6

How many ways are there to draw 4 consonants from the bag with replacement, if order does not matter?

? Exercise 7.12.5.7

Give the sample space of the experiment that asks you to draw 2 tiles from the bag with replacement and note their point values, where order doesn't matter. Give the outcomes as ordered pairs.

? Exercise 7.12.5.8

Give the sample space of the experiment that asks you to draw 2 tiles from the bag with replacement and note their point values, where order doesn't matter. Give the outcomes as ordered pairs.

? Exercise 7.12.5.9

If you draw a single tile from the bag, what is the probability that it's an E?

? Exercise 7.12.5.10

If you draw a single tile from the bag, what is the probability that it's *not* an A?

? Exercise 7.12.5.11

If you draw 3 tiles from the bag without replacement, what is the probability that they spell RED, in order?

? Exercise 7.12.5.12

If you draw 3 tiles from the bag without replacement, what is the probability that they spell RED, in *any* order?

? Exercise 7.12.5.13

What are the odds against drawing a vowel?

? Exercise 7.12.5.14

Use your answer to question 12 to find the odds against drawing three tiles without replacement and being able to spell RED.

? Exercise 7.12.5.15

If you draw one tile, what is the probability of drawing a J or a K?

? Exercise 7.12.5.16

If you draw one tile, what is the probability that it's a vowel or that it's worth more than 4 points?

? Exercise 7.12.5.17

Suppose you're about to draw one tile from the bag. Find $P(\text{the letter is R})$ and $P(\text{the letter is R}|\text{the point value is 1})$.

? Exercise 7.12.5.18

If you draw 2 tiles with replacement, what is the probability of drawing a consonant first and then a vowel?

? Exercise 7.12.5.19

If you draw 2 tiles *without* replacement, what is the probability of drawing a consonant first and then a vowel?

? Exercise 7.12.5.20

If you draw 10 tiles with replacement, what is the probability that you draw exactly 3 vowels? Round to 3 decimal places.

? Exercise 7.12.5.21

If you draw 100 tiles with replacement, what is the probability that you draw fewer than 35 vowels? Round to 4 decimal places.

? Exercise 7.12.5.22

Find and interpret the expected number of points on the tile, assuming you draw 1 tile from the bag.

? Exercise 7.12.5.23

Find the expected sum of points on 2 tiles, selected without replacement.

? Exercise 7.12.5.24

If your friend offers you a bet where they pay you $10 if you draw a vowel from the bag, but you owe them $5 if you draw a consonant, should you take it? How do you know?

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CHAPTER OVERVIEW

8: Statistics

8.0: Introduction
8.1: Gathering and Organizing Data
  8.1.0: Exercises
8.2: Visualizing Data
  8.2.0: Exercises
8.3: Mean, Median and Mode
  8.3.0: Exercises
8.4: Range and Standard Deviation
  8.4.0: Exercises
8.5: Percentiles
  8.5.0: Exercises
8.6: The Normal Distribution
  8.6.0: Exercises
8.7: Applications of the Normal Distribution
  8.7.0: Exercises
8.8: Scatter Plots, Correlation, and Regression Lines
  8.8.0: Exercises
8.9: Chapter Summary
  8.9.0: Key Terms
  8.9.1: Key Concepts
  8.9.2: Videos
  8.9.3: Formula Review
  8.9.4: Projects
  8.9.5: Chapter Review
  8.9.6: Chapter Test

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8.0: Introduction

Before the 2021 WNBA season, professional basketball player Candace Parker signed a contract with the Chicago Sky, which entitled her to a salary of $190,000. This amount was the 23rd highest in the league at the time. How did the team's management decide on her salary? They likely considered some intangible qualities, like her leadership skills. However, much of their deliberations probably took into account her performance on the court. For example, Parker led the league in rebounds in the 2020 season (214 of them) and scored 14.7 points per game (which ranked her 18th among all WNBA players). Further, Parker brought 13 seasons of experience to the team. All of these factors played a role in deciding the terms of her contract.

JACKSONVILLE
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JU
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4

Figure $8.0.1$: Statistics can be used to decide on a fair salary for sports stars. (credit: "Jasmine Powell goes up for a shot in a game against Jacksonville" by Lorie Shaull/Flickr, CC BY 2.0)

Estimating the value of one variable (like salary) based on other, measurable variables (points per game, experience, rebounds, etc.) is among the most important applications of statistics, which is the mathematical field devoted to gathering, organizing, summarizing, and making decisions based on data.

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8.1: Gathering and Organizing Data

Figure 8.1.1: Surveys are commonly used to gather data. (credit: “survey” by Donnell King/Flickr, CC0 1.0 Public Domain)

Learning Objectives

1. Distinguish among sampling techniques.
2. Organize data using an appropriate method.
3. Create frequency distributions.

When a polling organization wants to try to establish which candidate will win an upcoming election, the first steps are to write questions for the survey and to choose which people will be asked to respond to the survey. These can seem like simple steps, but they have far-reaching implications in the analysis the pollsters will later carry out. The process by which samples (or groups of units from which we collect data) are chosen can strongly affect the data that are collected. Units are anything that can be measured or surveyed (such as people, animals, objectives, or experiments) and data are observations made on units.

One of the most famous failures of good sampling occurred in the first half of the twentieth century. *The Literary Digest* was among the most respected magazines of the early twentieth century. Despite the name, the *Digest* was a weekly newsmagazine. Starting in 1916, the *Digest* conducted a poll to try to predict the winner of each US Presidential election. For the most part, their results were good; they correctly predicted the outcome of all five elections between 1916 and 1932. In 1936, the incumbent President Franklin Delano Roosevelt faced Kansas governor Alf Landon, and once again the *Digest* ran their famous poll, with results published the week before the election. Their conclusion? Landon would win in a landslide, 57% to 43%. Once the actual votes had been counted, though, Roosevelt ended up with 61% of the popular vote, 18% more than the poll predicted. What went wrong?

The short answer is that the people who were chosen to receive the survey (over ten million of them!) were not a good representation of the population of voting adults. The sample was chosen using the *Digest's* own base of subscribers as well as publicly available lists of people that were likely adults (and therefore eligible to vote), mostly phone books and vehicle registration records. The pollsters then mailed every single person on these lists a survey. Around a quarter of those surveys were returned; this constituted the sample that was used to make the *Digest*’s disastrously incorrect prediction. However, the *Digest* made an error in failing to consider that the election was happening during the Great Depression, and only the wealthy had disposable income to spend on telephone lines, automobiles, and magazine subscriptions. Thus, only the wealthy were sent the *Digest*’s survey. Since Roosevelt was extremely popular among poorer voters, many of Roosevelt’s supporters were excluded from the *Digest*’s sample.

Another more complicated factor was the low response rate; only around 25% of the surveys were returned. This created what’s called a non-response bias.

Sampling and Gathering Data

The *Digest's* failure highlights the need for what is now considered the most important criterion for sampling: randomness. This randomness can be achieved in several ways. Here we cover some of the most common.

A simple random sample is chosen in a way that every unit in the population has an equal chance of being selected, and the chances of a unit being selected do not depend on the units already chosen. An example of this is choosing a group of people by drawing names out of a hat (assuming the names are well-mixed in the hat).

A systematic random sample is selected from an ordered list of the population (for example, names sorted alphabetically or students listed by student ID). First, we decide what proportion of the population will be in our sample. We want to express that proportion as a fraction with 1 in the numerator. Let's call that number D. Next, we'll choose a random number between one and D. The unit at that position will go into our sample. We'll find the rest of our sample by choosing every Dth unit in the list, starting with our random number.

To walk through an example, let's say we want to sample 2% of the population: $2\% = \frac{2}{100} = \frac{1}{50}$. (Note: If the number in the denominator isn't a whole number, we can just round it off. This part of the process doesn't have to be precise.) We can then use a random number generator to find a random number between 1 and 50; let's use 31. In our example, our sample would then be the units in the list at positions 31, 81 (31 + 50), 131 (81 + 50), and so forth.

A stratified sample is one chosen so that particular groups in the population are certain to be represented. Let's say you are studying the population of students in a large high school (where the grades run from 9th to 12th), and you want to choose a sample of 12 students. If you use a simple or systematic random sample, there's a pretty good chance that you'll miss one grade completely. In a stratified sample, you would first divide the population into groups (the *strata*), then take a random sample within each *stratum* (that's the singular form of "strata"). In the high school example, we could divide the population into grades, then take a random sample of three students within each grade. That would get us to the 12 students we need while ensuring coverage of each grade.

A cluster sample is a sample where clusters of units are chosen at random, instead of choosing individual units. For example, if we need a sample of college students, we may take a list of all the course sections being offered at the college, choose three of them at random (the sections are the clusters), and then survey all the students in those sections. A sample like this one has the advantage of convenience: If the survey needs to be administered in person, many of your sample units will be located in one place at the same time.

✓ Example ✓ 8.1.1: Random Sampling

For each of the following situations, identify whether the sample is a simple random sample, a systematic random sample, a stratified random sample, a cluster random sample, or none of these.

1. A postal inspector wants to check on the performance of a new mail carrier, so she chooses four streets at random among those that the carrier serves. Each household on the selected streets receives a survey.
2. A hospital wants to survey past patients to see if they were satisfied with the care they received. The administrator sorts the patients into groups based on the department of the hospital where they were treated (ICU, pediatrics, or general), and selects patients at random from each of those groups.
3. A quality control engineer at a factory that makes smartphones wants to figure out the proportion of devices that are faulty before they are shipped out. The phones are currently packed in boxes for shipping, each of which holds 20 devices. The engineer wants to sample 100 phones, so he selects five crates at random and tests every phone in those five crates.
4. A newspaper reporter wants to write a story on public perceptions on a project that will widen a congested street. She stands on the side of the street in question and interviews the first five people she sees there.
5. An executive at a streaming video service wants to know if her subscribers would support a second season of a new show. She gets a list of all the subscribers who have watched at least one episode of the show, and uses a random number generator to select a sample of 50 people from the list.
6. An agent for a state's Department of Revenue is in charge of selecting 100 tax returns for audit. He has a list of all of the returns eligible for audit (about 12,000 in all), sorted by the taxpayer's ID number. He asks a computer to give him a random number between 1 and 120; it gives him 15. The agent chooses the 15th, 135th, 255th, 375th, and every 120th return after that to be audited.

Answer

To decide which type of random sample is being used in each of these, we need to focus on how the randomization is being incorporated.

1. The surveys are being given to households, so households are the units in this case. But households aren't being chosen randomly; instead, *streets* are being chosen at random. These form clusters of units, so this is a cluster random sample.
2. In this case, the administrator isn't selecting patients at random from the entire list of patients. Instead, she is choosing at random from the patients who were in each of the departments (ICU, pediatrics, general) separately. The departments form *strata*, so this is a stratified random sample.
3. The engineer is testing whether the phones are faulty, so those are the units. But the random process is being used to select the *crates* of phones. Those crates form clusters, so this is a cluster random sample.
4. The reporter isn't using a random process at all, so this sample doesn't belong to any of the types we have been talking about. A sample like this one is sometimes described as a *convenience sample*, and shouldn't be used in a statistical setting.
5. The executive is choosing her sample completely at random from the full population, so this is a simple random sample.
6. The agent is choosing from the full population, but is only choosing the first unit for the sample at random; the rest are chosen by skipping down the list systematically. Thus, this is a systematic random sample.

Your Turn 8.1.1

For each of the following situations, identify whether the sample is a simple random sample, a systematic random sample, a stratified random sample, a cluster random sample, or none of these.

The chairperson of the University Chess Club is trying to decide on a time for the club's regular meetings, so she emails all of the members of the club to find their preferences.

The registrar at a small college wants to use a survey to determine if their office could do a better job of serving students. They choose three students at random from each major to take the survey.

A civic club is organizing a raffle as a fundraiser. To determine the three winners, each of the tickets is put into a large drum, then the tickets are thoroughly mixed. A blindfolded club member pulls three tickets out of the drum.

People in Mathematics: George Gallup

Figure 8.1.2: George Gallup was a founder of survey sampling techniques, and his legacy lives on to this day. (credit: "George Gallup at the National Press Club, Washington, D.C., 1969" by Bernard Gotfryd/Library of Congress Prints & Photographs Division, public domain)

George Gallup (1901–1984) rose to fame in 1936 when his prediction of the percentage of the vote going to each candidate in that year's U.S. Presidential election was more accurate than the one published in *Literary Digest*, and he did so using a sample that was much smaller than the *Digest*. He even took it one step farther, predicting with high accuracy the erroneous results of the poll that the *Literary Digest* would end up publishing! Gallup's theories on public opinion polling essentially created that field. In 1948, Gallup's reputation took a bit of a hit, when he famously, but incorrectly, predicted that Thomas Dewey would

> beat incumbent Harry Truman in that year’s Presidential election. Over the following decades, however, public trust in Gallup’s polls recovered and even steadily increased. The company Gallup founded continues to conduct daily public opinion polls, as well as provides consulting services for businesses.

Organizing Data

Once data have been collected, we turn our attention to analysis. Before we analyze, though, it’s useful to reorganize the data into a format that makes the analysis easier. For example, if our data were collected using a paper survey, our *raw data* are all broken down by respondent (represented by an individual response sheet). To perform an analysis on all the responses to an individual question, we need to first group all the responses to each question together. The way we organize the data depends on the type of data we’ve collected.

There are two broad types of data: categorical and quantitative. **Categorical data** classifies the unit into a group (or category). Examples of categorical data include a response to a yes-or-no question, or the color of a person’s eyes. **Quantitative data** is a numerical measure of a property of a unit. Examples of quantitative data include the time it takes for a rat to run through a maze or a person’s daily calorie intake. We’ll look at each type of data in turn when considering how best to organize.

Categorical Data Organization

The best way to organize categorical data is using a categorical frequency distribution. A categorical frequency distribution is a table with two columns. The first contains all the categories present in the data, each listed once. The second contains the *frequencies* of each category, which are just a count of how often each category appears in the data.

✓ Example $8.1.2$: Creating a Categorical Frequency Distribution

A teacher records the responses of the class (28 students) on the first question of a multiple choice quiz, with five possible responses (A, B, C, D, and E):

A	A	C	A	B	B	A	E	A	C	A	A	A	C
E	A	B	A	A	C	A	B	E	E	A	A	C	C

Create a categorical frequency distribution that organizes the responses.

Answer

Step 1: For each possible response, count the number of times that response appears in the data. In the responses for this class, “A” appears 14 times, “B” 4 times, “C” 6 times, “D” 0 times, and “E” 4 times.

Step 2: Make a table with two columns. The first column should be labeled so that the reader knows what the responses mean, and the second should be labeled “Frequency.”

Response to First Question	Frequency
A	14
B	4
C	6
D	0
E	4

Step 3: Check your work. If you add up your frequencies, you should get the same number as the total number of responses. Twenty-eight students answered that first question, and $14+4+6+0+4=28$.

Your Turn 8.1.2

Students in a statistics class who were asked to provide their majors provided the data below:

Undecided	Biology	Biology	Sociology
Political Science	Sociology	Undecided	Undecided
Undecided	Biology	Biology	Education
Biology	Biology	Political Science	Political Science

Create a categorical frequency distribution to organize these responses.

Quantitative Data

We have a couple of options available for organizing quantitative data. If there are just a few possible responses, we can create a frequency distribution just like the ones we made for categorical data above. For example, if we're surveying a group of high school students and we ask for each student's age, we'll likely only get whole-number responses between 13 and 19. Since there are only around seven (and likely fewer) possible responses, we can treat the data as if they're categorical and create a frequency distribution as before.

Example 8.1.3: Creating a Quantitative Frequency Distribution

Attendees of a conflict resolution workshop are asked how many siblings they have. The responses are as follows:

1	0	1	1	2	0	3	1	1	4	1	2	0	1	3
1	2	1	2	4	1	0	1	3	0	1	2	2	1	5

Create a frequency distribution to organize the responses.

Answer

Step 1: Count the number of times you see each unique response: "0" appears 5 times, "1" appears 13 times, "2" appears 6 times, "3" appears 3 times, "4" appears twice, and "5" appears once.

Step 2: Make a table with two columns. The first column should be labeled so that the reader knows what the responses mean, and the second should be labeled "Frequency." Then fill in the results of our count.

Number of Siblings	Frequency	Number of Siblings	Frequency
0	5	3	3
1	13	4	2
2	6	5	1

Step 3: Check your work. If you add up your counts, you should get the same number as the total number of responses. Looking back at the raw data, there were 30 responses, and $5+13+6+3+2+1=30$.

Your Turn 8.1.3

A question on a community survey asked each respondent to give the number of people who shared their residence, and the data from the responses was as follows:

1	3	2	2	1	3	3	4	2	2	2	4	1	1	2	3	1	1	5	2
1	4	3	2	1	2	2	1	3	1	3	3	4	1	4	2	2	2	1	4

Create a frequency distribution to organize the responses.

If there are many possible responses, a frequency distribution table like the ones we've seen so far isn't really useful; there will likely be many responses with a frequency of one, which means the table will be no better than looking at the raw data. In these cases, we can create a binned frequency distribution. A binned frequency distribution groups the data into ranges of values called *bins*, then records the number of responses in each bin.

For example, if we have height data for individuals measured in centimeters, we might create bins like 150–155 cm, 155–160 cm, and so forth (making sure that every data value falls into a bin). We must be careful, though; in this scenario, it's not clear which bin would contain a response of 155 cm. Usually, responses on the edge of a bin are placed in the higher bin, but it's good practice to make that clear. In cases where responses are rounded off, you can avoid this issue by leaving a gap between the bins that couldn't contain any responses. In our example, if the measurements were all rounded off to the nearest centimeter, we could make bins like 150–154 cm, 155–159 cm, etc. (since a response like 154.2 isn't possible). We'll use this method going forward. How do we decide what the boundaries of our bins should be? There's no one right way to do that, but there are some guidelines that can be helpful.

1. Every data value should fall into exactly one bin. For example, if the lowest value in our data is 42, the lowest bin should not be 45–49.
2. Every bin should have the same width. Note that if we shift the upper limits of our bins down a bit to avoid ambiguity (like described above), we can't simply subtract the lower limit from the upper limit to get the bin width; instead, we subtract the lower limit of the bin from the lower limit of the *next* bin. For example, if we're looking at GPAs rounded to the nearest hundredth, we might choose bins like 2.00–2.24, 2.25–2.49, 2.50–2.74, etc. These bins all have a width of 0.25.
3. If the minimum or maximum value of the data falls right on the boundary between two bins, then it's OK to bend the rule just a little in order to avoid having an additional bin containing just that one value. We'll see an example of this in just a moment.
4. If we have too many or too few bins, it can be difficult to get a good sense of the distribution. Seven or eight bins is ideal, but that's not a firm rule; anything between five and twelve is fine. We often choose the number of bins so that the widths are round numbers.

✓ Example 8.1.4: Creating a Binned Frequency Distribution

The GPAs of students enrolled in an advanced sociology class are listed in the following table. At this institution, 4.00 is the maximum possible GPA.

3.93	3.43	2.87	2.51	2.70	1.91	2.32	2.85	3.06	3.03	3.49	1.84	3.72	2.56
1.99	3.40	3.74	3.23	1.98	3.05	1.43	2.90	1.20	3.72	3.56	3.07	2.58	4.00
2.79	3.81	2.60	3.69	2.88	3.34	1.51	3.63	3.45	1.89	2.30	2.98	3.04	2.70

Create a binned frequency distribution for the data.

Answer

Step 1: Identify the max and min values in your bins. Looking at the dataset, you can see that the lowest value is 1.20, and the highest is 4.00.

Step 2: Get a rough idea of bin widths. Aim for seven or eight bins, give or take a couple. For eight bins, the minimum width can be found by taking the difference between the largest and smallest data values and dividing by the number of bins:

$$\frac{\text{maximum} - \text{minimum}}{\#\text{ of bins}} = \frac{4.00 - 1.20}{8} = 0.35.$$

If we use 0.35 for our widths, starting at our minimum value of 1.20, we'll get bins with these boundaries: 1.20, 1.55, 1.90, 2.25, 2.60, 2.95, 3.30, 3.65, 4.00.

Step 3: Consider the context of the values. Because these are GPAs, there are natural breaks at 2.00 and 3.00 that are important. (People like whole numbers!) Since 0.35 is very close to $\frac{1}{3}$, let's use that for our bin width instead, and make sure that whole numbers fall on the boundaries. That means our first bin needs to start at 1.00 and go up to 1.33 to make sure our minimum value is included. The next bin will run from 1.34 to 1.66, and so forth.

Step 4: Create the distribution table. We start our distribution table by filling in the bins:

GPA Range	Frequency	GPA Range	Frequency	GPA Range	Frequency
1.00–1.33		2.00–2.33		3.00–3.33	
1.34–1.66		2.34–2.66		3.34–3.66	
1.67–1.99		2.67–2.99		3.67–4.00	

Notice that the last bin doesn't follow the pattern; since our maximum data value is right on the upper boundary of that last bin, this is a case where we can bend that rule just a little to avoid creating a bin for 4.00–4.33 (which wouldn't really make sense in the context of these GPAs anyway, since 4.00 is the maximum possible GPA).

Step 5: Complete the table with the frequencies. Finish the table by counting the number of data values that fall in each bin, and recording them in the frequency column:

GPA Range	Frequency	GPA Range	Frequency	GPA Range	Frequency
1.00–1.33	1	2.00–2.33	2	3.00–3.33	6
1.34–1.66	2	2.34–2.66	4	3.34–3.66	7
1.67–1.99	5	2.67–2.99	8	3.67–4.00	7

Step 6: Check your work. Add up the frequencies to make sure all the data values are included. We started with forty-two data values, and $1+2+5+2+4+8+6+7+7=42$.

Your Turn 8.1.4

The following table displays the ages of a sample of customers who have shopped at a new boutique.

56	39	35	32	26	53	55	47	70	43
33	33	43	41	26	40	31	34	33	53

Create a binned frequency distribution to summarize these data.

Check Your Understanding

For the following problems, decide whether randomization is being used in the selection of these samples. If it is, identify the type of random sample (simple, systematic, cluster, or stratified).

1. High school guidance counselors want to know the proportion of the school's seniors who intend to apply for college. They choose four senior homerooms at random, then visit each one and ask every student in those homerooms whether they intend to apply.
2. A quality control technician wants to ensure that the sandals being made in his factory are up to specifications, so they check the first five pairs they see coming off the line.
3. A college athletic department wants to check up on the mental wellness of its student-athletes. The department wants to ensure every varsity sport is represented, so they survey three randomly selected members of each team.
4. The purchasing manager for a chain of bookstores wants to make sure they're buying the right types of books to put on the shelves, so they take a sample of 20 books that customers bought in the last five days and record the genres. Use the raw data below to create a categorical frequency distribution.

Nonfiction	Young Adult	Romance	Cooking	Young Adult
Young Adult	Thriller	Young Adult	Nonfiction	True Crime
Romance	Nonfiction	Thriller	True Crime	Romance

True Crime	Thriller	Romance	Young Adult	Young Adult

5. A survey of college students asked how many courses those students were currently taking. Create a quantitative frequency distribution to summarize the raw data given below:

3	4	4	3	5	4	4	3	2	3	5	5	3	3	4	3
2	4	3	3	4	3	5	3	3	3	2	3	1	3	4	3

6. The World Bank provides data on every country in the world. The following is a sample of twenty-five countries, along with the number of cell phone subscriptions registered in that country per hundred residents. Create a binned frequency distribution for the cell phone data.

Country	Cell	Country	Cell
Cameroon	83.7	Benin	78.5
Vanuatu	82.5	Eritrea	13.7
Georgia	140.7	Mauritania	92.2
Kazakhstan	146.6	Czech Republic	119
Bermuda	105.9	Qatar	151.1
Russia	157.9	Pakistan	73.4
Hungary	113.5	Egypt	105.5
Costa Rica	180.2	Nepal	123.2
Algeria	111	Turkey	96.4
Somalia	48.3	Congo	43.5
Fiji	114.2	Venezuela	78.5
El Salvador	156.5	Germany	133.6
Angola	44.7		

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8.1.0: Exercises

For the following exercises, data are collected on a sample of items found in a grocery store. Classify each of these datasets obtained from that sample as being categorical or quantitative.

? Exercise $8.1.0.1$

Price

? Exercise $8.1.0.2$

Calories per serving

? Exercise $8.1.0.3$

Whether the product is gluten-free

? Exercise $8.1.0.4$

Package weight

? Exercise $8.1.0.5$

Country of origin

For the following exercises, decide whether random samples are being selected. If they are, decide whether they are simple, systematic, cluster, or stratified.

? Exercise $8.1.0.6$

A newspaper asks its readers to answer an online poll about proposed zoning changes in their city.

? Exercise $8.1.0.7$

An electronics retailer uses a computer to randomly select customers in its rewards club to take a survey about their interest in a new product.

? Exercise $8.1.0.8$

The student affairs office at a university wants to make sure students who live on campus are satisfied with their access to laundry facilities. They select five students at random from each residence hall to take the survey.

? Exercise $8.1.0.9$

A professor wants to gauge how much time her students spend on homework, so she asks that question of each student who comes to her office hours that day.

? Exercise $8.1.0.10$

The management at a restaurant wants feedback about its new menu. They choose ten tables at random, and survey each person seated at that table.

Exercise 8.1.0.11

The transit authority in a large city wants to know about usage on a particular train route. They choose a number between 1 and 5 at random, and get 4. They then count the number of people on the fourth train to pass through the station, and then count every fifth train after that.

Exercise 8.1.0.12

A candidate for a seat in the U.S. Congress wants to learn which issues are most important to her potential constituents. She chooses 50 people at random from each zip code in her district to survey.

For the following exercises, you have been tasked with surveying a sample of 100 registered voters who live in your town. You have access to a spreadsheet containing the following data on every registered voter: name, address, age, phone number. The spreadsheet also can generate a unique random number for each person.

Exercise 8.1.0.13

Describe how you might choose a simple random sample from this population.

Exercise 8.1.0.14

Describe how you might choose a stratified random sample from this population to ensure that all age groups are represented.

Exercise 8.1.0.15

Assume that there are 50,000 registered votes on your list. Describe how you might choose a systematic random sample from this population.

Exercise 8.1.0.16

A sample of students was asked, “Which social media platform, if any, do you use most frequently?” The raw responses are given in this table:

None	Twitter	Snapchat	Snapchat	Twitter	Facebook
Instagram	Snapchat	Twitter	None	Snapchat	Instagram
Instagram	Facebook	None	Instagram	Snapchat	Twitter
Snapchat	Instagram	Instagram	Twitter	Snapchat	Twitter
Facebook	None	Instagram	Instagram	Twitter	Instagram

Create a categorical frequency distribution to summarize these data.

Exercise 8.1.0.17

A sample of students at a large university were asked whether they were full-time students living on campus (Full-Time Residential, FTR), full-time students who commuted (FTC), or part-time students (PT). The raw data are in the table below:

FTR	FTR	FTC	PT	FTR	PT	FTR	FTC	FTC	PT	FTC	FTC	PT	FTR	FTC	PT
FTR	FTC	FTC	FTR	FTR	PT	FTC	FTC	FTC	PT	FTR	PT	FTC	FTC	FTR	PT

Give the categorical frequency distribution for these data.

Exercise 8.1.0.18

A survey of students in a math class asked for the respondents' birth months. The table below lists the responses:

Dec	Feb	Apr	Sep	Nov	Dec	Aug	Feb	Feb	Sep	Oct	Feb	Jun	Jan
Jul	May	May	Jan	Mar	Feb	Nov	Oct	Apr	Oct	Aug	Jan	May	Jan

Give the categorical frequency distribution of the birth months.

Exercise 8.1.0.19

Students in a statistics class were asked how many countries (besides their home countries) they had visited. The table below gives the raw responses:

0	2	1	1	3	2	0	0	0	2	1	1	0	1	1
0	2	0	1	0	1	0	2	0	1	1	0	0	1	0

Create a frequency distribution to summarize the data.

Exercise 8.1.0.20

The following table contains the top 25 receivers (by number of receptions) in the NFL during the 2020 season, along with their teams and the number of fumbles each made over the course of the season:

Player	Team	Fumbles	Player	Team	Fumbles
Stefon Diggs	BUF	0	Calvin Ridley	ATL	1
Davante Adams	GNB	1	Robert Woods	LAR	2
DeAndre Hopkins	ARI	3	Justin Jefferson	MIN	1
Darren Waller	LVR	2	Diontae Johnson	PIT	2
Travis Kelce	KAN	1	Tyreek Hill	KAN	1
Allen Robinson	CHI	0	Terry McLaurin	WAS	1
Keenan Allen	LAC	3	Alvin Kamara	NOR	1
Tyler Lockett	SEA	1	D.K. Metcalf	SEA	1
JuJu Smith-Schuster	PIT	3	Cole Beasley	BUF	0
Robby Anderson	CAR	1	Brandin Cooks	HOU	0
Amari Cooper	DAL	0	J.D. McKissic	WAS	3
Cooper Kupp	LAR	1	Tyler Boyd	CIN	1
Curtis Samuel	CAR	1			

Create a frequency distribution for the number of fumbles made by these players.

Exercise 8.1.0.21

A public opinion poll about an upcoming election asked respondents, “How many political advertisements do you recall seeing on television in the last 24 hours?” The responses were as follows

6	2	5	5	2	2	4	1	3	0	1	2	1	6	2
5	2	4	8	6	3	3	4	2	5	3	4	2	2	3

Create a frequency distribution for these data.

For the following exercises, use the following table of data on the top 15 receivers (by number of receptions) in the NFL during the 2020 season:

Player	Team	Age	Receptions	Yards	Yds/Rec	TD	Long
Stefon Diggs	BUF	27	127	1535	12.1	8	55
Davante Adams	GNB	28	115	1374	11.9	18	56
DeAndre Hopkins	ARI	28	115	1407	12.2	6	60
Darren Waller	LVR	28	107	1196	11.2	9	38
Travis Kelce	KAN	31	105	1416	13.5	11	45
Allen Robinson	CHI	27	102	1250	12.3	6	42
Keenan Allen	LAC	28	100	992	9.9	8	28
Tyler Lockett	SEA	28	100	1054	10.5	10	47
JuJu Smith-Schuster	PIT	24	97	831	8.6	9	31
Robby Anderson	CAR	27	95	1096	11.5	3	75
Amari Cooper	DAL	26	92	1114	12.1	5	69
Cooper Kupp	LAR	27	92	974	10.6	3	55
Calvin Ridley	ATL	26	90	1374	15.3	9	63
Robert Woods	LAR	28	90	936	10.4	6	56
Justin Jefferson	MIN	21	88	1400	15.9	7	71

(source)

Exercise 8.1.0.22

Make a binned frequency distribution for receiving yards (“Yards”) using bins of width 200.

Exercise $8.1.0.23$

Make another binned frequency distribution for receiving yards (“Yards”), but this time use bins of width 250.

Exercise $8.1.0.24$

Make a binned frequency distribution for number of yards per reception (“Yds/Rec”), using bins of width 1.

Exercise $8.1.0.25$

Make a binned frequency distribution for longest reception (“Long”), using bins of width 10.

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8.2: Visualizing Data

Figure 8.2.1: Data visualizations can help people quickly understand important features of a dataset. (credit: "Group of diverse people having a business meeting" by Rawpixel Ltd/Flickr, CC BY 2.0)

Learning Objectives

1. Create charts and graphs to appropriately represent data.
2. Interpret visual representations of data.
3. Determine misleading components in data displayed visually.

Summarizing raw data is the first step we must take when we want to communicate the results of a study or experiment to a broad audience. However, even organized data can be difficult to read; for example, if a frequency table is large, it can be tough to compare the first row to the last row. As the old saying goes: a picture is worth a thousand words (or, in this case, summary statistics)! Just as our techniques for organizing data depended on the type of data we were looking at, the methods we'll use for creating visualizations will vary. Let's start by considering categorical data.

Visualizing Categorical Data

If the data we're visualizing is categorical, then we want a quick way to represent graphically the relative numbers of units that fall in each category. When we created the frequency distributions in the last section, all we did was count the number of units in each category and record that number (this was the *frequency* of that category). Frequencies are nice when we're organizing and summarizing data; they're easy to compute, and they're always whole numbers. But they can be difficult to understand for an outsider who's being introduced to your data.

Let's consider a quick example. Suppose you surveyed some people and asked for their favorite color. You communicated your results using a frequency distribution. Jerry is interested in data on favorite colors, so he reads your frequency distribution. The first row shows that twelve people indicated green was their favorite color. However, Jerry has no way of knowing if that's a lot of people without knowing how many people total took your survey. Twelve is a pretty significant number if only twenty-five people took the survey, but it's next to nothing if you recorded a thousand responses. For that reason, we will often summarize categorical data not with frequencies, but with proportions. The proportion of data that fall into a particular category is computed by dividing the frequency for that category by the total number of units in the data.

Proportion of a category $= \frac{\text{Category frequency}}{\text{Total number of data units}}$

Proportions can be expressed as fractions, decimals, or percentages.

✓ Example ✓ 8.2.1: Finding Proportions

Recall the example in which a teacher recorded the responses on the first question of a multiple choice quiz, with five possible responses (A, B, C, D, and E). The raw data was as follows:

A	A	C	A	B	B	A	E	A	C	A	A	A	C
E	A	B	A	A	C	A	B	E	E	A	A	C	C

We computed a frequency distribution that looked like this:

Response to First Question	Frequency
A	14
B	4
C	6
D	0
E	4

Proportion of a category = Category frequency/Total number of data units

Now, let's compute the proportions for each category.

Answer

Step 1: In order to compute a proportion, we need the frequency (which we have in the table above) and the total number of units that are represented in our data. We can find that by adding up the frequencies from all the categories: $14+4+6+0+4=28$.

Step 2: To find the proportions, we divide the frequency by the total. For the first category ("A"), the proportion is $\frac{14}{28}=\frac{1}{2}=0.5=50\%$. We can compute the other proportions similarly, filling in the rest of the table:

Response to First Question	Frequency	Proportion
A	14	$\frac{14}{28}=50\%$
B	4	$\frac{4}{28}=14.3\%$
C	6	$\frac{6}{28}=21.4\%$
D	0	$\frac{0}{28}=0\%$
E	4	$\frac{4}{28}=14.3\%$

Step 3: Check your work: If you add up your proportions, you should get 1 (if you're using fractions or decimals) or 100% (if you're using percentages). In this case, $50\%+14.3\%+21.4\%+0\%+14.3\%=100\%$.

Checkpoint

If you need to round off the results of the computations to get your percentages or decimals, then the sum might not be exactly equal to 1 or 100% in the end due to that rounding error.

Your Turn 8.2.1

Students in a statistics class were asked to provide their majors. Those results are again listed below:

Undecided	Biology	Biology	Sociology
Political Science	Sociology	Undecided	Undecided
Undecided	Biology	Biology	Education
Biology	Biology	Political Science	Political Science

You created a frequency distribution:

Major	Frequency
Biology	6
Education	1
Political Science	3
Sociology	2
Undecided	4

Now, find the proportions associated with each category. Express your answers as percentages.

Now that we can compute proportions, let's turn to visualizations. There are two primary visualizations that we'll use for categorical data: bar charts and pie charts. Both of these data representations work on the same principle: If proportions are represented as areas, then it's easy to compare two proportions by assessing the corresponding areas. Let's look at bar charts first.

Bar Charts

A bar chart is a visualization of categorical data that consists of a series of rectangles arranged side-by-side (but not touching). Each rectangle corresponds to one of the categories. All of the rectangles have the same width. The height of each rectangle corresponds to either the number of units in the corresponding category or the proportion of the total units that fall into the category.

✓ Example 8.2.2: Building a Bar Chart

In Example ✓ 8.2.1, we computed the following proportions:

Response to First Question	Frequency	Proportion
A	14	50%
B	4	14.3%
C	6	21.4%
D	0	0%
E	4	14.3%

Draw a bar chart to visualize this frequency distribution.

Answer

Step 1: To start, we’ll draw axes with the origin (the point where the axes meet) at the bottom left:

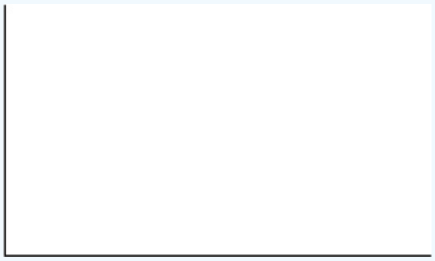


Figure ✓ 8.2.2

Step 2: Next, we’ll place our categories evenly spaced along the bottom of the horizontal axis. The order doesn’t really matter, but if the categories have some sort of natural order (like in this case, where the responses are labeled A to E), it’s best to maintain that order. We'll also label the horizontal axis:

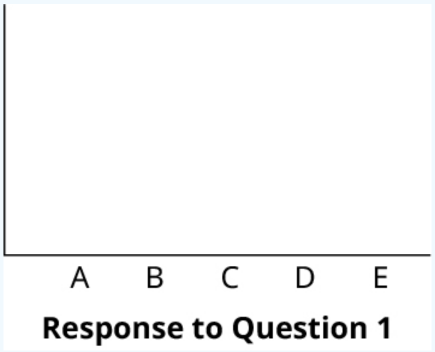


Figure ✓ 8.2.3

Step 3: Now, we have a decision to make: Will we use frequencies to define the height of our rectangles, or will we use proportions? Let’s try it both ways. First, let’s use frequencies. Notice that our frequencies run from zero to 14; this will correspond to the scale we put on the vertical axis. If we put a tick mark for every whole number between 0 and 14, the result will be pretty crowded; let’s instead put a mark on the multiples of 3 or 5:

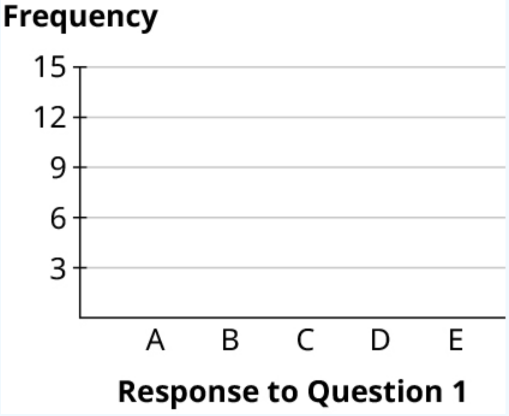


Figure ✓ 8.2.4

Step 4: Now, let’s draw in the first rectangle. The frequency associated with “A” is 14. So we’ll go to 14 on the vertical axis, and place a mark at that height above the “A” label:

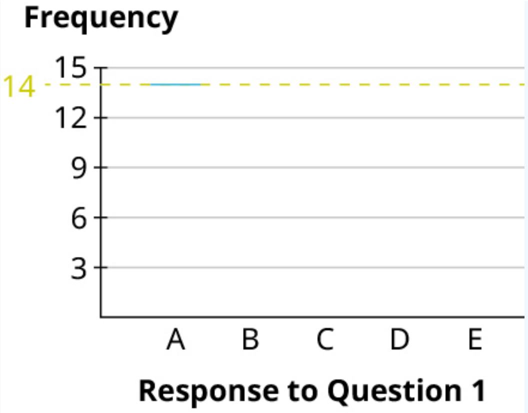


Figure ✓ 8.2.5

Step 5: Then, draw vertical lines straight down from the edges of your mark to make a rectangle:

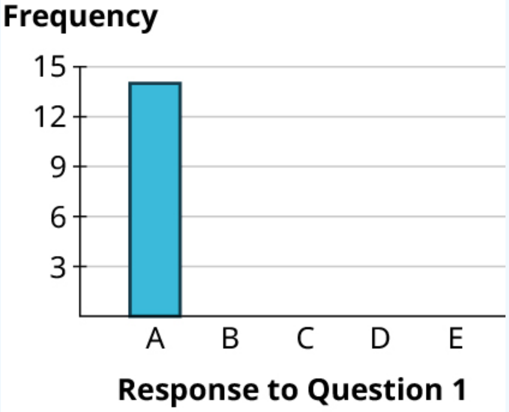


Figure ✓ 8.2.6

Step 6: Finally, we can build the rest of the rectangles, making sure that the bases all have the same length of the base = width of the rectangle, and the rectangles don't touch. Notice that, since the frequency for "D" is zero, that category has no rectangle (but we'll leave a space there so the reader can see that there is a category with frequency zero). Here's the result:

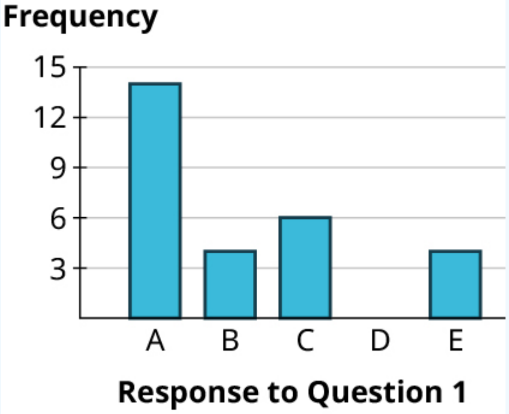


Figure ✓ 8.2.7

Step 7: That's it! Now, let's use proportions instead of frequencies. We'll label the vertical axis with evenly spaced numbers that run the full range of the percentages in our table: 0% to 50%. We can divide that into five equal parts (so that each has width 10%), and use that to label our vertical axis:

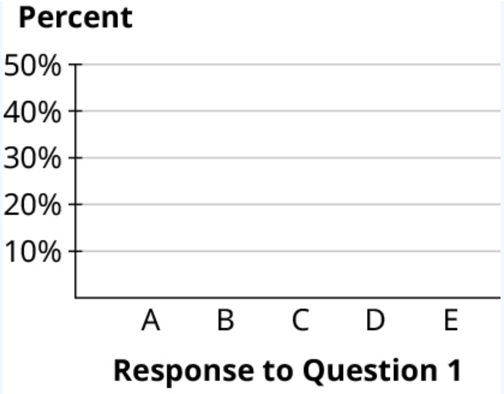


Figure ✔ 8.2.8

Step 8: Then, we can fill in the rectangles just as we did before. The height of the "A" rectangle is 50%, the "B" rectangle goes up to 14.3%, "C" goes to 21.4%, there is no rectangle for "D" (since its proportion is 0%), and the "E" rectangle also goes up to 14.3%:

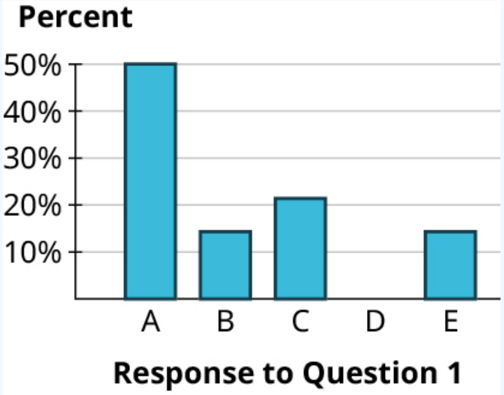


Figure ✔ 8.2.9

Step 9: Notice that the rectangles are basically identical in our two final bar charts. That's no coincidence! Bar charts that use proportions and those that use frequencies will always look identical (which is why it doesn't really matter much which option you choose). Here's why: look at the bars for "B" and "C". The frequencies for these are 4 and 6 respectively. Notice that 6 is 50% bigger than 4 (since $6 = 1.5 \times 4$), which means that the "C" bar will be 50% higher than the "B" bar. Now look at the same bars using proportions: since $21.4\% = 1.5 \times 14.3\%$, the bar for "C" will be 50% higher than the bar for "B." The same relationships hold for the other bars, too.

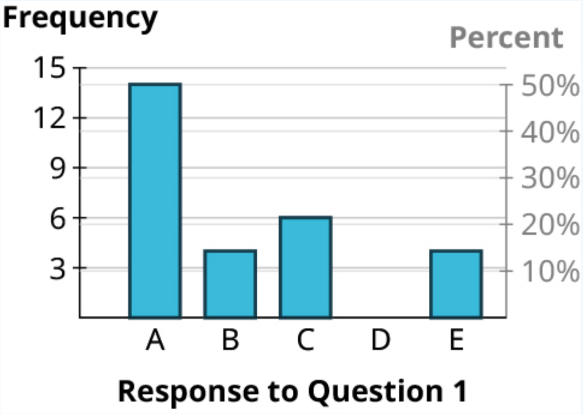


Figure 8.2.10

Your Turn 8.2.2

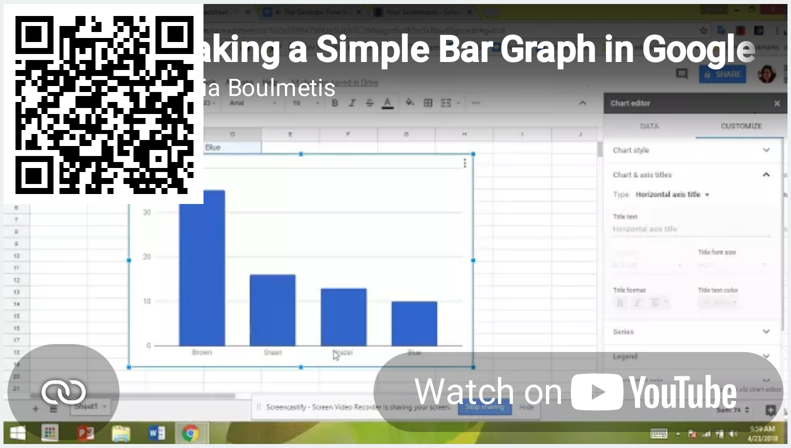
The students in a statistics class were asked to provide their majors. The computed proportions for each of the categories are as follows:

Major	Frequency	Proportion
Biology	6	37.5%
Education	1	6.3%
Political Science	3	18.8%
Sociology	2	12.5%
Undecided	4	25%

Create a bar graph to visualize these data. Use percentages to label the vertical axis.

In practice, most graphs are now made with computers. You can use Google Sheets, which is available for free from any web browser.

Video



Make a Simple Bar Graph in Google Sheets

Now that we've explored how bar graphs are made, let's get some practice reading bar graphs.

Example 8.2.3: Reading Bar Graphs

The bar graph shown gives data on 2020 model year cars available in the United States. Analyze the graph to answer the following questions.

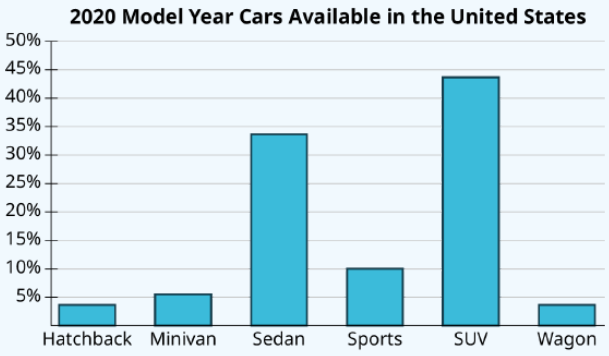


Figure ✓ 8.2.11 (data source: consumerreports.org/cars)

1. What proportion of available cars were sports cars?
2. What proportion of available cars were sedans?
3. Which categories of cars each made up less than 5% of the models available?

Answer

1. The bar for sports cars goes up to 10%, so the proportion of models that are considered sports cars is 10%.
2. The bar corresponding to sedan goes up past 30% but not quite to 35%. It looks like the proportion we want is between 33% and 34%.
3. We're looking for the bars that don't make it all the way to the 5% line. Those categories are hatchback and wagon.

Your Turn 8.2.3

The bar graph shows the region of every institution of higher learning in the United States (except for the service academies, like West Point).

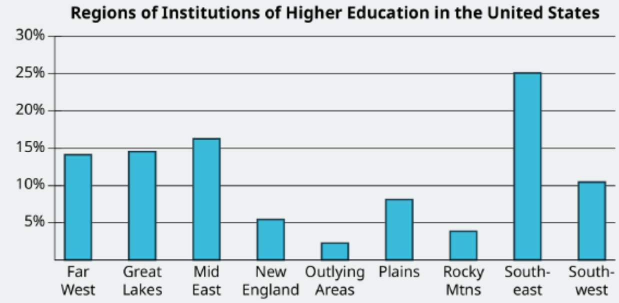


Figure 8.2.12

Analyze the bar chart to answer the following questions.

1. Which region contains the largest number of institutions of higher learning?
2. What proportion of all institutions of higher learning can be found in the Southwest?
3. Which regions each have under 5% of the total number of institutions of higher learning?

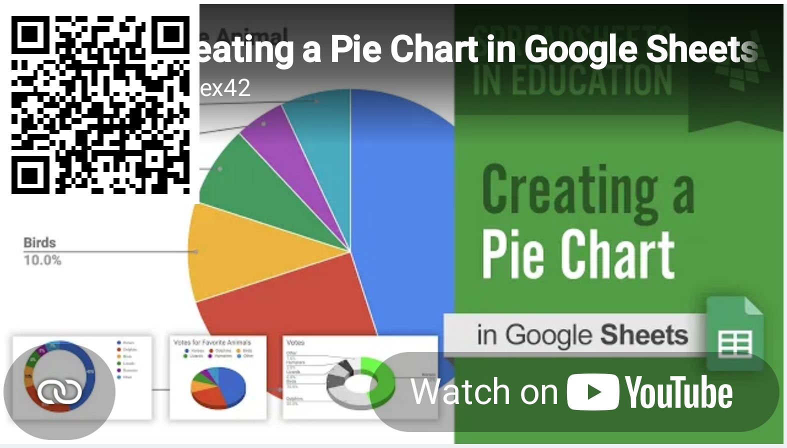
WORK IT OUT: Candy Color: Frequency and Distribution

M&Ms, Skittles, and Reese's Pieces are all candies that have pieces that are uniformly shaped, but which have different colors. Do the colors in each bag appear with the same frequency? Get a bag of one of these candies and make a bar chart to visualize the color distribution.

Pie Charts

A pie chart consists of a circle divided into wedges, with each wedge corresponding to a category. The proportion of the area of the entire circle that each wedge represents corresponds to the proportion of the data in that category. Pie charts are difficult to make without technology because they require careful measurements of angles and precise circles, both of which are tasks better left to computers.

Video


Creating a Pie Chart in Google Sheets
ex42
Birds
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IN EDUCATION
Creating a
Pie Chart
in Google Sheets
Watch on YouTube

Create Pie Charts Using Google Sheets

Pie charts are sometimes embellished with features like labels in the slices (which might be the categories, the frequencies in each category, or the proportions in each category) or a legend that explains which colors correspond to which categories. When making your own pie chart, you can decide which of those to include. The only rule is that there has to be some way to connect the slices to the categories (either through labels or a legend).

✓ Example 8.2.4: Making Pie Charts

Use the data that follows to generate a pie chart.

Table ✓ 8.2.1 (data source: www.consumerreports.org/cars)

Type	Percent	Type	Percent
SUV	43.6%	Minivan	5.5%
Sedan	33.6%	Hatchback	3.6%
Sports	10.0%	Wagon	3.6%

Answer

First, enter the chart above into a new sheet in Google Sheets. Next, click and drag to select the full table (including the header row). Click on the “Insert” menu, then select “Chart.” The result may be a pie chart by default; if it isn’t, you can change it to a pie chart using the “Chart type” drop-down menu in the Chart Editor.

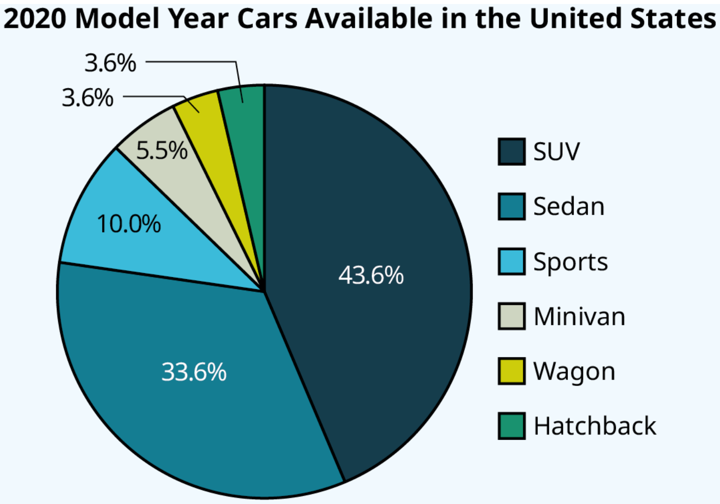


Figure ✓ 8.2.13 (data source: consumerreports.org/cars)

You can choose to use a legend to identify the categories, as well as label the slices with the relevant percentages.

Your Turn 8.2.4

In Your Turn 8.2.2, you created a bar chart using data on reported majors from students in a class. Here are those proportions again (sorted from largest to smallest):

Major	Proportion
Biology	37.5%
Undecided	25.0%
Political Science	18.8%
Sociology	12.5%
Education	6.3%

Create a pie graph using those data.

People in Mathematics: Florence Nightingale

Florence Nightingale (1820–1910) is best remembered today for her contributions in the medical field; after witnessing the horrors of field hospitals that tended to the wounded during the Crimean War, she championed reforms that encouraged sanitary conditions in hospitals. For those efforts, she is today considered the founder of modern nursing.

Figure $8.2.14$: Florence Nightingale's significant contribution to the field of statistical graphics cannot be understated. (credit: "Florence Nightingale" by Library of Congress Prints and Photographs Division/http://hdl.loc.gov/loc.pnp/pp.print, public domain)

Nightingale is also remembered for her contributions in statistics, especially in the ways we visualize data. She developed a version of the pie chart that is today known as a *polar area diagram*, which she used to visualize the causes of death among the soldiers in the war, highlighting the number of preventable deaths the British Army suffered in that conflict.

In 1859, the Royal Statistical Society honored her for her contributions to the discipline by electing her to join the organization. She was the first woman to be so honored. She was later named an honorary member of the American Statistical Association. Nightingale's status as a revered pioneer in both nursing and statistics is a complex one, because some of her writings and opinions demonstrate a colonialist mindset and disregard for those who lost their lives and lands at the hands of the British. Her core statistical writings indicated that she felt superior to the Indigenous people she was treating. Members of both fields continue to debate her near-iconic role.

Visualizing Quantitative Data

There are several good ways to visualize quantitative data. In this section, we’ll talk about two types: stem-and-leaf plots and histograms.

Stem-and-Leaf Plots

Stem-and-leaf plots are visualization tools that fall somewhere between a list of all the raw data and a graph. A stem-and-leaf plot consists of a list of stems on the left and the corresponding leaves on the right, separated by a line. The stems are the numbers that make up the data only up to the next-to-last digit, and the leaves are the final digits. There is one leaf for every data value (which means that leaves may be repeated), and the leaves should be evenly spaced across all stems. These plots are really nothing more than a fancy way of listing out all the raw data; as a result, they shouldn’t be used to visualize large datasets.

This concept can be difficult to understand without referencing an example, so let’s first look at how to read a stem-and-leaf plot.

✓ Example $8.2.5$: Reading a Stem-and-Leaf Plot

A collector of trading cards records the sale prices (in dollars) of a particular card on an online auction site, and puts the results in a stem-and-leaf plot:

Table $8.2.2$

0	5 8 9
1	0 0 0 3 4 4 5 5 5 5 5 6 9 9
2	0 0 0 0 5 5 9 9
3	0 0 0 5 5
4	0 0 5

5	
6	0

Answer the following questions about the data:

1. How many prices are represented?
2. What prices represent the five most expensive cards? The five least expensive?
3. What is the full set of data?

Answer

1. Each leaf (the numbers on the right side of the bar) represents one data value. So, on the first row (which looks like 0 | 5 8 9), there are three data values (one for each leaf: 5, 8, and 9). The next row has thirteen leaves, then eight, five, three, zero, and one. Adding those up, we get $3+13+8+5+3+0+1=33$ data points or prices.
2. The most expensive card is the last one listed. Its stem is 6 and its leaf is 0, so the price is \$60. There are no leaves associated with the 5 stem, so there were no cards sold for \$50 to \$59. The next most expensive cards are then on the 4 stem: \$45, \$40, and \$40 (remember, repeated leaves mean repeated values in the dataset). So, we have our four most expensive cards. The fifth would be on the next stem up. The biggest leaf on the 3 stem is a 5, so the fifth-most expensive card sold for \$35.
   As for the five least-expensive cards, the smallest stem is 0, with leaves 5, 8, and 9. So, the three least expensive cards sold for \$5, \$8, and \$9 (notice that we don't write down that leading 0 from the stem in the tens place). The next two least-expensive cards will be the two smallest leaves on the next stem: \$10 and \$10.
3. The full list of data is: 5, 8, 9, 10, 10, 10, 13, 14, 14, 15, 15, 15, 15, 16, 19, 19, 20, 20, 20, 24, 25, 25, 29, 29, 30, 30, 30, 35, 35, 40, 40, 45, 60.

Your Turn 8.2.5

The stem-and-leaf plot below shows data collected from a sample of employed people who were asked how far (in miles) they commute each day:

0	4 6 7
1	0 0 0 2 2 2 4 5 8 8
2	0 5 5 5
3	0 0 5 5 6
4	
5	0
6	0

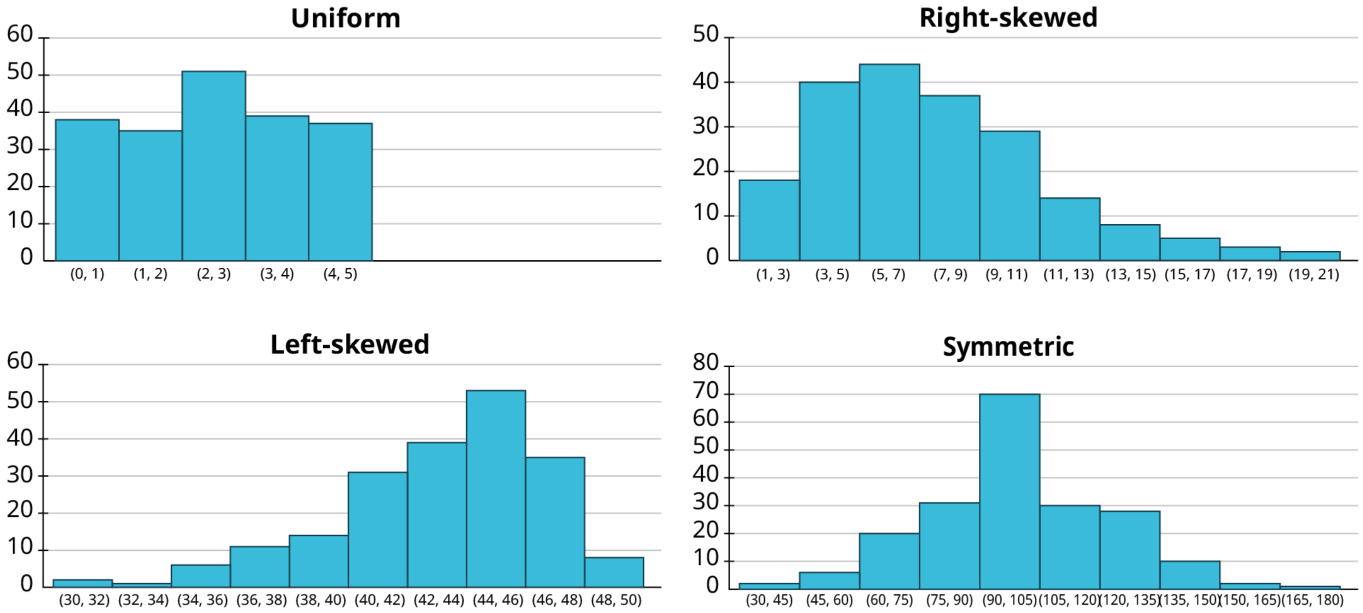
How many data points are represented?

What are the three longest and shortest commutes?

What is the full list of data?

Stem-and-leaf plots are useful in that they give us a sense of the shape of the data. Are the data evenly spread out over the stems, or are some stems "heavier" with leaves? Are the heavy stems on the low side, the high side, or somewhere in the middle? These are questions about the distribution of the data, or how the data are spread out over the range of possible values.

Some words we use to describe distributions are *uniform* (data are equally distributed across the range), *symmetric* (data are bunched up in the middle, then taper off in the same way above and below the middle), *left-skewed* (data are bunched up at the high end or larger values, and taper off toward the low end or smaller values), and *right-skewed* (data are bunched up at the low end, and taper off toward the high end). See below figures.



8.2.15

Looking back at the stem-and-leaf plot in the previous example, we can see that the data are bunched up at the low end and taper off toward the high end; that set of data is right-skewed. Knowing the distribution of a set of data gives us useful information about the property that the data are measuring.

Now that we have a better idea of how to read a stem-and-leaf plot, we're ready to create our own.

✓ Example 8.2.6: Constructing a Stem-and-Leaf Plot

An entomologist studying crickets recorded the number of times different crickets (of differing species, genders, etc.) chirped in a one-minute span. The raw data are as follows:

89	97	82	102	84	99	93	103	120	91
115	105	89	109	107	89	104	82	106	92
101	109	116	103	100	91	85	104	104	106

Construct a stem-and-leaf plot to visualize these results.

Answer

Step 1: Before we can create the plot, we need to sort the data in order from smallest to largest:

82	82	84	85	89	89	89	91	91	92
93	97	99	100	101	102	103	103	104	104
104	105	106	106	107	109	109	115	116	120

Step 2: Next, we identify the stems. To do that, we cut off the final digit of each number, which leaves us with stems of 8, 9, 10, 11, and 12. Arrange the stems vertically, and add the bar to separate these from the leaves:

8
9
10
11

12

Step 3: Write down the leaves on the right side of the bar, giving just the final digit (that we cut off to make the stems) of each data value. List these in order, and make sure they line up vertically:

Table 8.2.1

8	2 2 4 5 9 9 9
9	1 1 2 3 7 9
10	0 1 3 3 4 4 4 5 6 6 7 9 9
11	5 6
12	0

Your Turn 8.2.6

This table gives the records of the Major League Baseball teams at the end of the 2019 season:

Table 8.2.2 (source: www.mlb.com)

Team	Wins	Losses	Team	Wins	Losses
HOU	107	55	PHI	81	81
LAD	106	56	TEX	78	84
NYY	103	59	SFG	77	85
MIN	101	61	CIN	75	87
ATL	97	65	CHW	72	89
OAK	97	65	LAA	72	90
TBR	96	66	COL	71	91
CLE	93	69	SDP	70	92
WSN	93	69	PIT	69	93
STL	91	71	SEA	68	94
MIL	89	73	TOR	67	95
NYM	86	76	KCR	59	103
ARI	85	77	MIA	57	105
BOS	84	78	BAL	54	108
CHC	84	78	DET	47	114

Create a stem-and-leaf plot for the number of wins.

As we mentioned above, stem-and-leaf plots aren't always going to be useful. For example, if all the data in your dataset are between 20 and 29, then you'll just have one stem, which isn't terribly useful. (Although there are methods like *stem splitting* for addressing that particular problem, we won't go into those at this time.) On the other end of the spectrum, the data may be so spread out that every stem has only one leaf. (This problem can sometimes be addressed by rounding off the data values to the tens, hundreds, or some other place value, then using that place for the leaves.) Finally, if you have dozens or hundreds (or more) of data values, then a stem-and-leaf plot becomes too unwieldy to be useful. Fortunately, we have other tools we can use.

Histograms

Histograms are visualizations that can be used for any set of quantitative data, no matter how big or spread out. They differ from a categorical bar chart in that the horizontal axis is labeled with numbers (not ranges of numbers), and the bars are drawn so that they touch each other. The heights of the bars reflect the frequencies in each bin. Unlike with stem-and-leaf plots, we cannot recreate the original dataset from a histogram. However, histograms are easy to make with technology and are great for identifying the distribution of our data. Let’s first create one histogram without technology to help us better understand how histograms work.

Example 8.2.7: Constructing a Histogram

We built a stem-and-leaf plot for the number of chirps made by crickets in one minute. Here are the raw data that we used:

89	97	82	102	84	99
115	105	89	109	107	89
101	109	116	103	100	91
93	103	120	91	85	104
104	82	106	92	104	106

Construct a histogram to visualize these results.

Answer

Step 1: Add data to bins. Histograms are built on binned frequency distributions, so we’ll make that first. Luckily, the stem-and-leaf plot we made earlier can help us do this much more quickly:

8	2 2 4 5 9 9 9
9	1 1 2 3 7 9
10	0 1 2 3 3 4 4 4 5 6 6 7 9 9
11	5 6
12	0

If we’re using bins of width 10, we can compute the frequencies by counting the numbers of leaves associated with the corresponding stem:

Bin	Frequency
80-89	7
90-99	6
100-109	14
110-119	2
120-129	1

(Note that, when we made binned frequency diagrams in the last module, we noted that if the biggest data value was right on the border between two bins, it was OK to lump it in with the lower bin. That's not recommended when building histograms, so the data value 120 is all alone in the 120-129 bin.)

Step 2: Create the axes. On the horizontal axis, start labeling with the lower end of the first bin (in this case, 80), and go up to the higher end of the last bin (120). Mark off the other bin boundaries, making sure they're all evenly spaced. On the vertical axis, start with zero and go up at least to the greatest frequency you see in your bins (14 in this example), making sure that the labels you make are evenly spaced and that the difference between those numbers is the same. Let's count off our vertical axis by threes:

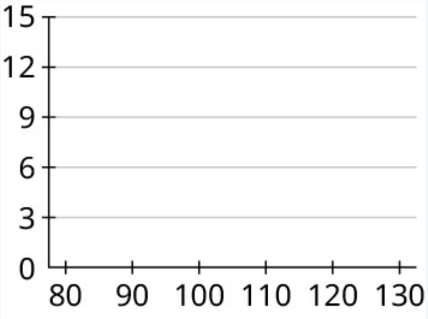


Figure ✓ 8.2.16:

Step 3: Draw in the bars. Remember that the bars of a histogram touch, and that the heights are determined by the frequency. So, the first bar will cover 80 to 90 on the horizontal axis, and have a height of 7:

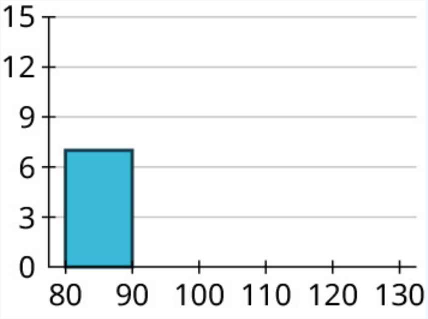


Figure ✓ 8.2.17

Now, we can fill in the others:

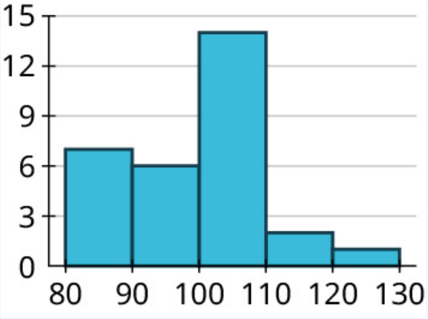


Figure ✓ 8.2.18:

Step 4: Let's compare the histogram we just created to the stem-and-leaf plot we made earlier:

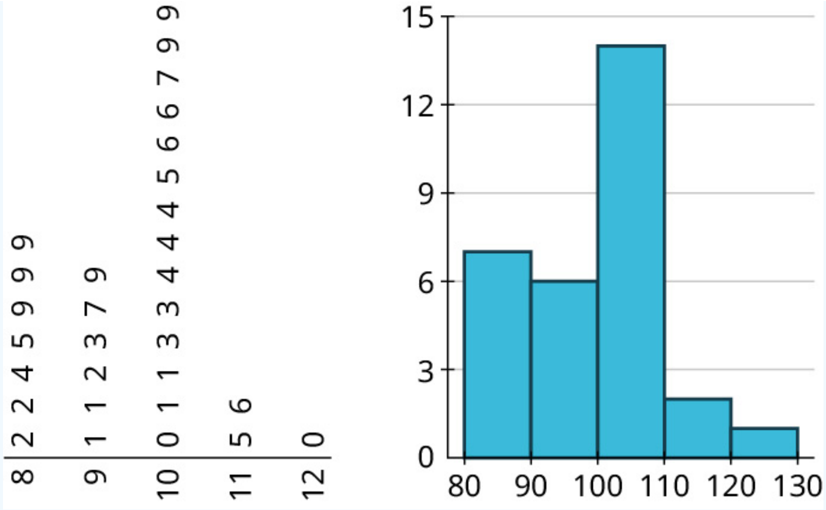


Figure ✓ 8.2.19:

Notice that the leaves on the rotated stem-and-leaf plot match the bars on our histogram! We can view stem-and-leaf plots as sideways histograms. But, as we'll see soon, we can do much more with histograms.

Your Turn 8.2.7

Previously you made a stem-and-leaf plot of the number of wins for each MLB team in 2019, using this set of data:

Table 8.2.3

Team	Wins	Losses
HOU	107	55
LAD	106	56
NYY	103	59
MIN	101	61
ATL	97	65
OAK	97	65
TBR	96	66
CLE	93	69
WSN	93	69
STL	91	71
MIL	89	73
NYM	86	76
ARI	85	77
BOS	84	78
CHC	84	78
PHI	81	81
TEX	78	84
SFG	77	85

Team	Wins	Losses
CIN	75	87
CHW	72	89
LAA	72	90
COL	71	91
SDP	70	92
PIT	69	93
SEA	68	94
TOR	67	95
KCR	59	103
MIA	57	105
BAL	54	108
DET	47	114

Create a histogram for the number of wins. Use bins of width 10, starting with a bin for 40-49 (so that your histogram reflects the stem-and-leaf plot you made earlier).

Now that we've seen the connection between stem-and-leaf plots and histograms, we are ready to look at how we can use Google Sheets to build histograms.

Video

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140.00
150.00
200.00
Watch on YouTube
Rozon's Bowling Scores

Make a Histogram Using Google Sheets

Let's use Google Sheets to create a histogram for a large dataset.

✓ Example 8.2.8: Creating a Histogram in Google Sheets

The data in "AvgSAT" contains the average SAT score for students attending every institution of higher learning in the US for which data is available. Create a histogram in Google Sheets of the average SAT scores. Use bins of width 50. Are the data uniformly distributed, symmetric, left-skewed, or right-skewed?

Answer

Using the procedure described in the video above, we get this:

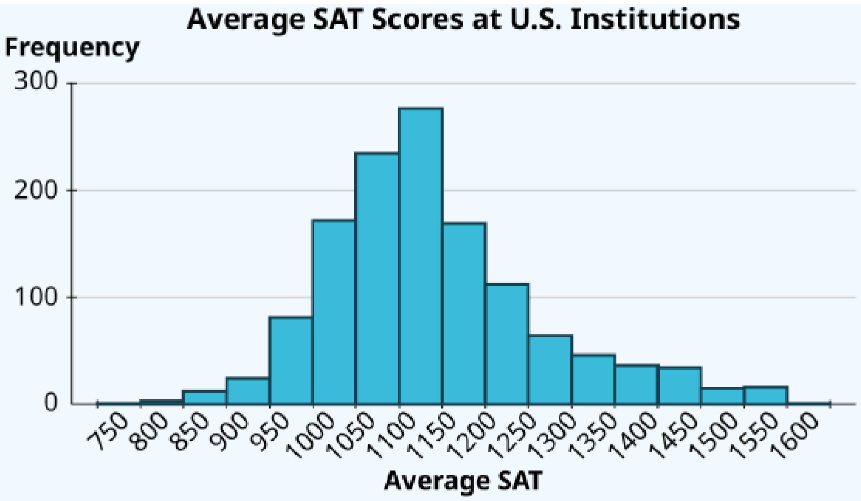

Figure ✓ 8.2.20 (data source: https://data.ed.gov)

The data are fairly symmetric, but slightly right-skewed.

Your Turn 8.2.8

The file "InState" contains in-state tuition costs (in dollars) for every institution of higher learning in the United States for which data is available (data from data.ed.gov). Create a histogram in Google Sheets of in-state tuition costs. Choose a bin size that you think works well. Are the data uniformly distributed, symmetric, left-skewed, or right-skewed?

Bar Charts for Labeled Data

Sometimes we have quantitative data where each value is labeled according to the source of the data. For example, in the Your Turn above, you looked at in-state tuition data. Every value you used to create that histogram was associated with a school; the schools are the labels. In YOUR TURN 8.11, you found a histogram of the wins of every Major League Baseball team in 2019. Each of those win totals had a label: the team. If we're interested in visualizing differences among the different teams, or schools, or whatever the labels are, we create a different version of the bar graph known as a **bar chart for labeled data**.

These graphs are made in Google Sheets in exactly the same way as regular bar graphs. The only change is that the vertical axis will be labeled with the units for your quantitative data instead of just "Frequency."

✓ Example ✓ 8.2.9: Building a Bar Chart for Labeled Data

The following table shows the gross domestic product (GDP) for the United States for the years 2010 to 2019:

Table 8.2.4 (source: https://data.worldbank.org)

Year	GDP (in $ trillions)	Year	GDP (in $ trillions)
2010	14.992	2015	18.225
2011	15.543	2016	18.715
2012	16.197	2017	19.519
2013	16.785	2018	20.580
2014	17.527	2019	21.433

Construct a histogram that represents these data.

Answer

In this case, the years are the labels, and the data we are interested in are the GDP numbers. Once you have the table above (including the labels) entered into a spreadsheet, click and drag to select the full table. Then, in the “Insert” menu, click “Chart.” The result may not be a bar chart; if it’s not, select “Column chart” in the drop-down menu “Chart type” in the Chart Editor. If you want, you can edit things like the chart title in the “Customize” tab in the Chart Editor.

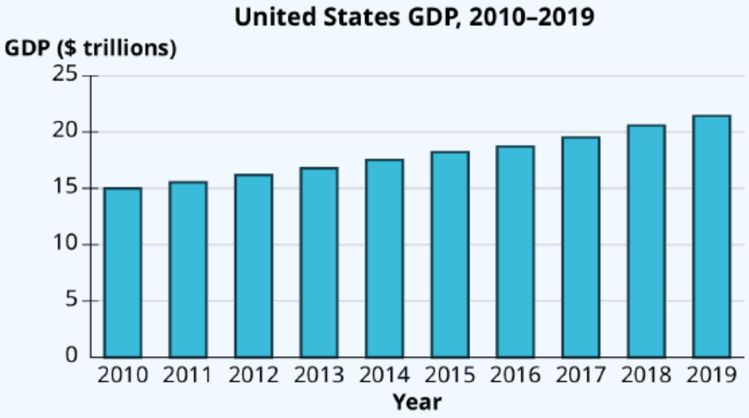

Figure ✓ 8.2.21 (data source: https://data.worldbank.org)

Your Turn 8.2.9

The following table shows the world record times (as of February 2020) of the various 100m women’s swimming events in international competition:

Table 8.2.5 (source: https://swimswam.com/records/womens-...d-records-lcm/)

Event	Time	Name	Nationality
Freestyle	51.71	Sarah Sjöström	Sweden
Backstroke	57.57	Regan Smith	United States
Breaststroke	64.10	Lilly King	United States
Butterfly	55.48	Sarah Sjöström	Sweden

Make a visualization of these times using the events as the labels.

Misleading Graphs

Graphical representations of data can be manipulated in ways that intentionally mislead the reader. There are two primary ways this can be done: by manipulating the scales on the axes and by manipulating or misrepresenting areas of bars. Let’s look at some examples of these.

✓ Example 8.2.10: Misleading Graphs

The table below shows the teams, and their payrolls, in the English Premier League, the top soccer organization in the United Kingdom.

Table 8.2.6 (source: www.spotrac.com)

Team	Salary (£1,000,000s)	Team	Salary (£1,000,000s)
Manchester United F.C.	175.7	Newcastle United F.C.	56.9
Manchester City F.C.	136.5	Aston Villa F.C.	52.3
Chelsea F.C.	132.8	Fulham F.C.	52.1
Arsenal F.C.	130.7	Southampton F.C.	49.6

Team	Salary (£1,000,000s)	Team	Salary (£1,000,000s)
Tottenham Hotspur F.C.	129.2	Wolverhampton Wanderers F.C.	49.5
Liverpool F.C.	118.6	Brighton & Hove Albion	43.7
Crystal Palace	85.0	Burnley F.C.	35.5
Everton F.C.	82.5	West Bromwich Albion F.C.	23.8
Leicester City	73.7	Leeds United F.C.	22.5
West Ham United F.C.	69.2	Sheffield United F.C.	19.7

How might someone present this data in a misleading way?

Answer

Step 1: Let’s focus on the top five teams. Here’s a bar chart of their payrolls:

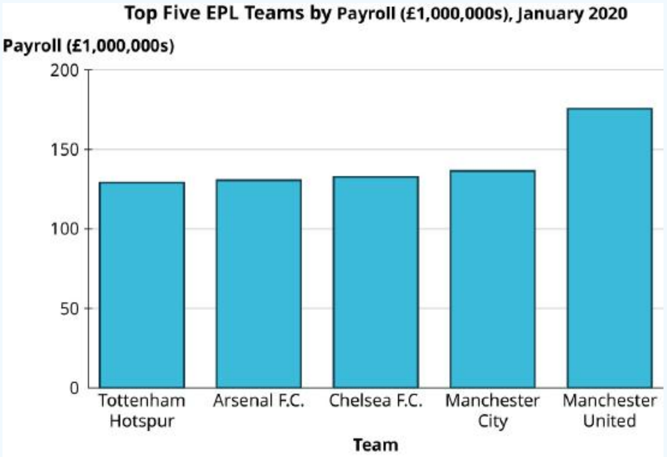

Figure ✓ 8.2.22 (data source: www.spotrac.com)

Step 2: Now, here’s another bar chart visualizing exactly the same data:

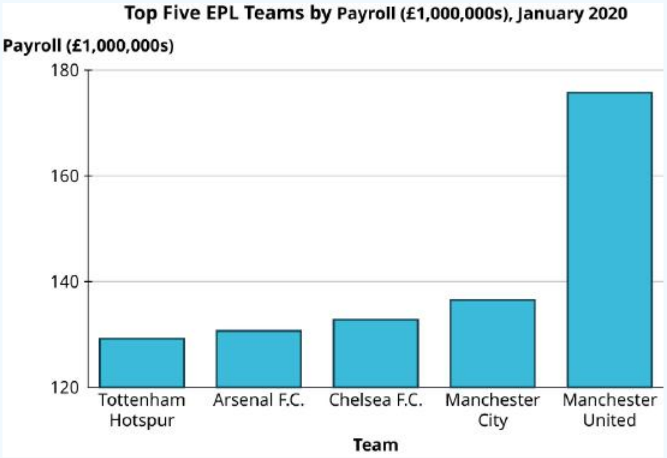

Figure ✓ 8.2.23: (data source: www.spotrac.com)

Step 3: You should notice that despite using the same data, these two graphs look strikingly different. In the second graph, the gap between Manchester United and the other four teams looks significantly larger than in the first graph. The scale on the vertical axis has been manipulated here. The first graph's axis starts at zero, while the lowest value on the second graph's axis is 120. This trick has a strong impact on the viewer’s perception of the data.

Step 4: To further emphasize the difference this creates in our perception, let's look at that data again, but this time using graphics instead of colored areas on our bar graph.

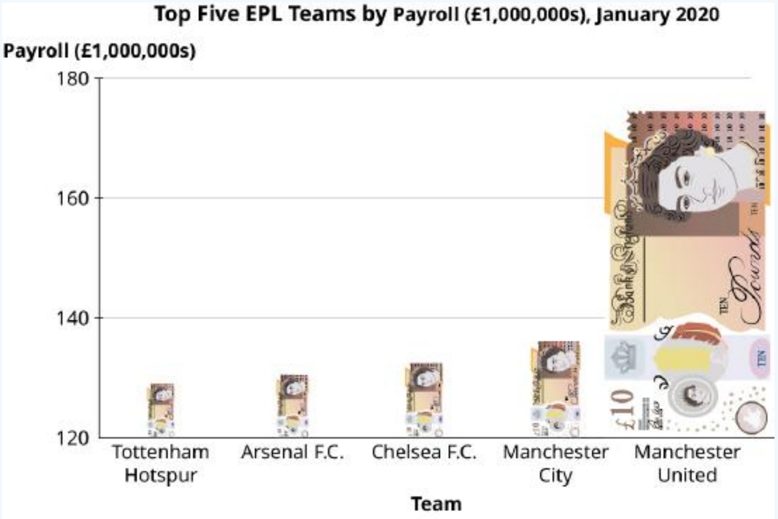

Figure ✓ 8.2.24: (data source: www.spotrac.com)

This graph uses an image of a £10 banknote in place of the bars. Using an image that evokes the context of the data in place of a standard, "boring" bar is a common tool that people use when creating *infographics*. However, this is generally not a good practice because it distorts the data. Notice that our "bars" (the banknotes) are just as tall here as they were in the previous figure. But, to maintain the right proportions, the *widths* had to be adjusted as well, which changes the area (height × width) of each bar. A key point is that when looking at rectangles, the human eye tends to process areas more easily than heights.

Checkpoint

Beware of infographics! Areas overemphasize a difference that should be measured with a height!

Step 5: Now, let's look at all 20 teams. This histogram indicates that the data are right-skewed, with the highest number of teams having a payroll between £40 million and £80 million:

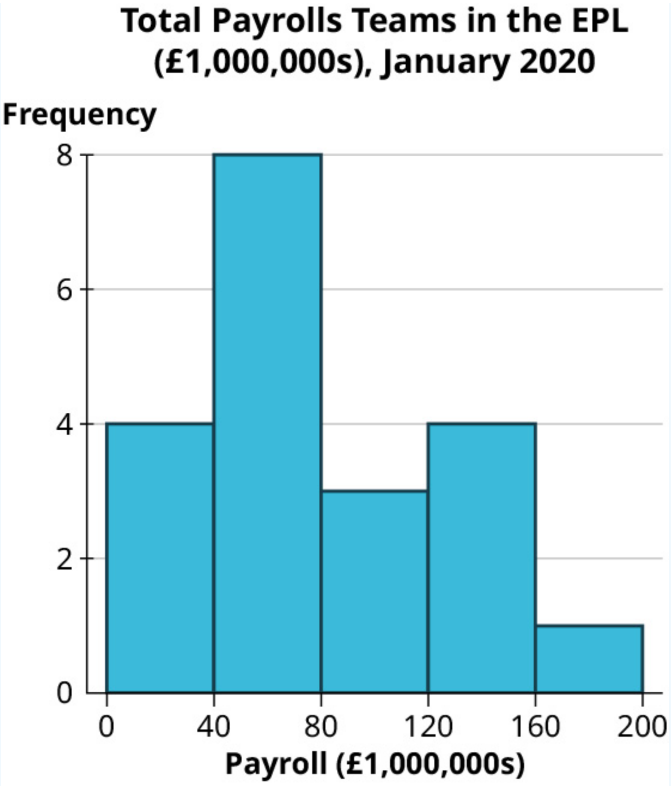

Figure ✓ 8.2.25: (data source: www.spotrac.com)

Step 6: Now let's view this same data in another chart:

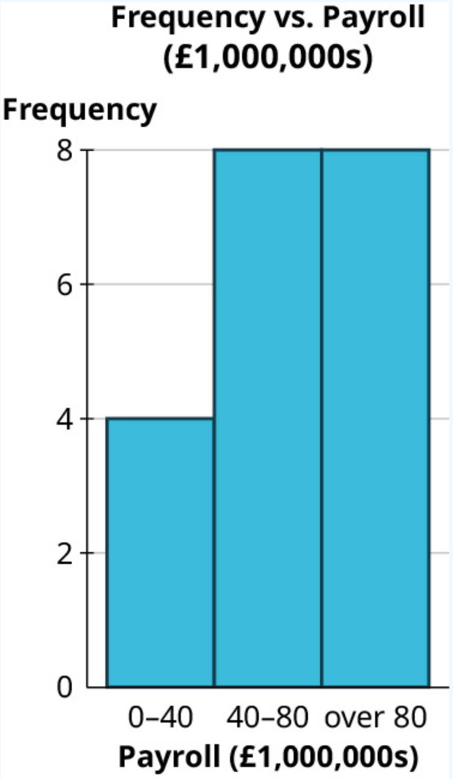

Figure ✓ 8.2.26: (data source: www.spotrac.com)

Step 7: Even though this chart uses the same data, the skew seems to be reversed. Why? Well, even though this graph *looks* like a histogram, it isn't. Look closely at the labels on the horizontal axis; they don't correspond to spots on the axis, but instead provide a range, meaning this is a bar graph based on a binned frequency distribution.

When we review these ranges, we can see that the last range is misleading as it consists of all data “over 80.” If the bins all had the same width, that last bin would run from 80 to 120. However, we can see from the histogram that the maximum value for this data is between 160 and 200. If the last bin in this bar graph were labeled honestly, it would read “80–200,” which would drive home the fact that the width of that bar is misleading.

Checkpoint

Always check the horizontal axis on histograms! The widths of all the bars should be equal.

Your Turn 8.2.10

Take a look again at the win totals for teams in Major League Baseball in 2019 :

Table 8.2.7 (source: https://www.espn.com/mlb/standings/_/season/2019/view)

Team	Wins	Team	Wins
HOU	107	PHI	81
LAD	106	TEX	78
NYY	103	SFG	77
MIN	101	CIN	75
ATL	97	CHW	72
OAK	97	LAA	72
TBR	96	COL	71
CLE	93	SDP	70
WSN	93	PIT	69
STL	91	SEA	68
MIL	89	TOR	67
NYM	86	KCR	59
ARI	85	MIA	57
BOS	84	BAL	54
CHC	84	DET	47

Make one good and one misleading chart showing the number of wins by the top ten teams. Then, looking at all the teams, make one good and one misleading histogram for the win totals.

Video

How to spot a misleading graph - Lea
-Ed
SPOT A
MISLEADING
GRAPH
YES
NO
Watch on YouTube

How to Spot a Misleading Graph

Who Knew?: Napoleon's Failed Invasion

One of the most famous data visualizations ever created is the cartographic depiction by Charles Joseph Minard of Napoleon's disastrous attempted invasion of Russia.

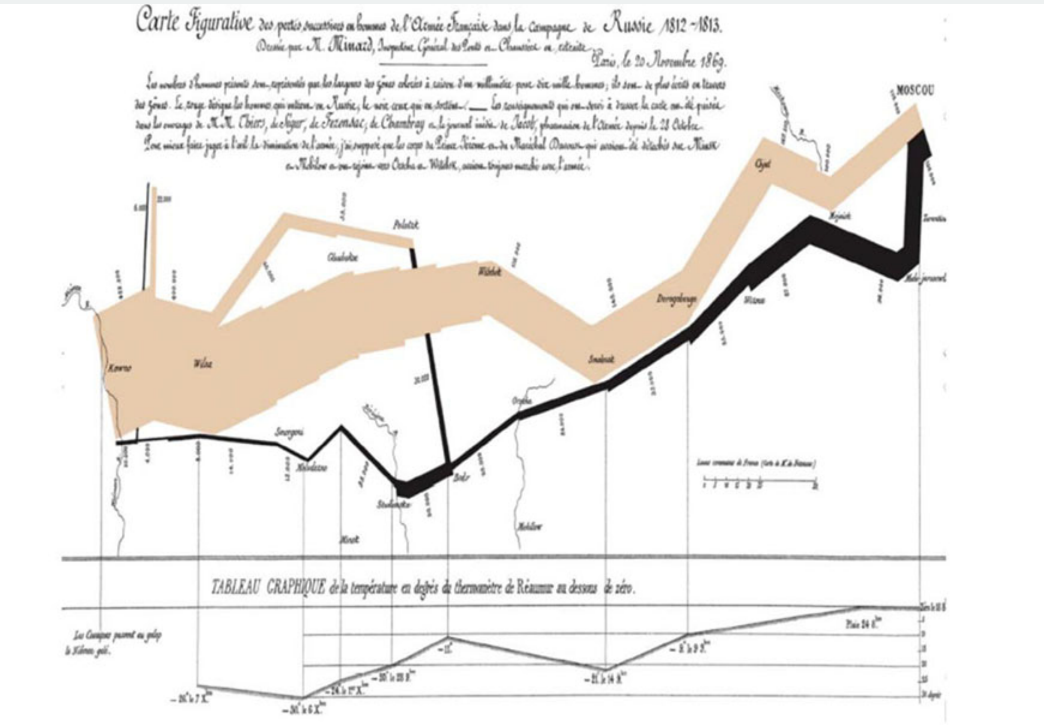
Carte Figurative des pertes successives en hommes de l'Armée Française dans la campagne de Russie 1812–1813.
Paris, le 20 Novembre 1869.
MOSCOU
TABLEAU GRAPHIQUE de la température en degrés du thermomètre de Réaumur au dessous de zéro.

Figure 8.2.27: Minard's Napoleon Map (credit: Carte de Charles Minard/Wikimedia, public domain)

Minard's chart is remarkable in that it shows not just how the size of Napoleon's army shrank drastically over time, but also the location on the map, the direction the army was traveling at the time, and the temperature during the retreat.

Check Your Understanding

The medical office at a zoo tracks the animals it treats each week. The table shows the classifications for a particular week:

Mammal	Mammal	Reptile	Bird	Mammal	Amphibian
Mammal	Mammal	Mammal	Reptile	Mammal	Bird

Mammal	Bird	Reptile	Reptile	Amphibian	Mammal
Bird	Mammal	Amphibian	Mammal	Mammal	Bird

1. Create a bar graph of the data without technology.
2. Create a pie chart of the data using technology.

Employees at a college help desk track the number of people who request assistance each week. The table gives a sample of the results :

142	153	158	156	141	143
139	158	156	146	137	153
136	127	157	148	132	139
155	167	143	168	133	157
138	156	164	130	148	136

3. Make a stem-and-leaf plot of the data.
4. Create a histogram of the data. Use bins of width 5.

The following are data on the admission rates of the different branch campuses in the University of California system, along with the out-of-state tuition and fee cost:

Campus	Admission Rate	Cost ($)
Berkeley	0.1484	43,176
Davis	0.4107	43,394
Irvine	0.2876	42,692
Los Angeles	0.1404	42,218
Merced	0.6617	42,530
Riverside	0.5057	42,819
San Diego	0.3006	43,159
Santa Barbara	0.322	43,383
Santa Cruz	0.4737	42,952

(source: https://data.ed.gov)

5. Create a bar graph that illustrates the differences in admission rates among the different campuses.
6. Create two bar graphs for the out-of-state tuition. One should give an unbiased perception of the differences among them, and the other should overemphasize those differences.

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8.2.0: Exercises

The table below shows the answers to the question, “Which social media platform, if any, do you use most frequently?”

None	Twitter	Snapchat	Snapchat	Twitter	Facebook
Instagram	Snapchat	Twitter	None	Snapchat	Instagram
Instagram	Facebook	None	Instagram	Snapchat	Twitter
Snapchat	Instagram	Instagram	Twitter	Snapchat	Twitter
Facebook	None	Instagram	Instagram	Twitter	Instagram

Exercise 8.2.0.1

Make a bar chart to visualize these responses.

Exercise 8.2.0.2

Make a pie chart to visualize these responses.

A sample of students at a large university were asked whether they were full-time students living on campus (Full-Time Residential, FTR), full-time students who commuted (FTC), or part-time students (PT). The raw data are in the table below:

FTR	FTR	FTC	PT	FTR	PT	FTR	FTC
FTR	FTC	FTC	FTR	FTR	PT	FTC	FTC
FTC	PT	FTC	FTC	PT	FTR	FTC	PT
FTC	PT	FTR	PT	FTC	FTC	FTR	PT

Exercise 8.2.0.3

Make a bar chart to visualize these responses.

Exercise 8.2.0.4

Make a pie chart to visualize these responses.

Students in a statistics class were asked how many countries (besides their home countries) they had visited; the table below gives the raw responses:

0	2	1	1	3	2	0	2	0	1
0	2	0	1	0	1	1	0	1	0
0	0	0	2	1	1	0	1	1	0

Exercise 8.2.0.5

Create a bar graph visualizing these data (treating the responses as categorical).

Exercise 8.2.0.6

Create a pie chart visualizing these data.

The purchasing department for a chain of bookstores wants to make sure they're buying the right types of books to put on the shelves, so they take a sample of 20 books that customers bought in the last five days and record the genres:

Nonfiction	Young Adult	Romance	Cooking	Young Adult
Young Adult	Thriller	Young Adult	Nonfiction	True Crime
Romance	Nonfiction	Thriller	True Crime	Romance
True Crime	Thriller	Romance	Young Adult	Young Adult

Exercise 8.2.0.7

Create a bar graph to visualize these data.

Exercise 8.2.0.8

Create a pie chart to visualize these data.

An elementary school class is administered a standardized test for which scores range from 0 to 100, as shown below:

60	54	71	80	63
72	70	88	88	67
74	79	50	99	64
98	55	64	86	92
72	65	88	80	65

(source: www.nwslsoccer.com)

Exercise 8.2.0.9

Make a stem-and-leaf plot to visualize these results.

Exercise 8.2.0.10

Make a histogram to visualize these results. Use bins of width 10.

The following table gives the final results for the 2021 National Women's Soccer League season. The columns are standings points (PTS; teams earn three points for a win and one point for a tie), wins (W), losses (L), ties (T), goals scored by that team (GF), and goals scored against that team (GA).

Team	PTS	W	L	T	GF	GA
Portland Thorns FC	44	13	6	5	33	17
OL Reign	42	13	8	3	37	24
Washington Spirit	39	11	7	6	29	26

Team	PTS	W	L	T	GF	GA
Chicago Red Stars	38	11	8	5	28	28
NJ/NY Gotham FC	35	8	5	11	29	21
North Carolina Courage	33	9	9	6	28	23
Houston Dash	32	9	10	5	31	31
Orlando Pride	28	7	10	7	27	32
Racing Louisville FC	22	5	12	7	21	40
Kansas City Current	16	3	14	7	15	36

(source: http://www.nwslsoccer.com)

? Exercise $8.2.0.11$

Make a stem-and-leaf plot for PTS.

? Exercise $8.2.0.12$

Make a histogram for PTS, using bins of width 5.

? Exercise $8.2.0.13$

Make a histogram for GF, using bins of width 5.

? Exercise $8.2.0.14$

Make a histogram for GA, using bins of width 5.

For the following exercises, use the "CUNY" dataset–which gives the location (borough) of each college in the City University of New York (CUNY) system, the highest degree offered, and the proportions of total degrees awarded in a partial list of disciplines–to identify the right visualization to address each question. Then, create those visualizations.

? Exercise $8.2.0.15$

What is the highest degree offered in colleges across the CUNY system?

? Exercise $8.2.0.16$

What is the distribution of the proportion of degrees awarded in Information Science across the CUNY system?

? Exercise $8.2.0.17$

In which boroughs are the CUNY colleges located?

? Exercise 8.2.0.18

What are the proportions of degrees awarded across the listed humanities fields (Foreign Language, English, Humanities, Philosophy & Religion, History) at City College?

? Exercise 8.2.0.19

What proportions of degrees are awarded in Social Service at the different institutions located in Manhattan?

For the following exercises, use the data found in the "Receivers" dataset on the top 25 receivers (by number of receptions; data collected from pro-football-reference.com) in the NFL during the 2020 season.

? Exercise 8.2.0.20

Make a stem-and-leaf plot for the longest receptions (“Long”).

? Exercise 8.2.0.21

Make a stem-and-leaf plot for receptions.

? Exercise 8.2.0.22

Make a histogram for yards.

? Exercise 8.2.0.23

Make a histogram for yards per reception (“Yds/Rec”).

? Exercise 8.2.0.24

Make a histogram for the longest receptions (“Long”).

? Exercise 8.2.0.25

Make a histogram for receptions.

? Exercise 8.2.0.26

Make a histogram for age.

? Exercise 8.2.0.27

Describe the distribution of age as left-skewed, symmetric, or right-skewed.

? Exercise 8.2.0.28

Describe the distribution of receptions as left-skewed, symmetric, or right-skewed.

? Exercise 8.2.0.29

Describe the distribution of yards as left-skewed, symmetric, or right-skewed.

? Exercise $8.2.0.30$

Describe the distribution of touchdowns (“TD”) as left-skewed, symmetric, or right-skewed.

? Exercise $8.2.0.31$

Describe the distribution of longest receptions as left-skewed, symmetric, or right-skewed.

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8.3: Mean, Median and Mode

Figure 8.3.1: What does it mean to say someone has average height? (credit: modification of work "I'm the tallest" by Jenn Durfey/Flickr, CC BY 2.0)

Learning Objectives

1. Calculate the mode of a dataset.
2. Calculate the median of a dataset.
3. Calculate the mean of a dataset.
4. Contrast measures of central tendency to identify the most representative average.
5. Solve application problems involving mean, median, and mode.

What exactly do we mean when we describe something as "average"? Is the height of an average person the height that more people share than any other? What if we line up every person in the world, in order from shortest to tallest, and find the person right in the middle: Is that person's height the average? Or maybe it's something more complicated.

Imagine a game where you and a friend are trying to guess the typical person's height. Once the guesses are made, you bring in every person and measure their height. You and your friend figure out how far off each of your guesses were from the actual value, then square that number. The result is the number of points you earn for that person. After we check every height and award points accordingly, the person with the lower score wins (because a lower score means that person's guess was, overall, closer to the actual values). Could we define the average height to be the number that you should guess to give you the smallest possible score?

Each of these three methods of determining the "average" is commonly used. They are all methods of measuring *centrality* (or *central tendency*). Centrality is just a word that describes the middle of a set of data. All give potentially different results, and all are useful for different reasons. In this section, we'll explore each of these methods of finding the "average."

The Mode

In our discussion of average heights, the first possible definition we offered was the height that more people share than any other. This is the mode, or the value that appears most often. If there are two modes, the data are bimodal.

Let's look at some examples.

✓ Example 8.3.1: Finding the Mode Using a Stem-and-Leaf Plot

Previously, we looked at a stem-and-leaf plot of the sale prices (in dollars) of a particular collectible trading card:

0	5 8 9
1	0 0 0 3 4 4 5 5 5 5 6 9 9
2	0 0 0 0 5 5 9 9
3	0 0 0 5 5
4	0 0 5
5	
6	0

What is the mode price?

Answer

The mode is the price that appears most often. Both 15 and 20 appear 4 times, more than any other values. So, they are the modes (and we can conclude that this set of data is bimodal).

Your Turn 8.3.1

Previously, we constructed a stem-and-leaf plot for the number of times in one minute that different crickets chirped:

8	2 2 4 5 9 9 9
9	1 1 2 3 7 9
10	0 1 2 3 3 4 4 4 5 6 6 7 9 9
11	5 6
12	0

What is the mode of the number of chirps in a minute?

When we have a complete list of the data or a stem-and-leaf plot, it’s pretty straightforward to find the mode; we just need to find the number that appears most often. If we’re given a frequency distribution instead, the technique is different (but just as straightforward): we’re looking for the number with the highest frequency.

✓ Example 8.3.2: Finding the Mode Using a Frequency Distribution

Previously, we created a frequency distribution of the number of siblings of conflict resolution class attendees.

Number of Siblings	Frequency
0	5
1	13
2	6
3	3
4	2
5	1

What is the mode of the number of siblings?

Answer

The mode is the value that appears the most often, which means it has the greatest frequency. Thirteen of the respondents have one sibling, more than any other number. So, the mode is 1.

Your Turn 8.3.2

Previously, you found a frequency distribution for the number of people who shared a residence with people in a sample.

Number of People in the Residence	Frequency
1	12
2	13
3	8
4	6
5	1

What's the mode of the number of people in these residences?

What happens if there is no number in the data that appears more than once? In that case, by our definition, every data value is a mode. But according to some other definitions, the data would have no mode. In practice, though, it doesn't really matter; if no data value appears more than once, then the mode is not helpful at all as a measure of centrality.

The Median

Let's revisit our example of trying to identify the height of the "average" person. If we lined everyone up in order by height and found the person right in the middle, that person's height is called the median, or the value that is greater than no more than half and less than no more than half of the values.

Let's look at a really simple example. Consider the following list of numbers: 11, 12, 13, 13, 14. Is the first number on the list, 11, the median? There are no values less than 11 (that's 0%), and there are four values greater than 11 (that's 80%). Since more than 50% of the data are greater than 11, the definition is violated; it's not the median. Here's a chart with the rest of the data, with red shading to show where the definition is violated:

Table 8.3.1

Data Value	Number of Values Below	Percentage of Values Below	Number of Values Above	Percentage of Values Above
11	0	0%	4	**80%**
12	1	20%	3	**60%**
13	2	40%	1	20%
14	4	**80%**	0	0%

Only 13 has no violations, so it's the median according to the definition. In practice, we find the median just like we described in the average height example: by lining up all the data values in order from smallest to largest and picking the value in the middle. For our easy example (with data values 11, 12, 13, 13, 14), that first 13 is right in the middle; there are two values to the left and two values to the right. If there's not one value right in the middle, we pick the two closest, then choose the number exactly between them. For example, let's say we have the data 41, 44, 46, 53. Since there are an even number of data values in our list, we can't pick the one right in the middle. The two closest to the middle are 44 and 46, so we'll choose the number halfway between those to be the median: 45. As this example shows, the median (unlike the mode) doesn't have to be a number in our original set of data.

In the examples we’ve looked at so far, it’s been pretty easy to identify which number is right in the middle. If we had a very large dataset, though, it might be harder. Fortunately, we have some formulas to help us with that.

FORMULA

Suppose we have a set of data with n values, ordered from smallest to largest. If n is odd, then the median is the data value at position mfrac12n+12. If n is even, then we find the values at positions semanticsn2n2 and $\frac{n}{2}+1$n2+1. If those values are named a and b, then the median is defined to be mfracb2a+b2.

Let’s put those formulas to work in an example.

Example 8.3.3: Finding the Median Using a Stem-and-Leaf Plot

Previously, we looked at a stem-and-leaf plot that contained 33 sale prices (in dollars) of a particular collectible trading card:

0	5 8 9
1	0 0 0 3 4 4 5 5 5 5 5 6 9 9
2	0 0 0 0 5 5 9 9
3	0 0 0 5 5
4	0 0 5
5	
6	0

What is the median price?

Answer

Step 1: Since 33 is odd, the median is the data value at position $\frac{n+1}{2}$, where n is the number of values in the dataset. There are 33 total values, so our formula becomes $\frac{33+1}{2} = 17$. That means we want to look for the 17th number in the dataset.

Step 2: We'll want to count from the lowest value to the 17th number. We can use our stem and leaf

	1 2 3
0	5 8 9
	4 5 6 7 8 9 10 11 12 13 14 15 16
1	0 0 0 3 4 4 5 5 5 5 5 6 9 9
	17
2	0 0 0 0 5 5 9 9
3	0 0 0 5 5
4	0 0 5
5	
6	0

The seventeenth number is 20, so the median is 20.

Your Turn 8.3.3

Consider the data given in this stem-and-leaf plot (there are 17 data values):

12	1 2 2 5
13	0 3 4 4 6 8
14	2 5 9 9
15	0 3
16	
17	0

What is the median of this data?

Now, let's tackle an example with an even number of values.

Example 8.3.4: Finding the Median

Previously, we looked at the number of times different crickets (of differing species, genders, etc.) chirped in a one-minute span. That data is again provided below:

89	97	82	102	84	99
115	105	89	109	107	89
101	109	116	103	100	91
93	103	120	91	85	104
104	82	106	92	104	106

Find the median.

Answer

Step 1: In order to find the median, we first need to sort the data so that they're in order, smallest to largest:

82	82	84	85	89	89
89	91	91	92	93	97
99	100	101	102	103	103
104	104	104	105	106	106
107	109	109	115	116	120

Step 2: Next, we figure out how many data values we have. Counting them up, we see there are 30, which is even.

Step 3: Since we have an even number of data values, we need to find the values in positions $\frac{30}{2} = 15$ and $\frac{30}{2} + 1 = 16$. These are 101 and 102.

Step 4: We use the formula to compute the median: $\frac{101+102}{2} = 101.5$.

Your Turn 8.3.4

This table gives the records of the Major League Baseball teams at the end of the 2019 season:

Table 8.11 (source: www.mlb.com)

Team	Wins	Losses	Team	Wins	Losses

Team	Wins	Losses	Team	Wins	Losses
HOU	107	55	PHI	81	81
LAD	106	56	TEX	78	84
NYY	103	59	SFG	77	85
MIN	101	61	CIN	75	87
ATL	97	65	CHW	72	89
OAK	97	65	LAA	72	90
TBR	96	66	COL	71	91
CLE	93	69	SDP	70	92
WSN	93	69	PIT	69	93
STL	91	71	SEA	68	94
MIL	89	73	TOR	67	95
NYM	86	76	KCR	59	103
ARI	85	77	MIA	57	105
BOS	84	78	BAL	54	108
CHC	84	78	DET	47	114

What is the median number of wins?

✓ Example $8.3.5$: Finding the Median Using a Frequency Distribution

Previously, we created a frequency distribution of the number of siblings of the people who attended a conflict resolution class. Let's review that data again:

Number of Siblings	Frequency
0	5
1	13
2	6
3	3
4	2
5	1

What is the median of the number of siblings?

Answer

There are 30 data values total, so the median is between the 15th and 16th values in the ordered list. There are five 0s and thirteen 1s according to the frequency distribution, so items one through five are all 0s and items six through eighteen are all 1s. Since both items fifteen and sixteen are 1s, the median is 1.

Your Turn 8.3.5

Previously, you found a frequency distribution for the number of people who shared a residence with people in a sample. Let's review that data again:

Number of People in the Residence	Frequency
1	12
2	13
3	8
4	6
5	1

What’s the median of the number of people in these residences?

The Mean

Recall our example of ways we could identify the “average” height of an individual. The last method we discussed was also the most complicated. It involved a game where the player guesses a height, then figures out how far off that guess is from every single person’s height. Those differences get squared and added together to get a score. Our next measure of centrality gives the lowest possible score: No other guess would beat it in the game. Given a dataset containing *n* total values, the mean of the dataset is the sum of all the data values, divided by *n*.

This is a computation you have likely done before. In many places, including spreadsheet programs like Microsoft Excel and Google Sheets, this number is called the *average*. For statisticians, though, the word *average* has too many possible meanings, so they prefer the one we’ll use: mean.

✓ Example 8.3.6: Finding the Mean

Compute the mean of the numbers 12, 15, 17, 18, 18, and 19.

Answer

The mean is the sum of the values, divided by the number of values on the list.

$$\frac{12+15+17+18+18+19}{6} = \frac{99}{6} = 16.5.$$

Your Turn \(\PageIndex{6\)

Compute the mean of the numbers 5, 8, 11, 12, 12, 12, 15, 18, and 20. Round your answer to three decimal places.

✓ Example 8.3.7: Finding the Mean Using a Frequency Distribution

Refer again to the frequency distribution of the number of siblings people who attended a conflict resolution class reported:

Number of Siblings	Frequency
0	5
1	13
2	6
3	3

Number of Siblings	Frequency
4	2
5	1

What is the mean of the number of siblings?

Answer

Step 1: We compute the mean by adding up all the data values and then dividing by the number of data values on the list.

Step 2: Adding up the frequencies, we get $5+13+6+3+2+1=30$ data values in our list.

Step 3: Now, to find the sum of all the data values, we could simply reconstruct the raw data and add up all the numbers there. But, there's an easier way: Remember that repeatedly adding a number to itself is the definition of multiplication. So, for example, since there are six 2 s in our data, the sum of all those 2 s must be $6\times 2=12$.

Step 4: Let's add a column to our distribution for these products:

Number of Siblings	Frequency	(Number of Siblings) $\times$ (Frequency)
0	5	0
1	13	13
2	6	12
3	3	9
4	2	8
5	1	5

Step 5: So, the sum of all our data values is $0+13+12+9+8+5=47$. The mean is $\frac{47}{30}\approx 1.567$.

Your Turn 8.3.7

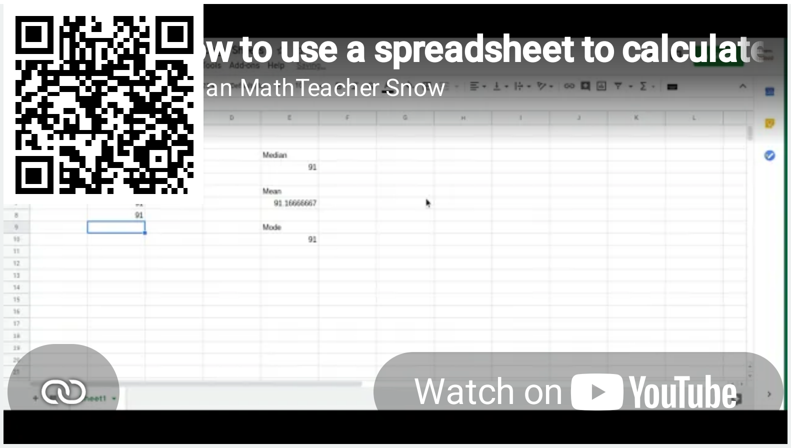
Refer again to the frequency distribution of the number of people who shared a residence with people in a sample:

Number of People in the Residence	Frequency
1	12
2	13
3	8

What’s the mean of the number of people in these residences?

As the number of data values we are considering grows, the computation for the mean gets more and more complicated. That’s why people generally trust technology to perform that computation.

Video



Compute Measures of Centrality Using Google Sheets

Note that a recent update to Google Sheets introduced a new function called "MODE.MULT," which will find every mode (not just the first one on the list).

✓ Example 8.3.8: Using Google Sheets to Compute Measures of Centrality

The dataset "InState" contains the in-state tuitions of every college and university in the country that reported that data to the Department of Education. Find the mode, median, and mean of those in-state tuition values.

Answer

Step 1: To find the mode, we select an empty cell type "=MODE(", click on the header of the column to insert a reference to the column into our formula, and then close the parentheses. When we hit the enter key, our formula is replaced with the mode: $13,380.

Step 2: We can find the median and mean using the same process, except using the functions "MEDIAN" and "AVERAGE" in place of "MODE". We find that the median is $11,207 and the mean is $15,476.79.

Your Turn 8.3.8

The dataset "MLB2019" gives the number of wins for every Major League Baseball team in the 2019 season. Use Google Sheets to find the mode(s), median, and mean.

Which Is Better: Mode, Median, or Mean?

If the mode, median, and mean all purport to measure the same thing (centrality), why do we need all three? The answer is complicated, as each measure has its own strengths and weaknesses. The mode is simple to compute, but there may be more than one. Further, if no data value appears more than once, the mode is entirely unhelpful. As for the mean and median, the main difference between these two measures is how each is affected by extreme values.

Consider this example: the mean and median of 1, 2, 3, 4, 5 are both 3. But what if the dataset is instead 1, 2, 3, 4, 10? The median is still 3, but the mean is now 4. What this example shows is that the mean is sensitive to extreme values, while the median isn't. This knowledge can help us decide which of the two is more relevant for a given dataset. If it is important that the really high or really low values are reflected in the measure of centrality, then the mean is the better option. If very high or low values are not important, however, then we should stick with the median.

The decision between mean and median only really matters if the data are skewed. If the data are symmetric, then the mean and median are going to be approximately equal, and the distinction between them is irrelevant. If the data are skewed, the mean gets pulled in the direction of the skew (i.e., if the data are right-skewed, then the mean will be bigger than the median; if the data are left-skewed, the opposite relation is true).

✓ Example 8.3.9: Choosing Which Measure of Centrality to Use

For the following situations, decide which measure(s) of centrality would be best:

1. You found a used car that you like, and you want to know if the price is too high or too low. You find a list of sale prices for that make and model, and you see that the distribution of those prices is skewed to the right. Some of the prices at the high end are close to the original sale price of the car, so you guess that those cars might have really low mileage, or have other enhancements added on that increased the value (but which don't apply to the car you found).
2. You are asked to analyze the responses to a survey. One of the questions asked, "How strongly do you agree with the statement, 'I believe my elected representatives have my best interests in mind when they vote'?" Responses are a number between 1 and 5, with 1 representing "strongly disagree" and 5 representing "strongly agree."
3. You are asked to find the "average" household income for a zip code. Those values are skewed right.
4. Thinking back to the situation at the beginning of this section: you want to find "average" height. The data you've collected seem to be distributed symmetrically.

Answer

1. In this situation, the high values are not comparable to the value of the car you found and we don't want them to affect the results. Also, we're unlikely to find many repeated values, so the mode is probably not useful. Median is best.
2. Here, we want to know what a typical result is. The mean doesn't really make sense; it involves adding the numbers together, so it would treat two "strongly disagree" and two "strongly agree" responses (those add to $1+1+5+5=12$) as exactly the same as four "neutral" responses ($3+3+3+3=12$). But those are really different situations; the first shows a strong polarization in the responses, while the second represents strong indifference. The mode is probably the best choice (because the data are actually categorical), but the median would be good too.
3. The mode isn't going to be useful in this situation because it's unlikely you will find many households that have exactly the same income. The mean and median will be different because of the skew, so the choice comes down to the extreme values. Remember that the data are skewed right, so high values are prevalent. Because these high values are important for our analysis, we want them to be reflected in the results. Thus, the mean is best. That being said, the median is also useful; it allows us to say something like "50% of the households surveyed make more than" the median.
4. Because we aren't likely to find many people with exactly the same height, the mode won't be useful. Since the data are symmetric, the mean and the median will be about the same. So, it doesn't really matter which of those two we choose.

Your Turn 8.3.9

For the following situations, decide which measure(s) of centrality would be best:

The data come from a survey question where the responses range from 1 ("not interested") to 10 ("very interested").

The data are fuel efficiency measures (in miles per gallon) for various vehicles. The data are right-skewed.

The data are scores on an exam, and they are left-skewed.

Check Your Understanding

1. Use the given stem-and-leaf plot to determine the mode, median, and mean:

8	8 9
9	0 0 7
10	2 5 6 7 8
11	1 2 2 2 4 5 7 9 9
12	0 0 3 5 7
13	0 1 1 4

2. A survey of college students asked how many courses those students were currently taking. The results are summarized in this frequency distribution:

Number of Classes	Frequency
1	1
2	3
3	16
4	8
5	4

3. Find the mode of the number of classes.
4. Find the median of the number of classes.
5. Find the mean of the number of classes.

6. Employees at a college help desk track the number of people who request assistance each week, as recorded below:

142	153	158	156	141	143	155	167	143	168
139	158	156	146	137	153	138	156	164	130
136	127	157	148	132	139	133	157	148	136

Compute the mode.

Compute the median.

Compute the mean.

7. The following table provides admission rates of the different branch campuses in the University of California system, along with the out-of-state tuition and fee cost:

Campus	Admission Rate	Cost ($)
Berkeley	0.1484	43,176
Davis	0.4107	43,394
Irvine	0.2876	42,692
Los Angeles	0.1404	42,218
Merced	0.6617	42,530
Riverside	0.5057	42,819
San Diego	0.3006	43,159
Santa Barbara	0.322	43,383
Santa Cruz	0.4737	42,952

(source: https://data.ed.gov)

Compute the mode, median, and mean admission rate.

Compute the mode, median, and mean cost.

Consider the data in “InState,”, which shows in-state tuition and fees for most institutions of higher learning in the United States. Compute the mode, median, and mean.

8. Workers for a particular company have complained that their wages are too low, while the management says wages are plenty high. If the company’s salaries are right-skewed, which measure of centrality will the workers choose to represent salary? Which will management choose?

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8.3.0: Exercises

Use the steam-and-leaf plot below to answer the questions. Round all answers to three decimal places.

0	1 1 1 2 6 6 7
1	0 0 0 0 2 5 5 9
2	0 0 5 9
3	0 5
4	0

Exercise 8.3.0.1

Find the mode.

Exercise 8.3.0.2

Find the median.

Exercise 8.3.0.3

Find the mean.

Use the stem-and-leaf plot below to answer the following questions. Round all answers to three decimal places.

7	8
8	0 6
9	1 1 4 8
10	0 0 3 5 9
11	0 0 1 1 1 2 6 8
12	0 1 5

Exercise 8.3.0.4

Find the mode.

Exercise 8.3.0.5

Find the median.

Exercise 8.3.0.6

Find the mean.

The following table shows the frequency distribution for a number of countries visited by students. Use this table to answer the questions, rounding all answers to three decimal places.

Number of Countries	Frequency
0	13

Number of Countries	Frequency
1	11
2	5
3	1

? Exercise 8.3.0.7

Find the mode of the number of countries visited.

? Exercise 8.3.0.8

Find the median of the number of countries visited.

? Exercise 8.3.0.9

Find the mean of the number of countries visited.

The following table shows the frequency distribution for the number of fumbles by a selection of football players. Use this table to answer the questions, rounding all answers to three decimal places.

Number of Fumbles	Frequency
0	5
1	13
2	3
3	4

? Exercise 8.3.0.10

Find the mode of the number of fumbles.

? Exercise 8.3.0.11

Find the median of the number of fumbles.

? Exercise 8.3.0.12

Find the mean of the number of fumbles.

A public opinion poll about an upcoming election asked respondents, "How many political advertisements do you recall seeing on television in the last 24 hours?" Use this data to answer the questions, rounding all answers to three decimal places.

6	2	5	5	2	2	4	1	3	0	1	2	1	6	2
5	2	4	8	6	3	3	4	2	5	3	4	2	2	3

? Exercise 8.3.0.13

Compute the mode.

? Exercise 8.3.0.14

Compute the median.

? Exercise 8.3.0.15

Compute the mean.

For the following exercises, use the data found in “TNSchools”, which has data on many institutions of higher education in the state of Tennessee. Here are what the columns represent:

Column Name	Description
AdmRate	Proportion of applicants that are admitted
UGEnr	Number of undergraduate students
PTUG	Proportion of undergraduates who attend part-time
InState	Tuition and fees for in-state students
OutState	Tuition and fees for out-of-state students
FacSal	Mean monthly faculty salary
Pell	Proportion of students receiving Pell Grants
MedDebt	Median student loan debt at degree completion
StartAge	Mean age at the time of entry
Female	Proportion of students who identify as female

(source: https://data.ed.gov)

? Exercise 8.3.0.16

What is the mean undergraduate enrollment at these schools?

? Exercise 8.3.0.17

What is the median undergraduate enrollment at these schools?

? Exercise 8.3.0.18

Generate a histogram for undergraduate enrollment, and use it to describe the effect of the distribution on the mean and the median as measures of centrality.

? Exercise 8.3.0.19

What is the mean proportion of part-time undergraduates at these schools?

? Exercise 8.3.0.20

What is the median proportion of part-time undergraduates at these schools?

? Exercise 8.3.0.21

Generate a histogram for proportion of part-time undergraduates, and use it to describe the effect of the distribution on the mean and the median as measures of centrality.

? Exercise 8.3.0.22

What is the mean of the monthly faculty salaries at these schools?

? Exercise 8.3.0.23

What is the median of the monthly faculty salaries at these schools?

? Exercise 8.3.0.24

Generate a histogram for the monthly faculty salaries, and use it to describe the effect of the distribution on the mean and the median as measures of centrality.

The dataset “AvgSAT” gives the average SAT score for incoming students (the use of “average” here stands in for “mean” to avoid confusion in the following problems) for every institution in the United States that reported the data (from data.ed.gov). Use that dataset to answer the following questions.

? Exercise 8.3.0.25

What is the mean of the average SAT scores?

? Exercise 8.3.0.26

What is the median of the average SAT scores?

? Exercise 8.3.0.27

Construct a histogram of average SAT scores, and describe how the distribution affects the mean and the median.

For the following exercises, use the table, which gives the final results for the 2021 National Women’s Soccer League season. The columns are standings points (PTS; teams earn three points for a win and one point for a tie), wins (W), losses (L), ties (T), goals scored by that team (GF), and goals scored against that team (GA).

Team	Points	W	L	T	GF	GA
Portland Thorns FC	44	13	6	5	33	17
OL Reign	42	13	8	3	37	24
Washington Spirit	39	11	7	6	29	26
Chicago Red Stars	38	11	8	5	28	28
NJ/NY Gotham FC	35	8	5	11	29	21
North Carolina Courage	33	9	9	6	28	23

Team	Points	W	L	T	GF	GA
Houston Dash	32	9	10	5	31	31
Orlando Pride	28	7	10	7	27	32
Racing Louisville FC	22	5	12	7	21	40
Kansas City Current	16	3	14	7	15	36

(source: http://www.nwslsoccer.com)

? Exercise 8.3.0.28

Compute the mode(s), median, and mean of points.

? Exercise 8.3.0.29

Compute the mode(s), median, and mean of wins.

? Exercise 8.3.0.30

Compute the mode(s), median, and mean of losses.

? Exercise 8.3.0.31

Compute the mode(s), median, and mean of ties.

? Exercise 8.3.0.32

Compute the mode(s), median, and mean of goals scored (GF).

? Exercise 8.3.0.33

Compute the mode(s), median, and mean of goals against (GA).

For the following exercises, use the data in “CUNY” (taken from data.ed.gov, which gives proportions of degrees awarded in various disciplines) to answer the given questions:

? Exercise 8.3.0.34

Compute the median and mean proportions of degrees awarded in Information Science.

? Exercise 8.3.0.35

Compute the median and mean proportions of degrees awarded in Education.

? Exercise 8.3.0.36

Compute the median and mean proportions of degrees awarded in Humanities.

Exercise $8.3.0.37$

Compute the median and mean proportions of degrees awarded in Biology.

Exercise $8.3.0.38$

Compute the median and mean proportions of degrees awarded in Physical Science.

Exercise $8.3.0.39$

Compute the median and mean proportions of degrees awarded in Social Science.

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8.4: Range and Standard Deviation

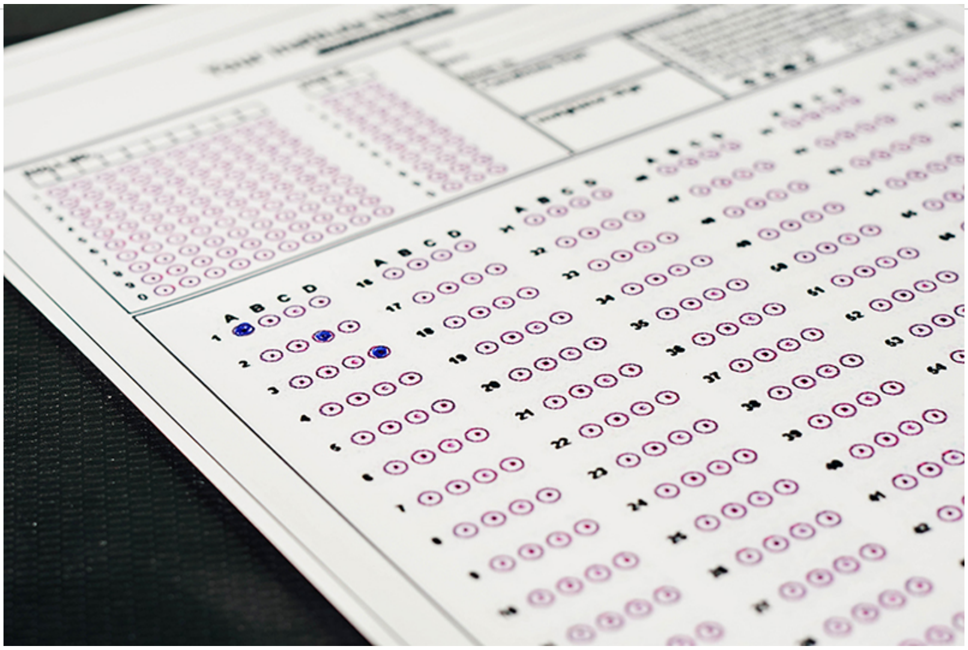


Figure 8.4.1: Measures of spread help us get a better understanding of test scores. (credit: "Standardized test exams form with answers bubbled" by Marco Verch Professional Photographer/Flickr, CC BY 2.0)

Learning Objectives

1. Calculate the range of a dataset
2. Calculate the standard deviation of a dataset

Measures of centrality like the mean can give us only part of the picture that a dataset paints. For example, let's say you've just gotten the results of a standardized test back, and your score was 138. The mean score on the test is 120. So, your score is above average! But how good is it *really*? If all the scores were between 100 and 140, then you know your score must be among the best. But if the scores ranged from 0 to 200, then maybe 140 is good, but not great (though still above average). Knowing information about how the data are spread out can help us put a particular data value in better context. In this section, we'll look at two numbers that help us describe the spread in the data: the range and the standard deviation. These numbers are called measures of dispersion.

The Range

Our first measure of dispersion is the **range**, or the difference between the maximum and minimum values in the set. It's the measure we used in the standardized test example above.

Let's look at a couple of examples.

Example 8.4.1: Finding the Range

You survey some of your friends to find out how many hours they work each week. Their responses are: 5, 20, 8, 10, 35, 12. What is the range?

Answer

The maximum value in the set is 35 and the minimum is 5, so the range is $35 - 5 = 30$.

Your Turn 8.4.1

On your morning commute, you decide to record how long you have to wait each time you get caught at a red light. Here are the times in seconds: 12, 58, 35, 79, 21. What is the range?

For large datasets, finding the maximum and minimum values can be daunting. There are two ways to do it in a spreadsheet. First, you can ask the spreadsheet program to sort the data from smallest to largest, then find the first and last numbers on the sorted list. The second method uses built-in functions to find the minimum and maximum.

Video

ogle Sheets: Min and Max Formula
nology Central
Min and Max
Functions
Watch on YouTube

Find the Minimum and Maximum Using Google Sheets

In either method, once you've found the maximum and minimum, all you have to do is subtract to find the range.

Example 8.4.2: Finding the Range with Google Sheets

The data in "AvgSAT" contains the average SAT score for students attending every institution of higher learning in the US for which data is available. What is the range of these average SAT scores?

Answer

Step 1: To find the maximum, click on an empty cell in the spreadsheet, type "=MAX(", and then click on the letter that marks the top of the column containing the AvgSAT data. That inserts a reference to the column into our function. Then we close the parentheses and hit the enter key. The formula is replaced with the maximum value in our data: 1566.
Step 2: Using the same process (but with "MIN" instead of "MAX"), we find the minimum value is 785.
Step 3: So, the range is $1566 - 785 = 781$.

Your Turn 8.4.2

The file "InState" contains in-state tuition costs (in dollars) for every institution of higher learning in the US for which data is available. What is the range of these costs?

The range is very easy to compute, but it depends only on two of the data values in the entire set. If there happens to be just one unusually high or low data value, then the range might give a distorted measure of dispersion. Our next measure takes every single data value into account, making it more reliable.

The Standard Deviation

The standard deviation is a measure of dispersion that can be interpreted as approximately the average distance of every data value from the mean. (This distance from the mean is the "deviation" in "standard deviation.")

FORMULA

The standard deviation is computed as follows:

$$s=\sqrt{\frac{\sum(x-x)^2}{n-1}}$$

Here, x represents each data value, $\overline{x}$ is the mean of the data values, n is the number of data values, and the capital sigma (Σ) indicates that we take a sum.

To compute the standard deviation using the formula, we follow the steps below:

1. Compute the mean of all the data values.
2. Subtract the mean from each data value.
3. Square those differences.
4. Add up the results in step 3.
5. Divide the result in step 4 by $n-1$
6. Take the square root of the result in step 5.

Let's see that process in action.

Example $8.4.3$: Computing the Standard Deviation

You surveyed some of your friends to find out how many hours they work each week. Their responses were: 5, 20, 8, 10, 35, 12. What is the standard deviation?

Answer

Let's follow the six steps mentioned previously to compute the standard deviation.

Step 1: Find the mean: $\overline{x}=\frac{5+20+8+10+35+12}{6}=15$.

Step 2: Subtract the mean from each data value. To help keep track, let's do this in a table. In the first row, we'll list each of our data values (and we'll label the row x); in the second, we'll subtract $\overline{x}=15$ from each data value.

x	5	20	8	10	35	12
$x-\overline{x}$	−10	5	–7	–5	20	–3

Step 3: Square the differences. Let's add a row to our table for those values:

x	5	20	8	10	35	12
$x-\overline{x}$	−10	5	–7	–5	20	–3
$(x-\overline{x})^2$	100	25	49	25	400	9

Step 4: Add up those squares: $100+25+49+25+400+9=608$

Step 5: Divide the sum by $n-1$. Since we have 6 data values, that gives us $\frac{608}{6-1}=121.6$.

Step 6: Take the square root of the result: $\sqrt{121.6}\approx 11.027$.

Thus, the standard deviation is $s\approx 11.027$.

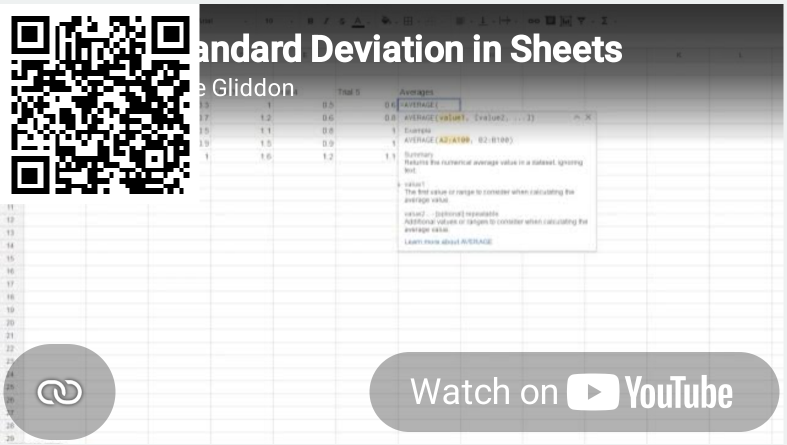
Your Turn $8.4.3$

On your morning commute, you decide to record how long you have to wait each time you get caught at a red light. Here are the times in seconds: 12, 58, 35, 79, 21.

What is the standard deviation?

The computation for the standard deviation is complicated, even for just a small dataset. We'd never want to compute it without technology for a large dataset! Luckily, technology makes this calculation easy.

Video



Find the Standard Deviation Using Google Sheets

Example 8.4.4: Finding the Standard Deviation with Google Sheets

The data in "AvgSAT" contains the average SAT score for students attending every institution of higher learning in the US for which data is available. What is the standard deviation of these average SAT scores?

Answer

To find the standard deviation, we click in an empty cell in our spreadsheet and then type "=STDEV(". Next, click on the letter at the top of the column containing our data; this will put a reference to that column into our formula. Then close the parentheses with and hit the enter key. The formula is replaced with the result: 125.517.

Your Turn 8.4.4

The file "InState" contains in-state tuition costs (in dollars) for every institution of higher learning in the US for which data is available. What is the standard deviation of these costs?

Check Your Understanding

1. Given the data 1, 4, 5, 5, and 10, find the range.
2. Given the data 1, 4, 5, 5, and 10, find the standard deviation using the process outlined in the definition.
3. Employees at a college help desk track the number of people who request assistance each week, as listed below:

142	153	158	156	141	143			
139	158	156	146	137	153			
136	127	157	148	132	143	168	133	157
138	156	164	130	148	136			

Compute the range.

Compute the standard deviation.

4. The following are data on the admission rates of the different branch campuses in the University of California system, along with the out-of-state tuition and fee cost.

Campus	Admission Rate	Cost ($)
Berkeley	0.1484	43,176
Davis	0.4107	43,394
Irvine	0.2876	42,692
Los Angeles	0.1404	42,218
Merced	0.6617	42,530
Riverside	0.5057	42,819
San Diego	0.3006	43,159
Santa Barbara	0.322	43,383
Santa Cruz	0.4737	42,952

(source: https://data.ed.gov)

Compute the range of the admission rate.

Compute the standard deviation of the admission rate.

5. Using the data above, find the:

Range of the cost.

Standard deviation of the cost.

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8.4.0: Exercises

Given the data 81, 84, 89, 85, 86, 84:

? Exercise 8.4.0.1

Compute the range.

? Exercise 8.4.0.2

Compute the standard deviation using the formula.

Given the data 127, 167, 156, 158, 156, 157:

? Exercise 8.4.0.3

Compute the range.

? Exercise 8.4.0.4

Compute the standard deviation using the formula.

For the following exercises, use the data found in "TNSchools", which has data on many institutions of higher education in the state of Tennessee. Here are what the columns represent:

Column Name	Description
AdmRate	Proportion of applicants that are admitted
UGEnr	Number of undergraduate students
PTUG	Proportion of undergraduates who attend part-time
InState	Tuition and fees for in-state students
OutState	Tuition and fees for out-of-state students
FacSal	Mean monthly faculty salary
Pell	Proportion of students receiving Pell Grants
MedDebt	Median student loan debt at degree completion
StartAge	Mean age at the time of entry
Female	Proportion of students who identify as female

(source: https://data.ed.gov)

? Exercise 8.4.0.5

Find the range of admissions rates.

? Exercise 8.4.0.6

Find the standard deviation of admissions rates.

? Exercise 8.4.0.7

Find the range of undergraduate enrollments.

? Exercise 8.4.0.8

Find the standard deviation of undergraduate enrollments.

? Exercise 8.4.0.9

Find the range of the proportions of undergraduates attending part-time.

? Exercise 8.4.0.10

Find the standard deviation of the proportions of undergraduates attending part-time.

? Exercise 8.4.0.11

Find the range of in-state tuition costs.

? Exercise 8.4.0.12

Find the standard deviation of in-state tuition costs.

? Exercise 8.4.0.13

Find the range of median debt.

? Exercise 8.4.0.14

Find the standard deviation of median debt.

For the following exercises, use the table below, which gives the final results for the 2021 National Women's Soccer League season. The columns are standings points (PTS; teams earn three points for a win and one point for a tie), wins (W), losses (L), ties (T), goals scored by that team (GF), and goals scored against that team (GA).

Team	PTS	W	L	T	GF	GA
Portland Thorns FC	44	13	6	5	33	17
OL Reign	42	13	8	3	37	24
Washington Spirit	39	11	7	6	29	26
Chicago Red Stars	38	11	8	5	28	28
NJ/NY Gotham FC	35	8	5	11	29	21
North Carolina Courage	33	9	9	6	28	23
Houston Dash	32	9	10	5	31	31
Orlando Pride	28	7	10	7	27	32

Team	PTS	W	L	T	GF	GA
Racing Louisville FC	22	5	12	7	21	40
Kansas City Current	16	3	14	7	15	36

? Exercise $8.4.0.15$

Compute the standard deviation and range of points.

? Exercise $8.4.0.16$

Compute the standard deviation and range of wins.

? Exercise $8.4.0.17$

Compute the standard deviation and range of losses.

? Exercise $8.4.0.18$

Compute the standard deviation and range of ties.

? Exercise $8.4.0.19$

Compute the standard deviation and range of goals scored (GF).

? Exercise $8.4.0.20$

Compute the standard deviation and range of goals against (GA).

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8.5: Percentiles

Figure $8.5.1$: Two students graduating with the same class rank could be in different percentiles depending on the school population. (credit: "graduation caps" by John Walker/Flickr, CC BY 2.0)

Learning Objectives

1. Compute percentiles.
2. Solve application problems involving percentiles.

A college admissions officer is comparing two students. The first, Anna, finished 12th in her class of 235 people. The second, Brian, finished 10th in his class of 170 people. Which of these outcomes is better? Certainly 10 is less than 12, which favors Brian, but Anna’s class was much bigger. In fact, Anna beat out 223 of her classmates, which is $\frac{223}{235} \approx 95\%$ of her classmates, while Brian bested 160 out of 170 people, or 94%. Comparing the proportions of the data values that are below a given number can help us evaluate differences between individuals in separate populations. These proportions are called percentiles. If $p\%$ of the values in a dataset are less than a number n, then we say that n is at the pth percentile.

Finding Percentiles

There are some other terms that are related to "percentile" with meanings you may infer from their roots. Remember that the word *percent* means “per hundred.” This reflects that percentiles divide our data into 100 pieces. The word quartile has a root that means “four.” So, if a data value is at the first quantile of a dataset, that means that if you break the data into four parts (because of the *quart*-), this data value comes after the first of those four parts. In other words, it’s greater than 25% of the data, placing it at the 25th percentile. Quintile has a root meaning “five,” so a data value at the third quintile is greater than three-fifths of the data in the set. That would put it at the 60th percentile. The general term for these is quantiles (the root *quant*– means “number”).

In Mean, Median, and Mode, we defined the *median* as a number that is greater than no more than half of the data in a dataset and is less than no more than half of the data in the dataset. With our new term, we can more easily define it: The median is the value at the 50th percentile (or second quartile).

Let’s look at some examples.

✓ Example ✓ 8.5.1: Finding Percentiles

Consider the dataset 5, 8, 12, 1, 2, 16, 2, 15, 20, 22.

1. At what percentile is the value 5?
2. What value is at the 60th percentile?

Answer

Before we can answer these two questions, we must put the data in increasing order: 1, 2, 2, 5, 8, 12, 15, 16, 20, 22.

1. There are three values (1, 2, and 2) in the set that are less than 5, and there are ten values in the set. Thus, 5 is at the $\frac{3}{10} \times 100 =$ percentile.
2. To find the value at the 60th percentile, we note that there are ten data values, and 60% of ten is six. Thus, the number we want is greater than exactly six of the data values. Thus, the 60th percentile is 15.

Your Turn 8.5.1

Consider the dataset 2, 5, 8, 16, 12, 1, 8, 6, 15, 4.

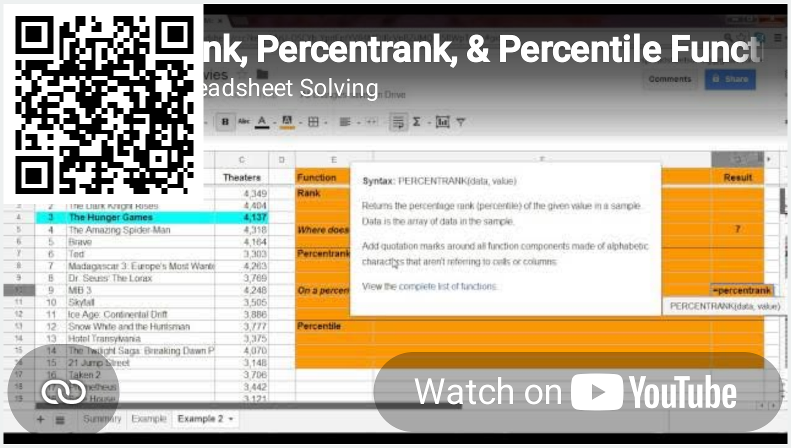
- What value is at the 80th percentile?

At what percentile is the value 12?

In each of the examples above, the computations were made easier by the fact that the we were looking for percentiles that “came out evenly” with respect to the number of values in our dataset. Things don’t always work out so cleanly. Further, different sources will define the term *percentile* in different ways. In fact, Google Sheets has *three* built-in functions for finding percentiles, none of which uses our definition. Some of the definitions you’ll see differ in the inequality that is used. Ours uses “less than or equal to,” while others use “less than” (these correspond roughly to Google Sheets’ ‘PERCENTILE.INC’ and ‘PERCENTILE.EXC’). Some of them use different methods for *interpolating* values. (This is what we did when we first computed medians without technology; if there were an even number of data values in our dataset, found the mean of the two values in the middle. This is an example of interpolation. Most computerized methods use this technique.) Other definitions don’t interpolate at all, but instead choose the closest actual data value to the theoretical value. Fortunately, for large datasets, the differences among the different techniques become very small.

So, with all these different possible definitions in play, what will we use? For small datasets, if you’re asked to compute something involving percentiles *without technology* , use the technique we used in the previous example. In all other cases, we’ll keep things simple by using the built-in ‘PERCENTILE’ and ‘PERCENTRANK’ functions in Google Sheets (which do the same thing as the ‘PERCENTILE.INC’ and ‘PERCENTRANK.INC’ functions; they’re “inclusive, interpolating” definitions).

Video


nk, Percentrank, & Percentile Funct
eadsheet Solving
Syntax: PERCENTRANK(data, value)
Watch on YouTube

Using RANK, PERCENTRANK, and PERCENTILE in Google Sheets

✓ Example 8.5.2: Using Google Sheets to Compute Percentiles: Average SAT Scores

The data in “AvgSAT” contains the average SAT score for students attending every institution of higher learning in the US for which data is available.

1. What score is at the 3rd quartile?
2. What score is at the 40th percentile?
3. At what percentile is Albion College in Michigan (average SAT: 1132)?
4. At what percentile is Oregon State University (average SAT: 1205)?

Answer

1. The 3rd quartile is the 75th percentile, so we’ll use the PERCENTILE function. Click on an empty cell, and type “=PERCENTILE(“. Next, enter the data: click on the letter at the top of the column containing the average SAT scores. Then, tell the function which percentile we want; it needs to be entered as a decimal. So, type a comma (to separate the two pieces of information we’re giving this function), then type “0.75” and close the parentheses with “)”. The result should look like this (assuming the data are in column C): “=PERCENTILE(C:C, 0.75)”. When you hit the enter key, the formula will be replaced with the average SAT score at the 75th percentile: 1199.
2. Using the PERCENTILE function, we find that an average SAT of 1100 is at the 40th percentile.
3. Since we want to know the percentile for a particular score, we’ll use the PERCENTRANK function. Like the PERCENTILE function, we need to give PERCENTRANK two pieces of information: the data, and the value we care about. So, click on an empty cell, type “=PERCENTRANK(“, and then click on the letter at the top of the column containing the data. Next, type a comma and then the value we want to find the percentile for; in this case, we’ll type “, 1132”. Finally, close the parentheses with “)” and hit the enter key. The formula will be replaced with the information we want: an average SAT of 1132 is at the 54th percentile.
4. Using the PERCENTRANK function, an average SAT of 1205 is at the 76.1th percentile.

Your Turn 8.5.2

Looking again at the “AvgSAT” dataset:

1. What score is at the 15th percentile?
2. What score is at the 90th percentile?
3. At what percentile is the University of Missouri (Columbia campus), whose average SAT score is 1244?
4. At what percentile is Rice University in Texas, whose average SAT score is 1513?

✓ Example 8.5.3: Using Google Sheets to Compute Percentiles: In-State Tuition

The dataset "InState" contains the in-state tuitions of every college and university in the country that reported that data to the Department of Education. Use that dataset to answer these questions.

1. What tuition is at the second quintile?
2. What tuition is at the 95th percentile?
3. At what percentile is Walla Walla University in Washington (in-state tuition: $28,035)?
4. At what percentile is the College of Saint Mary in Nebraska (in-state tuition: $20,350)?

Answer

1. The second quintile is the 40th percentile; using PERCENTILE in Google Sheets, we get $8,400.
2. Using PERCENTILE again, we get $44,866.
3. Using PERCENTRANK, we get the 81.6th percentile.
4. Using PERCENTRANK, we get the 74.8th percentile.

Your Turn 8.5.3

Looking again at the "InState" dataset, answer these questions.

1. What tuition is at the 10th percentile?
2. What tuition is at the fourth quintile?
3. At what percentile is the main campus of New Mexico State University (in-state tuition: $6,686)?
4. At what percentile is Bowdoin College in Maine (in-state tuition: $53,922)?

Check Your Understanding

1. Given the data 10, 12, 14, 18, 21, 23, 24, 25, 29, and 30, compute the following without technology:

The value at the 30th percentile

The value at the first quintile

At what percentile 29 falls

At what percentile 24 falls

For the remainder of these problems, use the dataset "MLB2019," which gives the number of wins for each Major League Baseball team in the 2019 season. Use Google Sheets to compute your answers.

2. How many wins is at the 30th percentile?
3. How many wins is at the 90th percentile?
4. How many wins is at the first quartile?
5. At what percentile are the Chicago Cubs (CHC, 84 wins)?
6. At what percentile are the Tampa Bay Rays (TBR, 96 wins)?
7. At what percentile are the Toronto Blue Jays (TOR, 67 wins)?

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8.5.0: Exercises

For the following exercises, use the following twenty data values to answer the questions *without technology*: 1, 4, 6, 7, 12, 15, 21, 25, 29, 30, 31, 33, 39, 43, 44, 45, 51, 55, 60, 63

? Exercise 8.5.0.1

What data value is at the 10th percentile?

? Exercise 8.5.0.2

What data value is at the 55th percentile?

? Exercise 8.5.0.3

What data value is at the 90th percentile?

? Exercise 8.5.0.4

What data value is at the 30th percentile?

? Exercise 8.5.0.5

What data value is at the first quartile?

? Exercise 8.5.0.6

What data value is at the third quintile?

? Exercise 8.5.0.7

At what percentile is 29?

? Exercise 8.5.0.8

At what percentile is 55?

? Exercise 8.5.0.9

At what percentile is 4?

? Exercise 8.5.0.10

At what percentile is 51?

For the following exercises, use the data in "TNSchools", which has data on many institutions of higher education in the state of Tennessee. Here are what the columns represent:

Column Name	Description
AdmRate	Proportion of applicants that are admitted
UGEnr	Number of undergraduate students
PTUG	Proportion of undergraduates who attend part-time

Column Name	Description
InState	Tuition and fees for in-state students
OutState	Tuition and fees for out-of-state students
FacSal	Mean monthly faculty salary
Pell	Proportion of students receiving Pell Grants
MedDebt	Median student loan debt at degree completion
StartAge	Mean age at the time of entry
Female	Proportion of students who identify as female

(source: https://data.ed.gov)

? Exercise $8.5.0.11$

What admission rate is at the second quintile?

? Exercise $8.5.0.12$

What admission rate is at the 80th percentile?

? Exercise $8.5.0.13$

What admission rate is at the 90th percentile?

? Exercise $8.5.0.14$

At what percentile is East Tennessee State University for admission rate?

? Exercise $8.5.0.15$

At what percentile is Rhodes College for admission rate?

? Exercise $8.5.0.16$

At what percentile is Freed-Hardeman University for admission rate?

? Exercise $8.5.0.17$

What proportion of part-time undergraduate enrollment is at the third quartile?

? Exercise $8.5.0.18$

What proportion of part-time undergraduate enrollment is at the 15th percentile?

? Exercise $8.5.0.19$

What proportion of part-time undergraduate enrollment is at the 40th percentile?

? Exercise 8.5.0.20

At what percentile is Lee University for proportion of part-time undergraduate enrollment?

? Exercise 8.5.0.21

At what percentile is Fisk University for proportion of part-time undergraduate enrollment?

? Exercise 8.5.0.22

At what percentile is Middle Tennessee State University for proportion of part-time undergraduate enrollment?

? Exercise 8.5.0.23

What median student loan debt is at the 10th percentile?

? Exercise 8.5.0.24

What median student loan debt is at the fourth quintile?

? Exercise 8.5.0.25

What median student loan debt is at the 85th percentile?

? Exercise 8.5.0.26

At what percentile is Carson-Newman College for median student loan debt?

? Exercise 8.5.0.27

At what percentile is Austin Peay State University for median student loan debt?

? Exercise 8.5.0.28

At what percentile is Belmont University for median student loan debt?

? Exercise 8.5.0.29

What mean starting age is at the first quartile?

? Exercise 8.5.0.30

What mean starting age is at the 67th percentile?

? Exercise 8.5.0.31

What mean starting age is at the 35th percentile?

? Exercise 8.5.0.32

At what percentile is the University of the South for mean starting age?

? Exercise 8.5.0.33

At what percentile is Lincoln Memorial University for mean starting age?

? Exercise 8.5.0.34

At what percentile is the University of Tennessee-Chattanooga for mean starting age?

? Exercise 8.5.0.35

What proportion of students who identify as female is at the third quintile?

? Exercise 8.5.0.36

What proportion of students who identify as female is at the 12th percentile?

? Exercise 8.5.0.37

What proportion of students who identify as female is at the 85th percentile?

? Exercise 8.5.0.38

At what percentile is Martin Methodist College for proportion of students who identify as female?

? Exercise 8.5.0.39

At what percentile is Tennessee Technological University for proportion of students who identify as female?

? Exercise 8.5.0.40

At what percentile is Maryville College for proportion of students who identify as female?

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8.6: The Normal Distribution

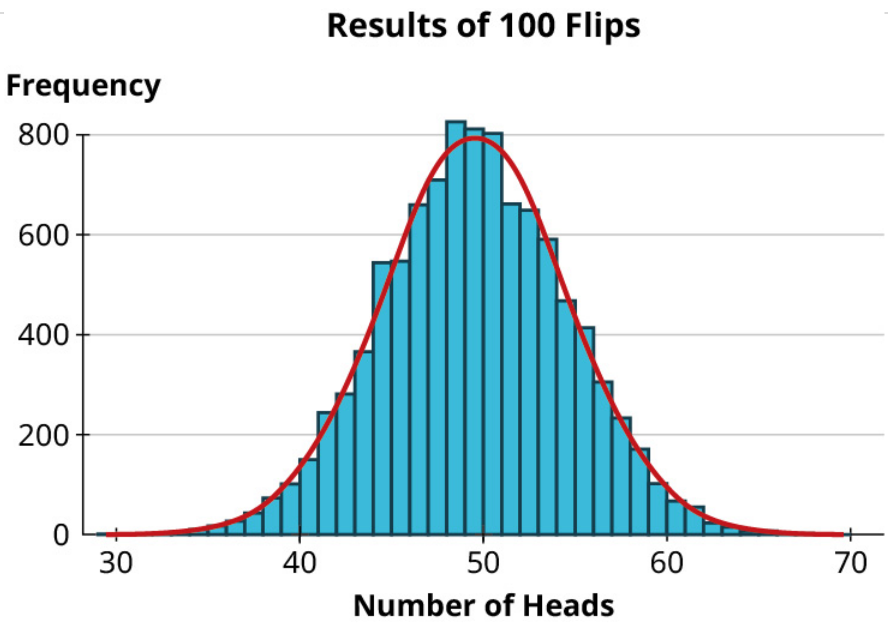

Figure 8.6.1: Symmetric, bell-shaped distributions arise naturally in many different situations, including coin flips.

Learning Objectives

1. Describe the characteristics of the normal distribution.
2. Apply the 68-95-99.7 percent groups to normal distribution datasets.
3. Use the normal distribution to calculate a z-score.
4. Find and interpret percentiles and quartiles.

Many datasets that result from natural phenomena tend to have histograms that are symmetric and bell-shaped. Imagine finding yourself with a whole lot of time on your hands, and nothing to keep you entertained but a coin, a pencil, and paper. You decide to flip that coin 100 times and record the number of heads. With nothing else to do, you repeat the experiment ten times total. Using a computer to simulate this series of experiments, here’s a sample for the number of heads in each trial:

54, 51, 40, 42, 53, 50, 52, 52, 47, 54

It makes sense that we’d get somewhere around 50 heads when we flip the coin 100 times, and it makes sense that the result won’t *always* be exactly 50 heads. In our results, we can see numbers that were generally near 50 and not always 50, like we thought.

Moving Toward Normality

Let’s take a look at a histogram for the dataset in our section opener:

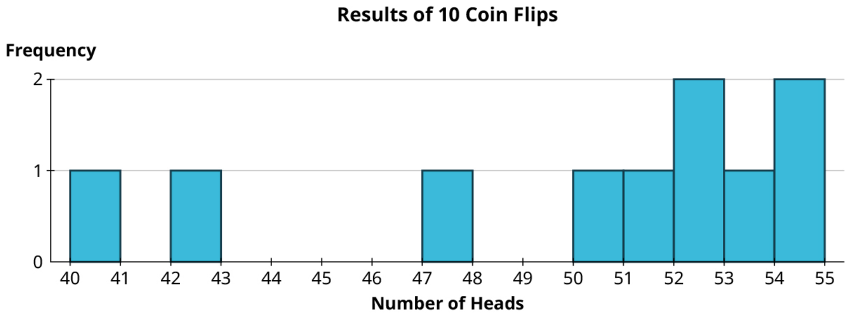

Figure 8.6.1

This is interesting, but the data seem pretty sparse. There were no trials where you saw between 43 and 47 heads, for example. Those results don’t seem impossible; we just didn’t flip enough times to give them a chance to pop up. So, let’s do it again, but this time we'll perform 100 coin flips 100 times. Rather than review all 100 results, which could be overwhelming, let's instead visualize the resulting histogram.

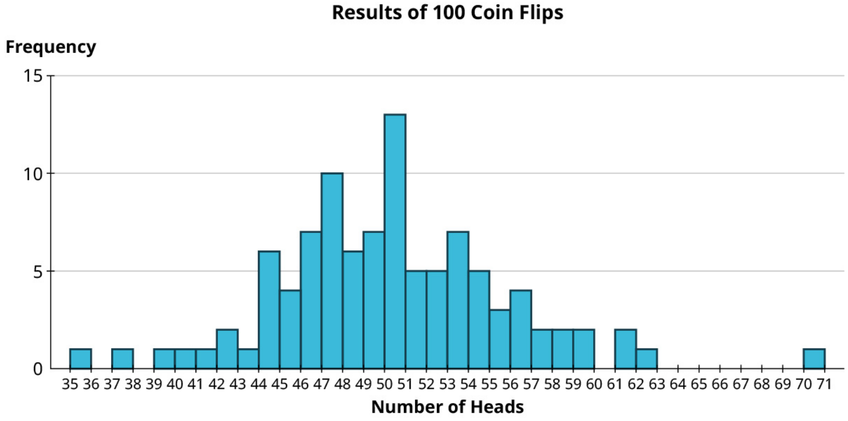

Figure 8.6.2

From the histogram, we see that most of the trials resulted in between, say, 44 and 56 heads. There were some more unusual results: one trial resulted in 70 heads, which seems really unlikely (though still possible!). But we're starting to maybe get a sense of the distribution. More data would help, though. Let's simulate another 900 trials and add them to the histogram!

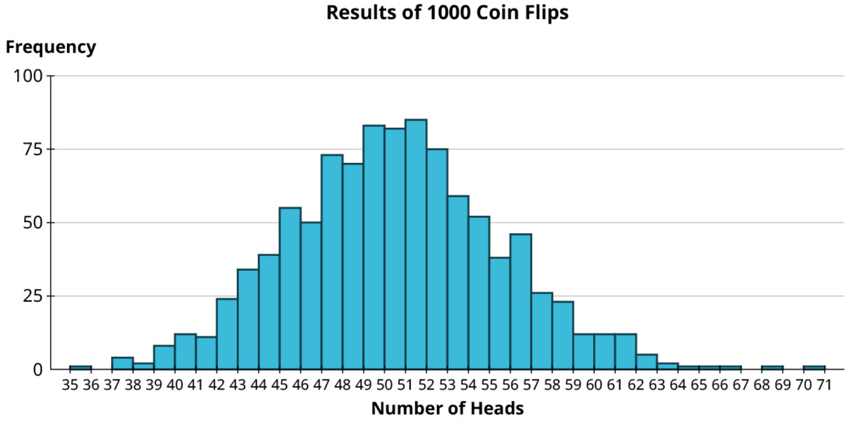

Figure 8.6.3

We can still see that 70 is a really unusual observation, though we came close in another trial (one that had 68 heads). Now, the distribution is coming more into focus: It looks quite symmetric and bell-shaped. Let's just go ahead and take this thought experiment to an extreme conclusion: 10,000 trials.

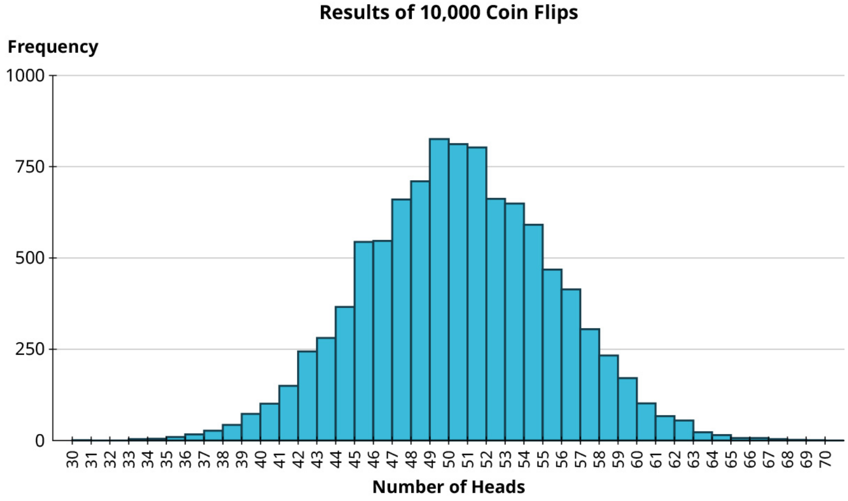

Figure 8.6.4

The distribution is pretty clear now. Distributions that are symmetric and bell-shaped like this pop up in all sorts of natural phenomena, such as the heights of people in a population, the circumferences of eggs of a particular bird species, and the numbers of leaves on mature trees of a particular species. All of these have bell-shaped distributions. Additionally, the results of many types of repeated experiments generally follow this same pattern, as we saw with the coin-flipping example; this fact is the basis for much of the work done by statisticians. It's a fact that's important enough to have its own name: the Central Limit Theorem.

People in Mathematics: John Kerrich

Having enough time on your hands to actually perform this coin-flipping experiment may sound far-fetched, but the English mathematician John Kerrich found himself in just such a situation. While he was studying abroad in Denmark in 1940, that country was invaded by the Germans. Kerrich was captured and placed in an internment camp, where he remained for the duration of the war. Kerrich knew that he had all kinds of time on his hands. He also studied statistics, so he knew what should happen *theoretically* if he flipped a coin many, many times. He also knew of nobody who had ever *tested* that theory with an actual, large-scale experiment. So, he did it: While he was incarcerated, Kerrich flipped a regular coin 10,000 times and recorded the results. Sure enough, the theory held up!

The Normal Distribution

In the coin flipping example above, the distribution of the number of heads for 10,000 trials was close to perfectly symmetric and bell-shaped:

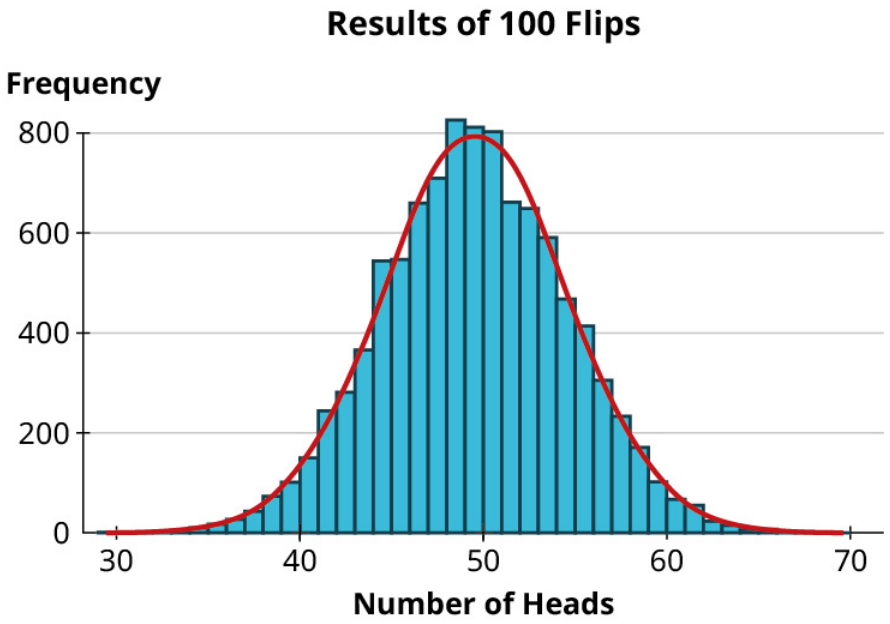

Figure 8.6.5

Because distributions with this shape appear so often, we have a special name for them: normal distributions. Normal distributions can be completely described using two numbers we've seen before: the *mean* of the data and the *standard deviation* of the data. You may remember that we described the mean as a measure of centrality; for a normal distribution, the mean tells us exactly where the center of the distribution falls. The peak of the distribution happens at the mean (and, because the distribution is symmetric, it's also the median). The standard deviation is a measure of dispersion; for a normal distribution, it tells us how spread out the histogram looks. To illustrate these points, let's look at some examples.

✓ Example 8.6.1: Identifying the Mean of a Normal Distribution

This graph shows three normal distributions. What are their means?

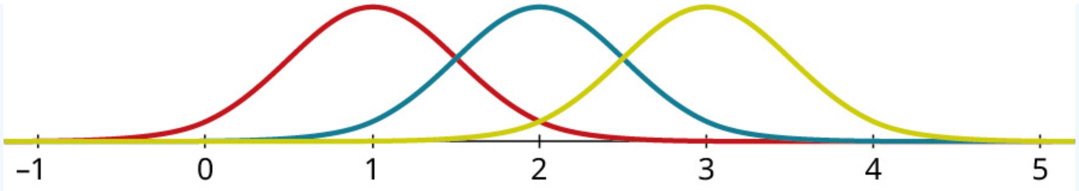

Figure ✓ 8.6.6

Answer

Step 1: Take a look at the three curves on the graph. Since the mean of a normal distribution occurs at the peak, we should look for the highest point on each distribution. Let’s draw a line from each curve's peak down to the axis, so we can see where these peaks occur:

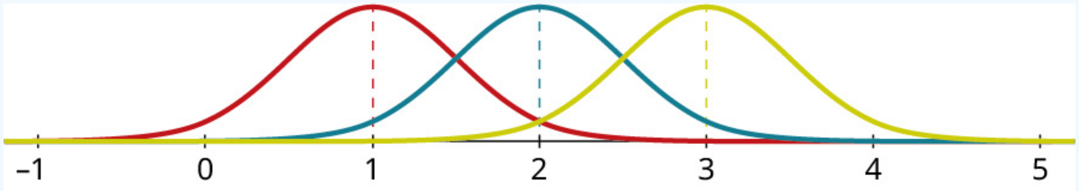

Figure ✓ 8.6.7

Step 2: The peak of the red (leftmost) distribution occurs over the number 1 on the horizontal axis. Thus, the mean of the red distribution is 1. Similarly, the mean of the blue (middle) distribution is 2, and the mean of the yellow (rightmost) distribution is 3.

Your Turn 8.6.1

Identify the means of these three distributions:

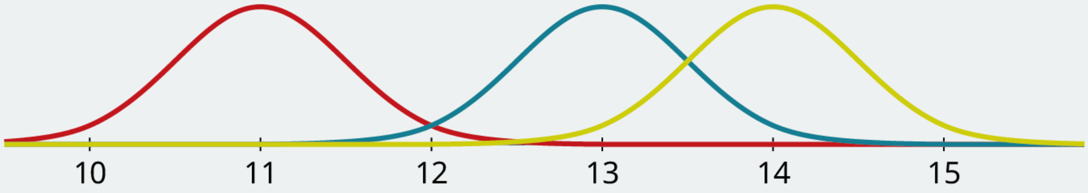

Figure 8.6.8

✓ Example 8.6.2: Identifying the Standard Deviation of a Normal Distribution

This graph shows three distributions, all with mean 2. What are their standard deviations?

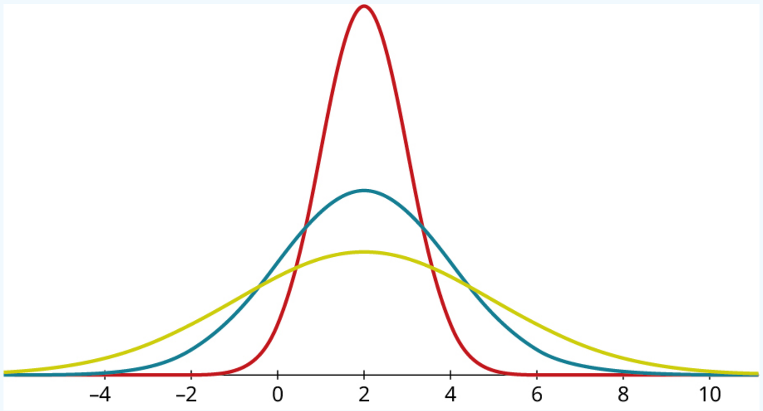

Figure ✓ 8.6.9

Answer

Step 1: Identifying the standard deviation from a graph can be a little bit tricky. Let's focus in on the yellow (lowest peaked) curve:

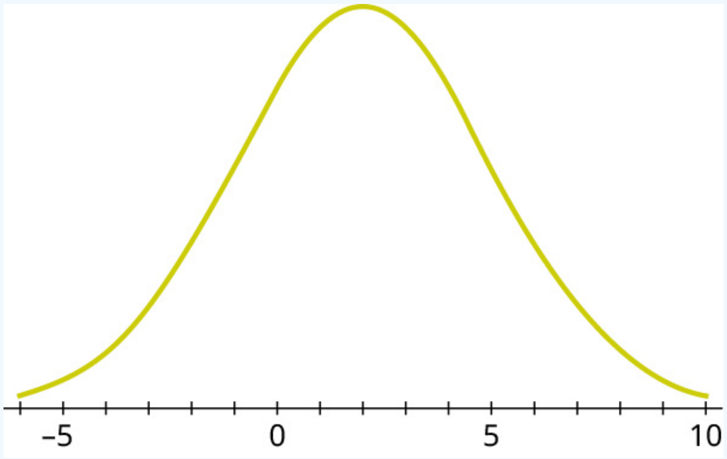

Figure ✓ 8.6.10

Step 2: Notice that the graph curves downward in the middle, and curves upwards on the ends. Highlighted in red is the part that curves downward and in green, the part that curves upward:

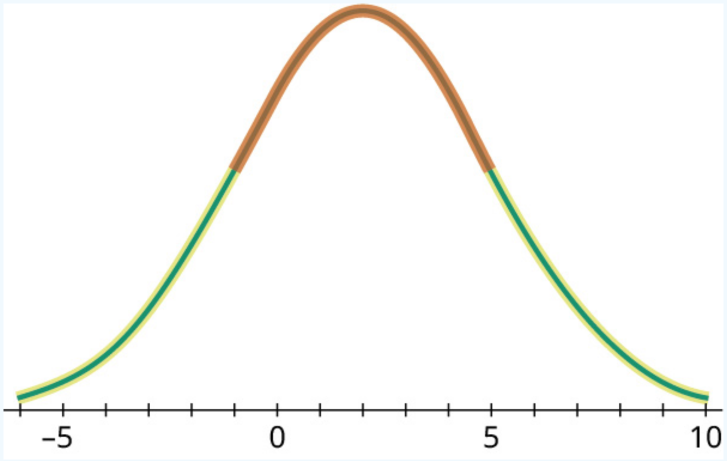

Figure ✓ 8.6.11

Step 3: The places where the graph changes from curving up to curving down (or vice versa) are called *inflection points*. Let's identify where those occur by dropping a line straight down from each:

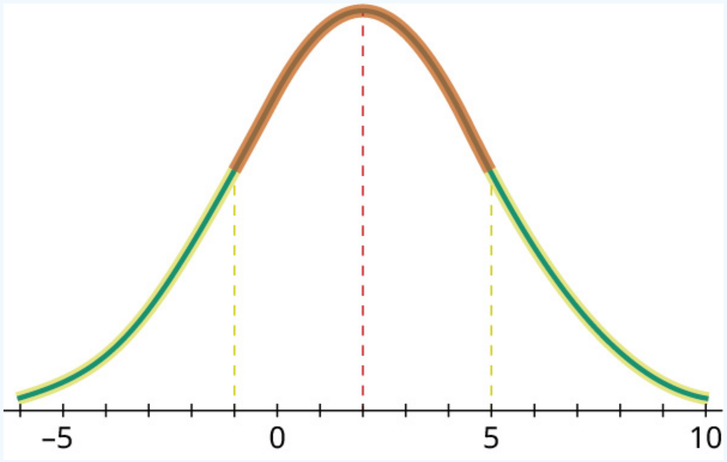

Figure ✓ 8.6.12

Step 4: We can estimate that the inflection points occur at $x = -1$ and $x = 5$; the mean is at $x = 2$ (as shown by the middle dotted line). The difference between the mean and the location of either inflection point is the standard deviation; since $5 - 2 = 2 - (-1) = 3$, we conclude that the standard deviation of the green distribution is 3.

Step 5: Now, looking at the other two graphs, let's first identify the inflection points:

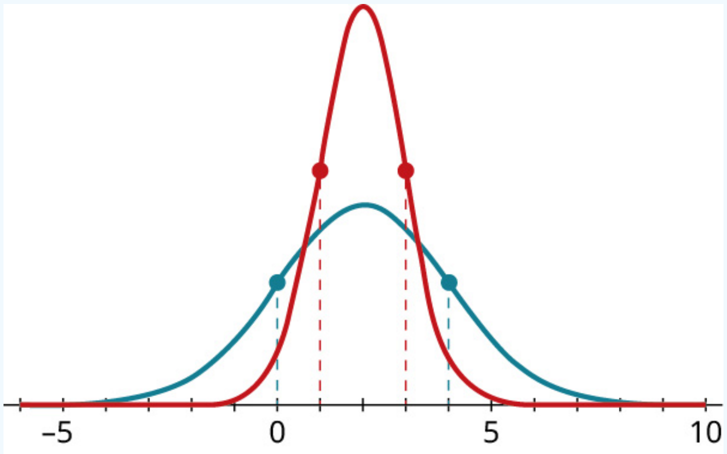

Figure ✔ 8.6.13

Step 6: The red (tallest peaked) distribution has inflection points at 1 and 3, and the mean is 2. Thus, the standard deviation of the red distribution is 3 - 2 = 2 - 1 = 1 3 - 2 = 2 - 1 = 1 . The blue (lower peaked) distribution has inflection points at 0 and 4, and its mean is also 2. So, the standard deviation of the blue distribution is 2.

📌 Your Turn 8.6.2

Estimate the standard deviations of this normal distribution, centered at 5:

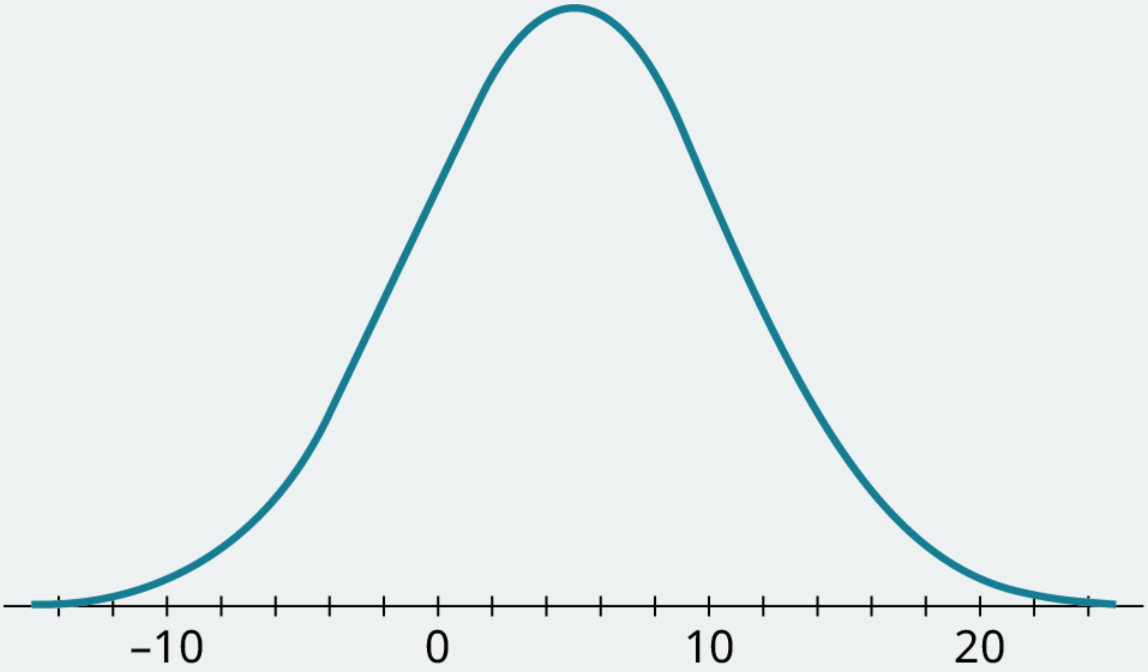

Figure 📌 8.6.14

Let’s put it all together to identify a completely unknown normal distribution.

✓ Example 8.6.3: Identifying the Mean and Standard Deviation of a Normal Distribution

Using the graph, identify the mean and standard deviation of the normal distribution.

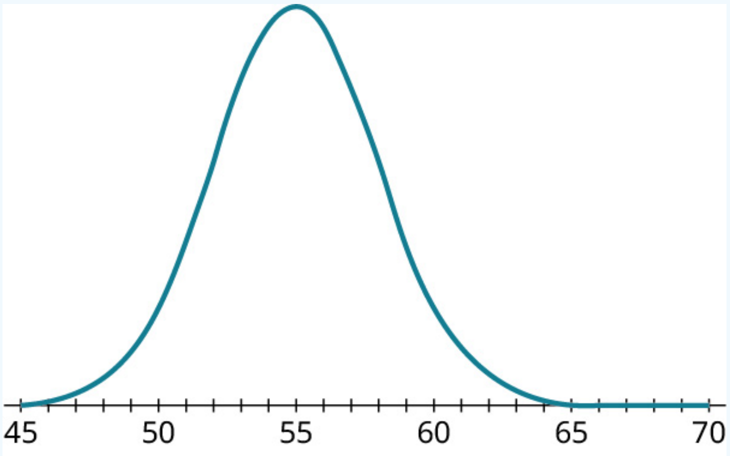

Figure ✓ 8.6.15

Answer

Step 1: Let's start by putting dots on the graph at the peak and at the inflection points, then drop lines from those points straight down to the axis:

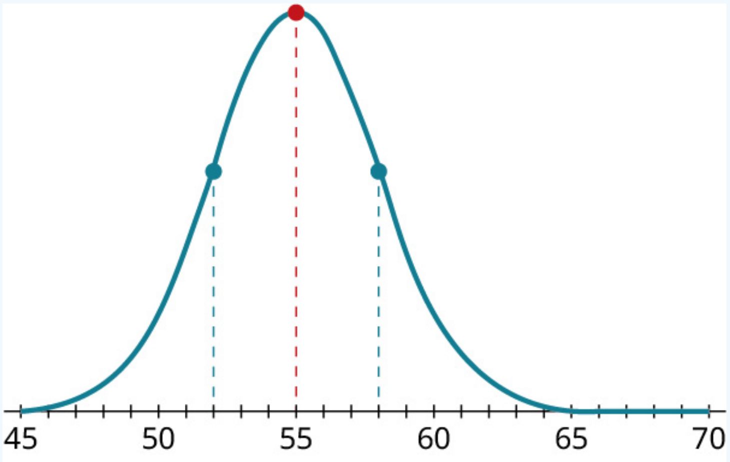

Figure ✓ 8.6.16

Step 2: From the red (middle) line, we can see that the mean of this distribution is 55. The blue (outermost) lines are each 3 units away from the mean (at 52 and 58), so the standard deviation is 3.

Your Turn 8.6.3

Identify the mean and standard deviation of this distribution. Anything within 5 on the standard deviation is acceptable.

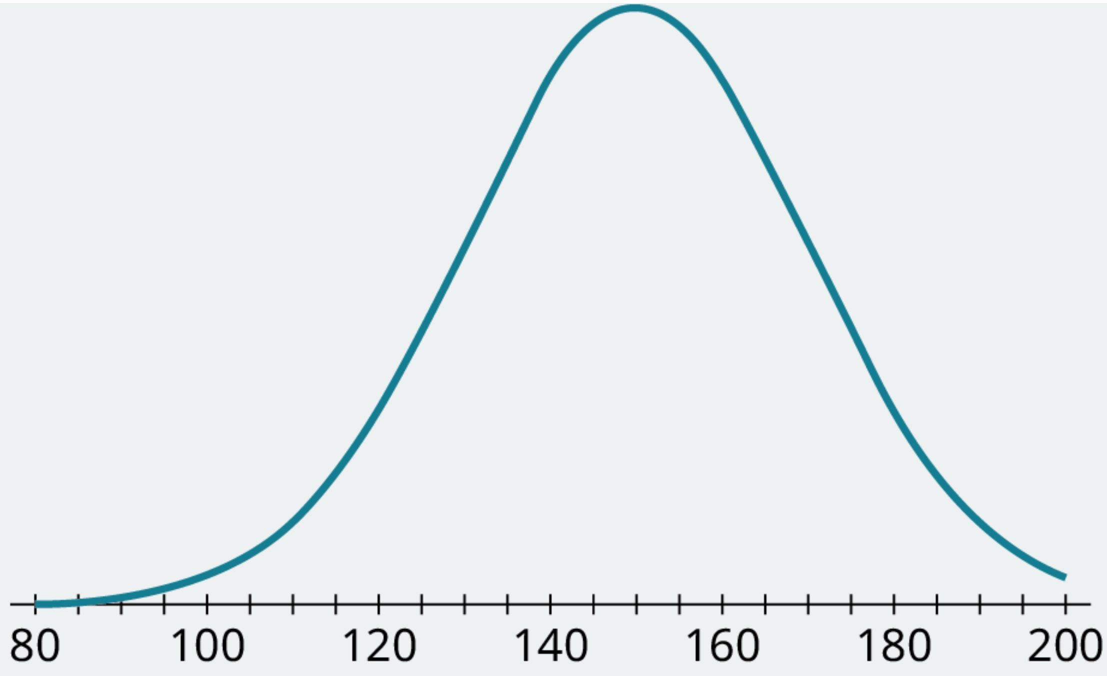

Figure 8.6.17

Properties of Normal Distributions: The 68-95-99.7 Rule

The most important property of normal distributions is tied to its standard deviation. If a dataset is perfectly normally distributed, then 68% of the data values will fall within one standard deviation of the mean. For example, suppose we have a set of data that follows the normal distribution with mean 400 and standard deviation 100. This means 68% of the data would fall between the values of 300 (one standard deviation *below* the mean: $400-100=300$) and 500 (one standard deviation *above* the mean: $400+100=500$). Looking at the histogram below, the shaded area represents 68% of the total area under the graph and above the axis:

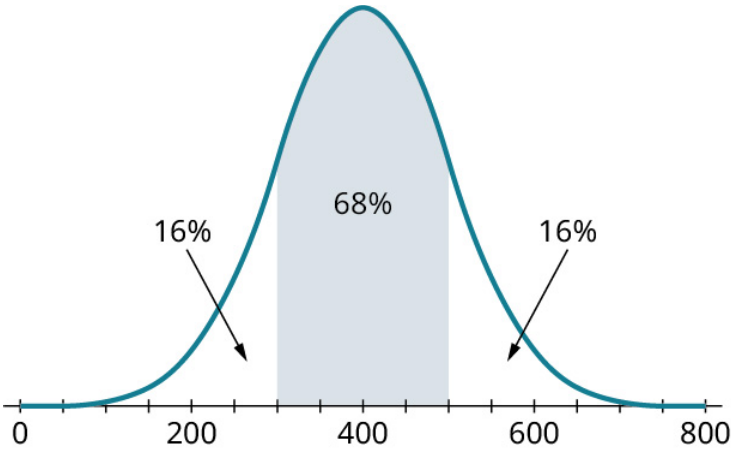

Figure 8.6.18

Since 68% of the area is in the shaded region, this means that $100\%-68\%=32\%$ of the area is found in the unshaded regions. We know that the distribution is symmetric, so that 32% must be divided evenly into the two unshaded tails: 16% in each.

Of course, datasets in the real world are never perfect; when dealing with actual data that seem to follow a symmetric, bell-shaped distribution, we'll give ourselves a little bit of wiggle room and say that *approximately* 68% of the data fall within one standard deviation of the mean.

The rule for one standard deviation can be extended to two standard deviations. Approximately 95% of a normally distributed dataset will fall within 2 standard deviations of the mean. If the mean is 400 and the standard deviation is 100, that means 95%

calculation describes the way we compute standardized scores. (two standard deviations *below* the mean: $400 - 2 \times 100 = 200$) and 600 (two standard deviations *above* the mean: $400 + 2 \times 100 = 600$). We can visualize this in the following histogram:

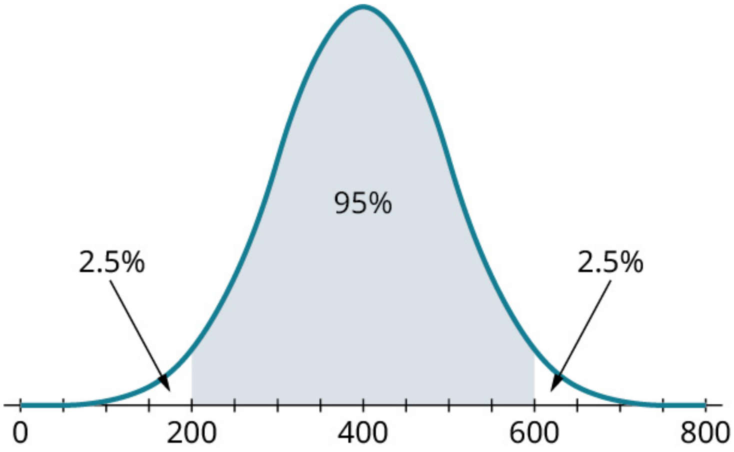

Figure 8.6.19

As before, since 95% of the data are in the shaded area, that leaves 5% of the data to go into the unshaded tails. Since the histogram is symmetric, half of the 5% (that's 2.5%) is in each.

We can even take this one step further: 99.7% of normally distributed data fall within 3 standard deviations of the mean. In this example, we'd see 99.7% of the data between 100 (calculated as $400 - 3 \times 100 = 100$) and 700 (calculated as $400 + 3 \times 100 = 700$). We can see this in the histogram below, although you may need to squint to find the unshaded bits in the tails!

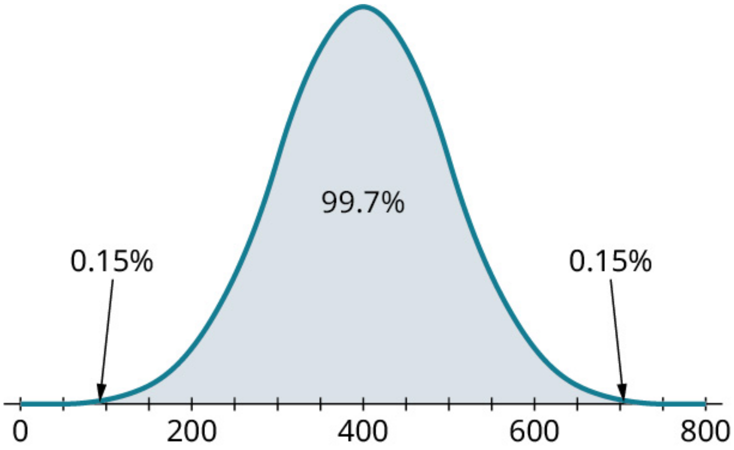

Figure 8.6.20

This observation is formally known as the 68-95-99.7 Rule.

1. If data are normally distributed with mean 8 and standard deviation 2, what percent of the data falls between 4 and 12?
2. If data are normally distributed with mean 25 and standard deviation 5, what percent of the data falls between 20 and 30?
3. If data are normally distributed with mean 200 and standard deviation 15, what percent of the data falls between 155 and 245?

Answer

1. **Let's look at a table that sets out the data values that are even multiples of the standard deviation (SD) above and below the mean:**

mean−3×SDmean-3×SD	mean−2×SDmean-2×SD	mean−1×SDmean-1×SD	Mean	mean+1×SDmean+1×SD	mean+2×SDmean+2×SD	mean+3×SDmean+3×SD
2	**4**	**6**	**8**	**10**	**12**	**14**

Since 4 and 12 represent two standard deviations above and below the mean, we conclude that 95% of the data will fall between them.

- Let's build another table:

mean−3×SDmean-3×SD	mean−2×SDmean-2×SD	mean−1×SDmean-1×SD	Mean	mean+1×SDmean+1×SD	mean+2×SDmean+2×SD	mean+3×SDmean+3×SD
10	15	20	25	30	35	40

We can see that 20 and 30 represent one standard deviation above and below the mean, so 68% of the data fall in that range.

- Let's make one more table:

mean−3×SDmean-3×SD	mean−2×SDmean-2×SD	mean−1×SDmean−1×SD	Mean	mean+1×SDmean+1×SD	mean+2×SDmean+2×SD	mean+3×SDmean+3×SD
155	170	185	200	215	230	245

Since 155 and 245 are three standard deviations above and below the mean, we know that 99.7% of the data will fall between them.

Your Turn 8.6.4

If data are distributed normally with mean 0 and standard deviation 3, what percent of the data fall between –9 and 9?

If data are distributed normally with mean 50 and standard deviation 10, what percent of the data fall between 30 and 70?

If data are distributed normally with mean 60 and standard deviation 5, what percent of the data fall between 55 and 65?

1. If data are distributed normally with mean 100 and standard deviation 20, between what two values will 68% of the data fall?
2. If data are distributed normally with mean 0 and standard deviation 15, between what two values will 95% of the data fall?
3. If data are distributed normally with mean 14 and standard deviation 2, between what two values will 99.7% of the data fall?

Answer

1. **The 68-95-99.7 Rule tells us that 68% of the data will fall within one standard deviation of the mean. So, to find the values we seek, we'll add and subtract one standard deviation from the mean:** $100 - 1 \times 20 = 80$ **and** $100 + 1 \times 20 = 120$**. Thus, we know that 68% of the data fall between 80 and 120.**
2. **Using the 68-95-99.7 Rule again, we know that 95% of the data will fall within 2 standard deviations of the mean. Let's add and subtract two standard deviations from that mean:** $0 - 2 \times 15 = -30$ **and** $0 + 2 \times 15 = 30$**. So, 95% of the data will fall between -30 and 30.**
3. **Once again, the 68-95-99.7 Rule tells us that 99.7% of the data will fall within three standard deviations of the mean. So, let's add and subtract three standard deviations from the mean: and** $14 + 3 \times 2 = 20$**. Thus, we conclude that 99.7% of the data will fall between 8 and 20.**

Your Turn 8.6.5

1. If data are distributed normally with mean 70 and standard deviation 5, between what two values will 68% of the data fall?
2. If data are distributed normally with mean 40 and standard deviation 7, between what two values will 95% of the data fall?
3. If data are distributed normally with mean 200 and standard deviation 30, between what two values will 99.7% of the data fall?

There are more problems we can solve using the 68-95-99.7 Rule. but first we must understand what the rule implies. Remember, the rule says that 68% of the data falls within one standard deviation of the mean. Thus, with normally distributed data with mean 100 and standard deviation 10, we have this distribution:

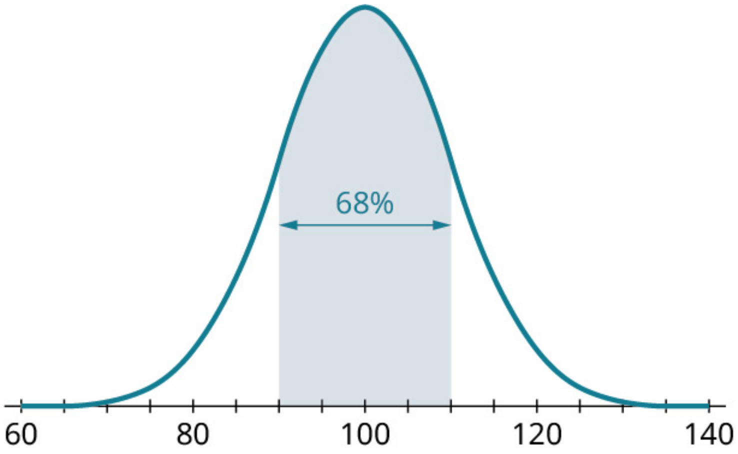

Figure $8.6.21$

Since we know that 68% of the data lie within one standard deviation of the mean, the implication is that 32% of the data must fall *beyond* one standard deviation away from the mean. Since the histogram is symmetric, we can conclude that half of the 32% (or 16%) is more than one standard deviation *above* the mean and the other half is more than one standard deviation *below* the mean:

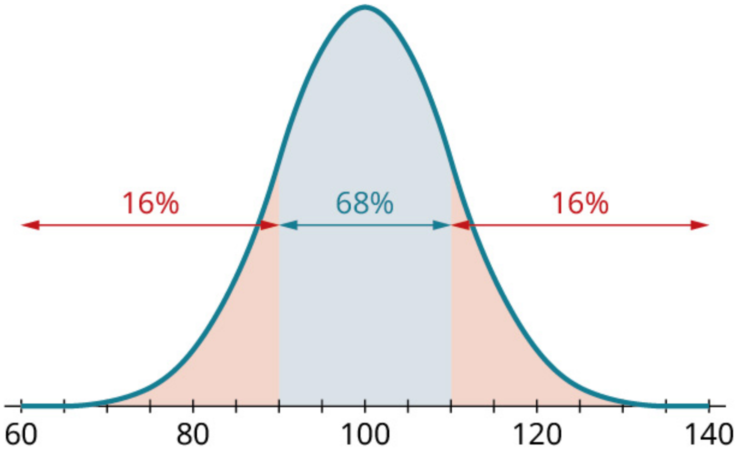

Figure $8.6.22$

Further, we know that the middle 68% can be split in half at the peak of the histogram, leaving 34% on either side:

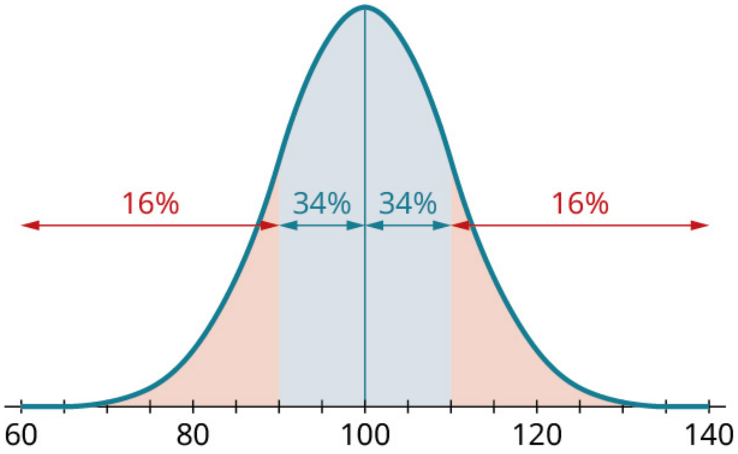

Figure $8.6.23$

So, just the “68” part of the 68-95-99.7 Rule gives us four other proportions in addition to the 68% in the rule. Similarly, the “95” and “99.7” parts each give us four more proportions:

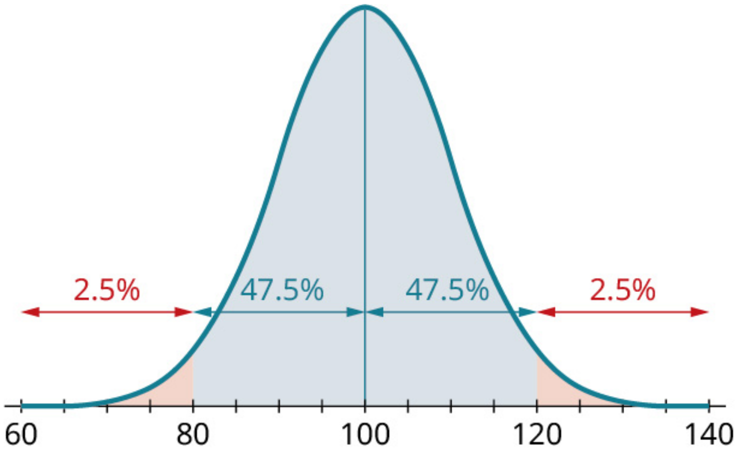

Figure $8.6.24$

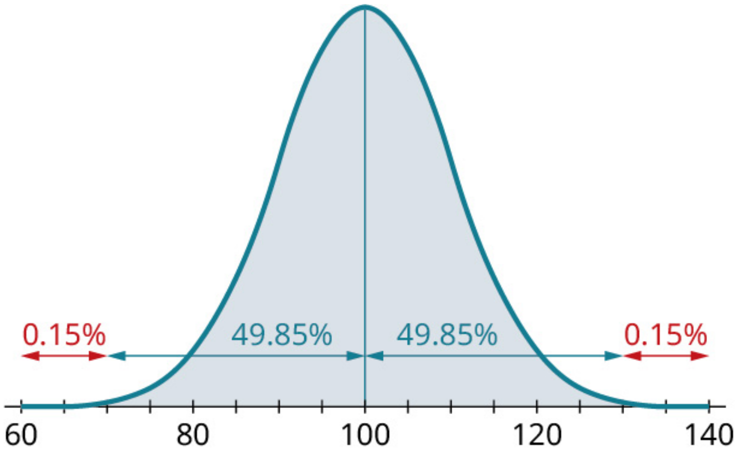

Figure $8.6.25$: Copy and Paste Caption here. (Copyright; author via source)

We can put all these together to find even more complicated proportions. For example, since the proportion between 100 and 120 is 47.5% and the proportion between 100 and 110 is 34%, we can subtract to find that the proportion between 110 and 120 is $47.5-34=13.5\%$:

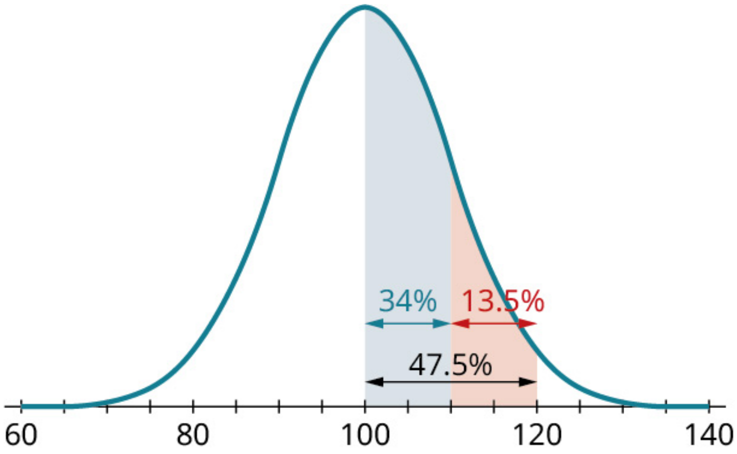

Figure $8.6.26$

Assume that we have data that are normally distributed with mean 80 and standard deviation 3.

1. What proportion of the data will be greater than 86?
2. What proportion of the data will be between 74 and 77?
3. What proportion of the data will be between 74 and 83?

Answer

Before we can answer these questions, we must mark off sections that are multiples of the standard deviation away from the mean:

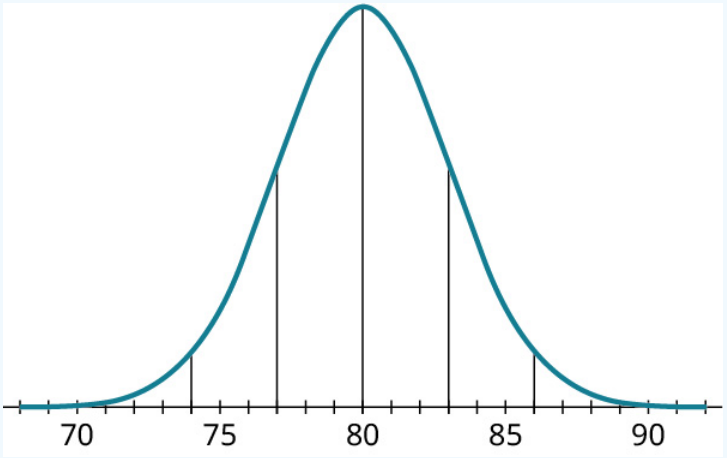

Figure ✓ 8.6.27

1. To figure out what proportion of the data will be greater than 86, let's start by shading in the area of data that are above 86 in our figure, or the data more than two standard deviations above the mean.

   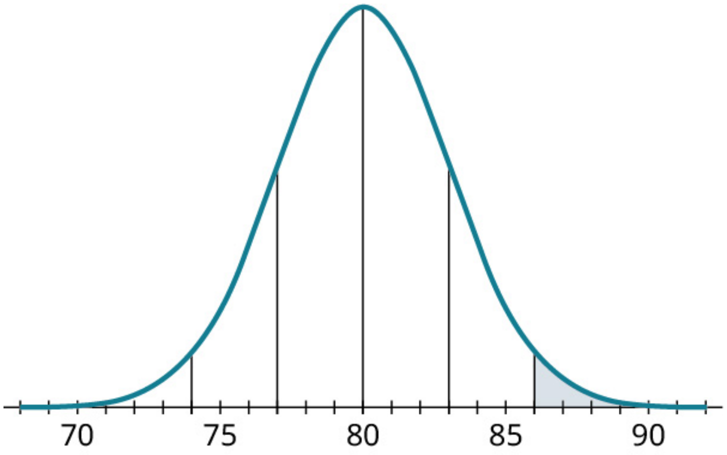

   Figure 8.6.28

   We saw in Figure 8.51 that this proportion is 2.5%.
2. To figure out what proportion of the data will be between 74 and 77, let's start by shading in that area of data. These are data that are more than one but less than two standard deviations below the mean.

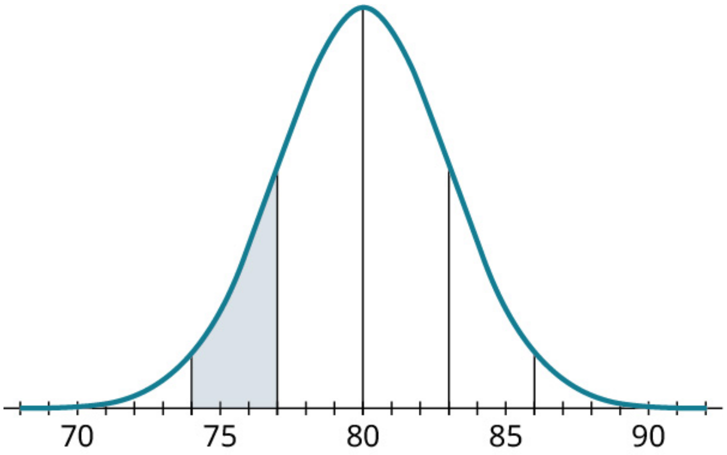

Figure 8.6.29

From Figure 8.51, we know that the proportion of data less than two standard deviations below the mean is 47.5%. And, from YOUR TURN 8.33, we know that 34% of the data is less than one standard deviation below the mean:

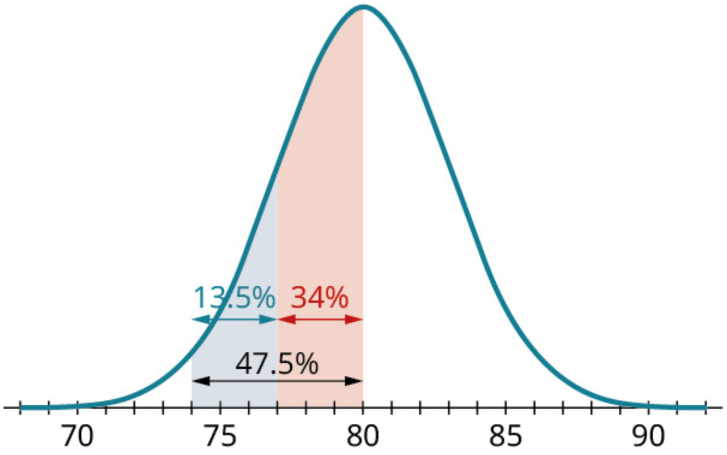

Figure 8.6.30

Subtracting, we see that the proportion of data between 74 and 77 is 13.5%.

3. To figure out what proportion of the data will be between 74 and 83, let's start by shading in that area of data in our figure.

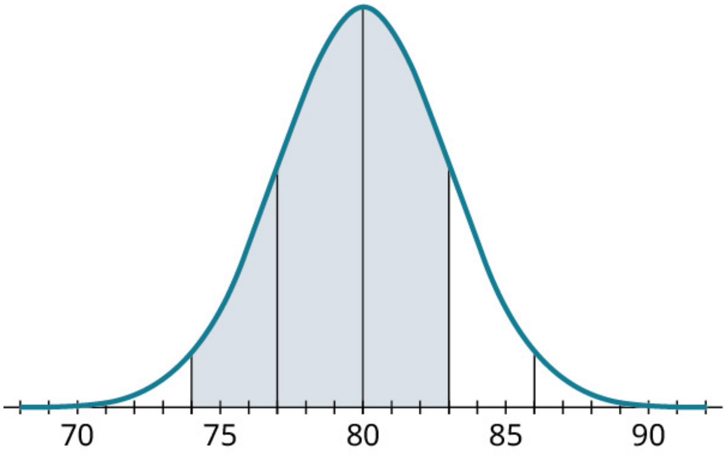

Figure 8.6.31

Next, we'll break this region into two pieces at the mean:

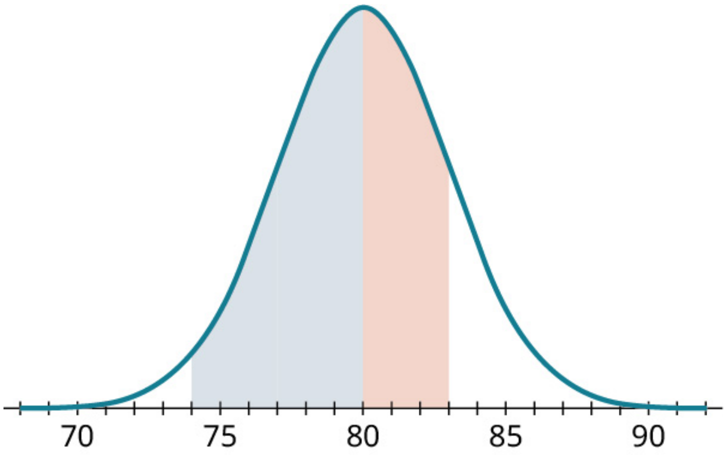

Figure $8.6.32$

From Figure 8.51, we know the blue (leftmost) region represents 47.5% of the data. And, using YOUR TURN 8.33, we get that the red (rightmost) region covers 34% of the data. Adding those together, the proportion we want is 81.5%.

Your Turn $8.6.6$

Suppose we have data that are normally distributed with mean 500 and standard deviation 100. What proportions of the data fall in these ranges?

300 to 500

600 to 800

400 to 700

Standardized Scores

When we want to apply the 68-95-99.7 Rule, we must first figure out how many standard deviations above or below the mean our data fall. This calculation is common enough that it has its own name: the standardized score. Values above the mean have positive standardized scores, while those below the mean have negative standardized scores. Since it's common to use the letter z to represent a standard score, this value is also often referred to as a ***z*-score**.

So far, we've only really considered z-scores that are whole numbers, but in general they can be any number at all. For example, if we have data that are normally distributed with mean 80 and standard deviation 6, the value 85 is five units above the mean, which is less than one standard deviation. Dividing by the standard deviation, we get $\frac{5}{6}$. Since 85 is $\frac{5}{6}$ of one standard deviation above the mean, we'd say that the standardized score for 85 is z =56z=56 (which is positive, since $85 > 80$). This calculation describes the way we compute standardized scores.

FORMULA

If x is a member of a normally distributed dataset with mean μ and standard deviation σ, then the standardized score for x is

$$z = \frac{x-\mu}{\sigma}.$$

If you know a z-score but not the original data value x, you can find it by solving the previous equation for x :

$$x = \mu + z \times \sigma.$$

z×σ.

Checkpoint

The symbols semanticsμμ and semanticsσσ are the Greek letters *mu* and *sigma*. They are the analogues of the English letters m and s, which stand for *mean* and *standard deviation*.

If you convert every data value in a dataset into its z-score, the resulting set of data will have mean 0 and standard deviation 1. This is why we call these *standardized scores*: the normal distribution with mean 0 and standard deviation 1 is often called the *standard normal distribution*.

Example 8.6.7: Standardizing Data

Suppose we have data that are normally distributed with mean 50 and standard deviation 6. Compute the standardized scores (rounded to three decimal places) for these data values:

1. 52
2. 40
3. 68

Answer

For each of these, we'll plug the given values into the formula. Remember, the mean is $\mu = 50$ and the standard deviation is $\sigma = 6$:

1. $z = \frac{x-\mu}{\sigma} = \frac{52-50}{6} = 0.333$
2. $z = \frac{x-\mu}{\sigma} = \frac{40-50}{6} = -1.667$
3. $z = \frac{x-\mu}{\sigma} = \frac{68-50}{6} = 3$

Your Turn 8.6.7

Suppose we have data that are normally distributed with mean 75 and standard deviation 5. Compute standardized scores for each of these data values: 66, 83, and 72.

Example 8.6.8: Converting Standardized Scores to Original Values

Suppose we have data that are normally distributed with mean 10 and standard deviation 2. Convert the following standardized scores into data values.

1. 1.4
2. −0.9
3. 3.5

Answer

We'll use the formula previously introduced to convert z-scores into x-values. In this case, the mean is $\mu = 10$ and the standard deviation is $\sigma = 2$:

1. $x = \mu + z \times \sigma = 10 + 1.4 \times 2 = 12.8$
2. $x = \mu + z \times \sigma = 10 + (-0.9) \times 2 = 8.2$
3. $x = \mu + z \times \sigma = 10 + 3.5 \times 2 = 17$ 17x=μ+z×σ=10+3.5×2=17

Your Turn 8.6.8

Suppose you have a normally distributed dataset with mean 2 and standard deviation 20. Convert these standardized scores to data values: –2.3, 1.4, and 0.2.

Using Google Sheets to Find Normal Percentiles

The 68-95-99.7 Rule is great when we're dealing with whole-number z-scores. However, if the z-score is not a whole number, the Rule isn't going to help us. Luckily, we can use technology to help us out. We'll talk here about the built-in functions in Google Sheets, but other tools work similarly.

Let's say we're working with normally distributed data with mean 40 and standard deviation 7, and we want to know at what percentile a data value of 50 would fall. That corresponds to finding the proportion of the data that are less than 50. If we create our histogram and mark off whole-number multiples of the standard deviation like we did before, we'll see why the 68-95-99.7 Rule isn't going to help:

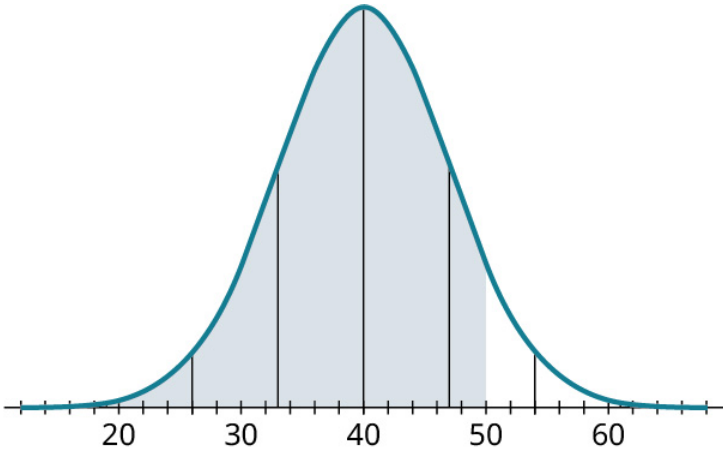

Figure 8.6.33

Since 50 doesn't line up with one of our lines, the 68-95-99.7 Rule fails us. Looking back at Figure 8.51 and Figure 8.52, the best we can say is that 50 is between the 84th and 99.5th percentiles, but that's a pretty wide range. Google Sheets has a function that can help; it's called NORM.DIST. Here's how to use it:

1. Click in an empty cell in your worksheet.
2. Type "=NORM.DIST("
3. Inside the parentheses, we must enter a list of four things, separated by commas: the data value, the mean, the standard deviation, and the word "TRUE". These have to be entered in this order!
4. Close the parentheses, and hit Enter. The result is then displayed in the cell; convert it to a percent to get the percentile.

So, for our example, we should type "=NORM.DIST(50, 40, 7, TRUE)" into an empty cell, and hit Enter. The result is 0.9234362745; converting to a percent and rounding, we can conclude that 50 is at the 92nd percentile. Let's walk through a few more examples.

Example 8.6.9: Using Google Sheets to Find Percentiles

Suppose we have data that are normally distributed with mean 28 and standard deviation 4. At what percentile do each of the following data values fall?

1. 30
2. 23
3. 35

Answer

1. By entering "=NORM.DIST(30, 28, 4, TRUE)" we find that 30 is at the 69th percentile.
2. By entering "=NORM.DIST(23, 28, 4, TRUE)" we find that 23 is at the 11th percentile.

3. By entering “=NORM.DIST(35, 28, 4, TRUE)” we find that 35 is at the 96th percentile.

Your Turn 8.6.9

Suppose you have data that are normally distributed with mean 20 and standard deviation 6. Determine at what percentiles these data values fall: 25, 12, and 31.

Google Sheets can also help us go the other direction: If we want to find the data value that corresponds to a given percentile, we can use the NORM.INV function. For example, if we have normally distributed data with mean 150 and standard deviation 25, we can find the data value at the 30th percentile as follows:

1. Click on an empty cell in your worksheet.
2. Type “=NORM.INV(“
3. Inside the parentheses, we’ll enter a list of three numbers, separated by commas: the percentile in question expressed as a decimal, the mean, and the standard deviation. These must be entered in this order!
4. Close the parentheses and hit Enter. The desired data value will be in the cell!

In our example, we want the 30th percentile; converting 30% to a decimal gives us 0.3. So, we’ll type “=NORM.INV(0.3, 150, 25)” to get 136.8899872; let’s round that off to 137.

✓ Example 8.6.10: Using Google Sheets to Find the Data Value Corresponding to a Percentile

Suppose we have data that are normally distributed with mean 47 and standard deviation 9. Find the data values (rounded to the nearest tenth) corresponding to these percentiles:

1. 75th (that’s the third quartile)
2. 12th
3. 90th

Answer

1. By entering “=NORM.INV(0.75, 47, 9)” we find that 53.1 is at the 75th percentile.
2. By entering “=NORM.INV(0.12, 47, 9)” we find that 36.4 is at the 12th percentile.
3. By entering “=NORM.INV(0.9, 47, 9)” we find that 58.5 is at the 90th percentile.

Your Turn 8.6.10

Suppose we have data that are normally distributed with mean 5 and standard deviation 1.6. Identify which data values (rounded to the nearest tenth) correspond to these percentiles: 25th (the first quartile), 80th, and 10th.

Check Your Understanding

For each of these problems, assume we’re working with normally distributed data with mean 100 and standard deviation 12.

1. What percentage of the data falls between 76 and 124? Use the 68-95-99.7 Rule.
2. What percentage of the data falls between 100 and 112? Use the 68-95-99.7 Rule.
3. At what percentile does 112 fall? Use the 68-95-99.7 Rule.
4. What’s the z-score of the data value 107? Round to three decimal places.
5. What data value’s z-score is –2.4?
6. At what percentile does 107 fall? Use Google Sheets (or another technology).
7. What data value is at the 90th percentile? Use Google Sheets (or another technology), and round to the nearest hundredth.

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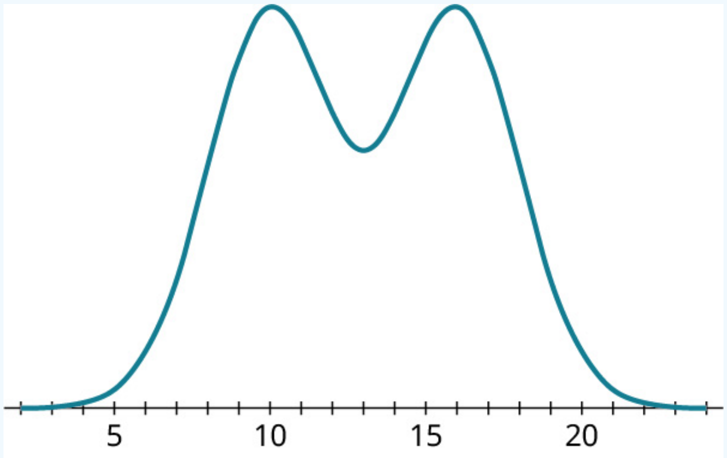
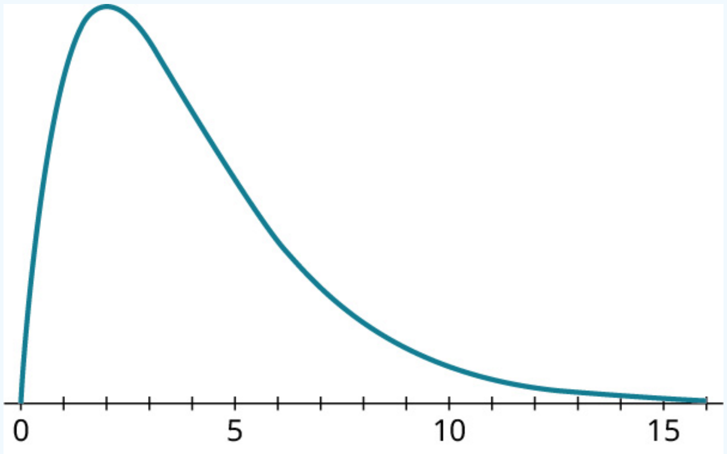
8.6.0: Exercises

For the following exercises, explain how you can tell the histogram does NOT represent normally distributed data.

Exercise 8.6.0.1



Exercise 8.6.0.2



For the following exercises, use the 68-95-99.7 Rule to answer the given questions about normally distributed data with mean 100 and standard deviation 5.

Exercise 8.6.0.3

What proportion of the data fall between 95 and 105?

Exercise 8.6.0.4

What proportion of the data fall between 90 and 110?

Exercise 8.6.0.5

what proportion of the data fall between 85 and 100?

? Exercise 8.6.0.6

What proportion of the data fall between 110 and 120?

? Exercise 8.6.0.7

What proportion of the data are less than 90?

? Exercise 8.6.0.8

What proportion of the data are greater than 105?

? Exercise 8.6.0.9

What proportion of the data fall between 90 and 105?

? Exercise 8.6.0.10

What proportion of the data are between 95 and 115?

For the following exercises, use the 68-95-99.7 Rule to answer the given questions about normally distributed data with mean 9 and standard deviation 1.

? Exercise 8.6.0.11

What proportion of the data are less than 7?

? Exercise 8.6.0.12

What proportion of the data are greater than 12?

? Exercise 8.6.0.13

What proportion of the data are between 6 and 12?

? Exercise 8.6.0.14

What proportion of the data are between 8 and 9?

? Exercise 8.6.0.15

What proportion of the data are between 6 and 8?

? Exercise 8.6.0.16

What proportion of the data are between 8 and 11?

? Exercise 8.6.0.17

What proportion of the data are less than 10?

? Exercise $8.6.0.18$

What proportion of the data are greater than 6?

In the following exercises, convert the given data values to standardized scores. Assume the data are distributed normally with mean 15 and standard deviation 3. Round to the nearest hundredth.

? Exercise $8.6.0.19$

$x = 17$

? Exercise $8.6.0.20$

$x = 11$

? Exercise $8.6.0.21$

$x = 10$

? Exercise $8.6.0.22$

$x = 21$

? Exercise $8.6.0.23$

$x = 16$

? Exercise $8.6.0.24$

$x = 24$

? Exercise $8.6.0.25$

$x = 8$

? Exercise $8.6.0.26$

$x = 7.2$

In the following exercises, convert the given z-scores to data values. Assume the data are distributed normally with mean 15 and standard deviation 3.

? Exercise $8.6.0.27$

$z = 1.2$

? Exercise $8.6.0.28$

$z = -0.4$

? Exercise $8.6.0.29$

$z = 3.6$

Exercise 8.6.0.30

$z = 2.1$

Exercise 8.6.0.31

$z = -2.8$

Exercise 8.6.0.32

$z = 4$

Exercise 8.6.0.33

$z = -0.4$

Exercise 8.6.0.34

$z = -3.4$

For the following exercises, answer the questions about normally distributed data with mean 200 and standard deviation 20. Round percentiles to the nearest whole number and round data values to the nearest tenth.

Exercise 8.6.0.35

At what percentile is $x = 225$?

Exercise 8.6.0.36

At what percentile is $x = 184$?

Exercise 8.6.0.37

At what percentile is $x = 192$?

Exercise 8.6.0.38

At what percentile is $x = 206$?

Exercise 8.6.0.39

At what percentile is $x = 239$?

Exercise 8.6.0.40

At what percentile is $x = 202$?

Exercise 8.6.0.41

At what percentile is $x = 190$?

Exercise $8.6.0.42$

At what percentile is $x = 175$?

Exercise $8.6.0.43$

What data value is at the 40th percentile?

Exercise $8.6.0.44$

What data value is at the 10th percentile?

Exercise $8.6.0.45$

What data value is at the 55th percentile?

Exercise $8.6.0.46$

What data value is at the 95th percentile?

Exercise $8.6.0.47$

What data value is at the 33rd percentile?

Exercise $8.6.0.48$

What data value is at the third quartile?

Exercise $8.6.0.49$

What data value is at the 65th percentile?

Exercise $8.6.0.50$

What data value is at the 99th percentile?

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8.7: Applications of the Normal Distribution

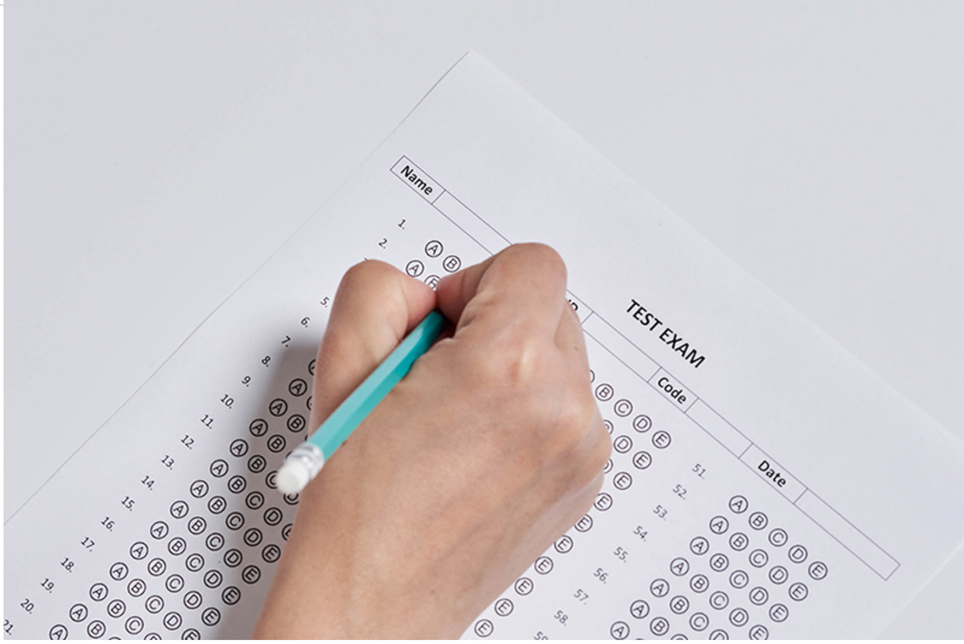

Figure 8.7.1: Standardized test results generally adhere to the normal distribution. (credit: “Taking a Test” by Marco Verch Professional Photographer/Flickr, CC BY 2.0)

Learning Objectives

Apply the normal distribution to real-world scenarios.

As we saw in The Normal Distribution, the word “standardized” is closely associated with the normal distribution. This is why tests like college entrance exams, state achievement tests for K–12 students, and Advanced Placement tests are often called “standardized tests”: scores are assigned in a way that forces them to follow a normal distribution, with a mean and standard deviation that are consistent from year to year. Standardization also allows people like college admissions officers to directly compare an applicant who took the ACT (a college entrance exam) to an applicant who instead chose to take the SAT (a different college entrance exam). Standardization allows us to compare individuals from different groups; this is among the most important applications of the normal distribution. We’ll explore this and other real-world uses of the normal distribution in this section.

College Entrance Exams

There are two good ways to compare two data values from different groups: using z-scores and using percentiles. The two methods will always give consistent results (meaning that we won’t find, for example, that the first value is better using z-scores but the second value is better using percentiles), so use whichever method is more comfortable for you.

Example 8.7.1: Evaluating College Entrance Exam Scores

According to the Digest of Education Statistics, composite scores on the SAT have mean 1060 and standard deviation 195, while composite scores on the ACT have mean 21 and standard deviation 5.

1. At what percentile would an SAT score of 990 fall?
2. What is the z-score of an ACT score of 27?
3. Which is better: a score of 1450 on the SAT or 29 on the ACT?

Answer

1. Using Google Sheets, we can answer this question with the formula "=NORM.DIST(990, 1060, 195, TRUE)". A score of 990 would fall at the 36 th percentile.

2. Using the formula $z = \frac{x-\mu}{s}$, we get $z = \frac{27-21}{5} = 1.2$.

3. Let's compare the values using both percentiles and z-values:

Percentiles: Using "=NORM.DIST(1450, 1060, 195, TRUE)" we find that an SAT score of 1450 is at the 98th percentile. Meanwhile, by entering "=NORM.DIST(29, 21, 5 , TRUE)" we see that an ACT score of 29 is around the 95th percentile. Since it's at a higher percentile, we can conclude that an SAT score of 1450 is better than an ACT score of 29.

z-scores: Using the formula, we see that the z-score for an SAT score of 1450 is $z = \frac{1450-1060}{195} = 2$, while the z-score for an ACT score of 29 is $z = \frac{29-21}{5} = 1.6$. Since it has a higher z-score, an SAT score of 1450 is better than an ACT score of 29.

Your Turn 8.7.1

According to the Graduate Management Admission Council (GMAC), the mean score on the GMAT (an entrance exam for graduate schools in business management) is 565, with standard deviation 116. For the LSAT (an entrance exam for law schools), Kaplan informs us that the mean score is 150 with standard deviation 10.

1. What is the z-score for a GMAT score of 715?
2. At what percentile is an LSAT score of 166?
3. Which is better: a GMAT score of 650 or an LSAT score of 161?

Coin flipping

In the opening of The Normal Distribution, we saw that the number of heads we get when we flip a coin 100 times is distributed normally. It can be shown that if n is the number of flips, then the mean of that distribution is $\frac{n}{2}$ and the standard deviation is $\frac{\sqrt{n}}{2}$ (as long as $n \geq$). So, for 100 flips, the mean of the distribution is 50 and the standard deviation is 5. In that opening example, one of our early runs gave us 70 heads in 100 flips, which we noted seemed unusual. Using the normal distribution, we can identify exactly how unusual that really is. Using Google Sheets, the formula “=NORM.DIST(70, 50, 5, TRUE)” gives us 0.999968, which is the 99.997th percentile! How is that useful? Suppose you need to test whether a coin is fair, and so you flip it 100 times. While we might be suspicious if we get 70 heads out of the 100 flips, we now have a numerical measure for how unusual that is: If the coin were fair, we would expect to see 70 heads (or more) only $100 - 99.9968 = 0.0032\%$ of the time. That’s really unlikely! Analysis like this is related to *hypothesis testing*, an important application of statistics in the sciences and social sciences.

✓ Example 8.7.2: Flipping a Coin

Let’s say we flip a coin 64 times and count the number of heads.

1. What would be the mean of the corresponding distribution?
2. What would be the standard deviation of the corresponding distribution?
3. Suppose we got 25 heads, which seems a little low. At what percentile would 25 heads fall?

Answer

1. Since $n = 64$, the mean is $\frac{64}{2} = 32$.

2. Again using $n = 64$, we get a standard deviation of $\frac{\sqrt{64}}{2} = 4$.

3. Using "=NORM.DIST(25, 32, 4, TRUE)", we see that 25 heads is at the 4th percentile. Reading and Interpreting Scatter Plots

Your Turn 8.7.2

You flip a coin 144 times and count the number of heads.

1. What is the mean of the corresponding distribution?
2. What is the standard deviation of the corresponding distribution?

3. You flipped 81 heads. At what percentile does that fall?

Analyzing Data That Are Normally Distributed

Whenever we're working with a dataset that has a distribution that looks symmetric and bell-shaped, we can use techniques associated with the normal distribution to analyze the data.

✓ Example 8.7.3: Using Normal Techniques to Analyze Data

The data in "AvgSAT" contains the average SAT score for students attending every institution of higher learning in the United States for which data is available. We previously created a histogram for these data:

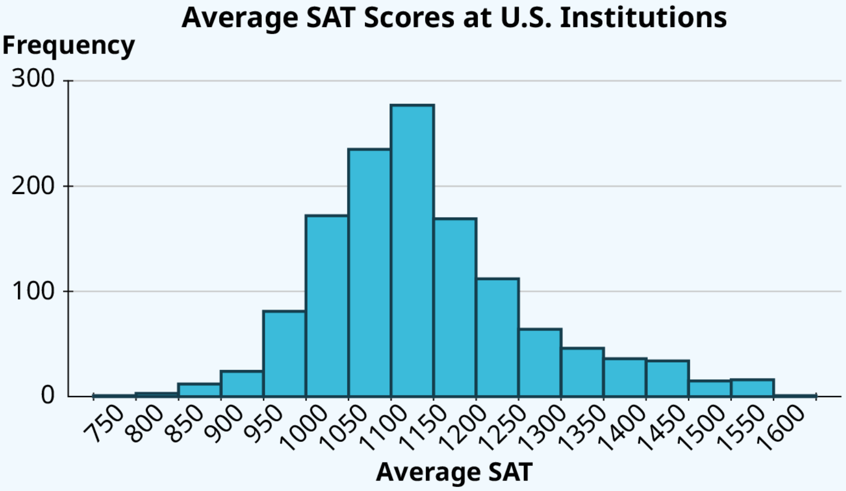

Figure ✓ 8.7.2 (data source: https://data.ed.gov)

This distribution is fairly symmetric (it's just a little right-skewed) and bell-shaped, so we can use normal distribution techniques to analyze the data.

1. What is the mean of these average SAT scores?
2. What is the standard deviation of these SAT scores?
3. Using the answers to the previous two questions, use NORM.DIST in Google Sheets to estimate at what percentile the University at Buffalo in New York (average SAT: 1250) falls.
4. Use PERCENTRANK to find the actual percentile of the University at Buffalo, and see how close the estimate in the previous question came.

Answer

1. Using the AVERAGE function in Google Sheets, we find that the mean is 1141.174.
2. Using the STDEV function, we get that the standard deviation is 125.517.
3. Entering "=NORM.DIST(1250, 1141, 125.517, TRUE)" into Google Sheets, we estimate that the University at Buffalo is at the 81st percentile.
4. Using PERCENTRANK, we find that the actual percentile is the 84th. These are close!

Your Turn 8.7.3

Again using the data in "AvgSAT", find the average SAT score of a school at the 35th percentile in two ways: using NORM.INV and using PERCENTILE.

Who Knew?: Political Meddling Exposed

The normal distribution pops up in some unusual places. Recently, a team at Duke University has been using statistics to help identify partisan gerrymandering, where electoral districts have been carefully drawn in a way that benefits one political party over another. In their analysis, they found that hypothetical election results in randomly drawn districts are normally

> distributed. By using techniques similar to the ones we used above, they can quantify precisely how biased a particular electoral map is by finding the percentile rank of the actual election result on the normal distribution of the hypothetical results. You can find out more about their work at the "Quantifying Gerrymandering" site here.

Check Your Understanding

For the following problems, recall that the SAT exam has mean 1060 and standard deviation 195, and that composite scores on the ACT have mean 21 and standard deviation 5.

1. At what percentile would an SAT score of 940 fall? Round to the nearest whole number.
2. What score would be at the 67th percentile on the ACT?
3. Which is a better score: 1300 on the SAT or 27 on the ACT?

For the following problems, recall that if we flip a coin at least 20 times, the distribution of the number of heads is approximately normal with mean equal to half the number of flips and standard deviation equal to half of the square root of the number of flips.

4. Suppose we flip a coin 120 times. What are the mean and standard deviation of the corresponding distribution of heads?
5. Let’s say you flip 70 heads in 120 flips. At what percentile would that fall?
6. How many heads would be at the 30th percentile? Round to the nearest whole number.

For the following problems, use the data in “World Tax,” which gives the tax revenue of many countries of the world in 2017, expressed as a percentage of their gross domestic products. You can assume that the distribution is approximately normal (but you can make a histogram to check, if you want).

7. What tax revenue percentage falls at the third quartile? Answer this question using Google Sheets in two ways: using PERCENTILE and using NORM.INV.
8. What tax revenue percentage falls at the 20th percentile? Answer this question using Google Sheets in two ways: using PERCENTILE and using NORM.INV.
9. At what percentile does the United Kingdom (25.62%, found on row 46 in the spreadsheet) fall? Answer this question using Google Sheets in two ways: using PERCENTRANK and using NORM.DIST.
10. At what percentile does Kiribati (21.97%, found on row 62 in the spreadsheet) fall? Answer this question using Google Sheets in two ways: using PERCENTRANK and using NORM.DIST.

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8.7.0: Exercises

For the following exercises, assume we're looking at results from two different standardized tests. The first, called the ABC, has mean 250 and standard deviation 50. The second, called the XYZ, has mean 80 and standard deviation 10.

? Exercise $8.7.0.1$

What would be the z-score of a result of 322 on the ABC?

? Exercise $8.7.0.2$

What would be the z-score of a result of 57 on the XYZ?

? Exercise $8.7.0.3$

At what percentile would a result of 211 on the ABC fall? Round your answer to the nearest percentile.

? Exercise $8.7.0.4$

At what percentile would a result of 94 on the XYZ fall? Round your answer to the nearest percentile.

? Exercise $8.7.0.5$

What score would fall at the first quartile on the ABC? Round your answer to the nearest whole number.

? Exercise $8.7.0.6$

What score would fall at the 60th percentile on the XYZ? Round your answer to the nearest whole number.

? Exercise $8.7.0.7$

Which score is better: 202 on the ABC or 72 on the XYZ?

? Exercise $8.7.0.8$

Which score is better: 324 on the ABC or 94 on the XYZ?

For the following exercises, recall that if we flip a coin at least 20 times, the distribution of the number of heads is approximately normal with mean equal to half the number of flips and standard deviation equal to half of the square root of the number of flips.

? Exercise $8.7.0.9$

What would be the mean and standard deviation for the number of heads in 80 coin flips?

? Exercise $8.7.0.10$

What would be the mean and standard deviation for the number of heads in 144 coin flips?

? Exercise $8.7.0.11$

How many heads would be at the 80th percentile for 80 coin flips? Round your answer to the nearest whole number.

? Exercise 8.7.0.12

How many heads would be at the 20th percentile for 144 coin flips? Round your answer to the nearest whole number.

? Exercise 8.7.0.13

What is the z-score for 51 heads in 80 coin flips? Round your answer to the nearest hundredth.

? Exercise 8.7.0.14

What is the z-score for 87 heads in 144 coin flips? Round your answer to the nearest hundredth.

? Exercise 8.7.0.15

Which would be more unusual: flipping 51 heads in 80 tries or flipping 87 heads in 144 tries? How do you know?

For the following exercises, use the data in "Wheat," which gives the yield of wheat (in bushels) produced over several equally sized plots. (These data were collected as part of an experiment reported in "The experimental error of field trials" by W.B. Mercer and A.D. Hall).

? Exercise 8.7.0.16

Create a histogram of Yield to check that it's approximately normally distributed.

? Exercise 8.7.0.17

Find the mean and standard deviation of Yield. Round answers to the nearest thousandth.

? Exercise 8.7.0.18

What proportion of yields does the normal distribution predict should fall below 4.56? Round to the nearest thousandth.

? Exercise 8.7.0.19

At what percentile does the normal distribution predict a yield of 3.6 would fall? Round to the nearest whole number.

? Exercise 8.7.0.20

What yield does the normal distribution estimate would fall at the 80th percentile? Round to the nearest hundredth.

? Exercise 8.7.0.21

What yield does the normal distribution estimate would fall at the 95th percentile? Round to the nearest hundredth.

? Exercise 8.7.0.22

If one of these plots yields 2.8 bushels, would that be a good or bad result? How do you know?

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8.8: Scatter Plots, Correlation, and Regression Lines

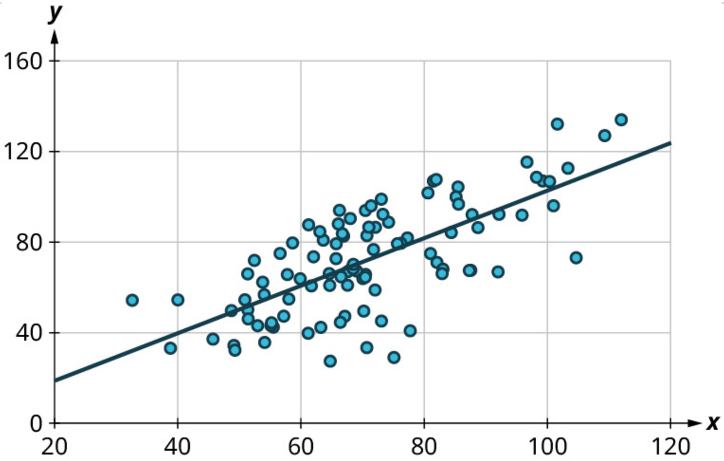


Figure 8.8.1: A scatter plot is a visualization of the relationship between quantitative dataset.

Learning Objectives

1. Construct a scatter plot for a dataset.
2. Interpret a scatter plot.
3. Distinguish among positive, negative and no correlation.
4. Compute the correlation coefficient.
5. Estimate and interpret regression lines.

One of the most powerful tools statistics gives us is the ability to explore relationships between two datasets containing quantitative values, and then use that relationship to make predictions. For example, a student who wants to know how well they can expect to score on an upcoming final exam may consider reviewing the data on midterm and final exam scores for students who have previously taken the class. It seems reasonable to expect that there is a relationship between those two datasets: If a student did well on the midterm, they were probably more likely to do well on the final than the average student. Similarly, if a student did poorly on the midterm, they probably also did poorly on the final exam.

Of course, that relationship isn't set in stone; a student's performance on a midterm exam doesn't cement their performance on the final! A student might use a poor result on the midterm as motivation to study more for the final. A student with a really good grade on the midterm might be overconfident going into the final, and as a result doesn't prepare adequately.

The statistical method of *regression* can find a formula that does the best job of predicting a score on the final exam based on the student's score on the midterm, as well as give a measure of the confidence of that prediction! In this section, we'll discover how to use regression to make these predictions. First, though, we need to lay some graphical groundwork.

Relationships Between Quantitative Datasets

Before we can evaluate a relationship between two datasets, we must first decide if we feel that one might depend on the other. In our exam example, it is appropriate to say that the score on the final depends on the score on the midterm, rather than the other way around: if the midterm depended on the final, then we'd need to know the final score *first*, which doesn't make sense.

Here's another example: if we collected data on home purchases in a certain area, and noted both the sale price of the house and the annual household income of the purchaser, we might expect a relationship between those two. Which depends on the other? In this case, sale price depends on income: people who have a higher income can afford a more expensive house. If it were the other way around, people could buy a new, more expensive house and then expect a raise! (This is very bad advice.)

It's worth noting that not every pair of related datasets has clear dependence. For example, consider the percent of a country's budget devoted to the military and the percent earmarked for public health. These datasets are generally related: as one goes up, the other goes down. However, in this case, there's not a preferred choice for dependence, as each could be seen as depending on the other. When exploring the relationship between two datasets, if one set seems to depend on the other, we'll say that dataset contains values of the response variable (or dependent variable). The dataset that the response variable depends on contains values of what we call the explanatory variable (or independent variable). If no dependence relationship can be identified, then we can assign either dataset to either role.

✓ Example ✓ 8.8.1: Identifying Explanatory and Response Variables

For each of the following pairs of related datasets, identify which (if any) should be assigned the role of response variable and which should be assigned to be the explanatory variable.

1. A person’s height and weight
2. A professional basketball player’s salary and their average points scored per game (which is a measure of how good they are at basketball)
3. The length and width of leaves on a tree

Answer

1. As people get taller, their weight tends to increase. But if a person goes on a diet and loses weight, we don’t expect them to also get shorter. So, weight depends on height. That means we’ll say that the response variable is *weight* and the explanatory variable is *height*.
2. The more points a basketball player scores, the more money they should make. But if a basketball player gets a raise, we wouldn’t expect them to get better at basketball as a result. So, the response is *salary* and the explanatory is *points per game*.
3. The way that the length and width of leaves are connected isn’t clear. It seems reasonable that as the width goes up, so would the length. But the other direction is also plausible: as the length goes up, so does the width. Without a clear dependence relationship, we’re free to declare either to be the response and the other to be the explanatory.

Your Turn 8.8.1

Given these pairs of datasets, identify which (if either) would be the best choice for the *response* variable.

1. A person’s age and their annual income
2. A student’s GPA and their score on the SAT
3. A student’s GPA and the number of hours they spend studying per week

Once we’ve assigned roles to our two datasets, we can take the first step in visualizing the relationship between them: creating a scatter plot.

Creating Scatter Plots

A scatter plot is a visualization of the relationship between two quantitative sets of data. The scatter plot is created by turning the datasets into ordered pairs: the first coordinate contains data values from the explanatory dataset, and the second coordinate contains the *corresponding* data values from the response dataset. These ordered pairs are then plotted in the semanticsxyxy-plane. Let's return to our exam example to put this into practice.

✓ Example 8.8.2: Creating Scatter Plots Without Technology

Students are exploring the relationship between scores on the midterm exam and final exam in their math course. Here are some of the scores reported by their classmates:

Name	Midterm grade	Final grade
Student 1	88	84
Student 2	71	80
Student 3	75	77
Student 4	94	95
Student 5	68	73

Create a scatter plot to visualize the data.

Answer

Step 1: Since it makes more sense to think of the final exam score as being dependent on the midterm exam score, we'll let the final grade be the response. So, let's think of these two datasets as a set of ordered pairs, midterm first, final second:

Name	(Midterm, Final)
Student 1	(88, 84)
Student 2	(71, 80)
Student 3	(75, 77)
Student 4	(94, 95)
Student 5	(68, 73)

Step 2: Next, let's make the axes. On the horizontal axis, make sure the range of values is sufficient to cover all of the explanatory data. For our data, that's 68 to 94. Similarly, the vertical axis should cover all of the response data (73 to 95):

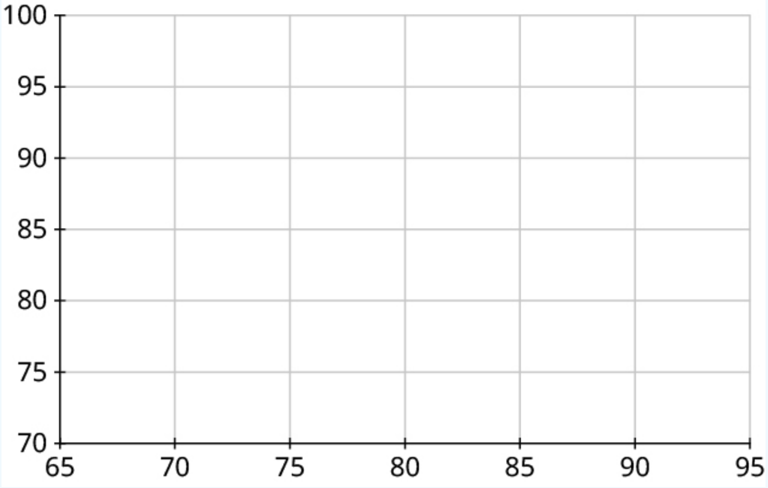


Figure ✓ 8.8.2

Step 3: Our first point is (88, 84). So, we'll locate 88 on the horizontal axis, 84 on the vertical axis, and identify the point that's directly above the first location and horizontally level with the second:

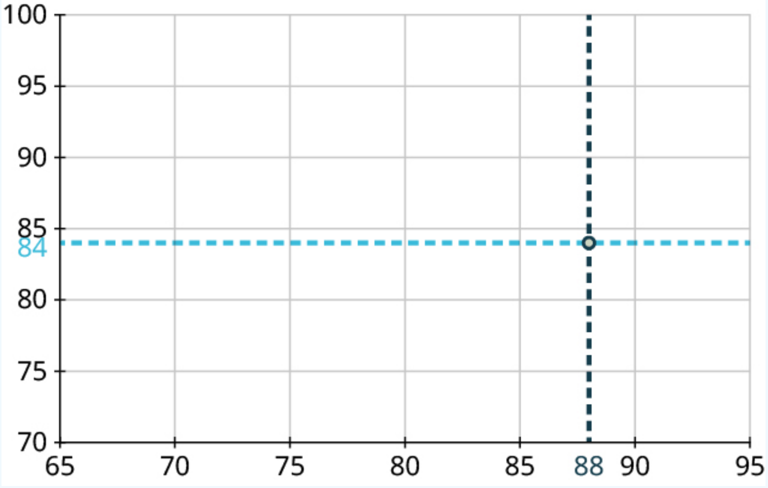


Figure ✓ 8.8.3

Step 4: Repeat this process to place the other four points on the graph:

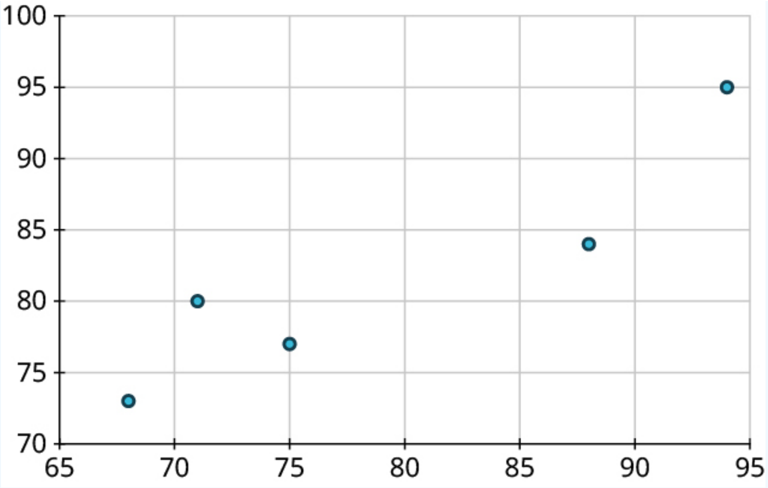


Figure ✓ 8.8.4

Step 5: Finally, label the axes:

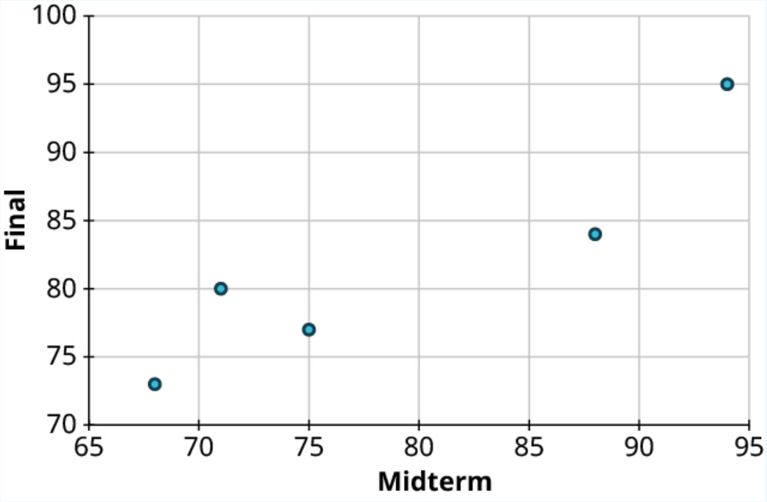


Figure ✓ 8.8.5

Your Turn 8.8.2

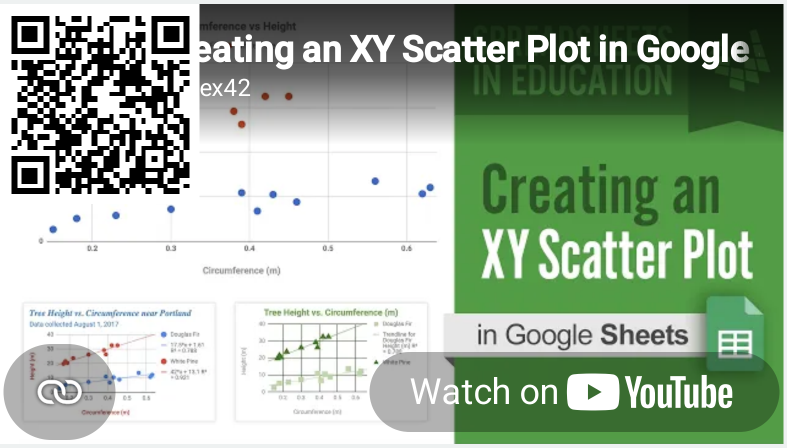
Create a scatter plot to visualize the following data, showing the top five NFL receivers by number of receptions for the 2019 season. Treat *Yards* as the response:

Table 8.12 (source: https://www.pro-football-reference.com/years/2019/)

Player	Receptions	Yards
Stefon Diggs	127	1535
Davante Adams	115	1374
DeAndre Hopkins	115	1407
Darren Waller	107	1196
Travis Kelce	105	1416

For large datasets, it’s impractical to create scatter plots manually. Luckily, Google Sheets automates this process for us.

Video


Creating an XY Scatter Plot in Google
Creating an XY Scatter Plot
in Google Sheets
Watch on YouTube

Making Scatter Plots in Google Sheets

Example $8.8.3$: Creating Scatter Plots in Google Sheets

The dataset “NHL19” gives the results of the 2018–2019 National Hockey League season. The columns are team, wins (W), losses (L), overtime losses (OTL), total points (PTS), goals scored by the team (GF), goals scored against the team (GA), and goal differential (the difference in GF and GA). Use Google Sheets to create a scatter plot for GF vs. GA.

Checkpoint

When we talk about plotting one set versus another, the first is the response and the second is explanatory.

Answer

Step 1: Open the dataset in Google Sheets, and click and drag to select the data we want to visualize (in this case, we want the columns for GF and GA; make sure you include those labels in the selection).

Step 2: Next, click on the “Insert” menu, then click “Chart.” Sheets will automatically choose a chart format; if the result isn’t a scatter plot, click on the drop-down menu under “Chart type” in the Chart Editor on the right side of the window and select “Scatter chart.”

Step 3: Next, check that the correct choices were made for the horizontal and vertical axes. In this case, we want to see GF on the vertical axis and GA on the horizontal axis. If Sheets made the wrong choice, we can fix it in the Chart Editor by clicking on the name of the dataset under “X-axis” to open up a dropdown menu, then selecting the variable that should go on the horizontal axis (GA in this case).

Step 4: Now, click on the variable under “Series” and select the one that should go on the vertical axis (GF). If you had to make that change, the axis labels in the graph may also need changing; those labels can be fixed using the “Customize” tab in the Chart Editor under “Chart & axis titles.” Your result should look like this:

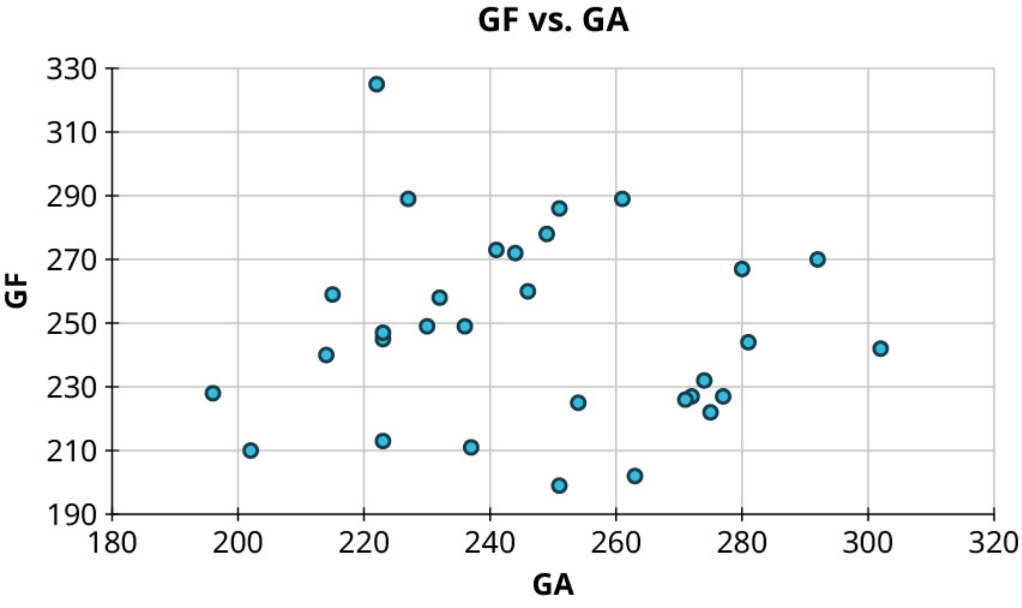


Figure ✓ 8.8.6 (data source: www.nhl.com)

Your Turn 8.8.3

With the data in "NHL19," use Google Sheets to create a scatter plot for points (PTS) vs. wins (W).

Reading and Interpreting Scatter Plots

Scatter plots give us information about the existence and strength of a relationship between two datasets. To break that information down, there are a series of questions we might ask to help us. First: Is there a curved pattern in the data? If the answer is "yes," then we can stop; none of the linear regression techniques from here to the end of this section are appropriate. Figure 8.75 and Figure 8.78 show several examples of scatter plots that can help us identify these curved patterns.

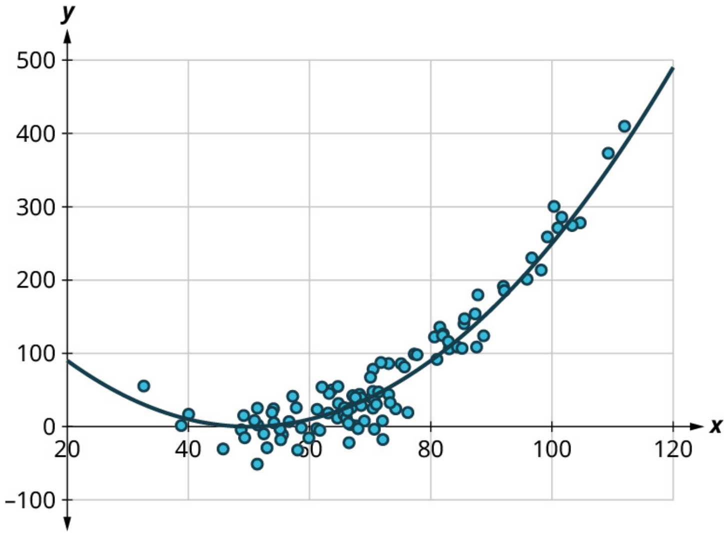


Figure 8.8.7: Curved pattern

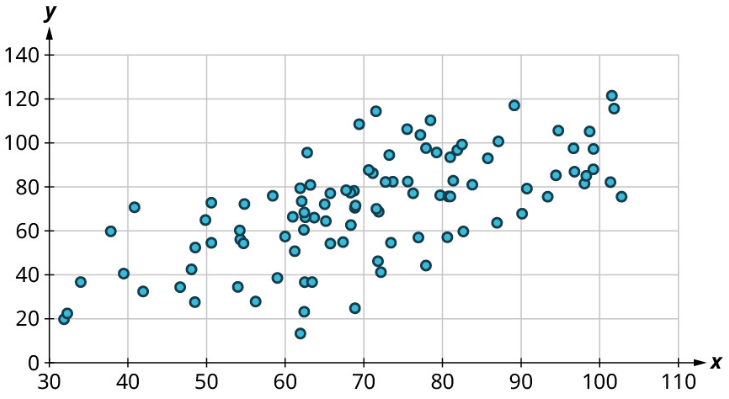


Figure 8.8.8: No curved pattern

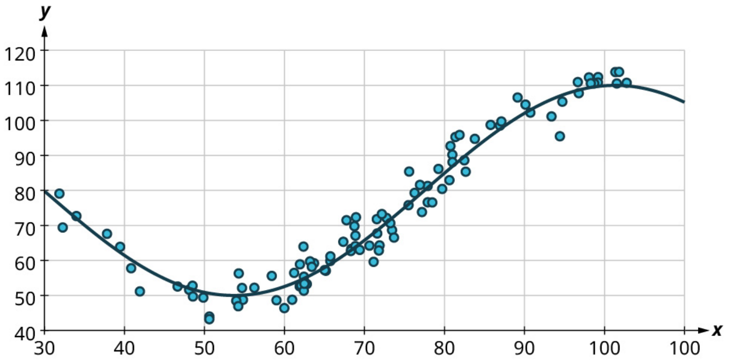


Figure 8.8.9: Curved pattern

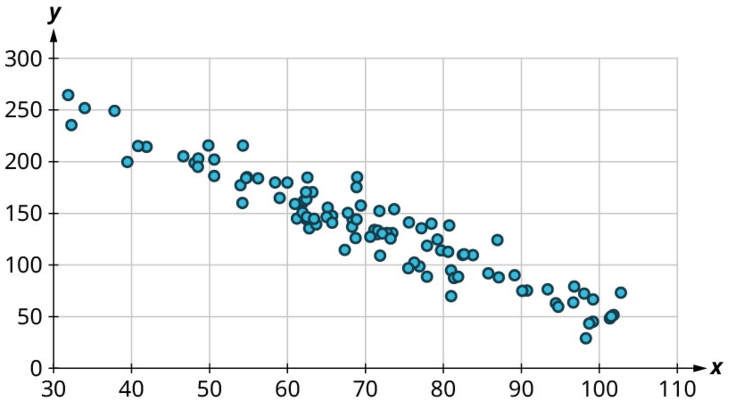


Figure 8.8.10: No curved pattern

Once we have confirmed that there is no curved pattern in our data, we can move to the next question: Is there a linear relationship? To answer this, we must look at different values of the explanatory variable and determine whether the corresponding response values are different, *on average*. It's important to look at the values "on average" because, in general, our scatter plots won't include just one corresponding response point for each value of the explanatory variable (i.e., there may be multiple response values for each explanatory value). So, we try to look for the center of those points. Let's look again at Figure 8.78, but consider some different values for the explanatory variable. Let's highlight the points whose x-values are around 50 and those that are around 80:

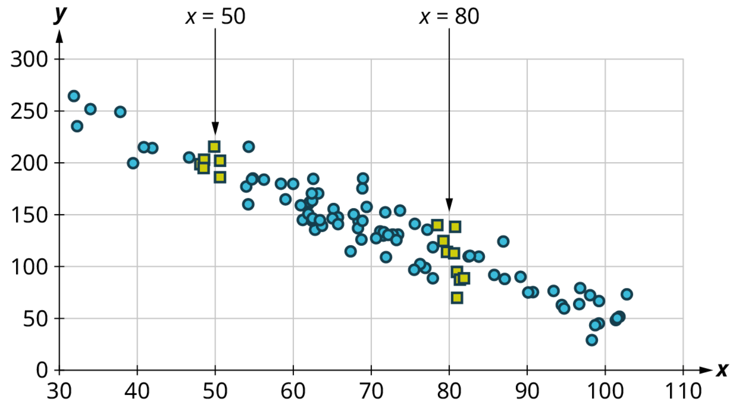


Figure 8.8.11

Now, we can estimate the middle of each group of points. Let's add our estimated averages to the plot as starred points:

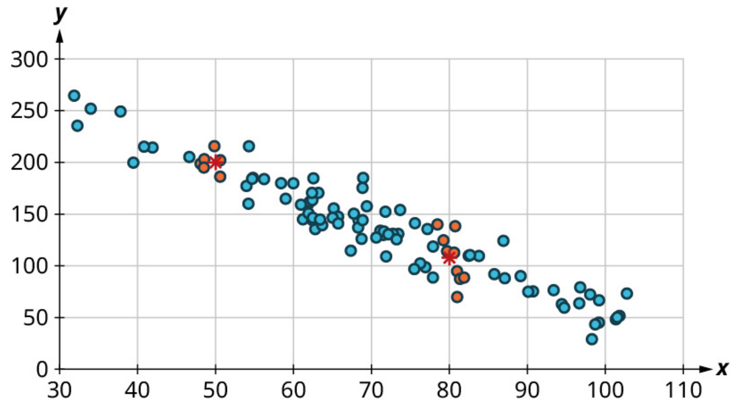


Figure 8.8.12

Since those two starred points occur at different heights, we can conclude that there’s likely a relationship worth exploring.

Here’s another example using a different set of data:

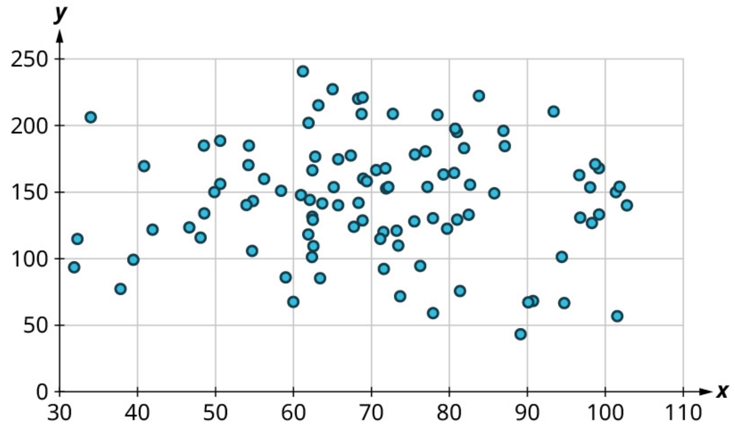


Figure 8.8.13

Let’s look again at the points near 50 and near 80, and estimate the middles of those clusters:

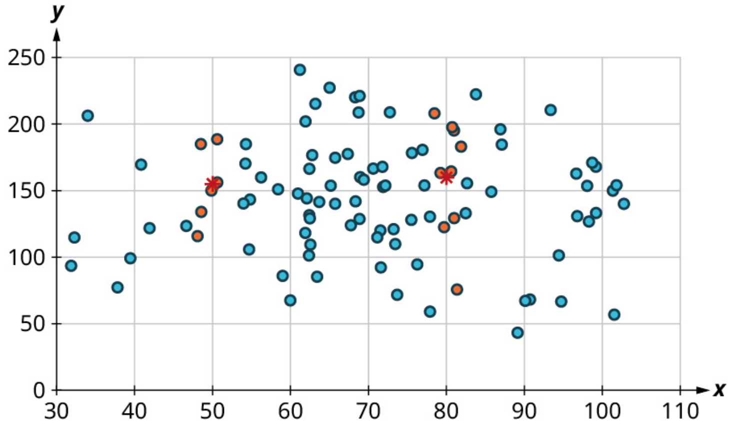


Figure 8.8.14

Notice that there’s not much vertical distance between our two starred points. This tells us that there’s not a strong relationship between these two datasets.

Positive and Negative Linear Relationships

Another way to assess whether there is a relationship between two datasets in a scatter plot is to see if the points seem to be clustered around a line (specifically, a line that’s not horizontal). The stronger the clustering around that line is, the stronger the relationship.

Once we’ve established that there’s a relationship worth exploring, it’s time to start quantifying that relationship. Two datasets have a positive linear relationship if the values of the response tend to increase, on average, as the values of the explanatory variable increase. If the values of the response *decrease* with increasing values of the explanatory variable, then there is a negative linear relationship between the two datasets. The strength of the relationship is determined by how closely the scatter plot follows a single straight line: the closer the points are to that line, the stronger the relationship. The scatter plots in Figure 8.78 to Figure 8.84 depict varying strengths and directions of linear relationships.

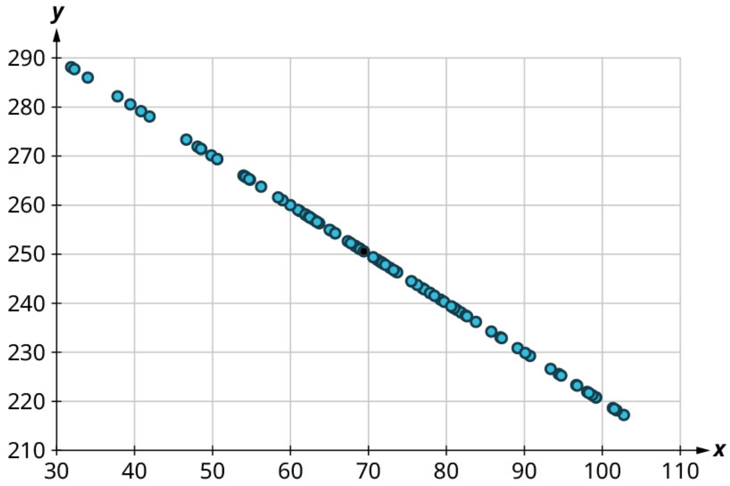


Figure 8.8.15: Perfect negative relationship

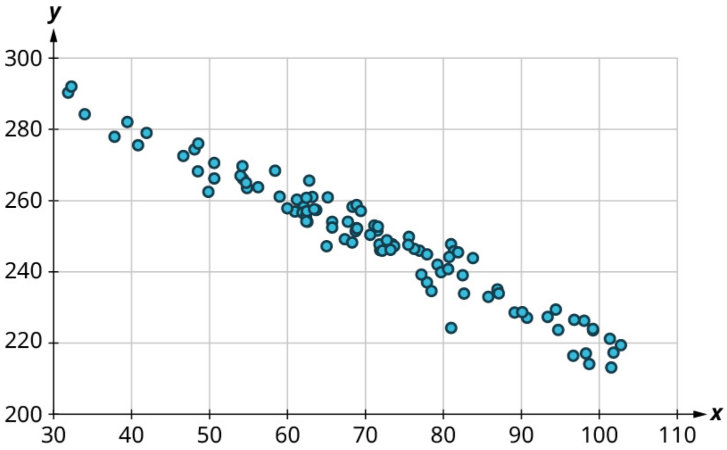


Figure 8.8.16: Strong negative relationship

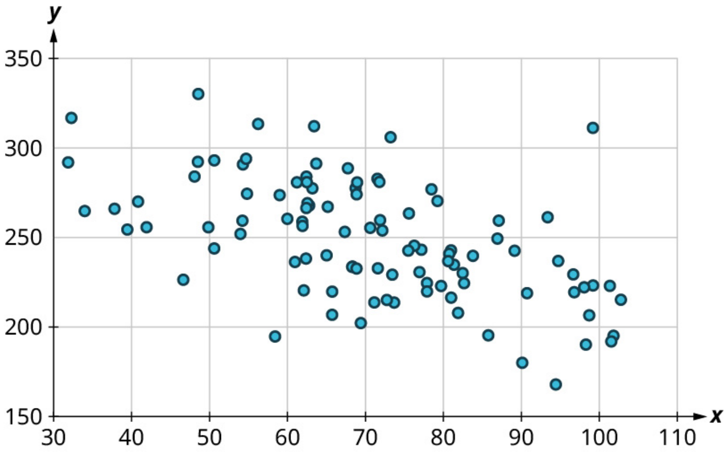


Figure 8.8.17: Weak negative relationship

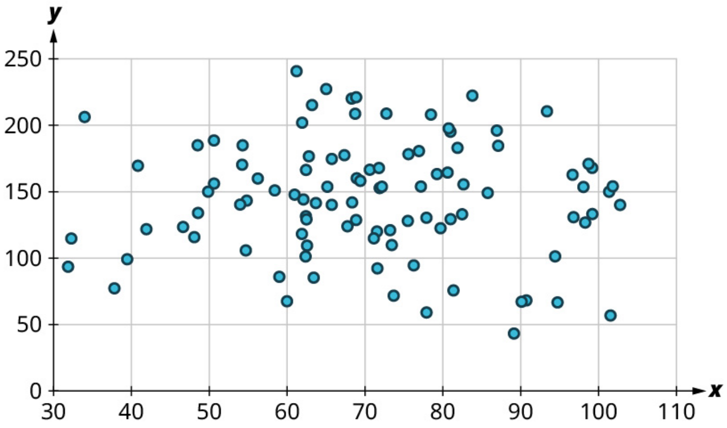


Figure 8.8.18: No relationship

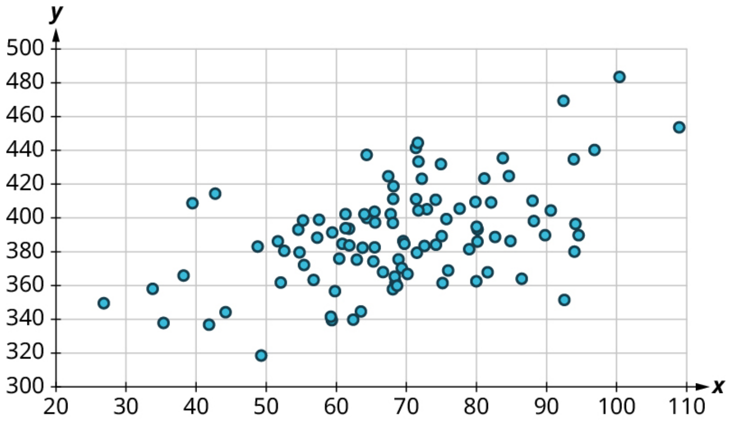


Figure $8.8.19$: Weak positive relationship

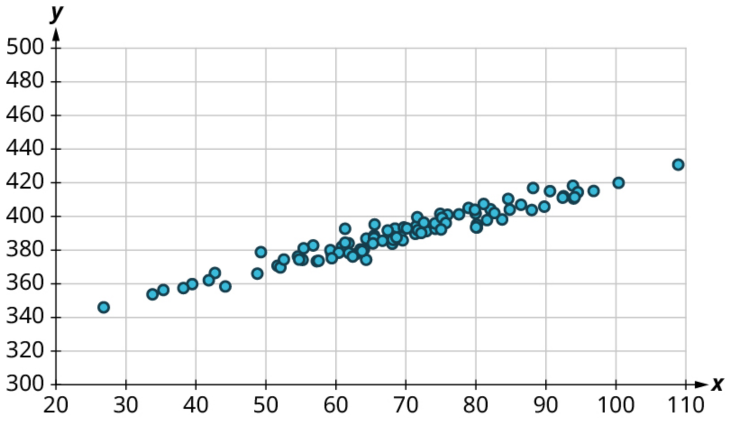


Figure $8.8.20$: Strong positive relationship

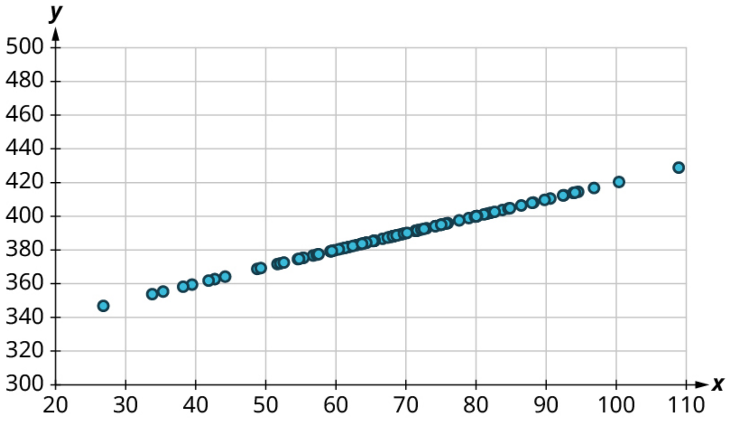


Figure $8.8.21$: Perfect positive relationship

The strength and direction (positive or negative) of a linear relationship can also be measured with a statistic called the correlation coefficient (denoted r

for Figure 8.78 to Figure 8.84, the correlation coefficients for each, in sequential order, are: −1, −0.97, −0.55, −0.03, 0.61, 0.97, and 1.

There's no firm rule that establishes a cutoff value of r to divide strong relationships from weak ones, but ± 0.7 is often given as the dividing line (i.e., if $r > 0.7$ or $r < -0.7$ the relationship is strong, and if $-0.7 < r < 0.7$ the relationship is weak).

The formula for computing r is very complicated; it's almost never done without technology. Google Sheets will do the computation for you using the CORREL function. The syntax works like this: if your explanatory values are in cells A2 to A50 and the corresponding response values are in B2 to B50, then you can find the correlation coefficient by entering "=CORREL(A2:A50, B2:B50)". (Note that the order doesn't matter for correlation coefficients; "=CORREL(B2:B50, A2:A50)" will give the same result.)

Let's put all of this together in an example.

✓ Example 8.8.4: Interpreting Scatter Plots

Consider the four scatter plots below:

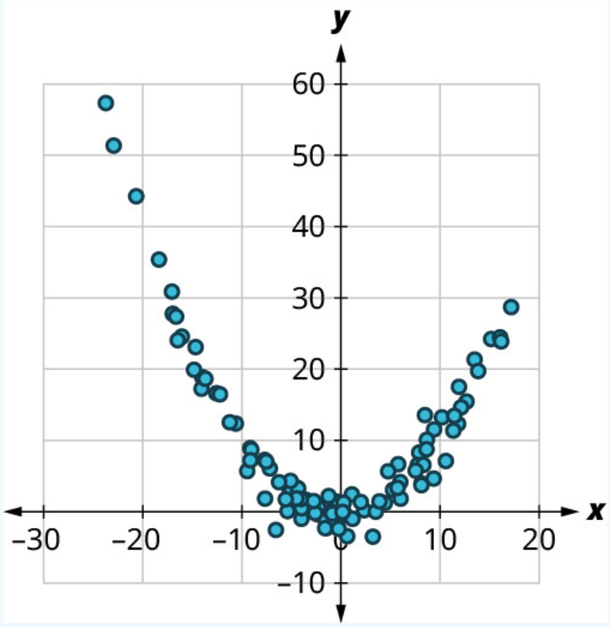
1. 

Figure ✓ 8.8.22

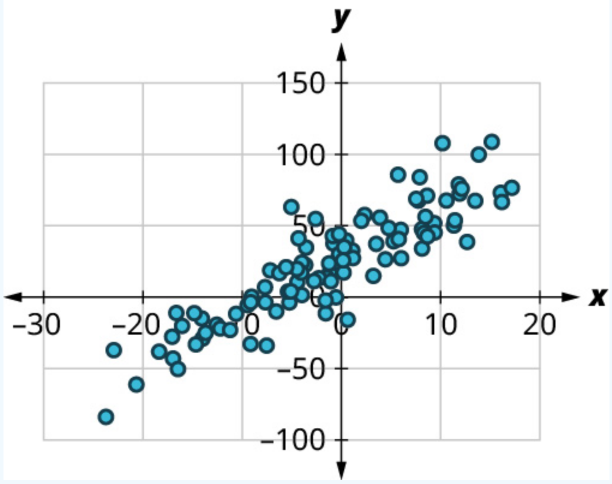
2. 

Figure ✓ 8.8.23

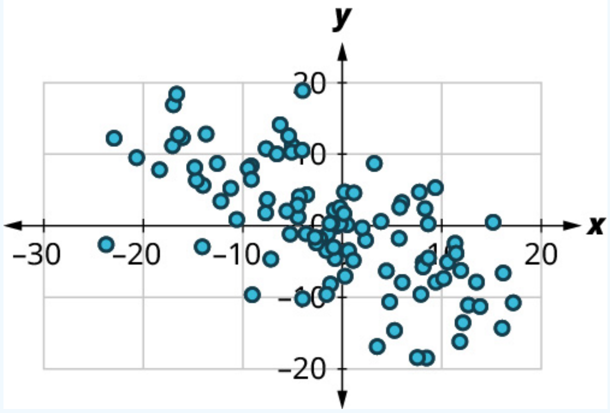
3. 

Figure ✓ 8.8.24

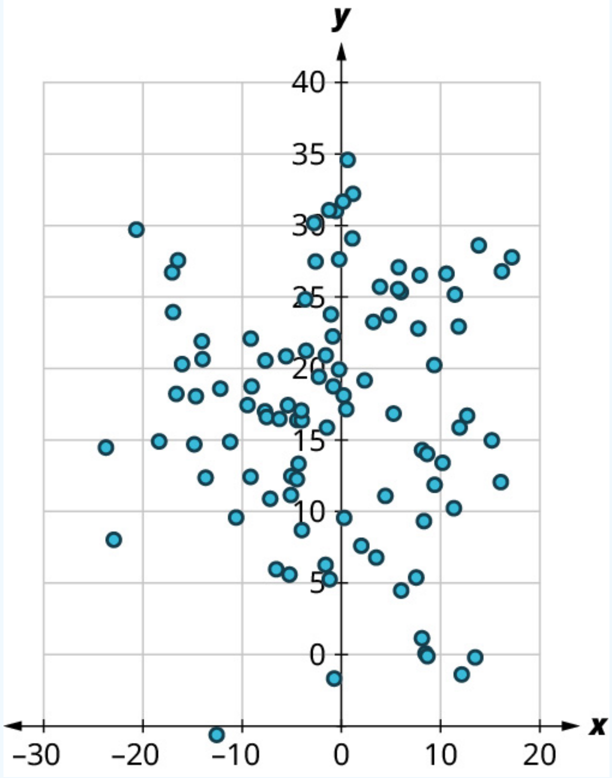
4. 

Figure ✓ 8.8.25

For each of these, answer the following questions:

a. Is there a curved pattern in the data? If yes, stop here. If no, continue to part b.
b. Classify the strength and direction of the relationship. Make a guess at the value of r.

Answer

1. a. Yes, there is a curved pattern.
2. a. No, there's no curved pattern.
   b. Since the points tend upward as we move from left to right, this is a positive relationship. The points seem pretty closely grouped around a line, so it's fairly strong. Comparing this scatter plot to those in Figure 8.83 to Figure 8.89, we can see that the relationship is stronger than the one in Figure 8.87 (Math input error
3. a. No, there's no curved pattern.
   b. Since the points tend downward as we move from left to right, this is a negative relationship. The points are not tightly grouped around a line, but the pattern is clear. It looks like it has approximately the same strength as the plot in Figure 8.87, just with the opposite sign. So, we might guess that $r = -0.6$.

4. a. No, there’s no curved pattern.
   b. Since the points don’t really tend upward or downward as we move from left to right, there is no real relationship here. Thus, $r \approx 0$.

Your Turn 8.8.4

For each of the plots below, answer the following questions:

a. Is there a curved pattern in the data? If yes, stop here. If no, continue to part b.
b. Classify the strength and direction of the relationship. Make a guess at the value of r.

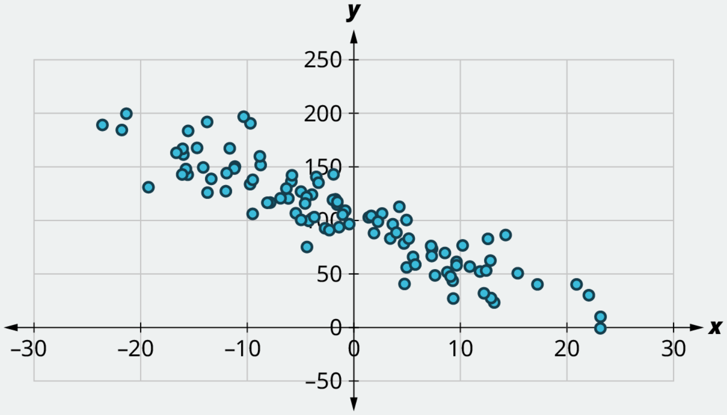


Figure 8.8.26

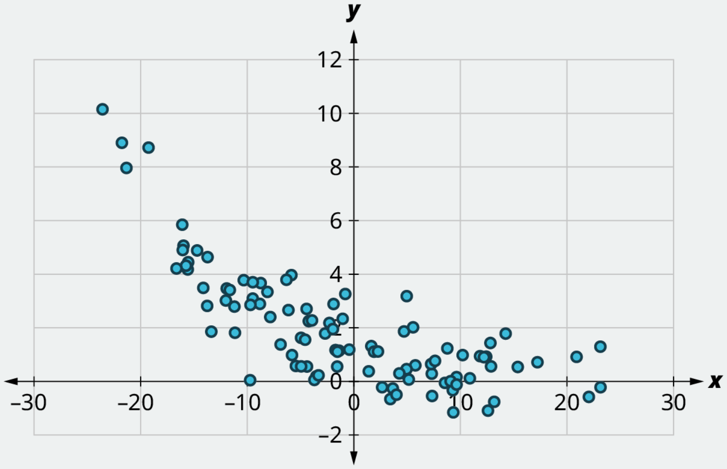


Figure 8.8.27

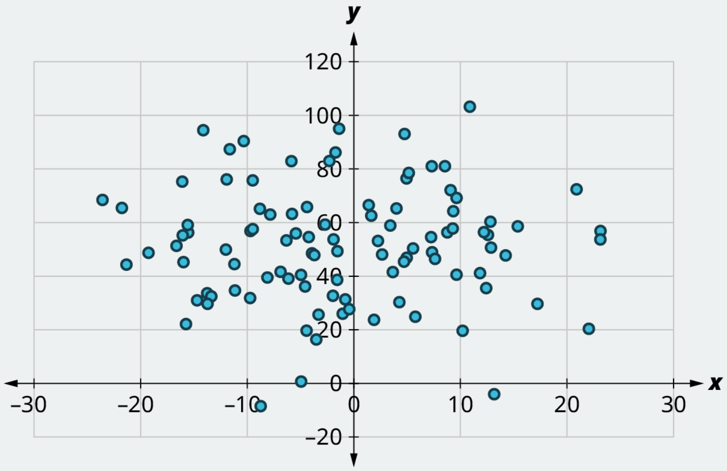


Figure 8.8.28

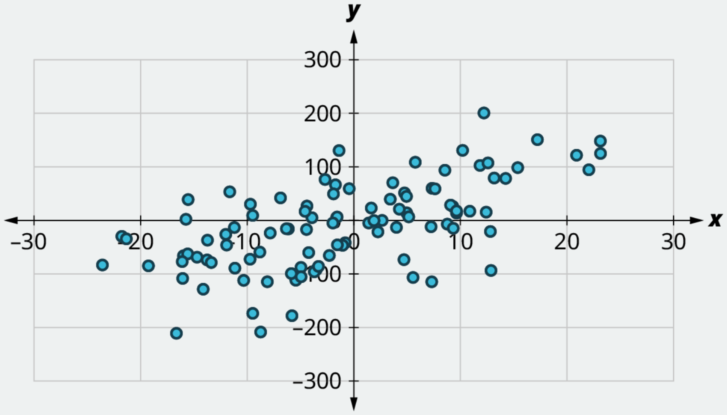


Figure 8.8.29

Example $8.8.5$: Finding the Correlation Coefficient

The data that were plotted in the previous example can be found in the dataset “correlationcoefficient1”. All of them share the same values for the explanatory variable x. The four responses are labeled semanticsy1y1 through $y4$. Compute the correlation coefficients for each, if appropriate, using Google Sheets. Round to the nearest hundredth.

Answer

Step 1: There is a curved pattern in the data, so the correlation coefficient isn’t meaningful.

Step 2: Using “=CORREL(A2:A101, C2:C101)” we get $r = 0.89$.

Step 3: Using “=CORREL(A2:A101, D2:D101)” we get $r = -0.66$.

Step 4: Using “=CORREL(A2:A101, E2:E101)” we get $r = -0.04$.

Your Turn $8.8.5$

The data that were plotted in Your Turn 8.47 can be found in the dataset “correlationcoefficient2”. All of them share the same values for the explanatory variable x. The four responses are labeled $y1$ through $y4$. Compute the correlation coefficients for each, if appropriate, using Google Sheets. Round to the nearest hundredth.

WORK IT OUT: Winning with Statistics

Billy Beane, the former general manager of the Oakland A’s baseball team, famously took his low budget team to unprecedented heights by using statistics to identify undervalued players; his story is recounted in the book *Moneyball* (which was later made into a movie, with Brad Pitt playing Beane). You can do the same thing: Take a look at team statistics in the sport of your choice and try to identify a statistic that’s most closely related to winning (meaning that it has the highest correlation coefficient with team wins).

Linear Regression

The final step in our analysis of the relationship between two datasets is to find and use the equation of the regression line. For a given set of explanatory and response data, the regression line (also called the **least-squares line** or line of best fit) is the line that does the best job of approximating the data.

What does it mean to say that a particular line does the “best job” of approximating the data? The way that statisticians characterize this “best line” is rather technical, but we’ll include it for the sake of satisfying your curiosity (and backing up the claim of "best"). Imagine drawing a line that looks like it does a pretty good job of approximating the data. Most of the points in the scatter plot will probably not fall exactly on the line; the distance above or below the line a given point falls is called that point’s *residual*. We could compute the residuals for every point in the scatter plot. If you take all those residuals and square them, then add the results together, you get a statistic called the *sum of squared errors* for the line (the name tells you what it is: “sum” because we’re adding, “squared” because we’re squaring, and “errors” is another word for “residuals”). The line that we choose to be the “best” is the one that has the smallest possible sum of squared errors. The implied minimization (“smallest”) is where the “least” in “least squares” comes from; the “squares” comes from the fact that we’re minimizing the sum of *squared* errors. This is very similar to the process we outlined in the "game" that we used to introduce the mean. Both the regression line and the mean are designed to minimize a sum of squared errors. Here ends the super technical part.

Finding the Equation of the Regression Line

So, how do we find the equation of the regression line? Recall the point-slope form of the equation of a line:

FORMULA

If a line has slope m and passes through a point (x_0, y_0), then the point-slope form of the equation of the line is:

$$y = m(x - x_0) + y_0$$

y0y=m(x-x0)+y0

The regression line has two properties that we can use to find its equation. First, it always passes through the *point of means*. If $\bar{x}$ and $\bar{y}$ are the means of the explanatory and response datasets, respectively, then the point of means is $(\bar{x}, \bar{y})$. We’ll use that as the point in the point-slope form of the equation. Second, if semanticssxsx and semanticssysy are the standard deviations of the explanatory and response datasets, respectively, and if r is the correlation coefficient, then the slope is m =r×sysxm=r×sysx. Putting all that together with the point-slope formula gives us this:

FORMULA

Suppose x and y are explanatory and response datasets that have a linear relationship. If their means are $\bar{x}$ and $\bar{y}$ respectively, their standard deviations are s_x and s_y respectively, and their correlation coefficient is r, then the equation of the regression line is:

$$y = r\left(\frac{s_y}{s_x}\right)(x - \bar{x}) + \bar{y}$$

Let's walk through an example.

✓ Example 8.8.6: Finding the Equation of the Regression Line from Statistics

Suppose you have datasets x and y with the following statistics: x has mean 21 and standard deviation 4, y has mean 8 and standard deviation 2, and their correlation coefficient is −0.4. What's the equation of the regression line?

Answer

Step 1: We're given $\bar{x} = 21, s_x = 4, \bar{y} = 8, s_y = 2$, and $r = -0.4$. Let's start with the formula for the equation of the regression line:

$$y = r\left(\frac{s_y}{s_x}\right)(x - \bar{x}) + \bar{y}$$

Step 2: Plugging in our values gives us:

$$y = -0.4\left(\frac{2}{4}\right)(x - 21) + 8$$

Step 3: Our final regression line equation is:

$$y = -0.2x + 12.2$$

Your Turn 8.8.6

Suppose you have datasets x and y with the following statistics: x has mean 100 and standard deviation 5, y has mean 200 and standard deviation 20, and their correlation coefficient is 0.75. What's the equation of the regression line?

As you can see, finding the equation of the regression line involves a lot of steps if you have to find all of the values of the needed quantities yourself. But, as usual, technology comes to our rescue. This video (which you actually watched earlier when learning how to create scatter plots) covers the regression line at around the 3:30 mark. Note that Google Sheets calls it the "trendline."

Let's put this into practice.

✓ Example

✓ 8.8.7: Finding the Equation of the Regression Line Using Google Sheets

We previously considered the relationship between goals scored (GF) and goals against (GA) using the dataset "NHL19". Recreate the scatter plot in Google Sheets, and use it to find the equation of the regression line.

Answer

Once we have recreated the scatter plot, we find the equation of the regression line by clicking the three dots at the top right of the plot, selecting "Edit chart," then clicking on "Customize" and "Series." We add the regression line by checking the box next to "Trendline," and then we show the equation by selecting "Use Equation" in the drop-down menu under "Label." The equation of the tangent line is $y = -0.0554x + 261$.

Your Turn 8.8.7

You previously created a scatter plot for points (PTS) vs. wins (W) using the dataset "NHL19". Recreate the scatter plot in Google Sheets, and use it to find the equation of the regression line.

Using the Equation of the Regression Line

Once we've found the equation of the regression line, what do we do with it? We'll look at two possible applications: making predictions and interpreting the slope.

We can use the equation of the regression line to predict the response value y for a given explanatory value x. All we have to do is plug that explanatory value into the formula and see what response value results. This is useful in two ways: first, it can be used to make a guess about an unknown data value (like one that hasn't been observed yet). Second, it can be used to evaluate performance (meaning, we can predict an outcome given a particular event). In Example 8.45, we created a scatter plot of final exam scores vs. midterm exam scores using this data:

Name	Midterm Grade	Final Grade
Allison	88	84
Benjamin	71	80
Carly	75	77
Daniel	94	95
Elmo	68	73

The equation of the regression line is $y = 0.687x + 27.4$, where y is the final exam score and x is the midterm exam score. If Frank scored 85 on the midterm, then our prediction for his final exam score is $0.687 \times 85 + 27.4 = 85.795$. To use the regression line to evaluate performance, we use a data value we've already observed. For example, Allison scored 88 on the midterm. The regression line predicts that someone who scores an 88 on the midterm will get $0.687 \times 88 + 27.4 = 87.856$ on the final. Allison actually scored 84 on the final, meaning she underperformed expectations by almost 4 points $(87.856 - 84)$.

The second application of the equation of the regression line is interpreting the slope of the line to describe the relationship between the explanatory and response datasets. For the exam data in the previous paragraph, the slope of the regression line is 0.687 . Recall that the slope of a line can be computed by finding two points on the line and dividing the difference in the y-values of those points by the difference in the x-values. Keeping that in mind, we can interpret our slope as $0.687 = \frac{\text{difference in final scores}}{\text{difference in midterm scores}}$. Multiplying both sides of that equation by the denominator of the fraction, we get $0.687\times$ difference in midterm scores $=$ difference in final scores. Thus, a one-point increase in the midterm score would result in a predicted increase in the final score of 0.687 points. A ten-point drop in the midterm score would give us a decrease in the predicted final score of 6.87 points. In general, the slope gives us the predicted change in the response that corresponds to a one unit increase in the explanatory variable.

✓ Example 8.8.8: Applying the Equation of the Regression Line

The data in "MLB2019Off" gives offensive team stats for the 2019 Major League Baseball season. Use that dataset to answer the following questions:

1. What is the equation of the regression line for runs (R) vs. hits (H)?
2. How many runs would we expect a team to score if the team got 1500 hits in a season?
3. Did the Kansas City Royals (KCR) overperform or underperform in terms of runs scored, based on their hit total? By how much?
4. Write a sentence to interpret the slope of the regression line.

Answer

1. Using Google Sheets, we find that the regression line equation is $y = 0.884x - 456$, where y is the number of runs scored and x is the number of hits.

2. Plugging 1500 into the equation of the regression line, we get $0.884 \times 1500 - 456 = 870$. We would predict that a team with 1500 hits would score 870 runs.

3. The Royals had 1356 hits, so we would predict their run total to be $0.884 \times 1356 - 456 \approx 743$. They actually scored 691 runs, so they underperformed expectations by 52 runs $(743 - 691)$.

4. The slope gives us the predicted change in the response that corresponds to a one unit increase in the explanatory variable. So, we expect one additional hit to result in 0.884 more runs. Since 0.884 runs doesn't really make sense, we can get a better interpretation by multiplying through by ten or one hundred: Ten additional hits will result in almost nine additional runs, or a hundred additional hits will yield on average just over 88 additional runs.

Your Turn 8.8.8

Using the “MLB2019Off” dataset, answer the following:

What is the equation of the regression line for the number of times a runner is caught stealing a base (CS) vs. the number of successful stolen bases (SB)?

How many times would we expect a team to be caught stealing if the team steals 70 bases in a season?

Did the Philadelphia Phillies (PHI) overperform or underperform in terms of getting caught stealing, based on their stolen base total? By how much?

Write a sentence to interpret the slope of the regression line.

Who Knew?: Math and the Movies

Statistics and regression are used by Hollywood movie producers to decide what movies to make, and to predict how much money they’ll earn at the box office. According to the American Statistical Association, not only do producers use statistics to identify the next potential blockbuster, but they’ve also pinned down how much money awards add to the bottom line. (An Academy Award is worth about $3 million!) In addition, studios use their streaming services to gather data about their customers and the types of movies they watch; this data helps them learn what kinds of entertainment their customers want more of.

WORK IT OUT: Collecting and Analyzing Your Own Data

This section has demonstrated many pairs of related quantitative datasets. Think about some quantitative variables that you can ask your classmates about, which might be related. Once you have some ideas, collect the data from your classmates. Then analyze the data by creating a scatter plot, finding the equation of the regression line (if appropriate), and interpreting it.

Extrapolation

A very common misuse of regression techniques involves extrapolation, which involves making a prediction about something that doesn't belong in the dataset.

✓ Example 8.8.9: More Applying the Equation of the Regression Line

The data in “WNBA2019” gives team statistics from the 2019 WNBA season. Use that dataset to answer these questions about team wins (W) and the proportion of team field goals made (FG%, the number of shots made divided by the number of shots attempted. Even though this column is labeled using a percent sign, the values are not expressed as percentages):

1. What is the equation of the regression line for wins vs. proportion of made field goals?
2. How many wins would we expect for a team that makes 42% of its shots?
3. Did the New York Liberty overperform or underperform in terms of wins, based on the team’s proportion of made field goals?
4. Write a sentence to interpret the slope of the regression line.

Answer

1. Using Google Sheets, we get the equation $y = 178.097x - 58.543$, where x is the proportion of field goals made and y is the number of wins.

2. Since 42% corresponds to a proportion of 0.42 , we'll plug 0.42 into the regression equation for x, which gives us $178.097 \times 0.42 - 58.543 \approx 16.26$. We would predict that a team that makes 42% of its shots would win about 16 games.

3. The New York Liberty made 41.4% of their shots, so we expect they would have $178.097 \times 0.414 - 58.543 \approx 15.19$ wins. In fact, they had only 10 wins, so they underperformed expectations by over 5 wins.

4. **Step 1**: The slope gives us the expected increase in the response that corresponds to a one unit increase in the explanatory variable. If we simply go with that interpretation, we would get a sentence like "We expect an increase in

proportion of field goals made of 1 would result in an additional 178 wins." However, that sentence doesn't make much sense. Let's consider why.

Step 2: First, proportions must be between 0 and 1, the proportion of made field goals can't be increased by 1 and still make sense. Second, the total number of games played is only 34 , so no team could get an additional 178 wins! So, we'll have to change the units.

Step 3: Since the proportions of made field goals are often expressed as a percentage, we could try to use that. If we express the slope as a fraction with 1 in the denominator (remember, the denominator represents the proportion of field goals made), then convert the denominator to a percentage and simplify, we get $\frac{178.097}{1} = \frac{178.097}{100\%} \approx \frac{1.78}{1\%}$.

Step 4: So, an increase in field goal percentage of 1% would result in an expected increase of 1.78 wins.

Your Turn 8.8.9

Use the data in “WNBA2019” to answer these questions about the relationship between the proportion of made field goals (FG%) and the proportion of made three-point field goals (3P%):

1. What is the equation of the regression line for proportion of made three-point field goals vs. proportion of made field goals?
2. What proportion of made three-point field goals would we expect for a team that makes 44% of its field goals?
3. Did the Dallas Wings overperform or underperform in terms of proportion of made three-point field goals, based on the team’s proportion of made field goals?
4. Write a sentence to interpret the slope of the regression line.

Correlation Does Not Imply Causation

One of the most common fallacies about statistics has to do with the relationship between two datasets. In the dataset “Public”, we find that the correlation coefficient between the 75th percentile math SAT score and the 75th percentile verbal SAT score is 0.92, which is really strong. The slope of the regression line that predicts the verbal score from the math score is 0.729, which we might interpret as follows: “If the 75th percentile math SAT score goes up by 10 points, we’d expect the corresponding verbal SAT score to go up by just over 7 points.”

Does the increasing math score *cause* the increase in the verbal score? Probably not. What’s really going on is that there’s a third variable that’s affecting them both: To raise the SAT math score by 10 points, a school will recruit students who do better on the SAT in general; these students will also naturally have higher SAT verbal scores. This third variable is sometimes called a *lurking variable* or a *confounding variable*. Unless all possible lurking variables are ruled out, we cannot conclude that one thing causes another.

People in Mathematics: Dr. Talithia Williams

Figure 8.8.30: A photo of Dr. Williams (credit: Used by permission of Talithia Williams)

Dr. Talithia Williams is a statistician on the faculty of Harvey Mudd College, and the first Black woman to achieve tenure at this university. She advocates for more women to become involved in the fields of engineering and science, and is on the board of directors for the EDGE Foundation, an organization that helps women obtain advanced degrees in mathematics (EDGE standing for Enhancing Diversity in Graduate Education). In 2018, Dr. Williams published the book Power in Numbers: The Rebel Women of Mathematics, a retrospective look at historical female figures who have contributed to the development of the field of mathematics.

Dr. Williams earned a Master’s degree in Mathematics from Howard University and a Master’s in Statistics from Rice University, and also went on to earn her Ph.D. in Statistics from Rice. She has held research appointments at the Jet Propulsion Laboratory, the National Security Agency, and NASA. Her research focuses on the environmental and medical applications of statistics. In 2014, she gave a popular TED talk titled “Own Your Body’s Data” that discussed the potential insights to be gained from collecting personal health data. She was even recently a host for the NOVA Wonders documentary series and a narrator for the NOVA Universe series on PBS.

To stay up-to-date on Dr. Williams’s accomplishments, you can follow her on Twitter or her Facebook account.

Who Knew?: Statistics and Eugenics

Some of the brightest minds in the history of statistics unfortunately decided to use their considerable intellects to further a pseudoscience known as eugenics. Eugenicists took Charles Darwin’s theories of evolution and ruthlessly applied them to the human race. Francis Galton (1822–1911), a cousin of Darwin and also the mathematician who invented the formula for standard deviation, claimed that people in the British upper classes possessed higher intelligence due to their superior breeding. Karl Pearson (1857–1936), who derived the formula for the correlation coefficient, argued in *National Life from the Standpoint of Science*, that, instead of providing social welfare programs, nations could better improve the fortunes of the poor by waging “war with inferior races.” Ronald Fisher (1890–1962) was possibly the most important statistician of the 20th century, having invented several new techniques (including the ubiquitous analysis of variance), and yet he also founded the Cambridge

> University Undergraduates Eugenics Society, whose self-prescribed goal was to evangelize “not by precept only, but by example, the doctrine of a new natural ability of worth and blood.”
>
> When eugenics took hold in the United States, it was used to justify terrible acts by the government, including the forced sterilization of individuals with mental illness, epilepsy, a physical impairment (like blindness), or a criminal history. The Nazi regime took these ideas to their ultimate, terrible conclusion: killing people who had mental or physical disabilities, or who were born into an “inferior” race. Over six million people died in this Holocaust, one of the darkest events in human history. To learn more, watch this video about Francis Galton and the legacy of eugenics

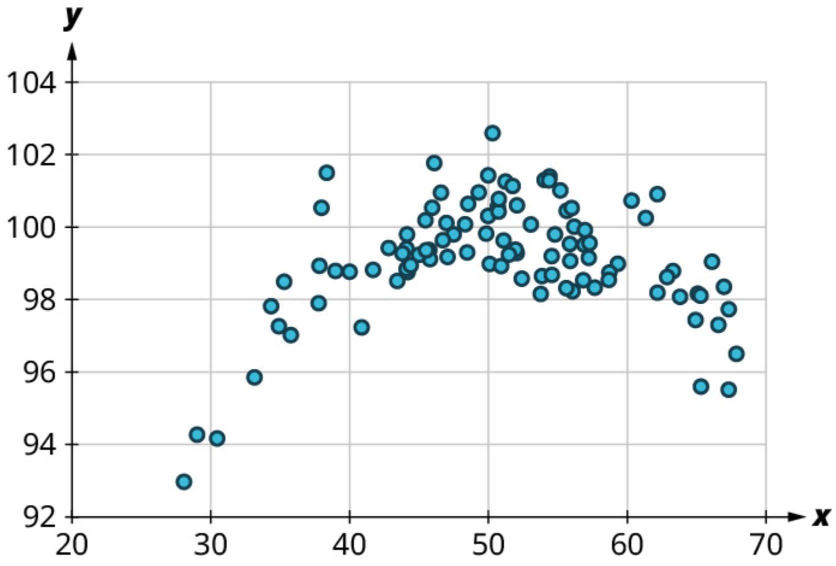
Check Your Understanding

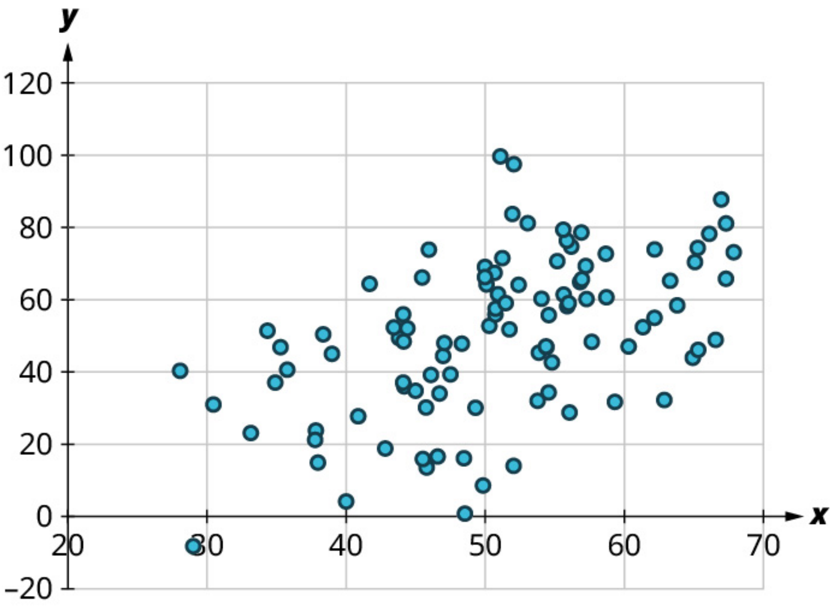
1. Make a scatter plot for the following data without technology:

x	20	11	8	22	25
y	13	15	17	13	10

For the following problems, answer these questions:

a. Is there a curved pattern in the data? If yes, stop here.
b. Classify the strength and direction of the relationship. Make a guess at the value of r.





Use the data in "MLB2019Off" to investigate the relationship between slugging percentage (SLG, explanatory) and runs scored (R, response).

2. What's the correlation coefficient? Round to the nearest hundredth.
3. What's the equation of the regression line? Round the slope and intercept to the nearest whole number.
4. The regression equation used to predict average monthly faculty salary (FacSal) from out-of-state tuition (OutState) using the data in "TNSchools" is $y = 0.161\ x + 2645$
5. Predict the average monthly faculty salary for a school that charges $30,000 in out-of-state tuition.
6. Maryville College charges $34,880 for out-of-state students, and their average monthly faculty salary is $6,765. Do they pay faculty more or less than expected? By how much?
7. Write a sentence to interpret the slope.

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8.8.0: Exercises

Exercise 8.8.0.1

This table contains data for the first five schools (alphabetically) that fielded an NCAA Division I men's basketball team in the 2018–2019 season. It shows the total number of points each team scored (PF) and the total number of points their opponents scored against them (PA). Create a scatter plot without technology of PA vs. PF.

School	PF	PA
Abilene Christian	2502	2161
Air Force	2179	2294
Akron	2271	2107
Alabama A&M	1938	2285
Alabama-Birmingham	2470	2370

Exercise 8.8.0.2

This table contains data for the first five schools (alphabetically) that fielded an NCAA Division I men's basketball team in the 2018–2019 season. It shows the total number of field goals each team scored (FG) and the total number of three-point field goals they scored (3P). Create a scatter plot without technology of 3P vs. FG.

School	FG	3P
Abilene Christian	897	251
Air Force	802	234
Akron	797	297
Alabama A&M	736	182
Alabama-Birmingham	906	234

For the following exercises, use the data in "MBB2019", on every school that fielded an NCAA Division I men's basketball team in the 2018–2019 season.

Exercise 8.8.0.3

Use Google Sheets to create a scatter plot of points scored against a team (PA) vs. points scored by the team (PF).

Exercise 8.8.0.4

Use Google Sheets to create a scatter plot of number of three-point field goals made (3P) vs. total field goals made (FG).

Exercise 8.8.0.5

Use Google Sheets to create a scatter plot of number of fouls (Fouls) vs. number of blocks (BLK).

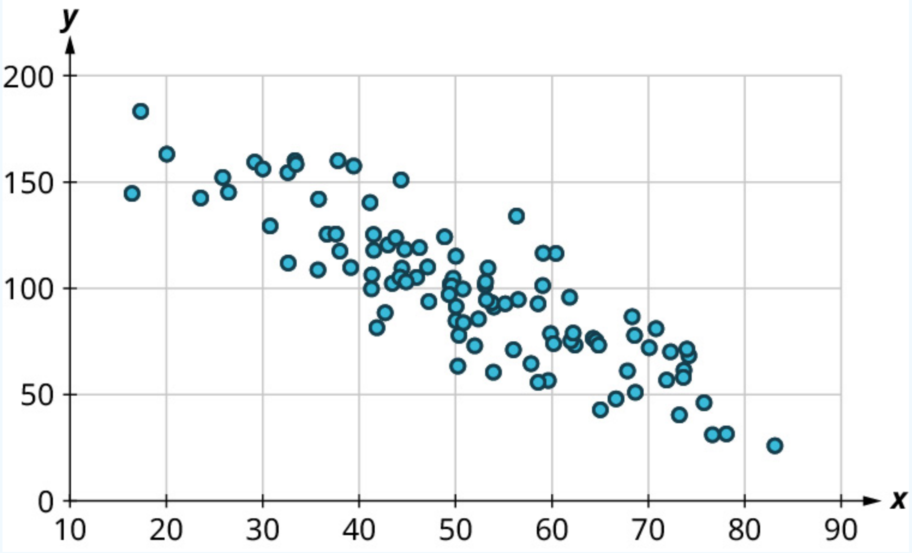
Exercise 8.8.0.6

Use Google Sheets to create a scatter plot of points scored (PF) vs. percent of three-point shots made (3P%).

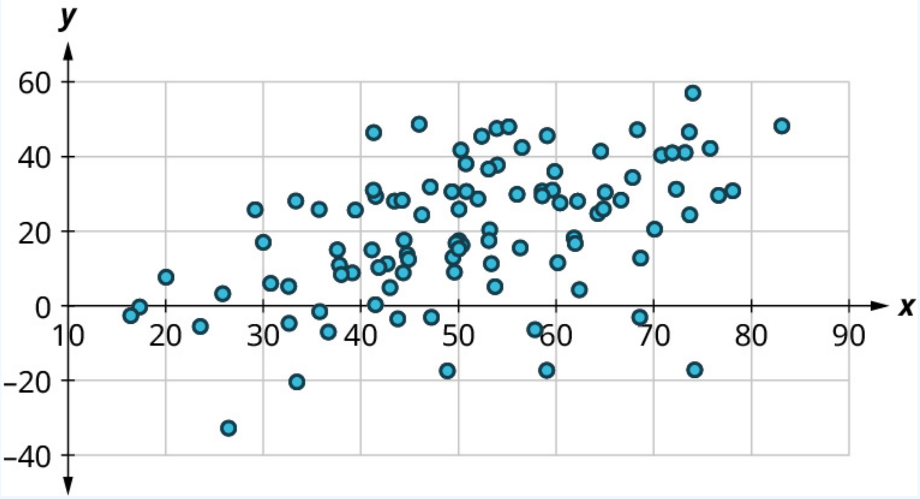
For the following exercises, answer the following questions:

a. Is there a curved pattern in the data? If yes, stop here. If no, continue to part b.
b. Classify the strength and direction of the relationship. Make a guess at the value of r.

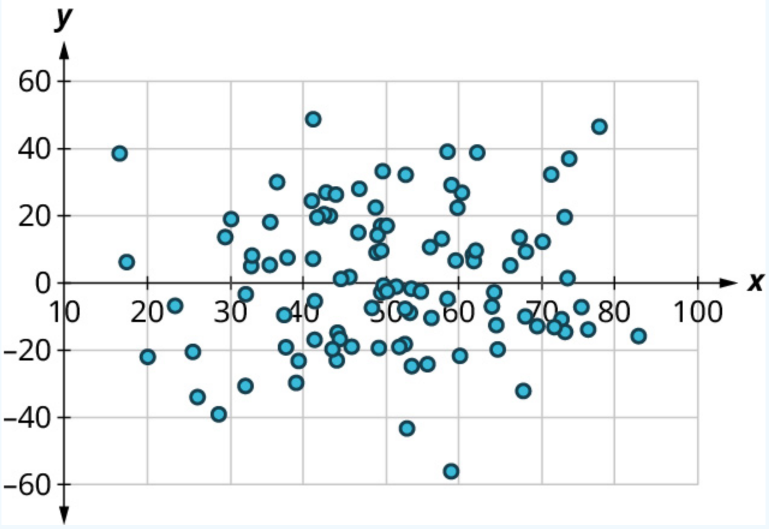
Exercise 8.8.0.7



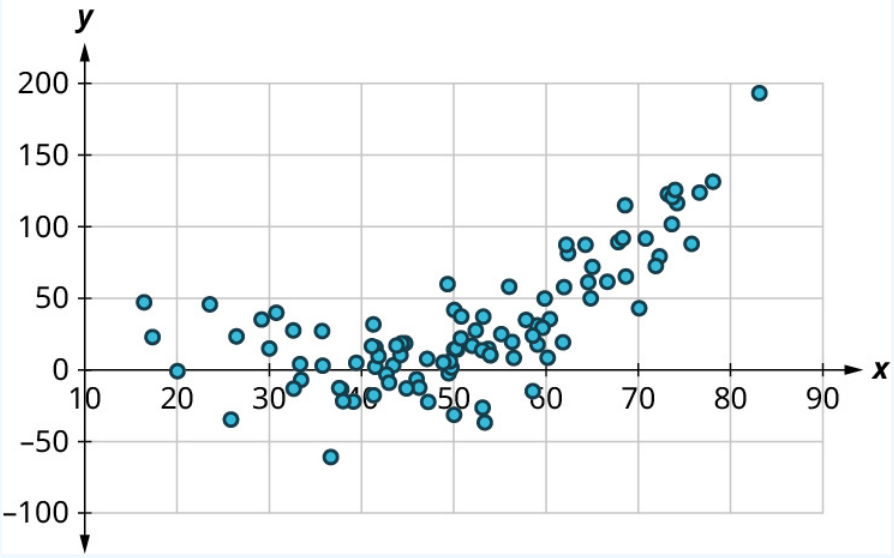
Exercise 8.8.0.8



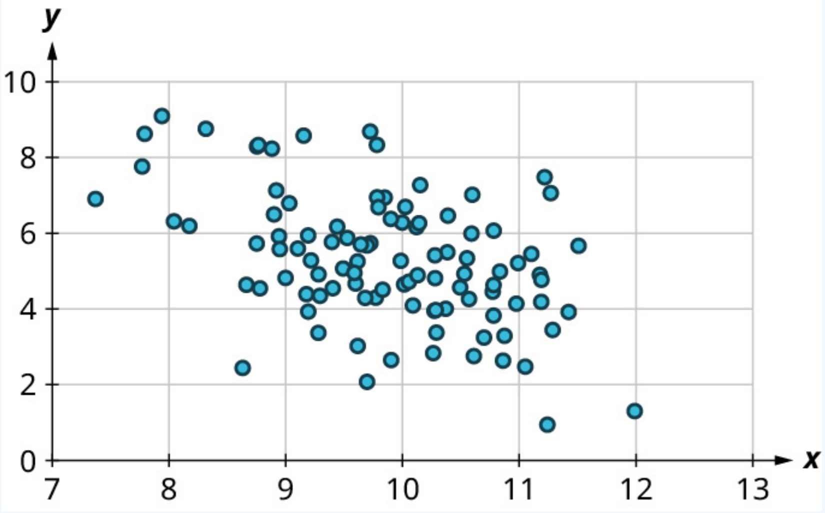
Exercise 8.8.0.9



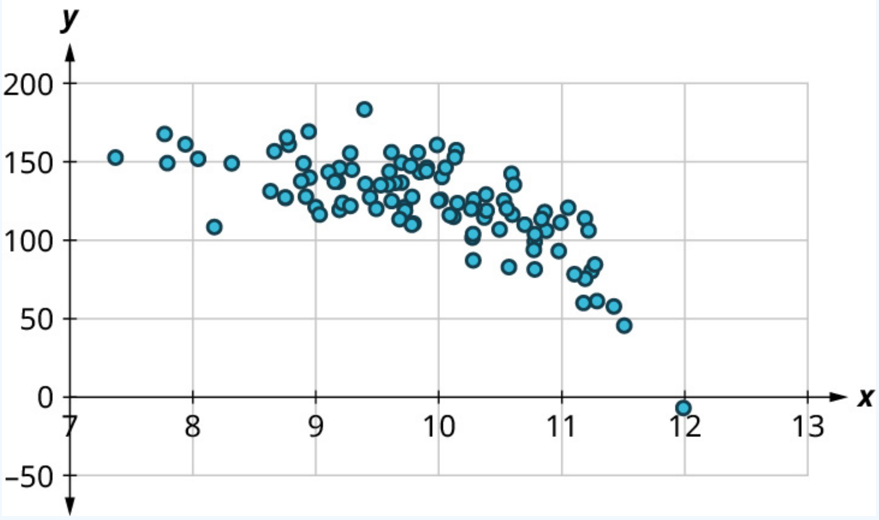
Exercise 8.8.0.10



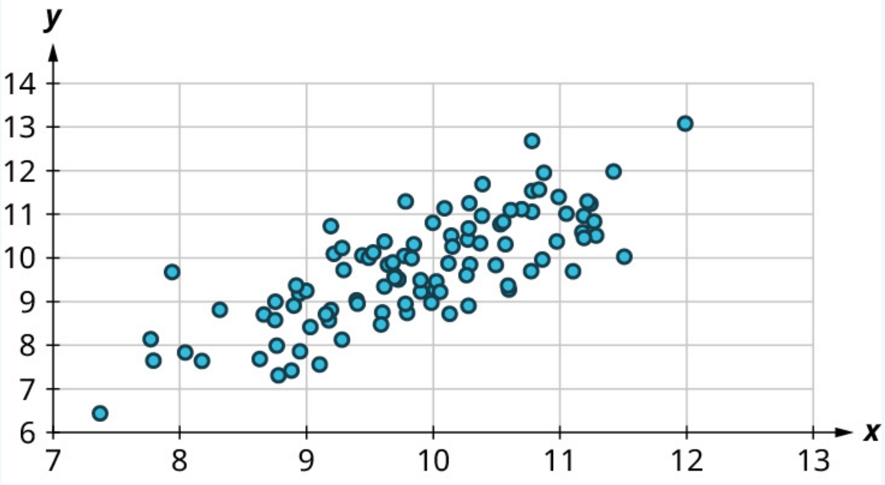
Exercise 8.8.0.11



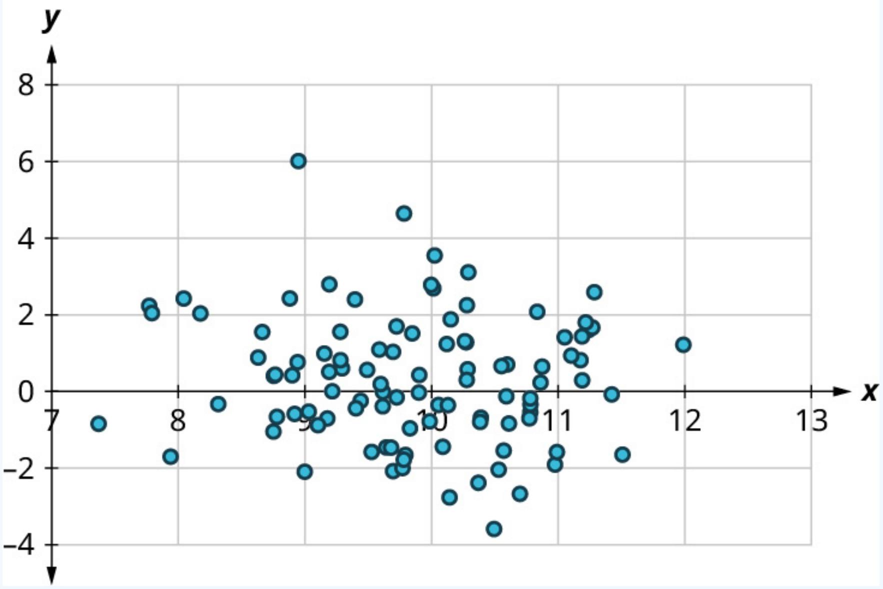
Exercise 8.8.0.12



Exercise 8.8.0.13



Exercise 8.8.0.14



For the following exercises, use the data in “MBB2019” on every school that fielded an NCAA Division I men’sbasketball team in the 2018–2019 season.

Exercise 8.8.0.15

What is the correlation coefficient for points scored against a team (PA) vs. points scored by the team (PF)? Round to the nearest hundredth.

Exercise 8.8.0.16

What is the equation of the regression line for PA vs. PF?

Exercise 8.8.0.17

Predict the total number of points scored against a team that itself scores 2200 points.

Exercise 8.8.0.18

Georgia Tech scored 2091 points, and had 2130 points scored against them. Is their PA higher or lower than expected? By how much?

Exercise 8.8.0.19

Write a sentence that interprets the slope of the regression line for PA vs. PF.

Exercise 8.8.0.20

What is the correlation coefficient for three-point field goals made (3P) vs. total field goals made (FG)? Round to the nearest hundredth.

? Exercise 8.8.0.21

What is the equation of the regression line for 3P vs. FG?

? Exercise 8.8.0.22

How many three-point field goals made would you expect for a team that made 1000 total field goals?

? Exercise 8.8.0.23

Seton Hall made 888 field goals; of those, 240 were three-point field goals. Did they make more or fewer three-point field goals than expected? How many more or fewer?

? Exercise 8.8.0.24

Write a sentence to interpret the slope of the regression line for 3P vs. FG.

For the following exercises, use the datasets “Public” and “Private”, which give many institutions of higher learning in the United States (public institutions in “Public” and private, non-profit institutions in “Private”), the schools’ 75th percentiles on the math section of the SAT (SATM75), the verbal section of the SAT (SATV75), the math section of the ACT (ACTM75), and the English section of the ACT (ACTE75). It also gives the schools’ admission rates (AdmRate) and total annual cost of attendance (Cost).

? Exercise 8.8.0.25

It might seem reasonable to expect the cost to attend a school to go down as the proportion of applicants admitted goes up. Create two scatter plots (one for private schools, one for public) to investigate that hunch. Can we use linear regression to describe that relationship for these? Why or why not?

? Exercise 8.8.0.26

Find the correlation between the 75th percentiles of the two sections of the SAT at public schools and at private schools. Which has a stronger relationship?

? Exercise 8.8.0.27

What score would we predict falls at the 75th percentile on the verbal section of the SAT at a public school where the 75th percentile on the math section of the SAT is 500?

? Exercise 8.8.0.28

What score would we predict falls at the 75th percentile on the verbal section of the SAT at a private school where the 75th percentile on the math section of the SAT is 500?

? Exercise 8.8.0.29

Find the slope of the regression line that we would use to predict the 75th percentile SAT math score from the 75th percentile ACT English score at public schools, and write a sentence to interpret that slope.

? Exercise 8.8.0.30

Predict the cost of attendance at a public school whose 75th percentile on the SAT verbal section is 700.

? Exercise 8.8.0.31

The cost of attendance at DePauw University, a private school, is \$62,567. The 75th percentile on the SAT math section is 680. Is DePauw more or less expensive that we would predict based on the SAT math score? By how much?

? Exercise 8.8.0.32

The cost of attendance at Coastal Carolina University, a public school, is \$24,599. The 75th percentile of ACT English scores at Coastal Carolina is 24. Is the cost higher or lower than we would expect based on the ACT English score? By how much?

? Exercise 8.8.0.33

Find the equation of the regression line that we would use to predict the 75th percentile ACT English score from the 75th percentile ACT math score at public institutions.

? Exercise 8.8.0.34

Find the equation of the regression line that we would use to predict cost of attendance at public schools using the 75th percentile ACT math score.

? Exercise 8.8.0.35

Does the University of Hawai'i at Hilo have a higher or lower 75th percentile verbal SAT score (590) than we'd expect based on its 75th percentile math SAT score (580)? By how much?

? Exercise 8.8.0.36

Find the slope of the regression line we would use to estimate cost from the 75th percentile SAT math scores at public institutions. Write a sentence to interpret that slope.

? Exercise 8.8.0.37

Find the slope of the regression line we would use to estimate cost from the 75th percentile SAT math scores at private institutions. Write a sentence to interpret that slope.

? Exercise 8.8.0.38

Look at the scatter plots that show the relationship between cost and the 75th percentiles of the various test scores at private institutions. Which (if any) of the four exhibit a pattern that rules out analysis using linear regression?

? Exercise 8.8.0.39

Look at the scatter plots that show the relationship between cost and the 75th percentiles of the various test scores at public institutions. Which (if any) of the four exhibit a pattern that rules out analysis using linear regression?

? Exercise 8.8.0.40

Looking at public institutions, rank the four test scores from highest to lowest in terms of the strength of their relationships to cost.

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SECTION OVERVIEW

8.9: Chapter Summary

8.9.0: Key Terms

8.9.1: Key Concepts

8.9.2: Videos

8.9.3: Formula Review

8.9.4: Projects

8.9.5: Chapter Review

8.9.6: Chapter Test

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8.9.0: Key Terms

8.1 Gathering and Organizing Data

- sample
- units
- data
- population
- simple random sample
- systematic random sample
- stratified sample
- cluster sample
- quantitative data
- categorical data
- categorical frequency Distribution
- binned frequency distribution

8.2 Visualizing Data

- proportion
- bar chart
- pie chart
- stem-and-leaf plot
- distribution (of quantitative data)
- histogram
- bar chart for labeled data

8.3 Mean, Median and Mode

- mode
- bimodal
- median
- mean

8.4 Range and Standard Deviation

- range
- standard deviation

8.5 Percentiles

- percentile
- quartile
- quintile
- quantile

8.6 The Normal Distribution

- normal distribution
- 68-95-99.7 Rule
- standardized score (z-score)

8.8 Scatter Plots, Correlation, and Regression Lines

- response variable (dependent variable)
- explanatory variable (independent variable)
- scatter plot
- positive linear relationship
- negative linear relationship

- correlation coefficient
- regression line (least-squares line, line of best fit)

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8.9.1: Key Concepts

8.1 Gathering and Organizing Data

- Categorical data places units into groups (categories), while quantitative data is a numerical measure of a property of a unit.
- The sampling method for a study depends on the way that randomization is used to select units for the sample.
- Frequency distributions help to summarize data by counting the number of units that fall into a particular category or range of quantitative values.

8.2 Visualizing Data

- Categorical data can be visualized using pie charts or bar charts; quantitative data can be visualized using stem-and-leaf plots or histograms.
- Areas in pie charts and bar charts represent proportions of the data falling into a particular category, while areas in histograms represent proportions of the data that fall into a given range of data values (or “bins”). Stem-and-leaf plots are visual representations of entire datasets.
- By manipulating the axes, changing widths of bars, or making bad choices for bins, we can create data visualizations that misrepresent the distribution of data.

8.3 Mean, Median and Mode

- The mode of a dataset is the value that appears the most frequently. The median is a value that is greater than or equal to no more than 50% of the data and less than or equal to no more than 50% of the data. The mean is the sum of all the data values, divided by the number of units in the dataset.
- The median of a dataset is not affected by outliers, but the mean will be biased toward outliers. This distinction might affect which measure of centrality is used to summarize a dataset.

8.4 Range and Standard Deviation

- The range of a dataset is the difference between its largest and smallest values. The standard deviation is approximately the mean difference (in absolute value) that individual units fall from the mean of the dataset.

8.5 Percentiles

- The percentile rank of a data value is the percentage of all values in the dataset that are less than or equal to the given value.

8.6 The Normal Distribution

- Normally distributed data follow a bell-shaped, symmetrical distribution.
- The mean of normally distributed data falls at the peak of the distribution. The standard deviation of normally-distributed data is the distance from the peak to either of the inflection points.
- Data that are normally distributed follow the 68-95-99.7 Rule, which says that approximately 68% of the data fall within one standard deviation of the mean, 95% fall within two standard deviations, and 99.7% fall within three standard deviations.
- The z-score for a data value is the number of standard deviations that value falls above (or below, if the z-score is negative) the mean.
- We can use the normal distribution to estimate percentiles.

8.7 Applications of the Normal Distribution

- We can use z-scores to compare data values from different datasets.

8.8 Scatter Plots, Correlation, and Regression Lines

- If one variable affects the value of another variable, we say the first is an explanatory variable and the second is a response variable.
- Scatter plots place a point in the xy-plane for each unit in the dataset. The x-value is the value of the explanatory variable, and the y-value is the value of the response variable.
- The correlation coefficient r gives us information about the strength and direction of the relationship between two variables. If r is positive, the relationship is positive: an increase in the value of the explanatory variable tends to correspond to an **increase** in the value of the response variable. If r is negative, the relationship is negative: an increase in the value of the explanatory

variable tends to correspond to a **decrease** in the value of the response variable. Values of r that are close to 0 indicate weak relationships, while values close to –1 or indicate strong relationships.
- The regression line for a relationship between two variables is the line that best represents the data. It can be used to predict values of the response variable for a given value of the explanatory variable.

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8.9.2: Videos

8.2 Visualizing Data

- Make a Simple Bar Graph in Google Sheets
- Create Pie Charts Using Google Sheets
- Make a Histogram Using Google Sheets
- How to Spot a Misleading Graph

8.3 Mean, Median and Mode

- Compute Measures of Centrality Using Google Sheets

8.4 Range and Standard Deviation

- Find the Minimum and Maximum Using Google Sheets
- Find the Standard Deviation Using Google Sheets

8.5 Percentiles

- Using RANK, PERCENTRANK, and PERCENTILE in Google Sheets

8.8 Scatter Plots, Correlation, and Regression Lines

- Making Scatter Plots in Google Sheets

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8.9.3: Formula Review

8.3 Mean, Median and Mode

Suppose we have a set of data with n values, ordered from smallest to largest. If n is odd, then the median is the data value at position $\frac{n+1}{2}$. If n is even, then we find the values at positions $\frac{n}{2}$ and $\frac{n}{2}+1$. If those values are named a and b, then the median is defined to be $\frac{a+b}{2}$.

8.4 Range and Standard Deviation

$$s=\sqrt{\frac{\sum(x-\bar{x})^2}{n-1}}$$

Here, s is the standard deviation, x represents each data value, $\bar{x}$ is the mean of the data values, n is the number of data values, and the capital sigma (Σ) indicates that we take a sum.

8.6 The Normal Distribution

If x is a member of a normally distributed dataset with mean μ and standard deviation σ, then the standardized score for x is

$$z=\frac{x-\mu}{\sigma}$$

If you know a z-score but not the original data value x, you can find it by solving the previous equation for x :

$$x=\mu+z\times\sigma$$

8.8 Scatter Plots, Correlation, and Regression Lines

If a line has slope m and passes through a point (x_0, y_0), then the point-slope form of the equation of the line is:

$$y=m\left(x-x_0\right)+y_0$$

Suppose x and y are explanatory and response datasets that have a linear relationship. If their means are $\bar{x}$ and $\bar{y}$ respectively, their standard deviations are s_x and s_y respectively, and their correlation coefficient is r, then the equation of the regression line is:

$$y=r\left(\frac{s_y}{s_x}\right)(x-\bar{x})+\bar{y}$$

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8.9.4: Projects

1. Browse through some news websites to find five stories that report on data and include data visualizations. Can you tell from the report how the data were collected? Was randomization used? Are the visualizations appropriate for the data? Are the visualizations presented in a way that might bias the reader?
2. We discussed three measures of centrality in this chapter: the mode, the median, and the mean. In a broader context, the mean as we discussed it is more properly called the *arithmetic mean*, to distinguish it from other types of means. Examples of these include the geometric mean, harmonic mean, truncated mean, and weighted mean. How are these computed? How do they compare to the arithmetic mean? In what situations would each of these be preferred to the arithmetic mean?
3. *Simpson's Paradox* is a statistical phenomenon that can sometimes appear when we observe a relationship within several subgroups of a population, but when the data for all thegroups are analyzed all together, the opposite relationship appears. Find some examples of Simpson's Paradox in real-world situations, and write a paragraph or two that would explain the concept to someone who had never studied statistics before.

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8.9.5: Chapter Review

Gathering and Organizing Data

Decide whether randomization is being used in the selection of these samples. If it is, identify the type of random sample (simple, systematic, cluster, or stratified).

? Exercise 8.9.5.1

College students want to gauge interest in a new club they're thinking about starting. They choose three residence halls at random and slide a survey form under the door to every room.

? Exercise 8.9.5.2

The managers of a campus dining hall want feedback on a new dish that is being served, so they ask the first 10 people who choose that dish what they think of it.

? Exercise 8.9.5.3

The administration of a large university system with five campuses wants to administer a survey to students in the system. The administration chooses 50 students at random from each campus; the 250 students selected form thesample.

? Exercise 8.9.5.4

A sample of college students were asked how many of their meals were provided by a campus dining hall on the previous day. Here are the results:

0	1	0	3	2	3	3
2	2	0	2	1	1	0
0	1	2	0	3	0	1
3	2	3	0	1	3	0

Create a frequency distribution for these data.

Visualizing Data

? Exercise 8.9.5.5

Create a bar chart to visualize the data in question 4.

? Exercise 8.9.5.6

Use the data in “MBB2019” on every school that fielded an NCAA Division I men’s basketball team in the 2018–2019 season, to create a histogram of the number of free throws made by each team (FT).

Mean, Median and Mode

? Exercise 8.9.5.7

The data below show the top ten scorers (by points per game) in the National Basketball Association for the 2018–2019 season. Find the mode, median, and mean age of these players without technology.

Table 8.13 (source: https://www.basketball-reference.com...9_leaders.html)

Rank	Player	Age
1	James Harden	29
2	Paul George	28
3	Giannis Antetokounmpo	24
4	Joel Embiid	24
5	Stephen Curry	30
6	Devin Booker	22
7	Kawhi Leonard	27
8	Kevin Durant	30
9	Damian Lillard	28
10	Bradley Beal	25

? Exercise 8.9.5.8

Using the data in “MBB2019”, find the mean, median, and mode of the number of free throws made by each team (FT).

Range and Standard Deviation

? Exercise 8.9.5.9

Using the data in “MBB2019”, find the range and standard deviation of the number of free throws made by each team (FT).

Percentiles

Use the data in “MBB2019”, to answer the following:

? Exercise 8.9.5.10

What number of free throws (FT) is at the 40th percentile?

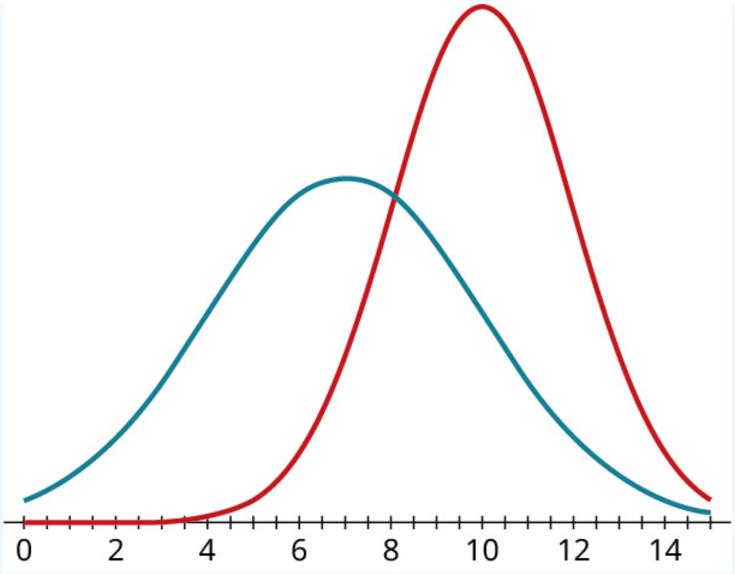
? Exercise 8.9.5.11

At what percentile is Syracuse University, which made 480 free throws?

The Normal Distribution

? Exercise 8.9.5.12

Identify the means and standard deviations of these normal distributions:



Answer the following about data that are distributed normally with mean 200 and standard deviation 20:

Exercise $8.9.5.13$

What proportion of the data are between 160 and 240?

Exercise $8.9.5.14$

What's the standardized score of the data value 235?

Exercise $8.9.5.15$

At what percentile would the data value 187 fall?

Exercise $8.9.5.16$

What data value would be at the 90th percentile?

Applications of the Normal Distribution

Use the data in "MBB2019" to answer the following:

Exercise $8.9.5.17$

What is the standardized score for Purdue Universitiy's number of free throws (FT = 461)?

Exercise $8.9.5.18$

At what percentile would we estimate Purdue's value of FT to fall, using the normal distribution?

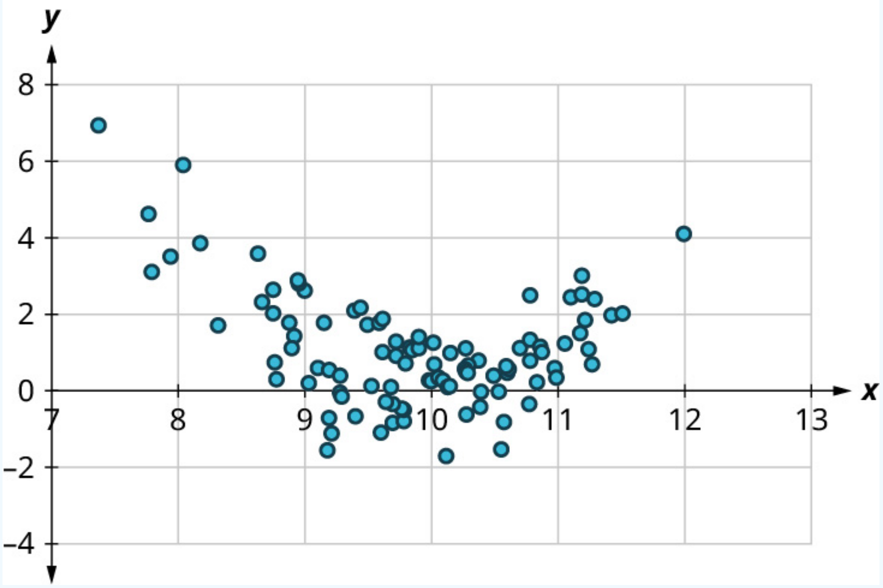
Exercise $8.9.5.19$

At what percentile does Purdue's value of FT actually fall?

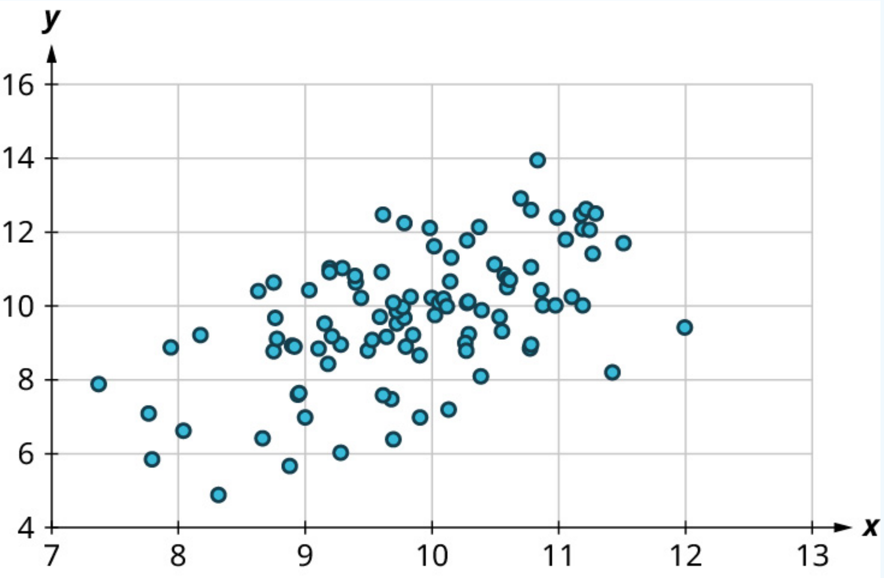
Scatter Plots, Correlation, and Regression Lines

For each of the following scatter plots, decide whether linear regression would be appropriate. If it is, classify the strength and direction of the relationship.

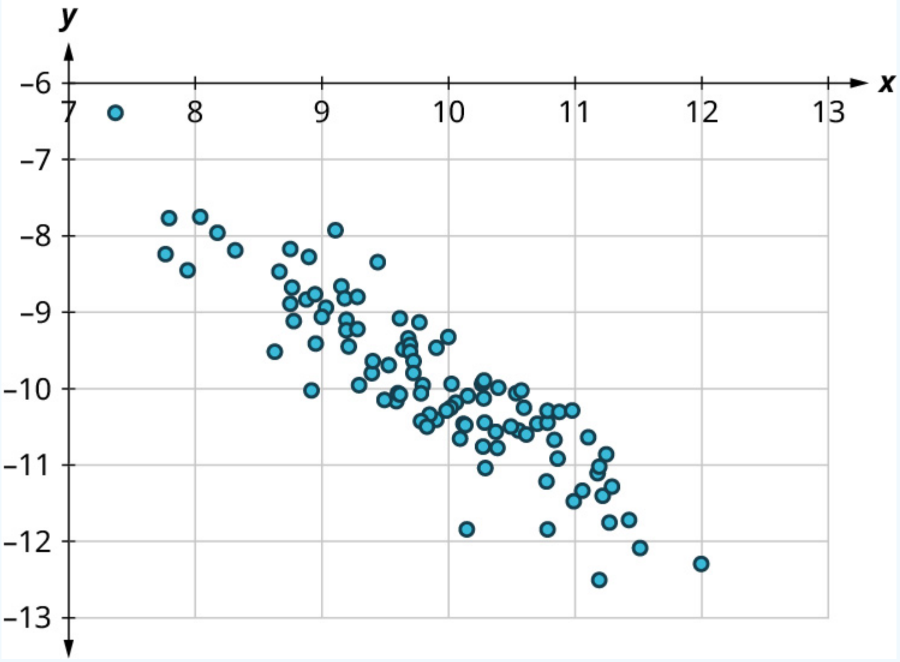
? Exercise $8.9.5.20$



? Exercise $8.9.5.21$



? Exercise $8.9.5.22$



Use the data in “NBA2019” to answer the following:

? Exercise 8.9.5.23

What is the correlation coefficient between the number of field goals attempted per game (FGA) and number of points scored per game (PTS)?

? Exercise 8.9.5.24

What is the equation of the regression line we would use to predict PTS from FGA?

? Exercise 8.9.5.25

Write a sentence to interpret the slope of the regression line.

? Exercise 8.9.5.26

Predict the number of points per game for a player who attempts 16 field goals per game.

? Exercise 8.9.5.27

Danilo Gallinari attempted 13 field goals per game, and averaged 19.8 points per game. Did he score more or fewer points per game than expected? By how much?

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8.9.6: Chapter Test

Decide whether randomization is being used in the selection of these samples. If it is, identify the type of random sample (simple, systematic, cluster, or stratified).

Exercise 8.9.6.1

A vinyl record dealer is trying to price a large collection she's thinking of buying. She looks at every tenth record on the shelf and notes the value.

Exercise 8.9.6.2

A court clerk is charged with identifying one hundred people for a jury pool for upcoming legal hearings. He has an alphabetized list of registered voters in his jurisdiction, so he uses a random number generator to pick one hundred names from the list.

Exercise 8.9.6.3

A state agricultural officer is worried about the spread of a parasitic disease among cattle. He chooses 30 cattle farms at random and tests each cow on those 30 farms for the disease.

Exercise 8.9.6.4

A sample of donations to a blood bank contained these blood types:

A	O	A	O	A	A	AB	O
O	A	AB	A	O	O	A	A
O	O	B	O	B	O	O	O
B	A	O	A	O	O	A	A

Create a frequency distribution for these data.

Exercise 8.9.6.5

Create a bar chart without technology to visualize the data in question 4.

Exercise 8.9.6.6

Use the data in "MBB2019" to create a histogram of the number of total rebounds by each team (TRB).

Exercise 8.9.6.7

The table below shows the number of players on the rosters of each English Premier League team. Find the mode, median, and mean without technology.

Table 8.14 (source: https://fbref.com/en/comps/9/Premier...quads_standard)

Team	Active Players	Team	Active Players
Manchester United F.C.	27	Newcastle United F.C.	28
Manchester City F.C.	24	Aston Villa F.C.	24
Chelsea F.C.	25	Fulham F.C.	27

Team	Active Players	Team	Active Players
Arsenal F.C.	28	Southampton F.C.	24
Tottenham Hotspur F.C.	27	Wolverhampton Wanderers F.C.	22
Liverpool F.C.	30	Brighton & Hove Albion	31
Crystal Palace	29	Burnley F.C.	22
Everton F.C.	23	West Bromwich Albion F.C.	26
Leicester City	25	Leeds United F.C.	25
West Ham United F.C.	24	Sheffield United F.C.	27

? Exercise 8.9.6.8

Using the data in “MBB2019”, find the mean, median, and mode of the total number of rebounds made by each team (TRB).

? Exercise 8.9.6.9

Using the data in “MBB2019”, find the range and standard deviation of the total number of rebounds made by each team (TRB).

Use the data in “MBB2019” to answer the following:

? Exercise 8.9.6.10

What number of total rebounds (TRB) is at the 3rd quartile?

? Exercise 8.9.6.11

At what percentile is the University of Evansville, which had 1095 total rebounds?

Answer the following about data that are distributed normally with mean 50 and standard deviation 5:

? Exercise 8.9.6.12

What proportion of the data are between 35 and 65?

? Exercise 8.9.6.13

What’s the standardized score of the data value 37?

? Exercise 8.9.6.14

At what percentile would the data value 58 fall?

? Exercise 8.9.6.15

What data value would be at the 40th percentile?

Use the data in “MBB2019” to answer the following:

? Exercise 8.9.6.16

What is the standardized score for the number of total rebounds (TRB) recorded by Mississippi State University (TRB = 1215)?

? Exercise 8.9.6.17

At what percentile would we estimate Mississippi State’s TRB value fall, using the normal distribution?

? Exercise 8.9.6.18

At what percentile does Mississippi State’s TRB value actually fall?

Use the data in “NBA2019” to answer the following:

? Exercise 8.9.6.19

What is the correlation coefficient between the number of two-point field goals made per game (2P) and number of points scored per game (PTS)?

? Exercise 8.9.6.20

What is the equation of the regression line we would use to predict PTS from 2P?

? Exercise 8.9.6.21

Write a sentence to interpret the slope of the regression line.

? Exercise 8.9.6.22

Predict the number of points per game for a player who makes 6 two-point field goals per game.

? Exercise 8.9.6.23

J.J. Redick made 2.8 field goals per game, and averaged 18.1 points per game. Did he score more or fewer points per game than expected? By how much?

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CHAPTER OVERVIEW

9: Metric Measurement

9.0: Introduction
9.1: The Metric System
  9.1.0: Exercises
9.2: Measuring Area
  9.2.0: Exercises
9.3: Measuring Volume
  9.3.0: Exercises
9.4: Measuring Weight
  9.4.0: Exercises
9.5: Measuring Temperature
  9.5.0: Exercises
9.6: Chapter Summary
  9.6.0: Key Terms
  9.6.1: Key Concepts
  9.6.2: Videos
  9.6.3: Formula Review
  9.6.4: Projects
  9.6.5: Chapter Review
  9.6.6: Chapter Test

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9.0: Introduction

You are planning a road trip from your home state to the sunny beaches of Mexico and need to prepare a budget. While in the United States, gasoline is sold in gallons and distances are measured in miles, but in almost any other country you will find that gasoline is sold in liters and distance is measured in kilometers.

Joutsenlampi 0,5
Tornikahvila 1,5
km

Figure $9.0.1$: A road sign in Finland, a country that uses the metric system. (credit: modification of work by Anna Järvenpää/Flickr, Public Domain)

Whether you’re traveling, baking, watching an international sporting event, working with machine tools, or using scientific equipment, it's important to understand the metric system, or the International System of Units (SI). The metric system is a decimal measuring system that uses meters, liters, and grams to quantify length, capacity, and mass. It is used in all but three countries in the world, including the United States.

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9.1: The Metric System

Learning Objectives

1. Identify units of measurement in the metric system and their uses.
2. Order the six common prefixes of the metric system.
3. Convert between like unit values.

Even if you don’t travel outside of the United States, many specialty grocery stores utilize the metric system. For example, if you want to make authentic tamales you might visit the nearest Hispanic grocery store. While shopping, you discover that there are two brands of masa for the same price, but one bag is marked 1,200 g and the other 1 kg. Which one is the better deal? Understanding the metric system allows you to understand that 1,200 grams is equivalent to 1.2 kg, so the 1,200 g bag is the better deal.

Figure 9.1.1: A scale that measures weight in both metric and customary units. (credit: “Weighing the homemade cheddar” by Ruth Hartnup/Flickr, CC BY 2.0)

Units of Measurement in the Metric System

Units of measurement provide common standards so that regardless of where or when an object or substance is measured, the results are consistent. When measuring distance, the units of measure might be feet, meters, or miles. Weight might be expressed in terms of pounds or grams. Volume or capacity might be measured in gallons, or liters. Understanding how metric units of measure relate to each other is essential to understanding the metric system:

- The metric unit for distance is the meter (m). A person’s height might be written as 1.8 meters (1.8 m). A meter slightly longer than a yard (3 feet), while a centimeter is slightly less than half an inch.
- The metric unit for area is the square meter (m^2). The area of a professional soccer field is 7,140 square meters (7,140 m^2).
- The metric base unit for volume is the cubic meter (m^3). However, the liter (L), which is a metric unit of capacity, is used to describe the volume of liquids. Soda is often sold in 2-liter (2 L) bottles.
- The gram (g) is a metric unit of mass but is commonly used to express weight. The weight of a paper clip is approximately 1 gram (1 g).
- The metric unit for temperature is degrees Celsius (°C). The temperature on a warm summer day might be 24 °C.

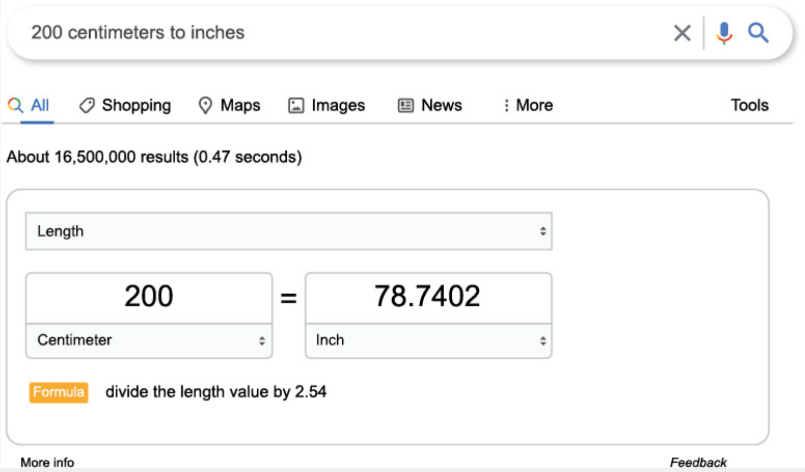
While the U.S. Customary System of Measurement uses ounces and pounds to distinguish between weight units of different sizes, in the metric system a *base unit* is combined with a prefix, such as *kilo*– in kilogram, to identify the relationship between smaller or larger units.

Checkpoint

When using abbreviations to represent metric measures, always separate the quantity and the units with a space, with no spaces between the letters or symbols in the units. For example, 7 millimeters is written as 7 mm, not 7mm.

Tech Check

It is important to be able to convert between the U.S. Customary System of Measurement and the metric system. However, in this chapter we'll focus on converting units within the metric system. Why? Typing "200 centimeters in inches" into any browser search bar will instantly convert those measures for you (Figure 9.1.2). You'll have an opportunity in the Projects to work between measurement systems.



200 centimeters to inches

About 16,500,000 results (0.47 seconds)

Length

200 = 78.7402

Centimeter

Inch

Formula divide the length value by 2.54

Figure 9.1.2 : (credit: Screenshot/Google)

Let's be honest. Most of us use computers or smartphones to perform many of the calculations and conversions we were taught in math class. But there is value in understanding the metric system since it exists all around us, and most importantly, knowing how the different metric units relate to each other allows you to compare prices, find the right tool in a workshop, or acclimate when in another country. No matter the circumstance, you cannot avoid the metric system.

✓ Example 9.1.1: Determining the Correct Base Unit

Which base unit would be used to express the following?

1. amount of water in a swimming pool
2. length of an electrical wire
3. weight of one serving of peanuts

Answer

1. Liquid volume is generally expressed in units of liters (L).
2. Length is measured in units of meters (m).
3. Weight is commonly expressed in units of grams (g).

Your Turn 9.1.1:

Determine the correct base measurement for each of the following.

- weight of a laptop
- width of a table
- amount of soda in a pitcher

While there are other base units in the metric system, our discussions in this chapter will be limited to units used to express length, area, volume, weight, and temperature.

Metric Prefixes

Unlike the U.S. Customary System of Measurement in which 12 inches is equal to 1 foot and 3 feet are equal to 1 yard, the metric system is structured so that the units within the system get larger or smaller by a power of 10. For example, a centimeter is 10^{-2}, or 100 times smaller than a meter, while the kilometer is 10^3, or 1,000 times larger than a meter.

The metric system combines base units and unit prefixes reasonable to the size of a measured object or substance. The most used prefixes are shown in Table $9.1.1$. An easy way to remember the order of the prefixes, from largest to smallest, is the mnemonic *King Henry Died From Drinking Chocolate Milk.*

Table $9.1.1$ Metric Prefixes

Prefix	kilo–	hecto–	deca–	base unit	deci–	cent–	milli–
Abbreviation	k	h	da		d	c	m
Magnitude	10^3	10^2	10^1	10^0 or 1	10^{-1}	10^{-2}	10^{-3}

Example $9.1.2$ Ordering the Magnitude of Units

Order the measures from smallest unit to largest unit: centimeter, millimeter, decimeter

Answer

Looking at the metric prefixes, we can see that the prefix order from smallest unit to largest unit is milli-, centi-, deci-, so the order of the units from smallest to largest is millimeter, centimeter, decimeter.

Your Turn $9.1.2$

Order the measures from largest unit to smallest unit.

hectogram, decagram, kilogram

Example $9.1.3$: Determining Reasonable Values for Length

What is a reasonable value for the length of a person’s thumb: 5 meters, 5 centimeters, or 5 millimeters?

Answer

Given that a meter is slightly longer than a yard, 5 meters is not a reasonable value for the length of a person’s thumb. Since a millimeter is 10 times smaller than a centimeter, which is approximately $\frac{1}{2}$ inch, 5 millimeters is not a reasonable estimate for the length of a person’s thumb. The correct answer is 5 centimeters.

Your Turn $9.1.3$

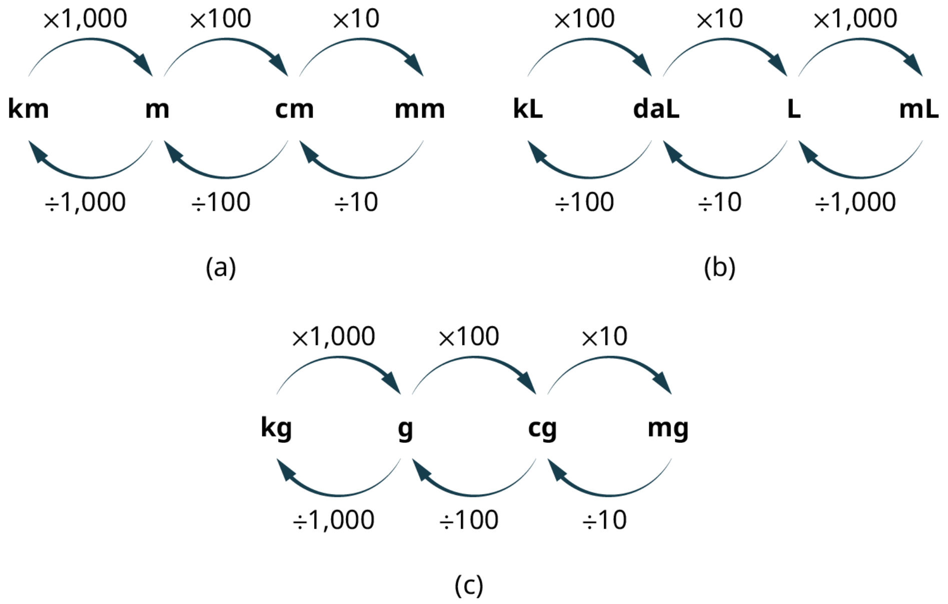
What is a reasonable estimate for the length of a hallway: 2.5 kilometers, 2.5 meters, or 2.5 centimeters?

Converting Metric Units of Measure

Imagine you order a textbook online and the shipping detail indicates the weight of the book is 1 kg. By attaching the letter “k” to the base unit of gram (g), the unit used to express the measure is 10^3or 1,000 times greater than a gram. One kilogram is equivalent to 1,000 grams.

The tip of a highlighter measures approximately 1 cm. The letter “c” attached to the base unit of meter (m) means the unit used to express the measure is 10^{-}2or110010−2or1100 of a meter. One meter is equivalent to 100 centimeters.

A **conversion factor** is used to convert from smaller metric units to bigger metric units and vice versa. It is a number that when used with multiplication or division converts from one metric unit to another, both having the same base unit. In the metric system, these conversion factors are directly related to the powers of 10. The most common used conversion factors are shown in Figure 9.1.3.



(a) (b)

(c)

Figure 9.1.3 : Common Metric Conversion Factors for (a) Meters, (b) Liters, and (c) Grams

✓ Example \(\PageIndex{4}\): Converting Metric Distances Using Multiplication

The firehouse is 13.45 km from the library. How many meters is it from the firehouse to the library?

Answer

When converting from a larger unit to a smaller unit, use multiplication. The conversion factor from kilometer to the base unit of meters is 1,000.

$$13.45 \times 1{,}000 = 13{,}450$$

So, the firehouse is 13,450 meters away from the firehouse.

Your Turn 9.1.4

The record for the men’s high jump is 2.45 m. What is the record when expressed in centimeters?

✓ Example 9.1.5: Converting Metric Capacity Using Division

How many liters is 3,565 milliliters?

Answer

When converting from a smaller unit to a larger unit, use division. The conversion factor from milliliter to the base unit of liters is 1,000.

$$3{,}565 \div 1{,}000 = 3.565$$

So, 3,565 milliliters is 3.565 liters.

Your Turn 9.1.5

A bottle of cleaning solution measures 7.6 liters. How many decaliters is that?

✓ Example 9.1.6: Converting Metric Units of Mass to Solve Problems

Caroline and Aiyana are working on a chemistry experiment together and must perform calculations using measurements taken during the experiment. Due to miscommunication, Caroline took measurements in centigrams and Aiyana used milligrams. Convert Caroline's measurement of 125 centigrams to milligrams.

Answer

When converting from a larger unit to a smaller unit, use multiplication. The conversion factor from centigrams to the milligrams is 10.

$$125 \times 10 = 1{,}250$$

So, the 125 centigrams are 1,250 milligrams.

Your Turn 9.1.6

Convert Aiyana's measurement of 1,457 mg to centigrams.

✓ Example 9.1.7: Converting Metric Units of Volume to Solve Problems

A bottle contains 500 mL of juice. If the juice is packaged in 24-bottle cases, how many liters of juice does the case contain?

Answer

Step 1: Multiply the amount of juice in each bottle by the number of bottles.

$$500 \text{ mL} \times 24 = 12{,}000 \text{ mL}$$

Step 2: Divide by 1,000 to convert from milliliters to liters.

$$\frac{12{,}000 \text{ mL}}{1{,}000} = 12 \text{ L}$$

So, there are 12 liters of juice in each case.

Your Turn 9.1.7

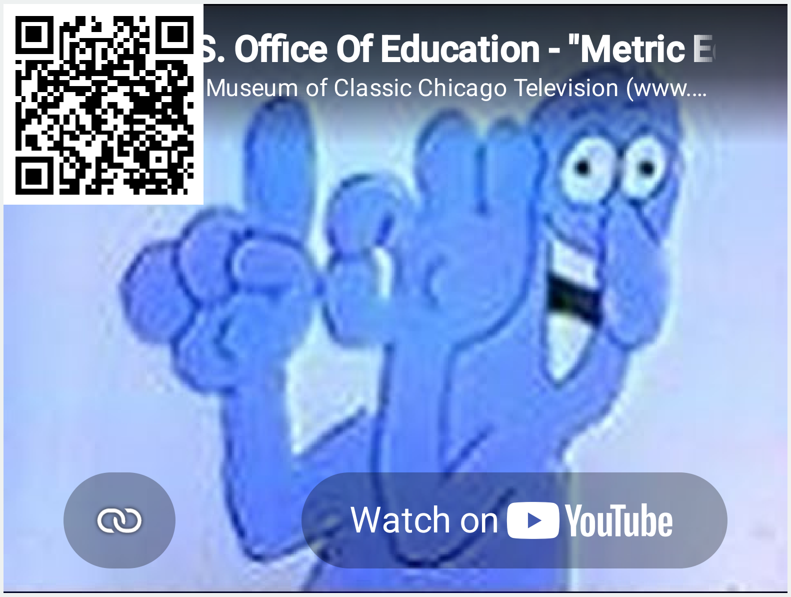
A hospital orders 250 doses of liquid amoxicillin. Each dose is 5 mL. How many liters of amoxicillin did the hospital order?

People in Mathematics: Valerie Antoine

In the 1970s, people were told that they must learn the metric system because the United States was soon going to convert to using metric measurements. Children and young adults probably watched educational cartoons about the metric system on Saturday mornings.

In 1975, President Gerald Ford signed the Metric Conversion Act and created a board of 17 people commissioned to coordinate the voluntary switch to the metric system in the United States. Among those 17 people was Valerie Antoine, an engineer who made it her life's work to push for this change. Despite President Ronald Reagan dissolving the board in 1982, effectively killing the move to the metric system at the time, Antoine continued the movement out of her own home as the executive director of the U.S. Metric Association. Reagan's decision followed intensive lobbying by American businesses whose factories used machinery designed to use customary measurements by workers trained in customary measurements. There was also intense public pressure from American citizens who didn't want to go through the time consuming and expensive process of changing the country's entire infrastructure. Fueled by a Congressional mandate in 1992 that required all federal agencies make the switch to the metric system, Antoine never gave up hope that the metric system would trickle down from the government and find its way into American schools, homes, and everyday life.

Video


S. Office Of Education - "Metric E
Museum of Classic Chicago Television (www....
Watch on YouTube

U.S. Office of Education: Metric Education

Example $9.1.8$: Converting Grams to Solve Problems

The nutrition label on a jar of spaghetti sauce indicates that one serving contains 410 mg of sodium. You have poured two servings over your favorite pasta before recalling your doctor’s advice about keeping your sodium consumption below 1 g per meal. Have you followed your doctor’s recommendation?

Answer

Step 1: Multiply the number of servings by the amount of sodium in each serving.

$$410 \text{ mg} \times 2 = 820 \text{ mg}$$

Step 2: Divide by 1,000 to convert from milligrams to grams.

$$\frac{820 \text{ mg}}{1{,}000} = 0.82 \text{ g}$$

You have followed doctor’s recommendation because 0.82 g is less than 1 gram.

Your Turn $9.1.8$

The FDA recommends that you consume less than 0.5 g of caffeine daily. A cup of coffee contains 95 mg of caffeine and a can of soda contains 54 mg. If you drink 2 cups of coffee and 3 cans of soda, have you kept your day’s caffeine consumption to the FDA recommendation? Explain.

Example $9.1.9$: Comparing Different Units

A student carefully measured 0.52 cg of copper for a science experiment, but their lab partner said they need 6 mg of copper total. How many more centigrams of copper does the student need to add?

Answer

Step 1: Convert these two measurements into a common unit. Since the question asks for the number of centigrams, convert 6 mg to centigrams, which is 0.6 cg.

Step 2: Find the difference by subtracting $0.6-0.52$ which is 0.08 cg. This means the student must add another 0.08 cg of copper.

Your Turn 9.1.9

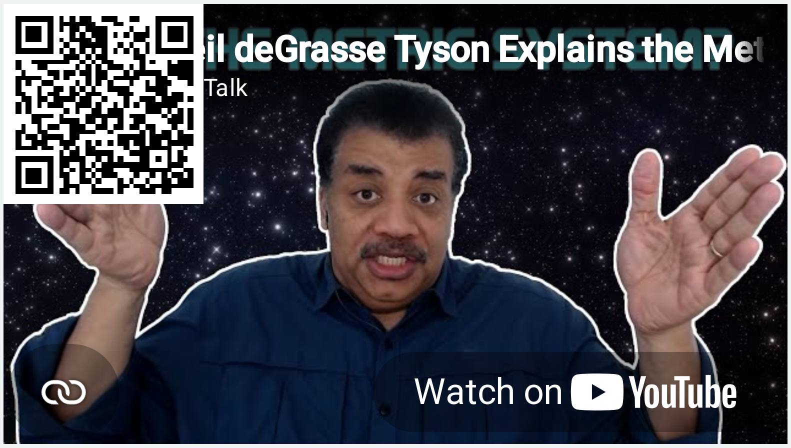
Kyrie boasted he jumped out of an airplane at an altitude of 3,810 meters on his latest skydive trip. His friend said they beat Kyrie because their jump was at an altitude of 3.2 km. Whose skydive was at a greater altitude?

Who Knew?: The United States and the Metric System

Did you know that the metric system pervades daily life in the United States already? While Americans still may purchase gallons of milk and measure house sizes in square feet, there are many instances of the metric system. Photographers buy 35 mm film and use 50 mm lenses. When you have a headache, you might take 600 mg of ibuprofen. And if you are eating a low-carb diet you probably restrict your carb intake to fewer than 20 g of carbs daily. Did you know even the dollar is metric? In the video, Neil DeGrasse Tyson and comedian co-host Chuck Nice provide an amusing perspective on the metric system.

The International System of Units (SI) is the current international standard metric system and is the most widely used system around the world. In most English-speaking countries SI units such as meter, liter, and metric ton are spelled metre, litre, and tonne.

Video



Neil deGrasse Tyson Explains the Metric System

WORK IT OUT: Get to Know the Metric System

Just how much is the metric system a part of your life now? Probably more than you think. For the next 24 hours, take notice as you move through your daily activities. When you are shopping, are the package sizes provided in metric units? Change the weather app on your phone to display the temperature in degrees Celsius. Are you able to tell what kind of day it will be now? While the United States is not officially using the metric system, you will still find the metric system all around you.

Exercise 9.1.1

Which metric base unit would be used to measure the height of a door?

? Exercise 9.1.2

Which metric base unit would be used to measure your weight?

? Exercise 9.1.3

Which is greater: 12 hectoliters or 12 centiliters?

? Exercise 9.1.4

Convert 1,520 cm to meters (m).

? Exercise 9.1.5

Convert 1.34 km to decameters (dam).

? Exercise 9.1.6

Convert 12,700 cg to hectograms (hg).

? Exercise 9.1.7

Convert 750 km to millimeters (mm).

? Exercise 9.1.8

Which is the larger measurement: 0.04 dam or 40 cm?

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9.1.0: Exercises

For the following exercises, determine the base unit of the metric system described. Choose from *liter*, *gram*, or *meter*.

? Exercise 9.1.0.1

amount of soda in a bottle

? Exercise 9.1.0.2

weight of a book you are mailing

? Exercise 9.1.0.3

amount of gasoline needed to fill a car’s tank

? Exercise 9.1.0.4

height of a computer desk

? Exercise 9.1.0.5

weight of a dog at the veterinarian’s office

? Exercise 9.1.0.6

dimensions of the newest HD TV

? Exercise 9.1.0.7

distance a student athlete ran on the treadmill during their latest workout

? Exercise 9.1.0.8

amount of water to add to bleach for mopping floors

? Exercise 9.1.0.9

Write the order of the metric prefixes from greatest to least.

For the following exercises, choose the smaller of the two units.

? Exercise 9.1.0.10

decagrams or decigrams

? Exercise 9.1.0.11

centimeters or millimeters

? Exercise 9.1.0.12

liters or kiloliters

? Exercise 9.1.0.13

decigrams or centigrams

? Exercise 9.1.0.14

decameters or hectometers

? Exercise 9.1.0.15

milliliters or deciliters

? Exercise 9.1.0.16

Convert 158 hectometers to meters (m).

? Exercise 9.1.0.17

Convert 12.3 cg to grams (g).

? Exercise 9.1.0.18

Convert 160 dam to kilometer (km).

? Exercise 9.1.0.19

You purchase 10 kg of candy. You divide the candy into 25 bags. How many grams of candy are in each bag?

? Exercise 9.1.0.20

An outdoor track is 400 m long. If you run 10 laps around the track, how many kilometers have you run?

? Exercise 9.1.0.21

An aspirin tablet is 650 mg. If you take 2 aspirin twice in one day, how many grams of aspirin have you taken?

? Exercise 9.1.0.22

A juice box contains 450 mL of juice. If there are 6 juice boxes in a package, how many liters of juice are in the package?

? Exercise 9.1.0.23

Carlos consumed 4 cans of his favorite energy drink. If each can contains 111 mg of caffeine, how many grams of caffeine did he consume?

? Exercise 9.1.0.24

Celeste ran a total distance of 72,548 m in 1 week while training for a 5K fun run. How many kilometers did she run?

? Exercise 9.1.0.25

During week one of their diet, Dakota consumed 413 g of carbs. After speaking to their doctor, they only consumed 210 g of carbs the second week. How many fewer milligrams of carbs did they consume the second week?

? Exercise 9.1.0.26

Kaylea gained 2.3 kg of muscle weight in 9 months of working out. Cho gained 250 decagrams of muscle during that same time. Who gained more muscle weight? How many grams more?

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9.2: Measuring Area

Learning Objectives

1. Identify reasonable values for area applications.
2. Convert units of measures of area.
3. Solve application problems involving area.

Area is the size of a surface. It could be a piece of land, a rug, a wall, or any other two-dimensional surface with attributes that can be measured in the metric unit for distance-meters. Determining the area of a surface is important to many everyday activities. For example, when purchasing paint, you'll need to know how many square units of surface area need to be painted to determine how much paint to buy.

Figure 9.2.1: A painter uses an extension roller to paint a wall. (credit: "Paint Rollers are effective" by WILLPOWER STUDIOS/Flickr, CC BY 2.0)

Square units indicate that two measures in the same units have been multiplied together. For example, to find the area of a rectangle, multiply the length units and the width units to determine the area in square units.

$$1\ \text{cm} \times 1\ \text{cm} = 1\ \text{cm}^2$$

Note that to accurately calculate area, each of the measures being multiplied must be of the same units. For example, to find an area in square centimeters, both length measures (length and width) must be in centimeters.

FORMULA

The formula used to determine area depends on the shape of that surface. Here we will limit our discussions to the area of rectangular-shaped objects like the one in Figure 9.6. Given this limitation, the basic formula for area is:

$$\text{Area} = \text{length}\,(l) \times \text{width}\,(w)$$

or

$$A =$$

l×wor A=l×w

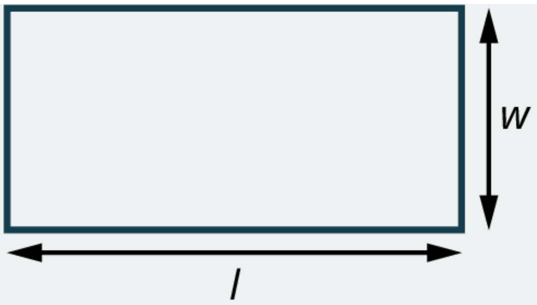


Figure $9.2.2$: Rectangle with Length (l) and Width (w) Labeled

Reasonable Values for Area

Because area is determined by multiplying two lengths, the magnitude of difference between different square units is exponential. In other words, while a meter is 100 times greater in length than a centimeter, a square meter (m^2) is 100×100, or 10, 000 times greater in area than a square centimeter (cm^2). The relationships between benchmark metric area units are shown in the following table.

Units	Relationship	Conversion Rate
km^2 to m^2	$\mathrm{km} \times \mathrm{km} = \mathrm{km}^2$ $1\ \mathrm{km} = 1,000\ \mathrm{m}$ $1,000\ \mathrm{m} \times 1,000\ \mathrm{m} = 1,000,000\ \mathrm{m}^2$	$1\ \mathrm{km}^2 = 1,000,000\ \mathrm{m}^2$
m^2 to cm^2	$\mathrm{m} \times \mathrm{m} = \mathrm{m}^2$ $1\ \mathrm{m} = 100\ \mathrm{cm}$ $100\ \mathrm{cm} \times 100\ \mathrm{cm} = 10,000\ \mathrm{cm}^2$	$1\ \mathrm{m}^2 = 10,000\ \mathrm{cm}^2$
cm^2 to mm^2	$\mathrm{cm} \times \mathrm{cm} = \mathrm{cm}^2$ $1\ \mathrm{cm} = 10\ \mathrm{mm}$ $10\ \mathrm{mm} \times 10\ \mathrm{mm} = 100\ \mathrm{mm}^2$	$1\ \mathrm{cm}^2 = 100\ \mathrm{mm}^2$

An essential understanding of metric area is to identify reasonable values for area. When testing for reasonableness you should assess both the unit and the unit value. Only by examining both can you determine whether the given area is reasonable for the situation.

✓ Example $9.2.1$: Determining Reasonable Units for Area

Which unit of measure is most reasonable to describe the area of a sheet of paper: km^2, cm^2, or mm^2 ?

In the U.S. Customary System of Measurement, the length and width of paper is usually measured in inches. In the metric system centimeters are used for measures usually expressed in inches. Thus, the most reasonable unit of measure to describe the area of a sheet of paper is square centimeters. Square kilometers is too large a unit and square millimeters is too small a unit.

Your Turn $9.2.1$

Which unit of measure is most reasonable to describe the area of a forest: km^2, cm^2, or mm^2?

✓ Example $9.2.2$: Determining Reasonable Values for Area

You want to paint your bedroom walls. Which represents a reasonable value for the area of the walls: 100 cm^2, 100 m^2, or 100 km^2?

Answer

An area of 100 cm^2 is equivalent to a surface of 10cm×10cm, which is much too small for the walls of a bedroom. An area of 100 km^2 is equivalent to a surface of 10km×10km, which is much too large for the walls of a bedroom. So, a reasonable value for the area of the walls is 100 m^2.

Your Turn 9.2.2

Which represents a reasonable value for the area of the top of a kitchen table: 1,800 mm^2, 1,800 cm^2, or $1{,}800 \mathrm{~m}^2$?

Example 9.2.3: Explaining Reasonable Values for Area

A landscaper is hired to resod a school's football field. After measuring the length and width of the field they determine that the area of the football field is 5,350 km^2. Does their calculation make sense? Explain your answer.

Answer

No, kilometers are used to determine longer distances, such as the distance between two points when driving. A football field is less than 1 kilometer long, so a more reasonable unit of value would be m^2. An area of 5,350 km^2 can be calculated using the dimensions 53.5 by 100, which are reasonable dimensions for the length and width of a football field. So, a more reasonable value for the area of the football field is 5,350 m^2.

Your Turn 9.2.3

A crafter uses four letter-size sheets of paper to create a paper mosaic in the shape of a square. They decide they want to frame the paper mosaic, so they measure and determine that the area of the paper mosaic is $2{,}412.89 \mathrm{~cm}^2$. Does their calculation make sense? Explain your answer.

Converting Units of Measures for Area

Just like converting units of measure for distance, you can convert units of measure for area. However, the conversion factor, or the number used to multiply or divide to convert from one area unit to another, is not the same as the conversion factor for metric distance units. Recall that the conversion factor for area is exponentially relative to the conversion factor for distance. The most frequently used conversion factors are shown in Figure 9.7.

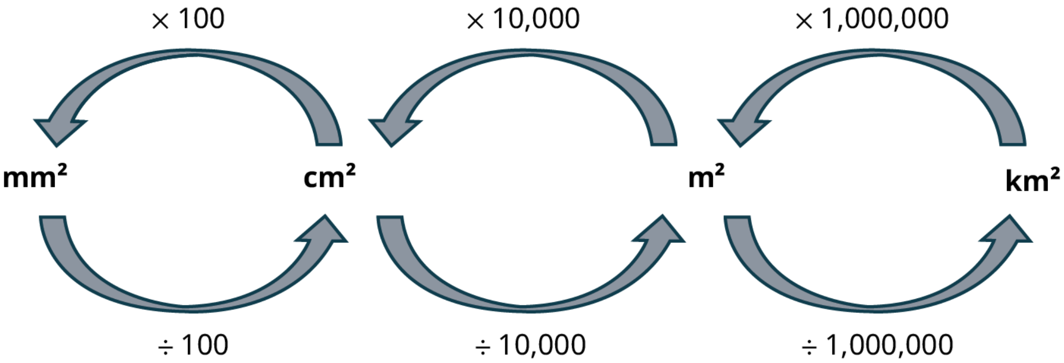
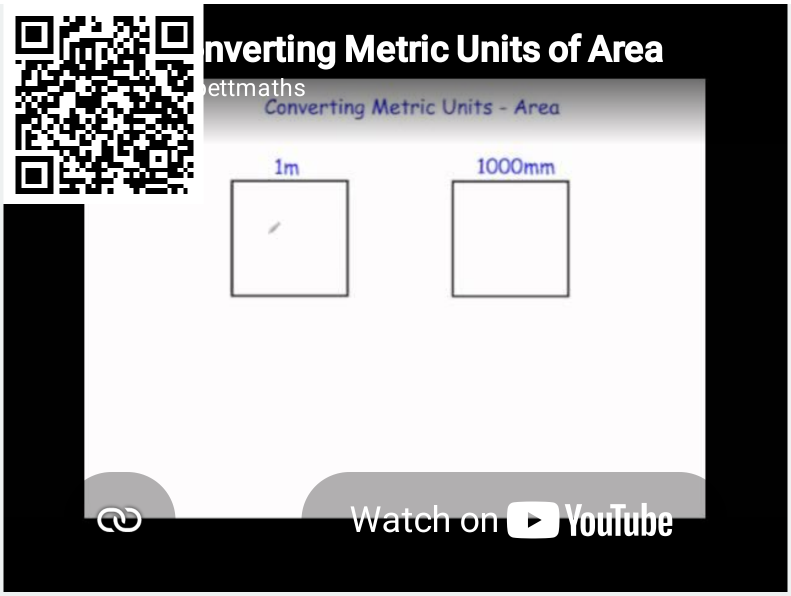


Figure 9.2.2: Common Conversion Factors for Metric Area Units

Video



Converting Metric Units of Area

✓ Example 9.2.4: Converting Units of Measure for Area Using Division

A plot of land has an area of 237,500,000 m^2. What is the area in square kilometers?

Answer

Use division to convert from a smaller metric area unit to a larger metric area unit. To convert from m^2 to km^2, divide the value of the area by 1,000,000.

$$\frac{237{,}500{,}000}{1{,}000{,}000} = 237.5$$

The plot of land has an area of 237.5 km^2.

Your Turn 9.2.4

A roll of butcher paper has an area of 1,532,900 cm^2. What is the area of the butcher paper in square meters?

✓ Example 9.2.5: Converting Units of Measure for Area Using Multiplication

A plot of land has an area of 0.004046 km^2. What is the area of the land in square meters?

Answer

Use multiplication to convert from a larger metric area unit to a smaller metric area unit. To convert from km^2 to m^2, multiply the value of the area by 1,000,000.

$$0.004046 \times 1{,}000{,}000 = 4{,}046$$

The plot of land has an area of 4,046 m^2.

Your Turn 9.2.5

A bolt of fabric has an area of 136.5 m^2. What is the area of the bolt of fabric in square centimeters?

✓ Example 9.2.6: Determining Area by Converting Units of Measure for Length First

A computer chip measures 10 mm by 15 mm. How many square centimeters is the computer chip?

Answer

Step 1: Convert the measures of the computer chip into centimeters

$$\begin{gathered} 10 \text{ mm} = 1 \text{ cm} \\ 15 \text{ mm} = 1.5 \text{ cm} \end{gathered}$$

Step 2: Use the area formula to determine the area of the chip.

$$1 \times 1.5 = 1.5$$

The computer chip has an area of 1.5 cm^2.

Your Turn 9.2.6

A piece of fabric measures 100 cm by 106 cm. What is the area of the fabric in square meters?

Solving Application Problems Involving Area

While it may seem that solving area problems is as simple as multiplying two numbers, often determining area requires more complex calculations. For example, when measuring the area of surfaces, you may need to account for portions of the surface that are not relevant to your calculation.

✓ Example 9.2.7: Solving for the Area of Complex Surfaces

One side of a commercial building is 12 meters long by 9 meters high. There is a rolling door on this side of the building that is 4 meters wide by 3 meters high. You want to refinish the side of the building, but not the door, with aluminum siding. How many square meters of aluminum siding are required to cover this side of the building?

Answer

Step 1: Determine the area of the side of the building.

$$12 \text{ m} \times 9 \text{ m} = 106 \text{ m}^2$$

Step 2: Determine the area of the door.

$$4 \text{ m} \times 3 \text{ m} = 12 \text{ m}^2$$

Step 3: Subtract the area of the door from the area of the side of the building.

$$106 \text{ m}^2 - 12 \text{ m}^2 = 92 \text{ m}^2$$

So, you need to purchase 92 m^2 of aluminum siding.

Your Turn 9.2.7

You want to cover a garden with topsoil. The garden is 5 meters by 8 meters. There is a path in the middle of the garden that is 8 meters long and 0.75 meters wide. What is the area of the garden you need to cover with topsoil?

When calculating area, you must ensure that both distance measurements are expressed in terms of the same distance units. Sometimes you must convert one measurement before using the area formula.

✓ Example $9.2.8$: Solving for Area with Distance Measurements of Different Units

A national park has a land area in the shape of a rectangle. The park measures 2.2 kilometers long by 1,250 meters wide. What is the area of the park in square kilometers?

Answer

Step 1: Use a conversion fraction to convert the information given in meters to kilometers.

$$1{,}250 \text{ m} \times \frac{1 \text{ km}}{1{,}000 \text{ m}} = 1.25 \text{ km}$$

Step 2: Multiply to find the area.

$$2.2 \text{ km} \times 1.25 \text{ km} = 2.75 \text{ km}^2$$

The park has an area of 2.75 km^2.

Your Turn $9.2.8$

An Olympic pool measures 50 meters by 2,500 centimeters. What is the surface area of the pool in square meters?

When calculating area, you may need to use multiple steps, such as converting units and subtracting areas that are not relevant.

✓ Example $9.2.9$: Solving for Area Using Multiple Steps

A kitchen floor has an area of 15 m^2. The floor in the kitchen pantry is 100 cm by 200 cm. You want to tile the kitchen and pantry floors using the same tile. How many square meters of tile do you need to buy?

Answer

Step 1: Determine the area of the pantry floor in square centimeters.

$$100 \text{ cm} \times 200 \text{ cm} = 20{,}000 \text{ cm}^2$$

Step 2: Divide the area in cm^2 by the conversion factor to determine the area in m^2 since the other measurement for the kitchen floor is in m^2.

$$\frac{20{,}000}{10{,}000} = 2$$

The area of the kitchen pantry floor is 2 m^2.

Step 3: Add the two areas of the pantry and the kitchen floors together.

$$15 \text{ m}^2 + 2 \text{ m}^2 = 17 \text{ m}^2$$

So, you need to buy 17 m^2 of tile.

Your Turn $9.2.9$

Your bedroom floor has an area of 25 m^2. The living room floor measures 600 cm by 750 cm. How many square meters of carpet do you need to buy to carpet the floors in both rooms?

Who Knew?: The Origin of the Metric System

The metric system is the official measurement system for every country in the world except the United States, Liberia, and Myanmar. But did you know it originated in France during the French Revolution in the late 18th century? At the time there were over 250,000 different units of weights and measures in use, often determined by local customs and economies. For example, land was often measured in days, referring to the amount of land a person could work in a day.

Video

Why the metric system matters - Matt
-Ed
WHY THE METRIC SYSTEM MATTERS
Watch on YouTube

Why the Metric System Matters

Check Your Understanding

Exercise 9.2.1

Determine the most reasonable value for area of a bedroom wall: 12 km^2, 12 m^2, 12 cm^2, or 12 mm^2

Exercise 9.2.2

Determine the most reasonable value for area of a city park: 1,200 km^2, 1,200 m^2, 1,200 cm^2, or 1,200 mm^2

Exercise 9.2.3

Determine the most reasonable value for area of a kitchen table: 2.5 km^2, 2.5 m^2, 2.5 cm^2, or 2.5 mm^2

Exercise 9.2.4

Convert the given area to the units shown.

20,000 cm^2 = __________ m^2

Exercise 9.2.5

Convert the given area to the units shown.

5.7 m^2 = __________ cm^2

Exercise 9.2.6

Convert the given area to the units shown.

217 cm^2 = __________ mm^2

Exercise 9.2.7

A wall measures 4 m by 2 m. A doorway in the wall measures 0.5 m by 1.6 m. What is the area of the wall not taken by the door in square meters?

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9.2.0: Exercises

For the following exercises, determine the most reasonable value for each area.

? Exercise 9.2.0.1

kitchen floor: 16 km^2, 16 m^2, 16 cm^2, or 16 mm^2

? Exercise 9.2.0.2

national Park: 1,000 km^2, 1,000 m^2, 1,000 cm^2, or 1,000 mm^2

? Exercise 9.2.0.3

classroom table: 5 km^2, 5 m^2, 5 cm^2, or 5 mm^2

? Exercise 9.2.0.4

window: 9,000 km^2, 9,000 m^2, 9,000 cm^2, or 9,000 mm^2

? Exercise 9.2.0.5

paper napkin: 10,000 km^2, 10,000 m^2, 10,000 cm^2, or 10,000 mm^2

? Exercise 9.2.0.6

parking lot: 45,000 km^2, 45,000 m^2, 45,000 cm^2, or 45,000 mm^2

For the following exercises, convert the given area to the units shown.

? Exercise 9.2.0.7

$700,000\,\text{cm}^2 = \_\_\_\_\_\_\_\ \text{m}^2$

? Exercise 9.2.0.8

$24\ \text{m}^2 = \_\_\_\_\_\,\text{cm}^2$

? Exercise 9.2.0.9

$985\ \text{cm}^2 = \_\_\_\_\_\,\text{mm}^2$

? Exercise 9.2.0.10

$4\ \text{km}^2 = \_\_\_\_\_\,\text{m}^2$

? Exercise 9.2.0.11

$0.00005\ \text{km}^2 = \_\_\_\_\_\,\text{cm}^2$

? Exercise 9.2.0.12

$68,500,000\text{ mm}^2 = \_\_\_\_\_\text{ m}^2$

? Exercise 9.2.0.13

$34.5\text{ m}^2 = \_\_\_\_\_\text{ cm}^2$

? Exercise 9.2.0.14

$7,800\text{cm}^2 = \_\_\_\_\_\text{ m}^2$

? Exercise 9.2.0.15

$34.5\text{ m}^2 = \_\_\_\_\_\text{ cm}^2$

? Exercise 9.2.0.16

$0.05\text{ km}^2 = \_\_\_\_\_\text{ m}^2$

? Exercise 9.2.0.17

$0.073\text{ m}^2 = \_\_\_\_\_\text{ mm}^2$

? Exercise 9.2.0.18

$27,750\text{ mm}^2 = \_\_\_\_\_\text{ m}^2$

For the following exercises, determine the area.

? Exercise 9.2.0.19

$l = 300$ cm, $w = 4$ m
$A = \_\_\_\_\_\text{ m}^2$

? Exercise 9.2.0.20

$l = 3,500$ mm, $w = 0.5$ m
$A = \_\_\_\_\_\text{ cm}^2$

? Exercise 9.2.0.21

$l = 5.5$ km, $w = 2,750$ m
$A = \_\_\_\_\_\text{ km}^2$

? Exercise 9.2.0.22

$l = 2,400$ cm, $w = 3.2$ m
$A = \_\_\_\_\_\text{ m}^2$

Exercise $9.2.0.23$

$l = 7$ cm, $w = 400$ mm
$A =$ _____ cm^2

Exercise $9.2.0.24$

$l = 2,300$ m, $w = 4$ km
$A =$ _____ km^2

Exercise $9.2.0.25$

$l = 4.2$ cm, $w = 320$ mm
$A =$ _____ cm^2

Exercise $9.2.0.26$

$l = 5,350$ mm, $w = 0.5$ m
$A =$ _____ cm^2

Exercise $9.2.0.27$

$l = 1.8$ km, $w = 2,300$ m
$A =$ _____ km^2

Exercise $9.2.0.28$

A notebook is 200 mm by 300 mm. A sticker on the notebook cover measures 2 cm by 2 cm. How many square centimeters of the notebook cover is still visible?

Exercise $9.2.0.29$

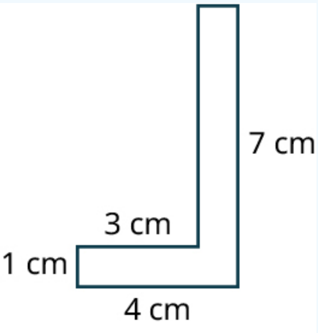
A bedroom wall is 4 m by 2.5 m. A window on the wall is 1 m by 2 m. How much wallpaper is needed to cover the wall?

Exercise $9.2.0.30$

A wall is 5 m by 2.5 m. A picture hangs on the wall that is 600 mm by 300 mm. How much of the wall, in m^2, is not covered by the picture?

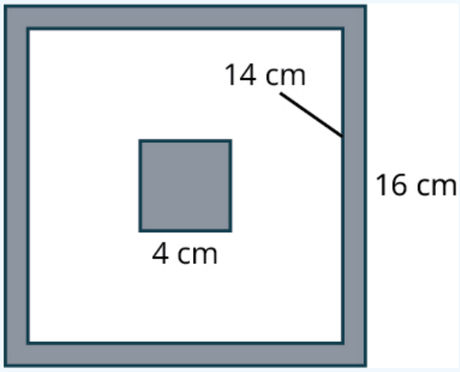
Exercise $9.2.0.31$

What is the area of the shape that is shown?



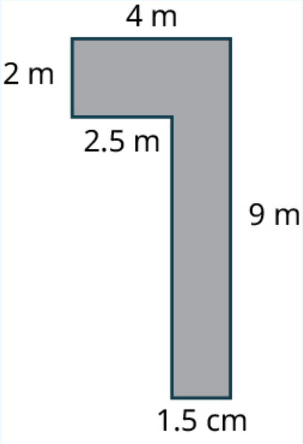
Exercise $9.2.0.32$

A quilter made a design using a small square, a medium square, and a large square. What is the area of the shaded parts of the design that is shown?



Exercise $9.2.0.33$

A landscaper makes a plan for a walkway in a backyard, as shown. How many square meters of patio brick does the landscaper need to cover the walkway?



Exercise $9.2.0.34$

A painter completed a portrait. The height of the portrait is 36 cm. The width is half as long as the height. What is the area of the portrait in square meters?

Exercise $9.2.0.35$

A room has an area of 137.5 m^2. The length of the room is 11 m. What is the width of the room?

Exercise $9.2.0.36$

A wall is 4.5 m long by 3 m high. A can of paint will cover an area of 10 m^2. How many cans of paint are needed if each wall needs 2 coats of paint?

? Exercise $9.2.0.37$

A soccer field is 110 meters long and 75 meters wide. If the cost of artificial turf is $30 per square meter, what is the cost of covering the soccer field with artificial turf?

? Exercise $9.2.0.38$

A window is 150 centimeters wide and 90 centimeters high. If three times the area of the window is needed for curtain material, how much curtain material is needed in square meters?

? Exercise $9.2.0.39$

A rectangular flower garden is 5.5 meters wide and 8.4 meters long. A path with a width of 1 meter is laid around the garden. What is the area of the path?

? Exercise $9.2.0.40$

A dining room is 8 meters wide by 6 meters long. Wood flooring costs $12.50 per square meter. How much will it costs to install wood flooring in the dining room?

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9.3: Measuring Volume

Learning Objectives

1. Identify reasonable values for volume applications.
2. Convert between like units of measures of volume.
3. Convert between different unit values.
4. Solve application problems involving volume.

Volume is a measure of the space contained within or occupied by three-dimensional objects. It could be a box, a pool, a storage unit, or any other three-dimensional object with attributes that can be measured in the metric unit for distance–meters. For example, when purchasing an SUV, you may want to compare how many cubic units of cargo the SUV can hold.

Figure $9.3.1$: Packing cartons sit on a loading dock ready to be filled. (credit: “boxing day” by Erich Ferdinand/Flickr, CC BY 2.0)

Cubic units indicate that three measures in the same units have been multiplied together. For example, to find the volume of a rectangular prism, you would multiply the length units by the width units and the height units to determine the volume in square units:

$$1 \text{ cm} \times 1 \text{ cm} \times 1 \text{ cm} = 1 \text{ cm}^3$$

Note that to accurately calculate volume, each of the measures being multiplied must be of the same units. For example, to find a volume in cubic centimeters, each of the measures must be in centimeters.

FORMULA

The formula used to determine volume depends on the shape of the three-dimensional object. Here we will limit our discussions to the area to rectangular prisms like the one in Figure $9.3.1$ Given this limitation, the basic formula for volume is:

$$\text{Volume} = \text{length}\,(l) \times \text{width}\,(w) \times \text{height}\,(h)$$

orV=lwhorV=lwh

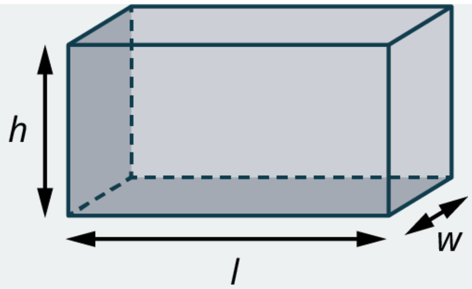

Figure $9.3.1$: Rectangular Prism with Height (h), Length (l), and Width (w) Labeled.

Reasonable Values for Volume

Because volume is determined by multiplying three lengths, the magnitude of difference between different cubic units is exponential. In other words, while a meter is 100 times greater in length than a centimeter, a cubic meter, m^3, is $100 \times 100 \times 100$, or $1{,}000{,}000$ times greater in area than a cubic centimeter, cm^3. This relationship between benchmark metric volume units is shown in the following table.

Units	Relationship	Conversion Rate
km^3 to m^3	$\text{km} \times \text{km} \times \text{km} = \text{km}^3$ $1 \text{ km} = 1,000 \text{ m}$ $1,000 \text{ m} \times 1,000 \text{ m} \times 1,000 \text{ m} = 1,000,000,000 \text{ m}^3$	1 km^3 = 1,000,000,000 m^3
m^3 to dm^3	$\text{m} \times \text{m} \times \text{m} = \text{m}^3$ $1 \text{ m} = 10\text{dm}$ $10\text{dm} \times 10\text{dm} \times 10\text{dm} = 1,000\text{dm}^3$	1 m^3 = 1,000 dm^3
dm^3 to cm^3	$\text{dm} \times \text{dm} \times \text{dm} = \text{dm}^3$ $1\text{dm} = 10 \text{ cm}$ $10 \text{ cm} \times 10 \text{ cm} \times 10 \text{ cm} = 1,000,000 \text{ cm}^3$	1 cm^3 = 1,000 dm^3
cm^3 to mm^3	$\text{cm} \times \text{cm} \times \text{cm} = \text{cm}^3$ $1 \text{ cm} = 10 \text{ mm}$ $10 \text{ mm} \times 10 \text{ mm} \times 10 \text{ mm} = 1,000 \text{ mm}^3$	1 cm^3 = 1,000 mm^3

To have an essential understanding of metric volume, you must be able to identify reasonable values for volume. When testing for reasonableness you should assess both the unit and the unit value. Only by examining both can you determine whether the given volume is reasonable for the situation.

✓ Example $9.3.1$: Determining Reasonable Values for Volume

A grandparent wants to send cookies to their grandchild away at college. Which represents a reasonable value for the volume of a box to ship the cookies:

- 3,375 km^3,
- 3,375 m^3, or
- 3,375 cm^3?

Answer

A volume of 3,375 km^3 is equivalent to a rectangular prism with dimensions of $15 \text{ km} \times 15 \text{ km} \times 15 \text{ km}$, which is far too large for a shipping box. An area of 3,375 m^3 is equivalent to a surface of $15 \text{ m} \times 15 \text{ m} \times 15 \text{ m}$, which is also too large. A reasonable value for the volume of the box is 3,375 cm^3.

Your Turn 9.3.1

Which represents a reasonable value for the volume of a storage area:
8 m^3, 8 cm^3, or 8 mm^3?

Example 9.3.2: Identifying Reasonable Values for Volume

A food manufacturer is prototyping new packaging for one of its most popular products. Which represents a reasonable value for the volume of the box:

- 2 dm^3,
- 2 cm^3, or
- 2 mm^3?

Answer

A decimeter is equal to 10 centimeters. A box with a volume of 2 dm^3 might have the dimensions $1\text{ dm}\times 1\text{ dm}\times 2$ dm1dm×1dm×2dm, or $10\text{ cm}\times 10\text{ cm}\times 20$, which is reasonable. A box with a volume of 2 cm^3 or 2 mm^3 would be too small.

Your Turn 9.3.2

Which represents a reasonable value for the volume of a fish tank:
40,000 mm^3, 40,000 cm^3, or 40,000 m^3?

Example 9.3.3: Explaining Reasonable Values for Volume

A farmer has a hay loft. They calculate the volume of the hayloft as 64 cm^3. Does the calculation make sense? Explain your answer.

Answer

No. Centimeters are used to determine smaller distances, such as the length of a pencil. A hayloft is more than 64 centimeters long, so a more reasonable unit of value would be m^2. A volume of 64 m^3 can be calculated using the dimensions 4 meters by 4 meters by 4 meters, which are reasonable dimensions for a hayloft. So, a more reasonable value for the volume of the hayloft is 64 m^3.

Your Turn 9.3.3

An artist creates a glass paperweight. They decide they want to box and ship the paperweight, so they measure and determine that the volume of the cubic box is 125,000 mm^3. Does their calculation make sense? Explain your answer.

Converting Like Units of Measures for Volume

Just like converting units of measure for distance, you can convert units of measure for volume. However, the conversion factor, the number used to multiply or divide to convert from one volume unit to another, is different from the conversion factor for metric distance units. Recall that the conversion factor for volume is exponentially relative to the conversion factor for distance. The most frequently used conversion factors are illustrated in Figure 9.10.

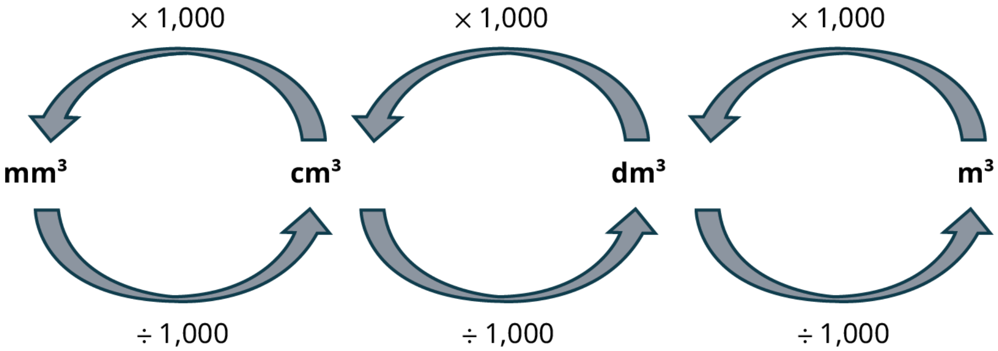
× 1,000
× 1,000
× 1,000
mm³
cm³
dm³
m³
÷ 1,000
÷ 1,000
÷ 1,000

Figure 9.3.3: Common Conversion Factors for Metric Volume Units

✓ Example 9.3.4: Converting Like Units of Measure for Volume Using Multiplication

A pencil case has a volume of 1,700 cm^3. What is the volume in cubic millimeters?

Answer

Use multiplication to convert from a larger metric volume unit to a smaller metric volume unit. To convert from cm^3 to mm^3, multiply the value of the volume by 1,000.

$$1{,}700 \times 1{,}000 = 1{,}700{,}000$$

The pencil case has a volume of 1,700,000 mm^3.

Your Turn 9.3.4

A jewelry box has a volume of 8 cm^3. What is the volume of the jewelry box in cubic millimeters?

Video

How To Convert Cubic Centimeters to
Organic Chemistry Tutor
m³ to m³
500,000 cm³
Watch on YouTube

How to Convert Cubic Centimeters to Cubic Meters

✓ Example 9.3.5: Converting Like Units of Measure for Volume Using Multi-Step Multiplication

A shipping container has a volume of 33.2 m^3. What is the volume in cubic centimeters?

Answer

Use multiplication to convert a larger metric volume unit to a smaller metric volume unit. To convert from m^3 to cm^3, first multiply the value of the volume by 1,000 to convert from m^3 to dm^3, and then multiply again by 1,000 to convert from dm^3 to cm^3.

$$33.2 \times 1{,}000 \times 1{,}000 = 33{,}200{,}000$$

The shipping container has a volume of 32,200,000 cm^3.

Your Turn 9.3.5

A gasoline storage tank has a volume of 37.854 m^3. What is the volume of the storage tank in cubic centimeters?

✓ Example 9.3.6: Converting Like Units of Measure for Volume Using Multi-Step Division

A holding tank at the local aquarium has a volume of 22,712,000,000 cm^3. What is the volume in cubic meters?

Answer

Figure 9.10 indicates that when converting from a smaller metric volume unit to a larger metric volume unit you divide using the given conversion factor. To convert from cm^3 to m^3, divide the value of the volume by 1,000 to first convert from cm^3 to dm^3, then divide again to convert from dm^3 to m^3.

$$\begin{aligned} \mathrm{cm}^3 \text{ to } \mathrm{dm}^3: \frac{22{,}712{,}000{,}000}{1{,}000} &= 22{,}712{,}000 \\ \mathrm{dm}^3 \text{ to } \mathrm{m}^3: \frac{22{,}712{,}000}{1{,}000} &= 22{,}712 \end{aligned}$$

The holding tank has a volume of 22,712 m^3.

Your Turn 9.3.6

A warehouse has a volume of 465,000,000 cm^3. What is the volume of the warehouse in cubic meters?

Understanding Other Metric Units of Volume

When was the last time you purchased a bottle of soda? Was the volume of the bottle expressed in cubic centimeters or liters? The liter (L) is a metric unit of capacity often used to express the volume of liquids. A liter is equal in volume to 1 cubic decimeter. A milliliter is equal in volume to 1 cubic centimeter. So, when a doctor orders 10 cc (cubic centimeters) of saline to be administered to a patient, they are referring to 10 mL of saline.

The most frequently used factors for converting from cubi eters to liters are listed in Table 9.2.

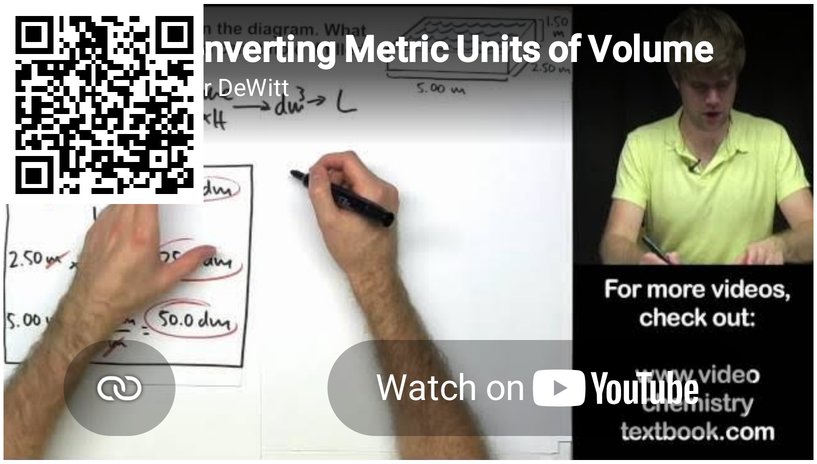
nverting Metric Units of Volume
r DeWitt
For more videos, check out:
www.video chemistry textbook.com
Watch on YouTube

Table 9.2 Relationships Between Metric Volume and Metric Capacity Units

m^3 to L	m^3 to mL
1 msupdm3=1L1dm3=1L	1 msupdm3=1,000mL1dm3=1,000mL

m^3 to L	m^3 to mL
1,000 msupcm3=1L1,000cm3=1L	1 msupcm3=1mL1cm3=1mL
1,000,000 msupmm3=1L1,000,000mm3=1L	1 msupmm3=0.001mL1mm3=0.001mL

Video

Converting Metric Units of Volume

Example 9.3.7: Converting Different Units of Measure for Volume

A holding tank at the local aquarium has a volume of 22,712,000,000 cm^3? What is the capacity of the holding tank in liters?

Answer

Use division to convert from cubic centimeters to liters. To determine the equivalent volume in liters, convert from cm^3 to L by dividing the value of the volume in cm^3 by 1,000.

$$\frac{22{,}712{,}000{,}000 \text{ cm}^3}{1{,}000} = 22{,}712{,}000 \text{ L}$$

The holding tank holds 22,712,000 L of water.

Your Turn 9.3.7

A gas can has a volume of 19,000 cm^3. How much gas, in liters, does the gas can hold?

Example 9.3.8: Converting Different Units of Measure for Volume Using Multiplication

An airplane used 150 m^3 of fuel to fly from New York to Hawaii. How many liters of fuel did the airplane use?

Answer

Because 1 liter is equivalent to 1 cubic decimeter, use multiplication to convert from m^3 to dm^3. Multiply the value of the volume by 1,000 to convert from m^3 to dm^3. Because 1 msupdm3=1L1dm3=1L, the resulting value is equivalent to the number of liters used.

$$150 \times 1{,}000 = 150{,}000 \text{ dm}^3 = 150{,}000 \text{ L}$$

The airplane used 150,000 liters of fuel.

Your Turn 9.3.8

A gasoline storage tank has a volume of 37.854 m^3. What is the volume of the storage tank in liters?

Example 9.3.9: Converting Different Units of Measure for Volume Using Multi-Step Division

How many liters can a pitcher with a volume of 8,000,000 mm^3 hold?

Answer

Use division to convert from a smaller metric volume unit to a larger metric volume unit. To convert from mm^3 to dm^3,

Step 1: Divide by 1,000 to convert from mm^3 to cm^3.

Step 2: Divide again by 1,000 to convert from cm^3 to dm^3.

Step 3: Use the unit value to express the volume in terms of liters.

mfrac

000mm31,000 = 8,000cm38,000cm31,000=8dm3=8L 8,000,000mm31,000 = 8,000cm38,000cm31,000=8dm3=8L

The pitcher can hold 8 liters of liquid.

Your Turn 9.3.9

A glass jar has a volume of 800,000 mm^3. How many mL of liquid can the glass jar hold?

Solving Application Problems Involving Volume

Knowing the volume of an object lets you know just how much that object can hold. When making a bowl of punch you might want to know the total amount of liquid a punch bowl can hold. Knowing how many liters of gasoline a car's tank can hold helps determine how many miles a car can drive on a full tank. Regardless of the application, understanding volume is essential to many every day and professional tasks.

Example 9.3.10: Using Volume to Solve Problems

A cubic shipping carton's dimensions measure semantics2m×2m×2m2m×2m×2m. A company wants to fill the carton with smaller cubic boxes that measure $10\text{ cm}\times 10\text{ cm}\times 10$ cm10cm×10cm×10cm. How many of the smaller boxes will fit in each shipping carton?

Answer

Step 1: Determine the volume of the shipping carton.

$$2\text{ m}\times 2\text{ m}\times 2\text{ m}=8\text{ m}^3$$

Step 2: Use the appropriate conversion factor to convert the volume of the shipping carton from m^3 to cm^3.

$$8\text{ m}^3\times 1{,}000=8{,}000\text{ dm}^3$$
$$8{,}000\text{ dm}^3\times 1{,}000=8{,}000{,}000\text{ cm}^3$$

Step 3: Determine the volume of the smaller boxes.

$$10\text{ cm}\times 10\text{ cm}\times 10\text{ cm}=1{,}000\text{ cm}^3$$

Step 4: Divide the volume of the shipping carton, in cm^3, by the volume of the smaller box, in cm^3.

$$\frac{8{,}000{,}000\text{ cm}^3}{1{,}000\text{ cm}^3}=8{,}000$$

The shipping carton will hold 8,000 smaller boxes.

Your Turn 9.3.10

A factory can mill 300 cubic meters of flour each day. They package the flour in boxes that measure $20\text{ cm}\times 5\text{ cm}\times 30\text{ cm}$. How many boxes of flour does the factory produce each day?

Example 9.3.11: Solving Volume Problems with Different Units

A carton of juice measures 6 cm long, 6 cm wide and 20 cm tall. A factory produces 28,800 liters of orange juice each day. How many cartons of orange juice are produced each day?

Answer

Step 1: Find the volume of the carton in cubic centimeters.

$$6\text{ cm}\times 6\text{ cm}\times 20\text{ cm}=720\text{ cm}^3$$

Step 2: Convert the volume in cm^3 to liters.

1cm3=1mL720cm3=720mL720mL1,000=0.72L 1cm3=1mL720cm3=720mL720mL1,000=0.72L

Step 3: Divide the number of liters of orange juice produced each day by the volume of each carton.

$$\frac{28{,}800 \text{ L}}{0.72 \text{ L}} = 40{,}000$$

The factory produces 40,000 cartons of orange juice each day.

Your Turn 9.3.11

An ice cream maker boxes frozen yogurt mix in boxes that measure 25 cm long, 8 cm wide and 35 cm tall. They produce 42,000 liters of frozen yogurt mix each day. How many boxes of frozen yogurt mix are produced each day?

✓ Example 9.3.12: Solving Complex Volume Problems

A fish tank measures 60 cm long, 15 cm wide and 34 cm tall (Figure 9.11). The tank is 25 percent full. How many liters of water are needed to completely fill the tank?

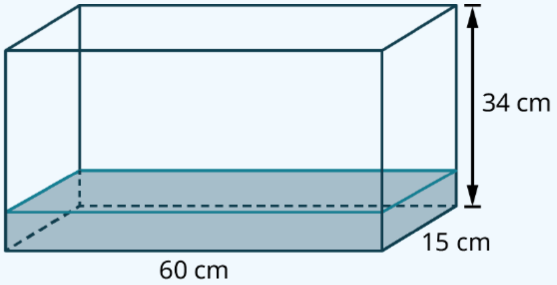

Figure 9.11

Answer

Step 1: Determine the volume of the fish tank in cubic centimeters.

$$60 \text{ cm} \times 15 \text{ cm} \times 34 \text{ cm} = 30{,}600 \text{ cm}^3$$

Step 2: Convert the volume in cm^3 to volume in liters.

1cm3 = 1mL30,600cm3 = 30,600mL30,600mL1,000=30.6L 1cm3 = 1mL30,600cm3 = 30,600mL30,600mL1,000=30.6L

Step 3: Since the tank is 25 percent full, the tank is 75 percent empty. Convert 75 percent to its decimal equivalent. Multiply the total volume by 75 percent expressed in decimal form to determine how many liters of water are required to fill the tank.

$$75\% = 0.75$$
$$30.6 \times 0.75 = 22.95$$

So, 22.95 liters of water are needed to fill the tank.

Your Turn 9.3.12

A fish tank measures 75 cm long, 20 cm wide and 25 cm tall. The tank is 50 percent full. How many liters of water are needed to completely fill the tank?

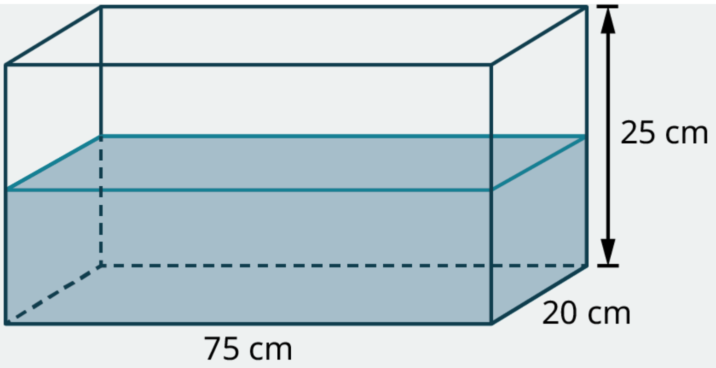

Figure 9.12

WORK IT OUT: How Does Shape Affect Volume?

Take two large sheets of card stock. Roll one piece to tape the longer edges together to make a cylinder. Tape the cylinder to the other piece of card stock which serves as the base of the cylinder. Fill the cylinder to the top with cereal. Pour the cereal from the cylinder into a plastic storage or shopping bag. Remove the cylinder from the base and the tape from the cylinder. Re-roll the cylinder along the shorter edges a tape together. Attach the new cylinder to the base. Pour the cereal from the plastic bag into the cylinder. What do you observe? How does the shape of a container affect its volume?

Check Your Understanding

For the following exercises, determine the most reasonable value for each volume.

- Terrarium: 50,000 km^3, 50,000 m^3, 50,000 cm^3, or 50,000 mm^3
- Milk carton: 236,000 L, 236 L, 236,000 mL, or 236 mL
- Box of crackers: 1,500 km^3, 1,500 m^3, 1,500 cm^3, or 1,500 mm^3

For the following exercises, Convert the given volume to the units shown.

- 42,500 mm^3 = __________ cm^3
- 1.5 dm^3 = __________ mL
- 6.75 cm^3 = __________ mm^3

For the following exercises, determine the volume of objects with the dimensions shown.

- $20\,\text{cm} \times 20\,\text{m} \times 20\,\text{cm}$
- $V =$ ________ L
- $12\,\text{mm} \times 5\,\text{cm} \times 1.7\,\text{cm}$
- $V =$ ________ mL
- $7.3\,\text{m} \times 3.2\,\text{m} \times 7\,\text{m}$
- $V =$ ________ m^3

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9.3.0: Exercises

For the following exercises, determine the most reasonable value for each volume.

Exercise 9.3.0.1

Fish tank:
71,120 km^3, 71,120 m^3, 71,120 cm^3, or 71,120 mm^3

Exercise 9.3.0.2

Juice box:
125,000 L, 125 L, 125,000 mL, or 125 mL

Exercise 9.3.0.3

Box of cereal:
2,700 km^3, 2,700 m^3, 2,700 cm^3, or 2,700 mm^3

Exercise 9.3.0.4

Water bottle:
5 L, 0.5 L, 5 mL, or 0.5 mL

Exercise 9.3.0.5

Shoe box:
3,600 km^3, 3.6 m^3, 3,600 cm^3, or 3,600 mm^3

Exercise 9.3.0.6

Swimming pool:
45 L, 45,000 L, 45 mL, or 45,000 mL

For the following exercises, convert the given volume to the units shown.

Exercise 9.3.0.7

38,861 mm^3 = __________ cm^3

Exercise 9.3.0.8

13 dm^3 = __________ mL

Exercise 9.3.0.9

874 cm^3 = __________ mm^3

Exercise 9.3.0.10

4 m^3 = __________ cm^3

? Exercise 9.3.0.11

0.00003 m^3 = __________ mm^3

? Exercise 9.3.0.12

57,500 mm^3 = ________ L

? Exercise 9.3.0.13

0.007 m^3 = ___________ L

? Exercise 9.3.0.14

8,600 cm^3 = __________ m^3

? Exercise 9.3.0.15

45.65 m^3 = ________ cm^3

? Exercise 9.3.0.16

0.06 m^3 = ___________ dm^3

? Exercise 9.3.0.17

0.081 m^3 = __________ mL

? Exercise 9.3.0.18

3,884,000 mm^3 = ________ m^3

For the following exercises, determine the volume of objects with the dimensions shown.

? Exercise 9.3.0.19

$30\,\text{cm} \times 20\,\text{m} \times 10\,\text{cm}$

$V =$ _________ L

? Exercise 9.3.0.20

$17\,\text{mm} \times 3\,\text{cm} \times 2.5\,\text{cm}$

$V =$ _________ mL

? Exercise 9.3.0.21

$3.4\,\text{m} \times 2.5\ \text{m} \times 10\,\text{m}$

$V =$ _________ m^3

? Exercise 9.3.0.22

$325\,\text{mm} \times 20\,\text{cm} \times 0.05\,\text{m}$

$V =$ _________ cm^3

Exercise 9.3.0.23

$3.7\,\text{m} \times 4\,\text{m} \times 5.5\,\text{m}$
$V =$ ________ m^3

Exercise 9.3.0.24

$18\,\text{dm} \times 0.8\,\text{m} \times 150\,\text{cm}$
$V =$ ________ L

Exercise 9.3.0.25

$15\,\text{cm} \times 400\,\text{mm} \times 3\,\text{dm}$
$V =$ ________ mL

Exercise 9.3.0.26

$3.5\,\text{cm} \times 200\,\text{mm} \times 0.7\,\text{dm}$
$V =$ ________ cm^3

Exercise 9.3.0.27

$35\,\text{m} \times 1.2\,\text{m} \times 0.007\,\text{km}$
$V =$ ________ m^3

Exercise 9.3.0.28

A box has dimensions of $20\,\text{cm} \times 15\,\text{cm} \times 30\,\text{cm}$. The box currently holds 1,250 cm^3 of rice. How many cubic centimeters of rice are needed to completely fill the box?

Exercise 9.3.0.29

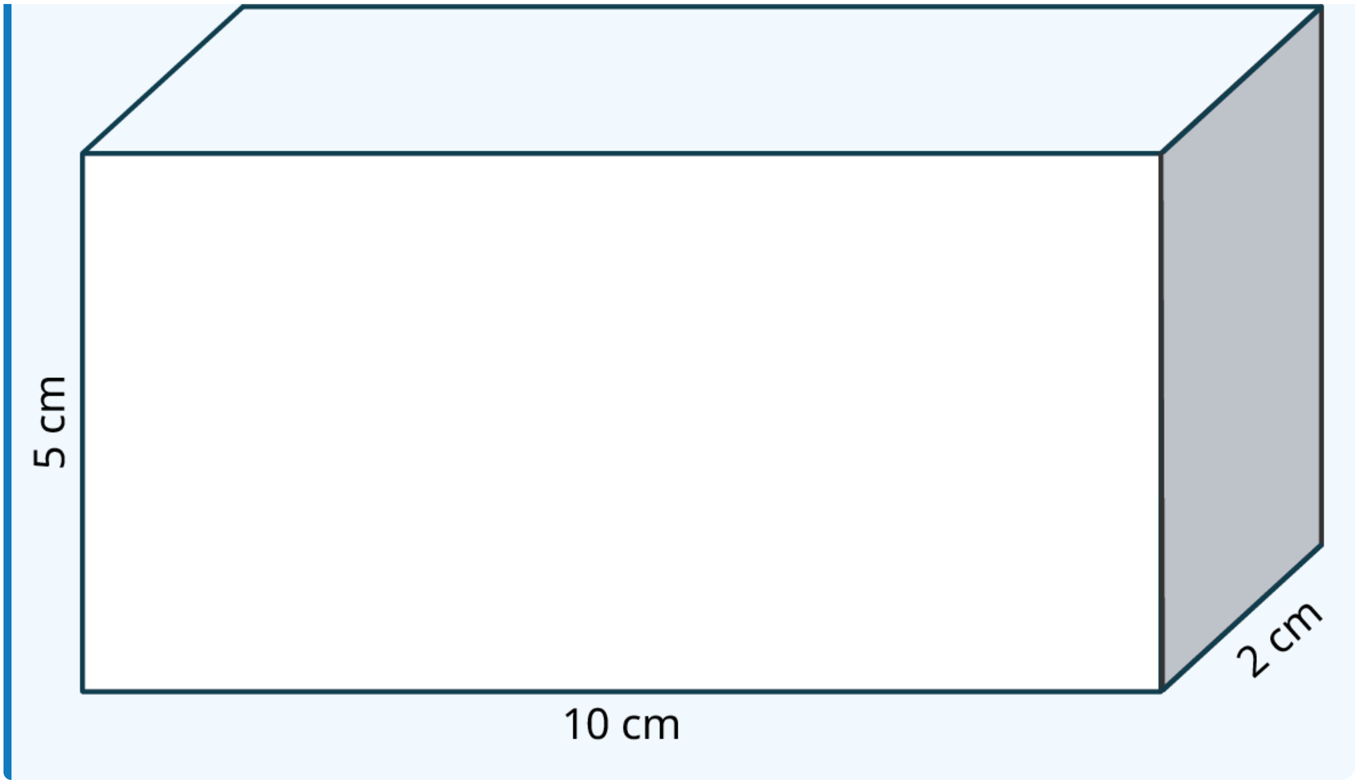
The dimensions of a medium storage unit are $4\,\text{m} \times 4\,\text{m} \times 8\,\text{m}$. What is the volume of a small storage area with dimensions half the size of the medium unit?

Exercise 9.3.0.30

How much liquid, in liters, can a container with dimensions of $40\,\text{cm} \times 20\,\text{cm} \times 120\,\text{cm}$ hold?

Exercise 9.3.0.31

What is the volume of the rectangular prism that is shown?



Exercise 9.3.0.32

A box is 15 centimeters long and 5 centimeters wide. The volume of the box is 225 cm^3. What is the height of the box?

Exercise 9.3.0.33

Kareem mixed two cartons of orange juice, three 2-liter bottles of soda water and six cans of cocktail fruits to make a fruit punch for a party. The cartons of orange juice and cans of cocktail fruits each have a volume of 500 cm^3. How much punch, in liters, did Kareem make?

Exercise 9.3.0.34

A holding tank has dimensions of $16\,\text{m} \times 8\,\text{m} \times 8\,\text{m}$. If the tank is half-full, how more liters of liquid can the tank hold?

Exercise 9.3.0.35

A large plastic storage bin has dimensions of $16\,\text{cm} \times 16\,\text{cm} \times 16\,\text{cm}$. A medium bin's dimensions are half the size of the large bin. A small bin's dimensions are the size of the medium bin. If the storage bins come in a set of 3—small, medium, and large—what is the total volume of the storage bin set in cubic centimeters?

Exercise 9.3.0.36

A soft serve ice cream machine holds a 19.2 liter bag of ice cream mix. If the average serving size of an ice cream cone is 120 mL, how many cones can be made from each bag of mix?

Exercise 9.3.0.37

A shipping carton has dimensions of $0.5\,\text{m} \times 0.5\,\text{m} \times 0.5\,\text{m}$. How many boxes with dimensions of $50\,\text{mm} \times 50\,\text{mm} \times 50\text{ mm}$ will fit in the shipping carton?

Exercise $9.3.0.38$

A recipe for chili makes 3.5 liters of chili. If a restaurant serves chili in 250 mL bowls, how many bowls of chili can they serve?

Exercise $9.3.0.39$

A contractor is building an in-ground pool. They excavate a pit that measures $12\text{ m} \times 9\text{ m} \times 2.5\text{ m}$. The dirt is being taken away in a truck that holds 30 m^3. How many trips will the truck have to make to cart away all of the dirt?

Exercise $9.3.0.40$

A juice dispenser measures $30\text{ cm} \times 30\text{ cm} \times 30\text{ cm}$. How many 375 mL servings will a full dispenser serve?

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9.4: Measuring Weight

DETECTO
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Figure 9.4.1: Weight scale at the local Antigua market (credit: “Weight scale at the local Antigua market” by Lucía García González/Flickr, CC0 1.0 Public Domain Dedication)

Learning Objectives

1. Identify reasonable values for weight applications.
2. Convert units of measures of weight.
3. Solve application problems involving weight.

In the metric system, weight is expressed in terms of grams or kilograms, with a kilogram being equal to 1,000 grams. A paper clip weighs about 1 gram. A liter of water weighs about 1 kilogram. In fact, in the same way that 1 liter is equal in volume to 1 cubic decimeter, the kilogram was originally defined as the mass of 1 liter of water. In some cases, particularly in scientific or medical settings where small amounts of materials are used, the milligram is used to express weight. At the other end of the scale is the metric ton (mt), which is equivalent to 1,000 kilograms. The average car weighs about 2 metric tons.

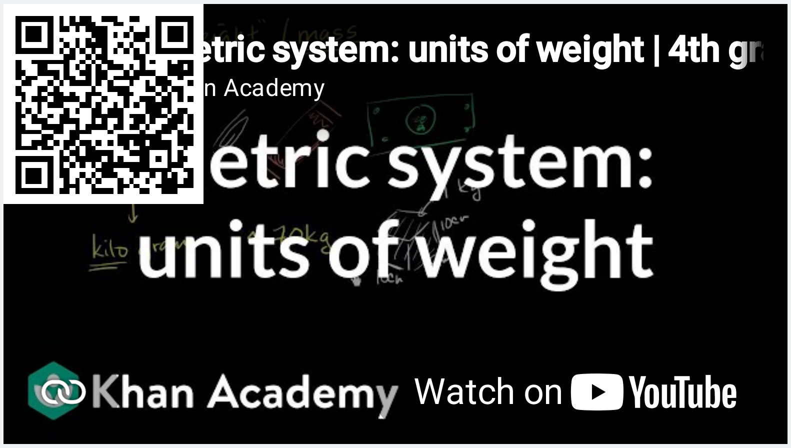
Any discussion about metric weight must also include a conversation about mass. Scientifically, mass is the amount of matter in an object whereas weight is the force exerted on an object by gravity. The amount of mass of an object remains constant no matter where the object is. Identical objects located on Earth and on the moon will have the same mass, but the weight of the objects will differ because the moon has a weaker gravitational force than Earth. So, objects with the same mass will weigh less on the moon than on Earth.

Since there is no easy way to measure mass, and since gravity is just about the same no matter where on Earth you go, people in countries that use the metric system often use the words mass and weight interchangeably. While scientifically the kilogram is only a unit of mass, in everyday life it is often used as a unit of weight as well.

Reasonable Values for Weight

To have an essential understanding of metric weight, you must be able to identify reasonable values for weight. When testing for reasonableness, you should assess both the unit and the unit value. Only by examining both can you determine whether the given weight is reasonable for the situation.

Video



Metric System: Units of Weight

Example 9.4.1: Identifying Reasonable Units for Weight

Which is the more reasonable value for the weight of a newborn baby:

- 3.5 kg or
- 3.5 g?

Answer

Using our reference weights, a baby weighs more than 3.5 paperclips, so 3.5 kilograms is a more reasonable value for the weight of a newborn baby.

Your Turn 9.4.1

Which represents a reasonable value for the weight of a penny:
2.5 g or 2.5 kg

Example 9.4.2: Determining Reasonable Values for Weight

Which of the following represents a reasonable value for the weight of three lemons?

- 250 g,
- 2,500 g, or
- 250 kg?

Answer

Because a kilogram is about 2.2 pounds, we can eliminate 250 kg as it is way too heavy. 2,500 grams is equivalent to 2.5 kilograms, or about five pounds, which is again, too heavy. So, a reasonable value for the weight of three lemons would be 250 grams.

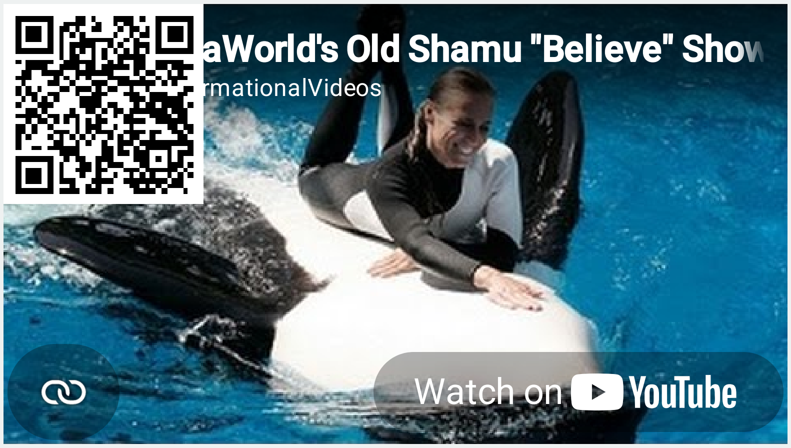
Your Turn 9.4.2

Which represents a reasonable value for the weight of a car:
1,300 g, 130 kg, or 1,300 kg?

Who Knew?: How Do You Measure the Weight of a Whale?

It is impossible to weigh a living whale. Fredrik Christiansen from the Aarhus Institute of Advanced Studies in Denmark developed an innovative way to measure the weight of whales. Using images taken from a drone and computer modeling, the weight of a whale can be estimated with great accuracy.

Video



Using Drones to Weigh Whales?

Example 9.4.3: Explaining Reasonable Values for Weight

The blue whale is the largest living mammal on Earth. Which of the following is a reasonable value for the weight of a blue whale: 149 g, 149 kg, or 149 mt? Explain your answer.

Answer

A reasonable value for the weight of a blue whale is 149 metric tons. Both 149 g and 149 kg are much too small a value for the largest living mammal on Earth.

Your Turn 9.4.3

The Etruscan shrew is one of the world’s smallest mammals. It has a huge appetite, eating almost twice its weight in food each day. Its heart beats at a rate of 25 beats per second! Which of the following is a reasonable value for the weight of an Etruscan shrew: 2 g, 2 kg, or 2 mt? Explain your answer.

Converting Like Units of Measures for Weight

Just like converting units of measure for distance, you can convert units of measure for weight. The most frequently used conversion factors for metric weight are illustrated in Figure 9.14.

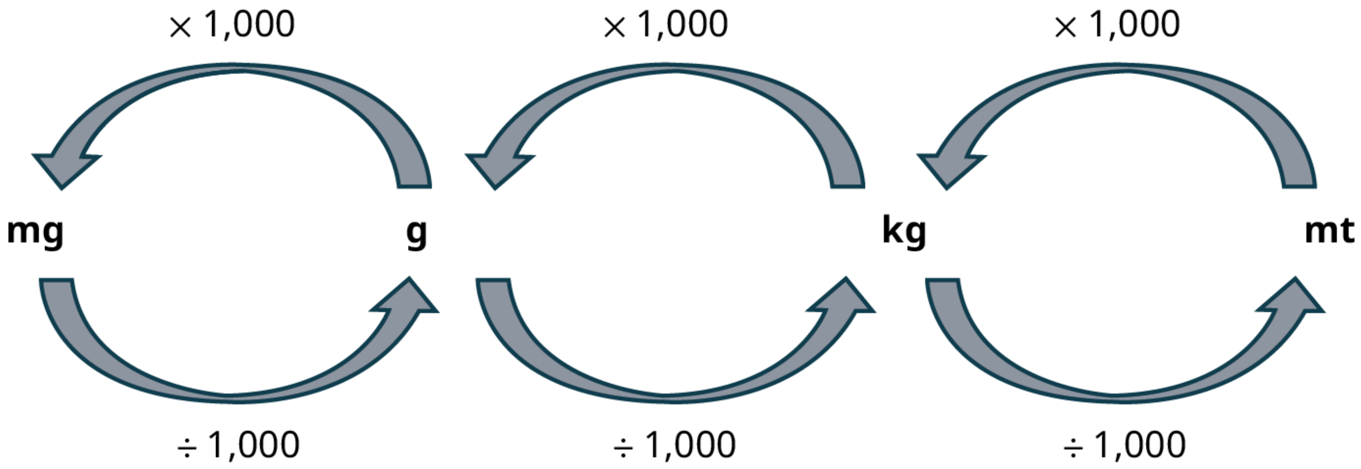


Figure 9.4.2: Common Conversion Factors for Metric Weight Units

✓ Example 9.4.4: Converting Metric Units of Weight Using Multistep Division

How many kilograms are in 24,300,000 milligrams?

Answer

Use division to convert from a smaller metric weight unit to a larger metric weight unit. To convert from milligrams to kilograms,

Step 1: Divide the value of the weight in milligrams by 1,000 to first convert from milligrams to grams.

Step 2: Divide by 1,000 again to convert from grams to kilograms.

mfrac

mg1,000=24,300g 24,300g1,000=24.3kg 24,300,000mg1,000=24,300g 24,300g1,000=24.3kg

So, 24,300,000 milligrams are equivalent to 24.3 kilograms.

Your Turn 9.4.4

How many kilograms are in 175,000 milligrams?

✓ Example9.4.5: Converting Metric Units of Weight Using Multiplication

The average ostrich weighs approximately 127 kilograms. How many grams does an ostrich weigh?

Answer

Use multiplication to convert from a larger metric weight unit to a smaller metric weight unit. To convert from kilograms to grams, multiply the value of the weight by 1,000.

$$127 \text{ kg} \times 1{,}000 = 127{,}000 \text{ g}$$

The average ostrich weighs 127,000 grams.

Your Turn 9.4.5

The world’s heaviest tomato weighed 4.869 kg when measured on July 15, 2020. How much did the tomato weigh in grams?

✓ Example 9.4.6: Converting Metric Units of Weight Using Multistep Multiplication

How many milligrams are there in 0.025 kilograms?

Answer

Use multiplication to convert from a larger metric weight unit to a smaller metric weight unit. To convert from kilograms to grams,

Step 1: Multiply the value of the weight by 1,000.

Step 2: Multiply the result by 1,000 to convert from grams to milligrams.

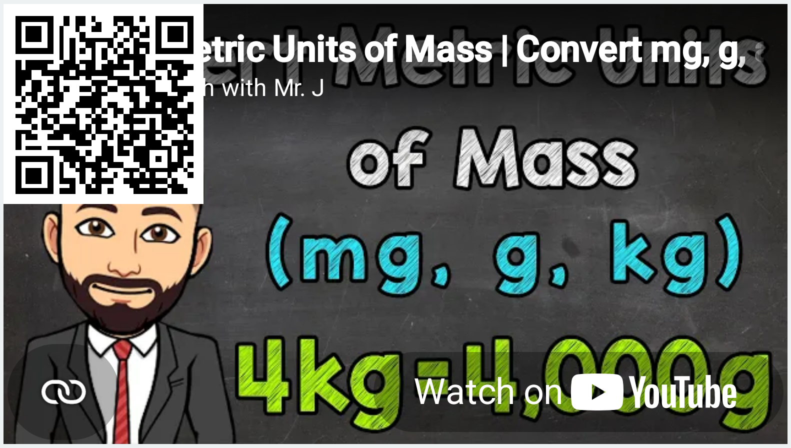
$$\begin{gathered} 0.025 \text{ kg} \times 1{,}000 = 25 \text{ g} \\ 25 \text{ g} \times 1{,}000 = 25{,}000 \text{ mg} \end{gathered}$$

So, 0.025 kilograms is equivalent to 25,000 milligrams.

Your Turn 9.4.6

How many milligrams are there in 1.23 kilograms?

Video



Metric Units of Mass: Convert mg, g, and kg

Solving Application Problems Involving Weight

From children's safety to properly cooking a pie, knowing how to solve problems involving weight is vital to everyday life. Let's review some ways that knowing how to work with metric weight can facilitate important decisions and delicious eating.

✓ Example 9.4.7: Comparing Weights to Solve Problems

The maximum weight for a child to safely use a car seat is 29 kilograms. If a child weighs 23,700 grams, can the child safely use the car seat?

Answer

Step 1: Convert the child's weight in grams to kilograms.

$$23{,}700 \text{ g} \div 1{,}000 = 23.7 \text{ kg}$$

Step 2: Compare the two weights.

$$23.7 \text{ kg} < 29 \text{ kg}$$

Yes, the child can safely use the car seat.

Your Turn $9.4.7$

The dosage recommendations for a popular brand of acetaminophen are listed in table below. What is the recommended dosage for a child who weighs 17,683 grams?

Weight	Dosage
11 kg to 15 kg	5 mL
16 kg to 21 kg	7.5 mL
22 kg to 27 kg	10 mL

✓ Example $9.4.8$: Solving Multistep Weight Problems

A recipe for scones calls for 350 grams of flour. How many kilograms of flour are required to make 4 batches of scones?

Answer

Step 1: Multiply the grams of flour need by 4 to determine the total amount of flour needed.

$$350 \text{ g} \times 4 = 1{,}400 \text{ g}$$

Step 2: Convert from grams to kilograms.

$$\frac{1{,}400 \text{ g}}{1{,}000} = 1.4 \text{ kg}$$

So, 1.4 kilograms of flour are needed to make four batches of scones.

Your Turn $9.4.8$

A croissant recipe calls for 500 g of flour. How many kilograms of flour are required to make 10 batches of croissants?

✓ Example $9.4.9$: Solving Complex Weight Problems

The average tomato weighs 140 grams. A farmer needs to order boxes to pack and ship their tomatoes to local grocery stores. They estimate that this year's harvest will yield 125,000 tomatoes. A box can hold 12 kilograms of tomatoes. How many boxes does the farmer need?

Answer

Step 1: Determine the total estimated weight of the harvested tomatoes.

$$140 \text{ g} \times 125{,}000 = 17{,}500{,}000 \text{ g}$$

Step 2: Convert the total weight from grams to kilograms.

$$17{,}500{,}000 \text{ g} \div 1{,}000 = 17{,}500 \text{ kg}$$

Step 3: Divide the weight of the tomatoes by the weight each box can hold.

$$17{,}500 \text{ kg} \div 12 \text{ kg} \approx 1{,}458$$

So, the farmer will need to order 1,458 boxes.

Your Turn 9.4.9

The average potato weighs 225 grams. A grocery chain orders 5,000 bags of potatoes. Each bag weighs 5 kg. Approximately how many potatoes did they order?

Check Your Understanding

For the following exercises, determine the most reasonable value for each weight.

- Candy bar: 50 kg, 50 g, or 50 mg
- Lion: 180 kg, 180 g, or 180 mg
- Basketball: 624 kg, 624 g, or 624 mg

For the following exercises, convert the given weight to the units shown.

- 8,900 g = __________ kg
- 17 g = __________ mg
- 0.07 kg = __________ g

For the following exercises, determine the total weight in the units shown.

- three 48 g granola bars ________ kg
- seven 28 g cheese slices ________ mg
- six 15 mg tea bags ________ g

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9.4.0: Exercises

For the following exercises, determine the most reasonable value for each weight.

? Exercise 9.4.0.1

Aspirin tablet: 300 kg, 300 g, or 300 mg

? Exercise 9.4.0.2

Elephant: 5,000 kg, 5,000 g, or 5,000 mg

? Exercise 9.4.0.3

Baseball: 145 kg, 145 g, or 145 mg

? Exercise 9.4.0.4

Orange: 115 kg, 115 g, or 115 mg

? Exercise 9.4.0.5

Pencil: 6 kg, 6 g, or 6 mg

? Exercise 9.4.0.6

Automobile: 1,300 kg, 1,300 g, or 1,300 mg

For the following exercises, convert the given weight to the units shown.

? Exercise 9.4.0.7

3,500 g = __________ kg

? Exercise 9.4.0.8

53 g = __________ mg

? Exercise 9.4.0.9

0.02 kg = __________ g

? Exercise 9.4.0.10

200 mg = __________ g

? Exercise 9.4.0.11

2.3 g = _________ mg

Exercise 9.4.0.12

20 kg = _______ mg

Exercise 9.4.0.13

2,300 kg = __________ g

Exercise 9.4.0.14

8,700 mg = _________ g

Exercise 9.4.0.15

9,730 mg = _______ kg

Exercise 9.4.0.16

0.0078 kg = __________ g

Exercise 9.4.0.17

2.34 g = _________ mg

Exercise 9.4.0.18

234.5 mg = _______ g

For the following exercises, determine the total weight in the units shown.

Exercise 9.4.0.19

Three 350 mg tablets ________ g

Exercise 9.4.0.20

Seven 115 g soap bars ________ kg

Exercise 9.4.0.21

Six 24 g batteries ________ mg

Exercise 9.4.0.22

Fifty 3.56 g pennies ________ mg

Exercise 9.4.0.23

Eight 2.25 kg bags of potatoes ________ g

? Exercise 9.4.0.24

Four 23 kg sacks of flour ________ g

? Exercise 9.4.0.25

Ten 2.5 kg laptops ________ g

? Exercise 9.4.0.26

Seven 1,150 g chickens ________ kg

? Exercise 9.4.0.27

Ninety 4,500 mg marbles ________ kg

? Exercise 9.4.0.28

There are 26 bags of flour. Each bag weighs 5 kg. What is the total weight of the flour?

? Exercise 9.4.0.29

The average female hippopotamus weighs 1,496 kg. The average male hippopotamus weighs 1,814 kg. How much heavier, in grams, is the male hippopotamus than the female hippopotamus?

? Exercise 9.4.0.30

Twelve pieces of cardboard weigh 72 grams. What is the weight of one piece of cardboard?

? Exercise 9.4.0.31

Miguel's backpack weighs 2.4 kg and Shanayl's backpack weighs 2,535 grams. Whose backpack is heavier and by how much?

? Exercise 9.4.0.32

A souvenir chocolate bar weighs 1.815 kg. If you share the candy bar equally with two friends, how many grams of chocolate does each person get?

? Exercise 9.4.0.33

You purchase 10 bananas that weigh 50 grams each. If bananas cost $5.50 per kilogram, how much did you pay?

? Exercise 9.4.0.34

A family-size package of ground meat costs $15.75. The package weighs 4.5 kg. What is the cost per gram of the meat?

? Exercise 9.4.0.35

A box containing 6 identical books weighs 7.2 kg. The box weighs 600 g. What is the weight of each book in grams?

? Exercise 9.4.0.36

A 2.316 kg bag of candy is equally divided into 12 party bags. What is the weight of the candy, in grams, in each party bag?

? Exercise 9.4.0.37

A store has 450 kg of flour at the beginning of the day. At the end of the day the store has 341 kg of flour. If flour costs $0.35 per kilogram, how much flour, in dollars, did the store sell that day?

? Exercise 9.4.0.38

A local restaurant offers lobster for $110 per kilogram. What is the price for a lobster that weighs 450 grams?

? Exercise 9.4.0.39

A student's backpack weighs 575 grams. Their books weigh 3.5 kg. If the student's weight while wearing their backpack is 58.25 kg, how much does the student weigh in kilograms?

? Exercise 9.4.0.40

The weight of a lamb is 41 kg 340 g. What is the total weight, in kilograms, of four lambs of the same weight?

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9.5: Measuring Temperature

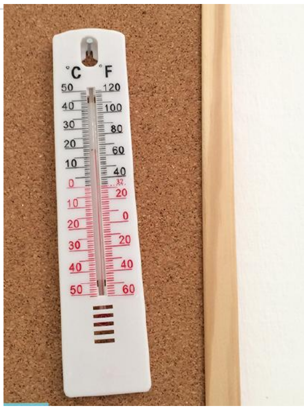

Figure 9.5.1: A thermometer that measures temperature in both customary and metric units. (credit: “Thermometer” by Jeff Djevdet/Flickr, CC BY 2.0)

Learning Objectives

1. Convert between Fahrenheit and Celsius.
2. Identify reasonable values for temperature applications.
3. Solve application problems involving temperature.

When you touch something and it feels warm or cold, what is that really telling you about that substance? Temperature is a measure of how fast atoms and molecules are moving in a substance, whether that be the air, a stove top, or an ice cube. The faster those atoms and molecules move, the higher the temperature.

In the metric system, temperature is measured using the Celsius (°C) scale. Because temperature is a condition of the physical properties of a substance, the Celsius scale was created with 100 degrees separating the point at which water freezes, 0 °C, and the point at which water boils, 100 °C. Scientifically, these are the points at which water molecules change from one state of matter to another—from solid (ice) to liquid (water) to gas (water vapor).

Checkpoint

When reading temperatures, it’s important to look beyond the degree symbol to determine which temperature scale the units express. For example, 13 °C reads “13 degrees Celsius,” indicating that the temperature is expressed using the Celsius scale, while 13 °F reads “13 degrees Fahrenheit,” indicating that the temperature is expressed using the Fahrenheit scale.

Video

sconceptions About Temperatur
tasium
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Misconceptions About Temperature

Who Knew?: How Many Temperature Scales Are There?

Did you know that in addition to Fahrenheit and Celsius, there is a third temperature scale widely used throughout the world? The Kelvin scale starts at absolute zero, the lowest possible temperature at which there is no heat energy present at all. It is primarily used by scientists to measure very high or very low temperatures when water is not involved.

Converting Between Fahrenheit and Celsius Temperatures

Understanding how to convert between Fahrenheit and Celsius temperatures is an essential skill in understanding metric temperatures. You likely know that below 32 °F means freezing temperatures and perhaps that the same holds true for 0 °C. While it may be difficult to recall that water boils at 212 °F, knowing that it boils at 100 °C is a fairly easy thing to remember.

But what about all the temperatures in between? What is the temperature in degrees Celsisus on a scorching summer day? What about a cool autumn afternoon? If a recipe instructs you to preheat the oven to 350 °F, what Celsius temperature do you set the oven at?

Figure $9.5.1$ lists common temperatures on both scales, because we don’t use Celsius temperatures daily it’s difficult to remember them. Fortunately, we don’t have to. Instead, we can convert temperatures from Fahrenheit to Celsius and from Celsius to Fahrenheit using a simple algebraic expression.

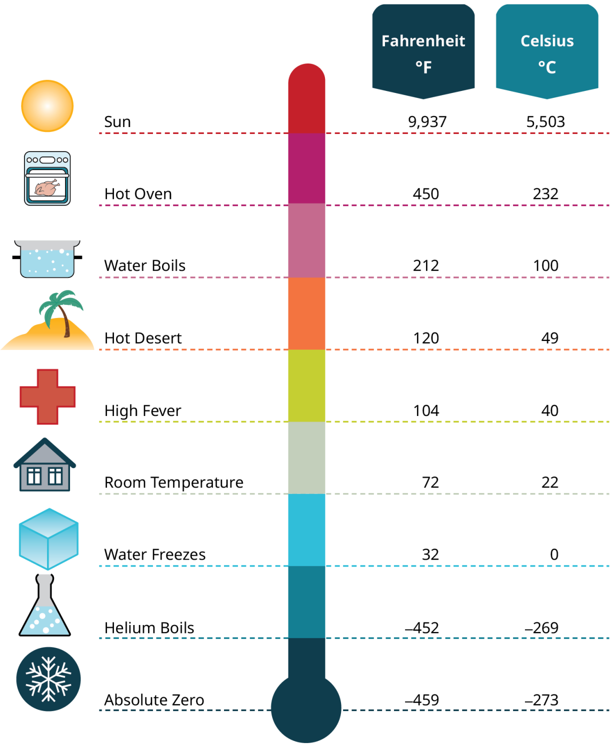


Figure $9.5.1$: Common Temperatures

📌 FORMULA

The formulas used to convert temperatures from Fahrenheit to Celsius or from Celsius to Fahrenheit are outlined in Table 📌 $9.5.1$

Table 📌 $9.5.1$ Temperature Conversion Formulas

Fahrenheit to Celsius	Celsius to Fahrenheit
$C = \frac{5}{9}(F-32)$	$F =$95C+32F

✓ Example $9.5.1$: Converting Temperatures from Fahrenheit to Celsius

A recipe calls for the oven to be set to 392 °F. What is the temperature in Celsius?

Answer

Use the formula in Table ✓ $9.5.1$ to convert from Fahrenheit to Celsius.

$$\begin{aligned} C &= \tfrac{5}{9}(F-32) \\ C &= \tfrac{5}{9}(392-32) \\ C &= \tfrac{5}{9}(360) \\ C &= 200 \end{aligned}$$

So, 392 °F is equivalent to 200 °C.

Your Turn 9.5.1

What is 482 °F in Celsius?

Example 9.5.2: Converting Temperatures from Celsius to Fahrenheit

On a sunny afternoon in May, the temperature in London was 20 °C. What was the temperature in degrees Fahrenheit?

Answer

Use the formula in Table 9.3 to convert from Celsius to Fahrenheit.

$$F =$$

95C+32 F=95(20)+32 F=36+32 F=68 F=95C+32 F=95(20)+32 F=36+32 F=68

The temperature was 68 °F.

Your Turn 9.5.2

What is 75 °C in Fahrenheit?

Example 9.5.3: Comparing Temperatures in Celsius and Fahrenheit

A manufacturer requires a vaccine to be stored in a refrigerator at temperatures between 36 °F and 46 °F. The refrigerator in the local pharmacy cools to 3 °C. Can the vaccine be stored safely in the pharmacy's refrigerator?

Answer

Use the formula in Table 9.3 to convert from Celsius to Fahrenheit.

$$F =$$

95C+32 F=95(3)+32 F=5.4+32 F=37.4 F=95C+32 F=95(3)+32 F=5.4+32 F=37.4

Then, compare the temperatures.

$$36\ ^{\circ}\text{F} < 37.4\ ^{\circ}\text{F} < 46\ ^{\circ}\text{F}$$

Yes. 37.4 °F falls within the acceptable range to store the vaccine, so it can be stored safely in the pharmacy's refrigerator.

Your Turn 9.5.3

In July, the average temperature in Madrid is 16.7 °C. The average temperature in Toronto in July is 57.4 °F. Which city has the higher temperature, and by how many degrees Celsius?

Reasonable Values for Temperature

While knowing the exact temperature is important in most cases, sometimes an approximation will do. When trying to assess the reasonableness of values for temperature, there is a quicker way to convert temperatures for an approximation using mental math. These simpler formulas are listed in Table 9.5.2

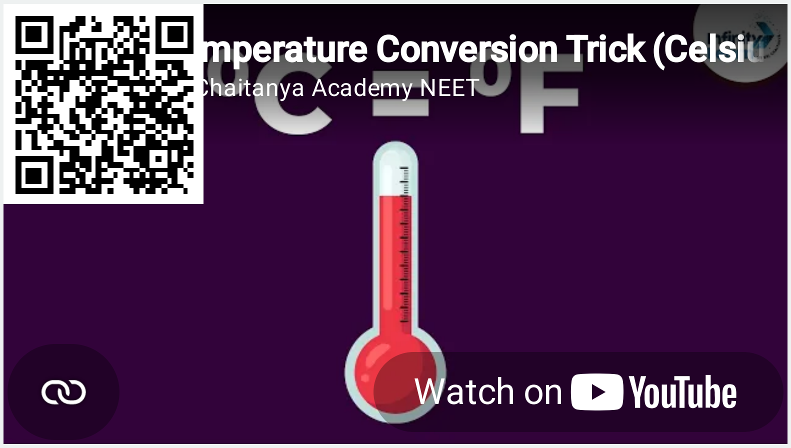
FORMULA

The formulas used to estimate temperatures from Fahrenheit to Celsius or from Celsius to Fahrenheit are outlined in Table 9.4.

Table 9.5.2 Estimate Temperature Conversion

Fahrenheit to Celsius	Celsius to Fahrenheit
$C = \frac{F-30}{2}$	$F = 2C + 30$

Video



Temperature Conversion Trick

Example 9.5.4: Using Benchmark Temperatures to Determine Reasonable Values for Temperatures

Which is the more reasonable value for the temperature of a freezer?

- 5 °C or
- –5 °C?

Answer

We know that water freezes at 0 °C. So, the more reasonable value for the temperature of a freezer is −5 °C, which is below 0 °C. At temperature of 5 °C is above freezing.

Your Turn 9.5.4

To make candy apples, you must boil the sugar mixture for about 20 minutes. Which is the more reasonable value for the temperature of the mixture when boiling:

- 48 °C or
- 148 °C?

Example 9.5.5: Using Estimation to Determine Reasonable Values for Temperatures

The average body temperature is generally accepted as 98.6 °F. What is a reasonable value for the average body temperature in degrees Celsius:

- 98.6 °C,
- 64.3 °C, or
- 34.3 °C?

Answer

To estimate the average body temperature in degrees Celsius, subtract 30 from the temperature in degrees Fahrenheit, and divide the result by 2.

$$\frac{(98.6-30)}{2}=\frac{68.6}{2}=34.3$$

A reasonable value for average body temperature is 34.3 °C.

Your Turn 9.5.5

Which represents a reasonable value for temperature of a hot summer day:

- 102 °C,
- 37 °C, or
- 20 °C?

✓ Example 9.5.6: Using Conversion to Determine Reasonable Values for Temperatures

Which is a reasonable temperature for storing chocolate:

- 28 °C,
- 18 °C, or
- 2 °C?

Answer

Use the formula in Table ✓ 9.5.2 to determine the temperature in degrees Fahrenheit.

$$\begin{aligned} &F=\frac{9}{5}C+32 \\ &F=\frac{9}{5}(28)+32=82.4 \\ &F=\frac{9}{5}(18)+32=64.4 \\ &F=\frac{9}{5}(2)+32=35.6 \end{aligned}$$

A temperature of $82.4^{\circ}\mathrm{F}$ would be too hot, causing the chocolate to melt. A temperature of $35.6^{\circ}\mathrm{F}$ is very close to freezing, which would affect the look and feel of the chocolate. So, a reasonable temperature for storing chocolate is $18^{\circ}\mathrm{C}$, or $64.4^{\circ}\mathrm{F}$.

Your Turn 9.5.6

Which represents a reasonable temperature for cooking chili:

- 240 °C,
- 60 °C, or
- 6 °C?

Solving Application Problems Involving Temperature

Whether traveling abroad or working in a clinical laboratory, knowing how to solve problems involving temperature is an important skill to have. Many food labels express sizes in both ounces and grams. Most rulers and tape measures are two-sided with one side marked in inches and feet and the other in centimeters and meters. And while many thermometers have both Fahrenheit and Celsius scales, it really isn't practical to pull out a thermometer when cooking a recipe that uses metric units. Let's review at few instances where knowing how to fluently use the Celsius scale helps solve problems.

✓ Example 9.5.7: Using Subtraction to Solve Temperature Problems

The temperature in the refrigerator is 4 °C. The temperature in the freezer is 21 °C lower. What is the temperature in the freezer?

Answer

Use subtraction to find the difference.

$$4-21=-17$$

So, the temperature in the freezer is −17 °C.

Your Turn 9.5.7

At 6 PM, the temperature was 4 °C. By 6 AM the temperature had fallen by 6 °C. What was the temperature at 6 AM?

✓ Example 9.5.8: Using Addition to Solve Temperature Problems

A scientist was using a liquid that was 35 °C. They needed to heat the liquid to raise the temperature by 6 °C. What was the temperature after the scientist heated it?

Answer

Use addition to find the new temperature.

$$35 + 6 = 41$$

The temperature of the liquid was 41 °C after the scientist heated it.

Your Turn 9.5.8

A hot dog was cooked in a microwave oven. The hot dog was 4 °C when it was put in the microwave and the temperature increased by 59 °C when it was taken out. What was the temperature of the cooked hot dog?

✓ Example 9.5.9: Solving Complex Temperature Problems

The optimum temperature for a chemical compound to develop its unique properties is 392 °F. When the heating process begins, the temperature of the compound is 20 °C. For safety purposes the compound can only be heated 9 °C every 15 minutes. How long until the compound reaches its optimum temperature?

Answer

Step 1: Determine the optimum temperature in degrees Celsius using the formula in Table 9.3.

$$\begin{aligned}C &= \tfrac{5}{9}(F-32)\\ C &= \tfrac{5}{9}(392-32)\\ C &= \tfrac{5}{9}(360)\\ C &= 200\end{aligned}$$

Step 2: Subtract the starting temperature.

$$200^{\circ}\mathrm{C} - 20^{\circ}\mathrm{C} = 180^{\circ}\mathrm{C}$$

Step 3: Determine the number of 15-minute cycles needed to heat the compound to its optimum temperature.

$$180 \div 9 = 20$$

Step 4: Multiply the number of cycles needed by 15 minutes and convert the product to hours and minutes.

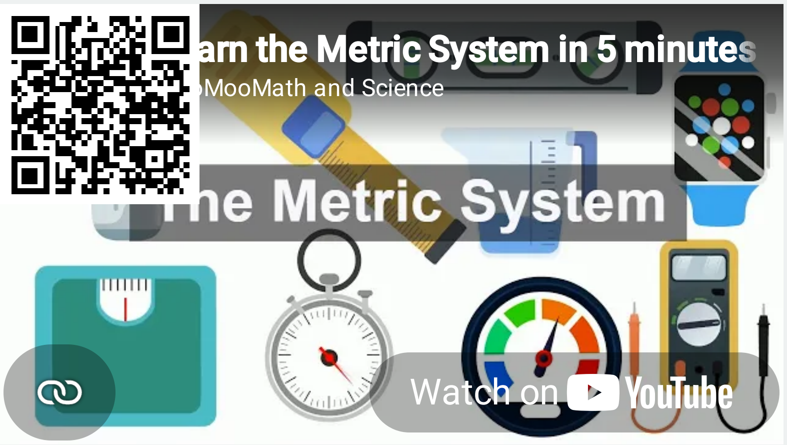
$$\begin{aligned}15 \text{ minutes } &\times 9 = 135 \text{ minutes}\\ 135 \text{ minutes } &= 2 \text{ hours } 15 \text{ minutes}\end{aligned}$$

So, it will take 2 hours and 15 minutes for the compound to reach its optimum temperature.

Your Turn 9.5.9

After reaching a temperature of 302 °F a chemical compound cools at the rate of 5 °C every 6 minutes. How long will it take until the compound has reached a temperature of zero degree Celsius.

Video


arn the Metric System in 5 minutes
MooMath and Science
he Metric System
Watch on YouTube

Learn the Metric System in 5 Minutes

Check Your Understanding

For the following exercises, determine the most reasonable value for each temperature.

Popsicle:

- 28.5 °C or
- 28.5 °F

Room temperature:

- 20 °F or
- 20 °C

Hot coffee:

- 71.1 °C or
- 71.1 °F

For the following exercises, use mental math to approximate the temperature.

- 450 °F = __________ °C
- 35 °C = __________ °F
- 100 °F = __________ °C

For the following exercises, convert the temperatures to the nearest degree.

- 225 °F = __________ °C
- 27 °C = __________ °F
- 750 °F = __________ °C

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9.5.0: Exercises

For the following exercises, determine the most reasonable value for each temperature.

? Exercise 9.5.0.1

Human body:

- 37 °C or
- 37 °F

? Exercise 9.5.0.2

An ice cube:

- 32 °C or
- 0 °C

? Exercise 9.5.0.3

Boiling water:

- 212 °C
- or 212 °F

? Exercise 9.5.0.4

Summer day:

- 8 °C,
- 23 °C, or
- 75 °C

? Exercise 9.5.0.5

Winter day:

- 33 °C,
- 30 °C, or
- 3 °C

? Exercise 9.5.0.6

Lava:

- 700 °C,
- 70 °C, or
- 7 °C

For the following exercises, use mental math to approximate the temperature.

? Exercise 9.5.0.7

500 °F = __________ °C

? Exercise 9.5.0.8

25 °C = __________ °F

? Exercise 9.5.0.9

350 °F = __________ °C

? Exercise 9.5.0.10

150 °C = __________ °F

? Exercise 9.5.0.11

–4 °F = __________ °C

? Exercise 9.5.0.12

73 °C = __________ °F

? Exercise 9.5.0.13

72 °F = __________ °C

? Exercise 9.5.0.14

10 °C = __________ °F

? Exercise 9.5.0.15

1,020 °F = __________ °C

For the following exercises, convert the temperatures to the nearest degree.

? Exercise 9.5.0.16

450 °F = __________ °C

? Exercise 9.5.0.17

35 °C = __________ °F

? Exercise 9.5.0.18

525 °F = __________ °C

? Exercise 9.5.0.19

140 °C = __________ °F

? Exercise 9.5.0.20

–40 °F = __________ °C

? Exercise 9.5.0.21

67 °C = __________ °F

? Exercise 9.5.0.22

85 °F = __________ °C

? Exercise 9.5.0.23

15 °C = __________ °F

? Exercise 9.5.0.24

1,200 °F = __________ °C

? Exercise 9.5.0.25

112 °F = __________ °C

? Exercise 9.5.0.26

105 °C = __________ °F

? Exercise 9.5.0.27

125 °F = __________ °C

? Exercise 9.5.0.28

A cup of tea is 30 °C. After adding ice, the temperature of the tea decreased 3 °C. What is the temperature of the tea now?

? Exercise 9.5.0.29

A liquid with a temperature of 15 °C is placed on a stovetop. The liquid is heated at the rate of 1.5 °C per minute. What is the temperature of the liquid after 10 minutes?

? Exercise 9.5.0.30

The temperature inside a store is 22 °C. Outside the temperature is 37 °C. How much cooler is it in the store than outside?

? Exercise 9.5.0.31

A boiling pan of water is 212 °F. As the water cools the temperature drops 6 °C every 2 minutes. How many minutes until the temperature of the water reaches 40 °C?

? Exercise 9.5.0.32

On May 1 the temperature was 79 °F. On June 1 the temperature was 25 °C. Which day was warmer?

? Exercise 9.5.0.33

For each log placed on a fire, the temperature increases 45 °C. How many logs are needed for the campfire to increase 315 °C?

? Exercise 9.5.0.34

The temperature of a Bunsen burner flame increases 572 °F each minute. About how many minutes does it take for the flame to increase 1,200 °C?

? Exercise 9.5.0.35

The instructions to cook a pizza say to set the oven at 425 °F. To the nearest degree, what is the temperature in degrees Celsius?

? Exercise 9.5.0.36

In the evening the temperature was 3 °C. By morning, the temperature had fallen 4 °C. What is the temperature now?

? Exercise 9.5.0.37

An 8,000 BTU air conditioner can cool a room 1 °C every 5 minutes. If the temperature in the room is 24 °C, how long will it take the air conditioner to cool the room to 20 °C?

? Exercise 9.5.0.38

What is 95 °F in degrees Celsius?

? Exercise 9.5.0.39

What is 50 °C in degrees Fahrenheit?

? Exercise 9.5.0.40

Which is a reasonable value for the temperature of a room:

- 60 °C,
- 40 °C, or
- 20 °C?

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SECTION OVERVIEW

9.6: Chapter Summary

9.6.0: Key Terms

9.6.1: Key Concepts

9.6.2: Videos

9.6.3: Formula Review

9.6.4: Projects

9.6.5: Chapter Review

9.6.6: Chapter Test

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9.6.0: Key Terms

9.1 The Metric System

- metric system
- meter (m)
- gram (g)
- liter (L)
- square meter (m^2)
- cubic meter (m^3)
- degrees Celsius (°C)
- conversion factor

9.2 Measuring Area

- area
- square units

9.3 Measuring Volume

- volume
- cubic units

9.4 Measuring Weight

- mass

9.5 Measuring Temperature

- temperature

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9.6.1: Key Concepts

9.1 The Metric System

- The metric system is decimal system of weights and measures based on base units of meter, liter, and gram. The system was first proposed in 1670 and has since been adopted as the International System of Units and used in nearly every country in the world.
- Each successive unit on the metric scale is 10 times larger than the previous one. To convert between units with the same base unit, you must either multiply or divide by a power of 10.
- The most used prefixes are listed in Table 9.1. An easy way to remember the order of the prefixes, from largest to smallest, is the mnemonic *King Henry Died from Drinking Chocolate Milk.*

9.2 Measuring Area

- Area describes the size of a two-dimensional surface. It is the amount of space contained within the lines of a two-dimensional space.
- Area is measured in square meters units; in the metric system the base unit for area is square meters (m^2).

9.3 Measuring Volume

- Volume is a measure of the space contained within or occupied by three-dimensional objects.
- Volume is measured in cubic units; the base unit for volume in the metric system is cubic meters (m^3)
- The liter (L) is a metric unit of capacity but is often used to express the volume of liquids. One liter is equivalent to one cubic decimeter, which is the volume of a cube measuring $10\ \mathrm{cm} \times 10\ \mathrm{cm} \times 10\ \mathrm{cm}$.

9.4 Measuring Weight

- Mass is the amount of matter in an object whereas weight is the force exerted on an object by gravity. The mass of an object never changes; the weight of an object changes depending on the force of gravity. An object with the same mass would weigh less on the moon than on Earth because the moon's gravity is less than that of Earth.
- In the metric system, weight and mass are often used interchangeably and are expressed in terms in grams or kilograms.

9.5 Measuring Temperature

- Temperature is a measure of how fast atoms and molecules are moving in a substance, whether that be the air, a stove top, or an ice cube. The faster those atoms and molecules move, the higher the temperature.
- In the metric system, temperature is measured using the Celsius (°C) scale.
- The Celsius scale was created with 100 degrees separating the point at which water freezes, 0 °C, and the point at which water boils, 100 °C. Scientifically, these are the points at which water molecules change from one state of matter to another—from solid (ice) to liquid (water) to gas (water vapor).

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9.6.2: Videos

9.1 The Metric System

- U.S. Office of Education: Metric Education
- Neil deGrasse Tyson Explains the Metric System

9.2 Measuring Area

- Converting Metric Units of Area
- Why the Metric System Matters

9.3 Measuring Volume

- How to Convert Cubic Centimeters to Cubic Meters
- Converting Metric Units of Volume

9.4 Measuring Weight

- Metric System: Units of Weight
- Using Drones to Weigh Whales?
- Metric Units of Mass: Convert mg, g, and kg

9.5 Measuring Temperature

- Misconceptions About Temperature
- Temperature Conversion Trick
- Learn the Metric System in 5 Minutes

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9.6.3: Formula Review

9.1 The Metric System

You can convert between unit sizes with the same base unit using the conversion factors shown in Figure 9.4.

9.2 Measuring Area

To determine the area of rectangular-shaped objects:

$$\text{Area } = \text{length}(l) \times \text{width}(w)$$

$$\text{or}$$

$$A = l \times w$$

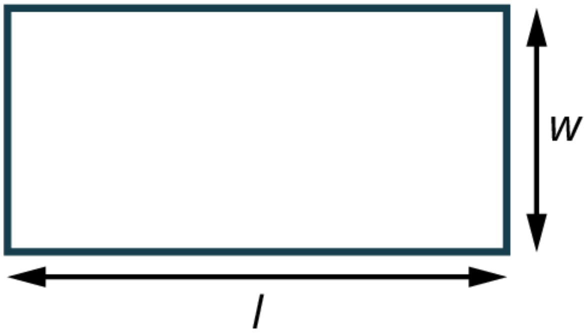

Figure 9.17: Rectangle with Length (l) and Width (w) Labeled

You can convert between metric area units using the conversion factors shown in Figure 9.7.

9.3 Measuring Volume

To determine the volume of a rectangular prism:

$$\text{Volume } = \text{ length } (l) \times \text{width}(w) \times \text{ height } (h)$$

$$\text{or}$$

$$V = lwh$$

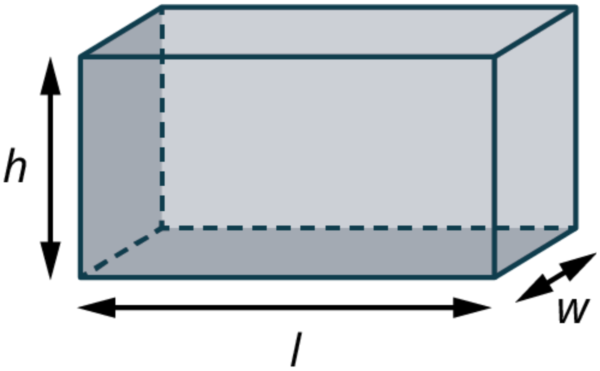

Figure 9.18: Rectangular Prism with Height (h), Length (l), and Width (w) Labeled

You can convert between metric volume units and metric capacity units using the relationships shown in Table 9.2.

9.4 Measuring Weight

You can convert between metric weight units using the conversion factors shown in this Table

9.5 Measuring Temperature

To convert temperature from Fahrenheit to Celsius:

$$C=\frac{5}{9}(F-32)$$

To convert temperature from Celsius to Fahrenheit:

$$F=\frac{9}{5}C+32$$

To estimate temperature from Fahrenheit to Celsius:

$$C=\frac{F-30}{2}$$

To estimate temperature from Celsius to Fahrenheit:

$$F=2C+30$$

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9.6.4: Projects

Cooking

1. Take a favorite recipe that uses customary measures and convert the measures and cooking temperature to the metric system.
2. Find a recipe that uses metric measures and convert the measures and cooking temperature to the U.S. Customary System of Measurement, using cups, tablespoons, or teaspoons as required.
3. What did you observe? Was it easier to convert from one system to another? Which system allows for more precise measurements? What kitchen tools would you need in your kitchen if you used the metric system?

Shopping

1. Compare the average gas price in California to the average gas price in Puerto Rico.
2. What conversions did you need to make to do the comparison?
3. Do you think that the price of the gasoline is affected by the units in which it is sold?

Sports

1. What system of measurement is used for track and field events? Why do you think this system is used?
2. What system of measurement is used for football? Why do you think this system is used?
3. Research various sports records. Which units of measurement is used? What do you think influenced the unit of measure used?

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9.6.5: Chapter Review

The Metric System

For the following exercises, determine the base unit of the metric system described: liter, gram, or meter.

? Exercise 9.6.5.1

amount of juice in a carton

? Exercise 9.6.5.2

weight of a puppy

? Exercise 9.6.5.3

height of a wall

For the following exercises, choose the smaller of the two units.

? Exercise 9.6.5.4

hectogram or centigram

? Exercise 9.6.5.5

decameter or kilometer

? Exercise 9.6.5.6

milliliter or deciliter

? Exercise 9.6.5.7

Convert 236 cm to meters (m).

? Exercise 9.6.5.8

Convert 137.5 mg to grams (g).

? Exercise 9.6.5.9

You purchase 15 kg of birdseed. You divide the birdseed into 10 bags. How many grams of birdseed are in each bag?

? Exercise 9.6.5.10

A track is 650 m long. If you run 8 laps around the track, how many kilometers did you run?

Measuring Area

For the following exercises, determine the most reasonable value for each area.

Exercise 9.6.5.11

bedroom floor
$12\ \text{km}^2$
$12\ \text{m}^2$
$12\ \text{cm}^2$
$12\ \text{mm}^2$

Exercise 9.6.5.12

nature preserve
$800\ \text{km}^2$
$800\ \text{m}^2$
$800\ \text{cm}^2$
$800\ \text{mm}^2$

Exercise 9.6.5.13

desk surface
$0.125\ \text{km}^2$
$0.125\ \text{m}^2$
$0.125\ \text{cm}^2$
$0.125\ \text{mm}^2$

For the following exercises, convert the given area to the units shown.

Exercise 9.6.5.14

$650{,}000\ \text{cm}^2 =$ __________ m^2

Exercise 9.6.5.15

$17\ \text{m}^2 =$ __________ cm^2

Exercise 9.6.5.16

$759\ \text{cm}^2 =$ __________ mm^2

Exercise 9.6.5.17

A laptop is 30 cm by 20 cm. A sticker on the laptop cover measures 15 mm by 15 mm. How many square centimeters of the laptop cover is still visible?

Exercise 9.6.5.18

A dining room wall is 2.75 m by 2.25 m. A window on the wall is 0.75 m by 1.5 m. How many square meters is the surface of the wall?

Exercise 9.6.5.19

A floor is 6 m by 4 m. A rug on the floor measures 200 cm by 100 cm. How much of the floor, in m^2, is not covered by the rug?

Exercise 9.6.5.20

A ceiling is 4 m by 2.5 m. A skylight in the ceiling measures 75 cm by 80 cm. How much of the ceiling, in m^2, would need to be painted if you decide to change the color of the ceiling?

Measuring Volume

For the following exercises, determine the most reasonable value for each volume.

Exercise 9.6.5.21

aquarium
65,750 km^3
65,750 m^3
65,750 cm^3
65,750 mm^3

Exercise 9.6.5.22

carton of soup
94,000 L
94 L
940,000 mL
940 mL

Exercise 9.6.5.23

gift box
6,250 km^3
6,250 m^3
6,250 cm^3
6,250 mm^3

For the following exercises, convert the given volume to the units shown.

Exercise 9.6.5.24

59,837 mm^3 = __________ cm^3

Exercise 9.6.5.25

12.5 dm^3 = __________ mL

Exercise 9.6.5.26

368 cm^3 = __________ mm^3

Exercise 9.6.5.27

A box has dimensions of $15\text{ cm} \times 10\text{ cm} \times 25\text{ cm}$. The box currently holds 1,750 cm^3 of black beans. How many cubic centimeters of black beans are needed to completely fill the box?

? Exercise 9.6.5.28

The dimensions of a small storage unit are $3\text{ m}\times 3\text{m}\times 3\text{ m}$. What is the volume of a medium storage area with dimensions twice the size of the small unit?

? Exercise 9.6.5.29

How much liquid, in liters, can a container with dimensions of $35\text{ cm}\times 15\text{ cm}\times 20\text{ cm}$ hold?

? Exercise 9.6.5.30

A box is 20 cm long and 15 cm wide. The volume of the box is 2,100 cm^3. What is the height of the box?

Measuring Weight

? Exercise 9.6.5.31

What is the most reasonable estimate for the weight of a small stack of coins?
120 mg
12 g
1.2 kg
0.12 mt

For the following exercises, convert the given weight to the units shown.

? Exercise 9.6.5.32

7,900 g = __________ kg

? Exercise 9.6.5.33

81 g = __________ mg

? Exercise 9.6.5.34

0.12 kg = __________ g

For the following exercises, determine the total weight in the units shown.

? Exercise 9.6.5.35

four 450 mg tablets ________ g

? Exercise 9.6.5.36

six 45 g candy bars ________ kg

? Exercise 9.6.5.37

seven 1.5 g buttons ________ mg

? Exercise 9.6.5.38

A box of chocolate weighs 2.75 kg. If you share the candy bar equally with 5 friends, how many grams of chocolate does each person get?

? Exercise 9.6.5.39

You purchase 15 tomatoes that weigh 100 grams each. If tomatoes cost $1.18 per kilogram, how much did you pay?

? Exercise 9.6.5.40

A value-size package of ground chicken costs $23.00. The package weighs 2.5 kg. What is the cost per gram of the chicken?

Measuring Temperature

For the following exercises, convert the temperatures to the nearest degree.

? Exercise 9.6.5.41

365 °F = __________ °C

? Exercise 9.6.5.42

42 °C = __________ °F

? Exercise 9.6.5.43

575 °F = __________ °C

? Exercise 9.6.5.44

135 °C = __________ °F

? Exercise 9.6.5.45

–20 °F = __________ °C

? Exercise 9.6.5.46

73 °C = __________ °F

? Exercise 9.6.5.47

The temperature of a welder’s torch increases 1,112 ºF every 3 minutes. About how many minutes does it take for the torch to increase 1,200 ºC?

? Exercise 9.6.5.48

A substance is heated to 302 ºF. As the substance cools the temperature drops 4 ºC every 3 minutes. How many minutes until the temperature of the water reaches 130 ºC?

? Exercise 9.6.5.49

On November 1 the temperature was 40 °F. On December 1 the temperature was 7 °C. Which day was cooler?

? Exercise 9.6.5.50

The instructions to cook lasagna say to set the oven at 475 °F. To the nearest degree, what is the temperature in degrees Celsius?

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9.6.6: Chapter Test

Exercise 9.6.6.1

A student consumed 10 cans of their favorite soda last week. If each can contained 39 g of sugar, how many milligrams of sugar did they consume?

Exercise 9.6.6.2

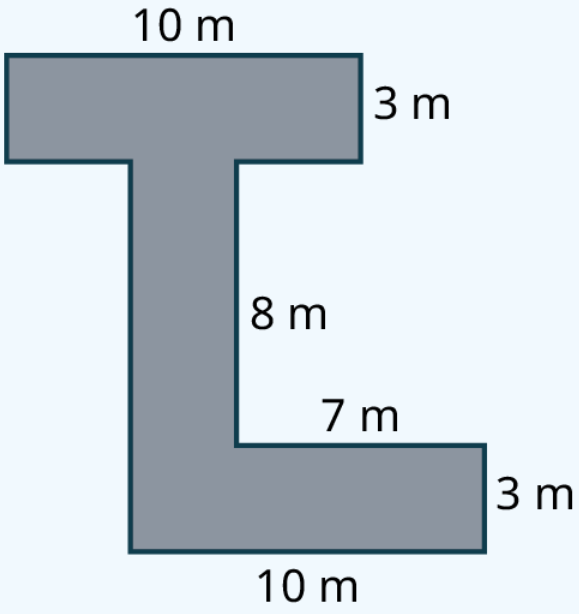
An athlete ran a total distance of 154 km in one month while training for a 5K fun run. How many meters did they run?

Exercise 9.6.6.3

A doctor recommends their patient drinks 5 liters of water on days of vigorous exercise. If the patient drinks 3,400 mL, how much more do they need to drink?

Exercise 9.6.6.4

A garden walkway has the dimensions shown. How many square meters of brick are needed to cover the walkway?



Exercise 9.6.6.5

A studio apartment has a length of 1,200 cm and a width that one half of the length. What is the area of the apartment in meters?

Exercise 9.6.6.6

A living room is 7 m long and 8 m wide. Carpet costs $9.50 per square meter. How much will it cost to carpet the room?

Exercise 9.6.6.7

Arif mixed two cartons of orange juice that each contain 600 cL, two bottles of cranberry juice that each contain 1,250 mL, and one 2-liter bottle of soda water to make punch for a party. How much punch, in liters, did Arif make?

Exercise 9.6.6.8

A fish tank has the dimensions $40\,\mathrm{cm} \times 20\,\mathrm{cm} \times 25\,\mathrm{cm}$. What is the capacity of the fish tank in liters?

Exercise 9.6.6.9

A recipe for chili makes 5.4 L of chili. If a restaurant serves chili in 300 mL bowls, how many bowls of chili can they serve?

Exercise 9.6.6.10

If a person weighs 76 kg but loses 3 kilograms, how much does the person weigh in grams?

Exercise 9.6.6.11

A car weighs approximately 2 metric tons (mt). If a used car dealer has 222,000 kg of cars in their lot, how many cars are in the lot?

Exercise 9.6.6.12

A restaurant has 126 kg of rice. If each serving contains 125 g, how many servings will they make?

Exercise 9.6.6.13

A boiling pan of water is 212 °F. As the water cools the temperature drops 12 °C every 3 minutes. How many minutes until the temperature of the water reaches 28 °C?

Exercise 9.6.6.14

On May 1 the temperature was 75 °F. On June 1 the temperature was 28 °C. Which day was warmer?

Exercise 9.6.6.15

For each log placed on a fire, the temperature increases 45 °C. How many logs are needed for the campfire to increase 315 °C?

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CHAPTER OVERVIEW

10: Geometry

- 10.0: Introduction
- 10.1: Points, Lines, and Planes
  - 10.1.0: Exercises
- 10.2: Angles
  - 10.2.0: Exercises
- 10.3: Triangles
  - 10.3.0: Exercises
- 10.4: Polygons, Perimeter, and Circumference
  - 10.4.0: Exercises
- 10.5: Tessellations
  - 10.5.0: Exercises
- 10.6: Area
  - 10.6.0: Exercises
- 10.7: Volume and Surface Area
  - 10.7.0: Exercises
- 10.8: Right Triangle Trigonometry
  - 10.8.0: Exercises
- 10.9: Chapter Summary
  - 10.9.0: Key Terms
  - 10.9.1: Key Concepts
  - 10.9.2: Videos
  - 10.9.3: Formula Review
  - 10.9.4: Projects
  - 10.9.5: Chapter Review
  - 10.9.6: Chapter Test

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10.0: Introduction

The painting *The School of Athens* presents great figures in history such as Plato, Aristotle, Socrates, Euclid, Archimedes, and Pythagoras. Other scientists are also represented in the painting.

Figure 10.0.1 *The School of Athens* by the Renaissance artist Raphael, painted between 1509 and 1511, depicts some of the greatest minds of ancient times. (credit: modification of work "School of Athens" by Raphael (1483–1520), Vatican Museums/Wikimedia, Public Domain)

To the ancient Greeks, the study of mathematics meant the study of geometry above all other subjects. The Greeks looked for the beauty in geometry and did not allow their geometrical constructions to be "polluted" by the use of anything as practical as a ruler. They permitted the use of only two tools—a compass for drawing circles and arcs, and an unmarked straightedge to draw line segments. They would mark off units as needed. However, they never could be sure of what the units meant. For instance, how long is an inch? These mathematicians defined many concepts.

The Greeks absorbed much from the Egyptians and the Babylonians (around 3000 BCE), including knowledge about congruence and similarity, area and volume, angles and triangles, and made it their task to introduce proofs for everything they learned. All of this historical wisdom culminated with Euclid in 300 BCE.

Euclid (325–265 BC) is known as the father of geometry, and his most famous work is the 13-volume collection known as *The Elements*, which are said to be "the most studied books apart from the Bible." Euclid brought together everything offered by the Babylonians, the Egyptians, and the more refined contributions by the Greeks, and set out, successfully, to organize and prove these concepts as he methodically developed formal theorems.

This chapter begins with a discussion of the most basic geometric tools: the point, the line, and the plane. All other topics flow from there. Throughout the eight sections, we will talk about how to determine angle measurement and learn how to recognize properties of special angles, such as right angles and supplementary angles. We will look at the relationship of angles formed by a transversal, a line running through a set of parallel lines. We will explore the concepts of area and perimeter, surface area and volume, and transformational geometry as used in the patterns and rigid motions of tessellations. Finally, we will introduce right-angle trigonometry and explore the Pythagorean Theorem.

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10.1: Points, Lines, and Planes

Learning Objectives

1. Identify and describe points, lines, and planes.
2. Express points and lines using proper notation.
3. Determine union and intersection of sets.

In this section, we will begin our exploration of geometry by looking at the basic definitions as defined by Euclid. These definitions form the foundation of the geometric theories that are applied in everyday life.

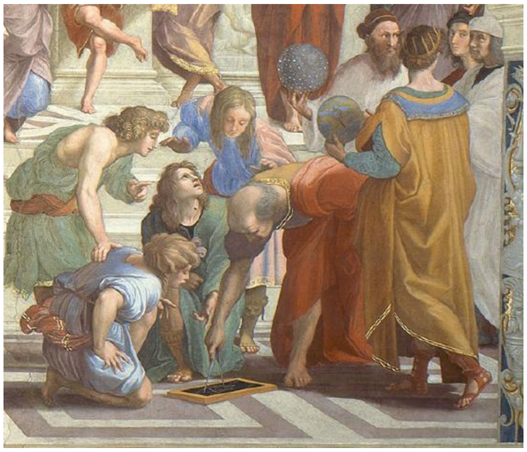


Figure 10.1.1 : The lower right-hand corner of *The School of Athens* depicts a figure representing Euclid illustrating to students how to use a compass on a small chalkboard. (credit: modification of work "School of Athens" by Raphael (1483–1520), Vatican Museums/Wikimedia, Public Domain)

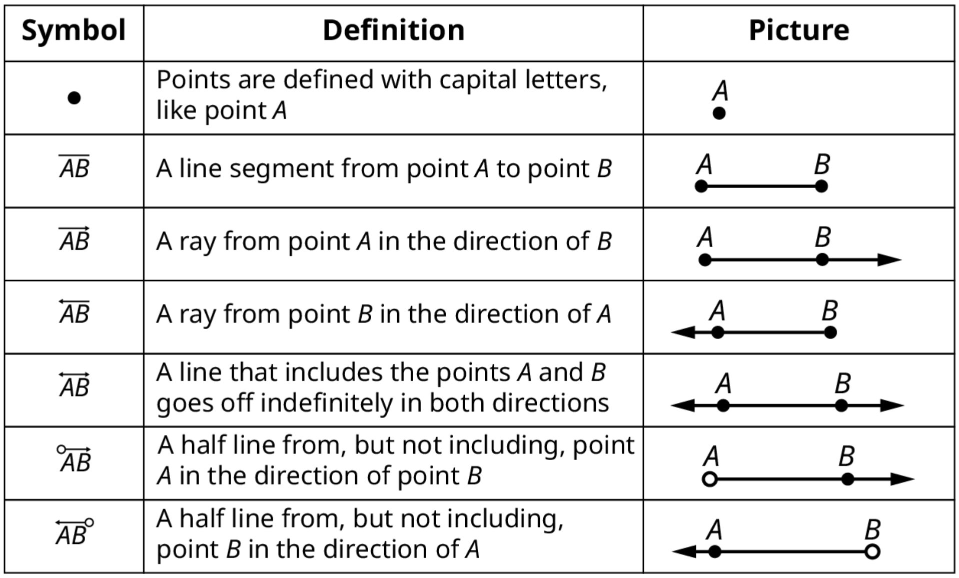
In *The Elements*, Euclid summarized the geometric principles discovered earlier and created an axiomatic system, a system composed of postulates. A *postulate* is another term for axiom, which is a statement that is accepted as truth without the need for proof or verification. There were no formal geometric definitions before Euclid, and when terms could not be defined, they could be described. In order to write his postulates, Euclid had to describe the terms he needed and he called the descriptions "definitions." Ultimately, we will work with theorems, which are statements that have been proved and can be proved.

Points and Lines

The first definition Euclid wrote was that of a point. He defined a point as "that which has no part." It was later expanded to "an indivisible location which has no width, length, or breadth." Here are the first two of the five postulates, as they are applicable to this first topic:

1. Postulate 1: A straight line segment can be drawn joining any two points.
2. Postulate 2: Any straight line segment can be extended indefinitely in a straight line.

Before we go further, we will define some of the symbols used in geometry in Figure 10.1.2



Symbol	Definition	Picture
●	Points are defined with capital letters, like point *A*	A
$\overline{AB}$	A line segment from point *A* to point *B*	A B
$\overrightarrow{AB}$	A ray from point *A* in the direction of *B*	A B
$\overleftarrow{AB}$	A ray from point *B* in the direction of *A*	A B
$\overleftrightarrow{AB}$	A line that includes the points *A* and *B* goes off indefinitely in both directions	A B
$\overset{\circ\rightarrow}{AB}$	A half line from, but not including, point *A* in the direction of point *B*	A B
$\overset{\leftarrow\circ}{AB}$	A half line from, but not including, point *B* in the direction of *A*	A B

Figure 10.1.2 : Basic Geometric Symbols for Points and Lines

From Figure 10.3, we see the variations in lines, such as line segments, rays, or half-lines. What is consistent is that two collinear points (points that lie on the same line) are required to form a line. Notice that a line segment is defined by its two endpoints showing that there is a definite beginning and end to a line segment. A ray is defined by two points on the line; the first point is where the ray begins, and the second point gives the line direction. A half-line is defined by two points, one where the line starts and the other to give direction, but an open circle at the starting point indicates that the starting point is not part of the half-line. A regular line is defined by any two points on the line and extends infinitely in both directions. Regular lines are typically drawn with arrows on each end.

✓ Example 10.1.1: Defining Lines

For the following exercises, use this line (Figure 10.1.3).

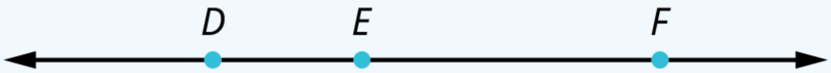


Figure 10.1.3

1. Define $\overline{DE}$.
2. Define F.
3. Define $\overleftrightarrow{DF}$.
4. Define $\overline{EF}$.

Answer

1. The symbol $\overline{DE}$, two letters with a straight line above, refers to the line segment that starts at point D and ends at point E.
2. The letter F alone refers to point F.
3. The symbol $\overleftrightarrow{DF}$, two letters with a line above containing arrows on both ends, refers to the line that extends infinitely in both directions and contains the points D and F.
4. The symbol $\overline{EF}$, two letters with a straight line above, refers to the line segment that starts at point E and ends at point F.

Your Turn 10.1.1

For the following exercises, use this line.

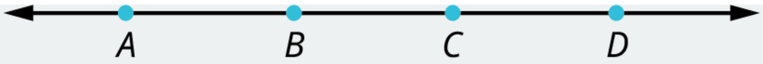


Figure 10.1.4

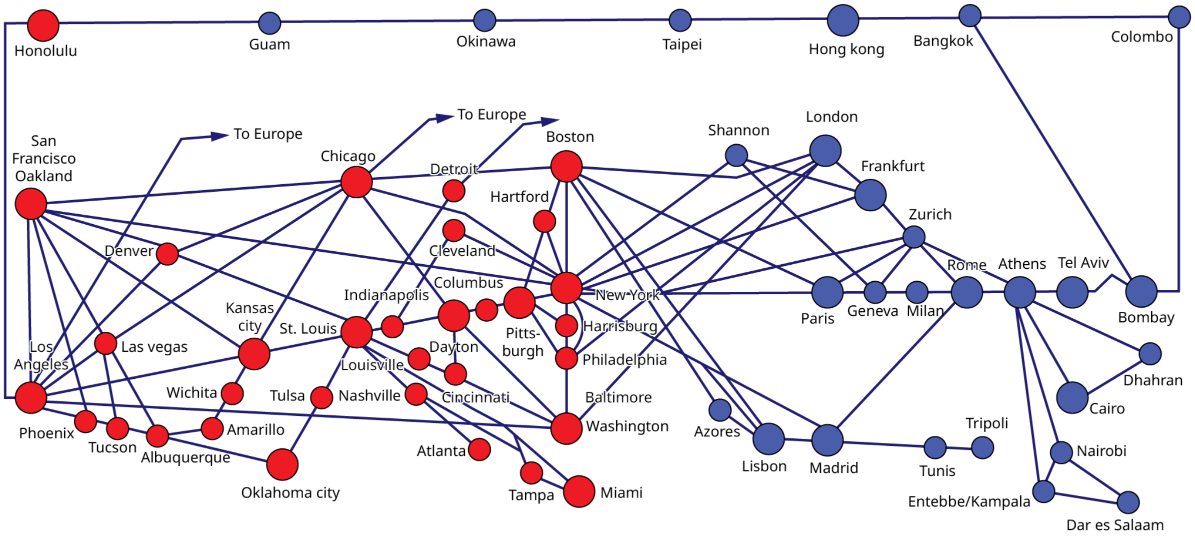
Define $\overrightarrow{BD}$

Define $\overline{AB}$

$\overleftarrow{BA}$

$\overleftrightarrow{AD}$

There are numerous applications of line segments in daily life. For example, airlines working out routes between cities, where each city's airport is a point, and the points are connected by line segments. Another example is a city map. Think about the intersection of roads, such that the center of each intersection is a point, and the points are connected by line segments representing the roads. See Figure 10.1.5.


Honolulu
Guam
Okinawa
Taipei
Hong kong
Bangkok
Colombo
To Europe
To Europe
San Francisco Oakland
Chicago
Detroit
Boston
Shannon
London
Frankfurt
Hartford
Zurich
Denver
Cleveland
Columbus
New York
Rome
Athens
Tel Aviv
Paris
Geneva
Milan
Bombay
Kansas city
St. Louis
Indianapolis
Pitts-burgh
Harrisburg
Los Angeles
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Dayton
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Dhahran
Wichita
Tulsa
Nashville
Cincinnati
Baltimore
Cairo
Tripoli
Phoenix
Tucson
Albuquerque
Amarillo
Atlanta
Washington
Azores
Lisbon
Madrid
Tunis
Nairobi
Oklahoma city
Tampa
Miami
Entebbe/Kampala
Dar es Salaam

Figure 10.1.5: Air Line Routes

Example 10.1.2 : Determining the Best Route

View the street map (Figure 10.1.6) as a series of line segments from point to point. For example, we have vertical line segments $\overline{AB}$, $\overline{BC}$, and $\overline{CD}$ on the right. On the left side of the map, we have vertical line segments $\overline{HI}$, $\overline{FG}$. The horizontal line segments are $\overline{HA}$, $\overline{EI}$, $\overline{IE}$, $\overline{EB}$, $\overline{FC}$, $\overline{CF}$, and $\overline{GB}$. There are two diagonal line segments, $\overline{AE}$ and $\overline{EF}$. Assume that each location is on a corner and that you live next door to the library.

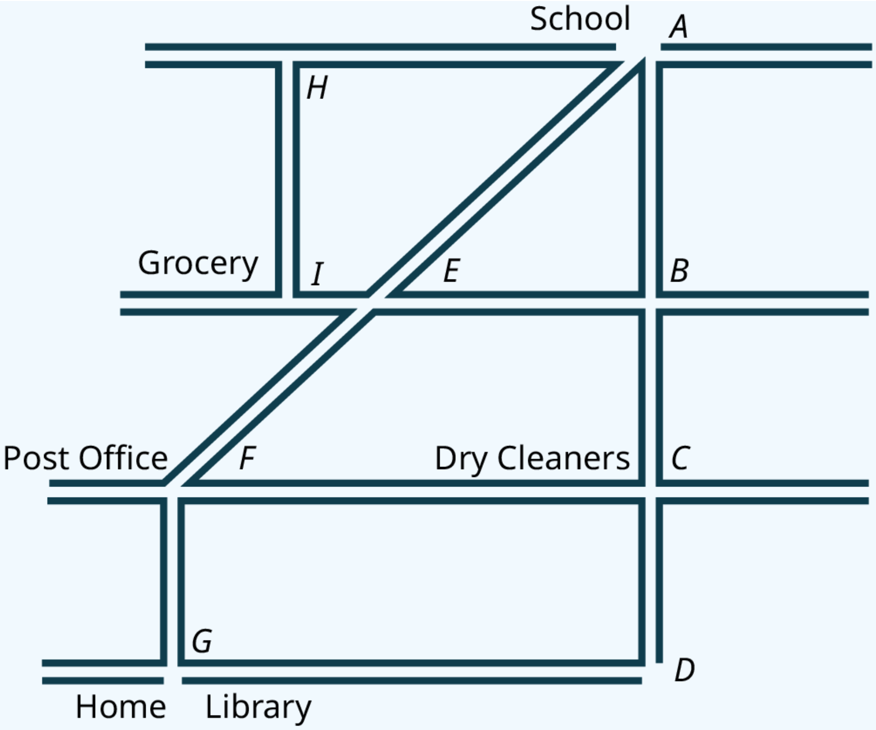


Figure 10.1.6 Street Map

1. Let's say that you want to stop at the grocery store on your way home from school. Come up with three routes you might take to do your errand and then go home. In other words, name the three ways by the line segments in the order you would walk, and which way do you think would be the most efficient route?
2. How about stopping at the library after school? Name four ways you might travel to the library and which way do you think is the most efficient?
3. Suppose you need to go to the post office and the dry cleaners on your way home from school. Name three ways you might walk to do your errands and end up at home. Which way do you think is the most efficient way to walk, get your errands done, and go home?

Answer

1. From school to the grocery store and home: first way $\overline{AH}, \overline{HI}, \overline{IE}, \overline{EF}, \overline{FG}$; second way $\overline{AB}, \overline{BE}, \overline{EI}, \overline{IE}, \overline{EF}, \overline{FG}$; third way $\overline{AE}, \overline{EI}, \overline{IE}, \overline{EF}, \overline{FG}$. It seems that the third way is the most efficient way.
2. From school to the library: first way $\overline{AE}, \overline{EF}, \overline{FG}$; second way $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DG}$; third way $\overline{AB}, \overline{BE}, \overline{EF}, \overline{FG}$; fourth way $\overline{AB}, \overline{BC}, \overline{CF}, \overline{FG}$. The first way should be the most efficient way.
3. From school to the post office or dry cleaners to home: first way $\overline{AE}, \overline{EF}, \overline{FC}, \overline{CD}, \overline{DG}$; second way $\overline{AB}, \overline{BE}, \overline{EF}, \overline{FC}, \overline{CD}, \overline{DG}$; third way $\overline{AB}, \overline{BC}, \overline{CF}, \overline{FG}$. The third way would be the most efficient way.

Your Turn 10.1.2

Using the street map in Figure 10.1.6, find two ways you would stop at the dry cleaners and the grocery store after school on your way home.

Parallel Lines

Parallel lines are lines that lie in the same plane and move in the same direction, but never intersect. To indicate that the line semanticsl1 Figure 10.1.7.

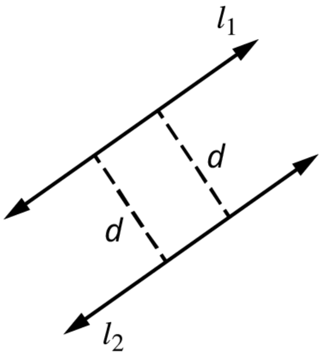


Figure $10.1.7$: Parallel Lines

Perpendicular Lines

Two lines that intersect at a 90°

see Figure $10.1.8$.

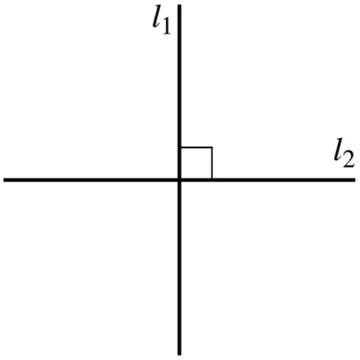


Figure $10.1.8$: Perpendicular Lines

✓ Example ✓ $10.1.3$ ✓ : Identifying Parallel and Perpendicular Lines

Identify the sets of parallel and perpendicular lines in Figure $10.1.9$.

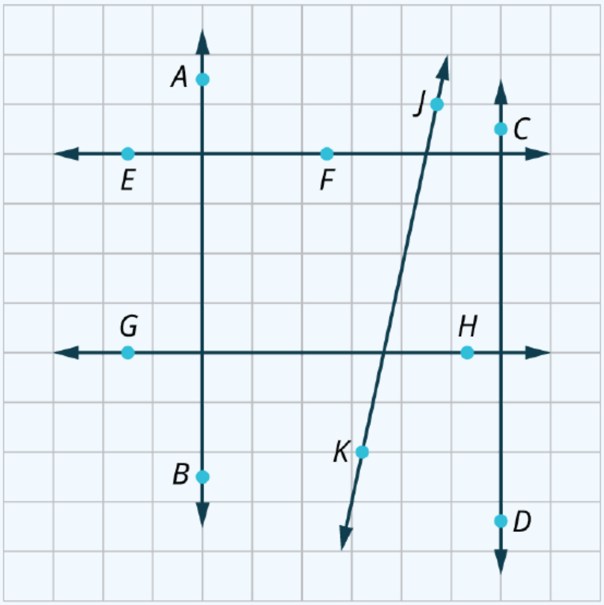


Figure $10.1.9$

Answer

Drawing these lines on a grid is the best way to distinguish which pairs of lines are parallel and which are perpendicular. Because they are on a grid, we assume all lines are equally spaced across the grid horizontally and vertically. The grid also tells us that the vertical lines are parallel and the horizontal lines are parallel. Additionally, all intersections form a 90° angle. Therefore, we can safely say the following:

$\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, the line containing the points A and B is parallel to the line containing the points C and D.

$\overleftrightarrow{EF} \parallel \overleftrightarrow{GH}$, the line containing the points E and F is parallel to the line containing the points G and H.

$\overleftrightarrow{AB} \perp \overleftrightarrow{EF}$, the line containing the points A and B is perpendicular to the line containing the points E and F. We know this because both lines trace grid lines, and intersecting grid lines are perpendicular.

We can also state that $\overleftrightarrow{AB} \perp \overleftrightarrow{GH}$; the line containing the points A and B is perpendicular to the line containing the points G and H because both lines trace grid lines, which are perpendicular by definition.

We also have $\overleftrightarrow{CD} \perp \overleftrightarrow{EF}$; the line containing the points C and D is perpendicular to the line containing the points E and F because both lines trace grid lines, which are perpendicular by definition.

Finally, we see that $\overleftrightarrow{CD} \perp \overleftrightarrow{GH}$; the line containing the points C and D is perpendicular to the line containing the points G and H because both lines trace grid lines, which are perpendicular by definition.

Your Turn 10.1.3

Identify the sets of parallel and perpendicular lines in the given figure.

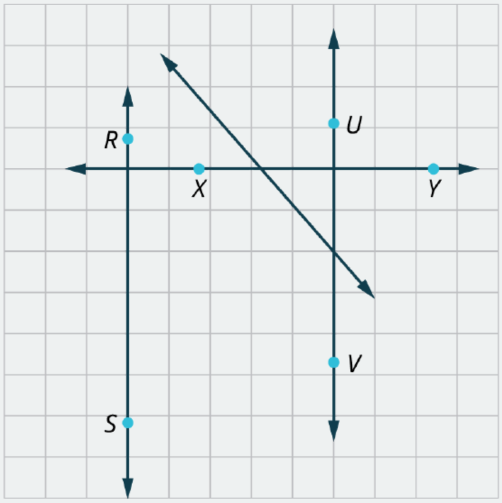


Figure 10.1.10

Defining Union and Intersection of Sets

Union and intersection of sets is a topic from set theory that is often associated with points and lines. So, it seems appropriate to introduce a mini-version of set theory here. First, a **set** is a collection of objects joined by some common criteria. We usually name sets with capital letters. For example, the set of odd integers between 0 and 10 looks like this: $A = \{1, 3, 5, 7, 9\}$. When it involves sets of lines, line segments, or points, we are usually referring to the **union** or **intersection** of set.

The union of two or more sets contains all the elements in either one of the sets or elements in all the sets referenced, and is written by placing this symbol $\cup$ in between each of the sets. For example, let set $A = \{1, 2, 3\}$, and let set $B = \{4, 5, 6\}$. Then, the union of sets A and B is $A \cup B = \{1, 2, 3, 4, 5, 6\}$.

The intersection of two or more sets contains only the elements that are common to each set, and we place this symbol $\cap$ in between each of the sets referenced. For example, let's say that set $A = \{1, 3, 5\}$, and let set $B = \{5, 7, 9\}$. Then, the intersection of sets A and B is $A \cap B = \{5\}$.

✓ Example ✓ 10.1.4: Defining Union and Intersection of Sets

Use the line (Figure 10.1.11) for the following exercises. Draw each answer over the main drawing.

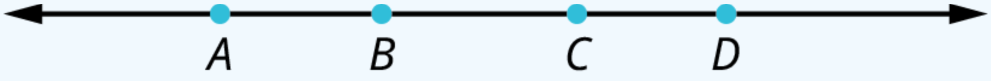


Figure 10.1.11

1. Find $\overrightarrow{BD} \cap \overleftarrow{CA}$.
2. Find $\overline{AB} \cup \overline{AD}$.
3. Find $\overleftrightarrow{AD} \cup \overline{BC}$.
4. Find $\overleftrightarrow{AD} \cap \overline{BC}$.
5. Find $\overleftarrow{BA} \cap \overline{CD}$.

Answer

1. Find $\overrightarrow{BD} \cap \overleftarrow{CA}$. This is the intersection of the ray $\overrightarrow{BD}$ and the ray $\overleftarrow{CA}$. Intersection includes only the elements that are common to both lines. For this intersection, only the line segment $\overline{BC}$ is common to both rays. Thus, $\overrightarrow{BD} \cap \overleftarrow{CA} = \overline{BC}$.

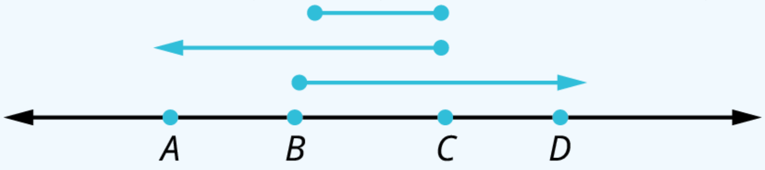


Figure 10.1.12

2. Find $\overline{AB} \cup \overline{AD}$. The problem is asking for the union of two line segments, $\overline{AB}$ and $\overline{AD}$. Union includes all elements in the first line and all elements in the second line. Since $\overline{AB}$ is part of $\overline{AD}$, $\overline{AB} \cup \overline{AD} = \overline{AD}$.

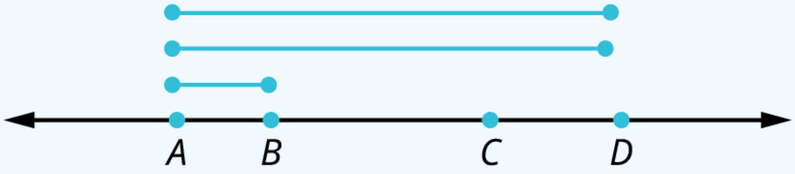


Figure 10.1.13

3. Find $\overleftrightarrow{AD} \cup \overline{BC}$. This is the union of the line $\overleftrightarrow{AD}$ with the line segment $\overline{BC}$. As the line segment $\overline{BC}$ is included on the line $\overleftrightarrow{AD}$, then the union of these two lines equals the line $\overleftrightarrow{AD}$.

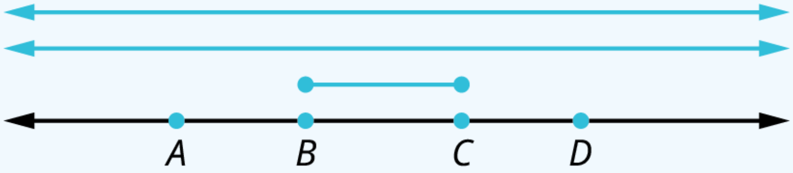


Figure 10.1.14

4. Find $\overleftrightarrow{AD} \cap \overline{BC}$. The intersection of the line $\overleftrightarrow{AD}$ with line segment $\overline{BC}$ is the set of elements common to both lines. In this case, the only element in common is the line segment $\overline{BC}$.

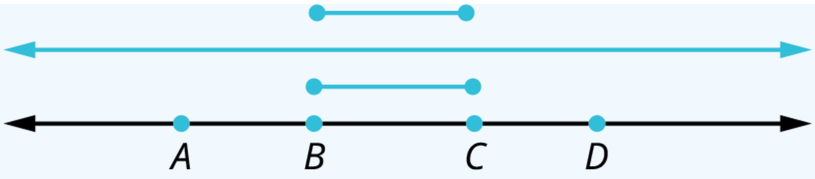


Figure 10.1.15

5. Find Math input error

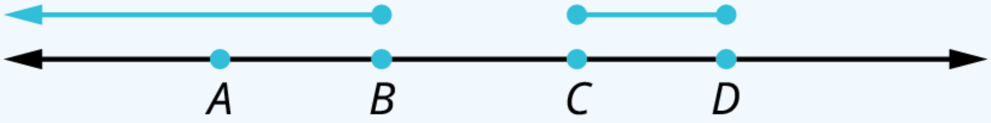


Figure 10.1.16

Your Turn 10.1.4

For the following exercises, use the line shown to identify and draw the union or intersection of sets.

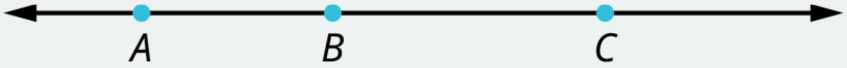


Figure 10.1.17

$\overline{AB} \cap \overline{BC}$

$\overrightarrow{BC} \cup \overleftarrow{CA}$

$\overline{BC} \cap \overline{AC}$

Planes

A **plane,** as defined by Euclid, is a "surface which lies evenly with the straight lines on itself." A plane is a two-dimensional surface with infinite length and width, and no thickness. We also identify a plane by three noncollinear points, or points that do not lie on the same line. Think of a piece of paper, but one that has infinite length, infinite width, and no thickness. However, not all planes must extend infinitely. Sometimes a plane has a limited area.

We usually label planes with a single capital letter, such as Plane semanticsPFigure 10.19, or by all points that determine the edges of a plane. In the following figure, Plane P contains points A and B, which are on the same line, and point C, which is not on that line. By definition, P is a plane. We can move laterally in any direction on a plane.

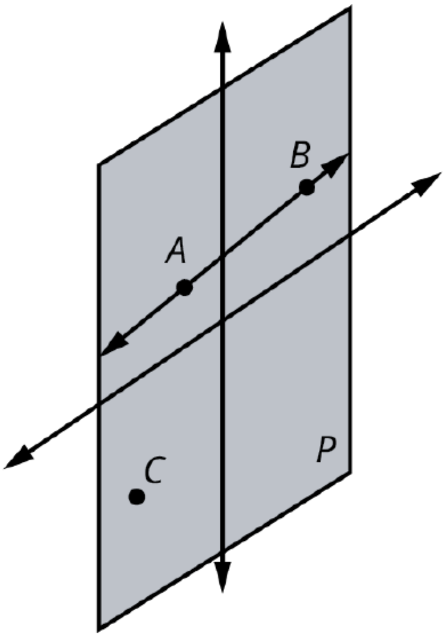


Figure 10.1.18: Plane P

One way to think of a plane is the Cartesian coordinate system with the semanticsxFigure 10.1.19.

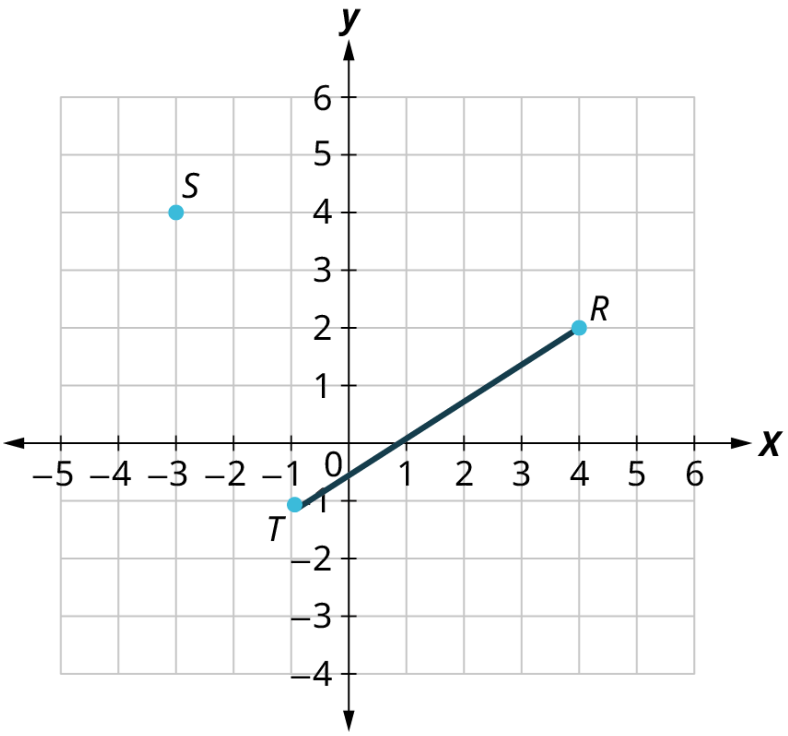


Figure 10.1.19: Cartesian Coordinate Plane

This plane contains points S, T, and R. Points T and R are colinear and form a line segment. Point S is not on that line segment. Therefore, this represents a plane.

To give the location of a point on the Cartesian plane, remember that the first number in the ordered pair is the horizontal move and the second number is the vertical move. Point R is located at $(4, 2)$; point S is located at $(-3, 4)$; and point T is located at $(-1, -1)$. We can also identify the line segment as $\overline{TR}$.

Two other concepts to note: Parallel planes do not intersect and the intersection of two planes is a straight line. The equation of that line of intersection is left to a study of three-dimensional space. See Figure 10.1.20.

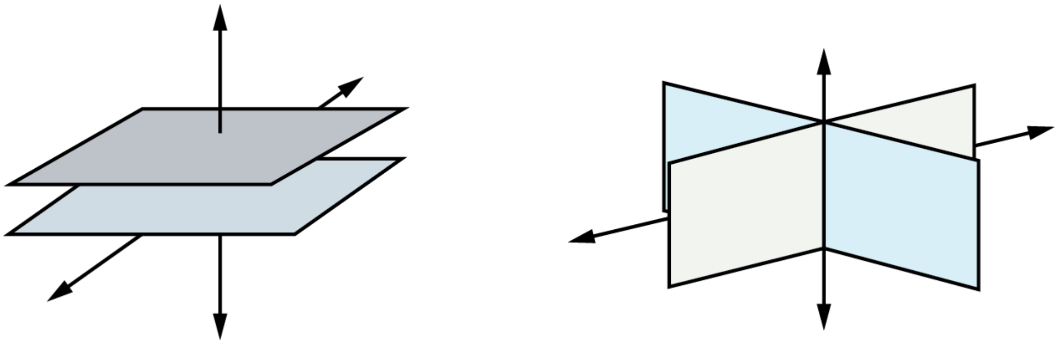


Figure 10.1.20: Parallel and Intersecting Planes

To summarize, some of the properties of planes include:

- Three points including at least one noncollinear point determine a plane.
- A line and a point not on the line determine a plane.
- The intersection of two distinct planes is a straight line.

Example 10.1.5: Identifying a Plane

For the following exercises, refer to Figure 10.1.21

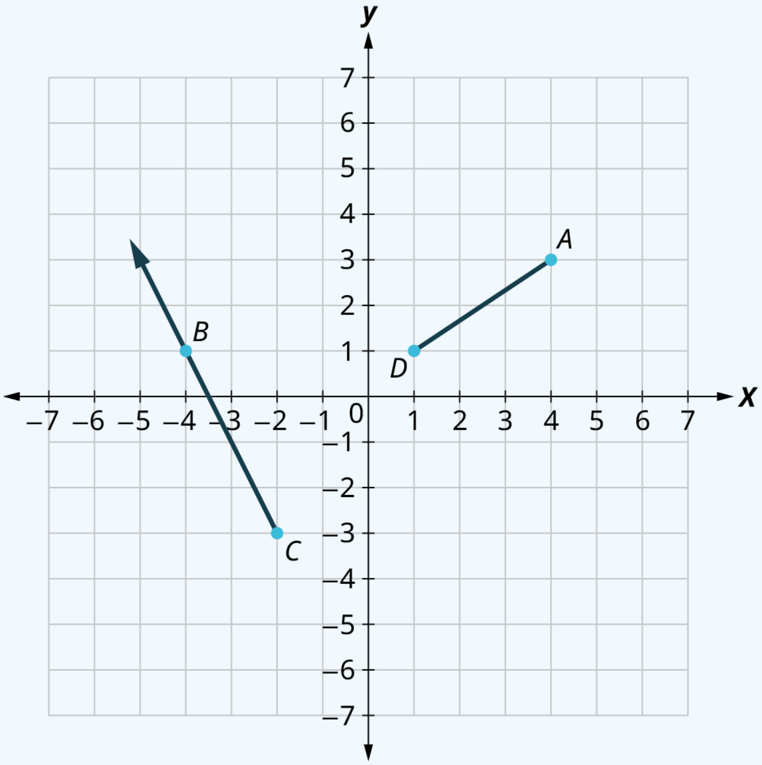


Figure 10.1.21

1. Identify the location of points A, B, C, and D.
2. Describe the line from point A to point D.
3. Describe the line from point C containing point B.
4. Does this figure represent a plane?

Answer

1. Point A is located at $(4, 3)$; point B is located at $(-4, 1)$; point C is located at $(-2, -3)$; point D is located at $(1, 1)$.
2. The line from point A to point D is a line segment $\overline{AD}$.

3. The line from point C containing point B is a ray $\overrightarrow{CB}$ starting at point C in the direction of B.
4. Yes, this figure represents a plane because it contains at least three points, points A and D form a line segment, and neither point B nor point C is on that line segment.

Your Turn 10.1.5

For the following exercises, refer to the given figure.

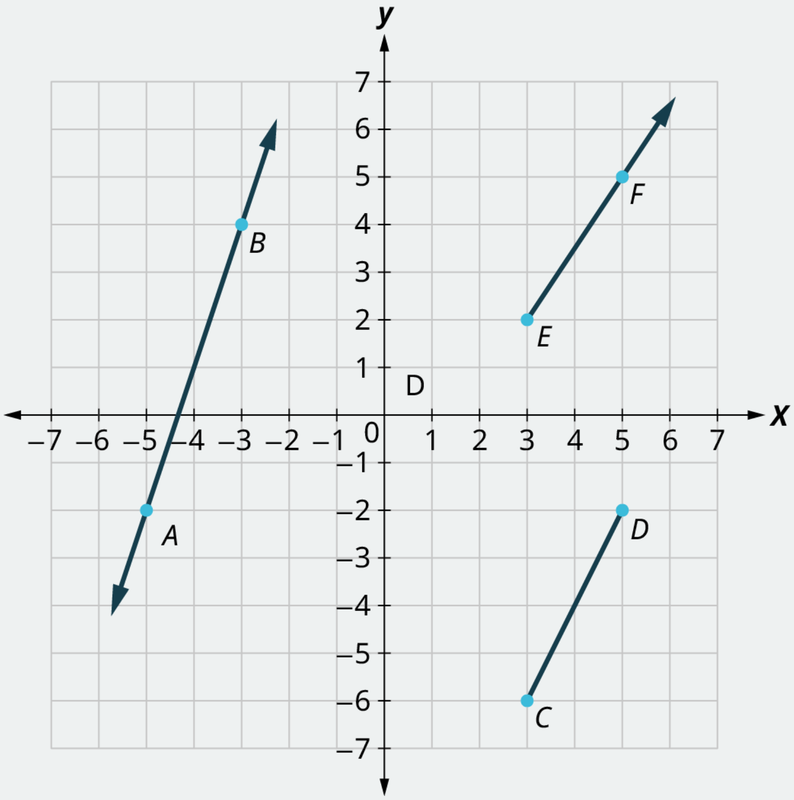


Figure 10.1.22

Identify the location of points A, B, C, D, E, and F.

Describe the line that includes point A and point B.

Describe the line from E to point F.

Describe the line from C to point D.

Does this figure represent a plane?

Example 10.1.6: Intersecting Planes

Name two pairs of intersecting planes on the shower enclosure illustration (Figure 10.1.23).

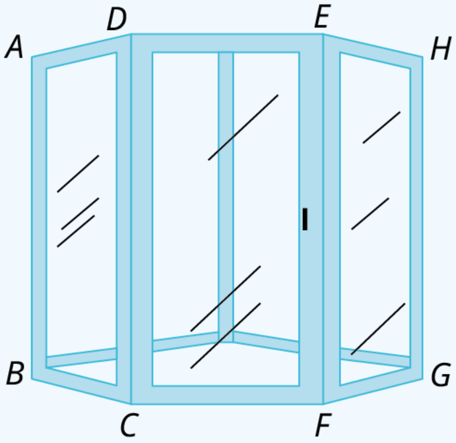


Figure
10.1.23

Answer

The plane semanticsABCDABCD intersects plane semanticsCDEFCDEF, and plane semanticsCDEFCDEF intersects plane semanticsEFGHEFGH.

Your Turn 10.1.6

Name two pairs of intersecting planes on the shower enclosure shown.

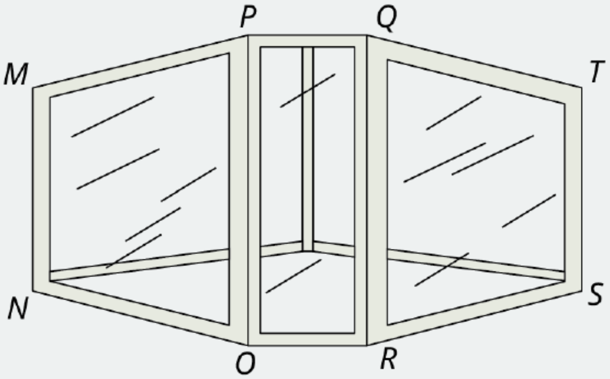


Figure
10.1.24

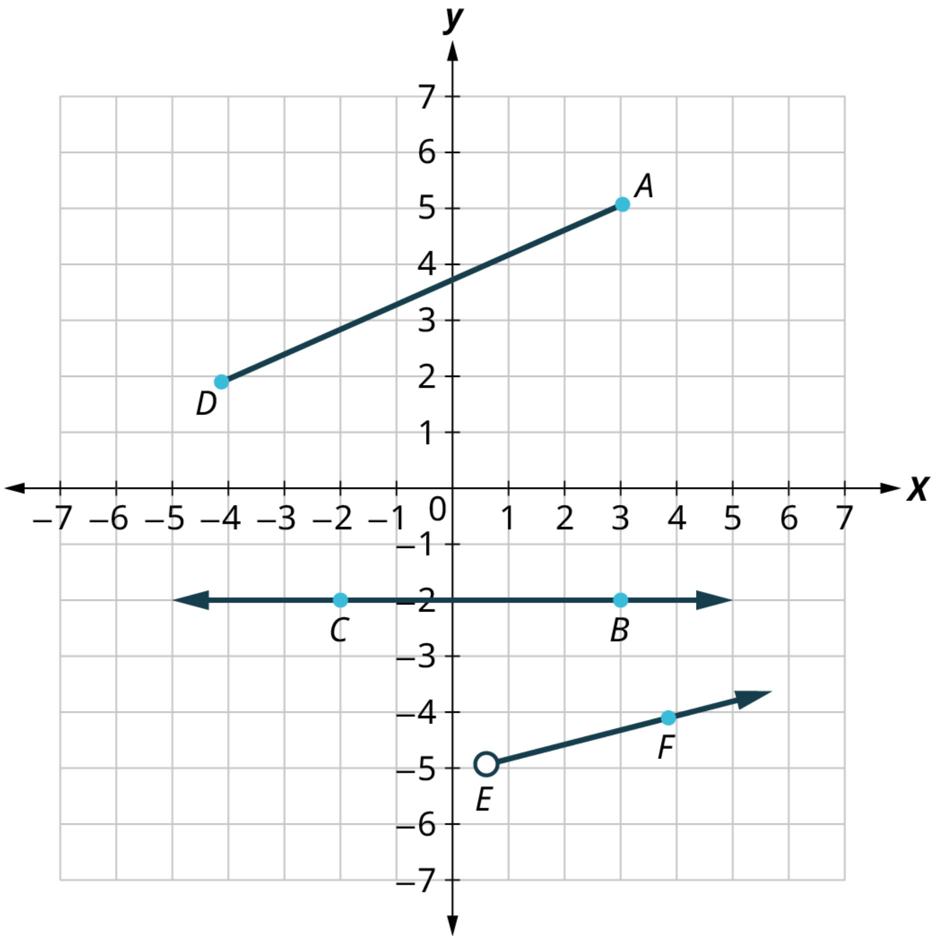
People in Mathematics: Plato

Part of a remarkable chain of Greek mathematicians, Plato (427–347 BC) is known as the teacher. He was responsible for shaping the development of Western thought perhaps more powerfully than anyone of his time. One of his greatest achievements was the founding of the Academy in Athens where he emphasized the study of geometry. Geometry was considered by the Greeks to be the “ultimate human endeavor.” Above the doorway to the Academy, an inscription read, “Let no one ignorant of geometry enter here.”

The curriculum of the Academy was a 15-year program. The students studied the exact sciences for the first 10 years. Plato believed that this was the necessary foundation for preparing students’ minds to study relationships that require abstract thinking. The next 5 years were devoted to the study of the “dialectic.” The dialectic is the art of question and answer. In Plato’s view, this skill was critical to the investigation and demonstration of innate mathematical truths. By training young students how to prove propositions and test hypotheses, he created a culture in which the systematic process was guaranteed. The Academy was essentially the world’s first university and held the reputation as the ultimate center of learning for more than 900 years.

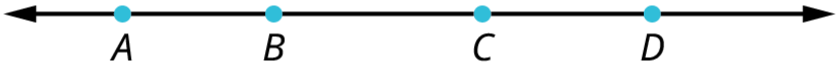
Check Your Understanding

Use the graph for the following exercises.

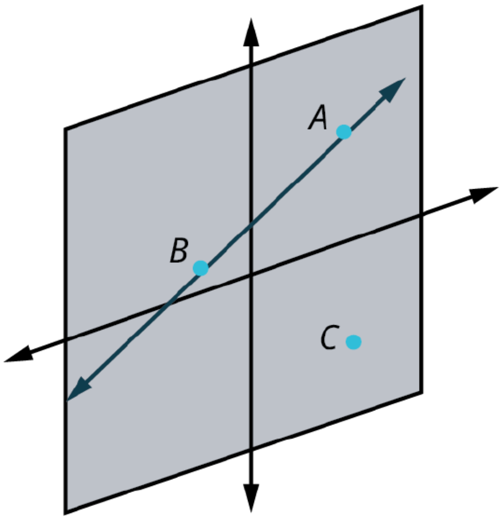


1. Identify the type of line containing point D and point A.
2. Identify the type of line containing points C and B.
3. Identify the type of line that references point E and contains point F.

Use the line for the following exercises.



4. Determine $\overline{AB} \cup \overline{BD}$.
5. Determine $\overleftarrow{BD} \cap \overline{BC}$.
6. Determine $\overleftarrow{BA} \cup \overrightarrow{BD}$.
7. How can you determine whether two lines are parallel?
8. How can you determine whether two lines are perpendicular?
9. Determine if the illustration represents a plane.



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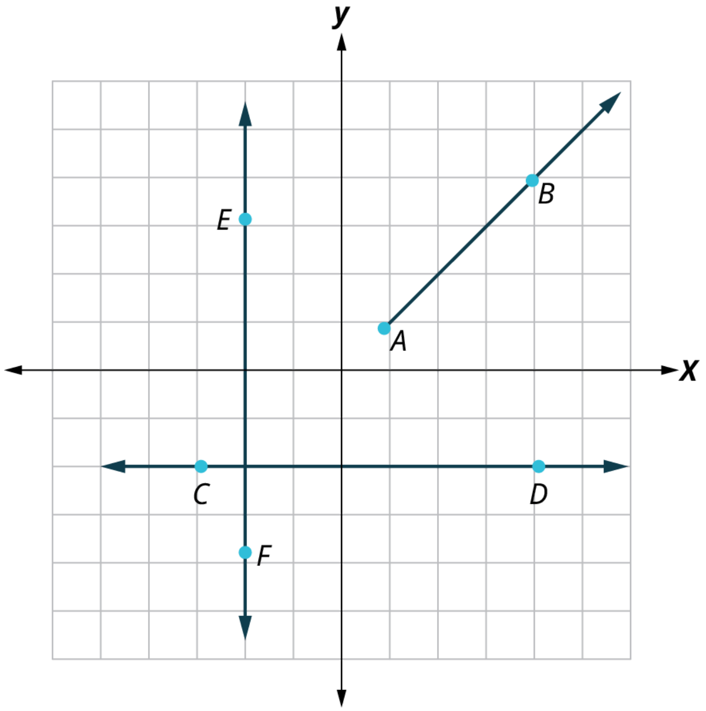
10.1.0: Exercises

Use this graph for the following exercises.



Exercise 10.1.0.1

Identify the kind of line passing through the points A and B.

Exercise 10.1.0.2

Identify the kind of line passing through the points C and D.

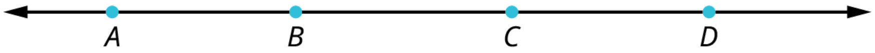
Exercise 10.1.0.3

Identify all sets of parallel lines.

Exercise 10.1.0.4

Identify all sets of perpendicular lines.

Use the line for the following exercises.



Exercise 10.1.0.5

Find $\overleftarrow{BA} \cup \overline{BC}$.

Exercise 10.1.0.6

Find $\overrightarrow{AD} \cup \overrightarrow{CD}$.

Exercise 10.1.0.7

Find $\overline{AC} \cap \overline{BD}$.

Exercise 10.1.0.8

Find $\overleftarrow{CA} \cap \overrightarrow{BD}$

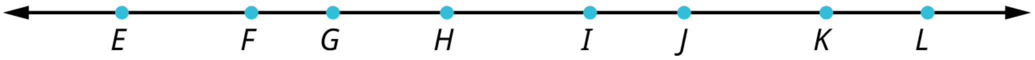
Exercise 10.1.0.9

Find $\overline{AB} \cup \overline{BD}$

Exercise 10.1.0.10

Find $\overleftrightarrow{BC} \cup \overline{BD}$

For the following exercises, reference this line.



Exercise 10.1.0.11

Name the points in the set $\overleftrightarrow{FH} \cap \overline{GH}$.

Exercise 10.1.0.12

Name the points contained in the set $\overrightarrow{HL} \cap \overline{JL}$.

Exercise 10.1.0.13

Name the points in the set $\overleftrightarrow{EL} \cup \overline{GH}$.

Exercise 10.1.0.14

Name the points in the set $\overleftarrow{GE} \cup \overrightarrow{KL}$.

Exercise 10.1.0.15

Name the points in the set $\overleftarrow{GE} \cap \overrightarrow{KL}$.

Exercise 10.1.0.16

Name the points in the set $\overrightarrow{HK} \cap \overleftarrow{JF}$.

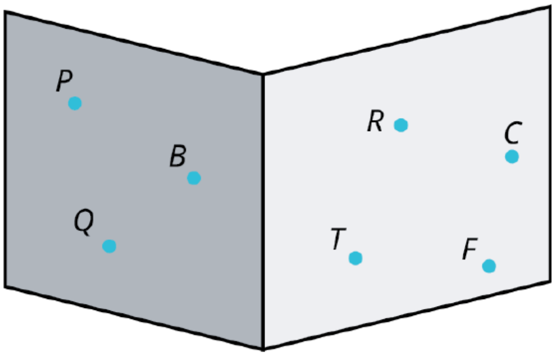
? Exercise 10.1.0.17

Name the points in the set $\overrightarrow{GL} \cap K$.

? Exercise 10.1.0.18

Name the points in the set $\overleftrightarrow{HE} \cup \overrightarrow{KL}$.

Use the given figure for the following exercises.



? Exercise 10.1.0.19

Describe the lighter plane (right) in the drawing.

? Exercise 10.1.0.20

Describe the darker plane (left) in the drawing.

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10.2: Angles

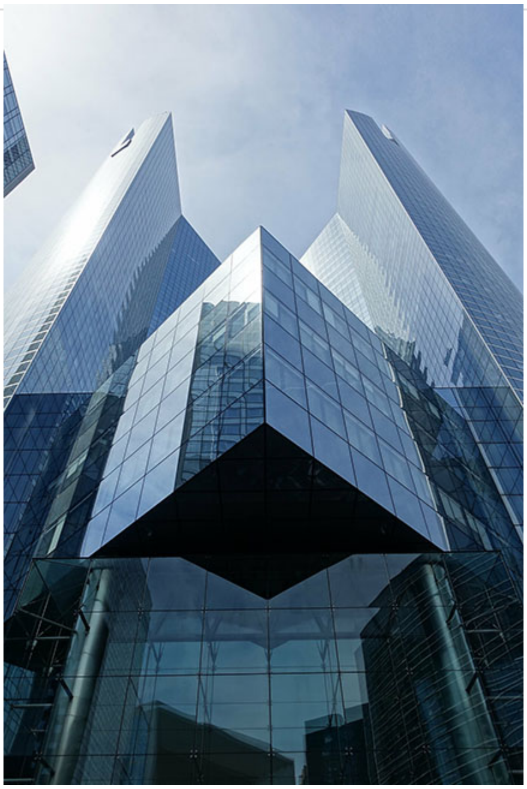


Figure 10.2.1: This modern architectural design emphasizes sharp reflective angles as part of the aesthetic through the use of glass walls. (credit: “Société Générale @ La Défense @ Paris” by Images Guilhem Vellut/Flickr, CC BY 2.0)

Learning Objectives

1. Identify and express angles using proper notation.
2. Classify angles by their measurement.
3. Solve application problems involving angles.
4. Compute angles formed by transversals to parallel lines.
5. Solve application problems involving angles formed by parallel lines.

Unusual perspectives on architecture can reveal some extremely creative images. For example, aerial views of cities reveal some exciting and unexpected angles. Add reflections on glass or steel, lighting, and impressive textures, and the structure is a work of art. Understanding angles is critical to many fields, including engineering, architecture, landscaping, space planning, and so on. This is the topic of this section.

We begin our study of angles with a description of how angles are formed and how they are classified. An angle is the joining of two rays, which sweep out as the sides of the angle, with a common endpoint. The common endpoint is called the vertex. We will often need to refer to more than one vertex, so you will want to know the plural of vertex, which is vertices.

In Figure 10.2.2 , let the ray $\overrightarrow{AB}$ stay put. Rotate the second ray $\overrightarrow{AC}$ in a counterclockwise direction to the size of the angle you want. The angle is formed by the amount of rotation of the second ray. When the ray $\overrightarrow{AC}$ continues to rotate in a counterclockwise direction back to its original position coinciding with ray $\overrightarrow{AB}$, the ray will have swept out 360°. We call the rays the “sides” of the angle.

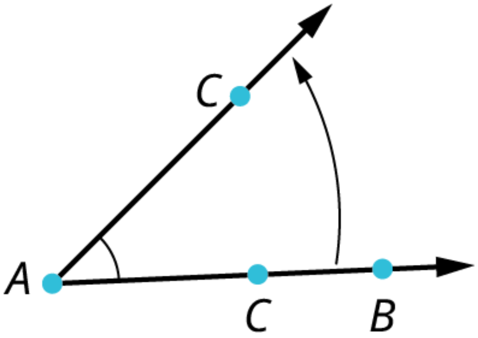


Figure 10.2.2: Vertex and Sides of an Angle

Classifying Angles

Angles are measured in radians or degrees. For example, an angle that measures π radians, or 3.14159 radians, is equal to the angle measuring 180°. An angle measuring $\frac{\pi}{2}$ radians, or 1.570796 radians, measures 90°. To translate degrees to radians, we multiply the angle measure in degrees by $\frac{\pi}{180}$. For example, to write 45° in radians, we have

$$45^\circ\left(\frac{\pi}{180}\right)=\frac{\pi}{4}=0.785398 \text{ radians.}$$

To translate radians to degrees, we multiply by $\frac{180}{\pi}$. For example, to write 2π radians in degrees, we have

$$2\pi\left(\frac{180}{\pi}\right)=360^\circ$$

Another example of translating radians to degrees and degrees to radians is $\frac{2\pi}{3}$. To write in degrees, we have $\frac{2\pi}{3}\left(\frac{180}{\pi}\right)=120^\circ$. To write 30° in radians, we have $30^\circ\left(\frac{\pi}{180}\right)=\frac{\pi}{6}$. However, we will use degrees throughout this chapter.

FORMULA

To translate an angle measured in degrees to radians, multiply by $\frac{\pi}{180}$.

To translate an angle measured in radians to degrees, multiply by $\frac{180}{\pi}$.

Several angles are referred to so often that they have been given special names. A straight angle measures 180°; a right angle measures 90°; an acute angle is any angle whose measure is less than 90°; and an obtuse angle is any angle whose measure is between 90° and 180°. See Figure 10.2.3.msup

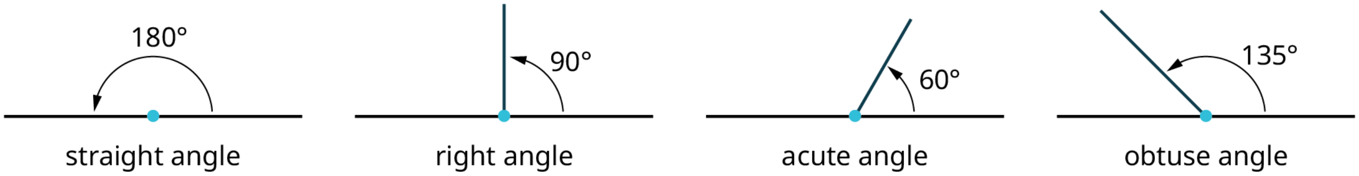


Figure 10.2.3: Classifying and Naming Angles

An easy way to measure angles is with a protractor (Figure 10.2.4). A protractor is a very handy little tool, usually made of transparent plastic, like the one shown here.

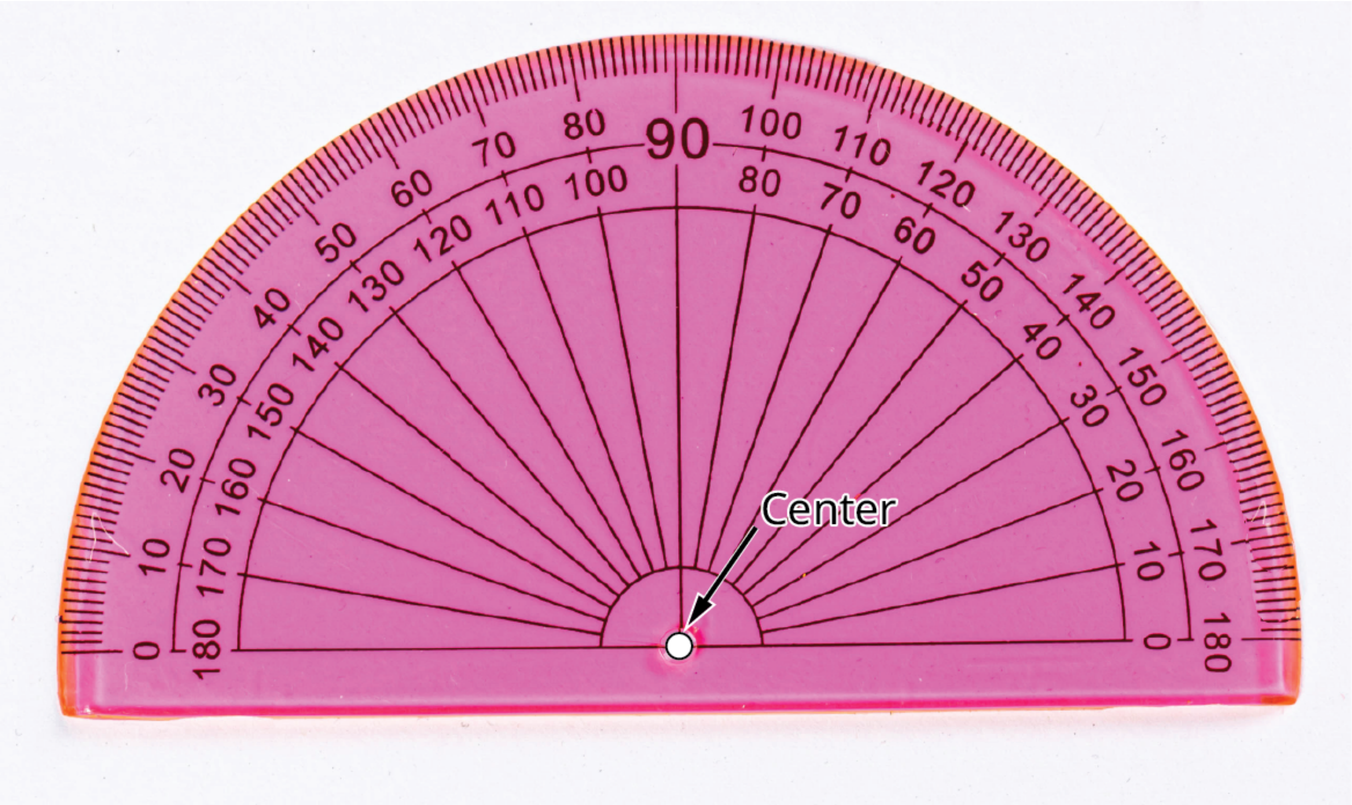


Figure 10.2.4: Protractor (credit: modification of work “School drawing tools” by Marco Verch/Flickr, CC BY 2.0)

With a protractor, you line up the straight bottom with the horizontal straight line of the angle. Be sure to have the center hole lined up with the vertex of the angle. Then, look for the mark on the protractor where the second ray lines up. As you can see from the image, the degrees are marked off. Where the second ray lines up is the measurement of the angle.

Checkpoint

Make sure you correctly match the center mark of the protractor with the vertex of the angle to be measured. Otherwise, you will not get the correct measurement. Also, keep the protractor in a vertical position.

Notation

Naming angles can be done in couple of ways. We can name the angle by three points, one point on each of the sides and the vertex point in the middle, or we can name it by the vertex point alone. Also, we can use the symbols $\angle$ or $\measuredangle$ before the points. When we are referring to the measure of the angle, we use the symbol $m\measuredangle$. See Figure 10.2.5.

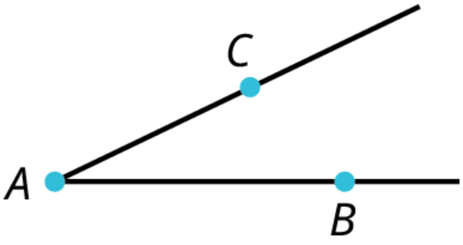


Figure 10.2.5: Naming an Angle

We can name this angle $\measuredangle BAC$, or $\measuredangle CAB$, or $\measuredangle A$.

✓ Example 10.2.1: Classifying Angles

Determine which angles are acute, right, obtuse, or straight on the graph (Figure ✓ 10.2.6). You may want to use a protractor for this one.

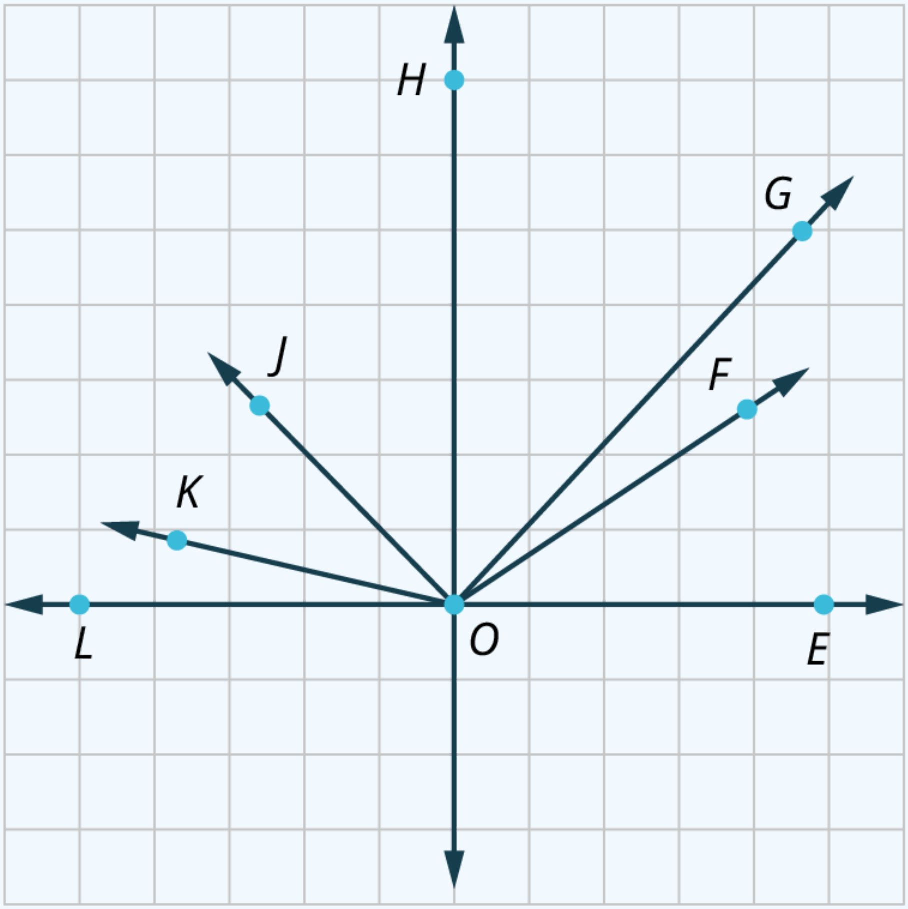


Figure ✓ Runaway argument for \eatSpaces?

Answer

Acute angles measure less than 90°.	Obtuse angles measure between 90° and 180°.	Right angles measure 90°.	Straight angles measure 180°.
∠EOF ∠EOG ∠FOG ∠GOH ∠FOH ∠HOJ ∠HOK ∠JOK ∠KOL ∠JOL	∠EOJ ∠EOK ∠FOJ ∠FOK ∠FOL ∠GOK ∠GOL	∠EOH ∠HOL ∠GOJ	∠EOL

Most angles can be classified visually or by description. However, if you are unsure, use a protractor.

Your Turn 10.2.1

Determine which angles are acute, obtuse, right, and straight in the graph.

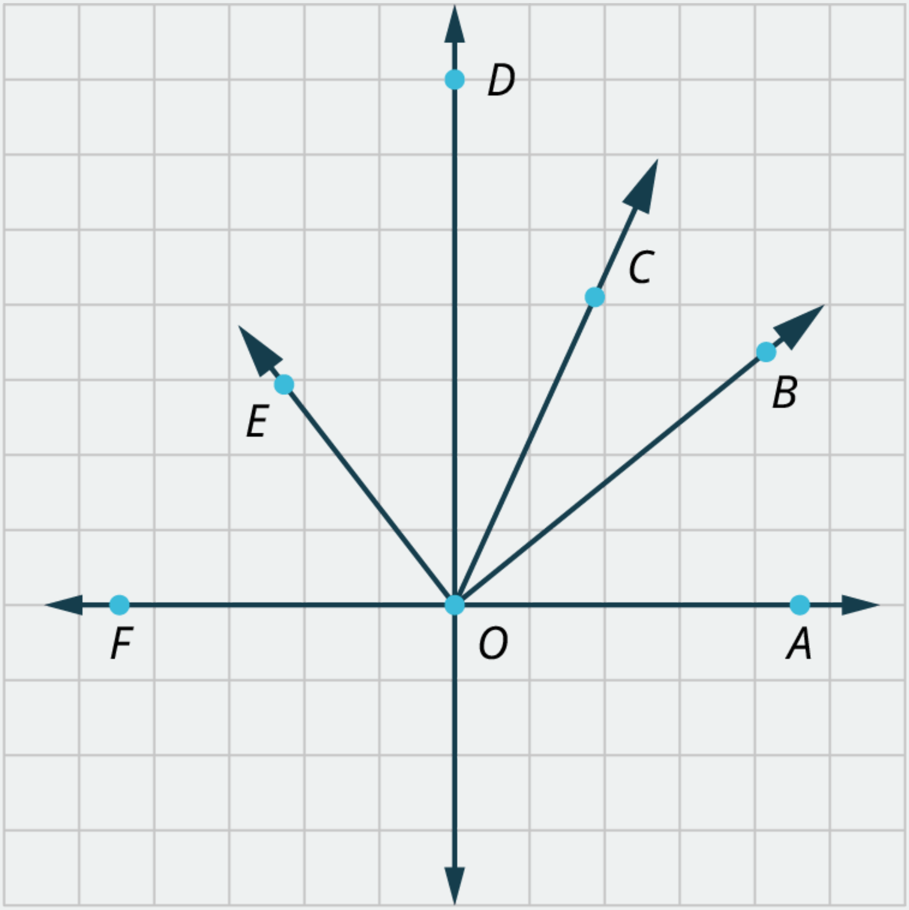


Figure 10.2.7

Adjacent Angles

Two angles with the same starting point or vertex and one common side are called adjacent angles. In Figure 10.2.8, angle $\angle DBC$ is adjacent to $\angle CBA$. Notice that the way we designate an angle is with a point on each of its two sides and the vertex in the middle.

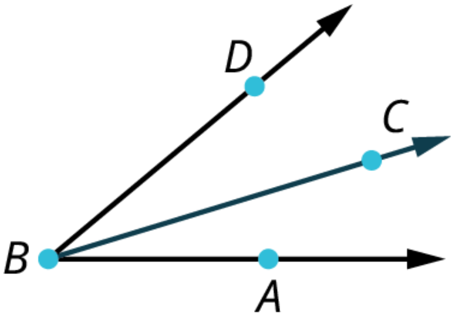


Figure 10.2.8: Adjacent Angles

Supplementary Angles

TTwo angles are supplementary if the sum of their measures equals 180°. In Figure m∡ABE=145∘.

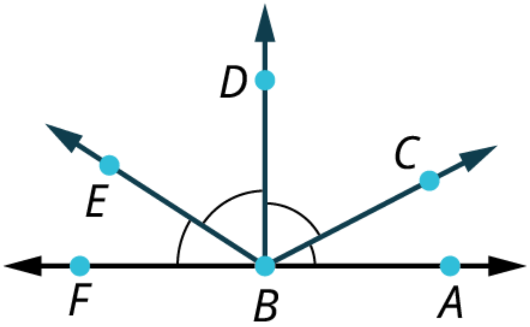


Figure 10.2.9: Supplementary Angles

✓ Example 10.2.2: Solving for Angle Measurements and Supplementary Angles

Solve for the angle measurements in Figure ✓ 10.2.10.

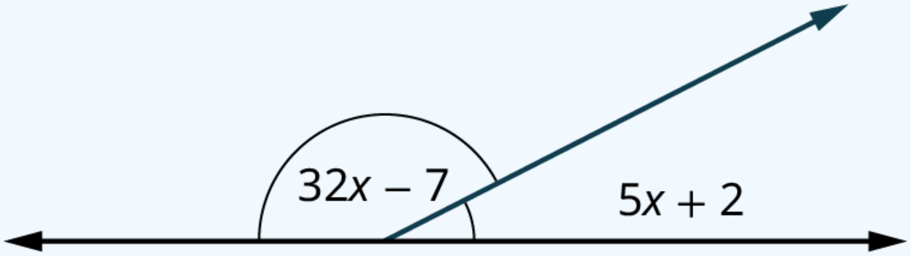


Figure ✓ 10.2.10

Answer

Step 1: These are supplementary angles. We can see this because the two angles are part of a horizontal line, and a horizontal line represents 180°. Therefore, the sum of the two angles equals 180°.

Step 2:

$$\begin{aligned}(32x-7)+(5x+2)&=180\\37x-5&=180\\37x&=185\\x&=5\end{aligned}$$

Step 3: Find the measure of each angle:

$$\begin{aligned}32x-7&=32(5)-7\\&=153^\circ\\5x+2&=5(5)+2\\&=27^\circ\end{aligned}$$

Step 4: We check: $153^\circ+27^\circ=180^\circ$.

Your Turn 10.2.2

Solve for the angle measurements in the figure shown.

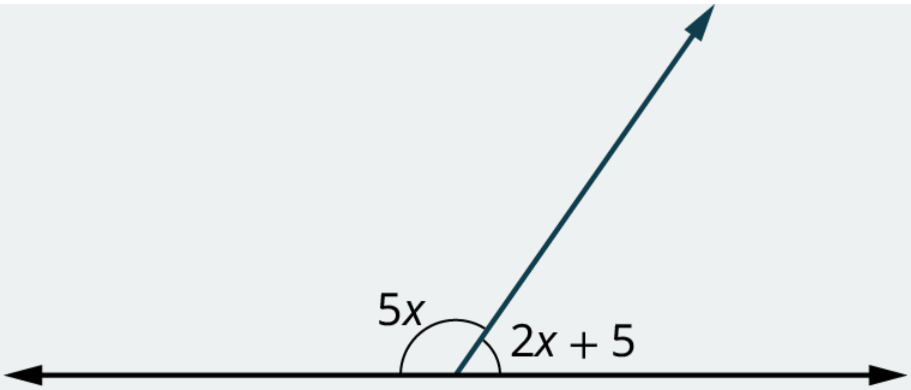


Figure 📌 10.2.11

Complementary Angles

Two angles are complementary if the sum of their measures equals 90°. In Figure we have $m\measuredangle ABC = 30^\circ$, and $m\measuredangle ABD = 90^\circ$. What is the $m\measuredangle CBD$? These are complementary angles. Therefore, because $90^\circ - 30^\circ = 60^\circ$, the $\measuredangle CBD = 60^\circ$.

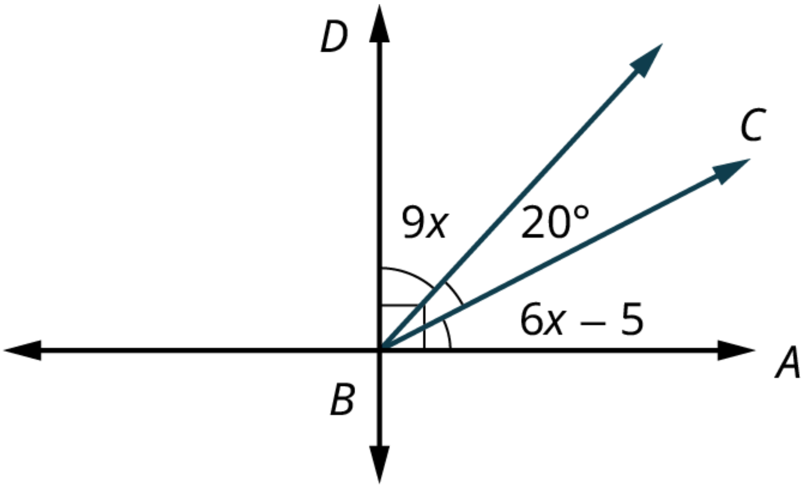


Figure 10.2.12: Complementary Angles

✓ Example 10.2.3: Solving for Angle Measurements and Complementary Angles

Solve for the angle measurements in Figure ✓ 10.2.13

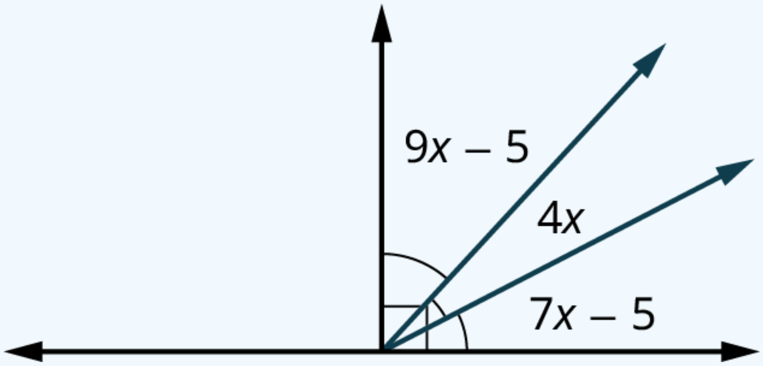


Figure ✓ 10.2.13

Answer

We have that

$$(9x - 5) + 4x + (7x - 5) = 90$$
$$20x = 100$$

$$20x = 100$$
$$x = 5$$

Then, $m\measuredangle(9x-5) = 40^{\circ}, m\measuredangle(4x) = 20^{\circ}$, and $m\measuredangle(7x-5) = 30^{\circ}$.

Your Turn $10.2.3$

Find the measure of each angle in the illustration.

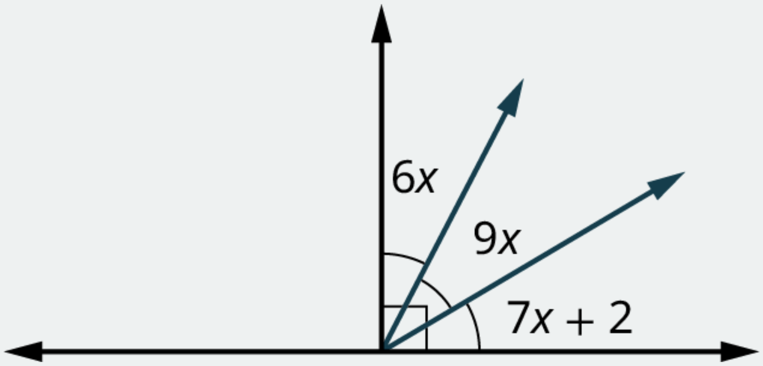


Figure $10.2.14$

Vertical Angles

When two lines intersect, the opposite angles are called vertical angles, and vertical angles have equal measure. For example, Figure $10.2.15$ shows two straight lines intersecting each other. One set of opposite angles shows angle markers; those angles have the same measure. The other two opposite angles have the same measure as well.

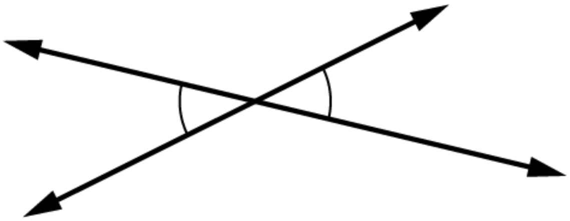


Figure $10.2.15$: Vertical Angles

Example $10.2.4$: Calculating Vertical Angles

In Figure $10.2.16$, one angle measures 40°. Find the measures of the remaining angles.

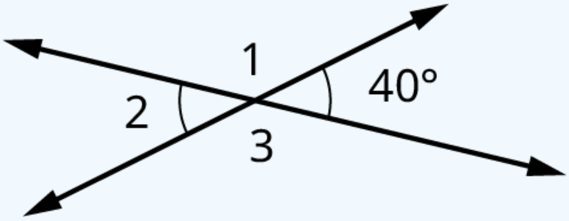


Figure $10.2.16$

Answer

The 40-degree angle and $\angle 2$ are vertical angles. Therefore, $m\measuredangle 2 = 40^{\circ}$.

Notice that $\measuredangle 2$ and $\measuredangle 1$ are supplementary angles, meaning that the sum of $m\measuredangle 2$ and $m\measuredangle 1$ equals 180°. Therefore, $m\measuredangle 1 = 180^{\circ} - 40^{\circ} = 140^{\circ}$.

Since $\measuredangle 1$ and $\measuredangle 3$ are vertical angles, then $m\measuredangle 3$ equals 140°.

Your Turn 10.2.4

Given the two intersecting lines in the figure shown and $m\measuredangle 2 = 67^\circ$, find the measure of the remaining angles.

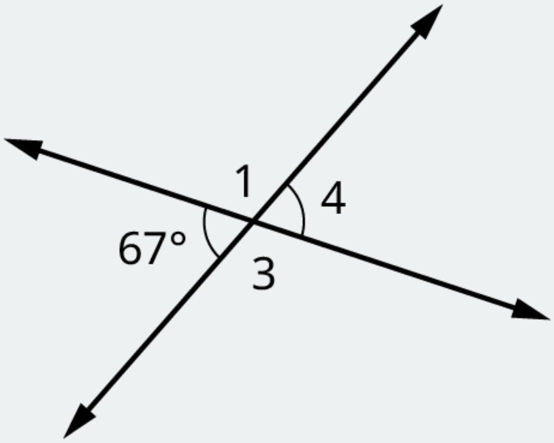


Figure 10.2.17

Transversals

When two parallel lines are crossed by a straight line or **transversal**, eight angles are formed, including alternate interior angles, alternate exterior angles, corresponding angles, vertical angles, and supplementary angles. See Figure 10.2.18. Angles 1, 2, 7, and 8 are called exterior angles, and angles 3, 4, 5, and 6 are called interior angles.

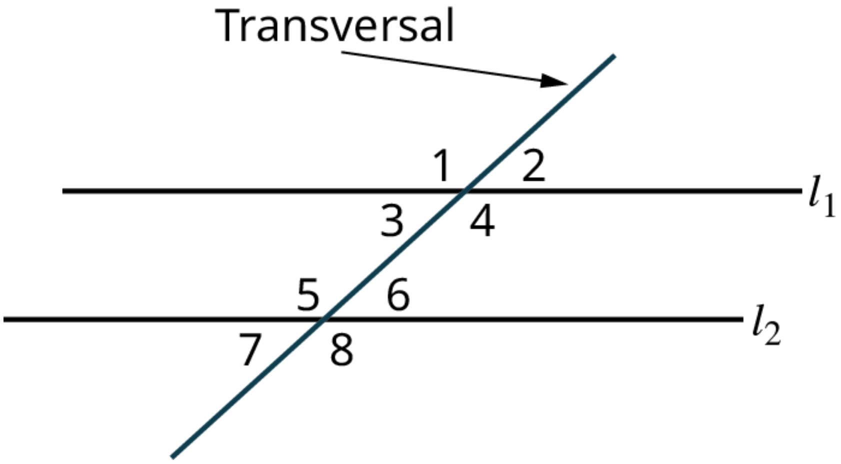


Figure 10.2.18: Transversal

Alternate Interior Angles

Alternate interior angles are the interior angles on opposite sides of the transversal. These two angles have the same measure. For example, $\measuredangle 3$ and $\measuredangle 6$ are alternate interior angles and have equal measure; $\measuredangle 4$ and $\measuredangle 5$ are alternate interior angles and have equal measure as well. See Figure 10.2.19.

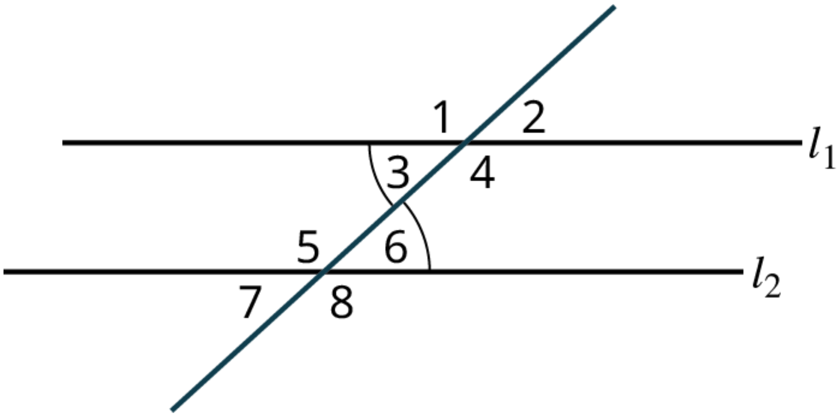


Figure 10.2.19: Alternate Interior Angles

Alternate Exterior Angles

Alternate exterior angles are exterior angles on opposite sides of the transversal and have the same measure. For example, in Figure 10.2.20, $\measuredangle 2$ and $\measuredangle 7$ are alternate exterior angles and have equal measures; $\measuredangle 1$ and $\measuredangle 8$ are alternate exterior angles and have equal measures as well.

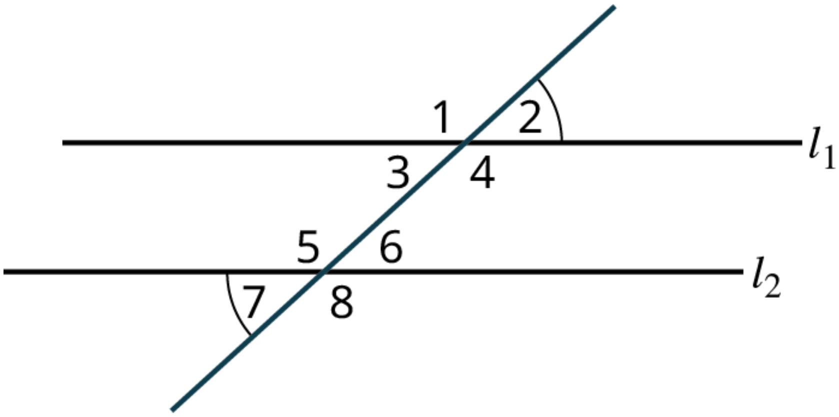


Figure 10.2.20: Alternate Exterior Angles

Corresponding Angles

Corresponding angles refer to one exterior angle and one interior angle on the same side as the transversal, which have equal measures. In Figure 10.2.21, $\measuredangle 1$ and $\measuredangle 5$ are corresponding angles and have equal measures; $\measuredangle 3$ and $\measuredangle 7$ are corresponding angles and have equal measures; $\measuredangle 2$ and $\measuredangle 6$ are corresponding angles and have equal measures; $\measuredangle 4$ and $\measuredangle 8$ are corresponding angles and have equal measures as well.

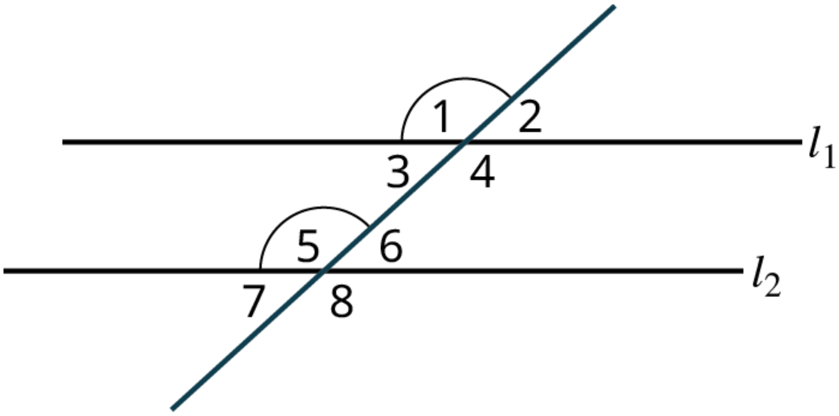


Figure 10.2.21: Corresponding Angles

✓ Example 10.2.5: Evaluating Space

You live on the corner of First Avenue and Linton Street. You want to plant a garden in the far corner of your property (Figure ✓ 10.2.22) and fence off the area. However, the corner of your property does not form the traditional right angle. You learned from the city that the streets cross at an angle equal to 150°. What is the measure of the angle that will border your garden?

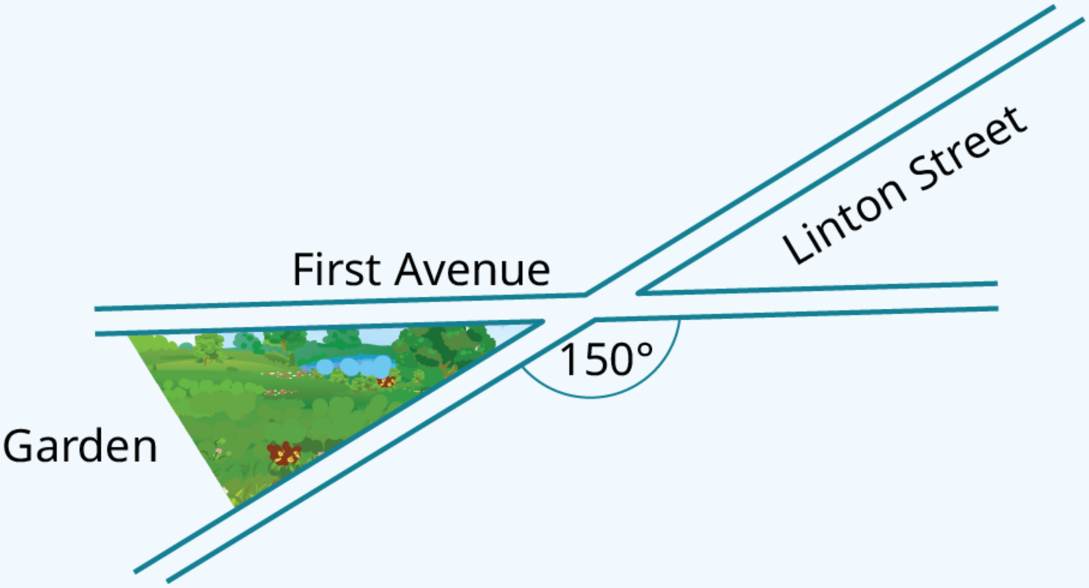


Figure ✓ 10.2.22

Answer

As the angle between Linton Street and First Avenue is 150°, the supplementary angle is 30°. Therefore, the garden will form a 30° angle at the corner of your property.

Your Turn 10.2.5

Suppose you have a similar property to the one in Figure 10.2.223, but the angle that corresponds to the garden corner is 50°. What is the measure between the two cross streets?

✓ Example 10.2.6: Determining Angles Formed by a Transversal

In Figure ✓ 10.2.23 given that angle 3 measures 40°, find the measures of the remaining angles and give a reason for your solution.

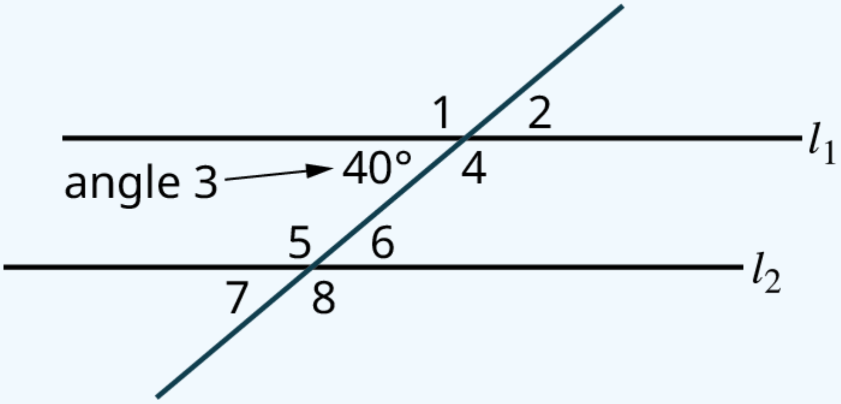


Figure ✓ 10.2.23

Answer

$m\measuredangle 2 = m\measuredangle 3 = 40^{\circ}$ by vertical angles.

$\measuredangle 3 = m\measuredangle 7$ by corresponding angles.

$m\measuredangle 7 = m\measuredangle 6 = 40^{\circ}$ by vertical angles.

$m\measuredangle 1 = 180 - 40 = 140^{\circ}$ by supplementary angles.

$m\measuredangle 4 = m\measuredangle 1 = 140^{\circ}$ by vertical angles.

$m\measuredangle 8 = m\measuredangle 1 = 140^{\circ}$ by alternate exterior angles.

$m\measuredangle 5 = m\measuredangle 8 = 140^{\circ}$ by vertical angles.

Your Turn 10.2.6

In the given figure if $m\measuredangle 1 = 120^{\circ}$, find the $m\measuredangle 5$, $m\measuredangle 4$, and $m\measuredangle 8$.

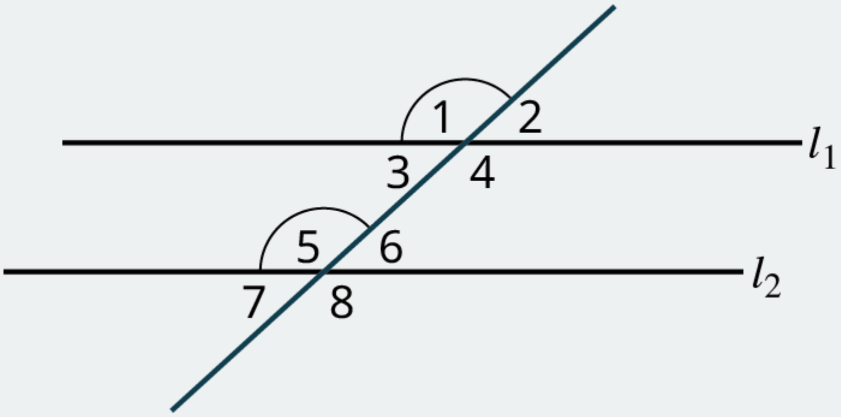


Figure 10.2.24

Example 10.2.7: Measuring Angles Formed by a Transversal

In Figure 10.2.25 given that angle 2 measures 23°, find the measure of the remaining angles and state the reason for your solution.

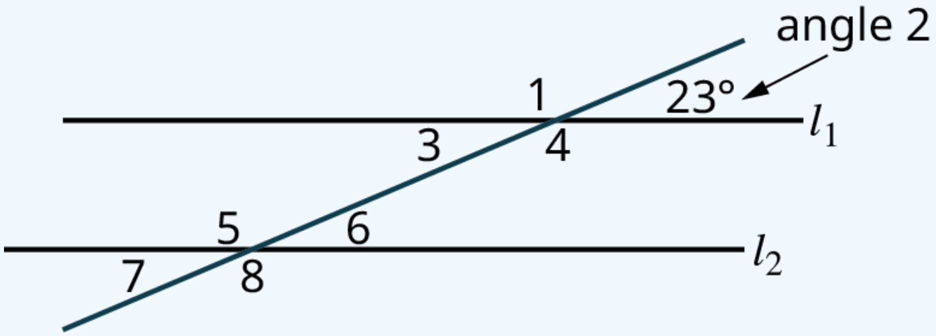


Figure 10.2.25

Answer

$m\measuredangle 2 = m\measuredangle 3 = 23^{\circ}$ by vertical angles, because $\measuredangle 2$ and $\measuredangle 3$ are the opposite angles formed by two intersecting lines.

$m\measuredangle 1 = 157^{\circ}$ by supplementary angles to $m\measuredangle 2$ or $m\measuredangle 3$. We see that $\measuredangle 1$ and $\measuredangle 2$ form a straight angle as does $\measuredangle 1$ and $\measuredangle 3$. A straight angle measures 180°, so $180^{\circ} - 23^{\circ} = 157^{\circ}$.

$m\measuredangle 4 = m\measuredangle 1 = 157^{\circ}$ by vertical angles, because $\measuredangle 4$ and $\measuredangle 1$ are the two opposite angles formed by two intersecting lines.

$m\measuredangle 5=m\measuredangle 1=157^{\circ}$ by corresponding angles because they are the same angle formed by the transversal crossing two parallel lines, one exterior and one interior.

$m\measuredangle 8=m\measuredangle 5=157^{\circ}$ by vertical angles because $\measuredangle 8$ and $\measuredangle 5$ are the two opposite angles formed by two intersecting lines.

$m\measuredangle 7=m\measuredangle 2=23^{\circ}$ by alternate exterior angles because, like vertical angles, these angles are the opposite angles formed by the transversal intersecting two parallel lines.

$m\measuredangle 6=m\measuredangle 7=23^{\circ}$ by vertical angles because these are the opposite angles formed by two intersecting lines.

Your Turn 10.2.7

In the provided figure given that the $m\measuredangle 2=48^{\circ}$, find $m\measuredangle 1$, and $m\measuredangle 5$.

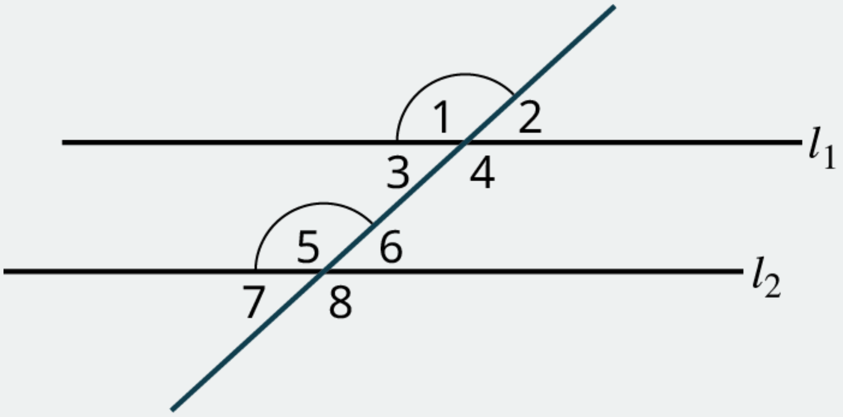


Figure 10.2.26

Example 10.2.8: Finding Missing Angles

Find the measures of the angles 1, 2, 4, 11, 12, and 14 in Figure 10.2.27 and the reason for your answer given that l_1 and l_2 are parallel.

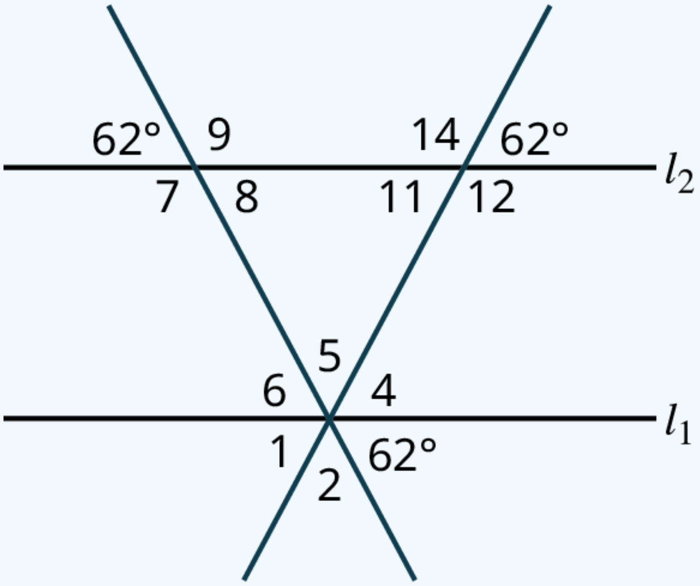


Figure 10.2.27

Answer

$m\measuredangle 12=118^{\circ}$, supplementary angles

$m\measuredangle 14 = 118^\circ$, vertical angles

$m\measuredangle 11 = 62^\circ$, vertical angles

$m\measuredangle 4 = 62^\circ$, corresponding angles

$m\measuredangle 1 = 62^\circ$, vertical angles

$m\measuredangle 2 = 56^\circ$, supplementary angles

Your Turn 10.2.8

Using Figure 10.2.27, find the measures of angles 5, 6, 7, 8, and 9.

Who Knew?: The Number 360

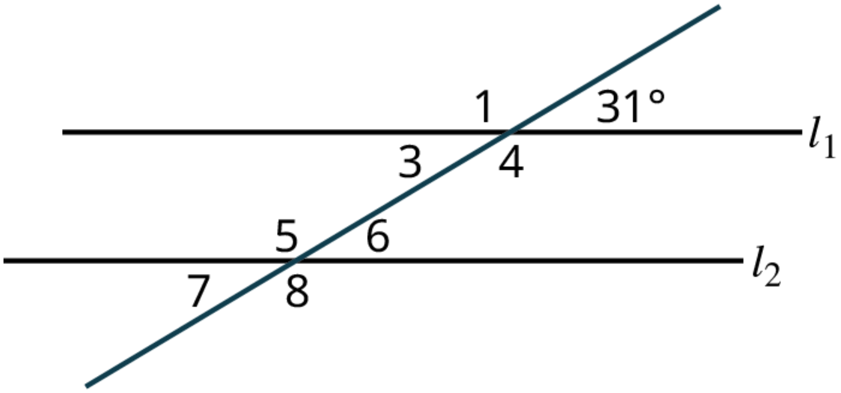
Did you ever wonder why there are 360° in a circle? Why not 100° or 500°? The number 360 was chosen by Babylonian astronomers before the ancient Greeks as the number to represent how many degrees in one complete rotation around a circle. It is said that they chose 360 for a couple of reasons: It is close to the number of days in a year, and 360 is divisible by 2, 3, 4, 5, 6, 8, 9, 10, …

Check Your Understanding

1.Classify the following angles as acute, right, obtuse, or straight.

- $m\measuredangle = 180^\circ$
- $m\measuredangle = 176^\circ$
- $m\measuredangle = 90^\circ$
- $m\measuredangle = 37^\circ$

2. For the following exercises, determine the measure of the angles in the given figure.



- Find the measure of $\measuredangle 1$ and state the reason for your solution.
- Find the measure of $\measuredangle 3$ and state the reason for your solution.
- Find the measure of $\measuredangle 5$ and state the reason for your solution.

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10.2.0: Exercises

Classify the angles in the following exercises as acute, obtuse, right, or straight.

? Exercise 10.2.0.1

$m\measuredangle = 32^\circ$

? Exercise 10.2.0.2

$m\measuredangle = 120^\circ$

? Exercise 10.2.0.3

$m\measuredangle = 180^\circ$

? Exercise 10.2.0.4

$m\measuredangle = 90^\circ$

? Exercise 10.2.0.5

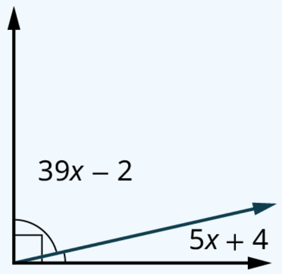
$m\measuredangle = 110^\circ$

? Exercise 10.2.0.6

$m\measuredangle = 45^\circ$

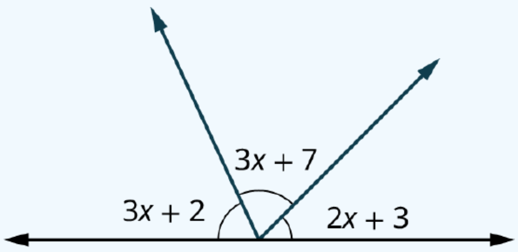
? Exercise 10.2.0.7

Use the given figure to solve for the angle measurements.



? Exercise 10.2.0.8

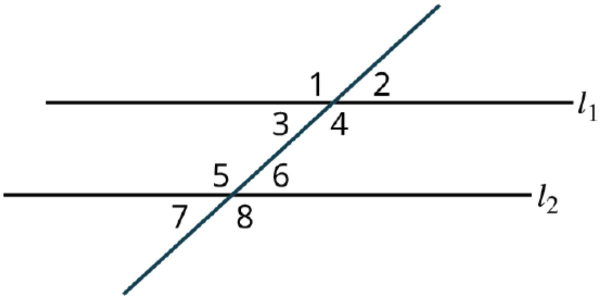
Use the given figure to solve for the angle measurements.



? Exercise 10.2.0.9

Give the measure of the supplement to 89°.

Use the given figure for the following exercises. Let angle 2 measure 35°.



? Exercise 10.2.0.10

Find the measure of angle 1 and state the reason for your solution.

? Exercise 10.2.0.11

Find the measure of angle 3 and state the reason for your solution.

? Exercise 10.2.0.12

Find the measure of angle 4 and state the reason for your solution.

? Exercise 10.2.0.13

Find the measure of angle 5 and state the reason for your solution.

? Exercise 10.2.0.14

Find the measure of angle 6 and state the reason and state the reason for your solution.

? Exercise 10.2.0.15

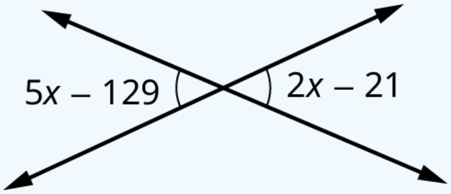
Find the measure of angle 7 and state the reason for your solution.

? Exercise 10.2.0.16

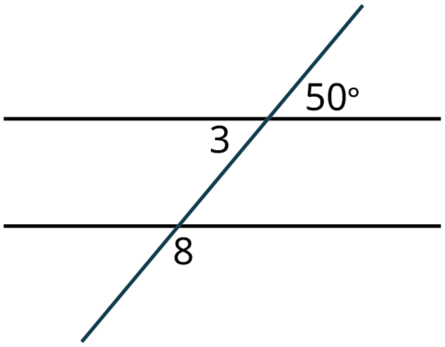
Find the measure of angle 8 and state the reason for your solution.

? Exercise 10.2.0.17

Use the given figure to solve for the angle measurements.



Use the given figure for the following exercises.



Exercise 10.2.0.18

Find the measure of angle 3 and explain the reason for your solution.

Exercise 10.2.0.19

Find the measure of angle 8 and explain the reason for your solution.

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10.3: Triangles

Learning Objectives

1. Identify triangles by their sides.
2. Identify triangles by their angles.
3. Determine if triangles are congruent.
4. Determine if triangles are similar.
5. Find the missing side of similar triangles.

How were the ancient Greeks able to calculate the radius of Earth? How did soldiers gauge their target? How was it possible centuries ago to estimate the height of a sail at sea? Triangles have always played a significant role in how we find heights of objects too high to measure or distances between objects too far away to calculate. In particular, the concept of similar triangles has countless applications in the real world, and we shall explore some of those applications in this section.

Figure $10.3.1$: The appearance of triangles in buildings is part of modern-day architectural design. (credit: "Inside Hallgrímskirkja church, Reykjavik, Iceland" by O Palsson/Flickr, CC BY 2.0)

Technology has given us instruments that allow us to find measurements of distant objects with little effort. However, it is all based on the properties of triangles discovered centuries ago. In this section, we will explore the various types of triangles and their special properties, as well as how to measure interior and exterior angles. We will also explore congruence theorems and similarity.

Identifying Triangles

Joining any three noncollinear points with line segments produces a triangle. For example, given points A, B, and C, connected by the line segments $\overline{AB}, \overline{BC}$, and $\overline{AC}$, we have a triangle, as shown in Figure $10.3.1$.

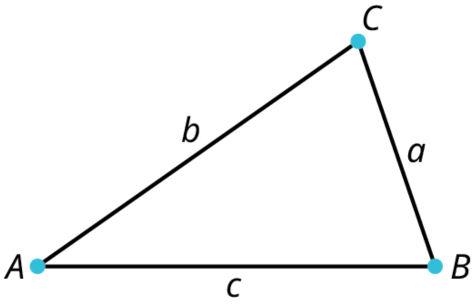


Figure $10.3.1$: Triangle

Triangles are classified by their angles and their sides. All angles in an acute triangle measure $< 90^\circ$. One of the angles in a right triangle measures 90°, symbolized by $\square$. One angle in an obtuse triangle measures between 90° and 180°. Sides that have equal length are indicated by the same hash marks. Figure $10.3.2$ illustrates the shapes of the basic triangles, their names, and their properties

A few other facts to remember as we move forward:

- The points where the line segments meet are called the **vertices** (plural for **vertex**).
- We often refer to sides of a triangle by the angle they are opposite. In other words, side a is opposite angle A, side b is opposite angle B, and side c is opposite angle C.

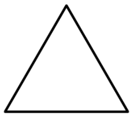
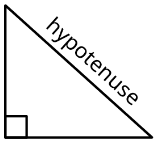
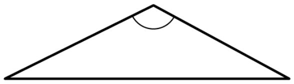
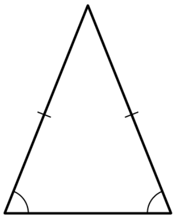
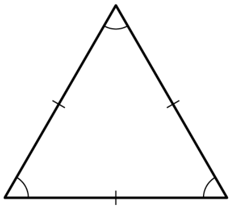
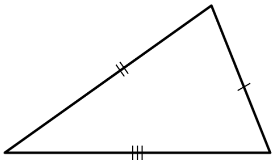
Acute Triangle	**Right Triangle**	**Obtuse Triangle**
All angles are acute. 	One 90° angle  	One angle is obtuse. 
Isosceles Triangle	**Equilateral Triangle**	**Scalene Triangle**
At least two angles have equal measure, and the sides opposite the equal angles are congruent. 	All sides are equal. All angles are equal. 	All sides have different lengths, all angles have different measure. 

Figure $10.3.2$: Types of Triangles

We want to add a special note about right triangles here, as they are referred to more than any other triangle. The side opposite the right angle is its longest side and is called the **hypotenuse**, and the sides adjacent to the right angle are called the legs.

One of the most important properties of triangles is that the sum of the interior angles equals 180°. Euclid discovered and proved this property using parallel lines. The completed sketch is shown in Figure $10.3.3$.

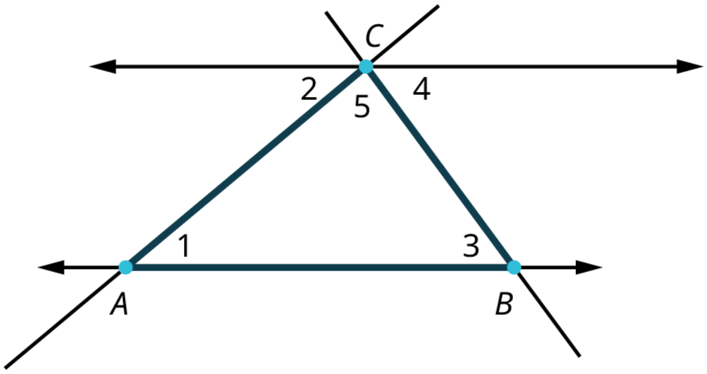


Figure $10.3.3$: Sum of Interior Angles

This is how the proof goes:

Step 1: Start with a straight line $\overleftrightarrow{AB}$ and a point C not on the line.

Step 2: Draw a line through point C parallel to the line $\overleftrightarrow{AB}$.

Step 3: Construct two transversals (a line crossing the parallel lines), one angled to the right and one angled to the left, to intersect the parallel lines.

Step 4: Because of the property that alternate interior angles inside parallel lines are equal, we have that

$$m\measuredangle 2 = m\text{\varangle}1 \quad \text{and} \quad m\measuredangle 3 = m\measuredangle 4.$$

Step 5: Notice that $m\measuredangle 2+m\measuredangle 5+m\measuredangle 4=180^\circ$ by the straight angle property.

Step 6: Therefore, by substitution, $m\measuredangle 1=m\measuredangle 2$, and $m\measuredangle 3=m\measuredangle 4$, we have that

$$m\measuredangle 1+m\measuredangle 3+m\backslash\text{varangle}5=180^\circ$$

Therefore, the sum of the interior angles of a triangle $=180^\circ$.

✓ Example 10.3.1: Finding Measures of Angles Inside a Triangle

Find the measure of each angle in the triangle shown (Figure ✓ 10.3.4). We know that the sum of the angles must equal 180°.

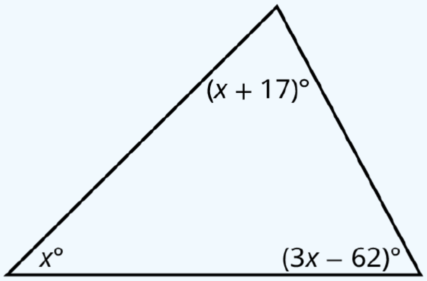


Figure ✓ 10.3.4

Answer

Step 1: As the sum of the interior angles equals 180°, we can use algebra to find the measures:

$$\begin{aligned} x+(x+17)+(3x-62)&=180\\ 5x-45&=180\\ 5x&=225\\ x&=\frac{225}{5}=45 \end{aligned}$$

Step 2: Now that we have the value of x, we can substitute 45 into the other two expressions to find the measure of those angles:

$$\begin{aligned} (x+17)&=45+17=62^\circ\\ (3x-62)&=3(45)-62=135-62=73^\circ \end{aligned}$$

Step 3: Then, $m\measuredangle x=45^\circ, m\measuredangle(x+17)=62^\circ$, and $m\measuredangle(3x-62)=73^\circ$.

📌 Your Turn 10.3.1

Find the measures of each angle in the triangle shown.

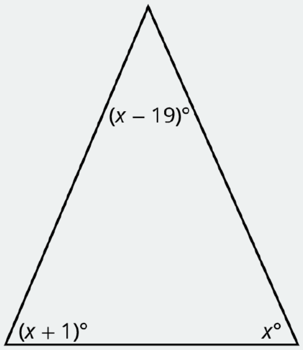


Figure 📌 10.3.5

✓ Example 10.3.12: Finding Angle Measures

Find the measure of angles numbered 1–5 in Figure ✓ 10.3.6.

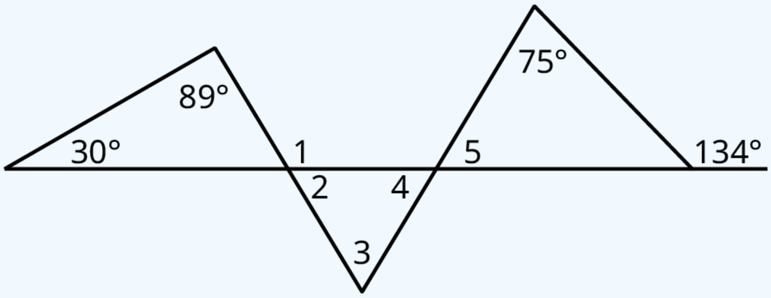


Figure ✓ 10.3.6

Answer

The $m\measuredangle 1 = 119^\circ$ because it is supplementary with the unknown angle of the adjacent triangle. The unknown angle measures 61°. The $m\measuredangle 2 = 61^\circ$ because of vertical angles. The $m\measuredangle 5 = 59^\circ$ because the angle that is supplementary to the 134° measures 46°, and angle 5 is the unknown angle in that triangle. The $m\measuredangle 4 = 59^\circ$ by vertical angles. Finally, $m\measuredangle 3 = 60^\circ$, as it is the third angle in the triangle with angles measuring 59° and 61°.

Your Turn 10.3.12

Find the measure of angles 1, 2, and 3 in the figure shown.

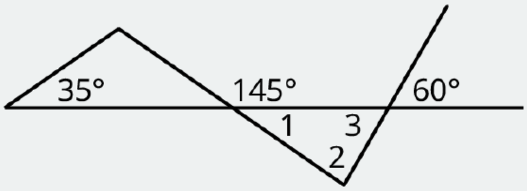


Figure 10.3.7

Congruence

If two triangles have equal angles and their sides lengths are equal, the triangles are congruent. In other words, if you can pick up one triangle and place it on top of the other triangle and they coincide, even if you have to rotate one, they are congruent.

✓ Example 10.3.3: Determining If Triangles Are Congruent

In Figure ✓ 10.3.8, is the triangle ABC congruent to triangle DEF?

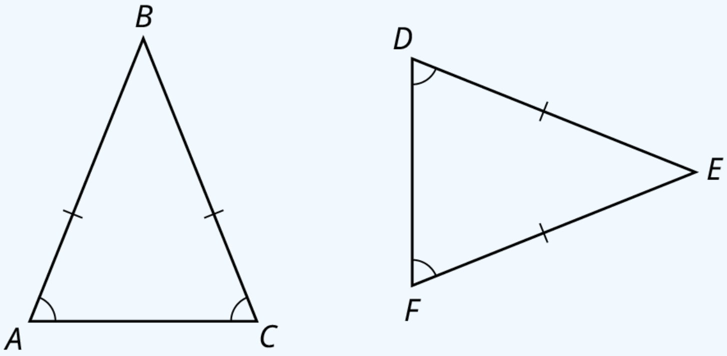


Figure ✓ 10.3.8

Answer

Triangle ABC is congruent to triangle DEF. Angles A and C are congruent to angles D and F, which implies that angle B is congruent to angle E. Side AB is congruent to side DE, and side CB is congruent to side FE, which implies that side AC is congruent to side DF.

Your Turn 10.3.3

In the figure shown, is triangle ABC congruent to triangle DEF?

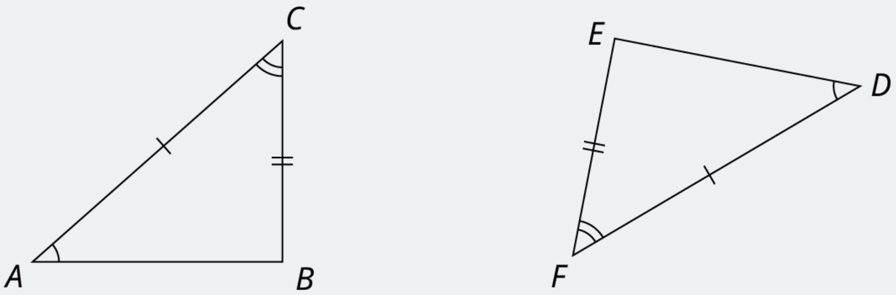


Figure 10.3.9

The Congruence Theorems

The following theorems are tools you can use to prove that two triangles are congruent. We use the symbol $\cong$ to define congruence. For example, $\triangle ABC \cong \triangle DEF$.

Side-Side-Side (SSS). If three sides of one triangle are equal to the corresponding sides of the second triangle, then the triangles are congruent. See Figure 10.3.10.

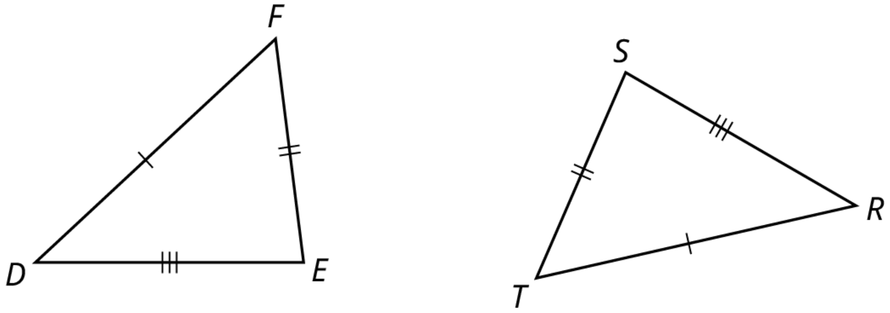


Figure 10.3.10: Side-Side-Side (SSS)

We have that $\overline{DF} \cong \overline{RT}$, $\overline{EF} \cong \overline{ST}$, and $\overline{DE} \cong \overline{RS}$, then $\Delta DEF \cong \Delta RST$.

We have that $\overline{DF} \cong \overline{RT}, \overline{EF} \cong \overline{ST}$, and $\overline{DE} \cong \overline{RS}$, then $\triangle DEF \cong \Delta RST$.

Side-Angle-Side (SAS). If two sides of a triangle and the angle between them are equal to the corresponding two sides and included angle of the second triangle, then the triangles are congruent. See Figure 10.3.11. We see that $\overline{AB} \cong \overline{A'B'}$ and $\overline{BC} \cong \overline{B'C'}, m\measuredangle B = m\measuredangle B'$, then $\Delta ABC \cong \Delta A'B'C'$

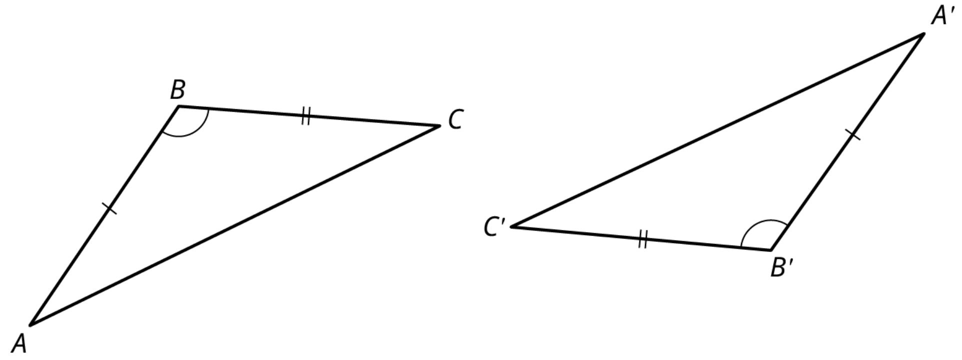


Figure 10.3.11: Side-Angle-Side (SAS)

Angle-Side-Angle (ASA). If two angles and the side between them in one triangle are congruent to the two corresponding angles and the side between them in a second triangle, then the two triangles are congruent. See Figure 10.3.12. Notice that $m\measuredangle \mathrm{A} \cong m\measuredangle \mathrm{F}$, and $m\measuredangle \mathrm{C} \cong m\measuredangle \mathrm{D}$, $\overline{AC} \cong \overline{DF}$, then $\Delta ABC \cong \Delta DEF$.

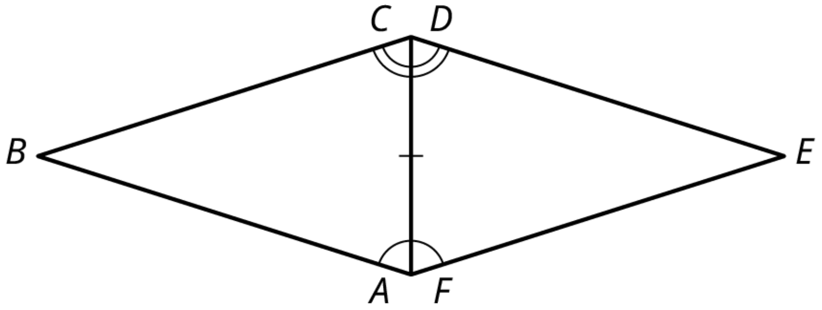


Figure 10.3.12: Angle-Side-Angle (ASA)

Angle-Side-Angle (ASA). If two angles and the side between them in one triangle are congruent to the two corresponding angles and the side between them in a second triangle, then the two triangles are congruent. See Figure 10.3.13. Notice that $m\measuredangle\ \mathrm{A} \cong m\measuredangle\ \mathrm{F}$, and $m\measuredangle \mathrm{C} \cong m\measuredangle \mathrm{D}$, $\overline{AC} \cong \overline{DF}$, then $\triangle ABC \cong \triangle DEF$.

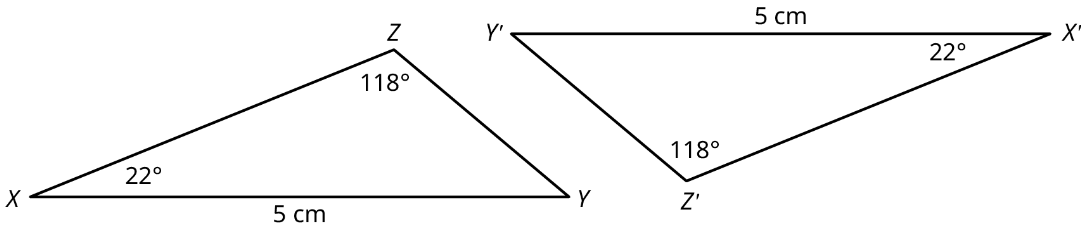


Figure 10.3.13: Angle-Angle-Side (AAS)

✓ Example 10.3.4: Identifying Congruence Theorems

What congruence theorem is illustrated in Figure ✓ 10.3.147?

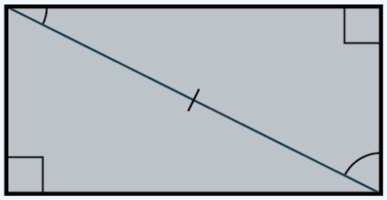


Figure ✓ 10.3.14

Answer

AAS: Two angles and a non-included side in one triangle are congruent to the corresponding angles and side in the second triangle.

Your Turn 10.3.4

Identify the congruence theorem being illustrated in the figure shown.

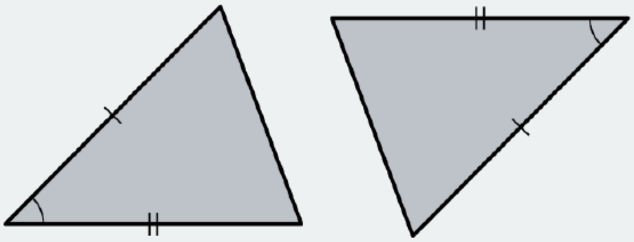


Figure 10.3.15

✓ Example 10.3.5: Determining the Congruence Theorem

What congruence theorem is illustrated in Figure ✓ 10.3.16?

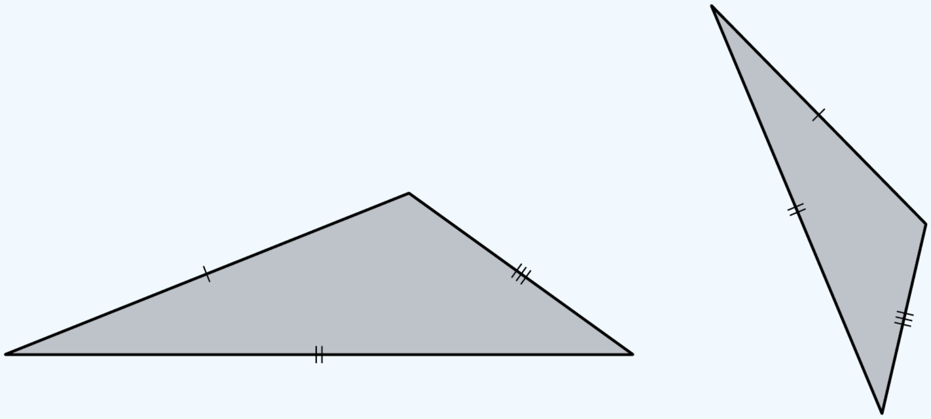


Figure ✓ 10.3.16

Answer

The SSS theorem.

Your Turn 10.3.5

What congruence theorem is being illustrated in the figure shown?

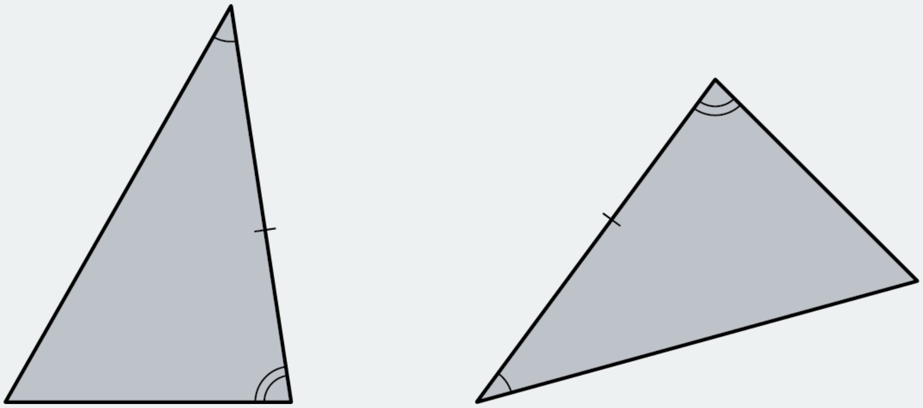


Figure 10.3.17

Similarity

If two triangles have the same angle measurements and are the same shape but differ in size, the two triangles are similar. The lengths of the sides of one triangle will be proportional to the corresponding sides of the second triangle. Note that a single fraction $\frac{a}{b}$ is called a ratio, but two fractions equal to each other is called a proportion, such as

$$\frac{a}{b}=\frac{c}{d}.$$

This rule of similarity applies to all shapes as well as triangles. Another way to view similarity is by applying a scaling factor, which is the ratio of corresponding measurements between an object or representation of the object, to an image that produces the second, similar image.

For example, why are the two images in Figure $10.3.18$ are similar? These two images have the same proportions between elements. Therefore, they are similar.

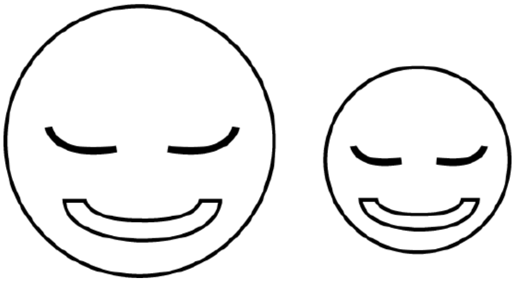


Figure $10.3.18$: Similarity

✓ Example $10.3.6$: Determining If Triangles Are Similar

Are the two triangles shown in Figure ✓ $10.3.19$ similar?

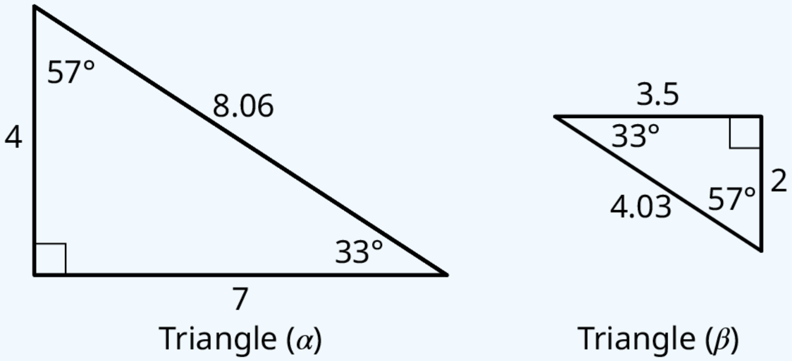


Figure ✓ $10.3.19$

Answer

Step 1: We will look at the proportions within each triangle. In triangle α (alpha), the side opposite the 57° angle measures 7 , and the side opposite the 33° angle measures 4 . Then, the measures of the corresponding sides in triangle β (beta) measures 3.5 and 2 , respectively. We have

$$\frac{4}{7}=0.5714 \quad \frac{2}{3.5}=0.5714$$

This is the proportion $\frac{4}{7}=\frac{2}{3.5}$. The scaling factor is 0.5714 .

Step 2: Let's try another correspondence. In triangle α, the hypotenuse measures 8.06 and the side opposite the 57° angle measures 7 . In triangle β, the hypotenuse measure 4.03 and the side opposite the 57° angle measures 3.5 . We have

$$\frac{7}{8.06}=0.8685 \quad \frac{3.5}{4.03}=0.8685$$

Step 3: Now, let's look at the proportions between triangle α and triangle β. The side measuring 2 in triangle β corresponds to the side measuring 4 in triangle α, the side measuring 3.5 in triangle β corresponds to the side measuring 7 in triangle α, and the hypotenuse in triangle β corresponds to the hypotenuse in triangle α. We have

$$\frac{2}{4}=0.5 \quad \frac{3.5}{7}=0.5 \quad \frac{4.03}{8.06}=0.5$$

Thus, the corresponding angles are equal and the proportions bet**ween each pair of corresponding sides equals 0.5 . In other words, the scaling factor is 0.5 . Therefore, the triangles are similar.**

Your Turn 10.3.6

Is triangle EFG similar to triangle JKL in the figure shown?

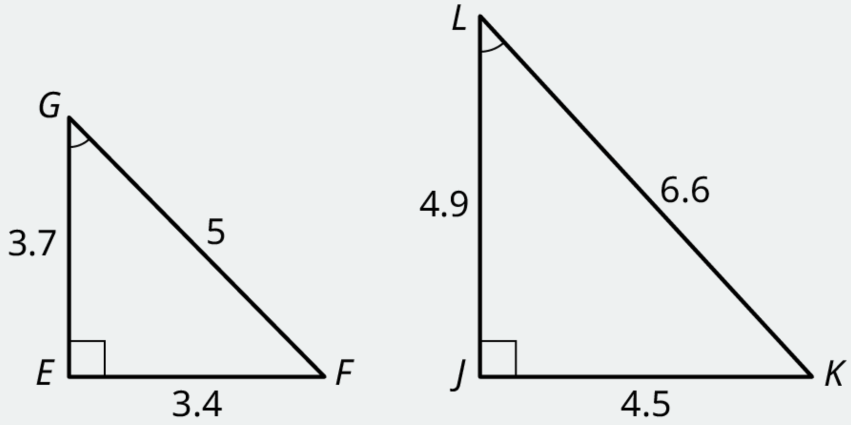


Figure 📌 10.3.20

Example 10.3.7: Proving Similarity

In Figure ✓ 10.3.21, is triangle δ (delta) similar to triangle ε (epsilon)? Find the lengths of sides x and y as part of your answer.

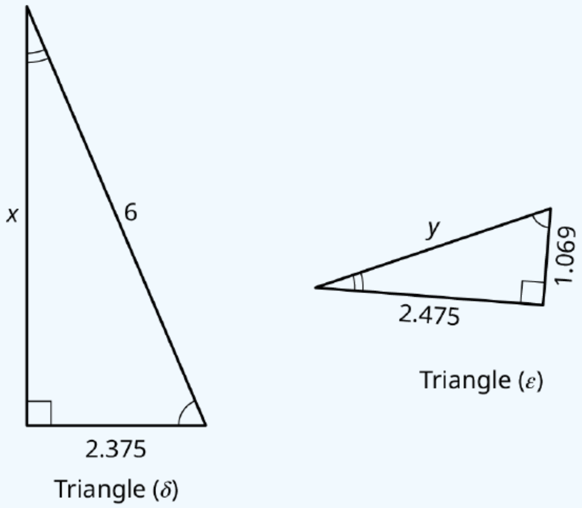


Figure ✓ 10.3.21

Answer

We can see that all three angles in triangle δ are equal to the corresponding angles in triangle ε. That is enough to determine similarity. However, we want to find the values of x and y to prove similarity.

Step 1: We have to do is set up the proportions between the corresponding sides. We have the side that measures 2.375 in triangle δ corresponding to the side measuring 1.069 in triangle ε. We have the hypotenuse/side in triangle δ measuring 6 corresponding to the hypotenuse/side labeled y in triangle ε. And, finally, the side labeled x in triangle δ corresponds to the side measuring 2.475 in triangle ε.

Each proportion should be equal. We start with the proportion of the shorter sides. Thus

$$\frac{1.069}{2.375}=0.45$$

Step 2: We solve for y using the first proportion. Set the two ratios equal to each other, cross-multiply, and solve for y. We have:

$$\begin{aligned}\frac{1.069}{2.375}&=\frac{y}{6}\\(6)(1.069)&=(2.375)(y)\\6.414&=2.375y\\\frac{6.414}{2.375}&=2.7=y\end{aligned}$$

So, $y=2.7$.

Step 3: Checking that length in the proportion factor of 0.45 , we have:

$$\frac{y}{6}=\frac{2.7}{6}=0.45$$

Step 4: Solving for x, we will use the same proportion we used to solve for y. We have:

$$\begin{aligned}\frac{1.069}{2.375}&=\frac{2.475}{x}\\1.069(x)&=2.475(2.375)\\1.069(x)&=5.878\\x&=\frac{5.878}{1.069}=5.5\\\frac{2.475}{x}&=\frac{2.475}{5.5}=0.45\end{aligned}$$

Step 5: We test the proportions. We have the following:

$$\frac{2.7}{6}=\frac{2.475}{5.5}=\frac{1.069}{2.375}=0.45$$

The proportions are all equal. Therefore, we have proven the property of similarity between triangle δ and triangle ε

Your Turn 10.3.7

Are these triangles similar? Find the lengths of sides x and y to prove your answer.

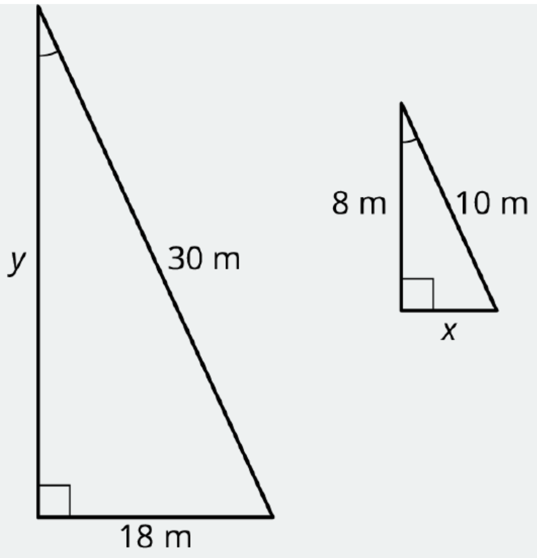


Figure 📌 $10.3.22$

✓ Example $10.3.8$: Applying Similar Triangles

A person who is 5 feet tall is standing 50 feet away from the base of a tree (Figure ✓ $10.3.23$). The tree casts a 57-foot shadow. The person casts a 7-foot shadow. What is the height of the tree?

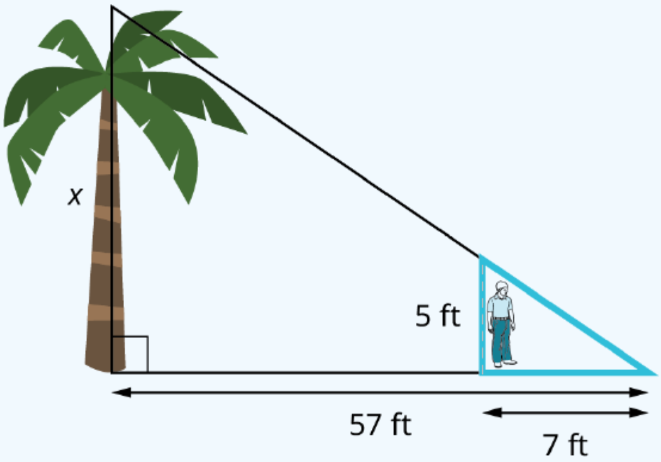


Figure ✓ $10.3.23$

Answer

The bigger triangle includes a tree at side x and the smaller triangle includes the person at the side labeled 5 ft . These two triangles are similar because the smaller triangle fits inside the larger triangle at the smallest angle. It would fit inside the larger triangle at either of the other two angles as well. That all angles are equal is one of the criteria for similar triangles, so we can solve using proportions:

$$\begin{aligned} \frac{5}{7} &= \frac{x}{57} \\ 5(57) &= 7x \\ \frac{285}{7} &= x = 40.7 \end{aligned}$$

The tree is 40.7 feet tall.

Your Turn 10.3.8

A person who is 6 feet tall is standing 100 feet away from the base of a tree. The tree casts a shadow 107.5-foot shadow. The person's shadow is 7.5 feet long. How tall is the tree?

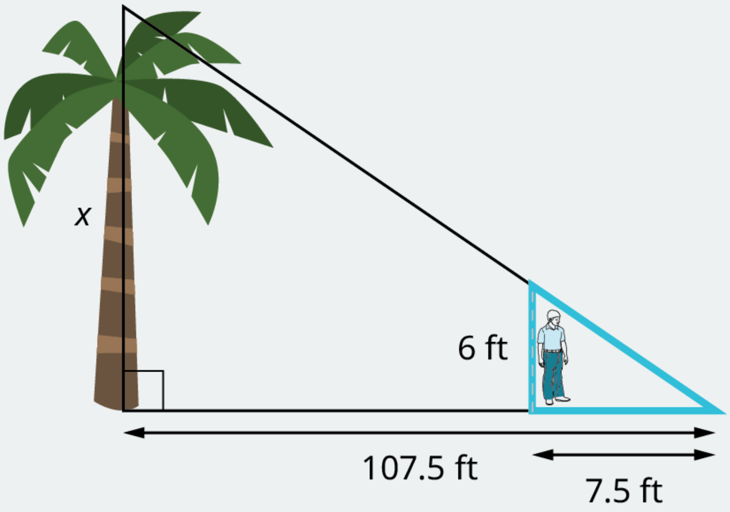


Figure 10.3.24

Example 10.3.9: Finding Missing Lengths

At a certain time of day, a radio tower casts a shadow 180 feet long (Figure ✓ 10.3.25). At the same time, a 9-foot truck casts a shadow 15 feet long. What is the height of the tower?

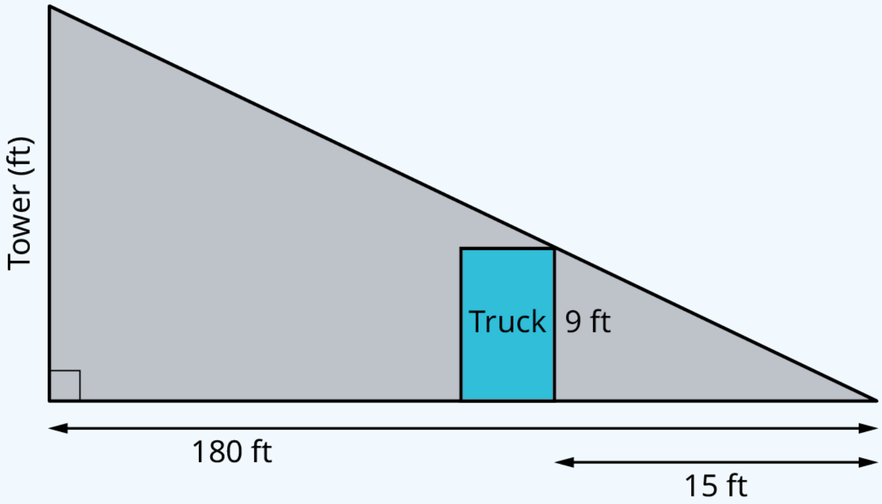


Figure ✓ 10.3.25

Answer

These are similar triangles and the problem can be solved by using proportions:

$$\begin{aligned}\frac{x}{180} &= \frac{9}{15}\\ 9(180) &= 15x\\ 1620 &= 15x\\ 108 &= x\end{aligned}$$

The height of the tower is 108 ft .

Your Turn 10.3.9

A tree casts a shadow of 180 feet early in the morning. A 10-foot high garage casts a shadow of 30 feet at the same time in the morning. What is the height of the tree?

People in Mathematics: Thales of Miletus

Thales of Miletus, sixth century BC, is considered one of the greatest mathematicians and philosophers of all time. Thales is credited with being the first to discover that the two angles at the base of an isosceles triangle are equal, and that the two angles formed by intersecting lines are equal—that is, vertical or opposite angles, are equal. Thales is also known for devising a method for measuring the height of the pyramids by similar right triangles. Figure 10.3.26 shows his method. He measured the length of the shadow cast by the pyramid at the precise time when his own shadow ended at the same place.

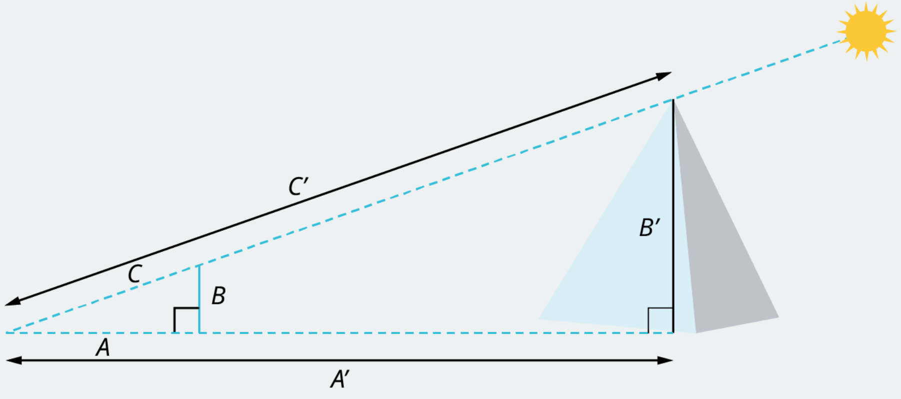


Figure 10.3.26: Thales and Similarity

He equated the vertical height of the pyramid with his own height; the horizontal distance from the pyramid to the tip of its shadow with the distance from himself and the tip of his own shadow; and finally, the length of the shadow cast off the top of the pyramid with length of his own shadow cast off the top of his head. Using proportions, as shown in Figure 10.3.26, he essentially discovered the properties of similarity for right triangles. That is, $\triangle ABC$ is similar to $\triangle A'B'C'$. Note that to be similar, all corresponding angles between the two triangles must be equal, and the proportions from one side to another side within each triangle, as well as the proportions of the corresponding sides between the two triangles must be equal.

Thales is also credited with discovering a method of determining the distance of a ship from the shoreline. Here is how he did it, as illustrated in Figure 10.3.27

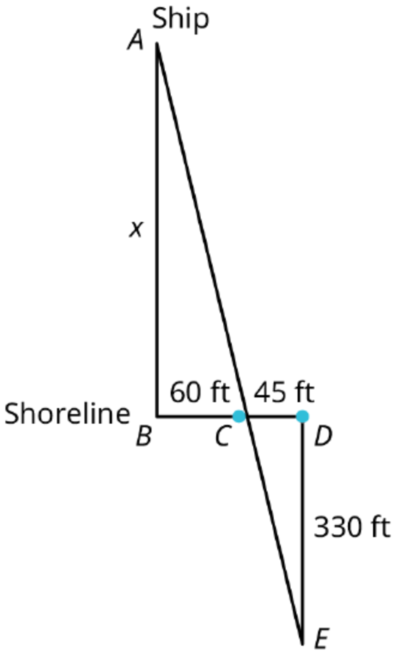
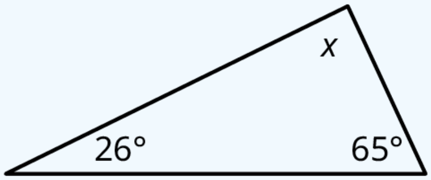


Figure $10.3.27$: Thales and Similar Triangles

Thales walked along the shoreline pointing a stick at the ship until it formed a 90° angle to the shore. Then he walked along the shot and placed the stick in the ground at point C. He continued walking until he reached point D. Then, he turned and walked away from the shore at a 90° angle until the stick he placed in the ground at point C lined up with the ship, point E. This is how he created similar triangles and estimated the distance of the ship to the shore by using proportions.

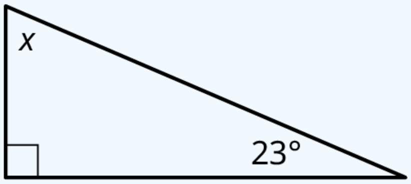
? Exercise $10.3.1$

Find the measure of the missing angle in the given figure.



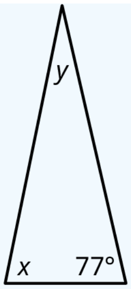
? Exercise $10.3.2$

Find the measure of the missing angle in the given figure.



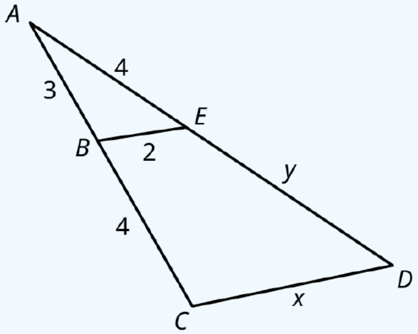
? Exercise $10.3.3$

In the isosceles triangle shown, find the missing angles.



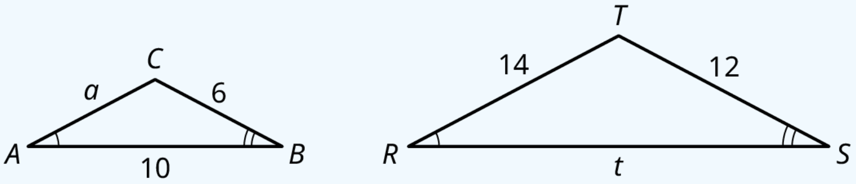
Exercise 10.3.4

In the figure shown given $\overline{BE}$ is parallel to $\overline{CD}$, find x and y.



Exercise 10.3.5

Find a and t in the given figure.



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10.3.0: Exercises

For the following exercises, classify the triangle with the listed angle measurements as acute, right, or obtuse.

Exercise 10.3.0.1

$m\measuredangle A=88^{\circ}$, $m\measuredangle B=32^{\circ}$, $m\measuredangle C=60^{\circ}$

Exercise 10.3.0.2

$m\measuredangle A=65^{\circ}$, $m\measuredangle B=90^{\circ}$, $m\measuredangle C=25^{\circ}$

Exercise 10.3.0.3

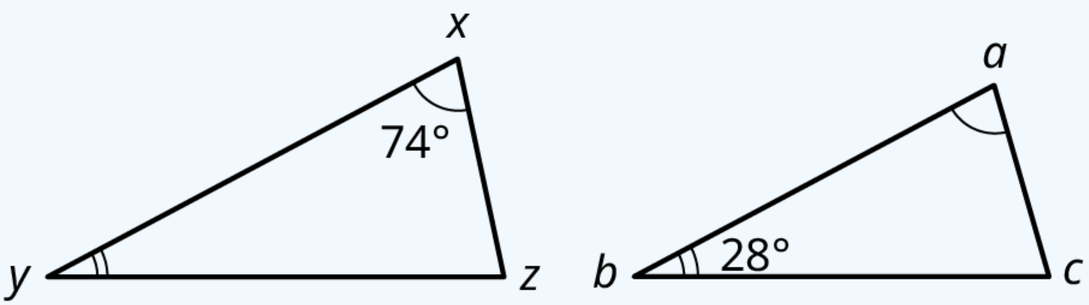
$m\measuredangle A=30^{\circ}$, $m\measuredangle B=120^{\circ}$, $m\measuredangle C=30^{\circ}$

Exercise 10.3.0.4

$m\measuredangle A=85^{\circ}$, $m\measuredangle B=50^{\circ}$, $m\measuredangle C=45^{\circ}$

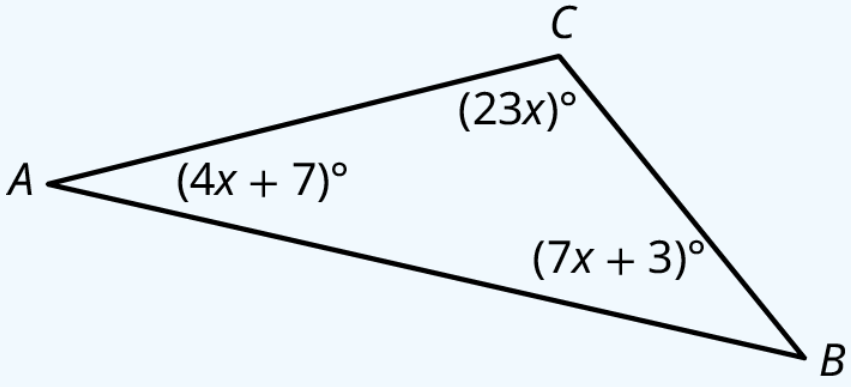
Exercise 10.3.0.5

Find the missing angles in the given figure.



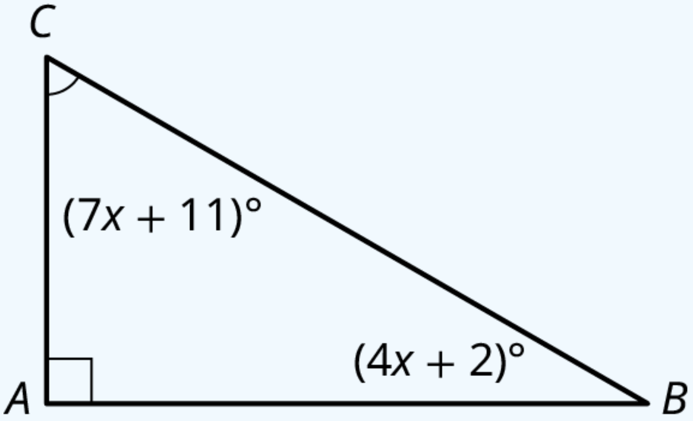
Exercise 10.3.0.6

What are the measurements of each angle in the given figure?



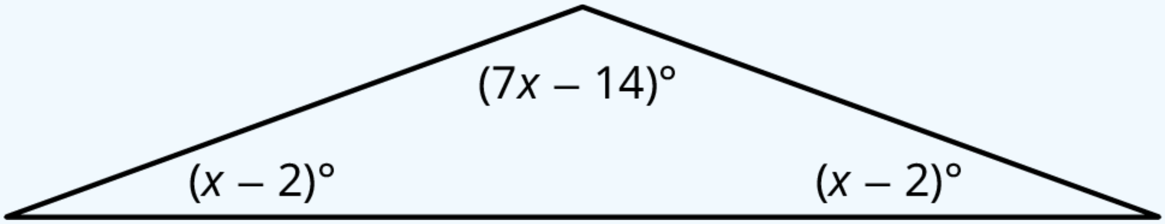
? Exercise 10.3.0.7

Find the angle measurements in the given figure.



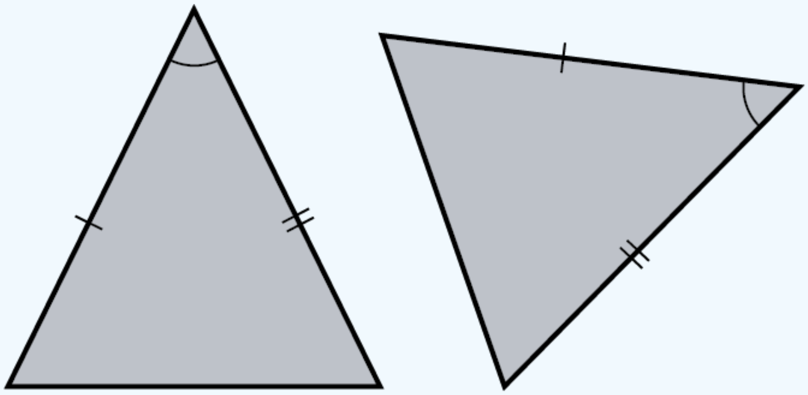
? Exercise 10.3.0.8

Find the angle measurements in the given figure.

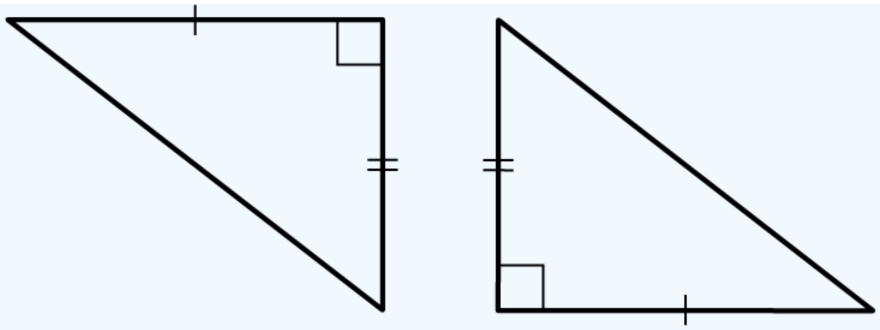


For the following exercises, determine which congruence theorem is used to show that the two triangles are congruent.

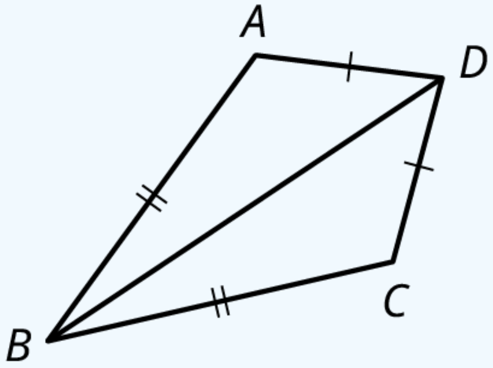
? Exercise 10.3.0.9



? Exercise 10.3.0.10

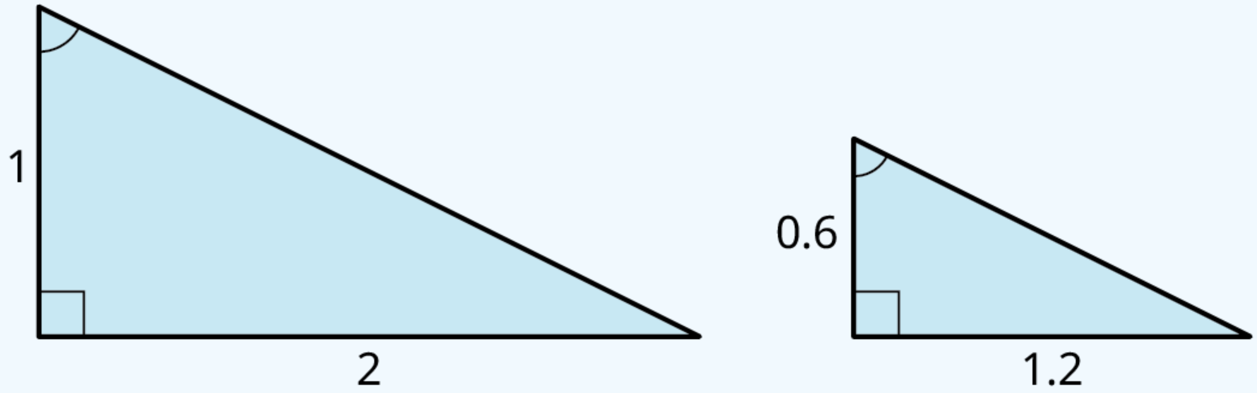


Exercise 10.3.0.11



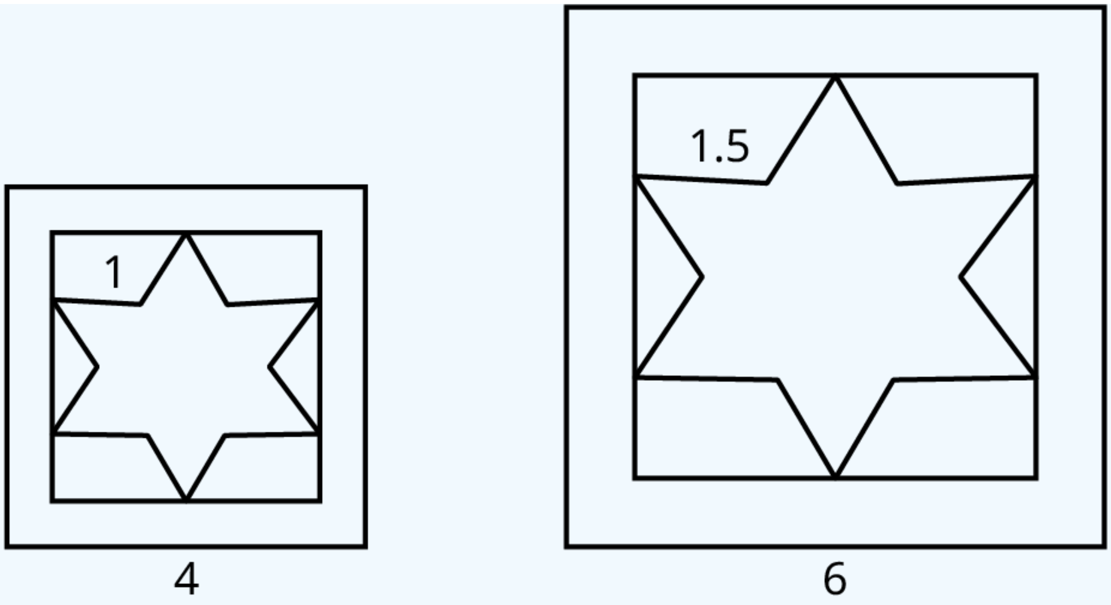
Exercise 10.3.0.12

Are these triangles similar? If so, what is the common proportion or the scaling factor?



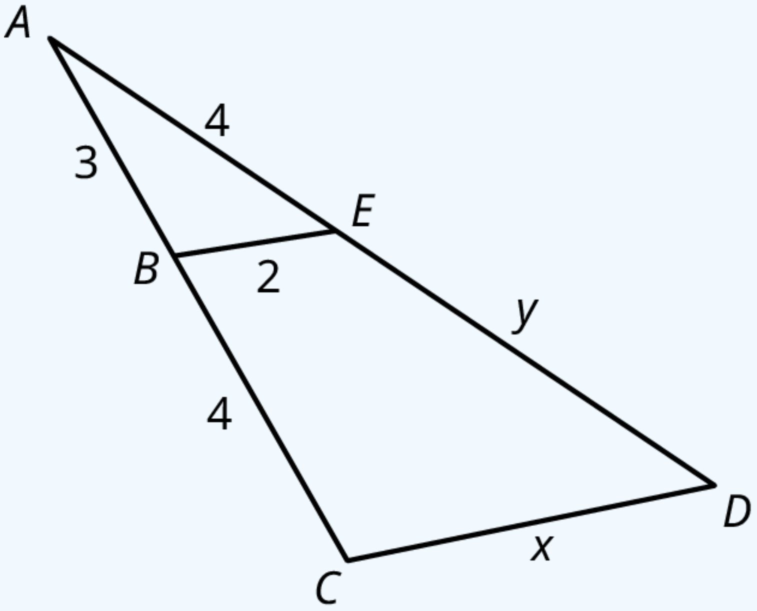
Exercise 10.3.0.13

Are the images in the given figure similar? If so, what is the common proportion or the scaling factor?



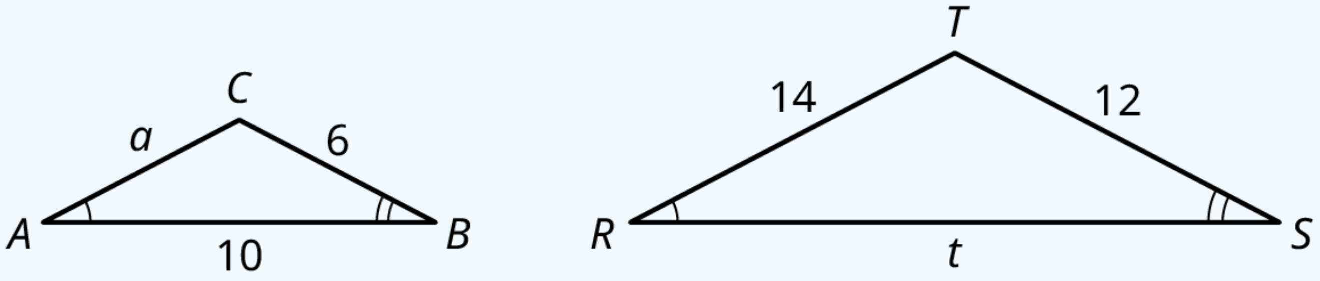
Exercise 10.3.0.14

In the figure shown, given $\overline{BE}$ is parallel to $\overline{CD}$, find x and y.



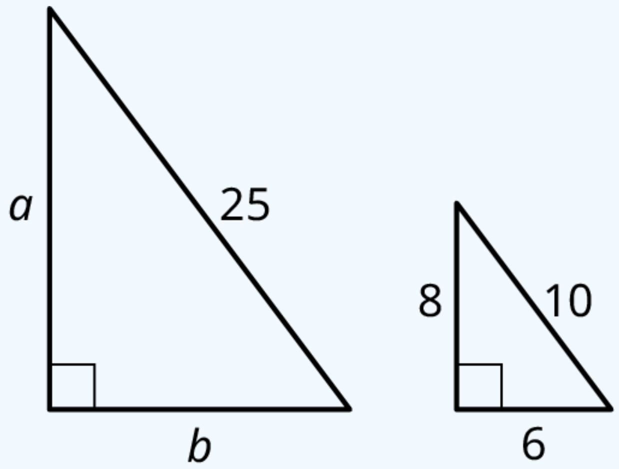
Exercise 10.3.0.15

Find a and t in the given figure.


C
a
6
A
10
B
T
14
12
R
t
S

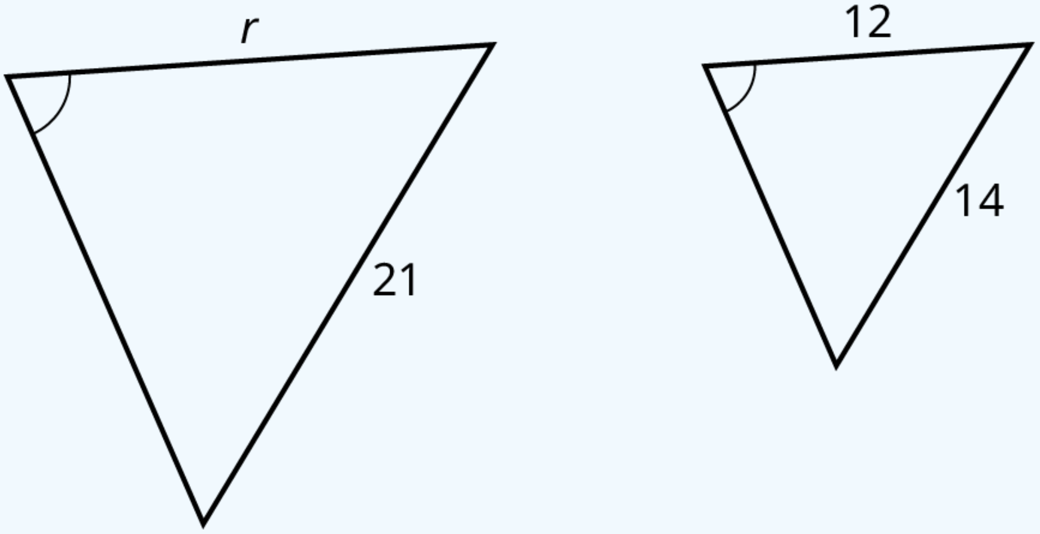
Exercise 10.3.0.16

In the given figure, find the proportions and solve for a and b.



Exercise 10.3.0.17

In the given figure, find the proportions and solve for r.



? Exercise 10.3.0.18

A triangular plot of land has a perimeter of 2,400 ft. The longest side is 200 ft less than twice the shortest side. The middle side is 200 ft less than the longest side. Find the lengths of the three sides.

? Exercise 10.3.0.19

The Orange Tree Hotel in Charleston, North Carolina has a fountain in the shape of a cylinder with a circular foundation. The circumference of the foundation is 6 times the radius increased by 12.88 ft. Find the radius of the circular foundation. (Use 3.14 as an approximation for π.)

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10.4: Polygons, Perimeter, and Circumference

Learning Objectives

1. Identify polygons by their sides.
2. Identify polygons by their characteristics.
3. Calculate the perimeter of a polygon.
4. Calculate the sum of the measures of a polygon's interior angles.
5. Calculate the sum of the measures of a polygon's exterior angles.
6. Calculate the circumference of a circle.
7. Solve application problems involving perimeter and circumference.

In our homes, on the road, everywhere we go, polygonal shapes are so common that we cannot count the many uses. Traffic signs, furniture, lighting, clocks, books, computers, phones, and so on, the list is endless. Many applications of polygonal shapes are for practical use, because the shapes chosen are the best for the purpose.

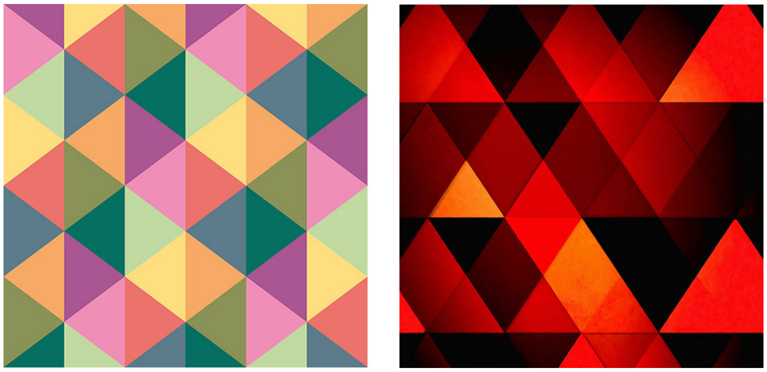


Figure $\{PageIndex1$: Geometric patterns are often used in fabrics due to the interest the shapes create. (credit: "Triangles" by Brett Jordan/Flickr, CC BY 2.0)

Modern geometric patterns in fabric design have become more popular with time, and they are used for the beauty they lend to the material, the window coverings, the dresses, or the upholstery. This art is not done for any practical reason, but only for the interest these shapes can create, for the pure aesthetics of design.

When designing fabrics, one has to consider the perimeter of the shapes, the triangles, the hexagons, and all polygons used in the pattern, including the circumference of any circular shapes. Additionally, it is the relationship of one object to another and experimenting with different shapes, changing perimeters, or changing angle measurements that we find the best overall design for the intended use of the fabric. In this section, we will explore these properties of polygons, the perimeter, the calculation of interior and exterior angles of polygons, and the circumference of a circle.

Identifying Polygons

A polygon is a closed, two-dimensional shape classified by the number of straight-line sides. See Figure 10.82 for some examples. We show only up to eight-sided polygons, but there are many, many more.

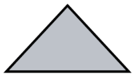
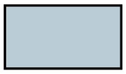
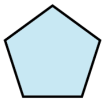
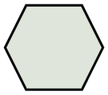
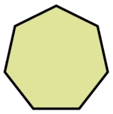
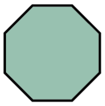
Number of Sides	Name	Shape
3	Triangle	
4	Quadrilateral	
5	Pentagon	
6	Hexagon	
7	Heptagon	
8	Octagon	

Figure $\{PageIndex2$: Types of Polygons

If all the sides of a polygon have equal lengths and all the angles are equal, they are called **regular polygons**. However, any shape with sides that are line segments can classify as a polygon. For example, the first two shapes, shown in Figure $\{PageIndex3$, are both pentagons because they each have five sides and five vertices. The third shape Figure $\{PageIndex31$ is a hexagon because it has six sides and six vertices. We should note here that the hexagon in Figure $\{PageIndex3$ is a concave hexagon, as opposed to the first two shapes, which are convex pentagons. Technically, what makes a polygon concave is having an interior angle that measures greater than 180°. They are hollowed out, or cave in, so to speak. Convex refers to the opposite effect where the shape is rounded out or pushed out.

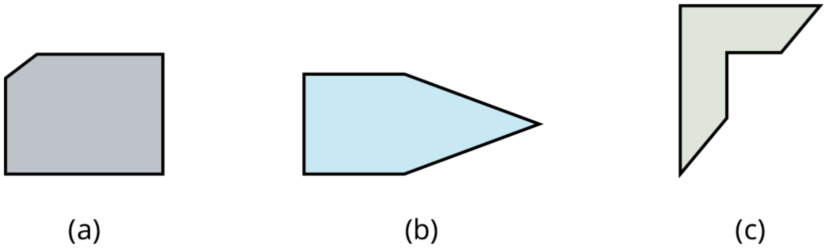


Figure $\{PageIndex3$: Polygons

While there are variations of all polygons, quadrilaterals contain an additional set of figures classified by angles and whether there are one or more pairs of parallel sides. See Figure $\{PageIndex4$.

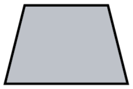
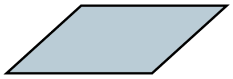
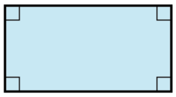
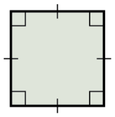
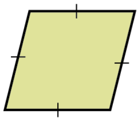
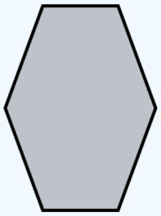
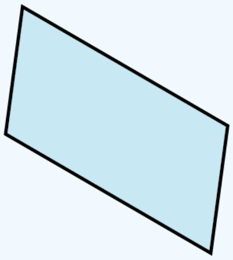
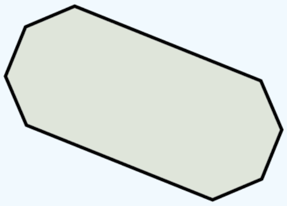
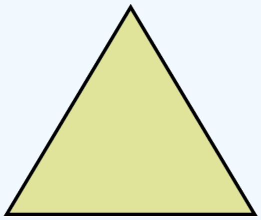
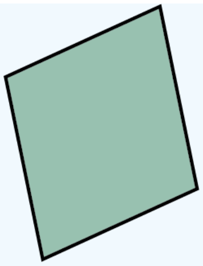
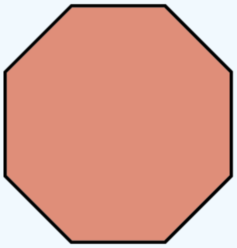
Description	Shape
A **trapezoid** has one pair of parallel sides.	
A **parallelogram** has two sets of parallel sides and no right angles.	
A **rectangle** is a parallelogram with four right angles and two sets of parallel sides.	
A **square** is a rectangle with four equal sides.	
A **rhombus** is a parallelogram with all sides equal, two sets of parallel sides.	

Figure $\{PageIndex4\}$: Types of Quadrilaterals

Example $\{PageIndex1\}$: Identifying Polygons

Identify each polygon.

1. 
2. 
3. 
4. 

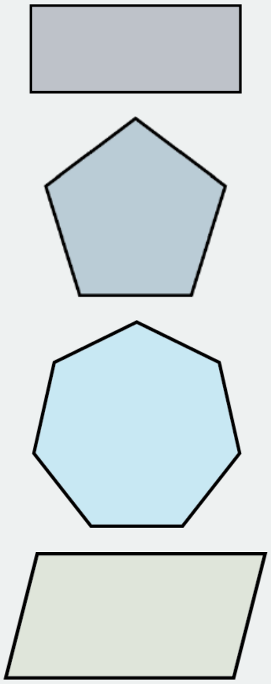
5. 
6. 

Answer

1. This shape has six sides. Therefore, it is a hexagon.
2. This shape has four sides, so it is a quadrilateral. It has two pairs of parallel sides making it a parallelogram.
3. This shape has eight sides making it an octagon.
4. This is an equilateral triangle, as all three sides are equal.
5. This is a rhombus; all four sides are equal.
6. This is a regular octagon, eight sides of equal length and equal angles.

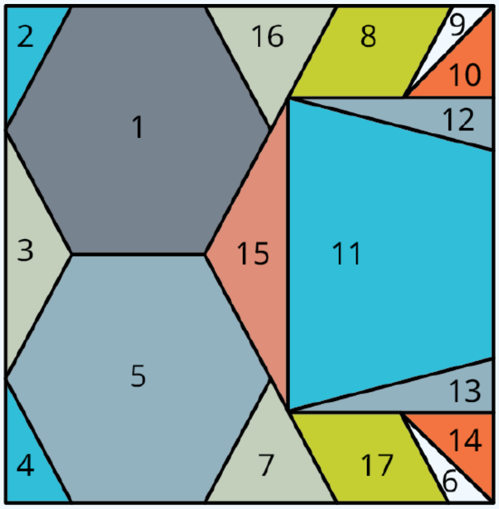
Your Turn $\{PageIndex1\}$

Identify the shape.



✓ Example $\{PageIndex2$: Determining Multiple Polygons

What polygons make up Figure 10.85?



Figure

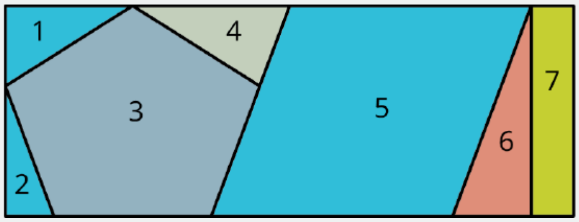
✓ $\{PageIndex5$

Answer

Shapes 1 and 5 are hexagons; shapes 2, 3, 4, 6, 7, 9, 10, 12, 13, 14, 15, and 16 are triangles; shapes 8 and 17 are parallelograms; and shape 11 is a trapezoid.

Your Turn $\{PageIndex2$

What polygons make up the figure shown?



Figure

$\{PageIndex6$

Perimeter

Perimeter refers to the outside measurements of some area or region given in linear units. For example, to find out how much fencing you would need to enclose your backyard, you will need the perimeter. The general definition of **perimeter** is the sum of the lengths of the sides of an enclosed region. For some geometric shapes, such as rectangles and circles, we have formulas. For other shapes, it is a matter of just adding up the side lengths.

A **rectangle** is defined as part of the group known as quadrilaterals, or shapes with four sides. A rectangle has two sets of parallel sides with four angles. To find the perimeter of a rectangle, we use the following formula:

FORMULA

The formula for the perimeter P of a rectangle is $P = 2L + 2W$, twice the length L plus twice the width W.

For example, to find the length of a rectangle that has a perimeter of 24 inches and a width of 4 inches, we use the formula. Thus,

$$24 = 2l + 2(4)$$

$$\begin{aligned} &= 2l + 8 \\ 24 - 8 &= 2l \\ 16 &= 2l \\ 8 &= l \end{aligned}$$

The length is 8 units.

The perimeter of a regular polygon with n sides is given as

$$P = n \cdot s.$$

For example, the perimeter of an equilateral triangle, a triangle with three equal sides, and a side length of 7 cm is $P = 3(7) = 21$ cm.

✓ Example $\{PageIndex3$: Finding the Perimeter of a Pentagon

Find the perimeter of a regular pentagon with a side length of 7 cm (Figure ✓ $\{PageIndex7$).

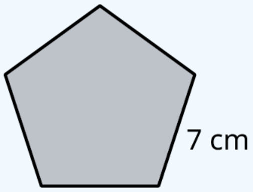


Figure ✓ $\{PageIndex7$

Answer

A regular pentagon has five equal sides. Therefore, the perimeter is equal to $P = 5(7) = 35 \text{ cm}$.

Your Turn $\{PageIndex3$

Find the perimeter of a square table that measures 30 inches across from one side to its opposite side.

✓ Example $\{PageIndex4$: Finding the Perimeter of an Octagon

Find the perimeter of a regular octagon with a side length of 14 cm (Figure ✓ $\{PageIndex8$).

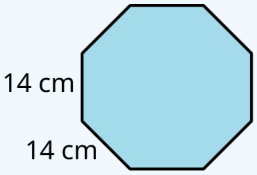


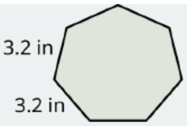
Figure ✓ $\{PageIndex8$

Answer

A regular octagon has eight sides of equal length. Therefore, the perimeter of a regular octagon with a side length of 14 cm is $P = 8(14) = 112 \text{ cm}$.

Your Turn $\{PageIndex1$

Find the perimeter of a regular heptagon with a side length of 3.2 as shown in the figure.



Figure

$\{PageIndex9$

Sum of Interior and Exterior Angles

To find the sum of the measurements of interior angles of a regular polygon, we have the following formula.

FORMULA

The sum of the interior angles of a polygon with n sides is given by

$$S=(n-2)180^{\circ}.$$

For example, if we want to find the sum of the interior angles in a parallelogram, we have

$$\begin{aligned}S&=(4-2)180^{\circ}\\&=2(180)=360^{\circ}\end{aligned}$$

Similarly, to find the sum of the interior angles inside a regular heptagon, we have

$$\begin{aligned}S&=(7-2)180^{\circ}\\&=5(180)\\&=900^{\circ}.\end{aligned}$$

To find the measure of each interior angle of a regular polygon with n sides, we have the following formula.

FORMULA

The measure of each interior angle of a regular polygon with n sides is given by

$$a=\frac{(n-2)180^{\circ}}{n}$$

For example, find the measure of an interior angle of a regular heptagon, as shown in Figure $\{PageIndex10$. We have

$$a=\frac{(7-2)180^{\circ}}{7}=128.57^{\circ}$$

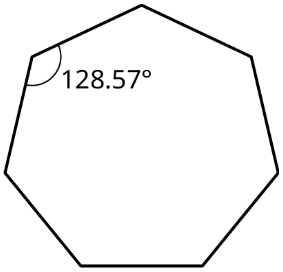


Figure $\{PageIndex10$: Interior Angles

✓ Example $\{PageIndex5$: Calculating the Sum of Interior Angles

Find the measure of an interior angle in a regular octagon using the formula, and then find the sum of all the interior angles using the sum formula.

Answer

An octagon has eight sides, so $n = 8$.

Step 1: Using the formula $a = \frac{(n-2)180^\circ}{8}$:

$$\begin{aligned} a &= \frac{(8-2)180^\circ}{8} \\ &= \frac{(6)180^\circ}{8} \\ &= 135^\circ. \end{aligned}$$

So, the measure of each interior angle in a regular octagon is 135°.

Step 2: The sum of the angles inside an octagon, so using the formula:

$$\begin{aligned} S &= (n-2)180^\circ \\ &= (8-2)180^\circ \\ &= 6(180) \\ &= 1,080^\circ. \end{aligned}$$

Step 3: We can test this, as we already know the measure of each angle is 135°. Thus, $8\left(135^\circ\right) = 1,080^\circ$.

Your Turn $\{PageIndex5$

Find the measure of each interior angle of a regular pentagon and then find the sum of the interior angles.

✓ Example $\{PageIndex6$: Calculating Interior Angles

Use algebra to calculate the measure of each interior angle of the five-sided polygon (Figure ✓ $\{PageIndex11$).

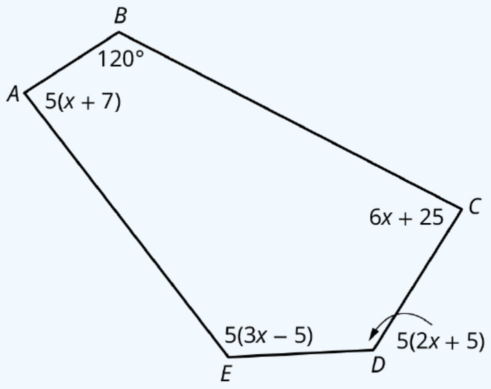


Figure ✓ $\{PageIndex11$

Answer

Step 1: Let us find out what the total of the sum of the interior angles should be. Use the formula for the sum of the angles in a polygon with n sides: $S = (n-2)180^\circ$. So, $S = (5-2)180^\circ = 540^\circ$.

Step 2: We add up all the angles and solve for x :

$$\begin{aligned} 5(x+7) + 120 + (6x+25) + 5(2x+5) + 5(3x-5) &= 540 \\ 5x + 6x + 10x + 15x + 180 &= 540 \\ 36x &= 360 \\ x &= 10 \end{aligned}$$

Step 3: We can then find the measure of each interior angle:

$$\begin{aligned}
&m\measuredangle A = 5(10+7) = 85^\circ \\
&m\measuredangle B = 120^\circ \\
&m\measuredangle C = 6(10) + 25 = 85^\circ \\
&m\measuredangle D = 5\left(2{*}10+5\right) = 125^\circ \\
&m\measuredangle E = 5\left(3{*}10-5\right) = 125^\circ
\end{aligned}$$

msup

📌 Your Turn $\{PageIndex6$

Find the sum of the measures of the interior angles and then find the measure of each interior angle in the figure shown.

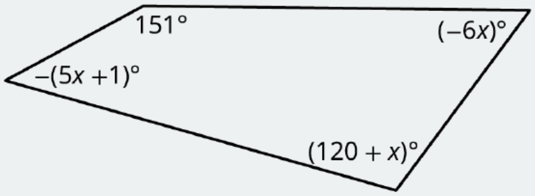


Figure 📌 $\{PageIndex12$

An **exterior angle** of a regular polygon is an angle formed by extending a side length beyond the closed figure. The measure of an exterior angle of a regular polygon with n sides is found using the following formula:

📌 FORMULA

To find the measure of an exterior angle of a regular polygon with n sides we use the formula

$$b = \frac{360^\circ}{n}.$$

In Figure $\{PageIndex13$, we have a regular hexagon $ABCDEF$. By extending the lines of each side, an angle is formed on the exterior of the hexagon at each vertex. The measure of each exterior angle is found using the formula, $b = \frac{360^\circ}{6} = 60^\circ$.

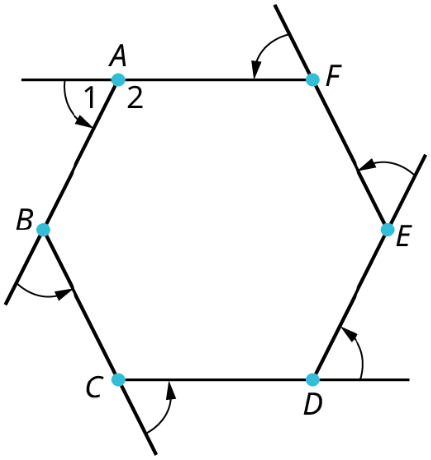


Figure $\{PageIndex13$: Exterior Angles

Now, an important point is that the sum of the exterior angles of a regular polygon with n sides equals 360°. This implies that when we multiply the measure of one exterior angle by the number of sides of the regular polygon, we should get 360°. For the example in Figure $\{PageIndex13$, we multiply the measure of each exterior angle, 60°, by the number of sides, six. Thus, the sum of the exterior angles is $6\left(60^\circ\right) = 360^\circ$.

✓ Example $\{PageIndex7$: Calculating the Sum of Exterior Angles

Find the sum of the measure of the exterior angles of the pentagon (Figure ✓ $\{PageIndex14$).

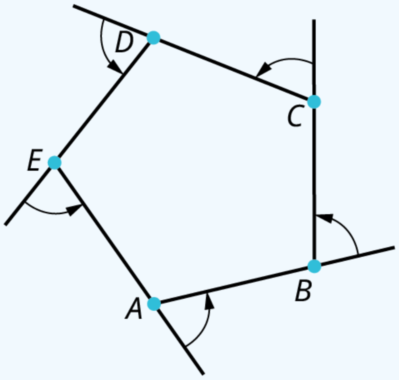


Figure ✓ $\{PageIndex14$

Answer

Each individual angle measures $\frac{360}{5} = 72^{\circ}$. Then, the sum of the exterior angles is $5\left(72^{\circ}\right) = 360^{\circ}$.

Your Turn $\{PageIndex7$

Find the sum of the measures of the exterior angles in the figure shown.

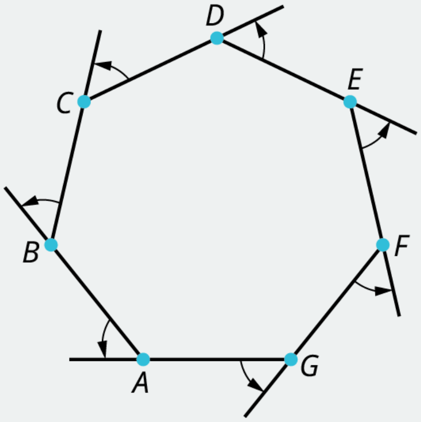


Figure $\{PageIndex15$

Circles and Circumference

The perimeter of a circle is called the **circumference**. To find the circumference, we use the formula

$$C = \pi d$$

where d is the diameter, the distance across the center, or

$$C = 2\pi r$$

where r is the radius.

FORMULA

The circumference of a circle is found using the formula $C = \pi d$, where d is the diameter of the circle, or $C = 2\pi r$, where r is the radius.

The radius is $1/2$ of the diameter of a circle. The symbol $\pi = 3.141592654\ldots$ is the ratio of the circumference to the diameter. Because this ratio is constant, our formula is accurate for any size circle. See Figure $\{PageIndex16$.

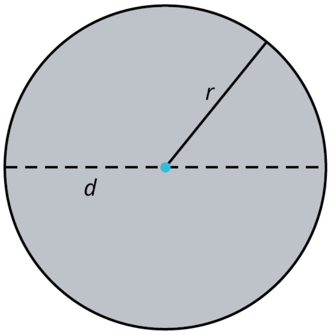


Figure $\{PageIndex16$: Circle Diameter and Radius

Let the radius be equal to 3.5 inches. Then, the circumference is

$$\begin{aligned} C &= 2\pi(3.5) \\ &= 21.99\text{in.} \end{aligned}$$

Example $\{PageIndex8$: Finding Circumference with Diameter

Find the circumference of a circle with diameter 10 cm.

Answer

If the diameter is 10 cm, the circumference is $C = 10\pi = 31.42\text{ cm}$.

Your Turn $\{PageIndex8$

Find the circumference of a circle with a radius of 2.25 cm.

Example $\{PageIndex9$: Finding Circumference with Radius

Find the radius of a circle with a circumference of 12 in.

Answer

If the circumference is 12 in, then the radius is

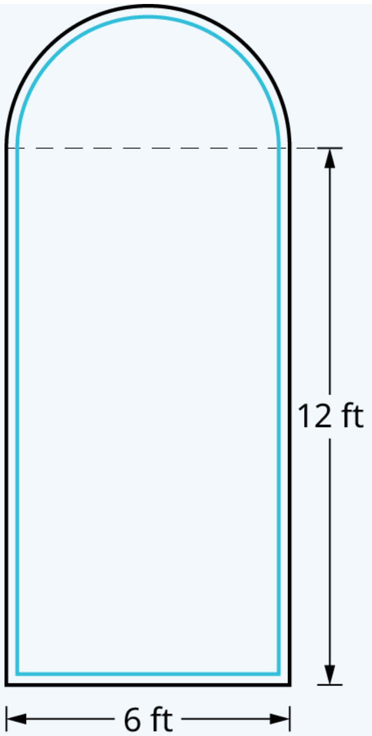
$$\begin{aligned} 12 &= 2\pi r \\ \frac{12}{2\pi} &= r = 1.91\text{in.} \end{aligned}$$

Your Turn $\{PageIndex9$

Find the radius of a circle with a circumference of 15.71 cm.

Example $\{PageIndex10$: Calculating Circumference for the Real World

You decide to make a trim for the window in Figure ✓ $\{PageIndex17$. How many feet of trim do you need to buy?



Figure

✓ $\{PageIndex17$

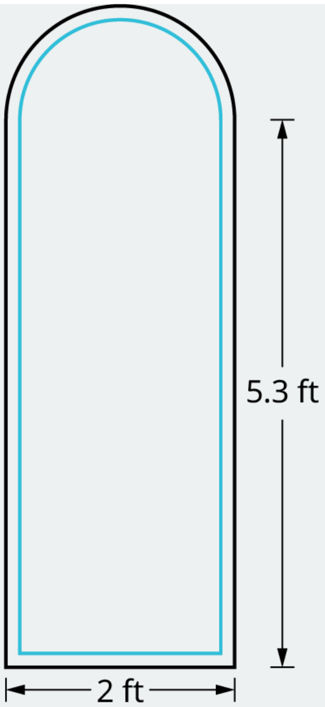
Answer

The trim will cover the 6 feet along the bottom and the two 12-ft sides plus the half circle on top. The circumference of a semicircle is ½ the circumference of a circle. The diameter of the semicircle is 6 ft. Then, the circumference of the semicircle would be 12πd=12π(6)=3πft=9.4ft.12πd=12π(6)=3πft=9.4ft.

Therefore, the total perimeter of the window is $6+12+12+9.4=39.4\text{ ft}$. You need to buy 39.4 ft of trim.

Your Turn $\{PageIndex10$

To make a trim for the window in the figure shown, you need the perimeter. How many feet of trim do you need to buy?



Figure

$\{PageIndex18$

People in Mathematics: Archimedes

The overwhelming consensus is that Archimedes (287–212 BCE) was the greatest mathematician of classical antiquity, if not of all time. A Greek scientist, inventor, philosopher, astronomer, physicist, and mathematician, Archimedes flourished in Syracuse, Sicily. He is credited with the invention of various types of pulley systems and screw pumps based on the center of gravity. He advanced numerous mathematical concepts, including theorems for finding surface area and volume. Archimedes anticipated modern calculus and developed the idea of the "infinitely small" and the method of exhaustion. The method of exhaustion is a technique for finding the area of a shape inscribed within a sequence of polygons. The areas of the polygons converge to the area of the inscribed shape. This technique evolved to the concept of limits, which we use today.

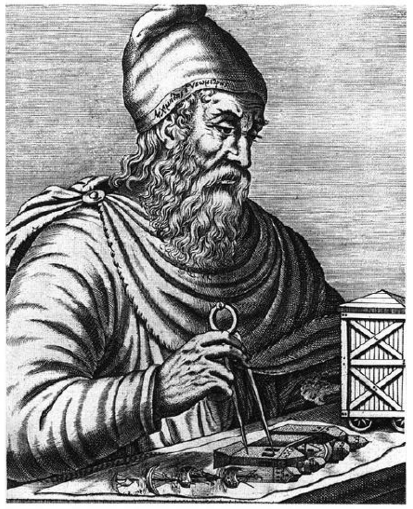
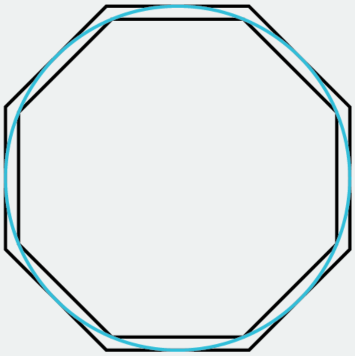


Figure $\{PageIndex19$: Archimedes (credit: "Archimedes" Engraving from the book Les vrais pourtraits et vies des hommes illustres grecz, latins et payens (1586)/Wikimedia Commons, Public Domain)

One of the more interesting achievements of Archimedes is the way he estimated the number pi, the ratio of the circumference of a circle to its diameter. He was the first to find a valid approximation. He started with a circle having a diameter of 1 inch.

His method involved drawing a polygon inscribed inside this circle and a polygon circumscribed around this circle. He knew that the perimeter of the inscribed polygon was smaller than the circumference of the circle, and the perimeter of the circumscribed polygon was larger than the circumference of the circle. This is shown in the drawing of an eight-sided polygon. He increased the number of sides of the polygon each time as he got closer to the real value of pi. The following table is an example of how he did this.

Sides	Inscribed Perimeter	Circumscribed Perimeter
4	2.8284	4.00
8	3.0615	3.3137
16	3.1214	3.1826
32	3.1365	3.1517
64	3.1403	3.1441



Archimedes settled on an approximation of $\pi \approx 3.1416$ after an iteration of 96 sides. Because pi is an irrational number, it cannot be written exactly. However, the capability of the supercomputer can compute pi to billions of decimal digits. As of 2002, the most precise approximation of pi includes 1.2 trillion decimal digits.

Who Knew?: The Platonic Solids

The Platonic solids (Figure 10.100) have been known since antiquity. A polyhedron is a three-dimensional object constructed with congruent regular polygonal faces. Named for the philosopher, Plato believed that each one of the solids is associated with one of the four elements: Fire is associated with the tetrahedron or pyramid, earth with the cube, air with the octahedron, and water with the icosahedron. Of the fifth Platonic solid, the dodecahedron, Plato said, "… God used it for arranging the constellations on the whole heaven."

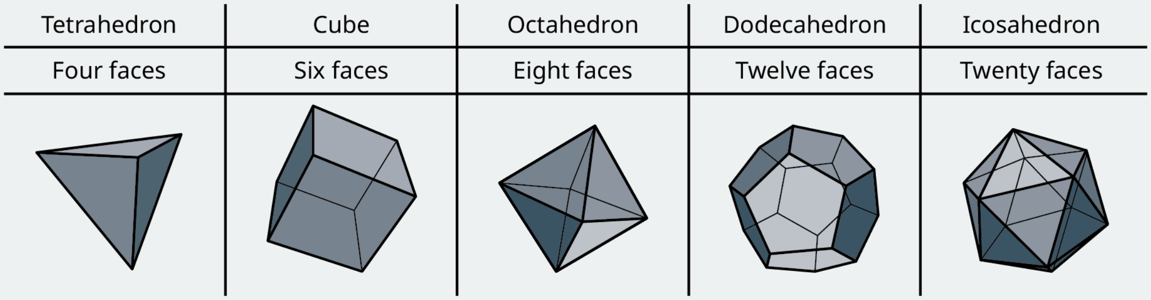
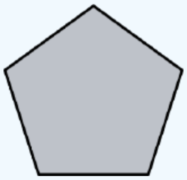


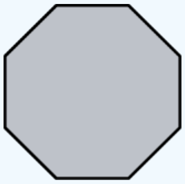
Figure $\{PageIndex20\}$: Platonic Solids

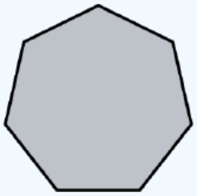
Plato believed that the combination of these five polyhedra formed all matter in the universe. Later, Euclid proved that exactly five regular polyhedra exist and devoted the last book of the *Elements* to this theory. These ideas were resuscitated by Johannes Kepler about 2,000 years later. Kepler used the solids to explain the geometry of the universe. The beauty and symmetry of the Platonic solids have inspired architects and artists from antiquity to the present.

Exercise 10.4.1

In the following exercises, identify the regular polygons.







Exercise 10.4.2

Find the perimeter of the regular pentagon with side length of 6 cm.

Exercise 10.4.3

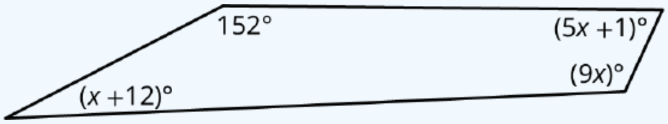
Find the sum of the interior angles of a regular hexagon.

Exercise 10.4.4

Find the measure of each interior angle of a regular hexagon.

Exercise 10.4.5

Find the measurements of each angle in the given figure.



Exercise 10.4.6

Find the circumference of the circle with radius equal to 3 cm.

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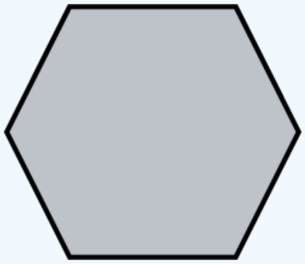
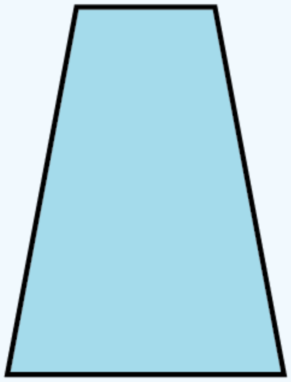
10.4.0: Exercises

In the following exercises, identify the polygons.

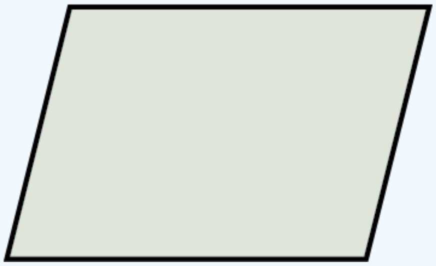
? Exercise 10.4.0.1



? Exercise 10.4.0.2



? Exercise 10.4.0.3



? Exercise 10.4.0.4

Find the perimeter of a regular hexagon with side length equal to 12 cm.

? Exercise 10.4.0.5

A regular quadrilateral has a perimeter equal to 72 in. Find the length of each side.

? Exercise 10.4.0.6

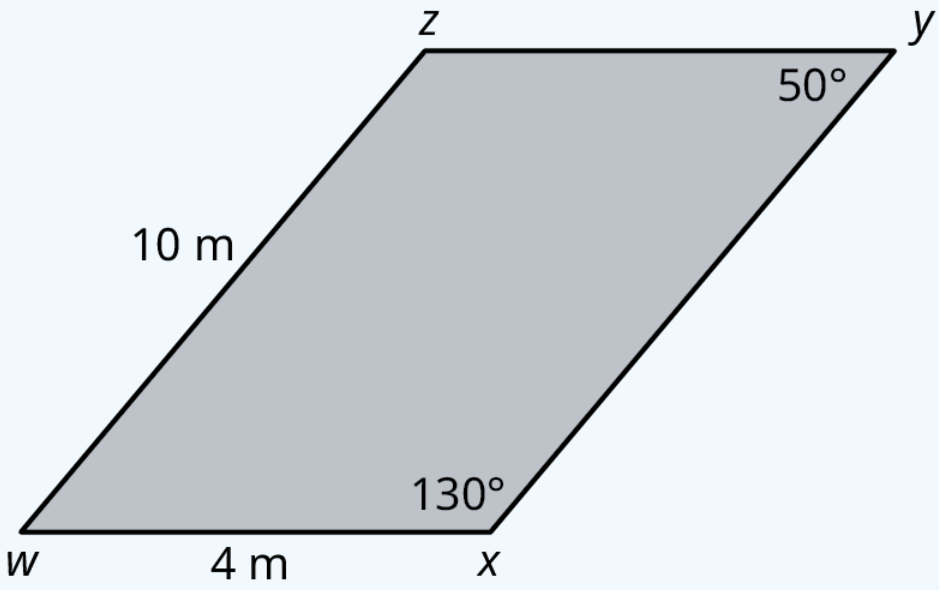
The perimeter of an equilateral triangle is 72 cm. Find the length of each side.

? Exercise 10.4.0.7

Find the dimensions of a rectangular region with perimeter of 34 m, where the shorter side is 13 less than twice the longer side. Let $x = \text{the longer side}$. Then, the shorter side is $(2x - 13)$.

? Exercise 10.4.0.8

Find the perimeter of the figure shown.

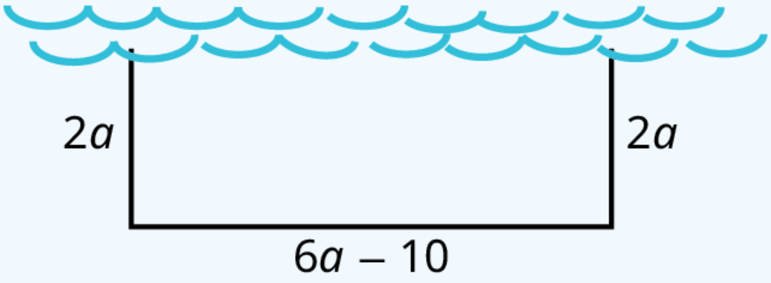


? Exercise 10.4.0.9

Find the perimeter of a fenced-in area where the length is 22 m, and the width is ½ of the length plus 3.

? Exercise 10.4.0.10

You have 140 ft of fencing to enclose a rectangular region that borders a river. You do not have to fence in the side that borders the river. The width is equal to $2a$, and the length is equal to six times the width less 10. Find the dimensions of the region to be fenced.



? Exercise 10.4.0.11

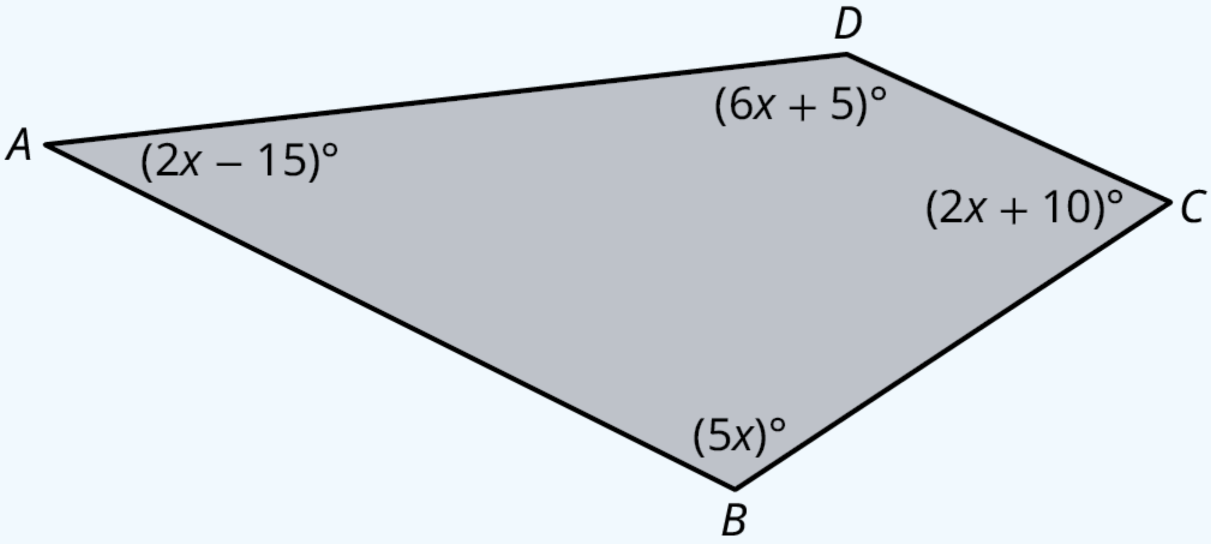
What is the measure of each interior angle of a regular hexagon?

Exercise 10.4.0.12

What is the sum of the interior angles of a triangle?

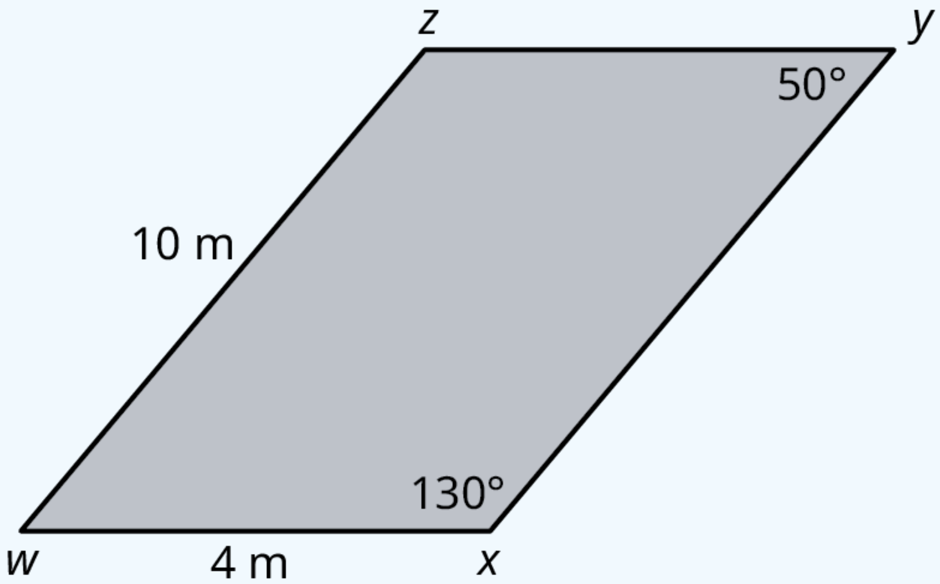
Exercise 10.4.0.13

Use algebra to find the measure of each angle of the quadrilateral shown.



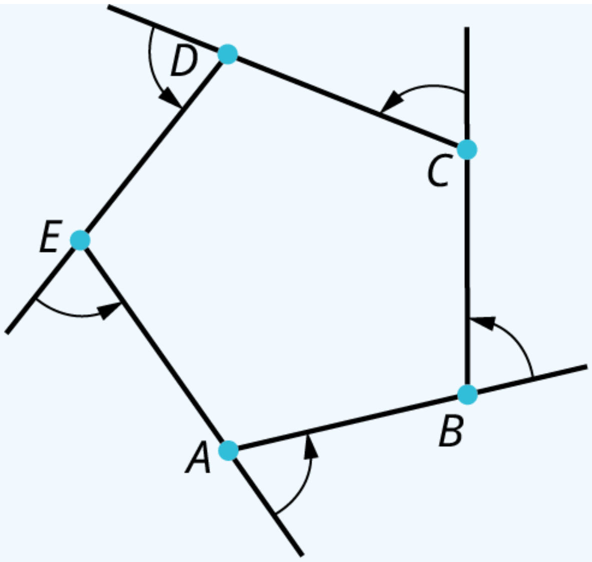
Exercise 10.4.0.14

Find the missing sides and angles of the parallelogram shown.



Exercise 10.4.0.15

Find the sum of the interior and exterior angles of the regular pentagon shown.



? **Exercise 10.4.0.16**

What is the measure of each exterior angle of a regular hexagon?

? **Exercise 10.4.0.17**

What is the sum of the exterior angles of a triangle?

? **Exercise 10.4.0.18**

What is the sum of the exterior angles of an octagon?

? **Exercise 10.4.0.19**

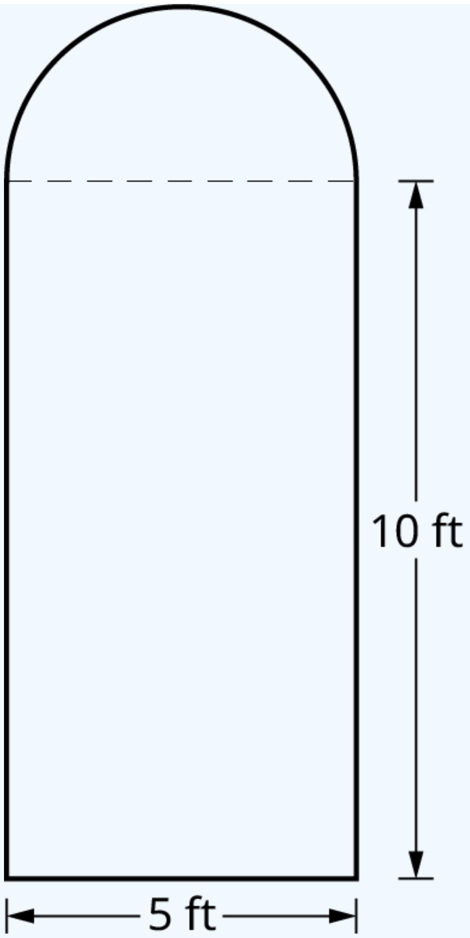
Find the circumference of a circle with radius 5.

? **Exercise 10.4.0.20**

Find the circumference of a circle with a diameter of 7.

? **Exercise 10.4.0.21**

Find the perimeter of the window in the figure shown.

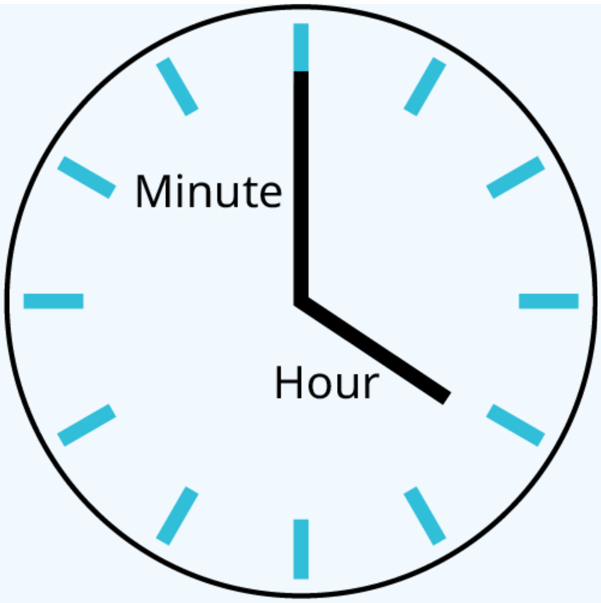


? Exercise 10.4.0.22

Find the circumference of a circle with a radius of 1.25 cm.

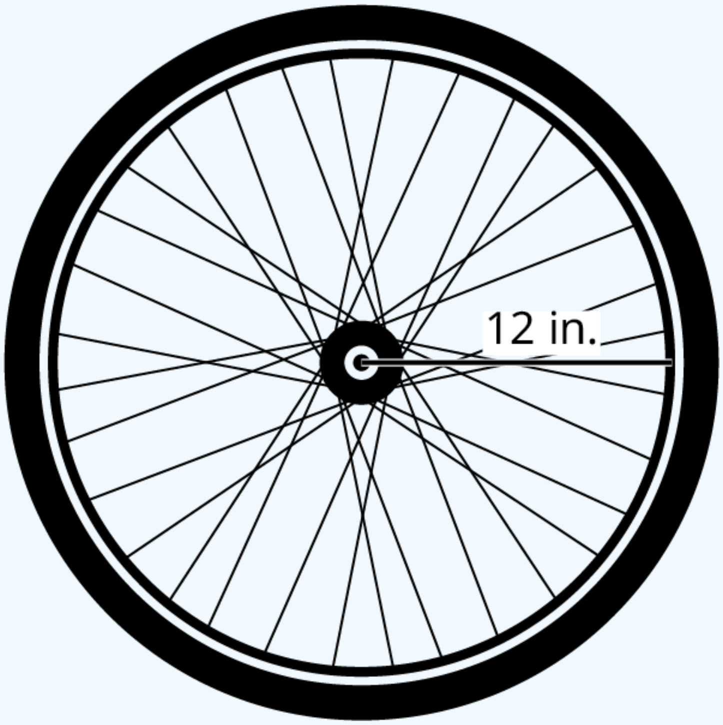
? Exercise 10.4.0.23

The hands of a clock vary in length. If the hour hand is 5 inches long and the minute hand is 7 inches long, how far does each hand travel in 12 hours?



? Exercise 10.4.0.24

A bicycle tire has a radius of 12 in. How many revolutions will the tire make if the bicycle travels approximately 377 ft?



? Exercise 10.4.0.25

When Chicago installed its Centennial Ferris wheel at Navy Pier in 1995, there was much discussion about how big it could be. The city wanted it to be as high as possible because the incredible views of the city would surely draw the tourists. It was decided that the height of the wheel could safely reach 150 ft. If the wheel begins 10 ft above ground level, what is the circumference of the Ferris wheel?

? Exercise $10.4.0.26$

Find the perimeter of an equilateral triangle with side length equal to 21 cm.

? Exercise $10.4.0.27$

What is the diameter of a circle whose circumference is 39.6 cm?

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10.5: Tessellations

Learning Objectives

1. Apply translations, rotations, and reflections.
2. Determine if a shape tessellates.

The illustration shown above (Figure 10.5.1) is an unusual pattern called a Penrose tiling. Notice that there are two types of shapes used throughout the pattern: smaller green parallelograms and larger blue parallelograms. What's interesting about this design is that although it uses only two shapes over and over, there is no repeating pattern.

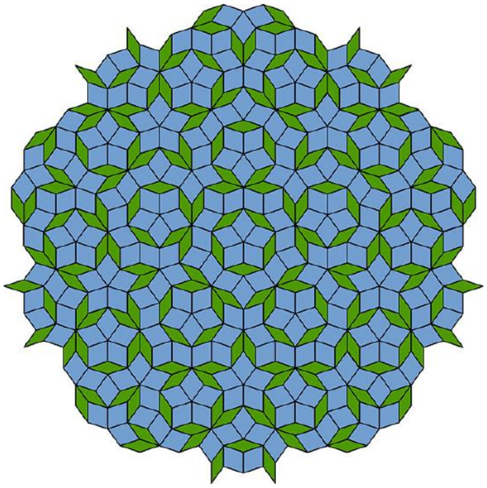


Figure 10.5.1: Penrose tiling represents one type of tessellation. (credit: "Penrose Tiling" by Inductiveload/Wikimedia Commons, Public Domain)

In this section, we will focus on patterns that do repeat. Repeated patterns are found in architecture, fabric, floor tiles, wall patterns, rug patterns, and many unexpected places as well. It may be a simple hexagon-shaped floor tile, or a complex pattern composed of several different motifs. These two-dimensional designs are called regular (or periodic) tessellations. There are countless designs that may be classified as regular tessellations, and they all have one thing in common—their patterns repeat and cover the plane.

We will explore how tessellations are created and experiment with making some of our own as well. The topic of tessellations belongs to a field in mathematics called transformational geometry, which is a study of the ways objects can be moved while retaining the same shape and size. These movements are termed *rigid motions* and *symmetries*.

Who Knew?: M. C. Escher

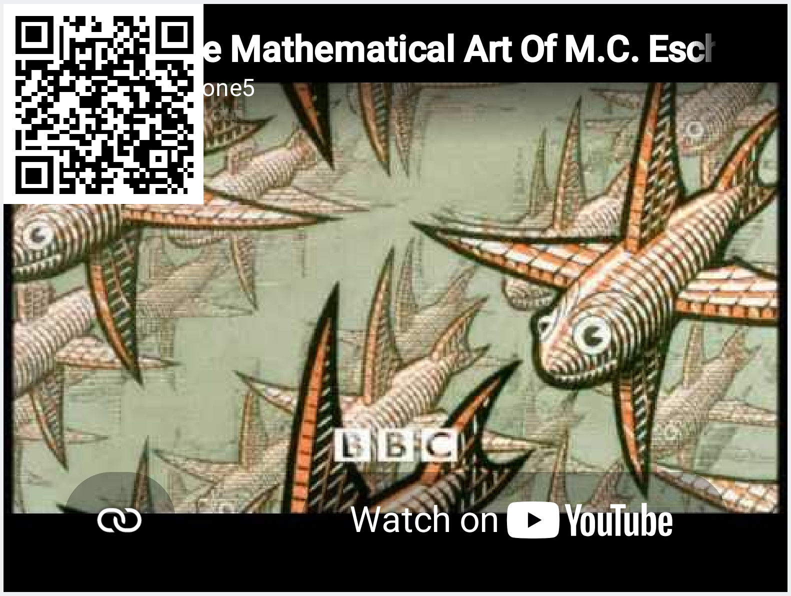
A good place to start the study of tessellations is with the work of M. C. Escher. The Dutch graphic artist was famous for the dimensional illusions he created in his woodcuts and lithographs, and that theme is carried out in many of his tessellations as well. Escher became obsessed with the idea of the “regular division of the plane.” He sought ways to divide the plane with shapes that would fit snugly next to each other with no gaps or overlaps, represent beautiful patterns, and could be repeated infinitely to fill the plane. He experimented with practically every geometric shape imaginable and found the ones that would produce a regular division of the plane. The idea is similar to dividing a number by one of its factors. When a number divides another number evenly, there are no remainders, like there are no gaps when a shape divides or fills the plane.

Escher went far beyond geometric shapes, beyond triangles and polygons, beyond irregular polygons, and used other shapes like figures, faces, animals, fish, and practically any type of object to achieve his goal; and he did achieve it, beautifully, and left it for the ages to appreciate.

Video

C. Escher - How To Create A Tes
al Impact Systems - Advertising and Marketing
Watch on YouTube

M.C. Escher: How to Create a Tessellation


e Mathematical Art Of M.C. Esc
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The Mathematical Art of M.C. Escher

Tessellation Properties and Transformations

A regular tessellation means that the pattern is made up of congruent regular polygons, same size and shape, including *some type of movement*; that is, some type of transformation or symmetry. Here we consider the rigid motions of translations, rotations, reflections, or glide reflections. A plane of tessellations has the following properties:

- Patterns are repeated and fill the plane.
- There are no gaps or overlaps. Shapes must fit together perfectly. (It was Escher who determined that a proper tessellation could have no gaps and no overlaps.)
- Shapes are combined using a transformation.
- All the shapes are joined at a vertex. In other words, if you were to draw a circle around a vertex, it would include a corner of each shape touching at that vertex.
- For a tessellation of regular congruent polygons, the sum of the measures of the interior angles that meet at a vertex equals 360°.

In Figure $10.5.2$, the tessellation is made up of squares. There are four squares meeting at a vertex. An interior angle of a square is 90° and the sum of four interior angles is 360°. In Figure $10.5.3$, the tessellation is made up of regular hexagons. There are three hexagons meeting at each vertex. The interior angle of a hexagon is 120°, and the sum of three interior angles is 360°. Both tessellations will fill the plane, there are no gaps, the sum of the interior angle meeting at the vertex is 360°, and both are achieved by translation transformations. These tessellations work because all the properties of a tessellation are present.

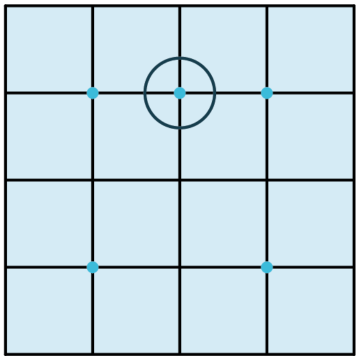


Figure $10.5.2$: Tessellation – Squares

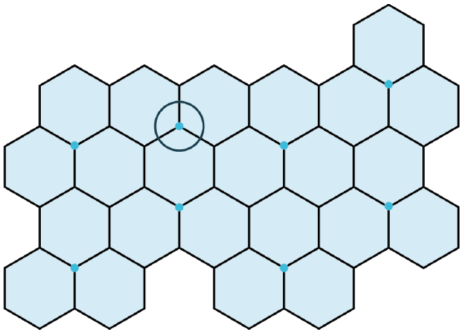


Figure $10.5.3$: Tessellation – Hexagons

The movements or rigid motions of the shapes that define tessellations are classified as translations, rotations, reflections, or glide reflections. Let’s first define these movements and then look at some examples showing how these transformations are revealed.

Translation

A translation is a movement that shifts the shape vertically, horizontally, or on the diagonal. Consider the trapezoid semanticsABCDFigure 10.104. We have translated it 3 units to the right and 3 units up. That means every corner is moved by the number of units and in the direction specified. Mathematicians will indicate this movement with a vector, an arrow that is drawn to illustrate the criteria and the magnitude of the translation. The location of the translated trapezoid is marked with the vertices, $A'B'C'D'$, but it is still the exact same shape and size as the original trapezoid semanticsABCDABCD.

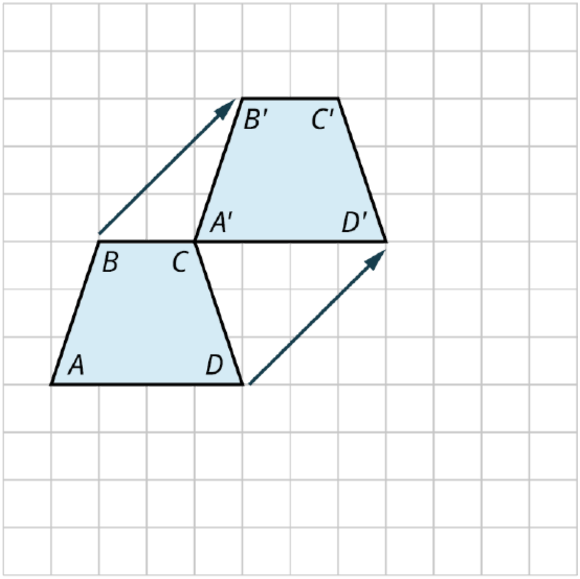


Figure 10.5.4: Translation

✓ Example 10.5.1: Creating a Translation

Suppose you have a hexagon on a grid as in Figure 10.105. Translate the hexagon 5 units to the right and 3 units up.

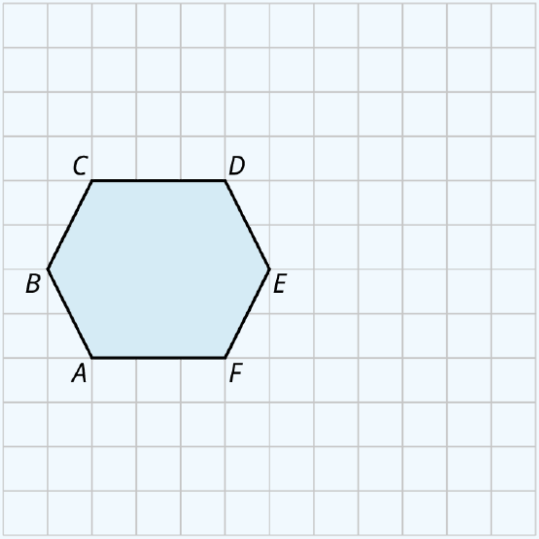
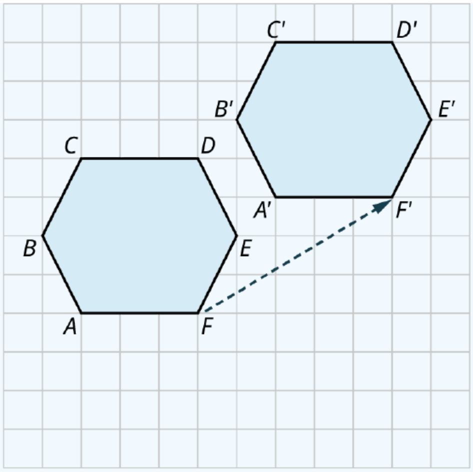


Figure ✓ 10.5.5

Answer

The best way to do this is by translating the individual points A, B, C, D, E, F. Once translated, the points become A', B', C', D', E', F'.

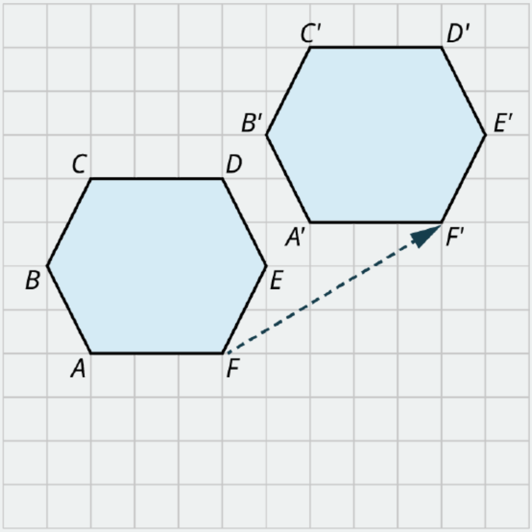


Figure

✓ 10.5.6

Your Turn 10.5.1

Translate the hexagon with points A', B', C', D', E', F' 6 units down.



Figure

10.5.7

Rotation

The rotation transformation occurs when you rotate a shape about a point and at a predetermined angle. In Figure $10.5.8$, the triangle is rotated around the rotation point by 90°, and then translated 7 units up and 4 units over to the right. That means that each corner is translated to the new location by the same number of units and in the same direction.

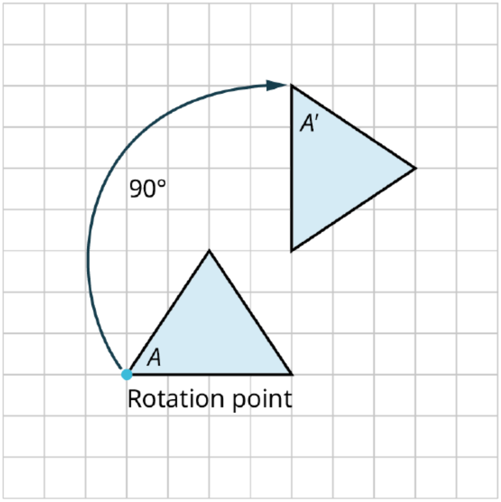


Figure $10.5.8$: Rotation

We can see that semanticsΔAΔA is mapped to ΔA′ΔA′ by a rotation of 90° up and to the right. If rotated again by 90°, the triangle would be upside down.

✓ Example $10.5.2$: Applying a Rotation

Figure ✓ $10.5.9$ illustrates a tessellation begun with an equilateral triangle. Explain how this pattern is produced.

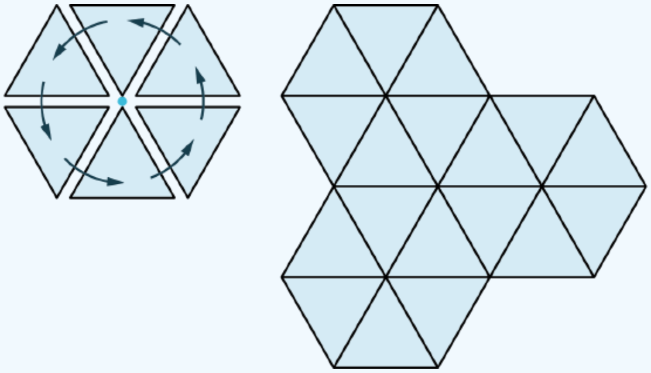


Figure
✓ $10.5.9$

Answer

A rotation to the right or to the left around the vertex by 60°, six times, produces the hexagonal shape. The sixth rotation brings the triangle back to its original position. Then, a reflection up and another one on the diagonal will reproduce the pattern. When a shape returns to its original position by a rotation, we say that it has rotational symmetry.

Your Turn $10.5.2$

Starting with the triangle in the figure shown, explain how the pattern on the right was achieved.

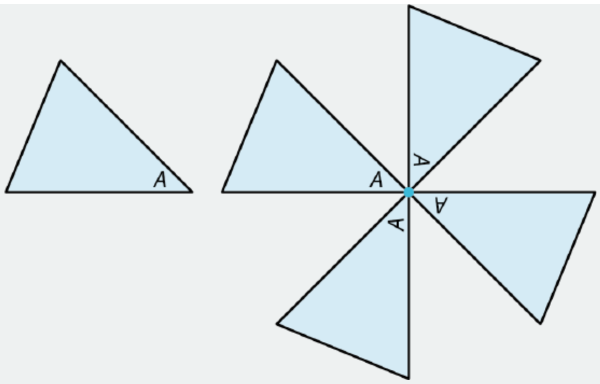


Figure 10.5.10

Reflection

A reflection is the third transformation. A shape is reflected about a line and the new shape becomes a mirror image. You can reflect the shape vertically, horizontally, or on the diagonal. There are two shapes in Figure 10.5.11. The quadrilateral is reflected horizontally; the arrow shape is reflected vertically.

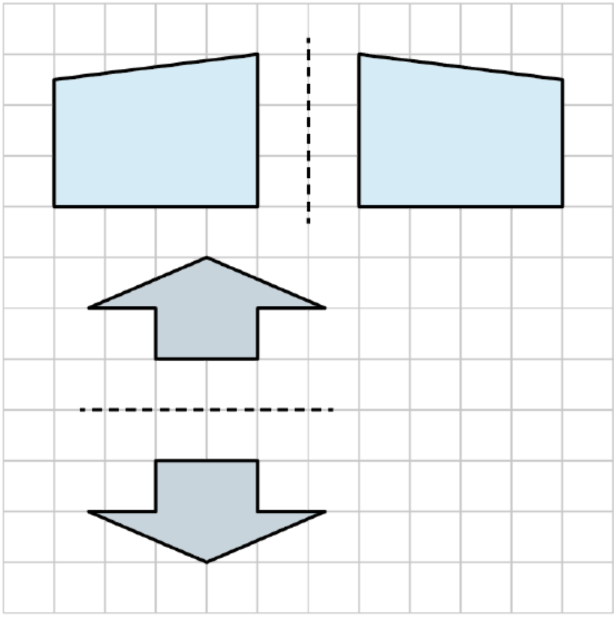


Figure 10.5.11: Reflection

Glide Reflection

The glide reflection is the fourth transformation. It is a combination of a reflection and a translation. This can occur by first reflecting the shape and then gliding or translating it to its new location, or by translating first and then reflecting. The example in Figure 10.5.12 shows a trapezoid, which is reflected over the dashed line, so it appears upside down. Then, we shifted the shape horizontally by 6 units to the right. Whether we use the glide first or the reflection first, the end result is the same in most cases. However, the tessellation shown in the next example can only be achieved by a reflection first and then a translation.

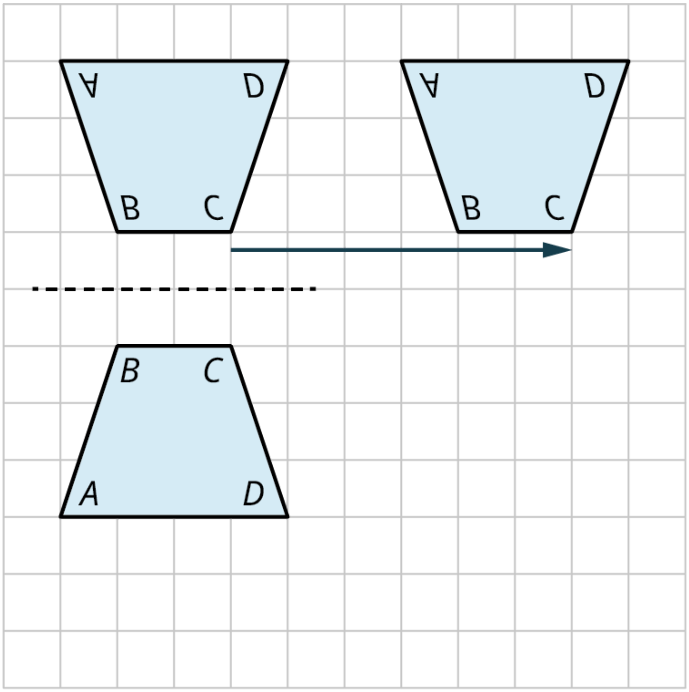


Figure 10.5.12: Glide Reflection

✓ Example 10.5.3: Applying the Glide Reflection

An obtuse triangle is reflected about the dashed line, and the two shapes are joined together. How does the tessellation shown in Figure ✓ 10.5.13 materialize?

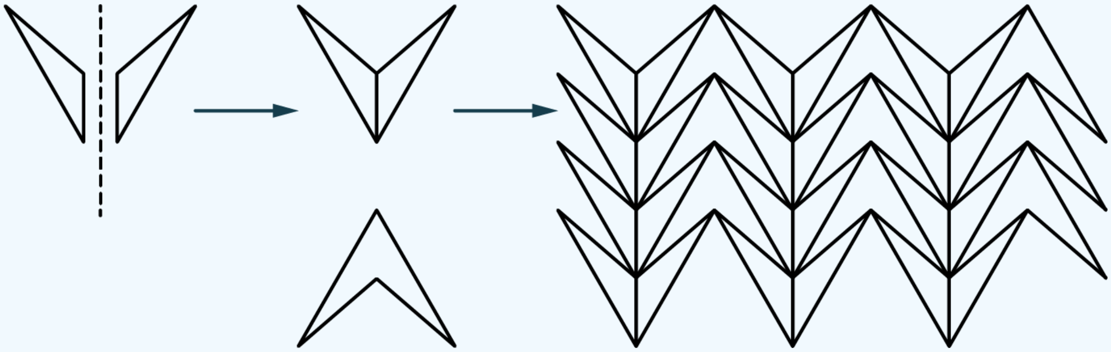


Figure ✓ 10.5.13

Answer

The new shape is reflected horizontally and joined with the original shape. It is then translated vertically and horizontally to make up the tessellation. Notice the blank spaces next to the vertical pattern. These areas are made up of the exact original shape rotated 180°, but with no line up the center. These rotated shapes are translated horizontally and vertically, and thus, the plane is tessellated with no gaps. This is an example of a glide reflection where the order of the transformations matters.

Your Turn 10.5.3

Explain how this tessellation of equilateral triangles could be produced.

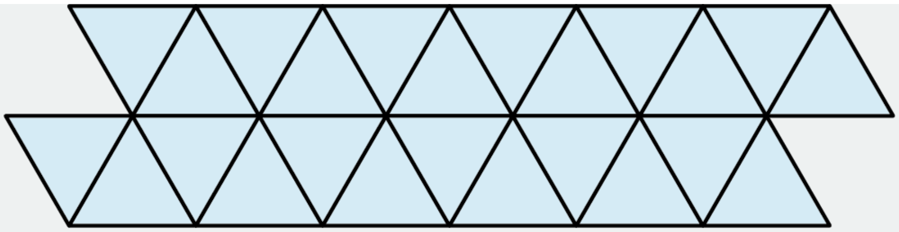


Figure 📌 10.5.14

✓ Example 10.5.4: Applying More Than One Tessellation

Show how this tessellation (Figure ✓ 10.5.15) can be achieved.

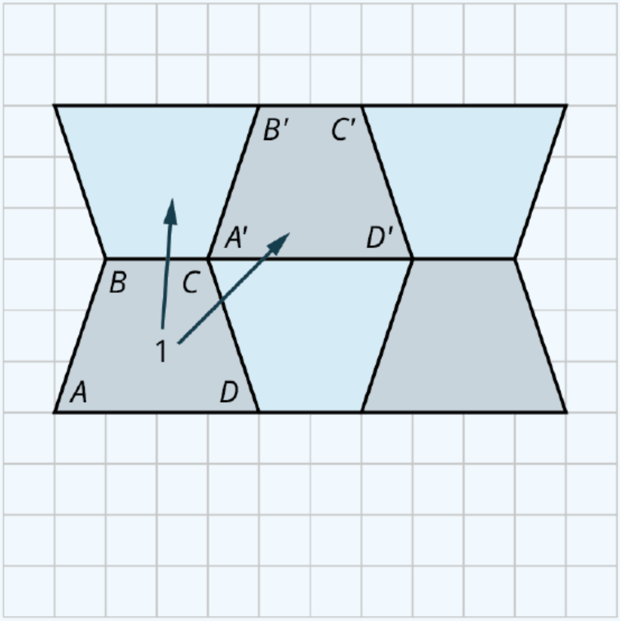


Figure ✓ 10.5.15

Answer

This is a tessellation that has one color on the front of the trapezoid and a different color on the back. There is a translation on the diagonal, and a reflection vertically. These are two separate transformations resulting in two new placements of the trapezoid. We can call this a combination of two transformations or a glide reflection.

📌 Your Turn 10.5.4

How does this tessellation of the squares come about?

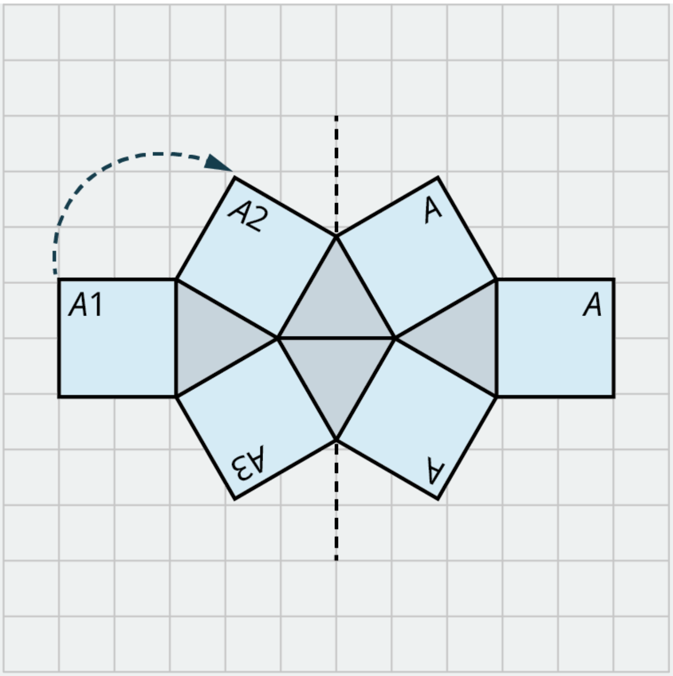


Figure 10.5.16

Interior Angles

The sum of the interior angles of a tessellation is 360°. In Figure $10.5.17$, the tessellation is made of six triangles formed into the shape of a hexagon. Each angle inside a triangle equals 60°, and the six vertices meet the sum of those interior angles, $6\left(60^{\circ}\right)=360^{\circ}$.

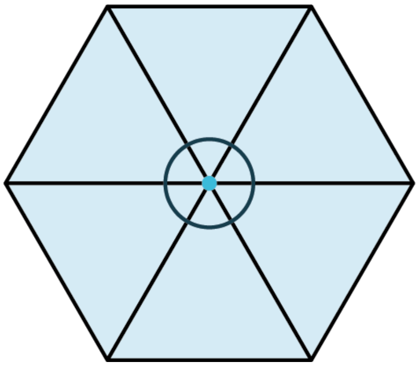


Figure $10.5.17$: Interior Angles at the Vertex of Triangles

In Figure $10.5.18$, the tessellation is made up of trapezoids, such that two of the interior angles of each trapezoid equals 75° and the other two angles equal 105°. Thus, the sum of the interior angles where the vertices of four trapezoids meet equals $105^{\circ}+75^{\circ}+75^{\circ}+105^{\circ}=360^{\circ}$.

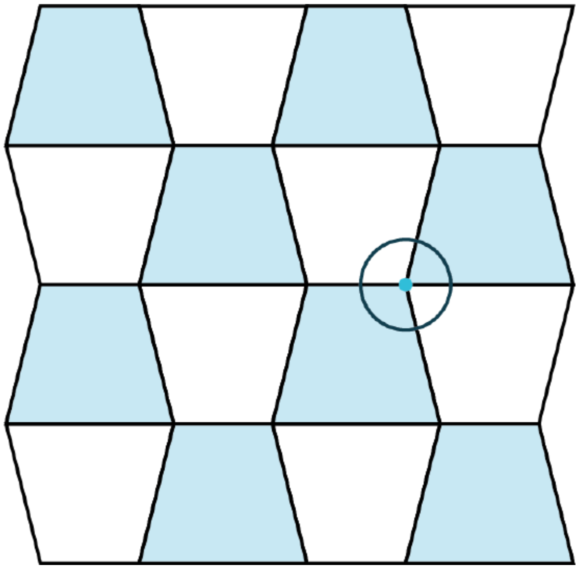


Figure $10.5.18$: Interior Angles at the Vertex of Trapezoids

These tessellations illustrate the property that the shapes meet at a vertex where the interior angles sum to 360°.

Tessellating Shapes

We might think that all regular polygons will tessellate the plane by themselves. We have seen that squares do and hexagons do. The pattern of squares in Figure $10.5.19$ is a translation of the shape horizontally and vertically. The hexagonal pattern in Figure $10.5.20$, is translated horizontally, and then on the diagonal, either to the right or to the left. This particular pattern can also be formed by rotations. Both tessellations are made up of congruent shapes and each shape fits in perfectly as the pattern repeats.

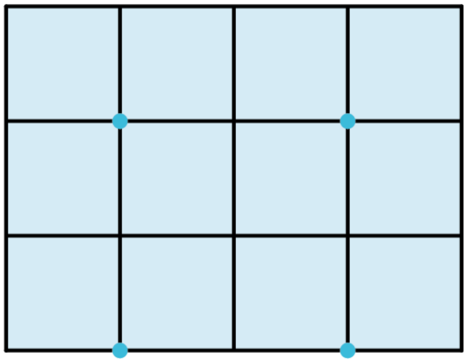


Figure $10.5.19$: Translation Horizontally and Vertically

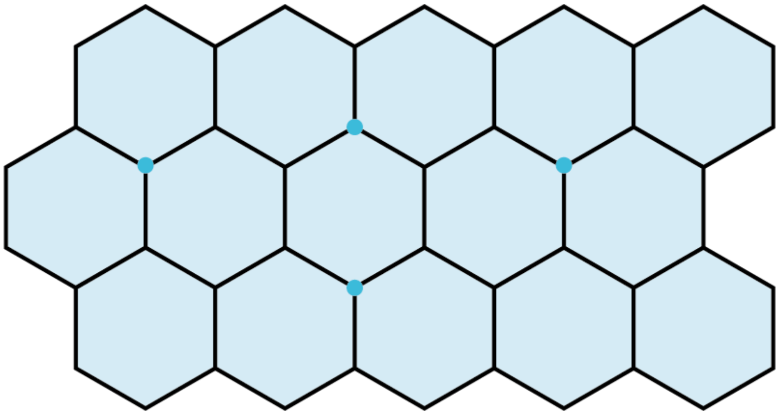


Figure $10.5.20$: Translation Horizontally and Slide Diagonally

We have also seen that equilateral triangles will tessellate the plane without gaps or overlaps, as shown in Figure $10.5.21$. The pattern is made by a reflection and a translation. The darker side is the face of the triangle and the lighter side is the back of the triangle, shown by the reflection. Each triangle is reflected and then translated on the diagonal.

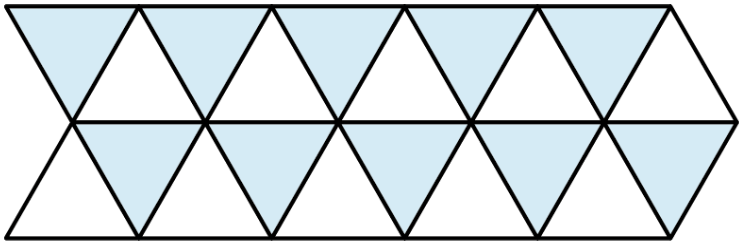


Figure 10.5.21: Reflection and Glide Translation

Escher experimented with all regular polygons and found that only the ones mentioned, the equilateral triangle, the square, and the hexagon, will tessellate the plane by themselves. Let’s try a few other regular polygons to observe what Escher found.

✓ Example 10.5.5: Tessellating the Plane

Do regular pentagons tessellate the plane by themselves (Figure ✓ 10.5.22)?

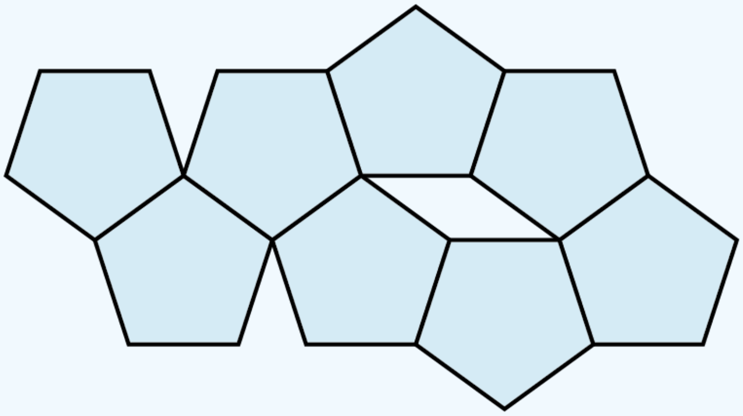


Figure ✓ 10.5.22

Answer

We can see that regular pentagons do not tessellate the plane by themselves. There is a gap, a gap in the shape of a parallelogram. We conclude that regular pentagons will not tessellate the plane by themselves.

Your Turn 10.5.5

Do regular heptagons tessellate the plane by themselves?

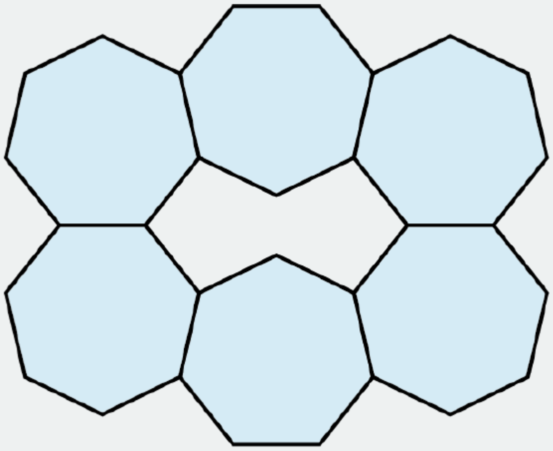


Figure 10.5.23

✓ Example 10.5.6: Tessellating Octagons

Do regular octagons tessellate the plane by themselves (Figure ✓ 10.5.24)?

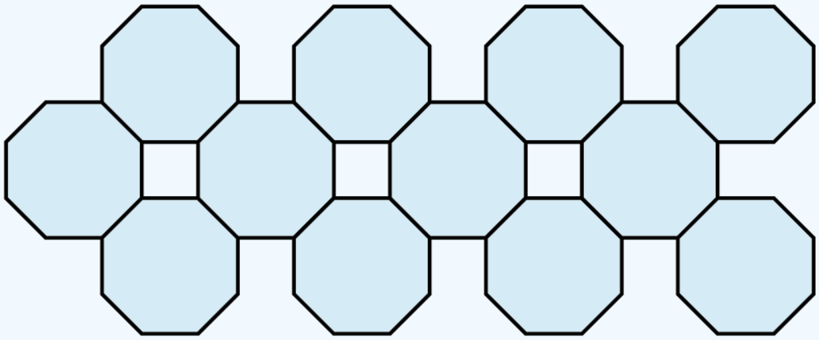


Figure ✓ 10.5.24

Answer

Again, we see that regular octagons do not tessellate the plane by themselves. The gaps, however, are squares. So, two regular polygons, an octagon and a square, do tessellate the plane.

Your Turn 10.5.6

Do regular dodecagons (12-sided regular polygons) tessellate the plane by themselves?

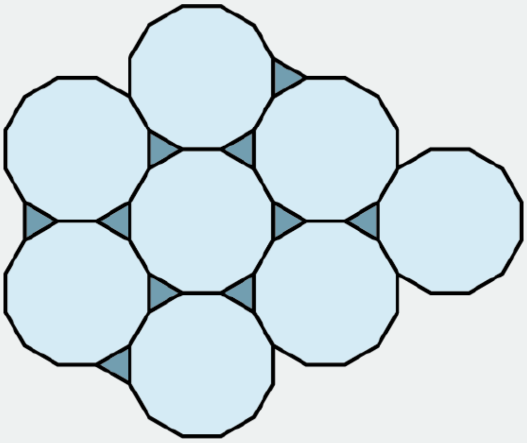


Figure 10.5.25

Just because regular pentagons do not tessellate the plane by themselves does not mean that there are no pentagons that tessellate the plane, as we see in Figure 10.5.26.

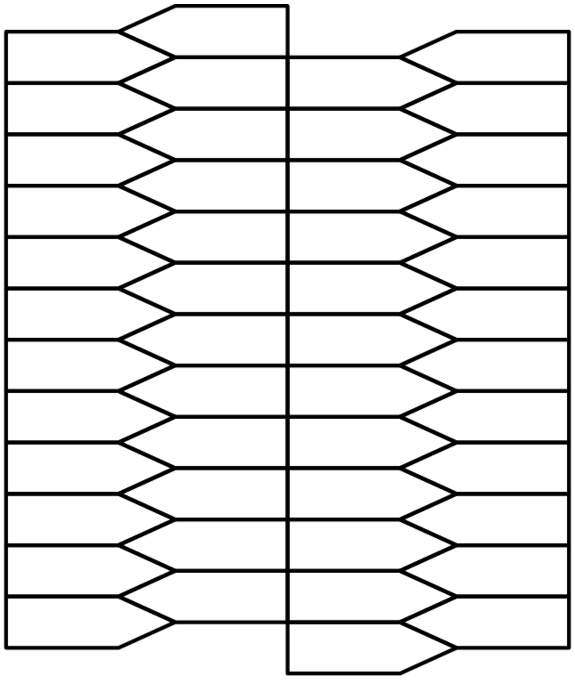


Figure $10.5.26$: Tessellation of Pentagons

Another example of an irregular polygon that tessellates the plane is by using the obtuse irregular triangle from a previous example. What transformations should be performed to produce the tessellation shown in Figure $10.5.27$?

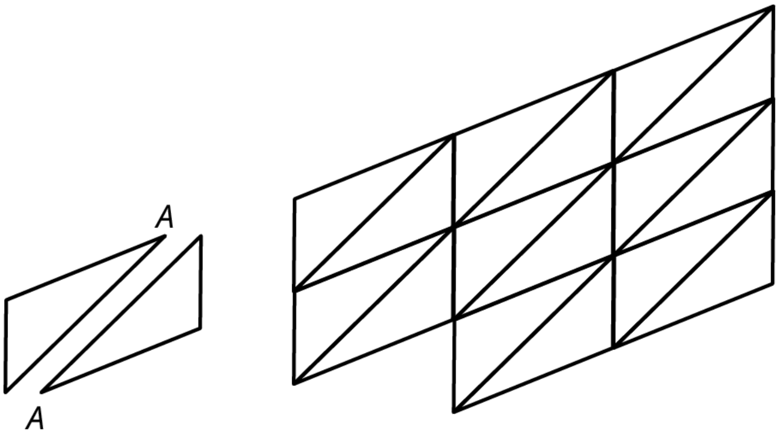


Figure 10.127: Tessellating with Obtuse Irregular Triangles

First, the triangle is reflected over the tip at point A, and then translated to the right and joined with the original triangle to form a parallelogram. The parallelogram is then translated on the diagonal and to the right and to the left.

Naming

A tessellation of squares is named by choosing a vertex and then counting the number of sides of each shape touching the vertex. Each square in the tessellation shown in Figure $10.5.28$ has four sides, so starting with square A, the first number is 4, moving around counterclockwise to the next square meeting the vertex, square B, we have another 4, square C adds another 4, and finally square D adds a fourth 4. So, we would name this tessellation a 4.4.4.4.

The hexagon tessellation, shown in Figure $10.5.29$ has six sides to the shape and three hexagons meet at the vertex. Thus, we would name this a 6.6.6. The triangle tessellation, shown in Figure $10.5.30$ has six triangles meeting the vertex. Each triangle has three sides. Thus, we name this a 3.3.3.3.3.3.

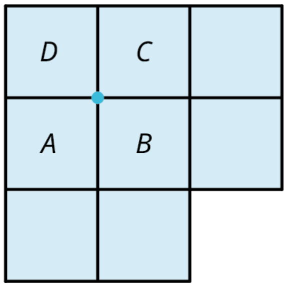


Figure 10.5.28: 4.4.4.4

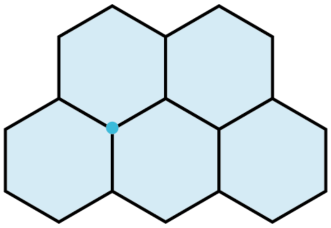


Figure 10.5.29: 6.6.6

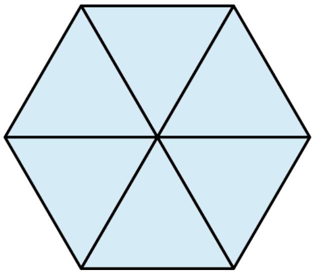


Figure 10.5.30: 3.3.3.3.3.3

✓ Example 10.5.7: Creating Your Own Tessellation

Create a tessellation using two colors and two shapes.

Answer

We used a parallelogram and an isosceles triangle. The parallelogram is reflected vertically and horizontally so that only every other corner touches. The triangles are reflected vertically and horizontally and then translated over the parallelogram. The result is alternating vertical columns of parallelograms and then triangles (Figure 10.131).

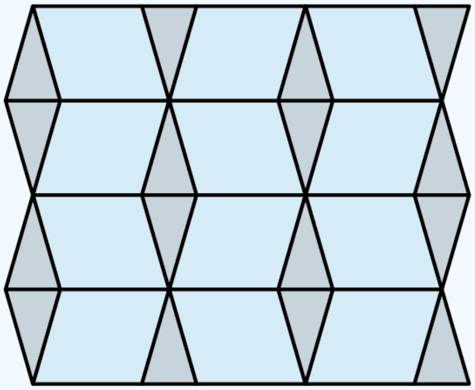


Figure ✓ 10.5.31

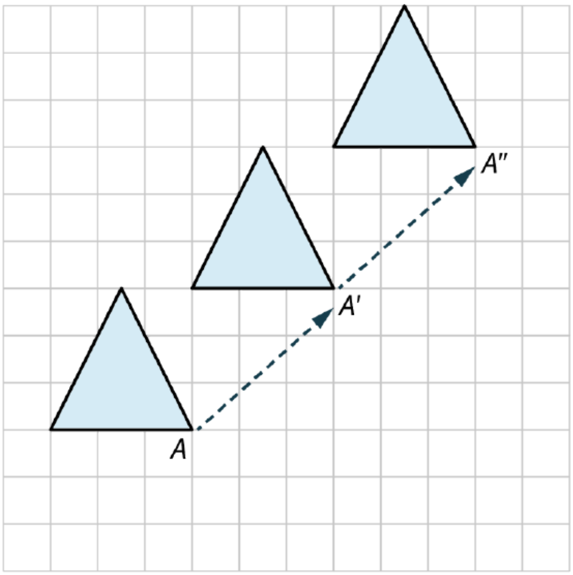
Your Turn 10.5.7

Create a tessellation using polygons, regular or irregular.

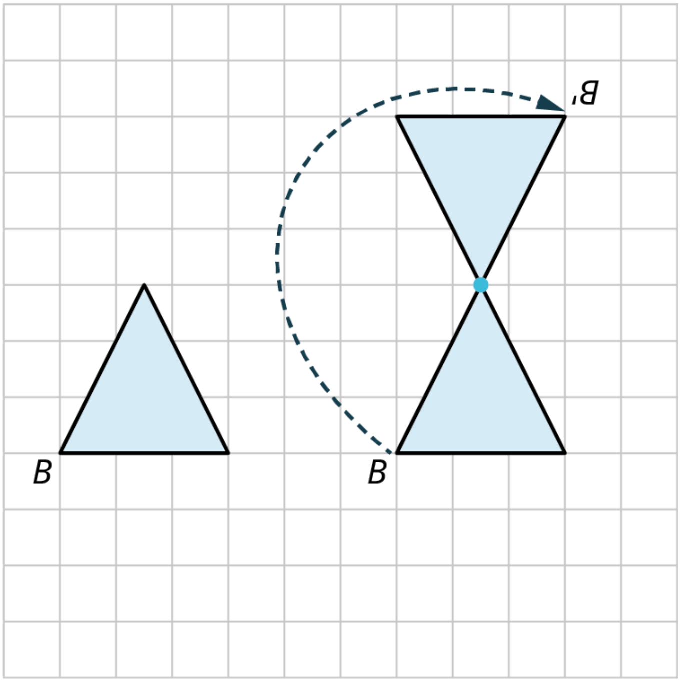
Check Your Understanding

1. What are the properties of repeated patterns that let them be classified as tessellations?

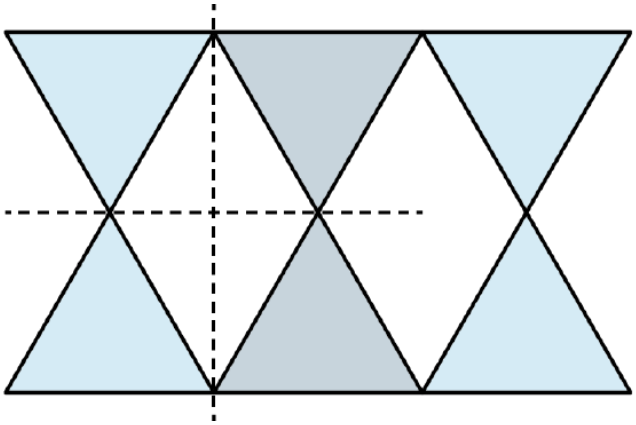
2. Explain how the using the transformation of a translation is applied to the movement of this shape starting with point A.



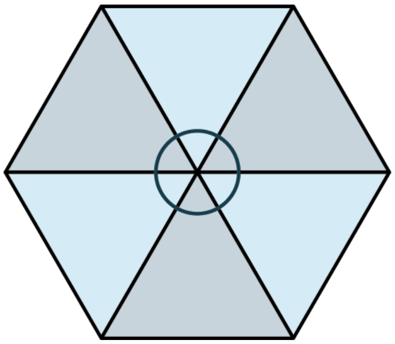
3. Starting with the triangle with vertex B, describe how the transformation in this drawing is achieved.



4. Starting with a triangle with a darker face and a lighter back, describe how this pattern came about.



5. Name the tessellation in the figure shown.



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10.5.0: Exercises

? Exercise 10.5.0.1

What type of movements are used to change the orientation and placement of a shape?

? Exercise 10.5.0.2

What is the name of the motion that renders a shape upside down?

? Exercise 10.5.0.3

What do we call the motion that moves a shape to the right or left or on the diagonal?

? Exercise 10.5.0.4

If you are going to tessellate the plane with a regular polygon, what is the sum of the interior angles that surround a vertex?

? Exercise 10.5.0.5

Does a regular heptagon tesselate the plane by itself?

? Exercise 10.5.0.6

What are the only regular polygons that will tessellate the plane by themselves?

? Exercise 10.5.0.7

What is the transformation called that revolves a shape about a point to a new position?

? Exercise 10.5.0.8

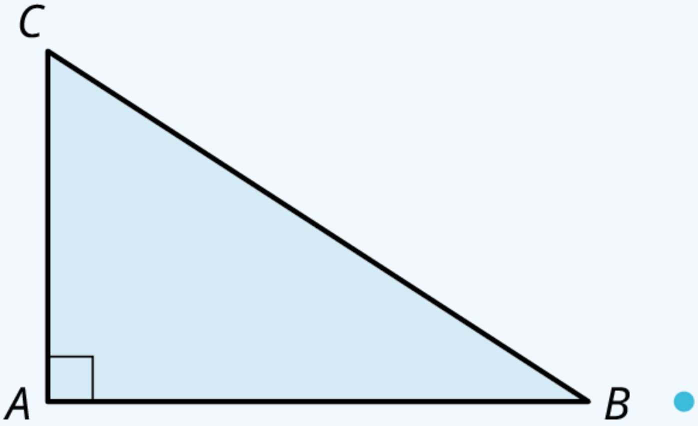
Transformational geometry is a study of what?

? Exercise 10.5.0.9

Describe how to achieve a rotation transformation.

? Exercise 10.5.0.10

Construct a 90° rotation of the triangle shown.



Exercise $10.5.0.11$

Shapes can be rotated around a point of rotation or a ____________.

Exercise $10.5.0.12$

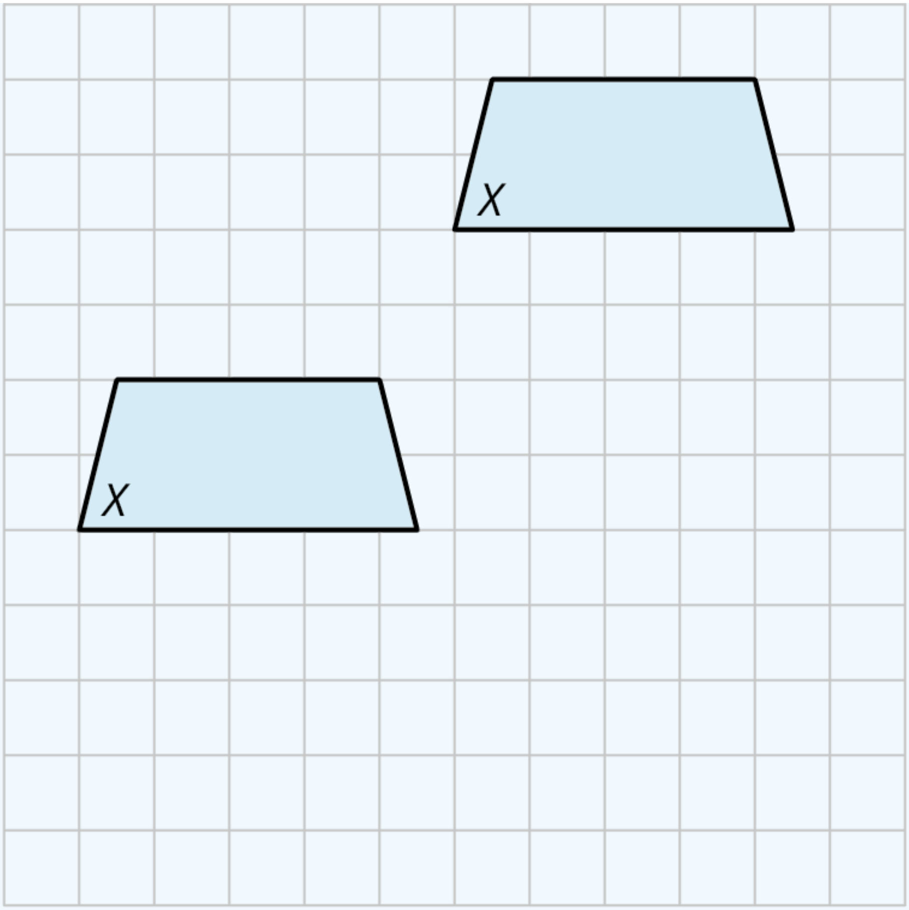
What is the name of the transformation that involves a reflection and a translation?

Exercise $10.5.0.13$

What can a tessellation *not* have between shapes?

Exercise $10.5.0.14$

Describe the transformation shown.

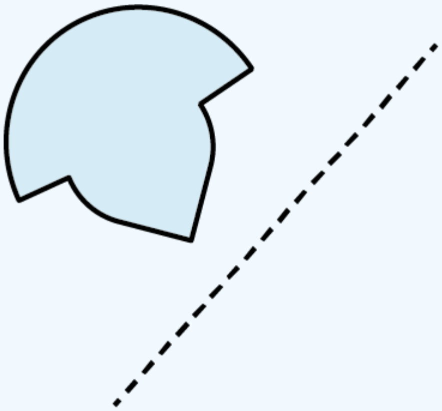


Exercise 10.5.0.15

What do we call a transformation that produces a mirror image?

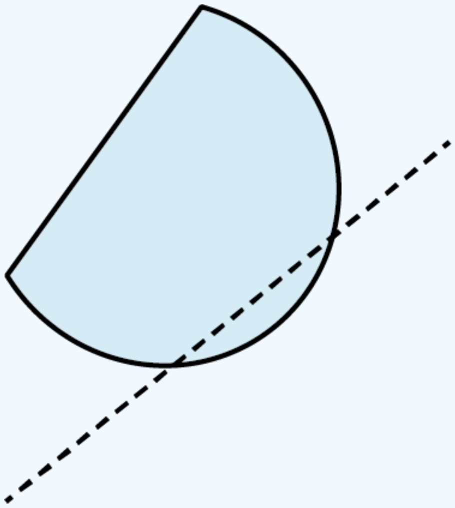
Exercise 10.5.0.16

Sketch the reflection of the shape about the dashed line.



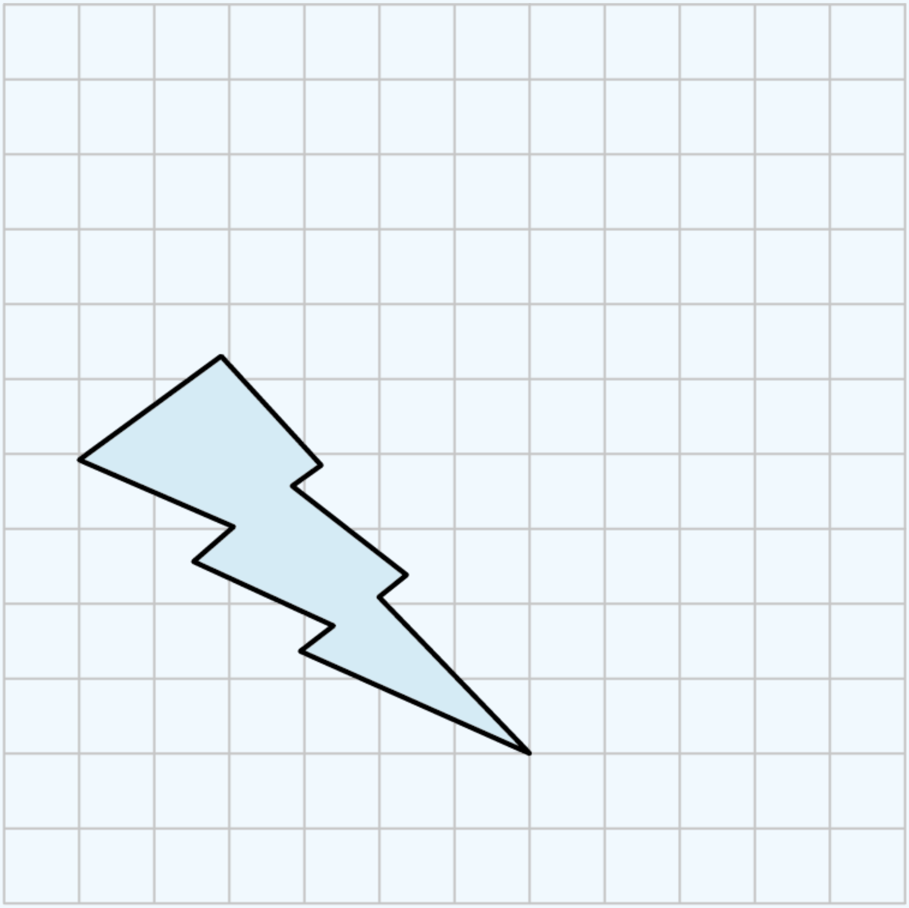
Exercise 10.5.0.17

Sketch the reflection of the shape about the dashed line.



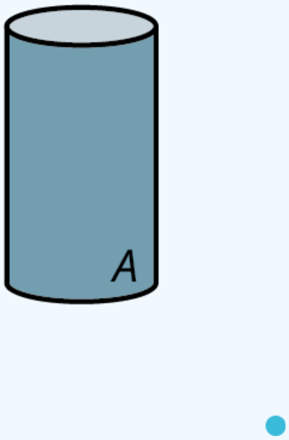
Exercise 10.5.0.18

Sketch the translation of the shape 3 units to the right and 3 units vertically.



Exercise 10.5.0.19

Rotate the shape 45° about the rotation point using point A as your guide.



Exercise 10.5.0.20

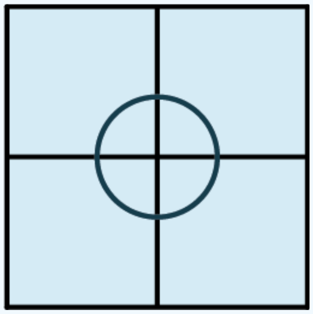
Do regular pentagons tessellate the plain by themselves?

Exercise 10.5.0.21

What do regular tessellations have in common?

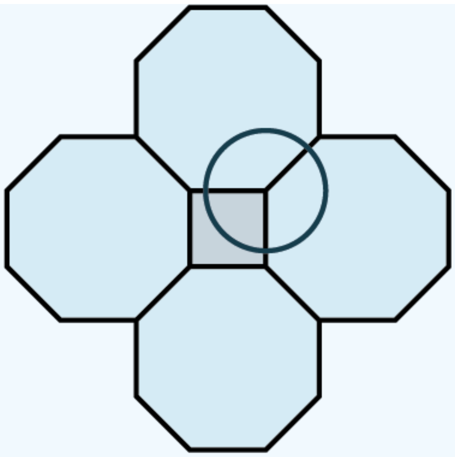
Exercise 10.5.0.22

How would we name a tessellation of squares as shown in the figure?



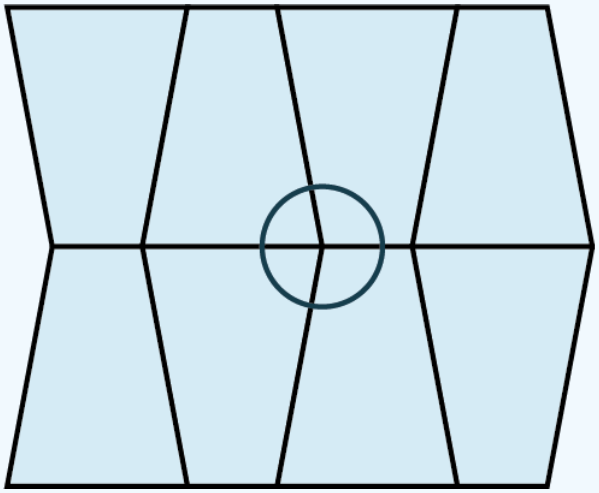
Exercise 10.5.0.23

How do we name a tessellation of octagons and squares as shown in the figure?



Exercise 10.5.0.24

How would we name a tessellation of trapezoids as shown in the figure?



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10.6: Area

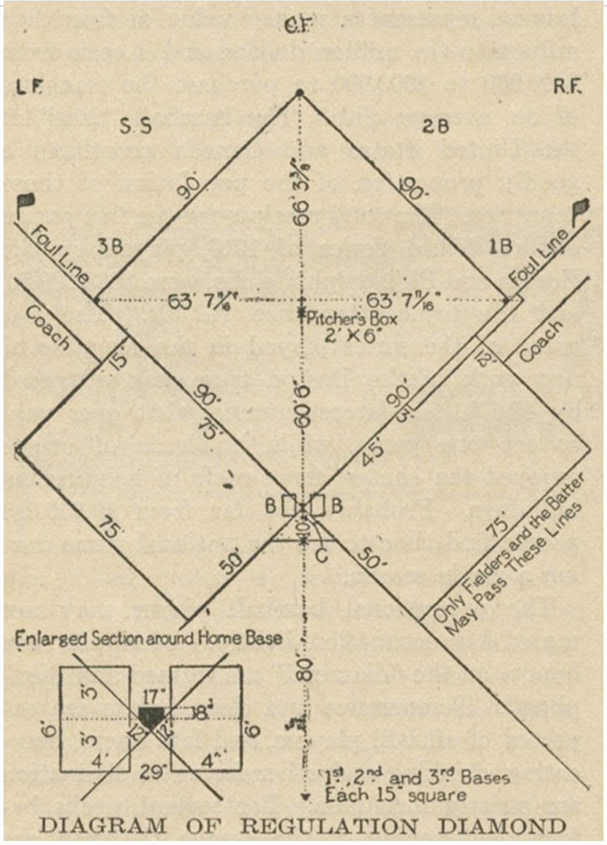

C.F.
L.F.
R.F.
S.S
2B
90'
66' 3⅜"
90'
3B
1B
Foul Line
Foul Line
63' 7 7/16"
63' 7 7/16"
Pitcher's Box 2'×6"
Coach
Coach
15'
12'
90'
75'
60' 6"
90'
15'
45'
B
B
10'
C
75'
50'
50'
75'
Only Fielders and the Batter May Pass These Lines
Enlarged Section around Home Base
80'
5'
5'
17"
18"
6'
6'
12"
12"
4'
4"
29"
1st, 2nd and 3rd Bases Each 15" square
DIAGRAM OF REGULATION DIAMOND

Figure 10.6.1: The area of a regulation baseball diamond must adhere to specific measurements to be legal. (credit: "Diagram of Regulation Diamond" by Erica Fischer from "Baseball" The World Book, 1920/Flickr, Public Domain, CC BY 2.0)

Learning Objectives

1. Calculate the area of triangles.
2. Calculate the area of quadrilaterals.
3. Calculate the area of other polygons.
4. Calculate the area of circles.

Some areas carry more importance than other areas. Did you know that in a baseball game, when the player hits the ball and runs to first base that he must run within a 6-foot wide path? If he veers off slightly to the right, he is out. In other words, a few inches can be the difference in winning or losing a game. Another example is real estate. On Manhattan Island, one square foot of real estate is worth far more than real estate in practically any other area of the country. In other words, we place a value on area. As the context changes, so does the value.

Area refers to a region measured in square units, like a square mile or a square foot. For example, to purchase tile for a kitchen floor, you would need to know how many square feet are needed because tile is sold by the square foot. Carpeting is sold by the square yard. As opposed to linear measurements like perimeter, which in in linear units. For example, fencing is sold in linear units, a linear foot or yard. Linear dimensions refer to an outline or a boundary. Square units refer to the area within that boundary. Different items may have different units, but either way, you must know the linear dimensions to calculate the area.

Many geometric shapes have formulas for calculating areas, such as triangles, regular polygons, and circles. To calculate areas for many irregular curves or shapes, we need calculus. However, in this section, we will only look at geometric shapes that have known area formulas. The notation for area, as mentioned, is in square units and we write sq in or sq cm, or use an exponent, such as in^2 or cm^2. Note that linear measurements have no exponent above the units or we can say that the exponent is 1.

Area of Triangles

The formula for the area of a triangle is given as follows.

FORMULA

The area of a triangle is given as $A=\frac{1}{2}bh$, where b represents the base and h represents the height.

For example, consider the triangle in Figure $10.6.2$

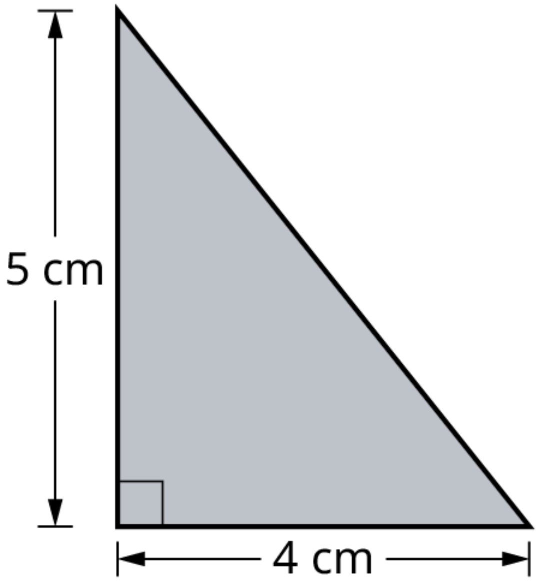


Figure $10.6.2$: Triangle 1

The base measures 4 cm and the height measures 5 cm . Using the formula, we can calculate the area:

$$A=\frac{1}{2}(4)(5)=\frac{1}{2}(20)=10\ \mathrm{cm}^2$$

In Figure 10.104, the triangle has a base equal to 7 cm and a height equal to 3.5 cm . Notice that we can only find the height by dropping a perpendicular to the base. The area is then

$$A=\frac{1}{2}(7)(3.5)=12.25\ \mathrm{cm}^2$$

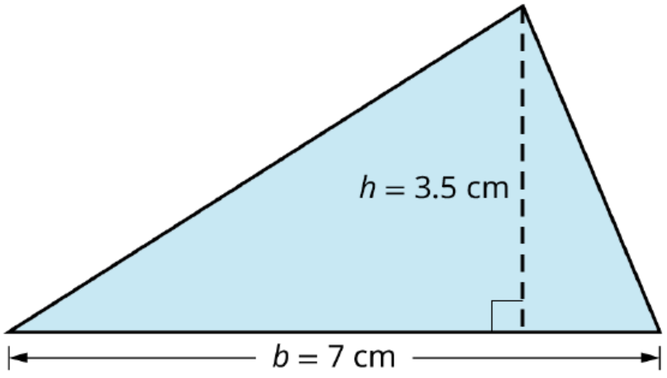


Figure $10.6.3$: Triangle 2

✓ Example 10.41: Finding the Area of a Triangle

Find the area of this triangle that has a base of 4 cm and the height is 6 cm (Figure ✓ $10.6.4$).

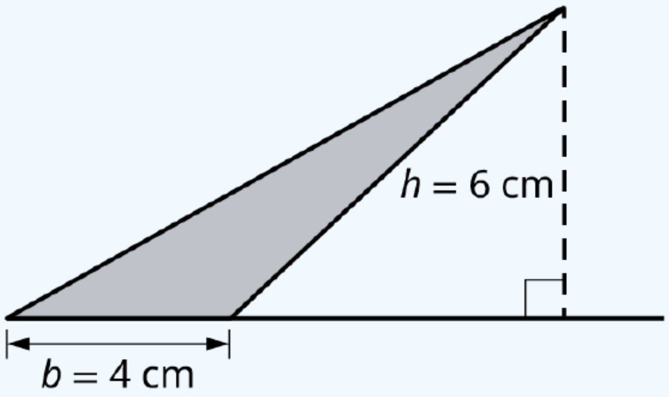


Figure ✓ $10.6.4$

Answer

Using the formula, we have

$$A = \frac{1}{2}(4)(6) = \frac{24}{2} = 12 \text{ cm}^2.$$

📌 Your Turn 10.41

Find the area of the triangle with a base equal to 4 cm and the height equal to 4 cm.

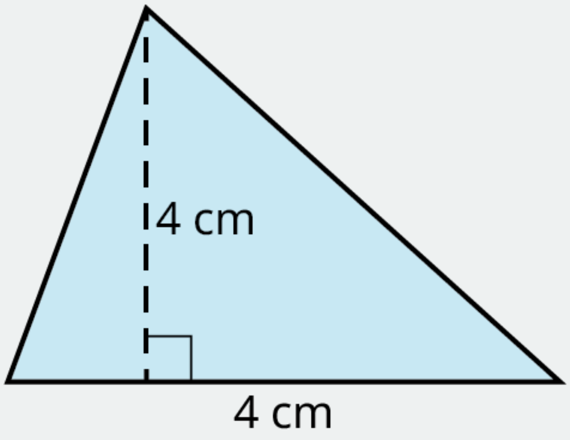


Figure 📌 $10.6.5$

Area of Quadrilaterals

A quadrilateral is a four-sided polygon with four vertices. Some quadrilaterals have either one or two sets of parallel sides. The set of quadrilaterals include the square, the rectangle, the parallelogram, the trapezoid, and the rhombus. The most common quadrilaterals are the square and the rectangle.

Square

In Figure $10.6.6$, a 12 in $\times 12$ in grid is represented with twelve 1 in $\times 1$ in squares across each row, and twelve 1 in $\times 1$ in squares down each column. If you count the little squares, the sum equals 144 squares. Of course, you do not have to count little squares to find area-we have a formula. Thus, the formula for the area of a square, where $s =$ length of a side, is $A = s \cdot s$. The area of the square in Figure $10.6.6$ is $A = 12$ in $\times 12$ in = 144in^2.

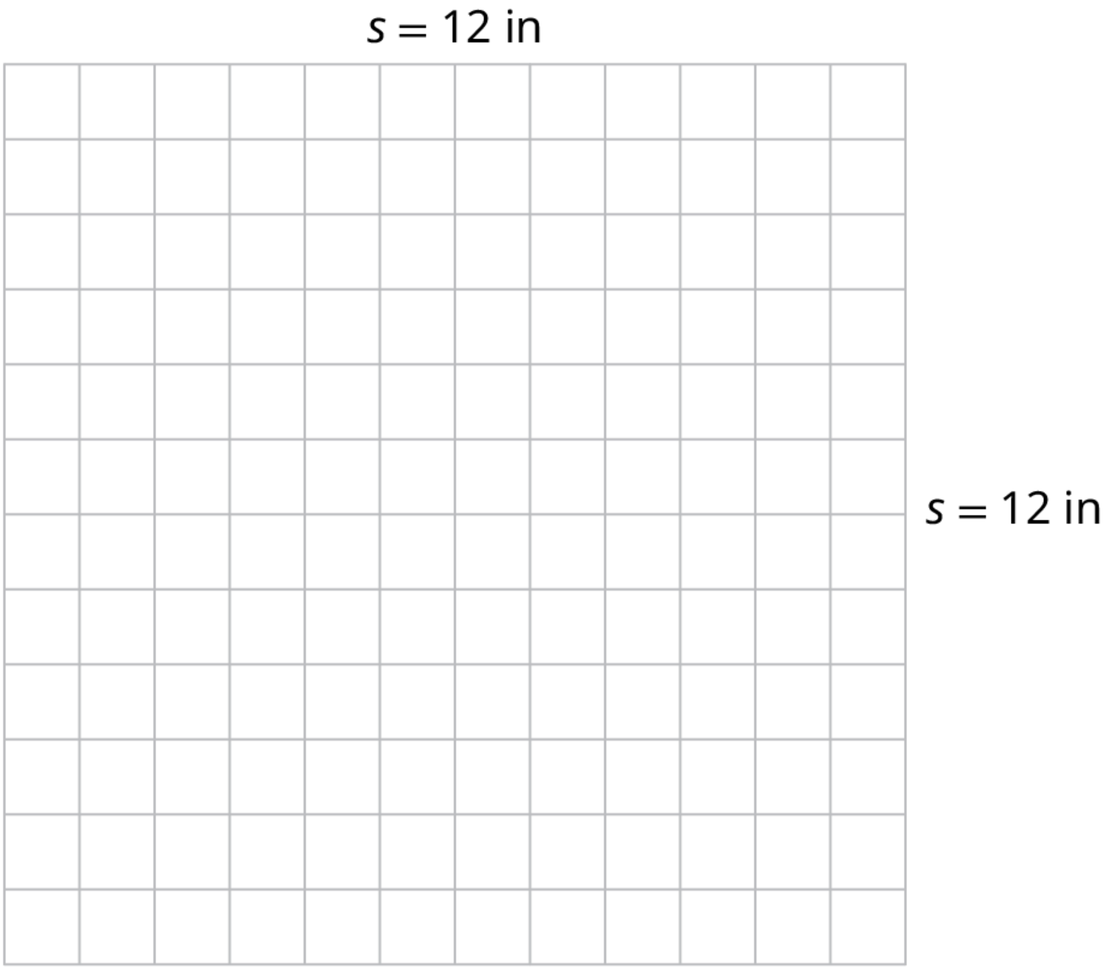


Figure 10.6.6: Area of a Square

FORMULA

The formula for the area of a square is $A = s \cdot s$ or $A = s^2$.

Rectangle

Similarly, the area for a rectangle is found by multiplying length times width. The rectangle in Figure 10.138 has width equal to 5 in and length equal to 12 in. The area is $A = 5(12) = 60 \text{ in}^2$.

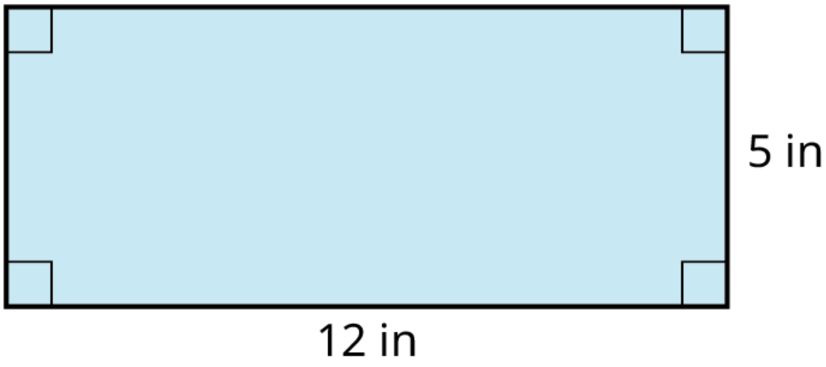


Figure 10.6.7: Area of a Rectangle

FORMULA

The area of a rectangle is given as $A = lw$.

Many everyday applications require the use of the perimeter and area formulas. Suppose you are remodeling your home and you want to replace all the flooring. You need to know how to calculate the area of the floor to purchase the correct amount of tile, or hardwood, or carpet. If you want to paint the rooms, you need to calculate the area of the walls and the ceiling to know how much paint to buy, and the list goes on. Can you think of other situations where you might need to calculate area?

✓ Example $10.6.1$: Finding the Area of a Rectangle

You have a garden with an area of 196 square feet. How wide can the garden be if the length is 28 feet?

Answer

The area of a rectangular region is $A = lw$. Letting the width equal w :

$$\begin{gathered}196 = 28w \\ \frac{196}{28} = w = 7\text{ft}\end{gathered}$$

Your Turn $10.6.1$

Find the area of a rectangular region with a length of 18 feet and a width 1/3 of the length.

✓ Example $10.6.2$: Determining the Cost of Floor Tile

Jennifer is planning to install vinyl floor tile in her rectangular-shaped basement, which measures 29 ft by 16 ft. How many boxes of floor tile should she buy and how much will it cost if one box costs \$50 and covers 20 ft^2?

Answer

The area of the basement floor is $A = 29(16) = 464\text{ft}^2$. We will divide this area by 20 ft . Thus, $\frac{464}{20} = 23.2$. Therefore, Jennifer will have to buy 24 boxes of tile at a cost $\$1,200$.

Your Turn $10.6.2$

You remodel your kitchen and decide to change out the tile floor. The floor measures 30 ft by 15 ft. One box of tile costs \$45 and covers 10 ft^2. How many boxes of tile should you buy, and what will it cost?

Parallelogram

The area of a parallelogram can be found using the formula for the area of a triangle. Notice in Figure $10.6.8$, if we cut a diagonal across the parallelogram from one vertex to the opposite vertex, we have two triangles. If we multiply the area of a triangle by 2 , we have the area of a parallelogram:

$$\begin{aligned}A &= 2\left(\frac{1}{2}bh\right) \\ A &= bh\end{aligned}$$

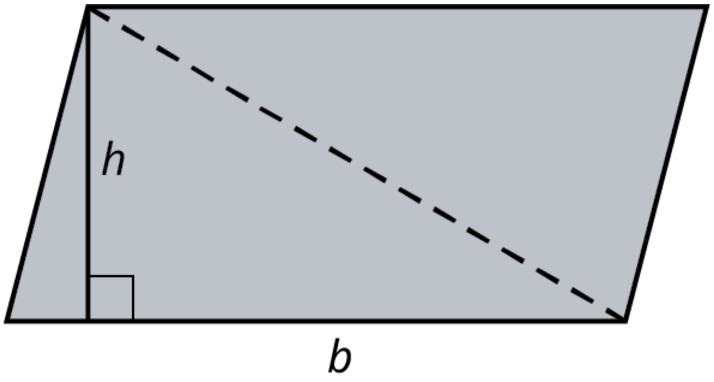


Figure 10.6.8: Area of a Parallelogram

FORMULA

The area of a parallelogram is $A = bh$.

For example, if we have a parallelogram with the base be equal to 10 inches and the height equal to 5 inches, the area will be $A = (10)(5) = 50 \text{ in}^2$.

✓ Example 10.6.3: Finding the Area of a Parallelogram

In the parallelogram (Figure ✓ 10.6.9), if $FB = 10$, $AD = 15$, find the exact area of the parallelogram.

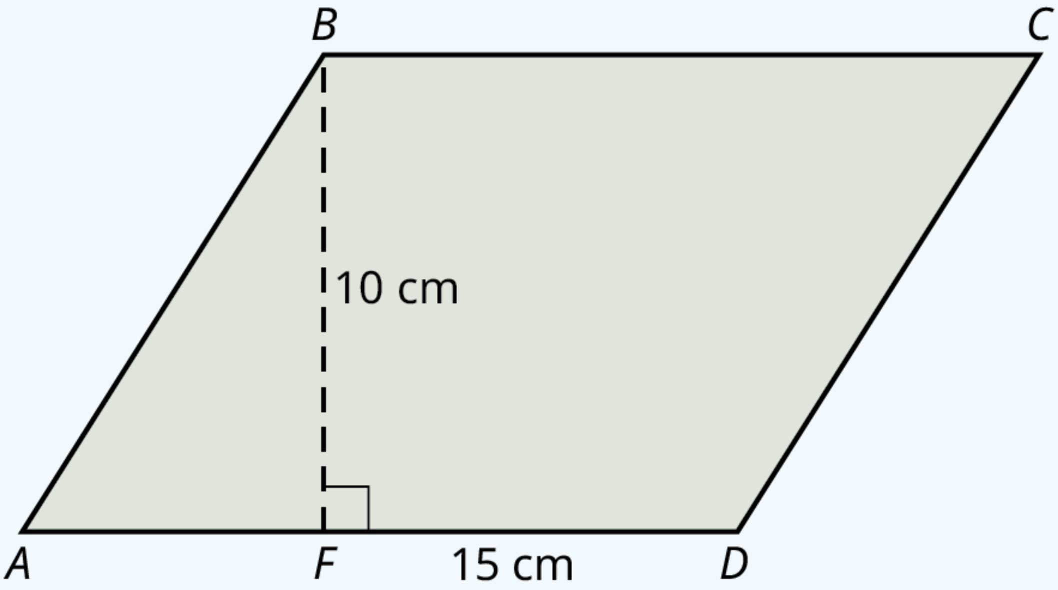


Figure ✓ 10.6.9

Answer

Using the formula of $A = bh$, we have

$$A = 10(15) = 150 \text{ cm}^2.$$

Your Turn 10.6.3

Find the area of the parallelogram.

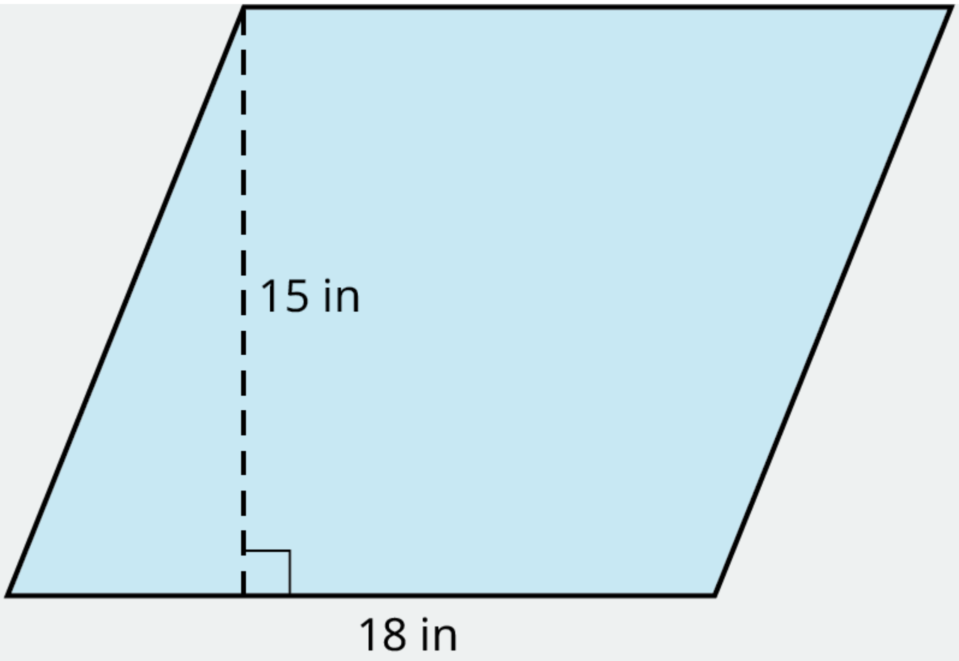


Figure 10.6.10

✓ Example 10.6.4: Finding the Area of a Parallelogram Park

The boundaries of a city park form a parallelogram (Figure ✓ 10.6.11). The park takes up one city block, which is contained by two sets of parallel streets. Each street measures 55 yd long. The perpendicular distance between streets is 39 yd. How much sod, sold by the square foot, should the city purchase to cover the entire park and how much will it cost? The sod is sold for \$0.50 per square foot, installation is \$1.50 per square foot, and the cost of the equipment for the day is \$100.

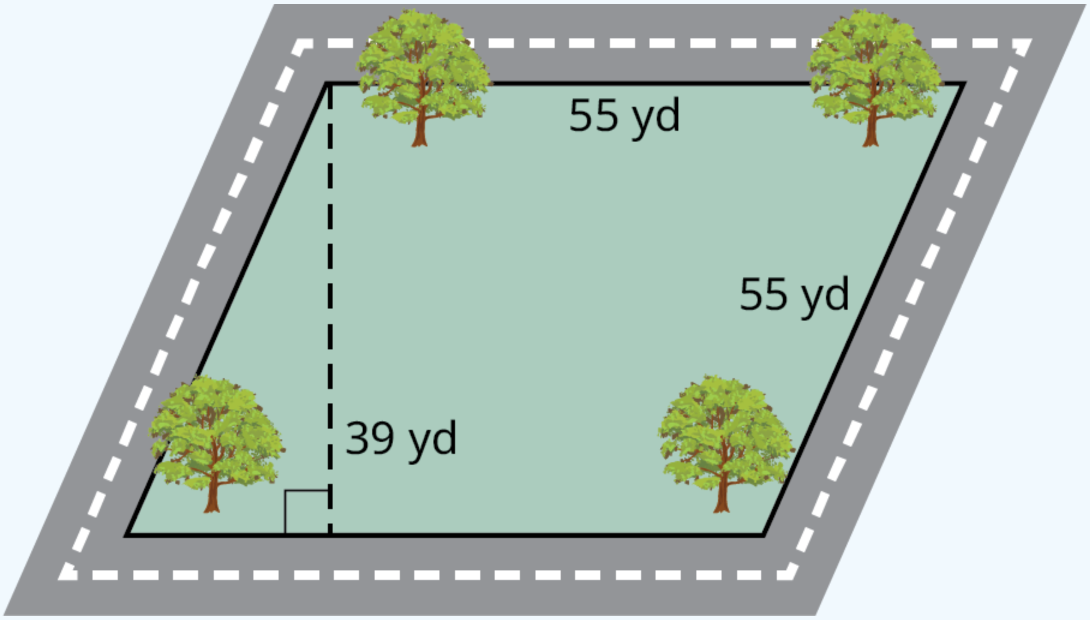


Figure ✓ 10.6.11

Answer

Step 1: As sod is sold by the square foot, the first thing we have to do is translate the measurements of the park from yards to feet. There are 3 ft to a yard, so 55 yd is equal to 165 ft, and 39 yd is equal to 117 ft.

Step 2: The park has the shape of a parallelogram, and the formula for the area is $A = bh$:

$$A = 165(117) = 19,305\text{ft}^2$$

Step 3: The city needs to purchase $19,305\text{ft}^2$ of sod. The cost will be $\$0.50$ per square foot for the sod and $\$1.50$ per square foot for installation, plus $\$100$ for equipment:

$$\begin{aligned} 19,305(\$0.50) + 19,305(\$1.50) &= \$9,652.5 + \$28,957.5 + \$100.00 \\ &= \$38,710.00 \end{aligned}$$

Your Turn 10.6.4

Suppose your city has a park just like the one in Example 10.6.11. The length of each street is 49 yd, and the perpendicular distance between streets is 31 yd. How much sod, sold by the square foot at $0.45 per square foot plus $1.00 per square foot for installation, and a flat $50 fee for equipment, should be purchased and what will the cost be?

Trapezoid

Another quadrilateral is the trapezoid. A trapezoid has one set of parallel sides or bases. The formula for the area of a trapezoid with parallel bases a and b and height h is given here.

FORMULA

The formula for the area of a trapezoid is given as $A = \frac{1}{2}h\,(a+b)$.

For example, find the area of the trapezoid in Figure 10.6.12 that has base a equal to 8 cm, base b equal to 6 cm, and height equal to 6 cm.

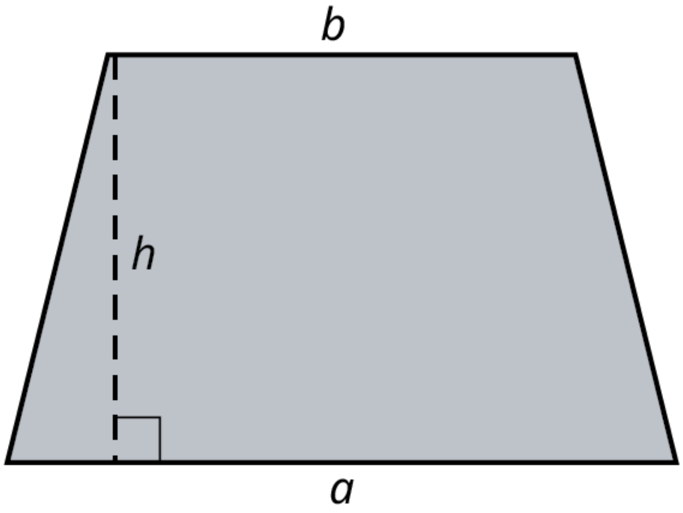


Figure 10.6.123: Area of a Trapezoid

The area is $A = \frac{1}{2}(6)(6+8) = 42\text{ cm}^2$.

✓ Example 10.6.5: Finding the Area of a Trapezoid

$ABCD$ (Figure ✓ 10.6.13) is a regular trapezoid with $\overline{AB} \| \overline{CD}$. Find the exact perimeter of $ABCD$, and then find the area.

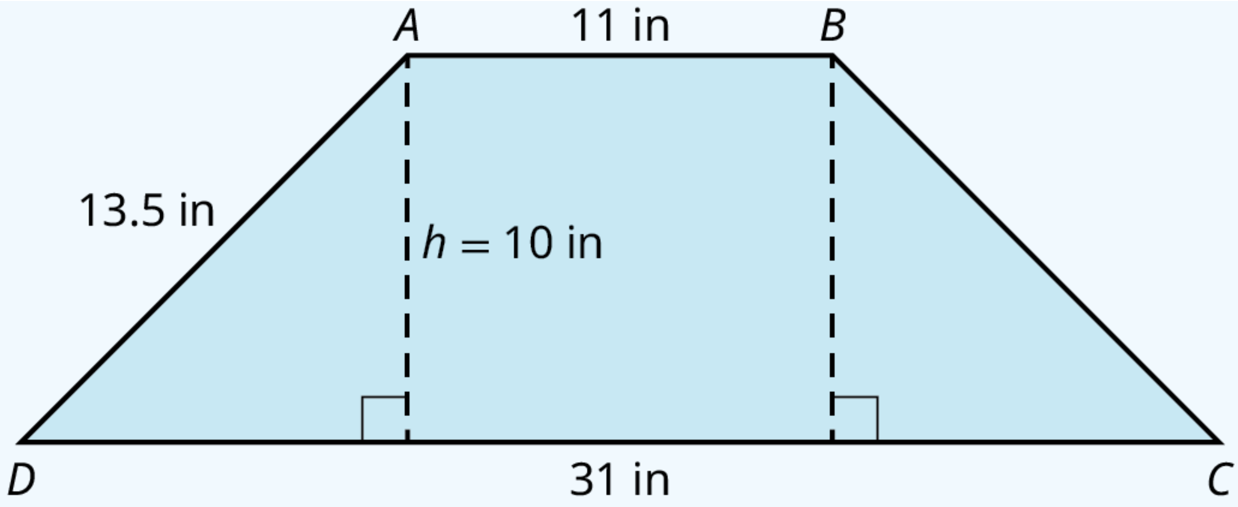


Figure ✓ 10.6.13

Answer

The perimeter is the measure of the boundary of the shape, so we just add up the lengths of the sides. We have $P = 31 + 13.5 + 11 + 13.5 = 69 \text{ in}$. Then, the area of the trapezoid using the formula is $A = \frac{1}{2}(10)(11 + 31) =$.

Your Turn 10.6.5

Find the area of the trapezoid shown.

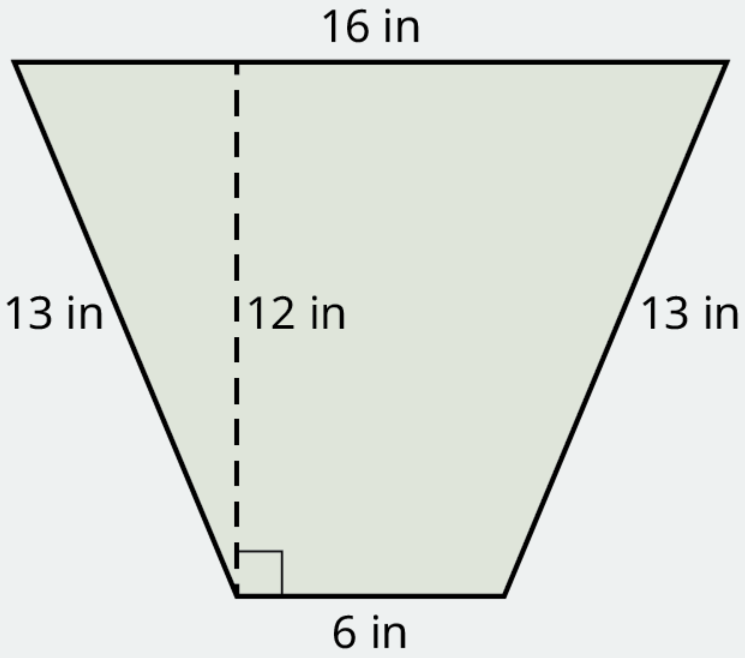


Figure 10.6.14

Rhombus

The rhombus has two sets of parallel sides. To find the area of a rhombus, there are two formulas we can use. One involves determining the measurement of the diagonals.

FORMULA

The area of a rhombus is found using one of these formulas:

- $A=\frac{d_1 d_2}{2}$, where d_1 and d_2 are the diagonals.
- $A=\frac{1}{2}bh$, where b is the base and h is the height.

For our purposes here, we will use the formula that uses diagonals. For example, if the area of a rhombus is $220\ \mathrm{cm}^2$, and the measure of $d_2=11$, find the measure of d_1. To solve this problem, we input the known values into the formula and solve for the unknown. See Figure $10.6.15$.

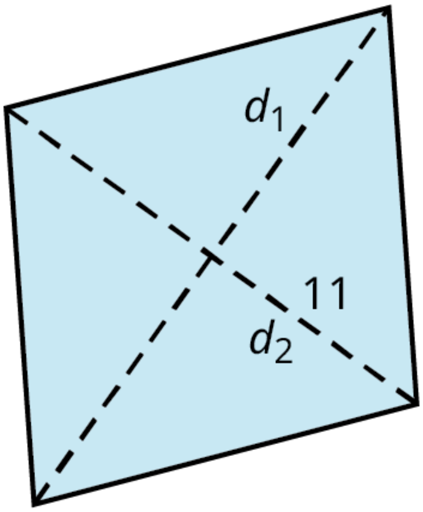


Figure $10.6.15$: Area of a Rhombus

We have that

$$\begin{aligned}220&=\frac{11d_1}{2}\\ 220(2)&=11d_1\\ \frac{440}{11}&=d_1=40\end{aligned}$$

Example $10.6.6$: Finding the Area of a Rhombus

Find the measurement of the diagonal d_1 if the area of the rhombus is $240\ \mathrm{cm}^2$, and the measure of $d_2=24\ \mathrm{cm}$

Answer

Use the formula with the known values:

$$\begin{aligned}240&=\frac{d_1(24)}{2}\\ 240(2)&=d_1(24)\\ \frac{480}{24}&=d_1=20\end{aligned}$$

Your Turn 10.6.6

A rhombus has an area of $40\ \text{in}^2$, the measure of $d_1 = 8$. Find the measure of d_2.

✓ Example 10.6.7: Finding the Area of a Rhombus

You notice a child flying a rhombus-shaped kite on the beach. When it falls to the ground, it falls on a beach towel measuring semantics36inFigure ✓ 10.6.16)?

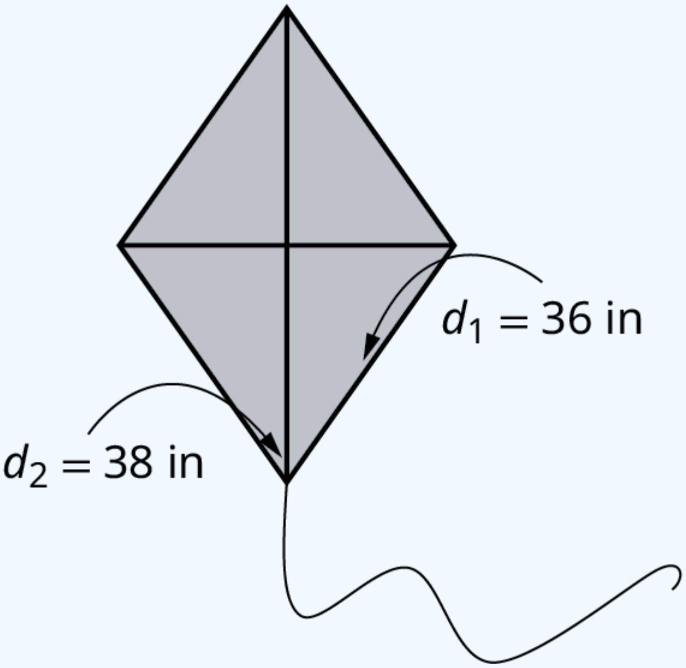


Figure ✓ 10.6.16

Answer

Using the formula, we have:

$$\begin{aligned} A &= \frac{d_1 d_2}{2} \\ &= \frac{36(38)}{2} = 684\text{in}^2 \end{aligned}$$

Your Turn 10.6.7

You have a kite that measures $500\,\text{in}^2$. If one of the diagonals measures 25 in, what is the length of the other diagonal?

Area of Polygons

To find the area of a regular polygon, we need to learn about a few more elements. First, the apothem a of a regular polygon is a line segment that starts at the center and is perpendicular to a side. The radius r of a regular polygon is also a line segment that starts at the center but extends to a vertex. See Figure 10.6.7.

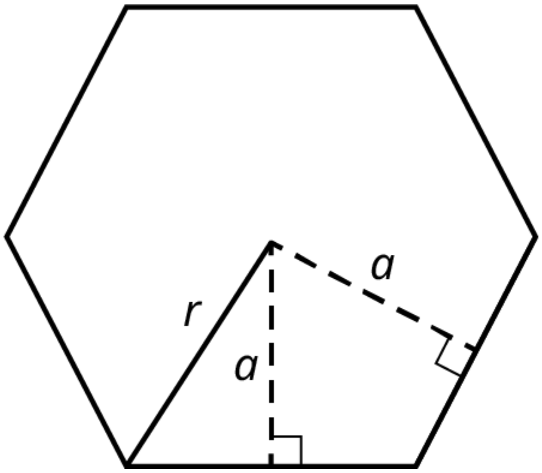


Figure $10.6.17$: Apothem and Radius of a Polygon

FORMULA

The area of a regular polygon is found with the formula $A=\frac{1}{2}ap$, where a is the apothem and p is the perimeter.

For example, consider the regular hexagon shown in Figure $10.6.18$ with a side length of 4 cm, and the apothem measures $a=2\sqrt{3}$.

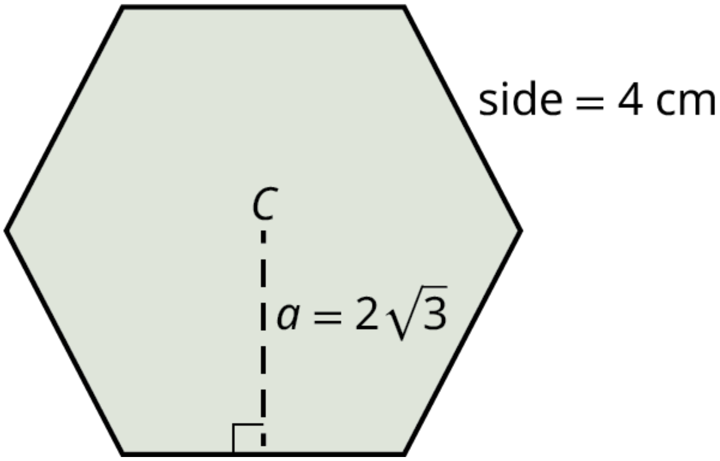


Figure $10.6.18$: Area of a Hexagon

We have the perimeter, $p=6(4)=24\text{ cm}$. We have the apothem as $a=2\sqrt{3}$. Then, the area is:

$$\begin{aligned}A&=\frac{1}{2}(2\sqrt{3})(24)\\&=24\sqrt{3}=41.57\text{ cm}^2\end{aligned}$$

msup

Example $10.6.8$: Finding the Area of a Regular Octagon

Find the area of a regular octagon with the apothem equal to 18 cm and a side length equal to 13 cm (Figure ✓ $10.6.190$).

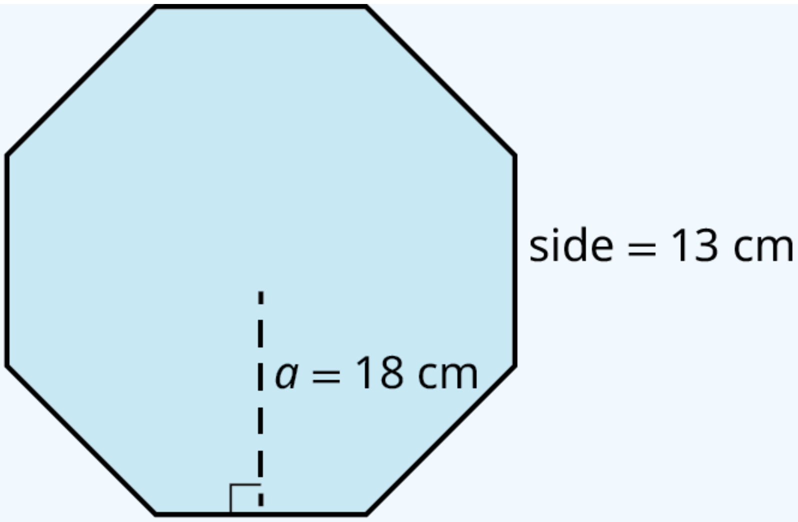


Figure ✓ 10.6.19

Answer

Using the formula, we have the perimeter $p = 8(13) = 104 \text{ cm}$. Then, the area is $A = \frac{1}{2}(10)(104) = 936 \text{ cm}^2$.

Your Turn 10.6.8

Find the area of the regular pentagon with the apothem equal to 5.5 cm and the side length equal to 7 cm.

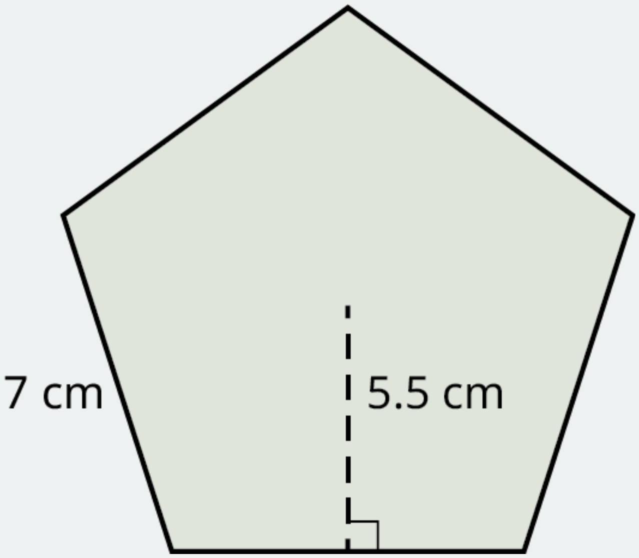


Figure 10.6.20

Changing Units

Often, we have the need to change the units of one or more items in a problem to find a solution. For example, suppose you are purchasing new carpet for a room measured in feet, but carpeting is sold in terms of yards. You will have to convert feet to yards to purchase the correct amount of carpeting. Or, you may need to convert centimeters to inches, or feet to meters. In each case, it is essential to use the correct equivalency.

Example 10.6.9: Changing Units

Carpeting comes in units of square yards. Your living room measures 21 ft wide by 24 ft long. How much carpeting do you buy?

Answer

We must convert feet to yards. As there are 3 ft in 1 yd, we have $21 \text{ ft} =$ and $24 \text{ ft} = 8 \text{ yds}$. Then, $7(8) = 56 \text{ yd}^2$.

Your Turn 10.6.9

You want to carpet your bedroom, which measures 15 ft wide by 18 ft long. Carpeting is sold by the square yard, so you must convert your measurements to yards. How much carpeting do you buy?

Area of Circles

Just as the circumference of a circle includes the number π, so does the formula for the area of a circle. Recall that π is a non-terminating, non-repeating decimal number: $\pi = 3.14159\ldots$. It represents the ratio of the circumference to the diameter, so it is a critical number in the calculation of circumference and area.

FORMULA

The area of a circle is given as $A = \pi r^2$, where r is the radius.

For example, to find the of the circle with radius equal to 3 cm, as shown in Figure $10.6.21$, is found using the formula $A = \pi r^2$.

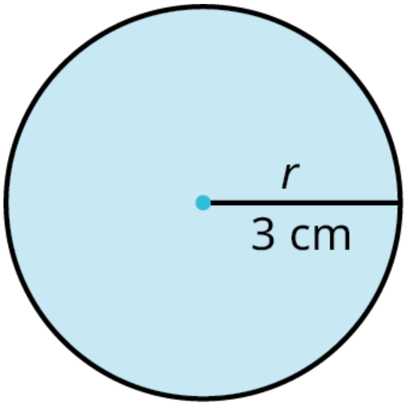


Figure $10.6.21$: Circle with Radius 3

We have

$$\begin{aligned} A &= \pi r^2 \\ &= \pi(3)^2 \\ &= 9\pi = 28.27 \text{ cm}^2 \end{aligned}$$

msup

Example 10.6.10: Finding the Area of a Circle

Find the area of a circle with diameter of 16 cm.

Answer

The formula for the area of a circle is given in terms of the radius, so we cut the diameter in half. Then, the area is

$$A = \pi(8)^2 = 201.1 \text{ cm}^2.$$

Your Turn 10.6.10

Find the area of the circle with a radius of 3 cm.

✓ Example 10.6.11: Determining the Better Value for Pizza

You decide to order a pizza to share with your friend for dinner. The price for an 8-inch diameter pizza is \$7.99. The price for 16-inch diameter pizza is \$13.99. Which one do you think is the better value?

Answer

The area of the 8 -inch diameter pizza is $A=\pi(4)^2=50.3\mathrm{in}^2$. The area of the 16-inch diameter pizza is $A=\pi(8)^2=201.1\mathrm{in}^2$. Next, we divide the cost of each pizza by its area in square inches. Thus, $\frac{13.99}{201.1}=\$0.07$ per square inch and $\frac{7.99}{50.3}=\$0.16$ per square inch. So clearly, the 16 -inch pizza is the better value.

Your Turn 10.6.1

You can buy a 9-inch diameter pizza for \$10.99, or a 15-inch diameter pizza for \$14.99. Which pizza is the better value?

✓ Example 10.6.12: Applying Area to the Real World

You want to purchase a tinted film, sold by the square foot, for the window in Figure ✓ 10.6.22. (This problem should look familiar as we saw it earlier when calculating circumference.) The bottom part of the window is a rectangle, and the top part is a semicircle. Find the area and calculate the amount of film to purchase.

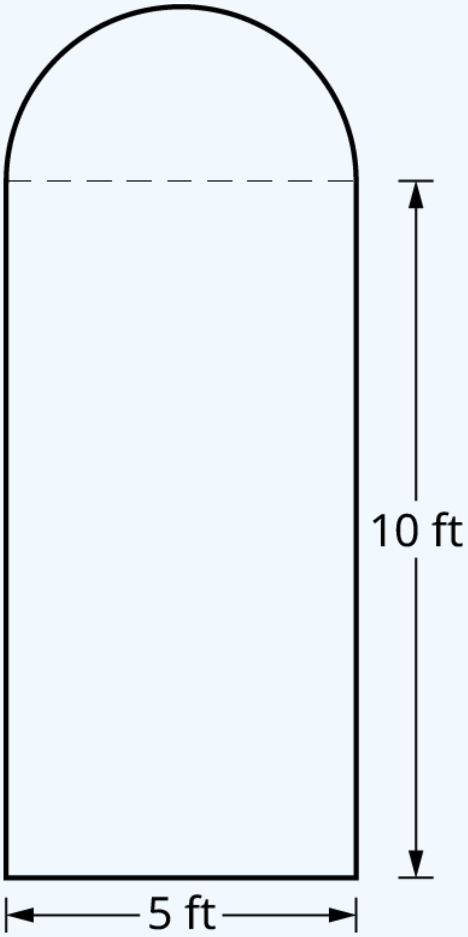


Figure ✓ 10.6.22

Answer

First, the rectangular portion has $A=5(10)=50 \mathrm{ft}^2$. For the top part, we have a semicircle with a diameter of 5 ft , so the radius is 2.5 ft . We want one half of the area of a circle with radius 2.5 ft , so the area of the top semicircle part is $A=\frac{1}{2} \pi(2.5)^2=9.8 \mathrm{ft}^2$. Add the area of the rectangle to the area of the semicircle. Then, the total area to be covered with the window film is

\[A=50+9.8=59.8 \mathrm{ft}^2\

Your Turn 10.6.12

You decide to install a new front door with a semicircle top as shown in the figure. How much area will the new door occupy?

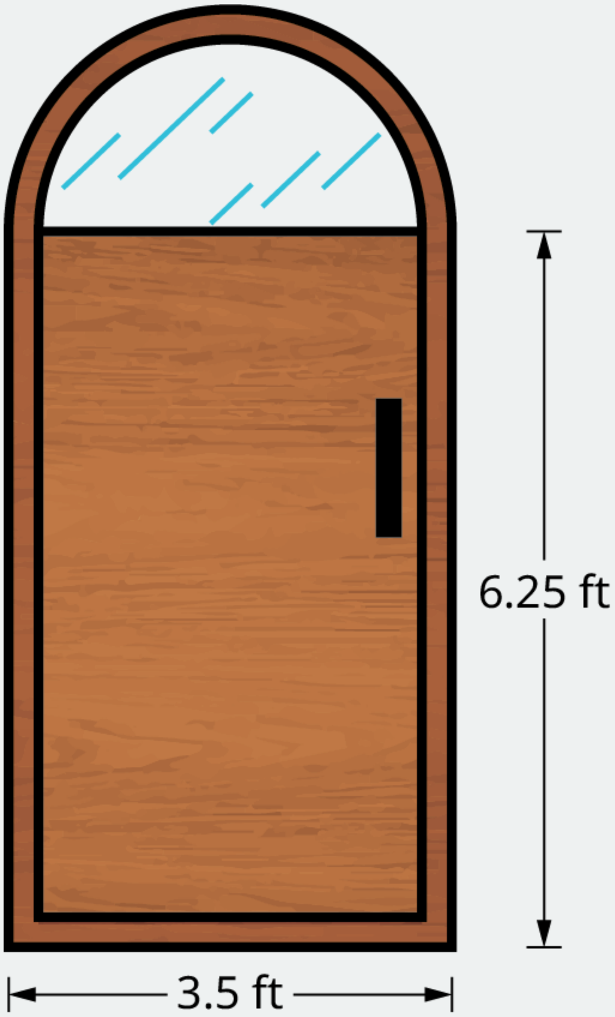


Figure 10.6.23

Area within Area

Suppose you want to install a round hot tub on your backyard patio. How would you calculate the space needed for the hot tub? Or, let’s say that you want to purchase a new dining room table, but you are not sure if you have enough space for it. These are common issues people face every day. So, let’s take a look at how we solve these problems.

✓ Example 10.6.13: Finding the Area within an Area

The patio in your backyard measures 20 ft by 10 ft (Figure ✓ 10.6.24). On one-half of the patio, you have a 4-foot diameter table with six chairs taking up an area of approximately 36 sq feet. On the other half of the patio, you want to install a hot tub measuring 6 ft in diameter. How much room will the table with six chairs and the hot tub take up? How much area is left over?

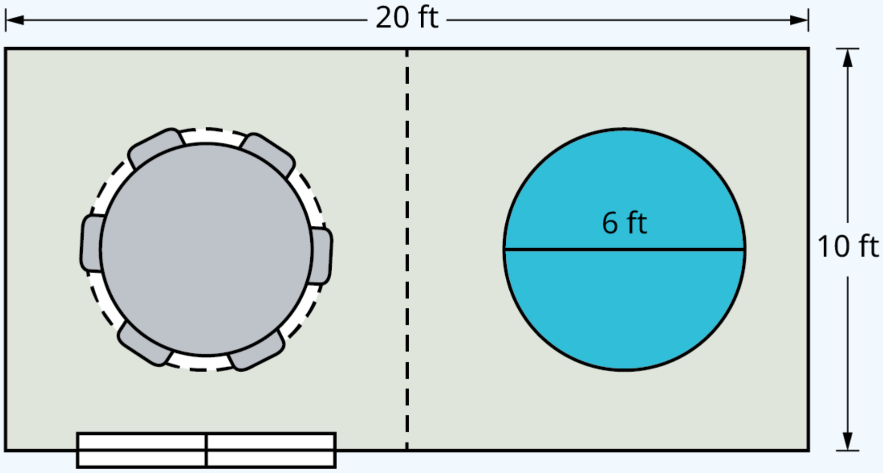


Figure ✓ 10.6.24

Answer

The hot tub has a radius of 3 ft . That area is then $A=\pi(3)^2=9\pi=28.27\mathrm{ft}^2$. The total square feet taken up with the table and chairs and the hot tub is $36+28.27=64.27\mathrm{ft}^2$. The area left over is equal to the total area of the patio, $200\mathrm{ft}^2$ minus the area for the table and chairs and the hot tub. Thus, the area left over is $200-64.27=135.7\mathrm{ft}^2$.

Your Turn 10.6.13

Find the area of the shaded region in the given figure.

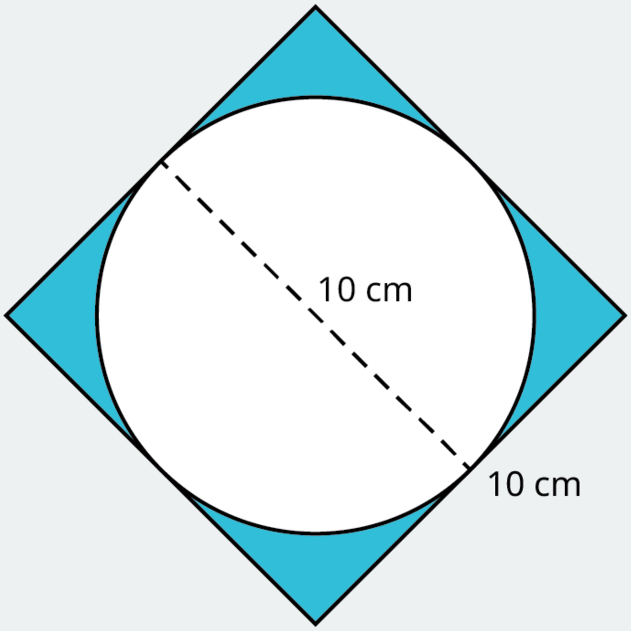


Figure 10.6.25

✓ Example 10.6.14: Finding the Cost of Fertilizing an Area

A sod farmer wants to fertilize a rectangular plot of land 150 ft by 240 ft. A bag of fertilizer covers 5,000ft25,000ft2 and costs $200. How much will it cost to fertilize the entire plot of land?

Answer

The plot of land is $36,000\text{ft}^2$. It will take 7.2 bags of fertilizer to cover the land area. Therefore, the farmer will have to purchase 8 bags of fertilizer at $\$200$ a bag, which comes to $1,600.

Your Turn 10.6.14

You want to install sod on one-half of your parallelogram-shaped backyard as shown. The patio covers the other half. Sod costs $\$50$ a bag and covers 25ft^2. How much will it cost to buy the sod?

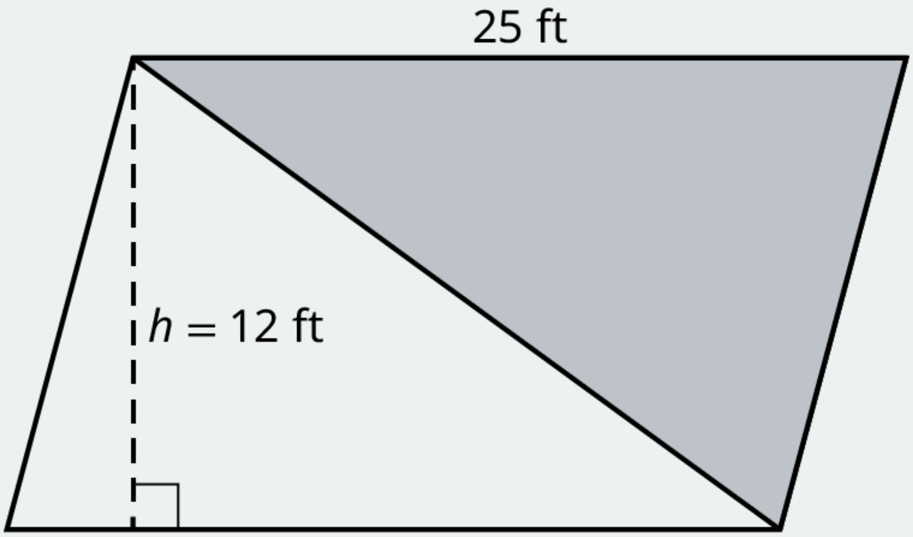


Figure 10.6.26

People in Mathematics: Heron of Alexandria

Figure 10.6.27: Heron of Alexandria (credit: "Heron of Alexandria" from 1688 German translation of Hero's Pneumatics/Wikimedia Commons, Public Domain) Heron of Alexandria, born around 20 A.D., was an inventor, a scientist, and an engineer. He is credited with the invention of the Aeolipile, one of the first steam engines centuries before the industrial revolution. Heron was the father of the vending machine. He talked about the idea of inserting a coin into a machine for it to perform a specific action in his book, *Mechanics and Optics*. His contribution to the field of mathematics was enormous. *Metrica*, a series of three books, was devoted to methods of finding the area and volume of three-dimensional objects like pyramids, cylinders, cones, and prisms. He also discovered and developed the procedures for finding square roots and cubic roots. However, he is probably best known for Heron's formula, which is used for finding the area of a triangle based on the lengths of its sides. Given a triangle

semantics

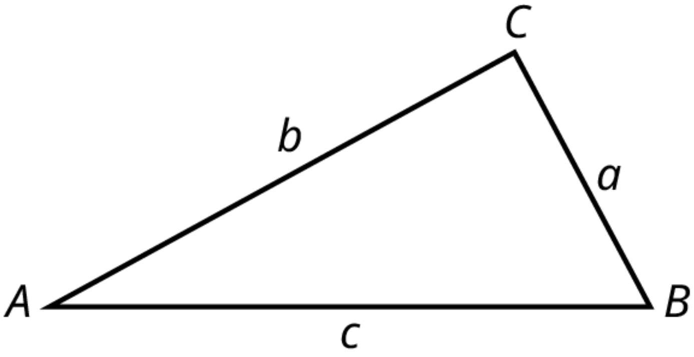


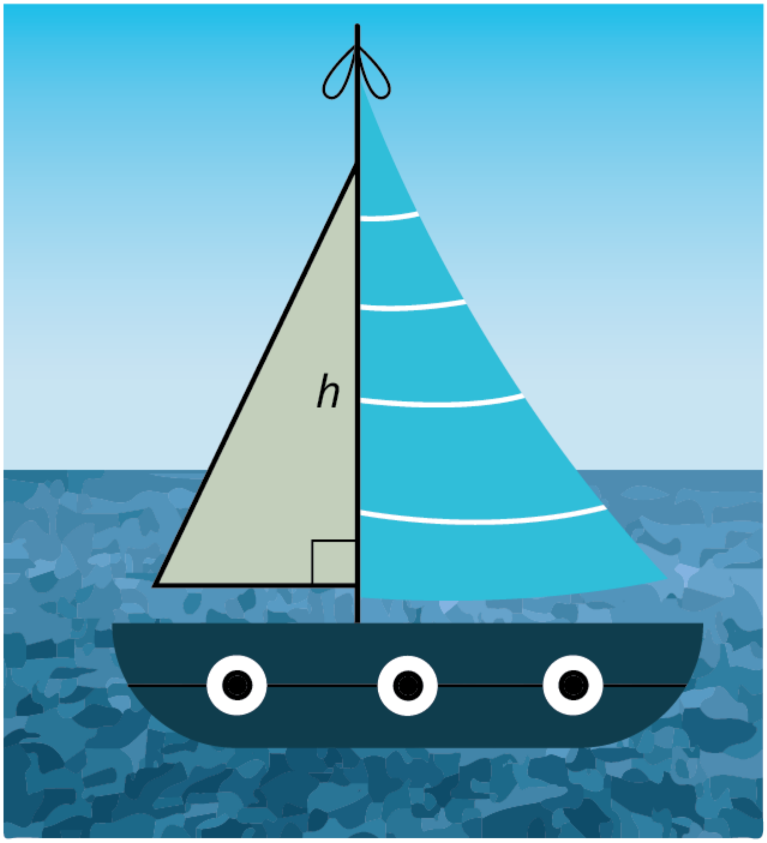
Figure 10.6.27

Heron's formula is $A=\sqrt{s(s-a)(s-b)(s-c)}$, where s is the semi-perimeter calculated as $s=\frac{a+b+c}{2}$.

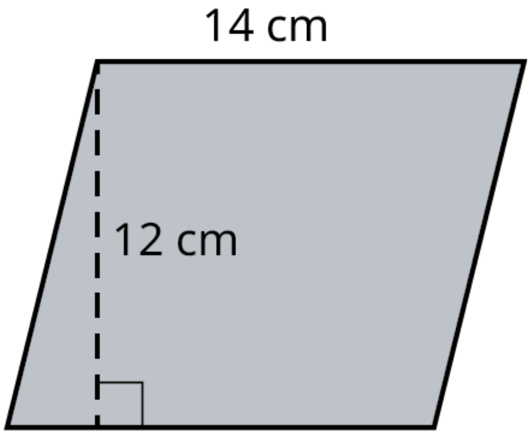
Check Your Understanding

1. Find the area of the triangle with base equal to 3 cm and height equal to 5 cm.

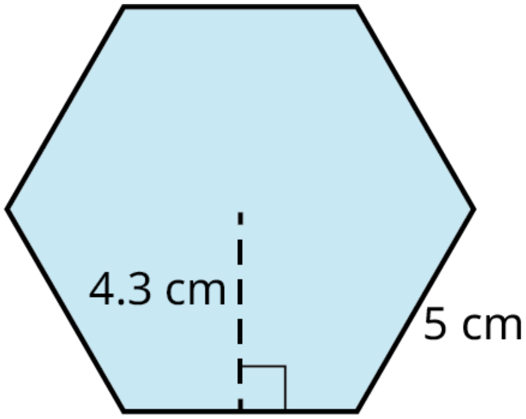
2. The area of the sail in the sailboat is $125\,\mathrm{ft}^2$. The shortest length is 10 ft. What is the height of the sail?



3. Find the area of this parallelogram.

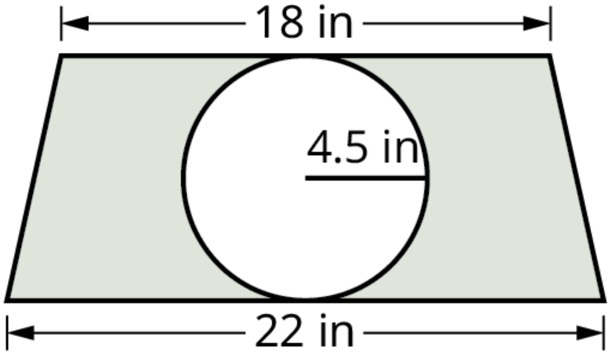


4. Find the area of a regular hexagon with side length of 5 and apothem equal to 4.3.



5. Find the area of circle with a diameter of 16 in.

6. Find the area of the shaded region in the given figure.



7. A round tray sits on top of the dining room table. The radius of the tray is 15 in. What is the area taken up by the tray?

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10.6.0: Exercises

For the following exercises, find the area of the figure with the given measurements.

? Exercise 10.6.0.1

Area of a triangle with base equal to 10 cm, and height equal to 15 cm.

? Exercise 10.6.0.2

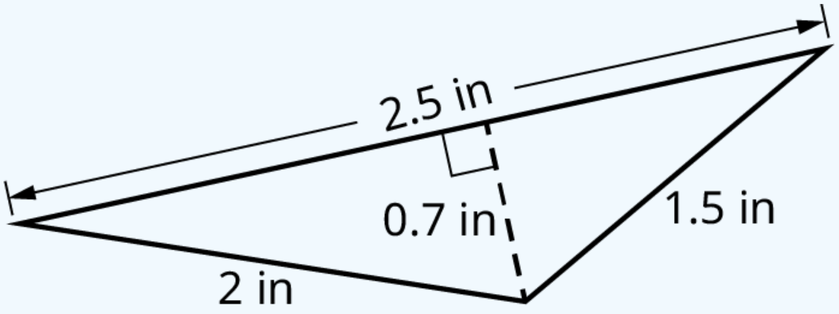
Area of right triangle with base 54 cm, and height equal to 72 cm.

? Exercise 10.6.0.3

A triangle has an area of $22.5\,\text{cm}^2$. If the base equals 5 cm, find the height.

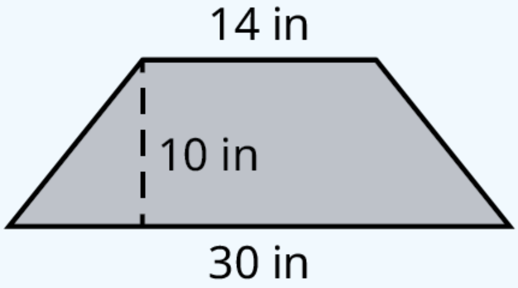
? Exercise 10.6.0.4

Find the area of the triangle with base equal to 2.5 in, height equal to 0.7 in as shown.



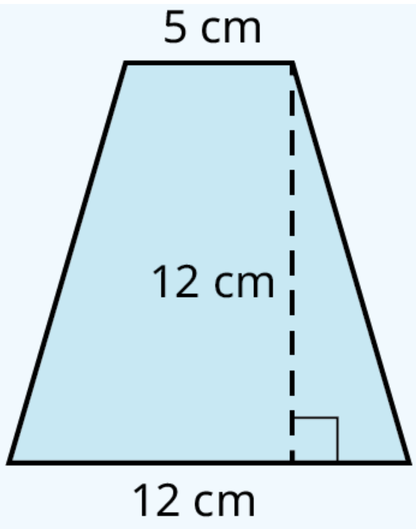
? Exercise 10.6.0.5

Find the area of the trapezoid with $b_1 = 30\,in,\ b_2 = 14\,in$, and $h = 10\,in$ as shown.



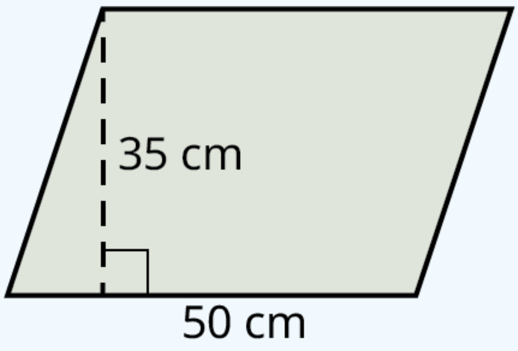
? Exercise 10.6.0.6

Find the area of the trapezoid shown.



Exercise 10.6.0.7

Find the area of a parallelogram with base equal to 50 cm and height equal to 35 cm as shown.

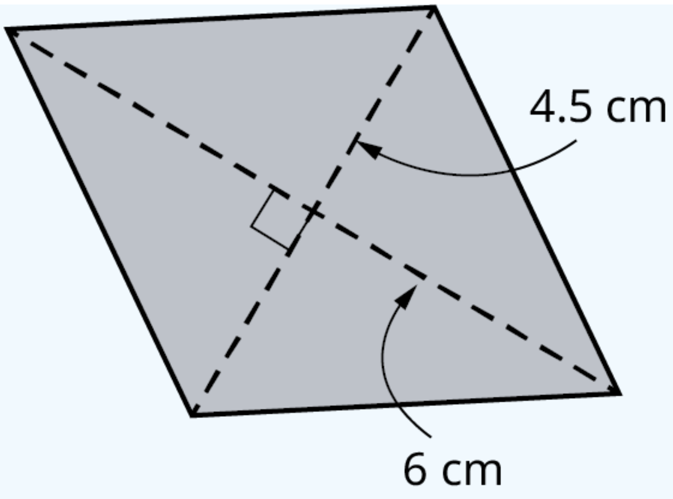


Exercise 10.6.0.8

Find the area of a parallelogram with base equal to 20 in, and height equal to 22 in.

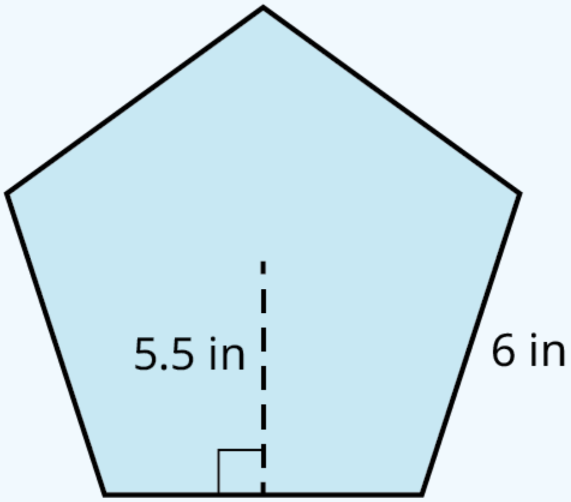
Exercise 10.6.0.9

Find the area of the rhombus shown.



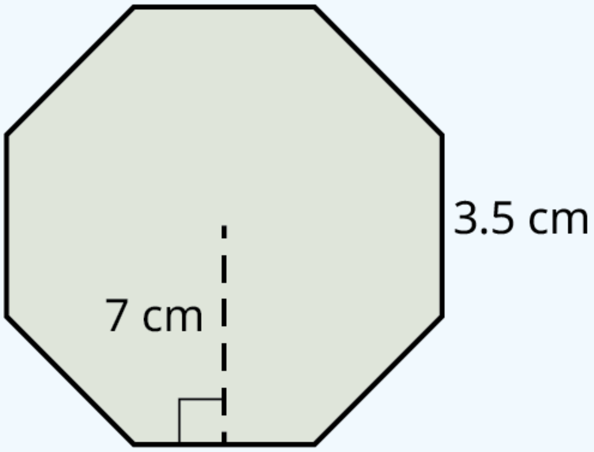
Exercise 10.6.0.10

Find the area of the regular pentagon with the apothem equal to 5.5 in and the side length equal to 6 in as shown.



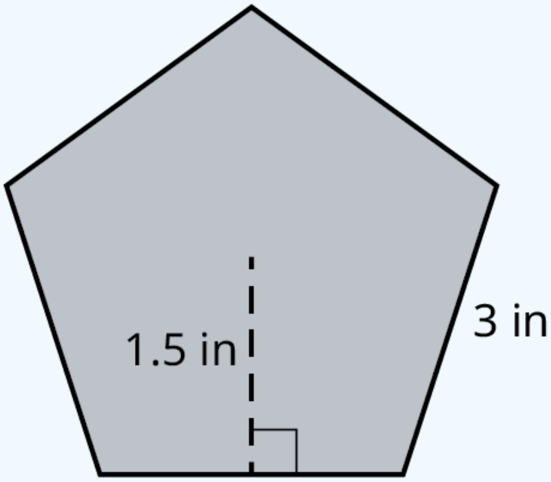
Exercise 10.6.0.11

Find the area of the regular octagon with apothem equal to 7 cm and the side length is 3.5 cm as shown.



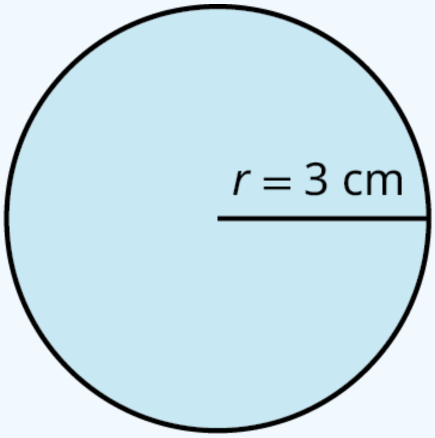
? Exercise 10.6.0.12

Find the area of a regular pentagon with apothem equal 1.5 in, and the sides are equal to 3 in as shown.



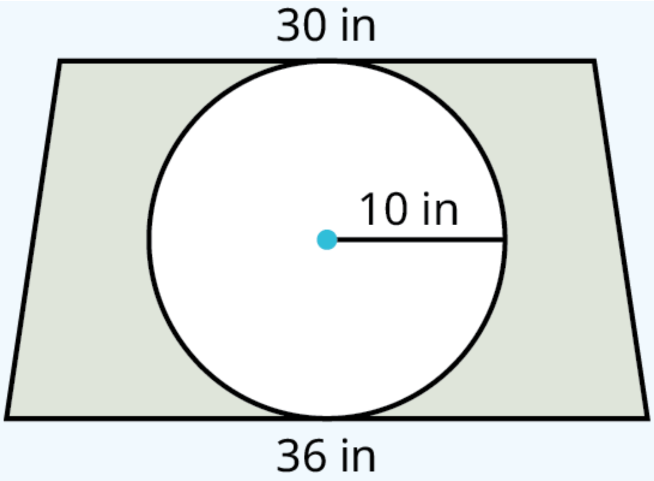
? Exercise 10.6.0.13

Find the area of a circle with radius of 3 cm as shown.



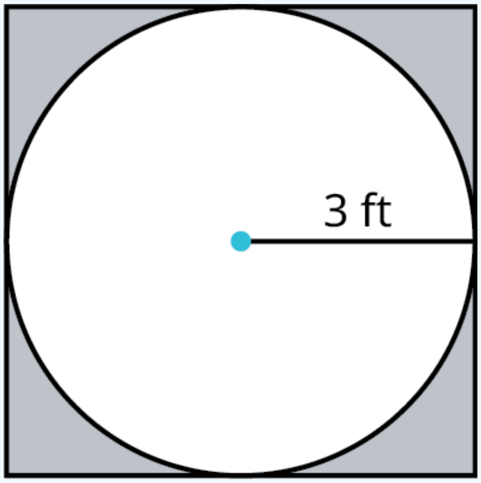
? Exercise 10.6.0.14

You are installing a countertop in the shape of a trapezoid with a round sink as shown in the figure. After the sink is installed, how much area is left on the countertop?



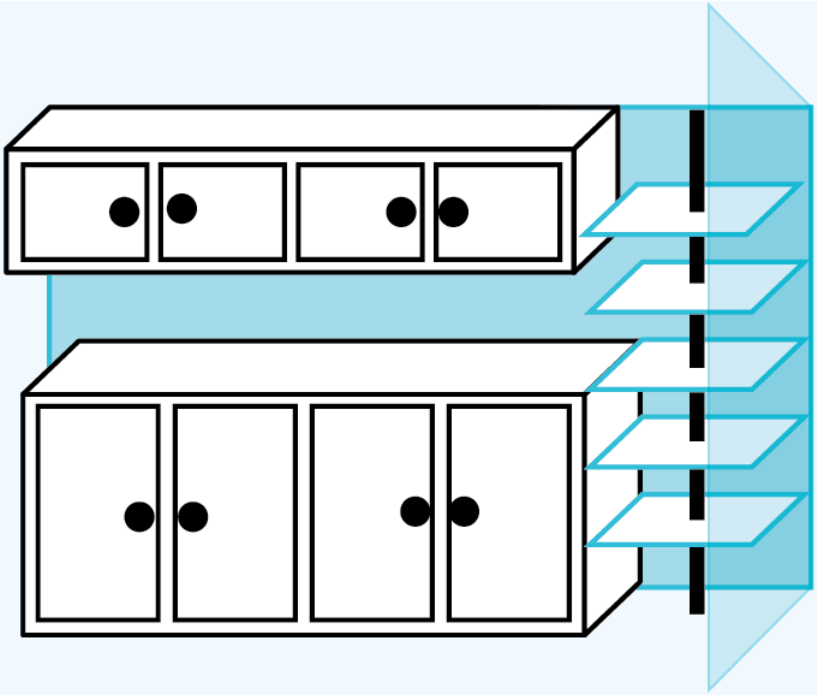
Exercise 10.6.0.15

Find the area of the shaded region in the given figure.



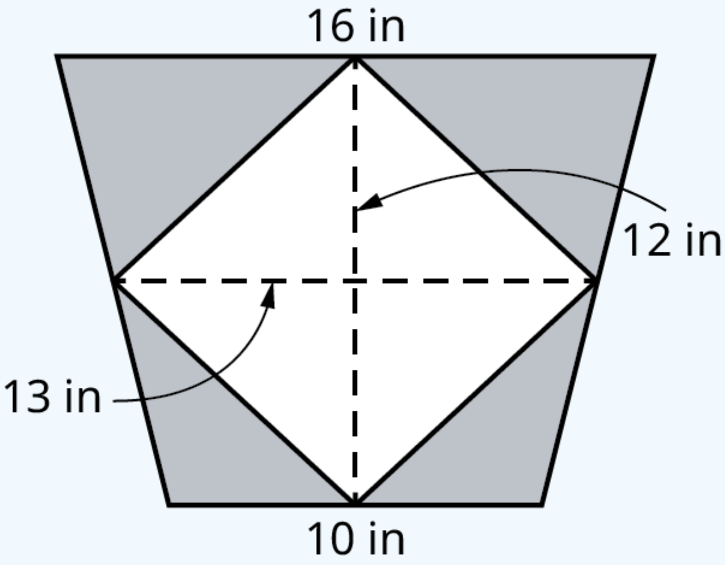
Exercise 10.6.0.16

You have a structural post in the corner of your kitchen. The adjacent room is the family room and the two rooms are separated by a transparent glass wall. To camouflage the post, you decide to have five shelves built that will fill the corner, as shown in the figure. The rectangular shelves are 26 inches wide and 24 inches deep. The diameter of the post is 6 inches. How much shelf area will be available after the shelves are built around the post? The laminate material for the shelves runs $25 per square foot. How much will the laminate shelves cost?



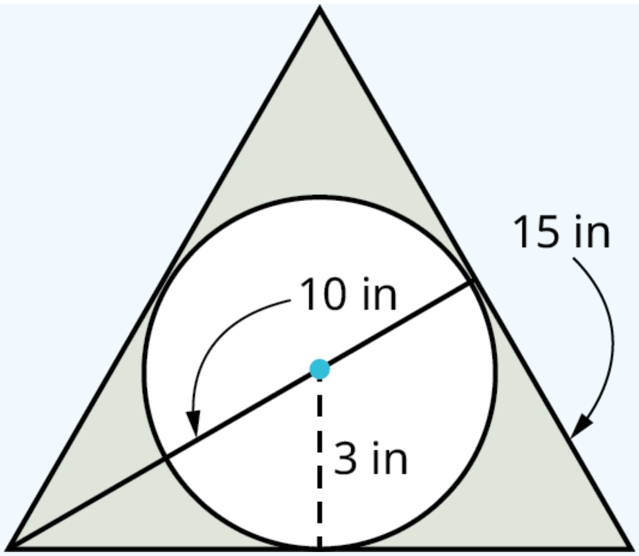
Exercise 10.6.0.17

Find the area of the shaded region in the given figure.



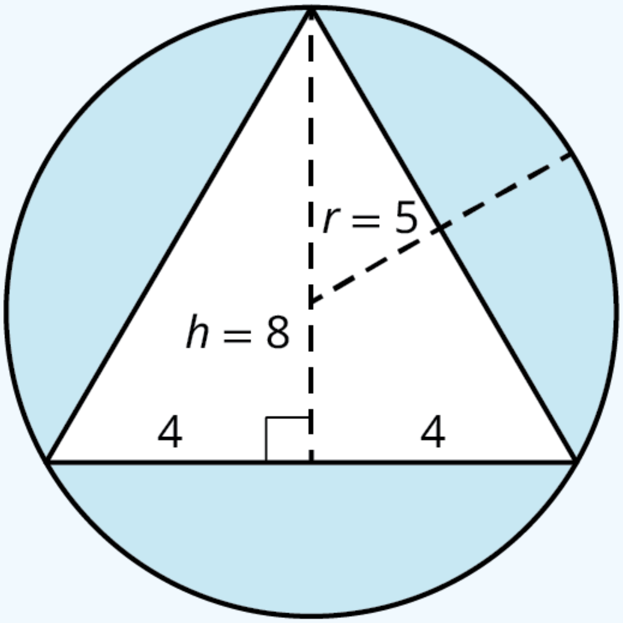
Exercise 10.6.0.18

Find the area of the shaded region in the given figure.



? Exercise 10.6.0.19

Find the area of the shaded region in the given figure. All measurements in centimeters.

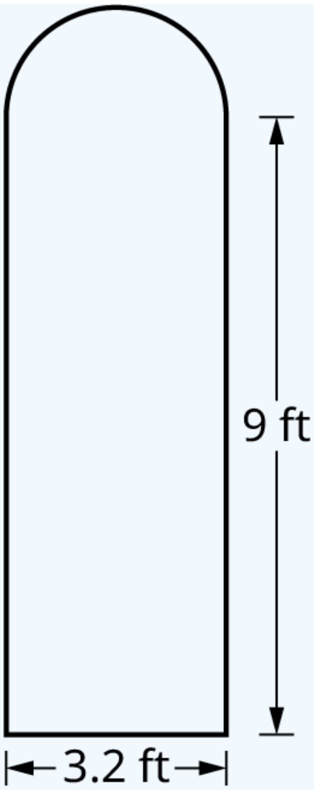


? Exercise 10.6.0.20

Your property measures 2 miles wide by 2.5 miles long. You want to landscape it but the landscaper charges by the square foot. How many square feet need to be landscaped?

? Exercise 10.6.0.21

Find the area of the window shown.



Exercise $10.6.0.22$

What is the area of a major league baseball diamond enclosed by the baselines if it forms a square with 90ft between bases?

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10.7: Volume and Surface Area

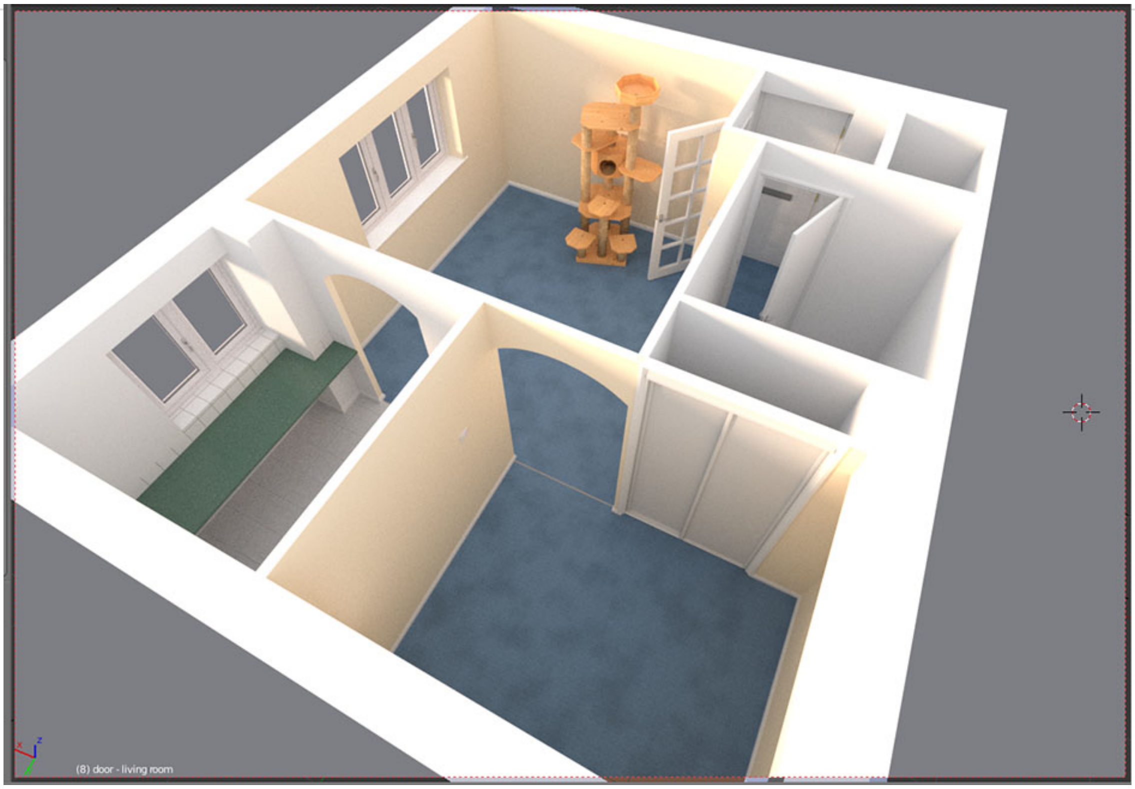


Figure 10.7.1: Volume is illustrated in this 3-dimensional view of an interior space. This gives a buyer a more realistic interpretation of space. (credit: "beam render 10 with sun and cat tree" by monkeywing/Flickr, CC BY 2.0)

Learning Objectives

1. Calculate the surface area of right prisms and cylinders.
2. Calculate the volume of right prisms and cylinders.
3. Solve application problems involving surface area and volume.

Volume and surface area are two measurements that are part of our daily lives. We use volume every day, even though we do not focus on it. When you purchase groceries, volume is the key to pricing. Judging how much paint to buy or how many square feet of siding to purchase is based on surface area. The list goes on. An example is a three-dimensional rendering of a floor plan. These types of drawings make building layouts far easier to understand for the client. It allows the viewer a realistic idea of the product at completion; you can see the natural space, the volume of the rooms. This section gives you practical information you will use consistently. You may not remember every formula, but you will remember the concepts, and you will know where to go should you want to calculate volume or surface area in the future.

We will concentrate on a few particular types of three-dimensional objects: right prisms and right cylinders. The adjective “right” refers to objects such that the sides form a right angle with the base. We will look at right rectangular prisms, right triangular prisms, right hexagonal prisms, right octagonal prisms, and right cylinders. Although, the principles learned here apply to all right prisms.

Three-Dimensional Objects

In geometry, three-dimensional objects are called geometric solids. Surface area refers to the flat surfaces that surround the solid and is measured in square units. Volume refers to the space inside the solid and is measured in cubic units. Imagine that you have a square flat surface with width and length. Adding the third dimension adds depth or height, depending on your viewpoint, and now you have a box. One way to view this concept is in the Cartesian coordinate three-dimensional space. The x-axis and the y-axis are, as you would expect, two dimensions and suitable for plotting two-dimensional graphs and shapes. Adding the z-axis, which shoots through the origin perpendicular to the xy-plane, and we have a third dimension. See Figure 10.7.2.

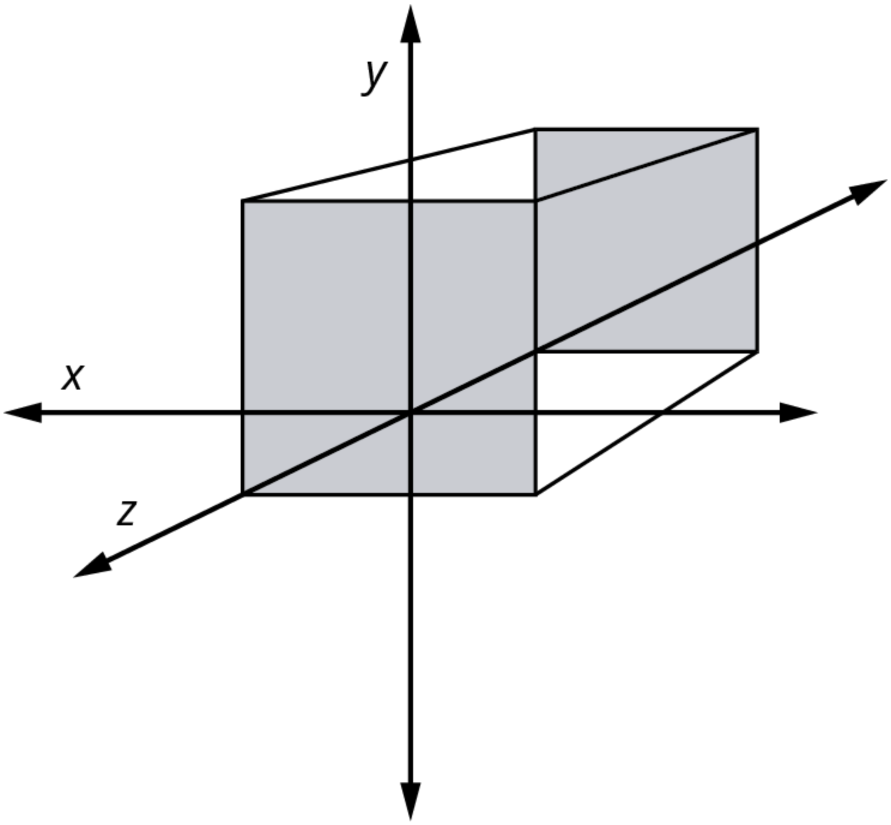


Figure $10.7.2$: Three-Dimensional Space

Here is another view taking the two-dimensional square to a third dimension. See Figure $10.7.3$.

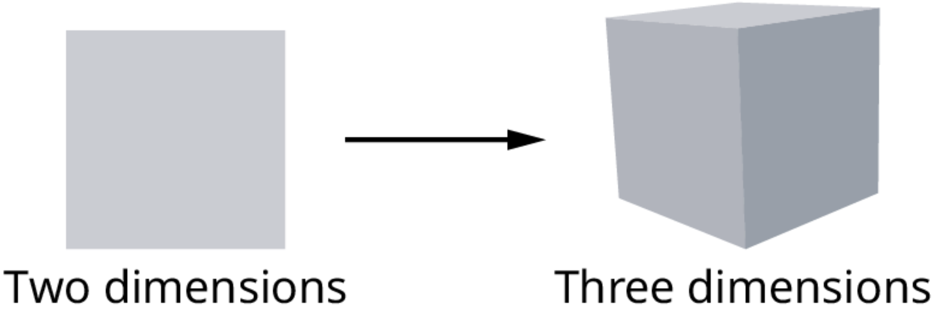


Figure $10.7.3$: Going from Two Dimensions to Three Dimensions

To study objects in three dimensions, we need to consider the formulas for surface area and volume. For example, suppose you have a box (Figure $10.7.4$) with a hinged lid that you want to use for keeping photos, and you want to cover the box with a decorative paper. You would need to find the surface area to calculate how much paper is needed. Suppose you need to know how much water will fill your swimming pool. In that case, you would need to calculate the volume of the pool. These are just a couple of examples of why these concepts should be understood, and familiarity with the formulas will allow you to make use of these ideas as related to right prisms and right cylinders.

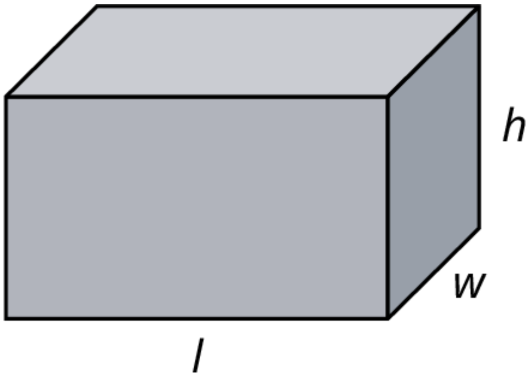


Figure 10.7.4

Right Prisms

A right prism is a particular type of three-dimensional object. It has a polygon-shaped base and a congruent, regular polygon-shaped top, which are connected by the height of its lateral sides, as shown in Figure $10.7.5$. The lateral sides form a right angle with the base and the top. There are rectangular prisms, hexagonal prisms, octagonal prisms, triangular prisms, and so on.

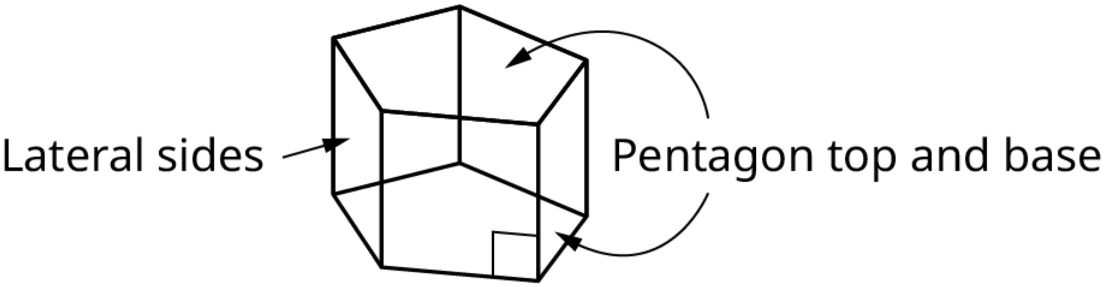


Figure $10.7.5$: Pentagonal Prism

Generally, to calculate surface area, we find the area of each side of the object and add the areas together. To calculate volume, we calculate the space inside the solid bounded by its sides.

FORMULA

The formula for the surface area of a right prism is equal to twice the area of the base plus the perimeter of the base times the height, $SA = 2B + ph$, where B is equal to the area of the base and top, p is the perimeter of the base, and h is the height.

FORMULA

The formula for the volume of a rectangular prism, given in cubic units, is equal to the area of the base times the height, $V = B \cdot h$, where B is the area of the base and h is the height.

✓ Example $10.7.1$: Calculating Surface Area and Volume of a Rectangular Prism

Find the surface area and volume of the rectangular prism that has a width of 10 cm, a length of 5 cm, and a height of 3 cm (Figure ✓ $10.7.6$).

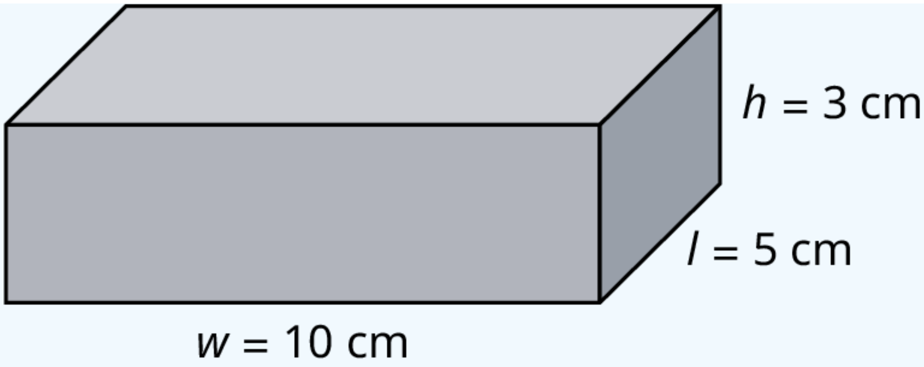


Figure ✓ 10.7.6

Answer

The surface area is $SA = 2(10)(5) + 2(5)(3) + 2(10)(3) = 190 \text{ cm}^2$.

The volume is $V = 10(5)(3) = 150 \text{ cm}^3$.

📌 Your Turn 10.7.1

A rectangular solid has a width of 6 cm, length of 15 cm, and height or depth of 6 cm. Find the surface area and the volume.

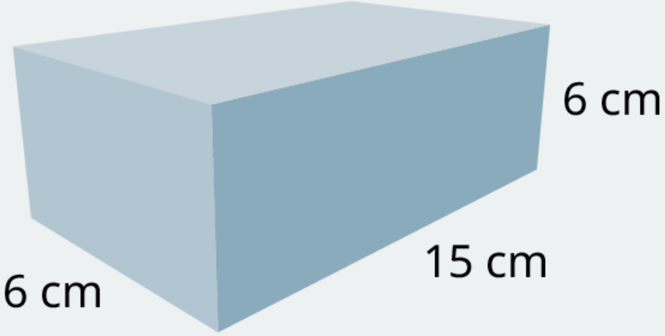


Figure 📌 10.7.7

In Figure 10.129, we have three views of a right hexagonal prism. The regular hexagon is the base and top, and the lateral faces are the rectangular regions perpendicular to the base. We call it a right prism because the angle formed by the lateral sides to the base is 90°. See Figure 10.7.8

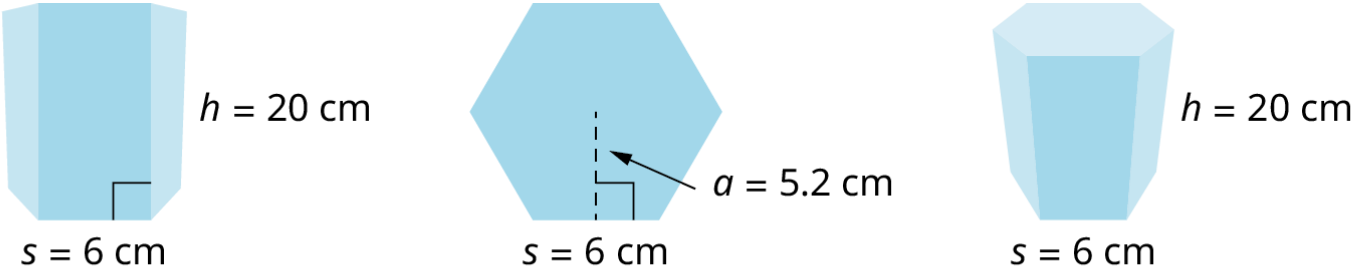


Figure 10.7.8: Right Hexagonal Prism

The first image is a view of the figure straight on with no rotation in any direction. The middle figure is the base or the top. The last figure shows you the solid in three dimensions. To calculate the surface area of the right prism shown in Figure 10.129, we first determine the area of the hexagonal base and multiply that by 2, and then add the perimeter of the base times the height. Recall the area of a regular polygon is given as $A = \frac{1}{2}ap$, where a is the apothem and p is the perimeter. We have that

$$A_{\text{base}} = \frac{1}{2}(5.2)(36) = 93.6 \text{ cm}^2$$

Then, the surface area of the hexagonal prism is

$$SA = 2(93.6) + 36(20) = 907.2 \text{ in }^2$$

To find the volume of the right hexagonal prism, we multiply the area of the base by the height using the formula $V = Bh$. The base is 93.6 cm^2, and the height is 20 cm . Thus,

$$V = 93.6(20) = 1872 \text{ cm}^3$$

✓ Example $10.7.2$: Calculating the Surface Area of a Right Triangular Prism

Find the surface area of the triangular prism (Figure ✓ $10.7.9$).

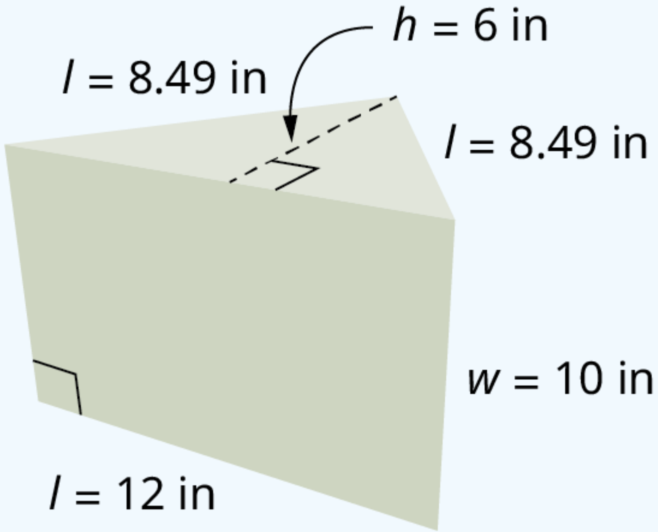


Figure ✓ $10.7.9$

Answer

The area of the triangular base is $A_{\text{base}} = \frac{1}{2}(12)(6) = 36\text{in}^2$. The perimeter of the base is $p = 12 + 8.49 + 8.49 = 28.98\text{in}$. Then, the surface area of the triangular prism is $SA = 2(36) + 28.98(10) = 361.8\text{in}^2$.

Your Turn $10.7.2$

Find the surface area of the triangular prism shown.

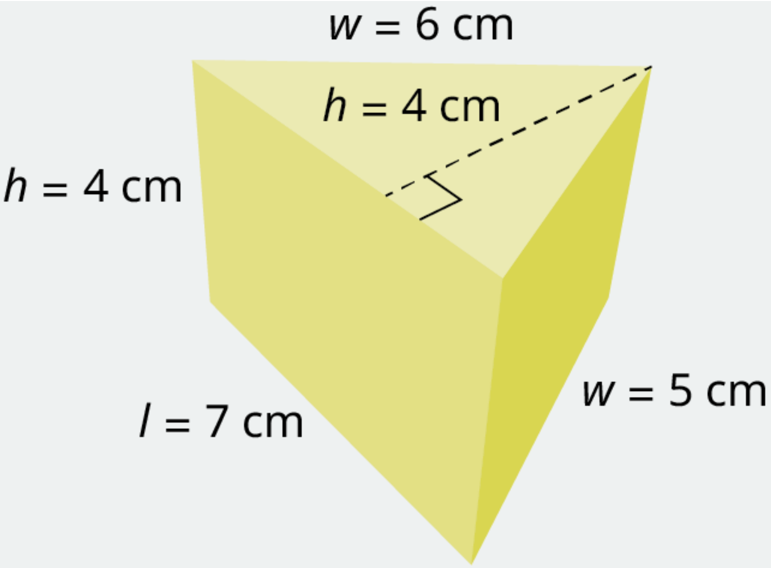


Figure 📌 $10.7.10$

✓ Example $10.7.3$: Finding the Surface Area and Volume

Find the surface area and the volume of the right triangular prism with an equilateral triangle as the base and height (Figure ✓ $10.7.11$).

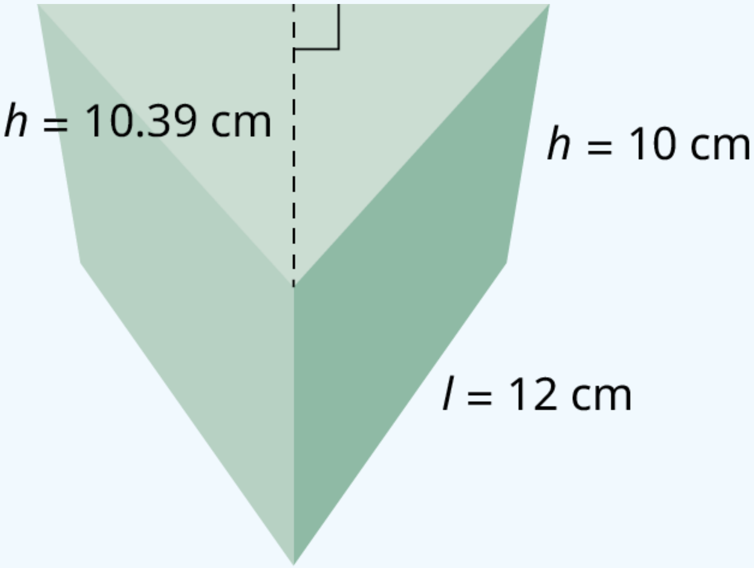


Figure ✓ $10.7.11$

Answer

The area of the triangular base is $A_{\text{base}} = \frac{1}{2}(6)(10.39) = 31.17\text{ cm}^2$. Then, the surface area is $SA = 2(31.17) + 36(10) = 422.34\text{ cm}^2$.

The volume formula is found by multiplying the area of the base by the height. We have that $V = B \cdot h = (31.17)(10) = 311.7\text{ cm}^3$.

Your Turn 10.7.3

Find the surface area and the volume of the octagonal figure shown.

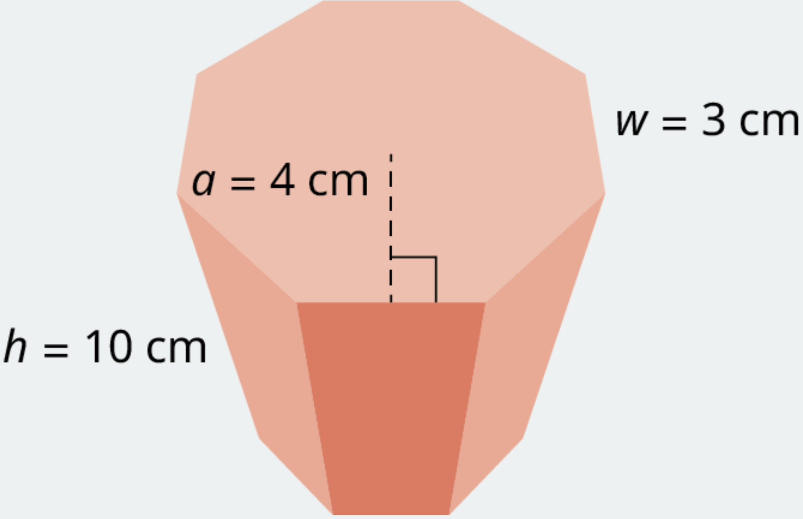


Figure 10.7.12

Example 10.7.4: Determining Surface Area Application

Katherine and Romano built a greenhouse made of glass with a metal roof (Figure 10.7.13). In order to determine the heating and cooling requirements, the surface area must be known. Calculate the total surface area of the greenhouse.

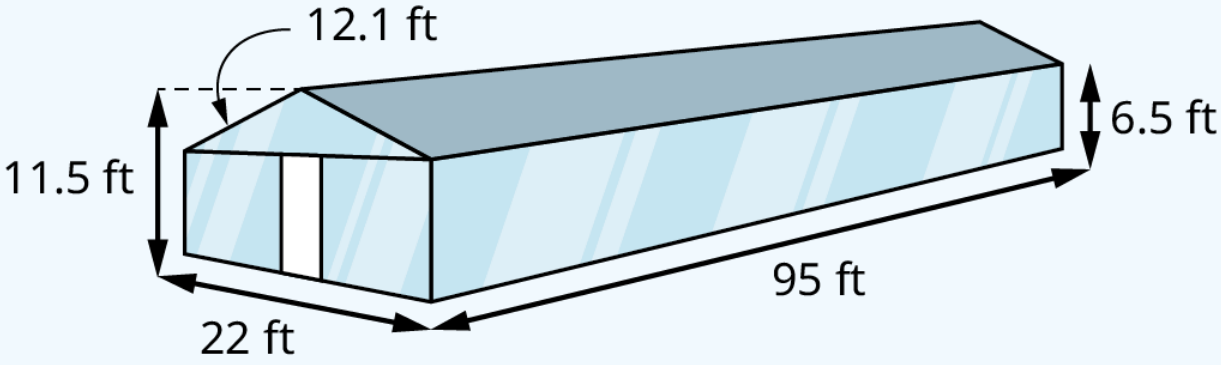


Figure 10.7.13

Answer

The area of the long side measures $95\text{ft} \times 6.5\text{ft} = 1,235\text{ft}^2$. Multiplying by 2 gives $2,470\text{ft}^2$. The front (minus the triangular area) measures $22\text{ft} \times 6.5\text{ft} = 286\text{ft}^2$. Multiplying by 2 gives 572ft^2. The floor measures $95\text{ft} \times 22\text{ft} = 2,090\text{ft}^2$. Each triangular region measures $A = \frac{1}{2}(22)(5) = 55\text{ft}^2$. Multiplying by 2 gives 110ft^2. Finally, one side of the roof measures $12.1\text{ft} \times 95\text{ft} = 1,149.5\text{ft}^2$. Multiplying by 2 gives 2299ft^2. Add them up and we have $SA = 2,470 + 572 + 2,090 + 110 + 2,299 = 7,541\text{ft}^2$.

Your Turn 10.7.4

Calculate the surface area of a greenhouse with a flat roof measuring 12 ft wide, 25 ft long, and 8 ft high.

Right Cylinders

There are similarities between a prism and a cylinder. While a prism has parallel congruent polygons as the top and the base, a right cylinder is a three-dimensional object with congruent circles as the top and the base. The lateral sides of a right prism make a 90°

angle with the polygonal base, and the side of a cylinder, which unwraps as a rectangle, makes a 90° angle with the circular base.

Right cylinders are very common in everyday life. Think about soup cans, juice cans, soft drink cans, pipes, air hoses, and the list goes on.

In Figure 10.7.14, imagine that the cylinder is cut down the 12-inch side and rolled out. We can see that the cylinder side when flat forms a rectangle. The semanticsSASA formula includes the area of the circular base, the circular top, and the area of the rectangular side. The length of the rectangular side is the circumference of the circular base. Thus, we have the formula for total surface area of a right cylinder.

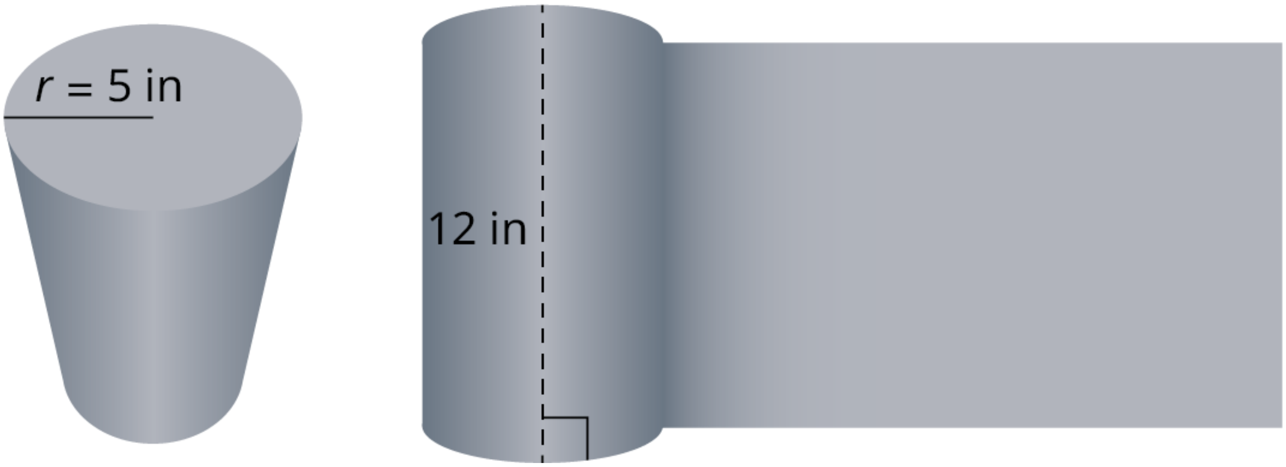


Figure 10.7.14: Right Cylinder

FORMULA

The surface area of a right cylinder is given as $SA = 2\pi r^2 + 2\pi rh$.

To find the volume of the cylinder, we multiply the area of the base with the height.

FORMULA

The volume of a right cylinder is given as $V = \pi r^2 h$.

✓ Example 10.7.5: Finding the Surface Area and Volume of a Cylinder

Given the cylinder in Figure ✓ 10.7.14, which has a radius of 5 inches and a height of 12 inches, find the surface area and the volume.

Answer

Step 1: We begin with the areas of the base and the top. The area of the circular base is

$$A_{base} = \pi(5)^2 = 25\pi = 78.5\text{in}^2$$

Step 2: The base and the top are congruent parallel planes. Therefore, the area for the base and the top is

$$A = 2(78.5) = 157\text{in}^2$$

Step 3: The area of the rectangular side is equal to the circumference of the base times the height:

$$A = 2\pi(5)(12) = 377\text{in}^2$$

Step 4: We add the area of the side to the areas of the base and the top to get the total surface area. We have

$$SA = 157 + 377 = 534 \text{ in }^2$$

Step 5: The volume is equal to the area of the base times the height. Then,

$$V = \pi(5)^2(12) = 942.48\text{in}^3$$

Your Turn 10.7.5

Find the surface area and volume of the cylinder with a radius of 7cm and a height of 5 cm.

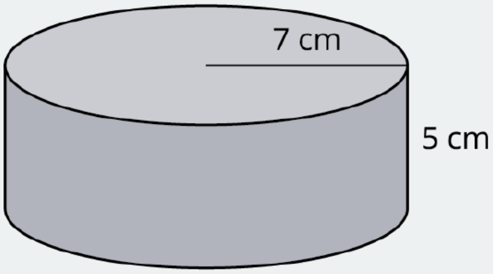


Figure 10.7.15

Applications of Surface Area and Volume

The following are just a small handful of the types of applications in which surface area and volume are critical factors. Give this a little thought and you will realize many more practical uses for these procedures.

Example 10.7.6: Applying a Calculation of Volume

A can of apple pie filling has a radius of 4 cm and a height of 10 cm. How many cans are needed to fill a pie pan (Figure 10.7.16) measuring 22 cm in diameter and 3 cm deep?

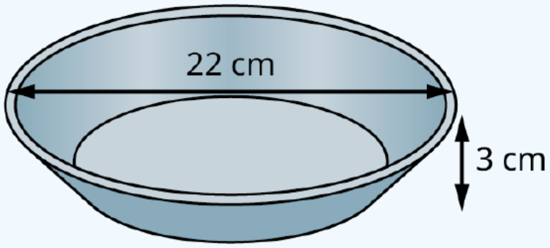


Figure 10.7.16

Answer

The volume of the can of apple pie filling is $V=\pi(4)^2(10)=502.7\text{ cm}^3$. The volume of the pan is $V=\pi(11)^2(3)=1140.4\text{ cm}^3$. To find the number of cans of apple pie filling, we divide the volume of the pan by the volume of a can of apple pie filling. Thus, $1140 \div 502.7=2.3$. We will need 2.3 cans of apple pie filling to fill the pan.

Your Turn 10.7.6

You are making a casserole that includes vegetable soup and pasta. The size of your cylindrical casserole dish has a diameter of 10 inches and is 4 inches high. The pasta will consume the bottom portion of the casserole dish about 1 inch high. The soup can has a diameter of 3 inches and is 4 inches high. After the pasta is added, how many cans of soup can you add?

Optimization

Problems that involve optimization are ones that look for the best solution to a situation under some given conditions. Generally, one looks to calculus to solve these problems. However, many geometric applications can be solved with the tools learned in this section. Suppose you want to make some throw pillows for your sofa, but you have a limited amount of fabric. You want to make the largest pillows you can from the fabric you have, so you would need to figure out the dimensions of the pillows that will fit these criteria. Another situation might be that you want to fence off an area in your backyard for a garden. You have a limited amount of fencing available, but you would like the garden to be as large as possible. How would you determine the shape and size of the garden? Perhaps you are looking for maximum volume or minimum surface area. Minimum cost is also a popular application of optimization. Let's explore a few examples.

✓ Example 10.7.7: Maximizing Area

Suppose you have 150 meters of fencing that you plan to use for the enclosure of a corral on a ranch. What shape would give the greatest possible area?

Answer

So, how would we start? Let's look at this on a smaller scale. Say that you have 30 inches of string and you experiment with different shapes. The rectangle in Figure ✓ 10.7.17 measures 12 in long by 3 in wide. We have a perimeter of $P = 2(12) + 2(3)$ =30inFigure 10.176, measures 8 in long and 7 in wide. The perimeter is $P = 2(8) + 2(7)$ =30inFigure 10.176, the square measures 7.5 in on each side. The perimeter is then $P = 4(7.5)$ =30inFigure 10.176, we would consider a circumference of 30 =2πr,30=2πr, which gives a radius of 30÷2π=4.77in30÷2π=4.77in and an area of $A = \pi(4.77)^2 = 71.5\text{ in}^2$.

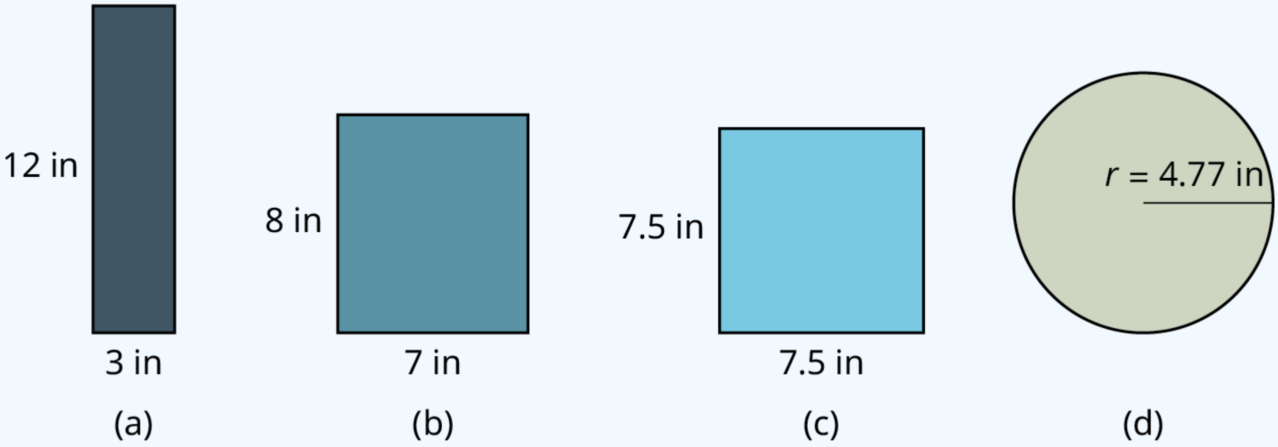


Figure ✓ 10.7.17

We see that the circular shape gives the maximum area relative to a circumference of 30 in. So, a circular corral with a circumference of 150 meters and a radius of 23.87 meters gives a maximum area of $1{,}790.5\text{ m}^2$.

Your Turn 10.7.7

You have 25 ft of rope to section off a rectangular-shaped garden. What dimensions give the maximum area that can be roped off?

✓ Example 10.7.8: Designing for Cost

Suppose you want to design a crate built out of wood in the shape of a rectangular prism (Figure ✓ 10.7.18). It must have a volume of 3 cubic meters. The cost of wood is \$15 per square meter. What dimensions give the most economical design while providing the necessary volume?

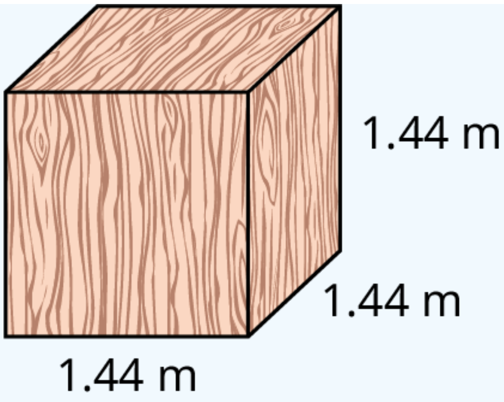


Figure ✓ 10.7.18

Answer

To choose the optimal shape for this container, you can start by experimenting with different sizes of boxes that will hold 3 cubic meters. It turns out that, similar to the maximum rectangular area example where a square gives the maximum area, a cube gives the maximum volume and the minimum surface area.

As all six sides are the same, we can use a simplified volume formula:

$$V = s^3$$

where s is the length of a side. Then, to find the length of a side, we take the cube root of the volume.
We have

$$\begin{aligned} 3 &= s^3 \\ \sqrt[3]{3} &= s \\ &= 1.4422 \text{ m} \end{aligned}$$

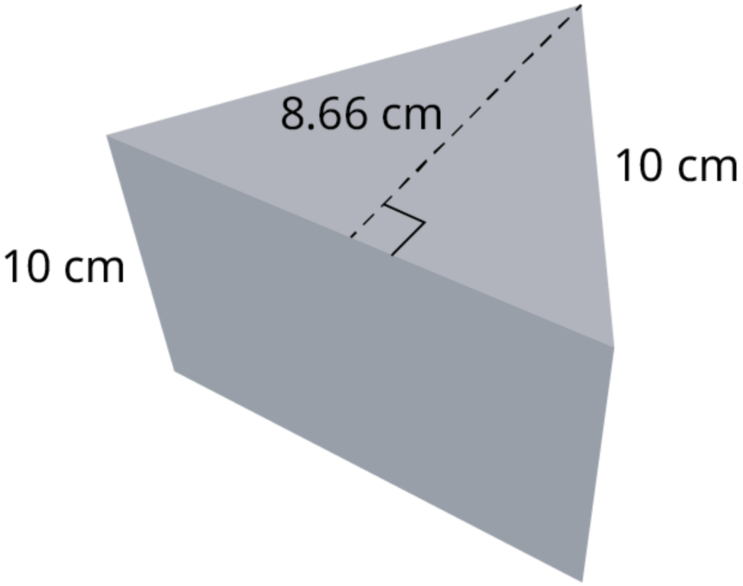
The surface area is equal to the sum of the areas of the six sides. The area of one side is $A = (1.4422)^2 = 2.08 \text{ m}^2$. So, the surface area of the crate is $SA = 6(2.08) = 12.5 \text{ m}^2$. At $\$15$ a square meter, the cost comes to $12.5(\$15) = \$187.50$. Checking the volume, we have $V = (1.4422)^3 = 2.99 \text{ m}^3$.

Your Turn 10.7.8

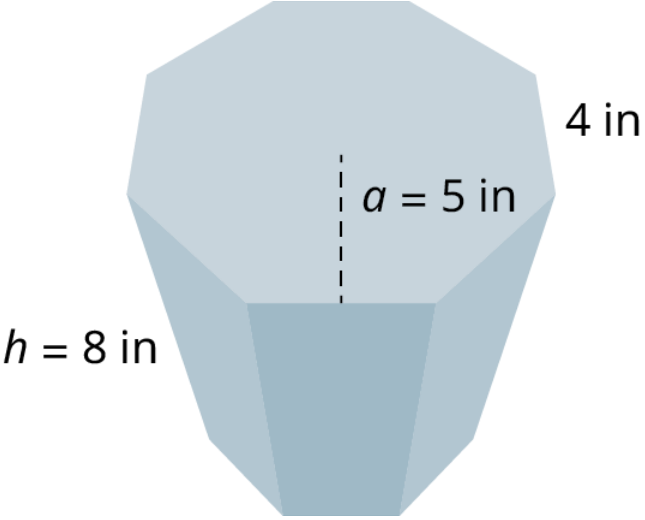
Suppose you want to a build a container to hold 2 cubic feet of fabric swatches. You want to cover the container in laminate costing $10 per square foot. What are the dimensions of the container that is the most economical? What is the cost?

Check Your Understanding

1. Find the surface area of the equilateral triangular prism shown.



2. Find the surface area of the octagonal prism shown.

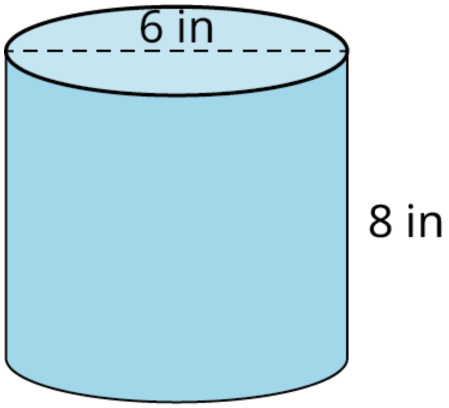


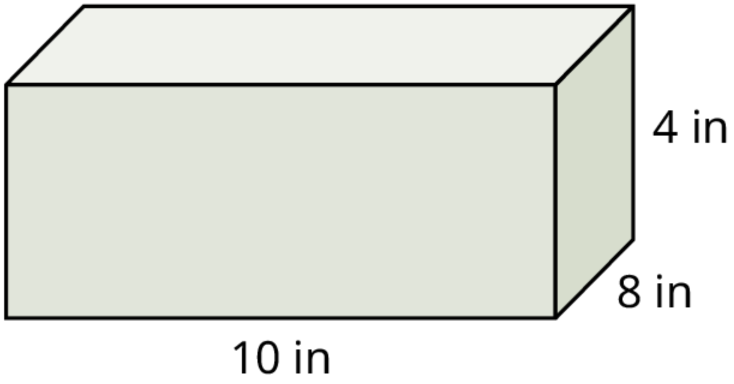
3. Find the volume of the octagonal prism shown with the apothem equal to 5 in, the side length equal to 4 in, and the height equal to 8 in.

4. Determine the surface area of the right cylinder where the radius of the base is 10 cm, and the height is 5 cm.

5. Find the volume of the cylinder where the radius of the base is 10 cm, and the height is 5 cm.

6. As an artist, you want to design a cylindrical container for your colored art pencils and another rectangular container for some other tools. The cylindrical container will be 8 inches high with a diameter of 6 inches. The rectangular container measures 10 inches wide by 8 inches deep by 4 inches high and has a lid. You found some beautiful patterned paper to use to wrap both pieces. How much paper will you need?





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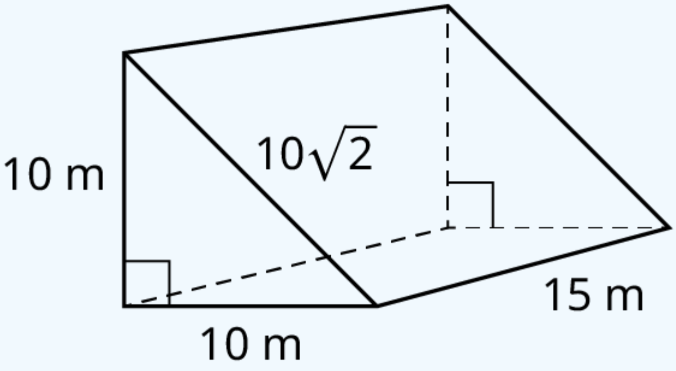
10.7.0: Exercises

Exercise 10.7.0.1

Find the volume of the right triangular prism with the two side legs of the base equal to 10 m, the hypotenuse equal to $10\sqrt{2}$, and the height or the length, depending on your viewpoint, is equal to 15 m.

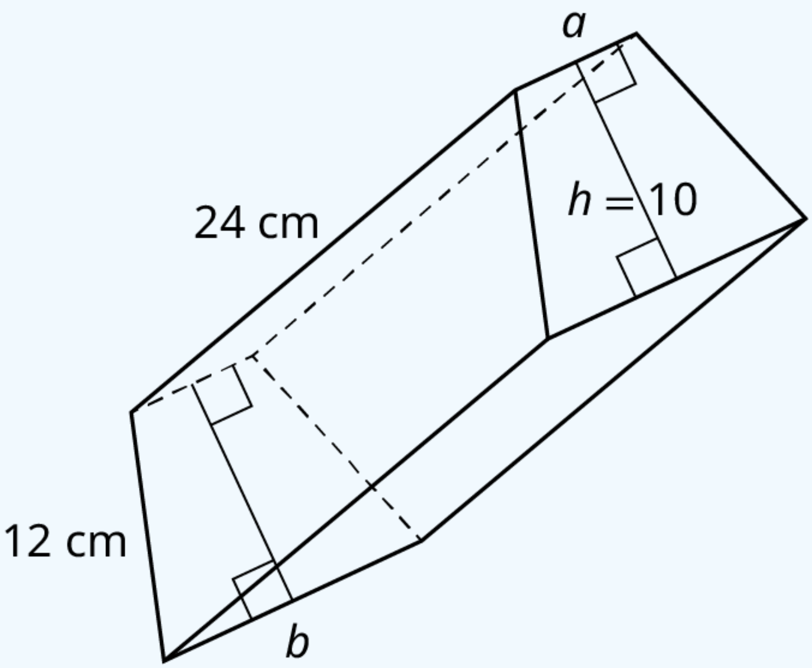


Exercise 10.7.0.2

Find the surface area of the right triangular prism in the Exercise 1 with the two legs of the base equal to 10 m, and the height equal to 15 m.

Exercise 10.7.0.3

Find the surface area of the right trapezoidal prism with side $\mathrm{a} = 5\,\mathrm{cm}$, side $\mathrm{b} = 12\,\mathrm{cm}$, the height is 10 cm, the slant length is 12 cm, and the length is 24 cm.

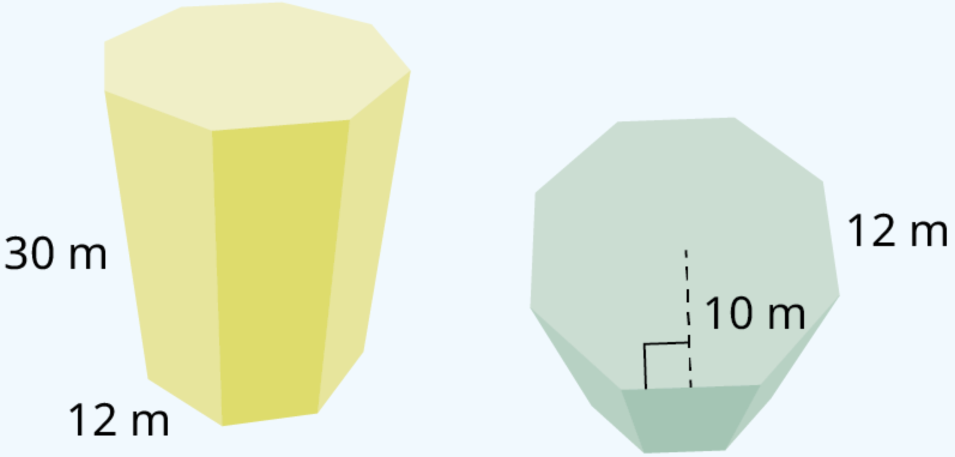


? Exercise 10.7.0.4

Find the volume of the trapezoidal prism in the exercise above where the base and top have the following measurements: side $a = 5\,\text{cm}$, side $b = 12\,\text{cm}$, the slant lengths are each $12\,\text{cm}$, and the height of the trapezoidal base =$10\,\text{cm}$. The height or length of the three-dimensional solid is $24\,\text{cm}$.

? Exercise 10.7.0.5

Find the surface area of the octagonal prism. The base and top are regular octagons with the apothem equal to 10 m, a side length equal to 12 m, and a height of 30 m.



? Exercise 10.7.0.6

Find the volume for the right octagonal prism, with the apothem equal to 10 m, the side length of the base is equal to 12 m, and the height equal to 30 m.

? Exercise 10.7.0.7

You decide to paint the living room. You will need the surface area of the 4 walls and the ceiling. The room measures 20 ft long and 14 ft wide, and the ceiling is 8 ft high.

For the following exercises, find the surface area of each right cylinder.

? Exercise 10.7.0.8

$r = 6\,\text{cm},\ h = 5\,\text{cm}$

? Exercise 10.7.0.9

$r = 9\,\text{cm},\ h = 15\,\text{cm}$

? Exercise 10.7.0.10

$r = 12.2\,\text{in},\ h = 30\,\text{in}$

? Exercise 10.7.0.11

$r = 3.4\,\text{m},\ h = 10.5\,\text{m}$

? Exercise 10.7.0.12

$d = 11.4 \text{ m}, \; h = 4.4 \text{ m}$

? Exercise 10.7.0.13

$d = 20 \text{ in}, \; h = 8.5 \text{ in}$

For the following exercises, find the volume of each right cylinder to the nearest tenth.

? Exercise 10.7.0.14

$r = 7 \text{ cm}, \; h = 8 \text{ cm}$

? Exercise 10.7.0.15

$r = 14 \text{ cm}, \; h = 19 \text{ cm}$

? Exercise 10.7.0.16

$r = 14 \text{ in}, \; h = 30 \text{ in}$

? Exercise 10.7.0.17

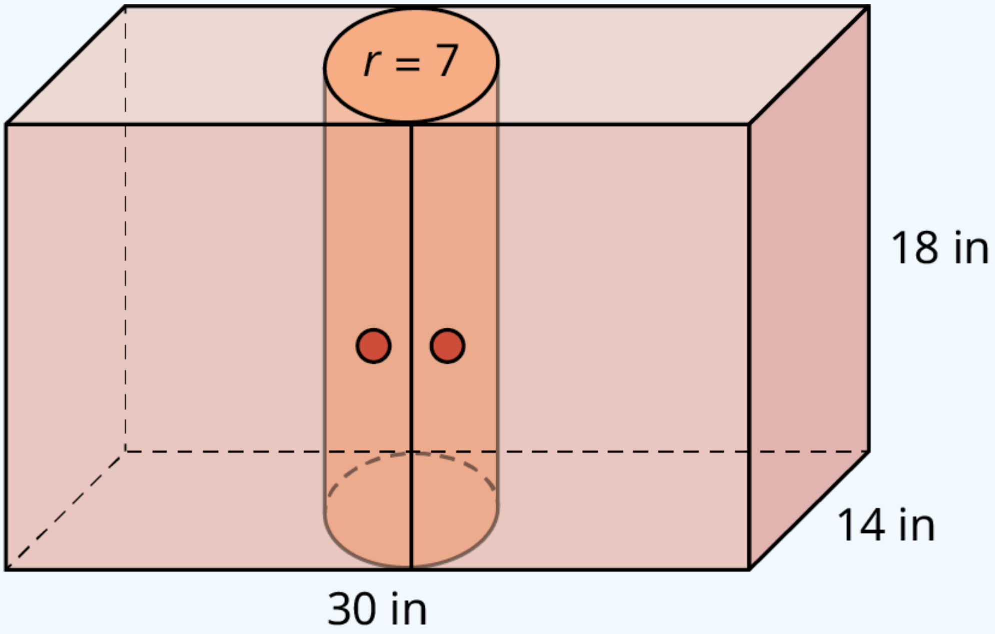
$r = 4.9 \text{ m}, \; h = 17 \text{ m}$

? Exercise 10.7.0.18

$r = 12.6 \text{ cm}, \; h = 16.2 \text{ cm}$

? Exercise 10.7.0.19

You have remodeled your kitchen and the exhaust pipe above the stove must pass through an overhead cabinet as shown in the figure. Find the volume of the remaining space in the cabinet.



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10.8: Right Triangle Trigonometry

Figure 10.8.1: In the lower left corner of the fresco *The School of Athens* by Raphael, the figure in white writing in the book represents Pythagoras. Alongside him, to the right, the figure with the long, light-brown hair is said to depict Archimedes. (credit: modification of work “School of Athens” by Raphael (1483–1520), Vatican Museums/Wikimedia, Public Domain)

Learning Objectives

1. Apply the Pythagorean Theorem to find the missing sides of a right triangle.
2. 2. Apply the $30^\circ-60^\circ-90^\circ$ and $45^\circ-45^\circ-90^\circ$ right triangle relationships to find the missing sides of a triangle.
3. Apply trigonometric ratios to find missing parts of a right triangle.
4. Solve application problems involving trigonometric ratios.

This is another excerpt from Raphael’s *The School of Athens*. The man writing in the book represents Pythagoras, the namesake of one of the most widely used formulas in geometry, engineering, architecture, and many other fields, the Pythagorean Theorem. However, there is evidence that the theorem was known as early as 1900–1100 BC by the Babylonians. The Pythagorean Theorem is a formula used for finding the lengths of the sides of right triangles.

Born in Greece, Pythagoras lived from 569–500 BC. He initiated a cult-like group called the Pythagoreans, which was a secret society composed of mathematicians, philosophers, and musicians. Pythagoras believed that everything in the world could be explained through numbers. Besides the Pythagorean Theorem, Pythagoras and his followers are credited with the discovery of irrational numbers, the musical scale, the relationship between music and mathematics, and many other concepts that left an immeasurable influence on future mathematicians and scientists.

The focus of this section is on right triangles. We will look at how the Pythagorean Theorem is used to find the unknown sides of a right triangle, and we will also study the *special* triangles, those with set ratios between the lengths of sides. By ratios we mean the relationship of one side to another side. When you think about ratios, you should think about fractions. A fraction is a ratio, the ratio of the numerator to the denominator. Finally, we will preview trigonometry. We will learn about the basic trigonometric functions, sine, cosine and tangent, and how they are used to find not only unknown sides but unknown angles, as well, with little information.

Pythagorean Theorem

The Pythagorean Theorem is used to find unknown sides of right triangles. The theorem states that the sum of the squares of the two legs of a right triangle equals the square of the hypotenuse (the longest side of the right triangle).

FORMULA

The Pythagorean Theorem states

$$a^2 + b^2 = c^2$$

where a and b are two sides (legs) of a right triangle and c is the hypotenuse, as shown in Figure 10.8.2

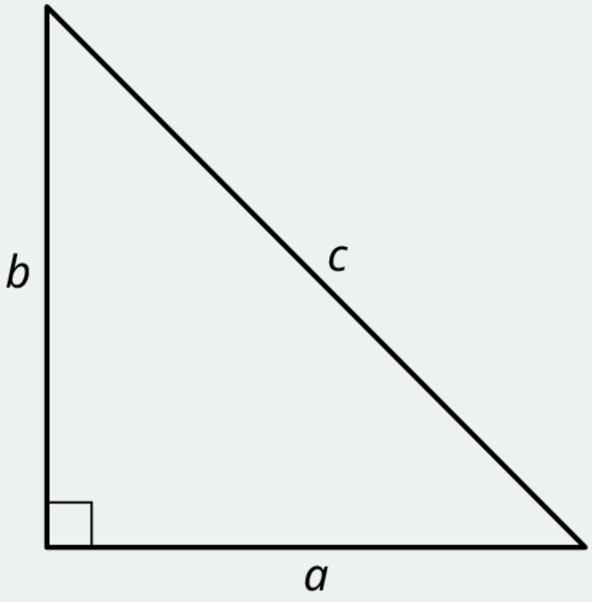


Figure 10.8.2: Pythagorean Right Triangle

For example, given that side $a = 6$, and side $b = 8$, we can find the measure of side c using the Pythagorean Theorem. Thus,

$$\begin{aligned} a^2 + b^2 &= c^2 \\ (6)^2 + (8)^2 &= c^2 \\ 36 + 64 &= c^2 \\ 100 &= c^2 \\ \sqrt{100} &= \sqrt{c^2} \\ 10 &= c \end{aligned}$$

Example 10.8.1: Using the Pythagorean Theorem

Find the length of the missing side of the triangle (Figure 10.8.3).

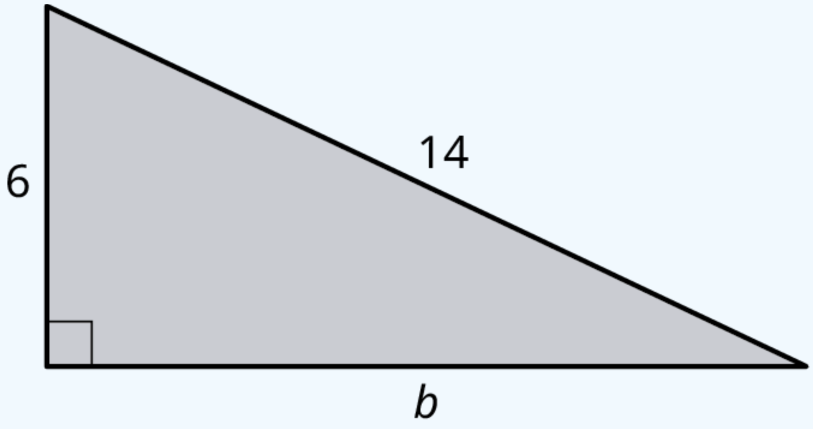


Figure 10.8.3

Answer

Using the Pythagorean Theorem, we have

\[\begin{aligned}
(6)^2+b^2 & =(14)^2 \
36+b^2 & =196 \
b^2 & =196-36 \
b^2 & =160 \
b & = \pm \sqrt{160} \
& =4 \sqrt{10}=12.65
\end{aligned \nonumber \]

When we take the square root of a number, the answer is usually both the positive and negative root. However, lengths cannot be negative, which is why we only consider the positive root.

Your Turn 10.8.1

Use the Pythagorean Theorem to find the missing side of the right triangle shown.

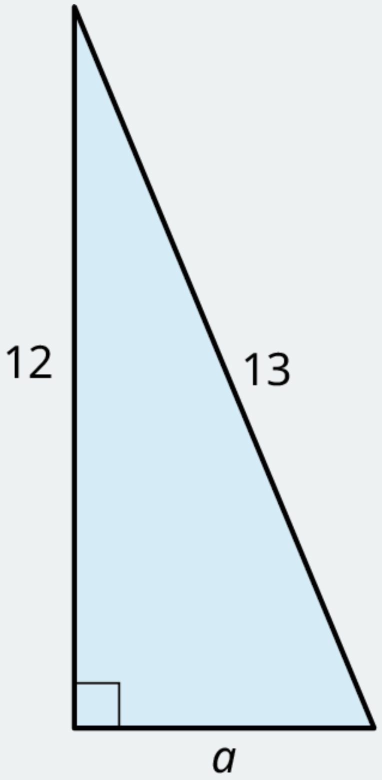


Figure 10.8.4

Distance

The applications of the Pythagorean Theorem are countless, but one especially useful application is that of distance. In fact, the distance formula stems directly from the theorem. It works like this:

In Figure

$$\begin{aligned} d &= \sqrt{(3-(-3))^2+(2-(-1))^2} \\ &= \sqrt{(6)^2+(3)^2} = \sqrt{36+9} \\ &= \sqrt{45} = 3\sqrt{5} = 6.7 \end{aligned}$$

Thus, $d = c$, the hypotenuse, in the Pythagorean Theorem.

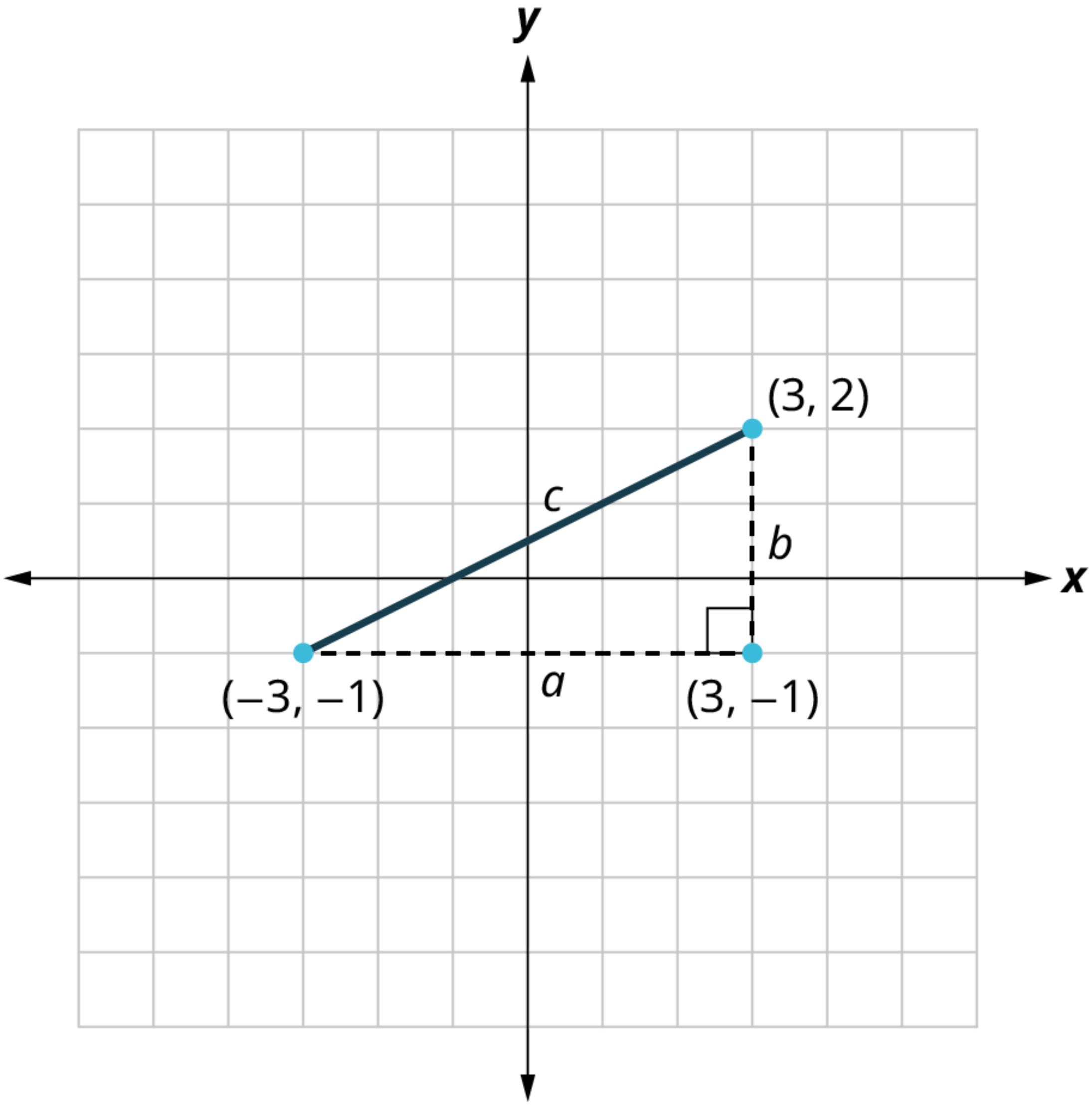


Figure $10.8.5$: Distance

✓ Example $10.8.2$: Calculating Distance Using the Distance Formula

You live on the corner of First Street and Maple Avenue, and work at Star Enterprises on Tenth Street and Elm Drive (Figure ✓ $10.8.6$). You want to calculate how far you walk to work every day and how it compares to the actual distance (as the crow flies). Each block measures 200 ft by 200 ft.

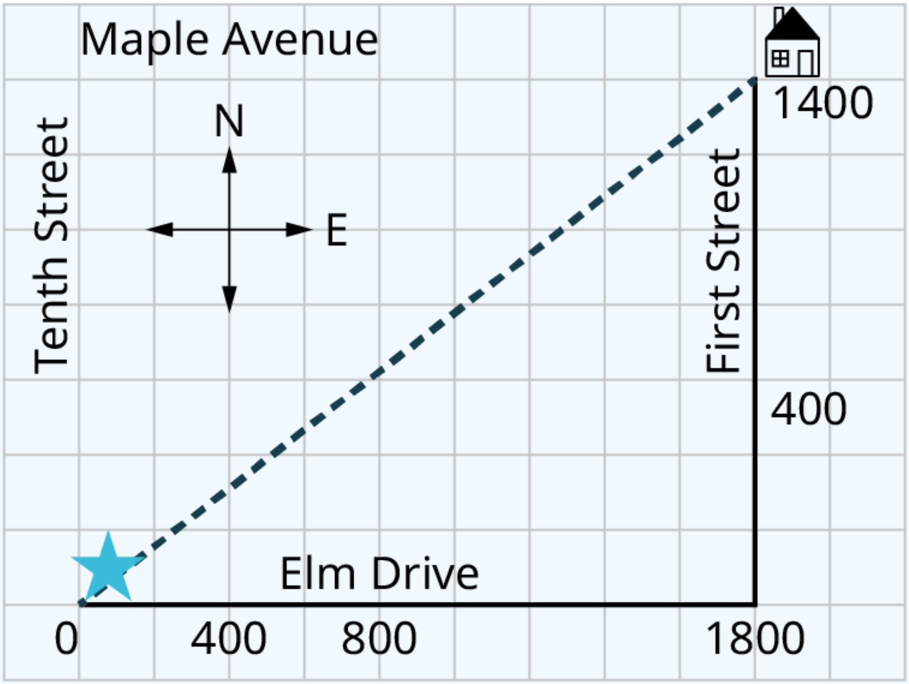


Figure ✓ 10.8.6

Answer

You travel 7 blocks south and 9 blocks west. If each block measures 200 ft by 200 ft , then $9(200)\ +7(200)\ =1,800\text{ft} +1,400\text{ft}=3,200\text{ft}$.

As the crow flies, use the distance formula. We have

$$\begin{aligned} d &= \sqrt{(1,800-0)^2+(1,400-0)^2} \\ &= \sqrt{3,240,000+1,960,000} \\ &= \sqrt{5,200,000} \\ &= 2280.4\text{ft} \end{aligned}$$

Your Turn 10.8.2

How far is it to your workplace (as the crow flies) if the blocks in the previous example measure 100 ft by 100 ft?

✓ Example 10.8.3: Calculating Distance with the Pythagorean Theorem

The city has specific building codes for wheelchair ramps. Every vertical rise of 1 in requires that the horizontal length be 12 inches. You are constructing a ramp at your business. The plan is to make the ramp 130 inches in horizontal length and the slanted distance will measure approximately 132.4 inches (Figure ✓ 10.8.7). What should the vertical height be?

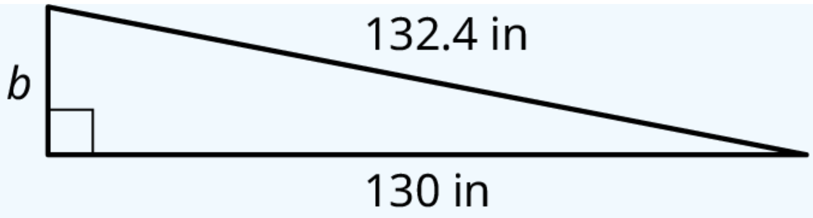


Figure ✓ 10.8.7

Answer

The Pythagorean Theorem states that the horizontal length of the base of the ramp, side *a*, is 130 in. The length of *c*, or the length of the hypotenuse, is

The Pythagorean Theorem states that the horizontal length of the base of the ramp, side a, is 130 in . The length of c, or the length of the hypotenuse, is 132.4 in . The length of the height of the triangle is side b.

Then, by the Pythagorean Theorem, we have:

$$\begin{aligned} a^2+b^2 &= c^2 \\ (130)^2+b^2 &= (132.4)^2 \\ 16,900+b^2 &= 17,529.76 \\ b^2 &= 17,529.76-16,900 \\ b^2 &= 629.8 \\ b &= \sqrt{629.8}=25 \end{aligned}$$

If you construct the ramp with a 25 in vertical rise, will it fulfill the building code? If not, what will have to change?

The building code states 12 in of horizontal length for each 1 in of vertical rise. The vertical rise is 25 in, which means that the horizontal length has to be $12(25)=300\text{in}$. So, no, this will not pass the code. If you must keep the vertical rise at 25 in , what will the other dimensions have to be? Since we need a minimum of 300 in for the horizontal length:

$$\begin{aligned} (300)^2+(25)^2 &= c^2 \\ 90,625 &= c^2 \\ \sqrt{90,625} &= c=301\text{in} \end{aligned}$$

The new ramp will look like Figure 10.185.

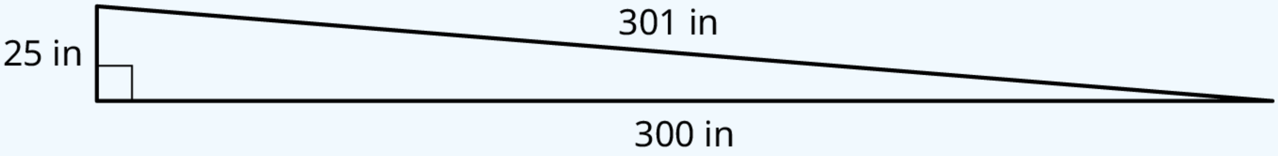


Figure ✓ 10.8.8

Your Turn 10.8.3

If 10 in is the maximum possible vertical rise as shown in the figure, how long would the ramp have to be to pass the building code rule of 12 horizontal inches to 1 vertical inch?

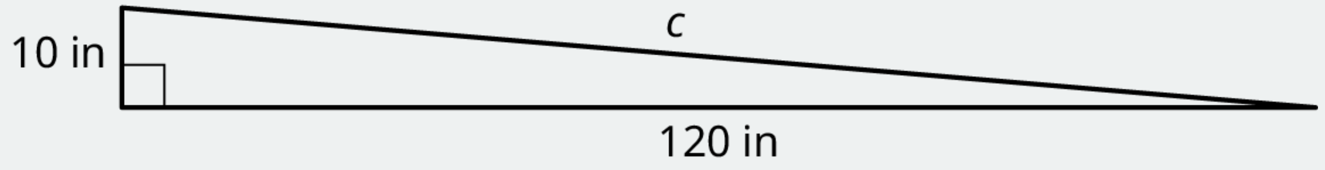


Figure 10.8.9

30°-60°-90° Triangles

In geometry, as in all fields of mathematics, there are always special rules for special circumstances. An example is the perfect square rule in algebra. When expanding an expression like $(2x+5y)^2$, we do not have to expand it the long way:

$$\begin{aligned}(2x+5y)^2 &= (2x+5y)(2x+5y)\\ &= (2x)^2+10xy+10xy+(5y)^2\\ &= 4x^2+20xy+25y^2\end{aligned}$$

If we know the perfect square formula, given as

$$(a+b)^2=a^2+2ab+b^2$$

we can skip the middle step and just start writing down the answer. This may seem trivial with problems like $(a+b)^2$. However, what if you have a problem like $(2\sqrt{3}+3\sqrt[3]{31.8c})^2$? That is a different story. Nevertheless, we use the same perfect square formula. The same idea applies in geometry. There are special formulas and procedures to apply in certain types of problems. What is needed is to remember the formula and remember the kind of problems that fit. Sometimes we believe that because a formula is labeled special, we will rarely have use for it. That assumption is incorrect. So, let us identify the $30^\circ-60^\circ-90^\circ$ triangle and find out why it is special. See Figure 10.8.10.

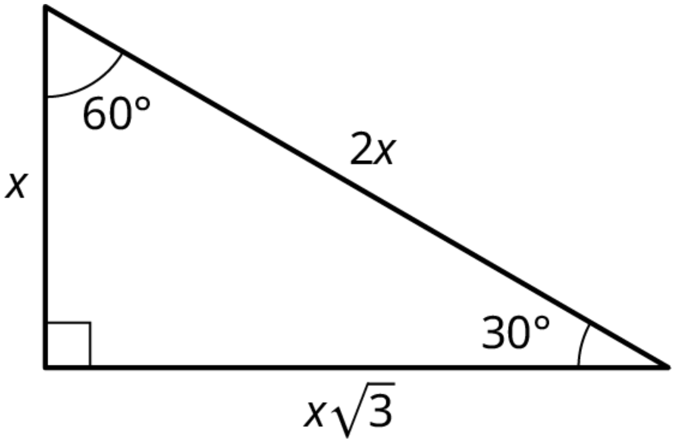


Figure 10.8.10: The 30°-60°-90°

We see that the shortest side is opposite the smallest angle, and the longest side, the hypotenuse, will always be opposite the right angle. There is a set ratio of one side to another side for the $30^\circ-60^\circ-90^\circ$ triangle given as $1:\sqrt{3}:2$, or $x:x\sqrt{3}:2x$. Thus, you only need to know the length of one side to find the other two sides in a $30^\circ-60^\circ-90^\circ$ triangle.

Example 10.8.4: Finding Missing Lengths in a 30°-60°-90° Triangle

Find the measures of the missing lengths of the triangle (Figure 10.8.11).

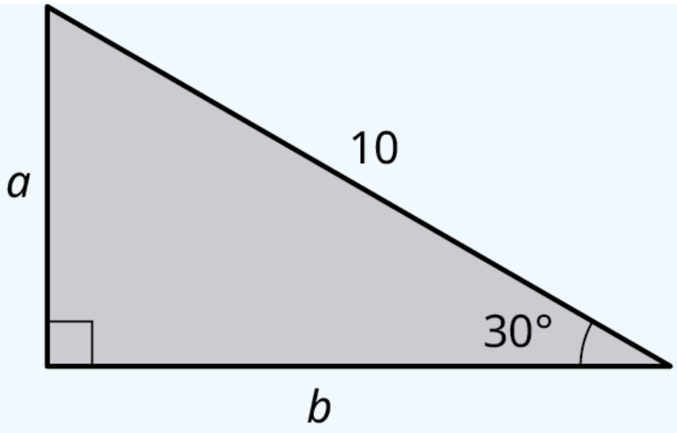


Figure ✓ 10.8.11

Answer

We can see that this is a $30^{\circ}-60^{\circ}-90^{\circ}$ triangle because we have a right angle and a 30° angle. The remaining angle, therefore, must equal 60°. Because this is a special triangle, we have the ratios of the sides to help us identify the missing lengths. Side a is the shortest side, as it is opposite the smallest angle 30°, and we can substitute $a=x$. The ratios are $x: x \sqrt{3}: 2x$. We have the hypotenuse equaling 10 , which corresponds to side c, and side c is equal to $2x$. Now, we must solve for x :

$$\begin{gathered}2x=10 \\ x=5\end{gathered}$$

Side b is equal to $x \sqrt{3}$ or $5 \sqrt{3}$. The lengths are $5, 5 \sqrt{3}, 10$.

Your Turn 10.8.4

Find the lengths of the missing sides in the given figure.

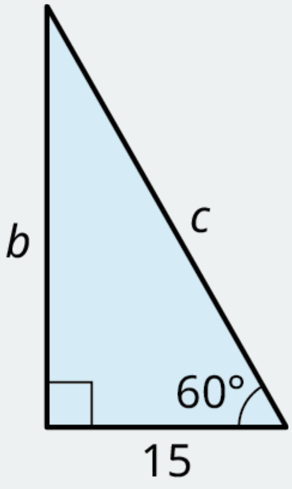


Figure 10.8.12

✓ Example 10.8.5: Applying 30°-60°-90° Triangle to the Real World

A city worker leans a 40-foot ladder up against a building at a 30° angle to the ground (Figure ✓ 10.8.13). How far up the building does the ladder reach?

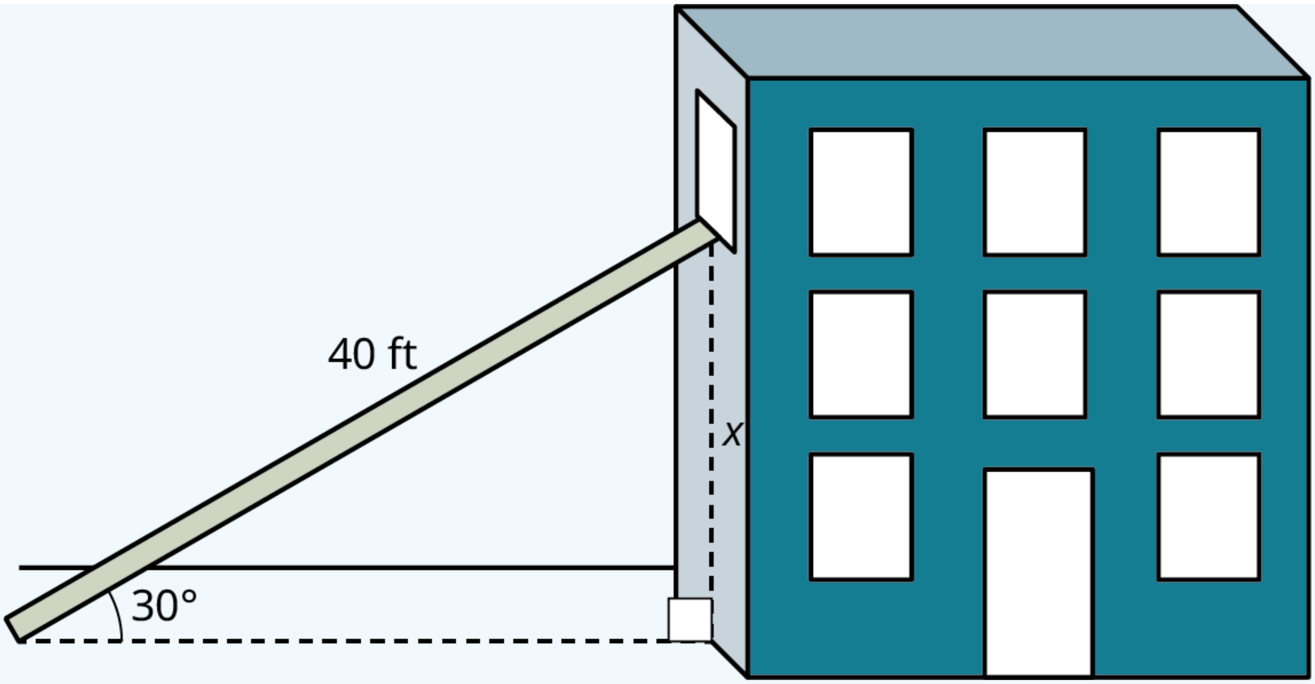


Figure ✓ 10.8.13

Answer

We have a $30^{\circ}-60^{\circ}-90^{\circ}$ triangle, and the hypotenuse is 40 ft . This length is equal to $2x$, where x is the shortest side. If $2x = 40$, then $x = 20$. The ladder is leaning on the wall 20 ft up from the ground.

Your Turn 10.8.5

You want to repair a window on the second floor of your home. If you place the ladder at a 30° angle to the ground, the ladder just about reaches the window. How far from the wall should you place the ladder? How far up will the ladder reach? Make a sketch as an aid.

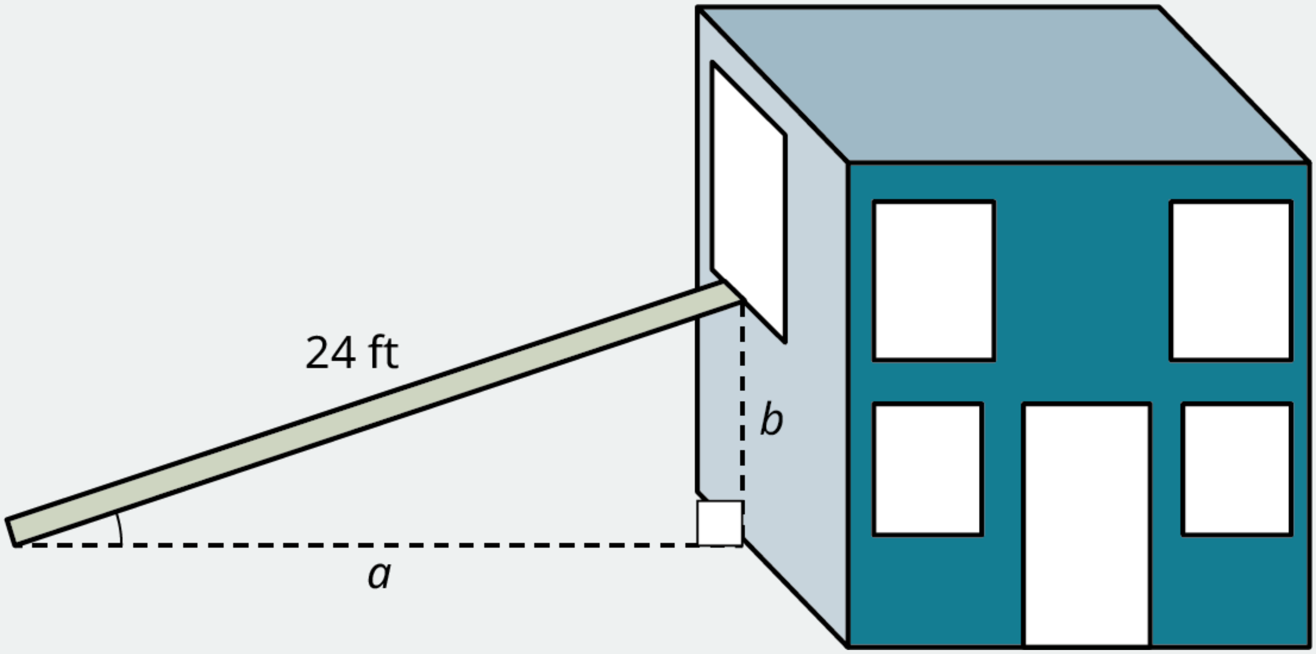


Figure 10.8.14

45°-45°-90° Triangles

The $45^{\circ}-45^{\circ}-90^{\circ}$ triangle is another special triangle such that with the measure of one side we can find the measures of all the sides. The two angles adjacent to the 90° angle are equal, and each measures 45°. If two angles are equal, so are their opposite sides. The ratio among sides is $1:1:\sqrt{2}$, or $x:x:x\sqrt{2}$, as shown in Figure $10.8.15$.

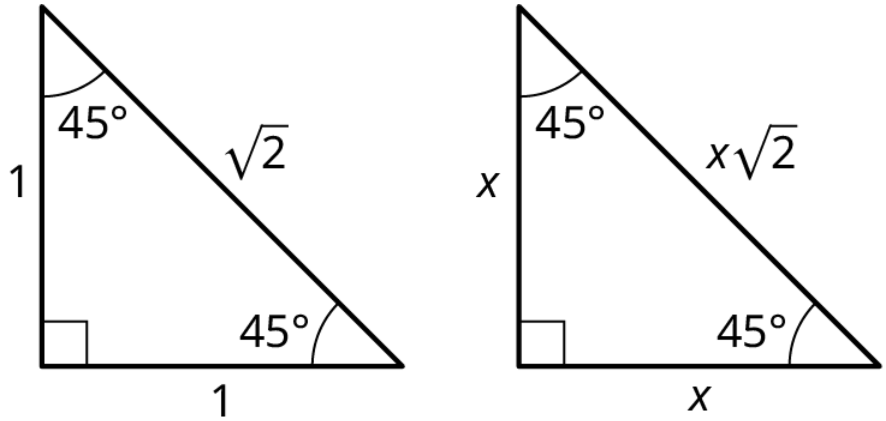


Figure $10.8.15$: 45°-45°-90° Triangles

✓ Example $10.8.6$: Finding Missing Lengths of a 45°-45°-90° Triangle

Find the measures of the unknown sides in the triangle (Figure 10.193).

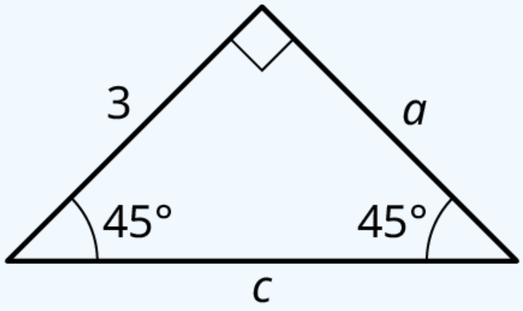


Figure ✓ 10.8.16

Answer

Because we have a $45^{\circ}-45^{\circ}-90^{\circ}$ triangle, we know that the two legs are equal in length and the hypotenuse is a product of one of the legs and $\sqrt{2}$. One leg measures 3 , so the other leg, a, measures 3 . Remember the ratio of $x:x:x\sqrt{2}$. Then, the hypotenuse, c, equals $3\sqrt{2}$.

Your Turn $10.8.6$

Find the measures of the unknown sides in the given figure.

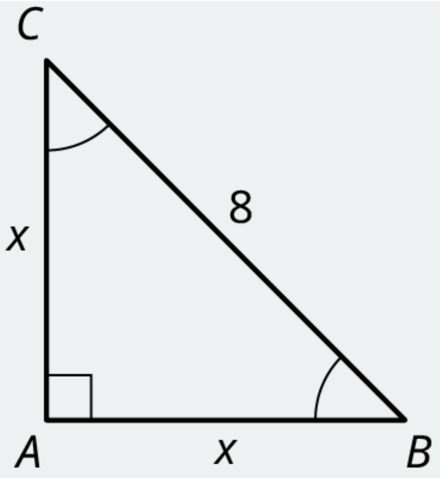


Figure $10.8.17$

Trigonometry Functions

Trigonometry developed around 200 BC from a need to determine distances and to calculate the measures of angles in the fields of astronomy and surveying. Trigonometry is about the relationships (or ratios) of angle measurements to side lengths in primarily right triangles. However, trigonometry is useful in calculating missing side lengths and angles in other triangles and many applications.

Checkpoint

NOTE: You will need either a scientific calculator or a graphing calculator for this section. It must have the capability to calculate trigonometric functions and express angles in degrees.

Trigonometry is based on three functions. We title these functions using the following abbreviations:

- $\sin=\text{sine}$
- $\cos=$ cosine
- $\tan=$ tangent

Letting $r=\sqrt{x^2+y^2}$, which is the hypotenuse of a right triangle, we have Table 10.1. The functions are given in terms of x, y, and r, and in terms of sides relative to the angle, like opposite, adjacent, and the hypotenuse.

Table $10.8.1$ Trigonometric Ratios

$\sin\theta=\frac{y}{r}=\frac{opp}{hyp}$	$\cos\theta=\frac{x}{r}=\frac{adj}{hyp}$	$\tan\theta=\frac{y}{x}=\frac{opp}{adj}$

We will be applying the sine function, cosine function, and tangent function to find side lengths and angle measurements for triangles we cannot solve using any of the techniques we have studied to this point. In Figure $10.8.18$, we have an illustration mainly to identify r and the sides labeled x and y.

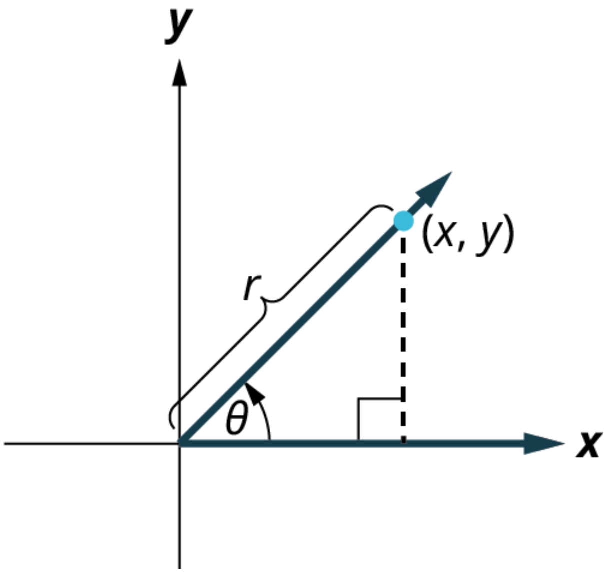


Figure 10.8.18: Angle semanticsθθ

An angle θ sweeps out in a counterclockwise direction from the positive x-axis and stops when the angle reaches the desired measurement. That ray extending from the origin that marks θ° is called the terminal side because that is where the angle terminates. Regardless of the information given in the triangle, we can find all missing sides and angles using the trigonometric functions. For example, in Figure 10.8.19, we will solve for the missing sides.

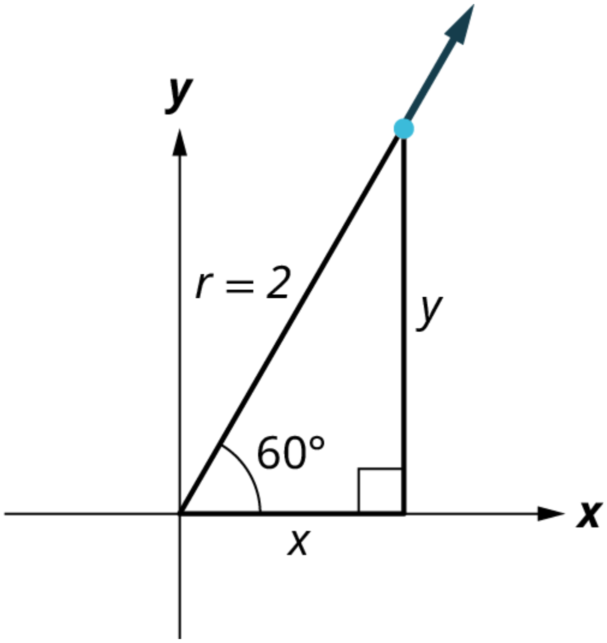


Figure 10.8.19: Solving for Missing Sides

Let’s use the trigonometric functions to find the sides x and y. As long as your calculator mode is set to degrees, you do not have to enter the degree symbol. First, let’s solve for y.

We have $\sin\theta = \frac{y}{r}$, and $\theta = 60^{\circ}$. Then,

$$\begin{aligned}\sin 60^{\circ} &= \frac{y}{2}\\ 2\sin 60^{\circ} &= y\\ 1.732 &= y\\ \sqrt{3} &= y\end{aligned}$$

Next, let's find x. This is the cosine function. We have $\cos\theta = \frac{x}{r}$. Then,

$$\begin{aligned} \cos 60^{\circ} &= \frac{x}{2} \\ 2\cos 60^{\circ} &= x \\ &= 1 \end{aligned}$$

Now we have all sides, $1, \sqrt{3}, 2$. You can also check the sides using the $30^{\circ}-60^{\circ}-90^{\circ}$ ratio of $1:\sqrt{3}:2$. Table 10.2 is a list of common angles, which you should find helpful. Table 10.2 is a list of common angles, which you should find helpful.

Table 10.2 Common Angles

$\sin 0^{\circ}=0$	$\cos 0^{\circ}=1$
$\sin 30^{\circ}=\frac{1}{2}$	$\cos 30^{\circ}=\frac{\sqrt{3}}{2}$
$\sin 45^{\circ}=\frac{\sqrt{2}}{2}$	$\cos 45^{\circ}=\frac{\sqrt{2}}{2}$
$\sin 60^{\circ}=\frac{\sqrt{3}}{2}$	$\cos 60^{\circ}=\frac{1}{2}$
$\sin 90^{\circ}=1$	$\cos 90^{\circ}=0$

✓ Example $10.8.7$: Using Trigonometric Functions

Find the lengths of the missing sides for the triangle (Figure ✓ $10.8.20$).

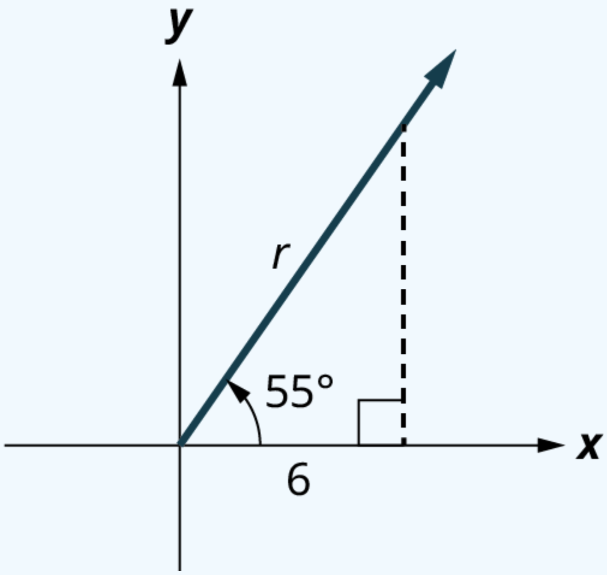


Figure ✓ $10.8.20$

Answer

We have a 55° angle, and the length of the triangle on the x-axis is 6 units.

Step 1: To find the length of r, we can use the cosine function, as $\cos\theta=\frac{x}{r}$. We manipulate this equation a bit to solve for r :

$$\begin{aligned} \cos(55^{\circ}) &= \frac{6}{r} \\ r\cos(55^{\circ}) &= 6 \\ r &= \frac{6}{\cos(55^{\circ})} \\ r &= \frac{6}{0.5736}=10.46 \end{aligned}$$

Step 2: We can use the Pythagorean Theorem to find the length of y. Prove that your answers are correct by using other trigonometric ratios:

$$\begin{aligned}6^2+y^2&=10.46^2\\ y^2&=109.4-36\\ y&=8.57\end{aligned}$$

Step 3: Now that we have y, we can use the sine function to prove that r is correct. We have $\sin\theta=\frac{y}{r}$.

$$\begin{aligned}\sin\left(55^\circ\right)&=\frac{8.57}{r}\\ r\sin\left(55^\circ\right)&=8.57\\ r&=\frac{8.57}{\sin\left(55^\circ\right)}\\ &=\frac{8.57}{0.819}=10.46\end{aligned}$$

Your Turn 10.8.7

Find the lengths of the missing sides in the given figure.

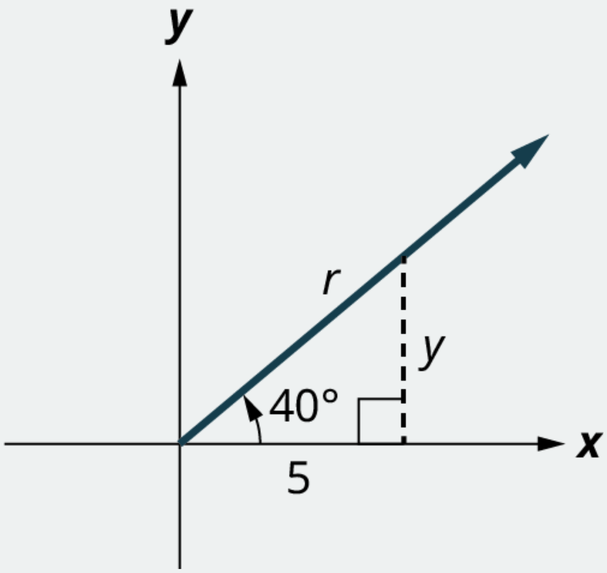


Figure 10.8.21

To find angle measurements when we have two side measurements, we use the inverse trigonometric functions symbolized as $\sin^{-1}$, $\cos^{-1}$, or $\tan^{-1}$. The -1 looks like an exponent, but it means inverse. For example, in the previous example, we had $x=6$ and $r=10.46$. To find what angle has these values, enter the values for the inverse cosine function $\cos^{-1}\left(\frac{x}{r}\right)$ in your calculator:

$$\cos^{-1}\left(\frac{6}{10.46}\right)=55^\circ$$

You can also use the inverse sine function and enter the values of $\sin^{-1}\left(\frac{y}{r}\right)$ in your calculator given $y=8.57$ and $r=10.46$. We have

$$\sin^{-1}\left(\frac{8.57}{10.46}\right)=55^\circ$$

Finally, we can also use the inverse tangent function. Recall $\tan\theta=\frac{y}{x}$. We have

$$\tan^{-1}\left(\frac{8.57}{6}\right)=55^\circ$$

✓ Example 10.8.8: Solving for Lengths in a Right Triangle

Solve for the lengths of a right triangle in which $\theta = 30^{\circ}$ and $r = 6$ (Figure ✓ 10.8.22).

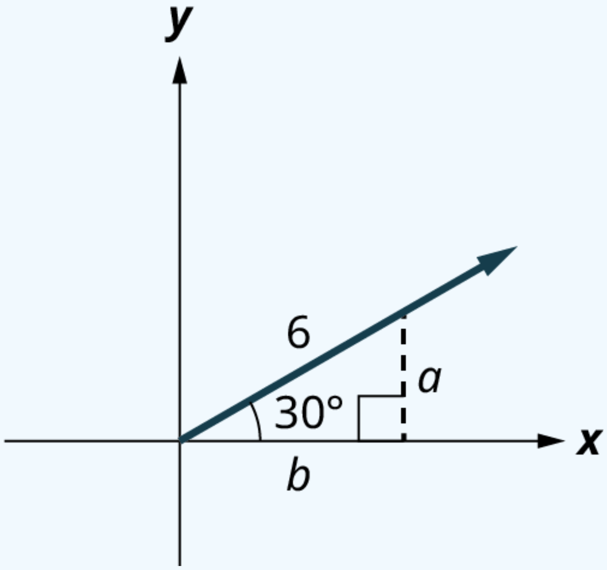


Figure ✓ 10.8.22

Answer

Step 1: To find side a, we use the sine function:

$$\begin{aligned} \sin 30^{\circ} &= \frac{a}{6} \\ 6 \sin 30^{\circ} &= a = 3 \end{aligned}$$

Step 2: To find b, we use the cosine function:

$$\begin{aligned} \cos 30^{\circ} &= \frac{b}{6} \\ 6 \cos 30^{\circ} &= b = 5.196 \end{aligned}$$

Step 3: Since this is a $30^{\circ} - 60^{\circ} - 90^{\circ}$ triangle and side b should equal $x\sqrt{3}$, if we input 3 for x, we have $b = 3\sqrt{3}$. Put this in your calculator and you will get $3\sqrt{3} = 5.196$.

Your Turn 10.8.8

Find the missing side and angles in the figure shown.

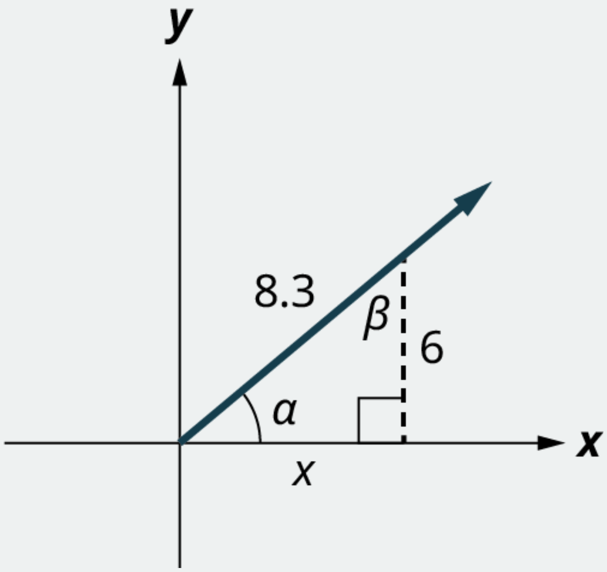


Figure 📌 10.8.23

✓ Example 10.8.9: Finding Altitude

A small plane takes off from an airport at an angle of 31.3° to the ground. About two-thirds of a mile $(3,520\text{ft})$ from the airport is an 1,100-ft peak in the flight path of the plane (Figure ✓ \(\PageIndex{24}\)). If the plane continues that angle of ascent, find its altitude when it is above the peak, and how far it will be above the peak.

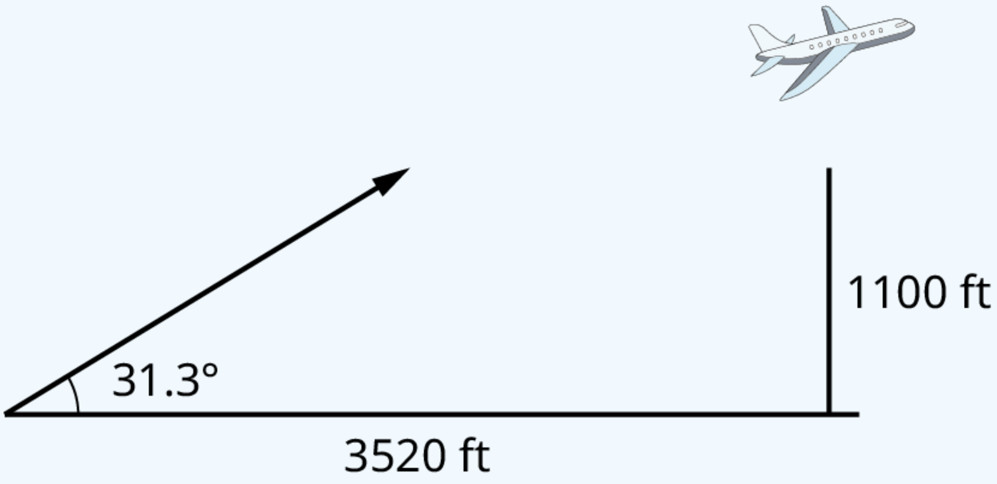


Figure ✓ 10.8.24

Answer

To solve this problem, we use the tangent function:

$$\begin{aligned}\tan 31.3^{\circ} &= \frac{x}{3,520}\\ 3,520\tan 31.3^{\circ} &= 2,140\end{aligned}$$

The plane's altitude when passing over the peak is $2,140\text{ft}$, and it is $1,040\text{ft}$ above the peak.

Your Turn 10.8.9

Suppose that the plane takes off at a 23° angle, and 1 mile from the airport is a 1,500-foot peak. At what altitude will the plane pass over the peak?

✓ Example 10.8.10: Finding Unknown Sides and Angles

Suppose you have two known sides, but do not know the measure of any angles except for the right angle (Figure ✓ 10.8.25)). Find the measure of the unknown angles and the third side.

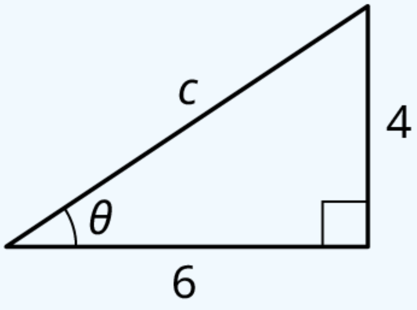


Figure ✓ 10.8.25

Answer

Step 1: We can find the third side using the Pythagorean Theorem:

$$\begin{aligned} 6^2+4^2&=c^2\\ 52&=c^2\\ 2\sqrt{13}&=c \end{aligned}$$

Now, we have all three sides.

Step 2: To find θ, we will first find $\sin\theta$.

$$\begin{aligned} \sin\theta&=\frac{opp}{hyp}\\ &=\frac{4}{2\sqrt{13}}\\ &=\frac{2}{\sqrt{13}} \end{aligned}$$

The angle θ is the angle whose sine is $\frac{2}{\sqrt{13}}$.

Step 3: To find θ, we use the inverse sine function:

$$\begin{aligned} \theta&=\sin^{-1}\left(\frac{2}{\sqrt{13}}\right)\\ &=33.7^{\circ} \end{aligned}$$

Step 4: To find the last angle, we just subtract: $180^{\circ}-90^{\circ}-33.7^{\circ}=56.3^{\circ}$**.**

Your Turn 10.8.10

You know the lengths of two sides and the right angle as shown in the figure. Find the length of the third side and the other angles.

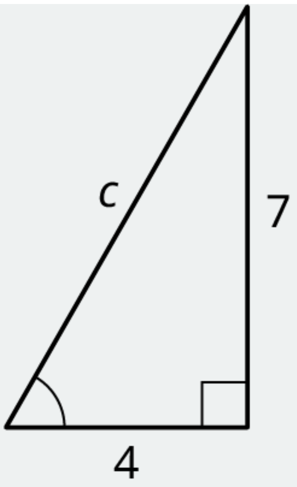


Figure 📌 $10.8.26$

Angle of Elevation and Angle of Depression

Other problems that involve trigonometric functions include calculating the angle of elevation and the angle of depression. These are very common applications in everyday life. The angle of elevation is the angle formed by a horizontal line and the line of sight from an observer to some object at a higher level. The angle of depression is the angle formed by a horizontal line and the line of sight from an observer to an object at a lower level.

✓ Example $10.8.11$: Finding the Angle of Elevation

A guy wire of length 110 meters runs from the top of an antenna to the ground (Figure ✓ $10.8.27$). If the angle of elevation of an observer to the top of the antenna is 43°, how high is the antenna?

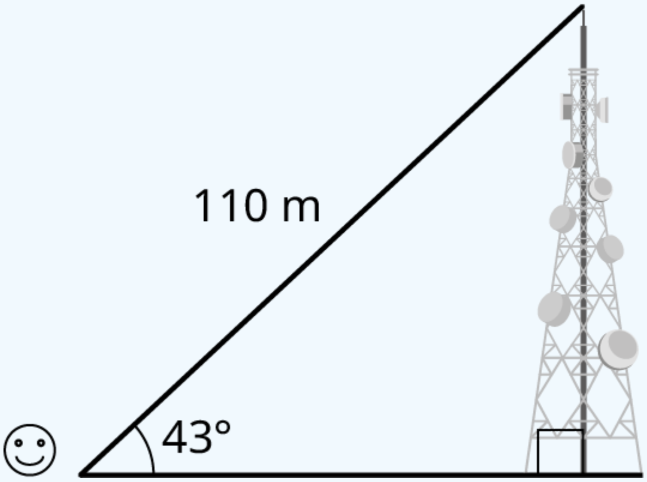


Figure ✓ \(\PageIndex{27}\)

Answer

We are looking for the height of the tower. This corresponds to the y-value, so we will use the sine function:

$$\begin{aligned} \sin 43^{\circ} &= \frac{y}{110} \\ 110 \sin 43^{\circ} &= y \\ 75 &= y \end{aligned}$$

The tower is 75 m high.

Your Turn 10.8.11

You travel to Chicago and visit the observation deck at Willis Tower, 1,450 ft above ground. You can see the Magnificent Mile to the northeast 6,864 ft away. What is the angle of depression from the observation deck to the Magnificent Mile?

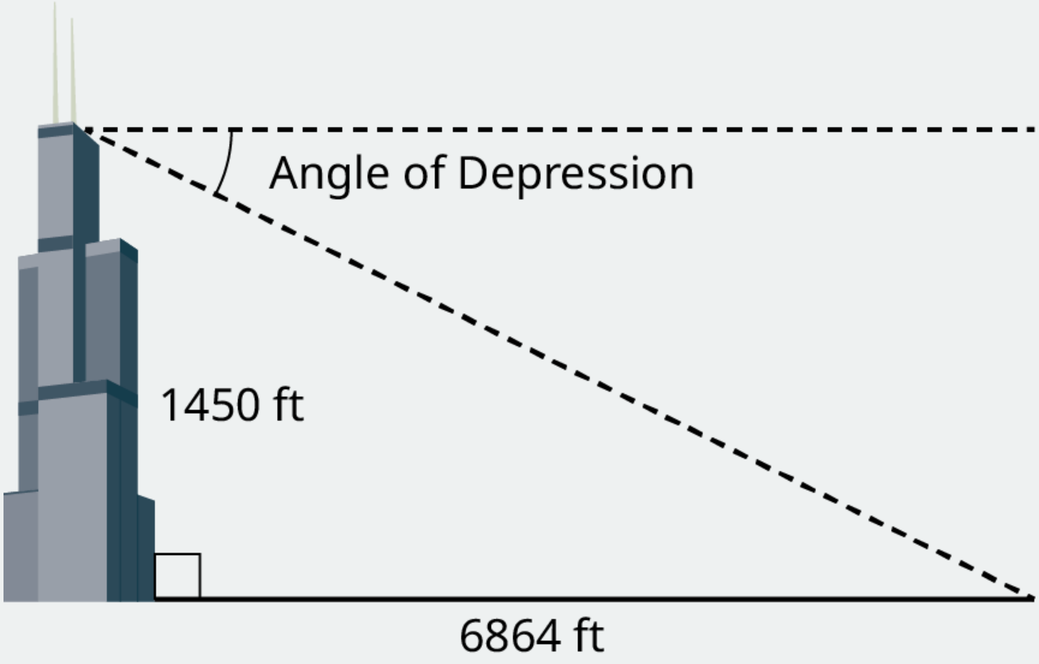


Figure \(\PageIndex{28}\)

Example 10.8.12: Finding Angle of Elevation

You are sitting on the grass flying a kite on a 50-foot string (Figure ✓ \(\PageIndex{29}\)). The angle of elevation is 60°. How high above the ground is the kite?

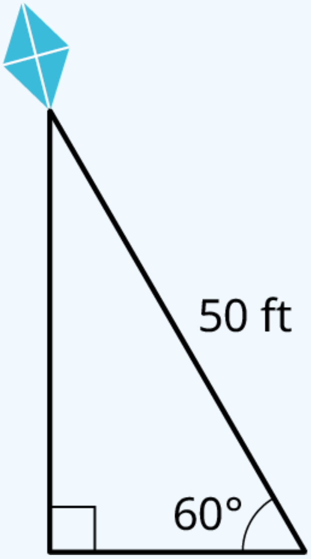


Figure ✓ \(\PageIndex{29}\)

Answer

We can solve this using the sine function, $\sin\theta = \frac{opp}{hyp}$.

$$\begin{aligned} \sin 60^{\circ} &= \frac{x}{50} \\ 50\sin 60^{\circ} &= x \\ &= 43.3\text{ft} \end{aligned}$$

Your Turn 10.8.12

You are flying a kite on a 60-foot string. The angle of elevation from the ground to the kite is 50°. How high above the ground is the kite?

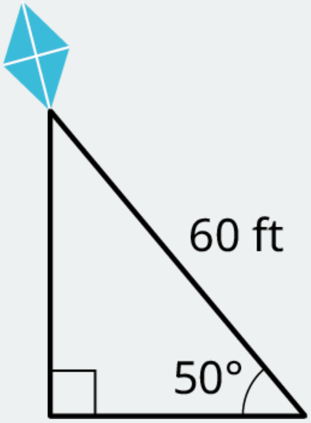


Figure \(\PageIndex{30}\)

People in Mathematics: Pythagoras and the Pythagoreans

The Pythagorean Theorem is so widely used that most people assume that Pythagoras (570–490 BC) discovered it. The philosopher and mathematician uncovered evidence of the right triangle concepts in the teachings of the Babylonians dating around 1900 BC. However, it was Pythagoras who found countless applications of the theorem leading to advances in geometry, architecture, astronomy, and engineering.

Among his accolades, Pythagoras founded a school for the study of mathematics and music. Students were called the Pythagoreans, and the school's teachings could be classified as a religious indoctrination just as much as an academic experience. Pythagoras believed that spirituality and science coexist, that the intellectual mind is superior to the senses, and that intuition should be honored over observation.

Pythagoras was convinced that the universe could be defined by numbers, and that the natural world was based on mathematics. His primary belief was *All is Number.* He even attributed certain qualities to certain numbers, such as the number 8 represented justice and the number 7 represented wisdom. There was a quasi-mythology that surrounded Pythagoras. His followers thought that he was more of a spiritual being, a sort of mystic that was all-knowing and could travel through time and space. Some believed that Pythagoras had mystical powers, although these beliefs were never substantiated.

Pythagoras and his followers contributed more ideas to the field of mathematics, music, and astronomy besides the Pythagorean Theorem. The Pythagoreans are credited with the discovery of irrational numbers and of proving that the morning star was the planet Venus and not a star at all. They are also credited with the discovery of the musical scale and that different strings made different sounds based on their length. Some other concepts attributed to the Pythagoreans include the properties relating to triangles other than the right triangle, one of which is that the sum of the interior angles of a triangle equals 180°. These geometric principles, proposed by the Pythagoreans, were proven 200 years later by Euclid.

Who Knew?: A Visualization of the Pythagorean Theorem

In Figure 10.208, which is one of the more popular visualizations of the Pythagorean Theorem, we see that square a is attached to side a; square b is attached to side b; and the largest square, square c, is attached to side c. Side a measures 3 cm in length, side b measures 4 cm in length, and side c measures 5 cm in length. By definition, the area of square a measures 9 square units, the area of square b measures 16 square units, and the area of square c measures 25 square units. Substitute the values given for the areas of the three squares into the Pythagorean Theorem and we have

$$\begin{aligned} a^2+b^2&=c^2\\ 3^2+4^2&=5^2\\ 9+16&=25 \end{aligned}$$

Thus, the sum of the squares of the two legs of a right triangle is equal to the square of the hypotenuse, as stated in the Pythagorean Theorem.

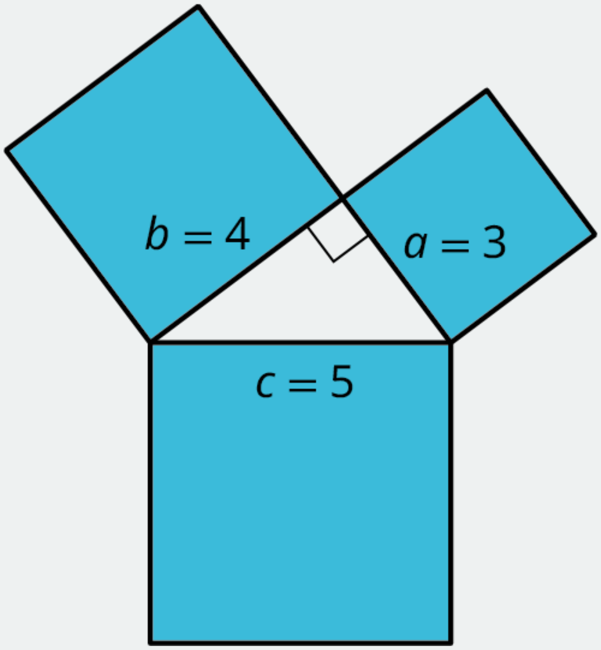
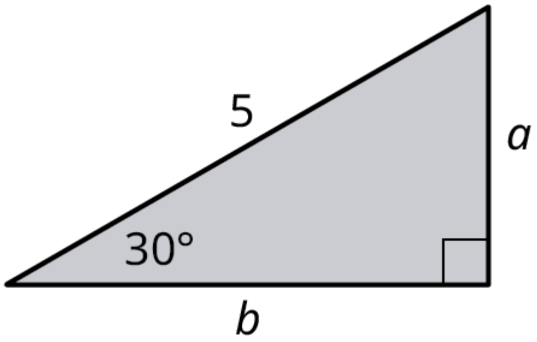


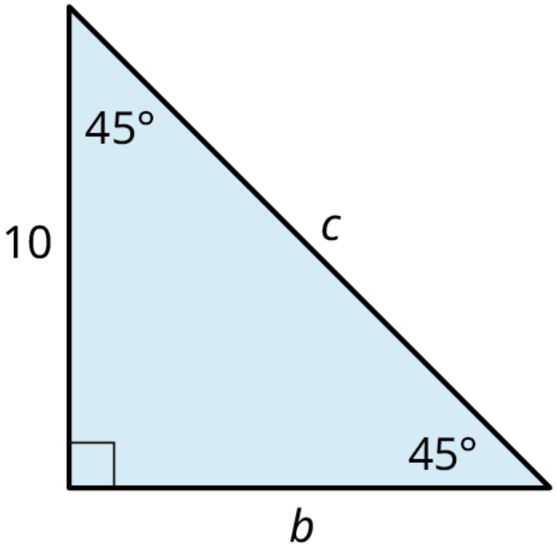
Figure \(\PageIndex{31}\)

Check Your Understanding

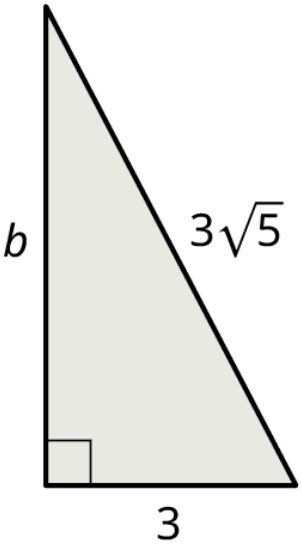
1. Find the lengths of the unknown sides of the 30°-60°-90° triangle shown..



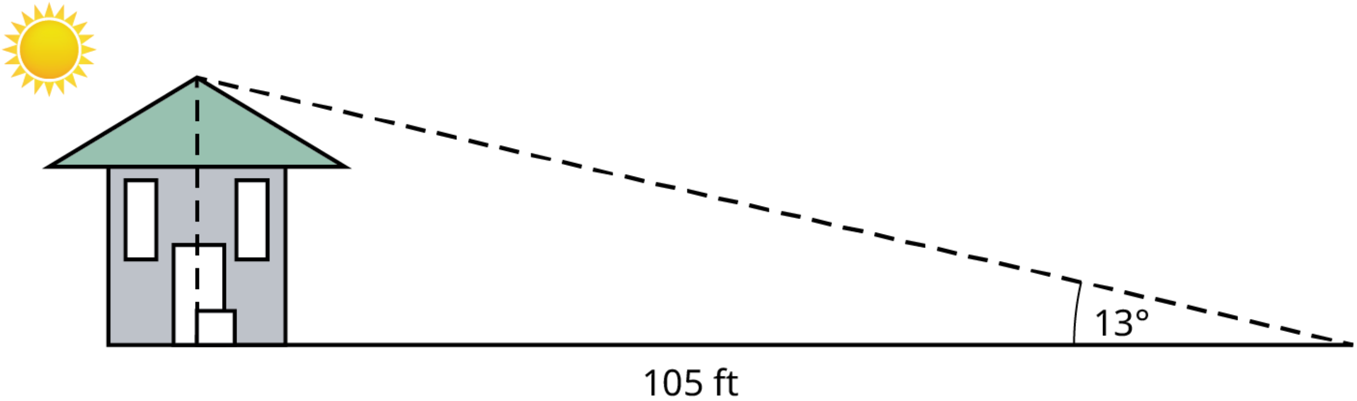
2. Find the missing lengths of the 45°-45°-90° triangle shown.



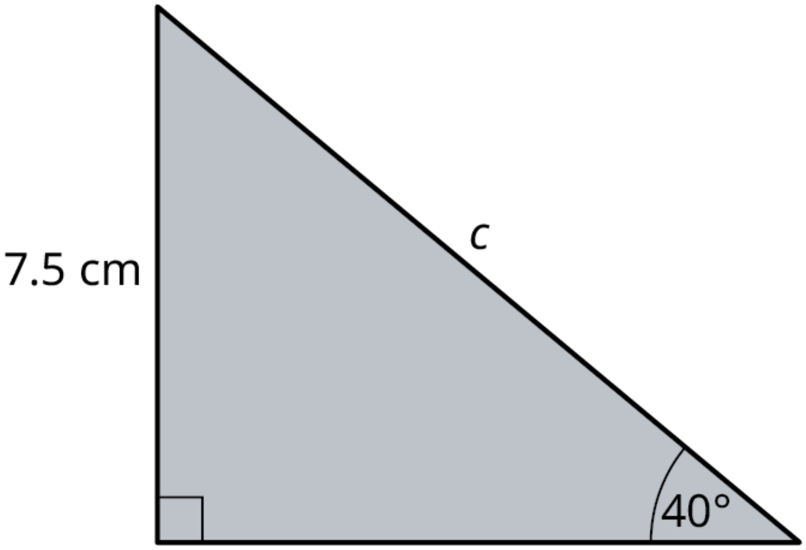
3. Use the Pythagorean theorem to find the missing length in the triangle shown.



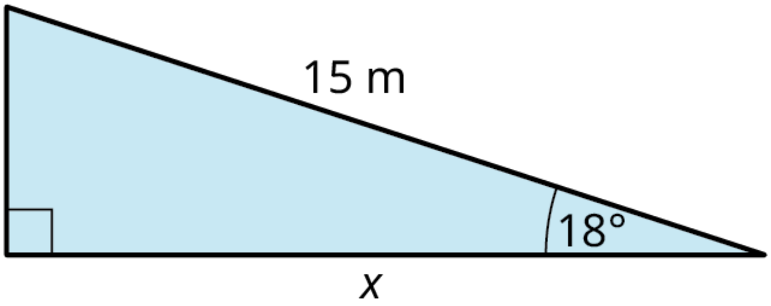
4. The sun casts a shadow over the roof of a house that ends 105 ft from the front door as shown in the figure. How high is the house to the tip of the roof?



5. Find the measure of side c in the given figure.



6. Find the measure of side x in the given figure.



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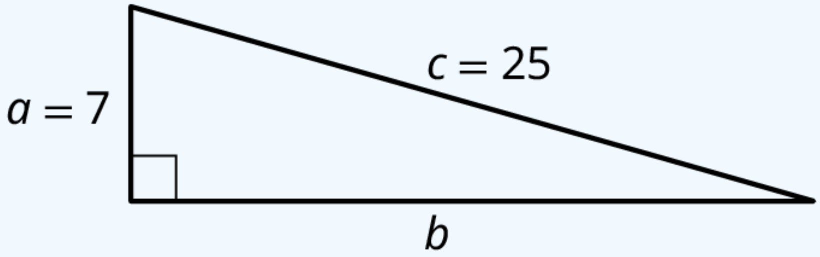
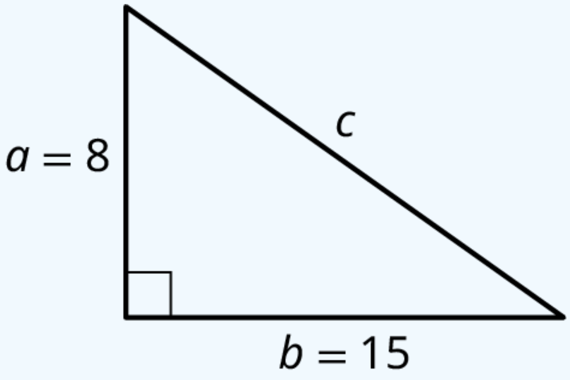
10.8.0: Exercises

Use the Pythagorean theorem to answer the following exercises. Let a and b represent two legs of a right triangle, and let c represent the hypotenuse. Find the lengths of the missing sides.

Exercise 10.8.0.1



Exercise 10.8.0.2



Exercise 10.8.0.3

If $b = 25$ and $c = 65$ ft, find a.

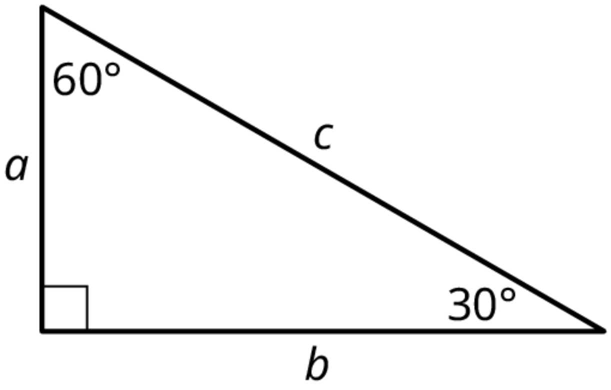
Exercise 10.8.0.4

If $a = 18$ ft and $c = 30$ ft, find b.

Exercise 10.8.0.5

If $b = 12$ cm and $c = 15$ cm, find a.

Refer to the 30°-60°-90° triangle shown for the following exercises.



Exercise 10.8.0.6

If $b = 10$, find c and a

Exercise 10.8.0.7

If $a = 2\sqrt{3}$ ft, find b and c.

Exercise 10.8.0.8

If $b = 7$ cm, find a and c

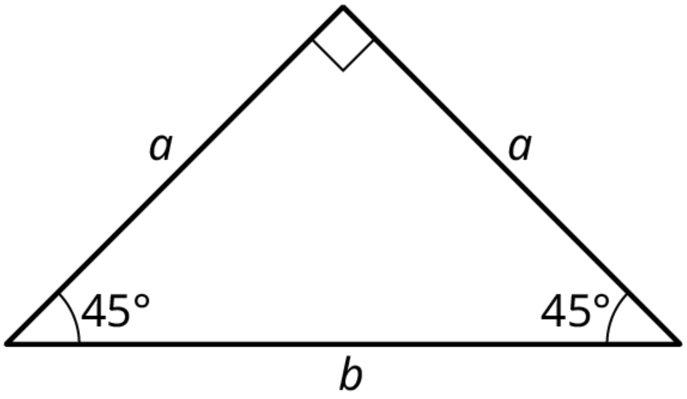
Exercise 10.8.0.9

If $c = \sqrt{3}$, find a and b.

Exercise 10.8.0.10

If $c = 8\sqrt{3}$, find a and b

Refer to the 45°-45°-90° triangle shown in the following exercises.



Exercise 10.8.0.11

If $a = 15$ cm, find b.

? Exercise 10.8.0.12

If $a = 3\sqrt{2}$ ft, find a.

? Exercise 10.8.0.13

If $a = 4\sqrt{5}$ cm, find b.

? Exercise 10.8.0.14

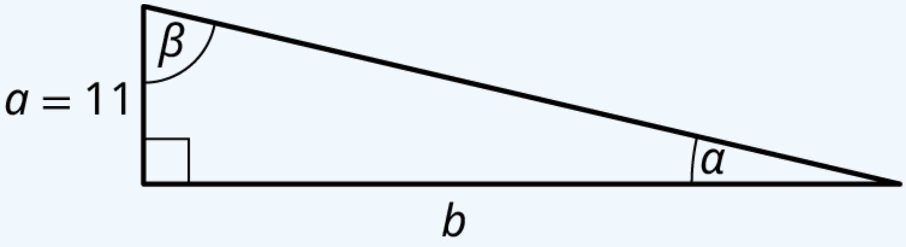
If $a = 10$ ft, find b.

? Exercise 10.8.0.15

If \(b = \frac
{2}{\text{ cm}}\), find a.

? Exercise 10.8.0.16

In the provided figure given $\alpha = 10^\circ$, find the unknown sides and angles.



Evaluate the expressions in the following exercises.

? Exercise 10.8.0.17

$\sin^{-1}\left(\frac{2}{5}\right)$

? Exercise 10.8.0.18

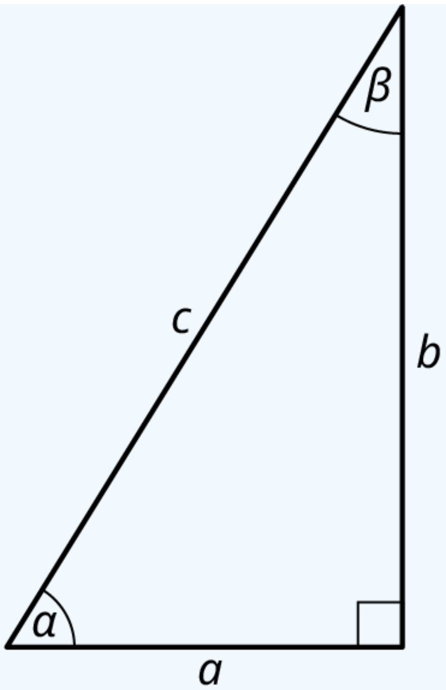
$\tan^{-1}(5.68)$

? Exercise 10.8.0.19

$\cos^{-1}\left(\frac{1}{9}\right)$

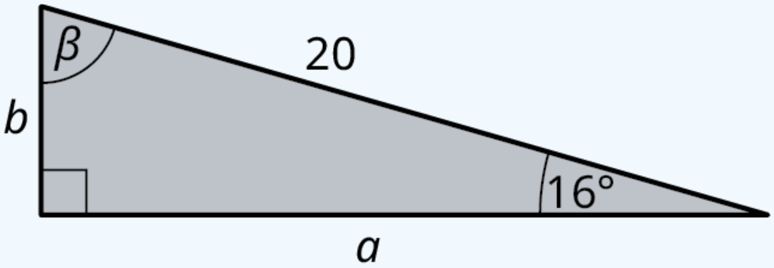
? Exercise 10.8.0.20

Use the figure shown to solve for all missing sides and angles given $a = 6,\ b = 8$.



? Exercise 10.8.0.21

Use the figure shown to solve for all missing sides and angles given $\alpha = 16^{\circ}$, $c = 20$.



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SECTION OVERVIEW

10.9: Chapter Summary

10.9.0: Key Terms

10.9.1: Key Concepts

10.9.2: Videos

10.9.3: Formula Review

10.9.4: Projects

10.9.5: Chapter Review

10.9.6: Chapter Test

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10.9.0: Key Terms

10.1 Points, Lines, and Planes

- line segment
- plane
- union
- intersection
- parallel
- perpendicular

10.2 Angles

- vertex
- right angle
- acute angle
- obtuse angle
- straight angle
- complementary
- supplementary

10.3 Triangles

- acute
- obtuse
- isosceles
- equilateral
- hypotenuse
- congruence
- similarity
- scaling factor

10.4 Polygons, Perimeter, and Circumference

- perimeter
- polygon
- pentagon
- hexagon
- heptagon
- octagon
- quadrilateral
- trapezoid
- parallelogram
- circumference

10.5 Tessellations

- tessellation
- translation
- reflection
- rotation
- glide reflection

10.6 Area

- triangle
- square
- rectangle

- rhombus
- apothem
- radius
- circle

10.7 Volume and Surface Area

- surface area
- volume
- right prism
- right cylinder

10.8 Right Triangle Trigonometry

- right triangle
- sine
- cosine
- tangent

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10.9.1: Key Concepts

10.1 Points, Lines, and Planes

- Modern-day geometry began in approximately 300 BCE with Euclid’s *Elements*, where he defined the principles associated with the line, the point, and the plane.
- Parallel lines have the same slope. Perpendicular lines have slopes that are negative reciprocals of each other.
- The union of two sets, A and B, contains all points that are in both sets and is symbolized as $A \cup B$.
- The intersection of two sets A and B includes only the points common to both sets and is symbolized as $A \cap B$.

10.2 Angles

- Angles are classified as acute if they measure less than 90°, obtuse if they measure greater than 90° and less than 180°, right if they measure exactly 90°, and straight if they measure exactly 180°.
- If the sum of angles equals 90°, they are complimentary angles. If the sum of angles equals 180°, they are supplementary.
- A transversal crossing two parallel lines form a series of equal angles: alternate interior angles, alternate exterior angles, vertical angles, and corresponding angles

10.3 Triangles

- The sum of the interior angles of a triangle equals 180°.
- Two triangles are congruent when the corresponding angles have the same measure and the corresponding side lengths are equal.
- The congruence theorems include the following: SAS, two sides and the included angle of one triangle are congruent to the same in a second triangle; ASA, two angles and the included side of one triangle are congruent to the same in a second triangle; SSS, all three side lengths of one triangle are congruent to the same in a second triangle; AAS, two angles and the non-included side of one triangle are congruent to the same in a second triangle.
- Two shapes are similar when the proportions between corresponding angles, sides or features of two shapes are equal, regardless of size.

10.4 Polygons, Perimeter, and Circumference

- Regular polygons are closed, two-dimensional figures that have equal side lengths. They are named for the number of their sides.
- The perimeter of a polygon is the measure of the outline of the shape. We determine a shape's perimeter by calculating the sum of the lengths of its sides.
- The sum of the interior angles of a regular polygon with n sides is found using the formula $S=(n-2) 180^{\circ}$. The measure of a single interior angle of a regular polygon with n sides is determined using the formula $a=\frac{(n-2) 180^{\circ}}{n}$.
- The sum of the exterior angles of a regular polygon is 360°. The measure of a single exterior angle of a regular polygon with n sides is found using the formula $b=\frac{360^{\circ}}{n}$.
- The circumference of a circle is $C=2 \pi r$, where r is the radius, or $C=\pi d$, and d is the diameter.

10.5 Tessellations

- A tessellation is a particular pattern composed of shapes, usually polygons, that repeat and cover the plane with no gaps or overlaps.
- Properties of tessellations include rigid motions of the shapes called transformations. Transformations refer to translations, rotations, reflections, and glide reflections. Shapes are transformed in such a way to create a pattern.

10.6 Area

- The area A of a triangle is found with the formula $A=\frac{1}{2} b h$, where b is the base and h is the height.
- The area of a parallelogram is found using the formula $A=b h$, where b is the base and h is the height.
- The area of a rectangle is found using the formula $A=l w$, where l is the length and w is the width.
- The area of a trapezoid is found using the formula $A=\frac{1}{2} h\left(b_1+b_2\right)$, where h is the height, b_1 is the length of one base, and b_2 is the length of the other base.
- The area of a rhombus is found using the formula $A=\frac{d_1 d_2}{2}$, where d_1 is the length of one diagonal and d_2 is the length of the other diagonal.The area of a regular polygon is found using the formula $A=\frac{1}{2} a p$, where a is the apothem and p is the perimeter.

- The area of a circle is found using the formula $A = \pi r^2$, where r is the radius.

10.7 Volume and Surface Area

- A right prism is a three-dimensional object that has a regular polygonal face and congruent base such that that lateral sides form a 90° angle with the base and top. The surface area SA of a right prism is found using the formula $SA = 2B + ph$, where B is the area of the base, p is the perimeter of the base, and h is the height. The volume V of a right prism is found using the formula $V = Bh$, where B is the area of the base and h is the height.
- A right cylinder is a three-dimensional object with a circle as the top and a congruent circle is the base, and the side forms a 90° angle to the base and top. The surface area of a right cylinder is found using the formula $SA = 2\pi r^2 + 2\pi rh$, where r is the radius and h is the height. The volume is found using the formula $V = \pi r^2 h$, where r is the radius and h is the height.

10.8 Right Triangle Trigonometry

- The Pythagorean Theorem is applied to right triangles and is used to find the measure of the legs and the hypotenuse according the formula $a^2 + b^2 = c^2$, where c is the hypotenuse.
- To find the measure of the sides of a special angle, such as a $30^\circ - 60^\circ - 90^\circ$ triangle, use the ratio $x : x\sqrt{3} : 2x$, where each of the three sides is associated with the opposite angle and $2x$ is associated with the hypotenuse, opposite the 90° angle.
- To find the measure of the sides of the second special triangle, the $45^\circ - 45^\circ - 90^\circ$ triangle, use the ratio $x : x : x\sqrt{2}$, where each of the three sides is associated with the opposite angle and $x\sqrt{2}$ is associated with the hypotenuse, opposite the 90° angle.
- The primary trigonometric functions are $\sin\theta = \frac{opp}{hyp}$, $\cos\theta = \frac{adj}{hyp}$, and $\tan\theta = \frac{opp}{adj}$.
- Trigonometric functions can be used to find either the length of a side or the measure of an angle in a right triangle, and in applications such as the angle of elevation or the angle of depression formed using right triangles.

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10.9.2: Videos

10.5 Tessellations

- M.C. Escher: How to Create a Tessellation
- The Mathematical Art of M.C. Escher

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10.9.3: Formula Review

10.2 Angles

To translate an angle measured in degrees to radians, multiply by $\frac{\pi}{180}$.
To translate an angle measured in radians to degrees, multiply by $\frac{180}{\pi}$.

10.4 Polygons, Perimeter, and Circumference

The formula for the perimeter P of a rectangle is $P = 2L + 2W$, twice the length L plus twice the width W.
The sum of the interior angles of a polygon with n sides is given by

$$S = (n-2)180^\circ.$$

The measure of each interior angle of a regular polygon with n sides is given by

$$a = \frac{(n-2)180^\circ}{n}.$$

To find the measure of an exterior angle of a regular polygon with n sides we use the formula

$$b = \frac{360^\circ}{n}.$$

The circumference of a circle is found using the formula $C = \pi d$, where d is the diameter of the circle, or $C = 2\pi r$, where r is the radius.

10.6 Area

The area of a triangle is given as $A = \frac{1}{2}bh$, where b represents the base and h represents the height.

The formula for the area of a square is $A = s \cdot s$ or $A = s^2$.

The area of a rectangle is given as $A = lw$.

The area of a parallelogram is $A = bh$.

The formula for the area of a trapezoid is given as $A = \frac{1}{2}h(a+b)$.

The area of a rhombus is found using one of these formulas:

- $A = \frac{d_1 d_2}{2}$, where d_1 and d_2 are the diagonals.
- $A = \frac{1}{2}bh$, where b is the base and h is the height.

The area of a regular polygon is found with the formula $A = \frac{1}{2}ap$, where a is the apothem and p is the perimeter.

The area of a circle is given as $A = \pi r^2$, where r is the radius.

10.7 Volume and Surface Area

The formula for the surface area of a right prism is equal to twice the area of the base plus the perimeter of the base times the height, $SA = 2B + ph$, where B is equal to the area of the base and top, p is the perimeter of the base, and h is the height.

The formula for the volume of a rectangular prism, given in cubic units, is equal to the area of the base times the height, $V = B \cdot h$, where B is the area of the base and h is the height.

The surface area of a right cylinder is given as $SA = 2\pi r^2 + 2\pi rh$.

The volume of a right cylinder is given as $V = \pi r^2 h$.

10.8 Right Triangle Trigonometry

The Pythagorean Theorem states

$$a^2 + b^2 = c^2$$

where a and b are two sides (legs) of a right triangle and c is the hypotenuse.

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10.9.4: Projects

1. One of the reasons so many formulas in geometry were discovered was because of the importance in finding measurements of lengths, areas, perimeter, and angles. Find at least five examples of how geometry can be used in practical applications today.
2. Who were the Pythagoreans? Why did this society exist? Explore what they did and discuss some of their beliefs.

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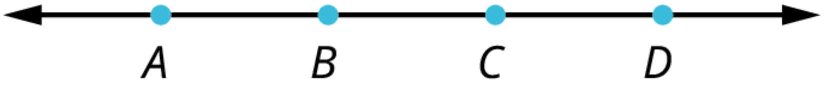
10.9.5: Chapter Review

Points, Lines, and Planes

Use the figure shown for the following exercises.



Exercise 10.9.5.1

Find $\overrightarrow{BC} \cup \overrightarrow{CD}$.

Exercise 10.9.5.2

Find $\overline{AC} \cap \overline{BC}$.

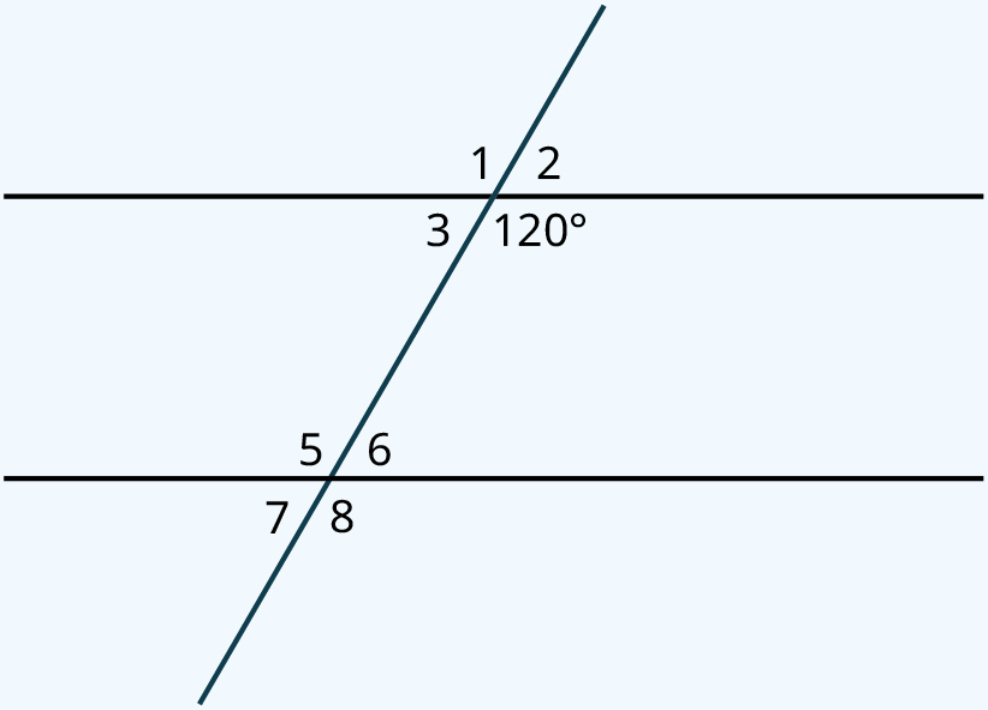
Exercise 10.9.5.3

Name the points in the set $\overrightarrow{BD} \cap \overleftarrow{CA}$.

Angles

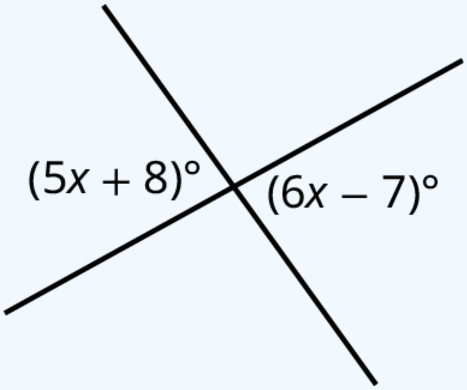
Exercise 10.9.5.4

Given that l_1 and l_2 are parallel lines, solve the angle measurements for all the angles in the figure shown.



Exercise $10.9.5.5$

Find the measure of the vertical angles in the figure shown.

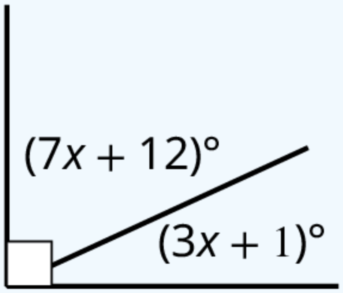


Exercise $10.9.5.6$

Classify the angle as acute, right, or obtuse: 79°.

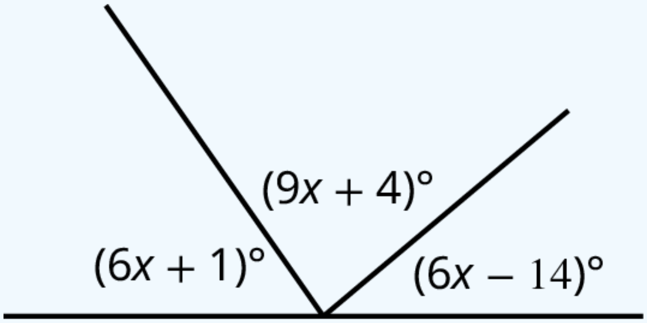
Exercise $10.9.5.7$

Use the given figure to solve for the angles.



Exercise $10.9.5.8$

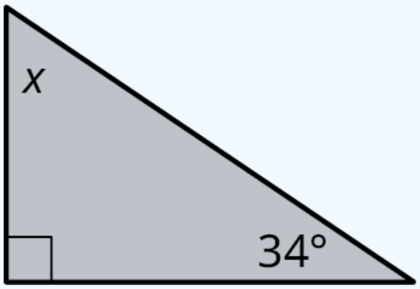
Use the given figure to solve for the angle measurements.



Triangles

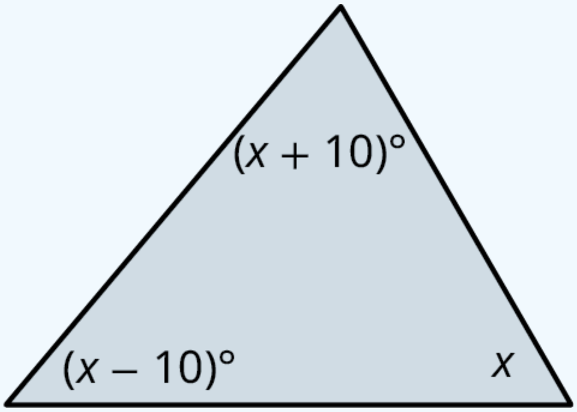
Exercise 10.9.5.9

Use the given figure to find the measure of the unknown angles.



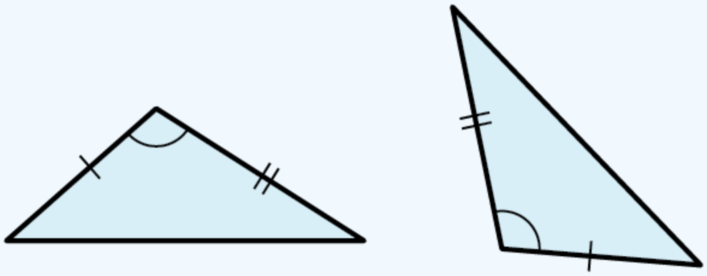
Exercise 10.9.5.10

Use algebra to find the measure of the angles in the figure shown.



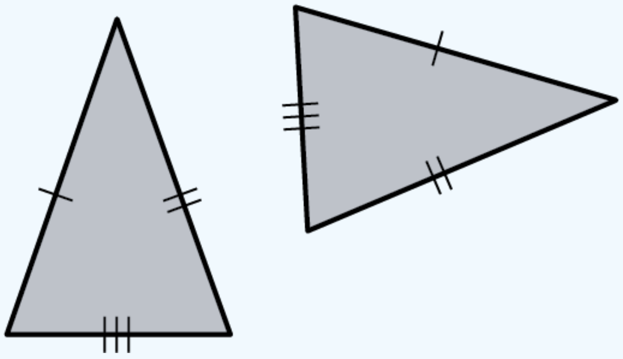
Exercise 10.9.5.11

The two triangles shown are congruent by what theorem?



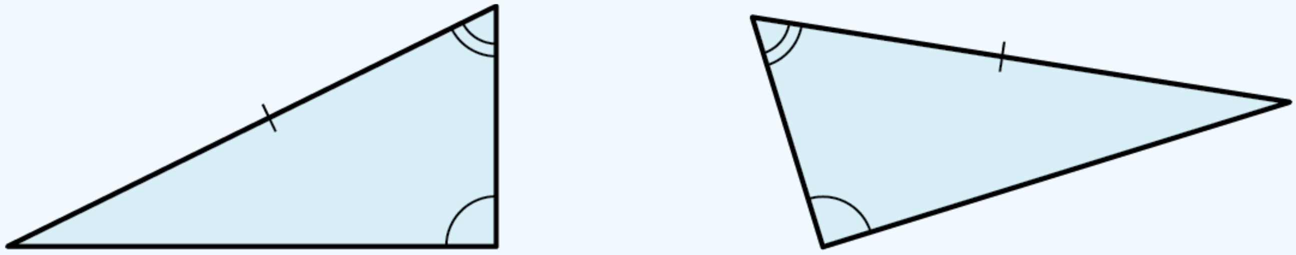
Exercise 10.9.5.12

The two triangles shown are congruent by what theorem?



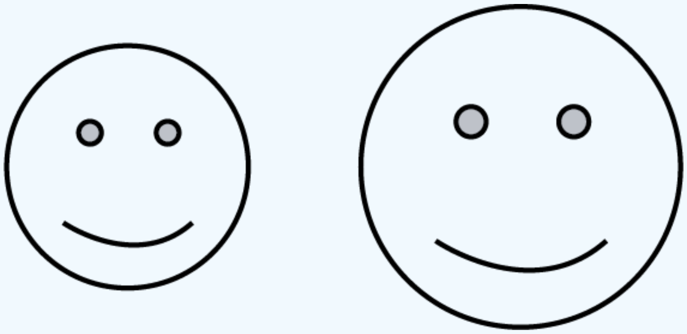
Exercise 10.9.5.13

The two triangles shown are congruent by what theorem?



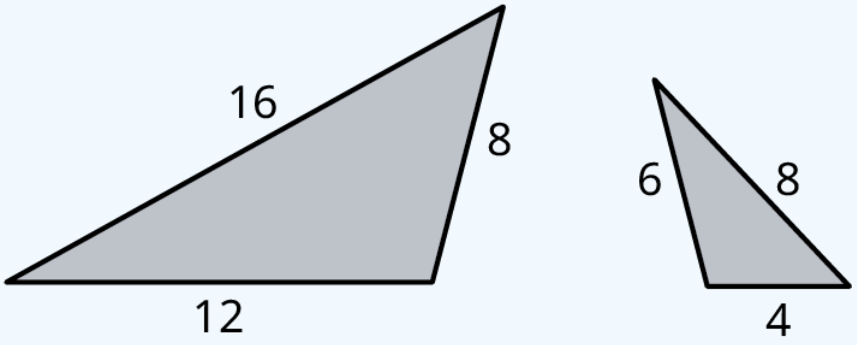
Exercise 10.9.5.14

Are the two figures shown similar?



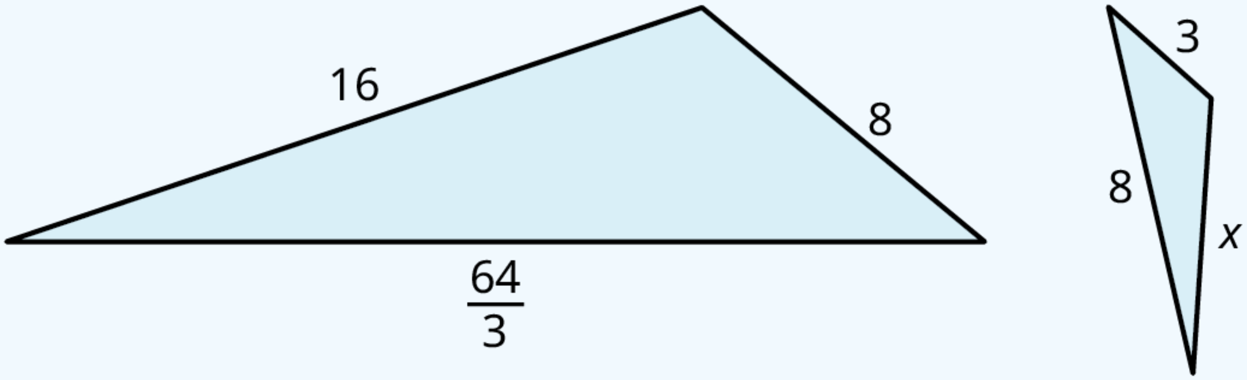
Exercise 10.9.5.15

Are the two triangles shown similar?



Exercise 10.9.5.16

Find the scaling factor of the two similar triangles in the given figure.



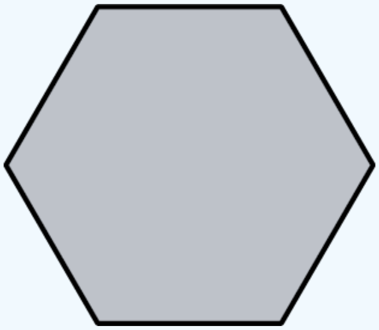
Exercise 10.9.5.17

Find the length of x in the figure shown.

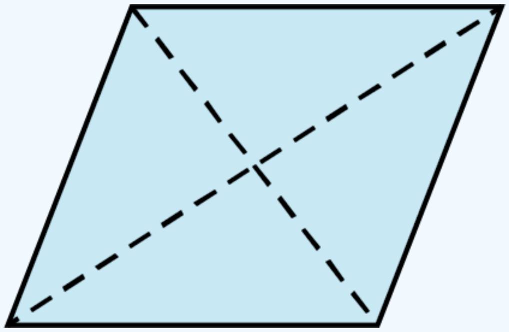
Polygons, Perimeter, and Circumference

Identify the polygons.

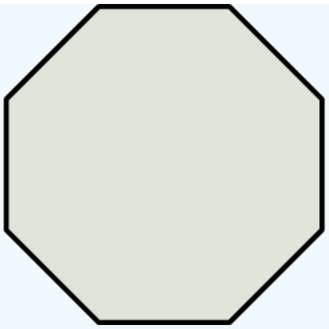
Exercise 10.9.5.18



Exercise 10.9.5.19



Exercise 10.9.5.20

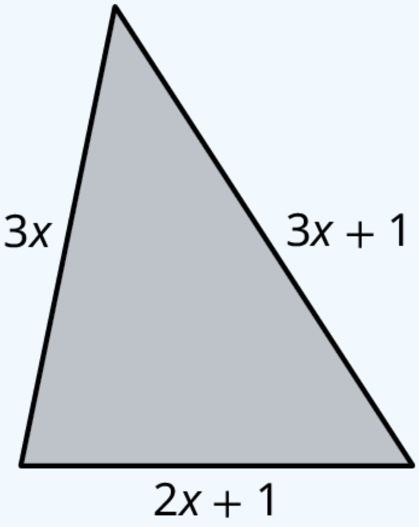


Exercise 10.9.5.21

Find the perimeter of a regular hexagon with side length 5 cm.

Exercise 10.9.5.22

The perimeter of a triangle is 18 in the given figure. Find the length of the sides.



Exercise 10.9.5.23

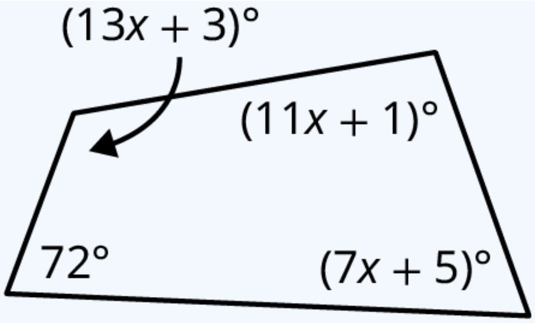
What is the measure of an interior angle of a regular heptagon?

Exercise 10.9.5.24

What is the measure of an interior angle of a regular octagon?

Exercise 10.9.5.25

Calculate the measure of each interior angle in the figure shown.



? Exercise 10.9.5.26

Find the measure of an exterior angle of a regular pentagon.

? Exercise 10.9.5.27

What is the sum of the measures of the exterior angles of a regular heptagon?

? Exercise 10.9.5.28

Find the circumference of a circle with radius 3.2 cm.

? Exercise 10.9.5.29

Find the diameter of a circle with a circumference of 35.6 in.

Tessellations

? Exercise 10.9.5.30

In what field of mathematics does the topic of tessellations belong?

? Exercise 10.9.5.31

Tessellations can have no _______or _________.

? Exercise 10.9.5.32

What type of transformation moves an object over horizontally by some number of units?

? Exercise 10.9.5.33

What type of transformation results in a mirror image of the shape?

? Exercise 10.9.5.34

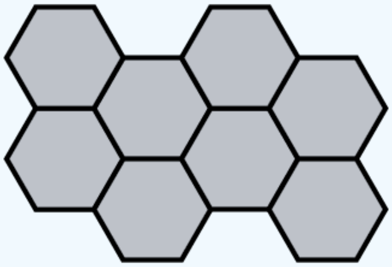
Which regular polygons will tessellate the plane by themselves?

? Exercise 10.9.5.35

The sum of the interior angles of the shapes meeting at a vertex is equal to how many degrees?

Exercise 10.9.5.36

How would you name this tessellation?



Area

Exercise 10.9.5.37

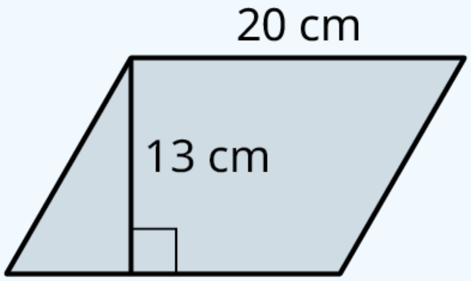
What is the area of a triangle with a base equal to 5 cm and a height equal to 12 cm?

Exercise 10.9.5.38

If the area of a triangle equals $24\ \text{cm}^2$ and the base equals 8, what is the height?

Exercise 10.9.5.39

Find the area of the parallelogram shown.



Exercise 10.9.5.40

If the area of a trapezoid equals $\frac{4}{5}2\ \text{cm}^2$, $b_1 = 3\ \text{cm}$, and the height equals $h = 5\ \text{cm}$, find the length of b_2.

Exercise 10.9.5.41

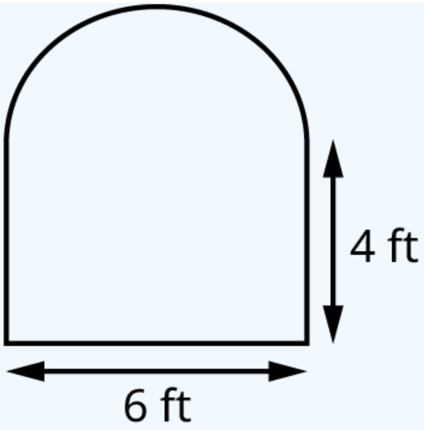
Find the area of an octagon if the apothem equals 10 cm and the side length is 12 cm.

Exercise 10.9.5.42

The area of a circle is $50.3\ \text{in}^2$. What is the radius?

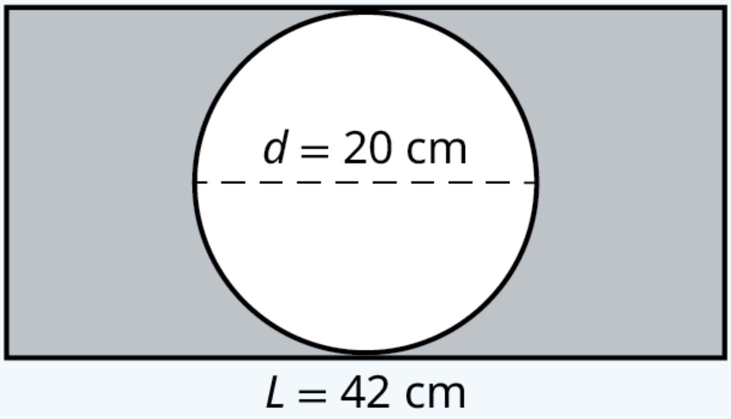
Exercise 10.9.5.43

You want to install a tinted protective shield on this window. How many square feet do you order?



Exercise 10.9.5.44

Find the area of the shaded region in the given figure.



Volume and Surface Area

Exercise 10.9.5.45

Find the surface area of a hexagonal prism with side length 5 cm, apothem 3.1 cm, and height 15 cm.

A triangular prism has an equilateral base and side lengths of 15 in, triangle height of 10 in, and prism length of 25 in.

Exercise 10.9.5.46

Find the surface area of the triangular prism.

Exercise 10.9.5.47

Find the volume of the triangular prism.

Exercise 10.9.5.48

Find the volume of a right cylinder with radius equal to 1.5 cm and height equal to 5 cm.

A right cylinder has a radius of 6 cm and a height of 10 cm.

Exercise $10.9.5.49$

Find the surface area of the right cylinder.

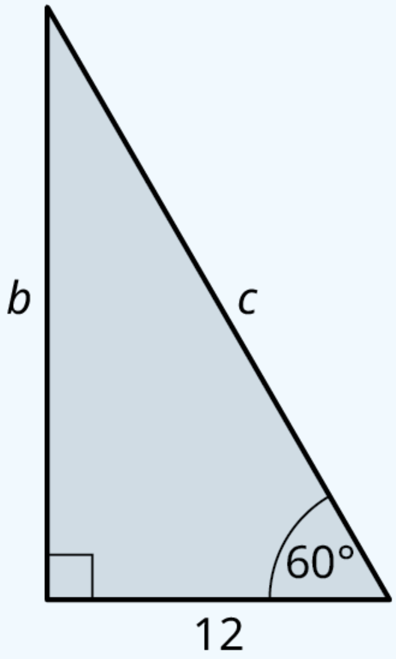
Exercise $10.9.5.50$

Find the volume of the right cylinder.

Right Triangle Trigonometry

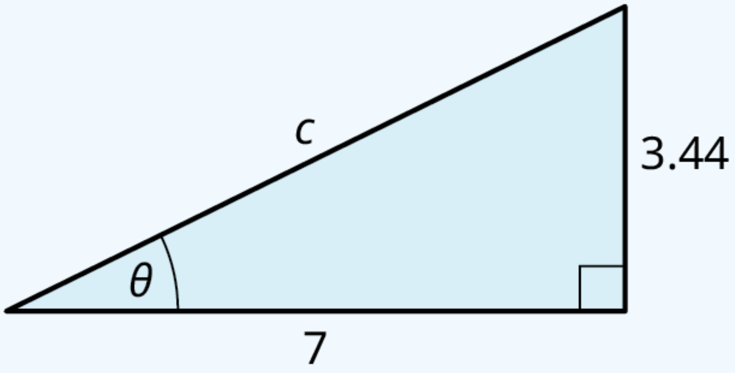
Exercise $10.9.5.51$

Find the lengths of the missing sides in the figure shown.



Exercise $10.9.5.52$

Find the measure of the unknown side and angle.



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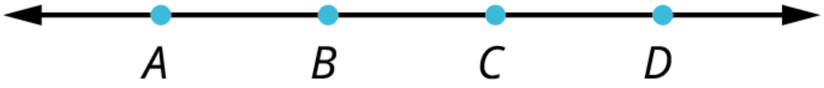
10.9.6: Chapter Test

Use the given figure for the following exercises.



Exercise 10.9.6.1

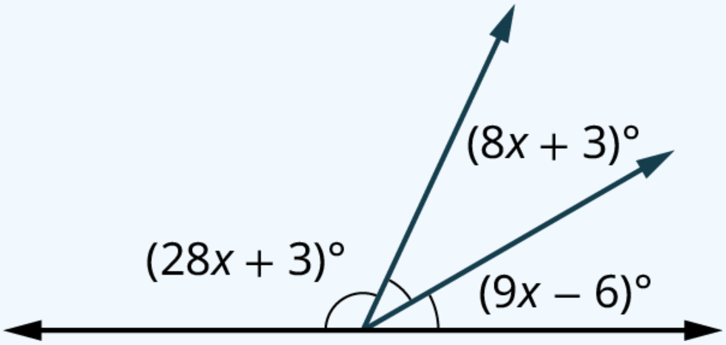
Find $\overleftrightarrow{AD} \cap \overline{CD}$.

Exercise 10.9.6.2

Find $\overline{AB} \cup \overline{BC}$.

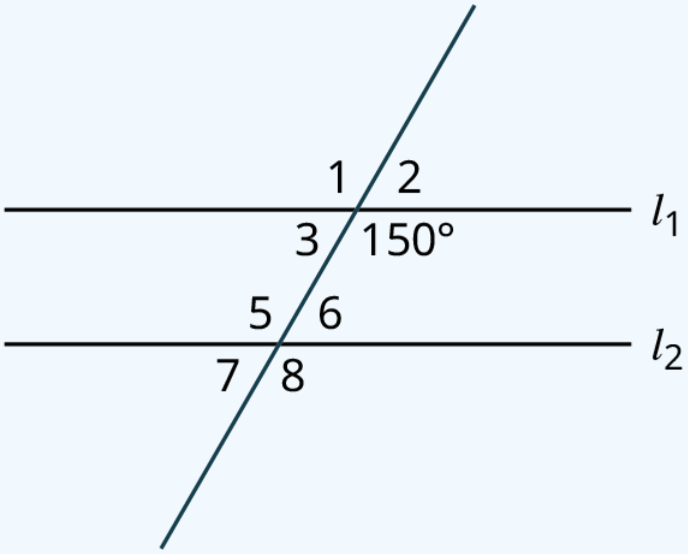
Exercise 10.9.6.3

Use the given figure to find the angle measurements.



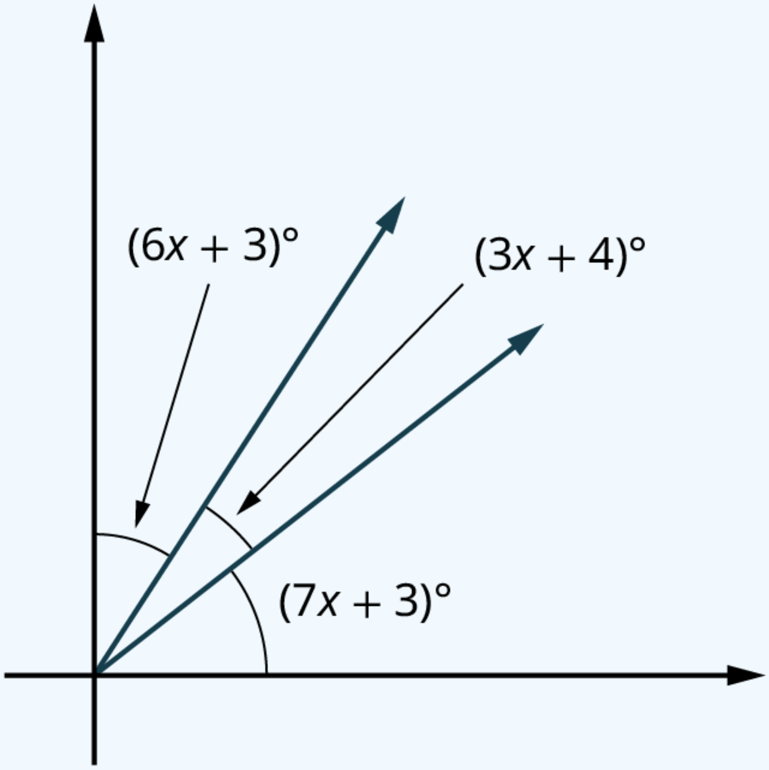
Exercise 10.9.6.4

Given that l_1 and l_2 are parallel lines, solve the angle measurements for all the angles in the given figure.



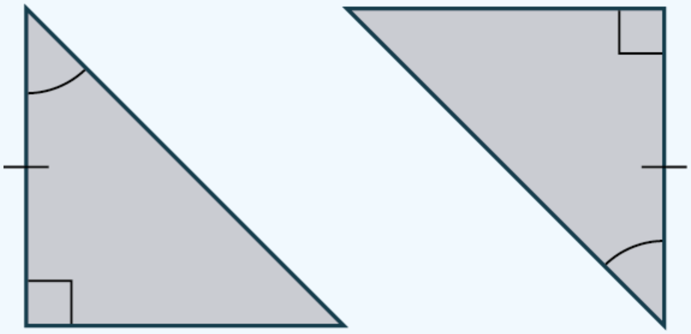
? Exercise 10.9.6.5

Find the angle measurements in the given figure.



? Exercise 10.9.6.6

The two triangles shown are congruent by what theorem?

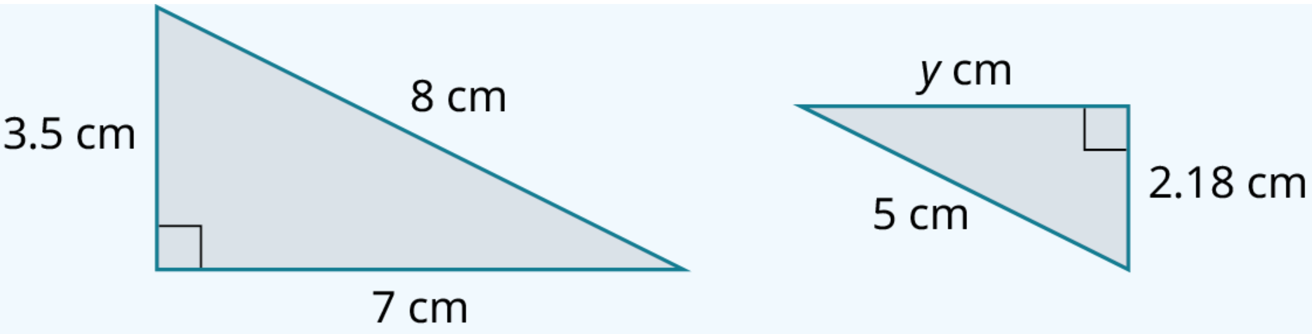


? Exercise 10.9.6.7

Find the sum of the interior angles of a regular heptagon.

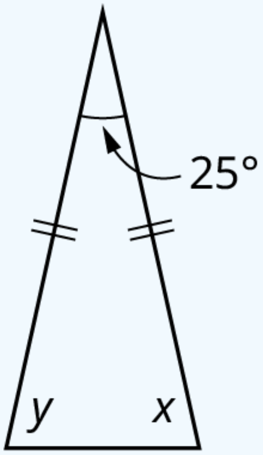
? Exercise 10.9.6.8

Determine the scaling factor between these two similar triangles.



? Exercise 10.9.6.9

Find the measure of the missing angles in the figure shown.



? Exercise 10.9.6.10

Calculate the perimeter of a regular octagon with a side length of 5 cm.

? Exercise 10.9.6.11

Find the measurement of an interior angle of a regular heptagon.

? Exercise 10.9.6.12

Find the sum of the interior angles of a regular pentagon.

? Exercise 10.9.6.13

Find the measure of an exterior angle of a regular pentagon.

? Exercise 10.9.6.14

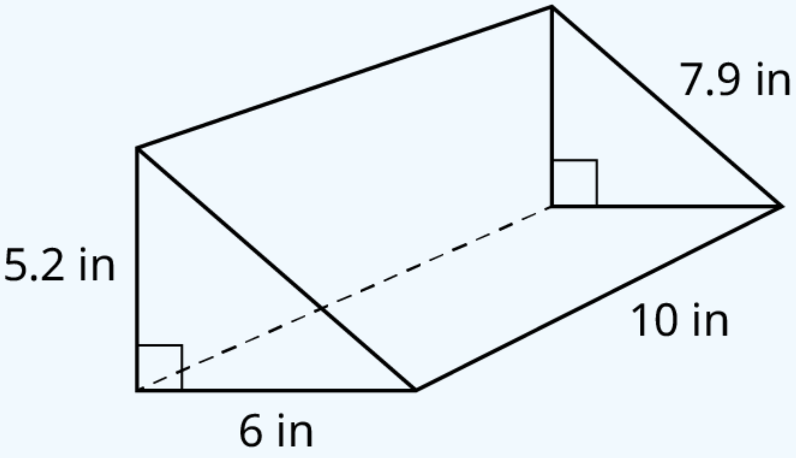
Find the circumference of the circle with a radius of 3.5 cm.

? Exercise 10.9.6.15

What are the four transformations that are used to produce tessellations?

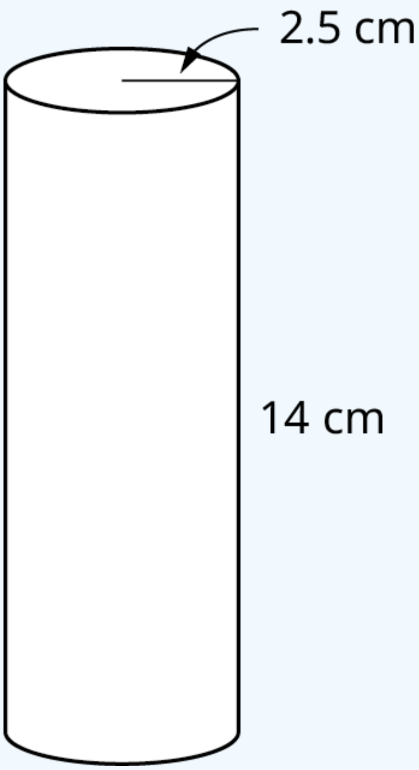
? Exercise 10.9.6.16

Find the surface area of the triangular prism shown.



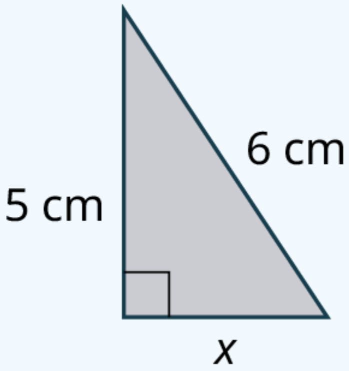
? Exercise 10.9.6.17

Find the volume of the right cylinder in the given figure.



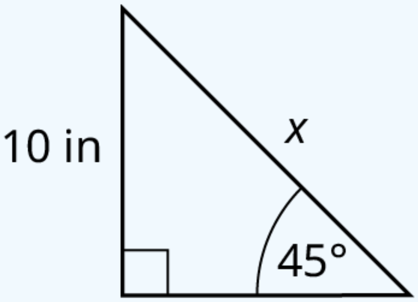
Exercise 10.9.6.18

Find the missing length in the given figure.



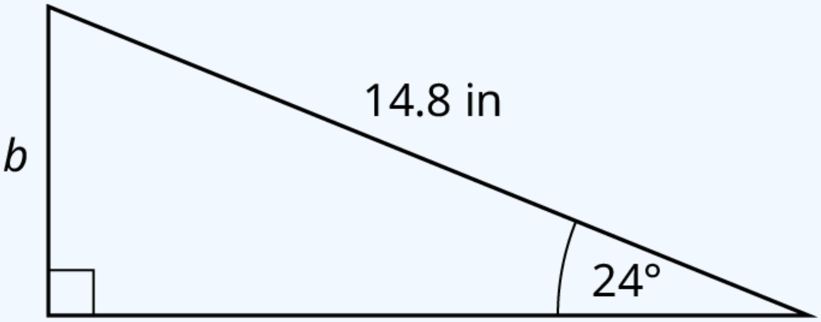
Exercise 10.9.6.19

Find the missing length in the given figure.



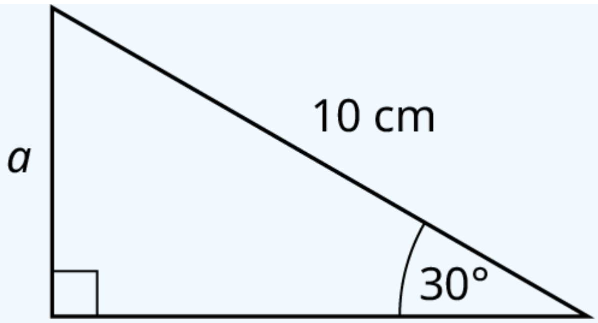
Exercise 10.9.6.20

Find the length of side b in the given figure.



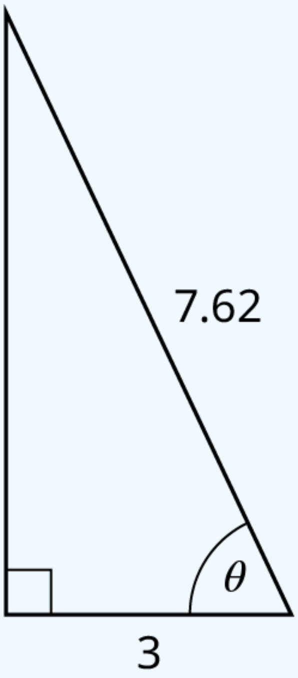
Exercise 10.9.6.21

Find the length of side a in the given figure.



Exercise 10.9.6.22

Find the measure of θ in the given figure.



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CHAPTER OVERVIEW

11: Voting and Apportionment

11.0: Introduction
11.1: Voting Methods
  11.1.0: Exercises
11.2: Fairness in Voting Methods
  11.2.0: Exercises
11.3: Standard Divisors, Standard Quotas, and the Apportionment Problem
  11.3.0: Exercises
11.4: Apportionment Methods
  11.4.0: Exercises
11.5: Fairness in Apportionment Methods
  11.5.0: Exercises
11.6: Chapter Summary
  11.6.0: Key Terms
  11.6.1: Key Concepts
  11.6.2: Videos
  11.6.3: Formula Review
  11.6.4: Projects
  11.6.5: Chapter Review
  11.6.6: Chapter Test

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11.0: Introduction

Suppose a friend asked you, “When did you last vote?” What would your answer be? Maybe you would tell your friend that the last time you voted was during the last presidential election, or perhaps you would tell your friend that you prefer not to vote. When thinking about voting, presidential campaigns or advertisements for reelections may come to mind, but you can cast your vote in many ways. Have you liked a post, followed a creator, friended a stranger, or clicked a heart online today? In the digital age, it's possible to vote several times a day. Voting systems are not only the machines that drive every democracy on Earth, but they are also the engines driving social media and many other aspects of life. A deeper understanding of these voting systems will enhance your ability to successfully engage with the world in which we live.

Figure 11.0.1 Voters cast their ballots in one of the world’s many democracies. (credit: “Governor Votes Early” by Maryland GovPics/Flickr, CC BY 2.0)

In this chapter, you will become one of the founders of the new democratic country of Imaginaria. You have a great responsibility to the people of this fledgling democracy. You have been tasked with writing the portion of the constitution that lays out voting procedures. In preparation for this important task, you will explore the various types of voting systems, from school board elections to Twitter wars. You will see how these types are alike, how they differ, and how they might be applied in Imaginaria. Most importantly, you will learn about the mathematically inherent advantages and disadvantages of various voting systems so that you can make informed choices to better the lives of the Imaginarians.

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11.1: Voting Methods

Figure 11.1.1: President Barack Obama votes in the 2012 election. (credit: Pete Souza/White House, Public Domain)

Learning Objectives

1. Apply plurality voting to determine a winner.
2. Apply runoff voting to determine a winner.
3. Apply ranked-choice voting to determine a winner.
4. Apply Borda count voting to determine a winner.
5. Apply pairwise comparison and Condorcet voting to determine a winner.
6. Apply approval voting to determine a winner.
7. Compare and contrast voting methods to identify flaws.

Today is the day that you begin your quest to collaborate on the constitution of Imaginaria! Let’s begin by thinking about the selection of a leader who can serve as president. It seems straightforward; if the majority of citizens prefer a particular candidate, that candidate should win. But not all votes are decided by a simple majority. Why not? What are the options?

Majority versus Plurality Voting

When an election involves only two options, a simple majority is a reasonable way to determine a winner. A majority is a number equaling more than half, or greater than 50 percent of the total.

Let’s take a look at the outcomes of U.S. presidential elections to understand more. Table 11.1 displays the results of the 2000 U.S. presidential election. Like most presidential elections, this election involved more than two options. If that is the case, is it possible that no single candidate will receive more than half of the votes cast?

Table 11.1 Results of the Popular Vote for the 2000 U.S. Presidential Election (source: https://www.fec.gov/introduction-cam...president2000/)

Candidate (Party Label)	Popular Vote Total
Al Gore (Democrat)	50,999,897
George W. Bush (Republican)	50,456,002
Ralph Nader (Green)	2,882,955
Patrick J. Buchanan (Reform/Independent)	448,895
Harry Browne (Libertarian)	384,431

Candidate (Party Label)	Popular Vote Total
Howard Phillips (Constitution)	98,020
Other	134,900
Total:	**105,405,100**

✓ Example 11.1.1: Majority of Popular Vote in the 2000 U.S. Presidential Election

Refer back to Table 11.1. Did any single candidate secure the majority of popular votes?

Answer

Step 1: Calculate 50 percent of 105,405,100 by multiplying the decimal form of 50 percent, which is 0.50, by 158,394,605: $0.50\,(105{,}405{,}100) = 52{,}702{,}550$

Step 2: Determine the minimum number of votes needed to have a majority. The minimum number of votes required is the lowest counting number that is larger than 50 percent of the votes. To have a majority, an individual candidate must have more than 52,702,550; so, a majority candidate must have 52,702,551 votes or more.

Step 3: Compare the number of votes each candidate received to 52,702,551. According to the data in Table 11.1, none of the candidates secured a majority.

Your Turn 11.1.1

According to the Cook Political Report, a total of 158,394,605 votes were cast in the 2020 U.S. presidential election. Of those, 81,281,502 were cast for Joe Biden, 74,222,593 for Donald Trump, and 2,890,510 for other candidates. Although U.S. presidential elections are not determined by popular vote, did either Joe Biden or Donald Trump secure a majority of the votes?

Unlike in the 2000 U.S. presidential election, a candidate won the majority of votes in the 2020 election (see Table 11.1). It is a common occurrence for no single candidate to receive a majority of the votes in an election with more than two candidates. When this occurs, the candidate with the largest portion of the votes is said to have a **plurality**.

✓ Example 11.1.2: Plurality of Popular Vote in the 2000 U.S. Presidential Election

Refer again to Table 11.1. In the 2000 U.S. presidential election, which candidate had a plurality of popular votes?

Answer

Al Gore secured 50,999,897 votes which was more than any other single candidate. Therefore, he had a plurality of the popular votes.

Your plans for Imaginarian elections will likely include primary elections, or preliminary elections to select candidates for a principal or general election. Table 11.2 displays the results of the 2018 U.S. Senate Republican primary for Maryland.

Table 11.2 (source: ballotpedia.org/United_State...lican_primary))

Top Four Republican Candidates	Votes	Percentage of Party Votes
Cambell, Tony	51,426	29.22%
Chaffee, Chris	42,328	24.05%
Grigorian, Christina J.	30,756	17.48%
Graziani, John R.	15,435	8.77%
Total Votes	175,981	100%

Your Turn 11.1.2

Consider the results of the 2018 U.S. Senate primary for Maryland. Determine which candidate won the primary for the Republicans (R). Did the candidate win a majority or a plurality of Republican votes?

Consider how election by plurality, not majority, is the most common method of selecting candidates for public office.

Who Knew?

The U.S. Electorial College: Winning the Presidency without a Plurality

In the 2000 U.S. presidential election, Al Gore had a plurality of the popular votes, but he did not win the election. Why? This occurred because the U.S. president and vice president are elected by electors rather than a direct vote by the citizens. The electors are part of the Electoral College, a body of people representing the states. Why was the Electoral College created? The Electoral College was created as a compromise between those authors of the U.S. Constitution who believed Congress should elect the president, and those who believed the citizens should vote directly. The popular vote was not recorded until the presidential election of 1824. Since then, only five presidents have been elected without winning a plurality of the popular vote: John Quincy Adams in 1824, Rutherford B. Hayes in 1876, Grover Cleveland in 1888, George Bush in 2000, and Donald Trump in 2016.

Runoff Voting

Has your family ever debated what to have for dinner? Suppose your family is deciding on a restaurant and exactly half of you want to have pizza but the other half want hamburgers. How do you decide when the result is a tie? You need a tiebreaker!

Will the new democracy of Imaginaria need tiebreakers? When no candidate satisfies the requirements to win the election, a runoff election, or second election, is held to determine a winner.

How would runoff voting work in Imaginaria? There are many types of **runoff voting systems**, which are voting systems that utilize a runoff election when the first round does not result in a winner. The method for implementing a runoff election can vary widely, particularly in the criteria used to determine whether a candidate will be on the ballot in the second election. For example, a two-round system is a runoff voting system in which only the top candidates advance to the runoff election. In some two-round systems, only the top two candidates are on the second ballot, or it may be any candidate who secures a certain percentage of the vote will advance. The Hare Method is another runoff voting system in which only the candidate(s) with the very least votes are eliminated. This can potentially result in several rounds of runoff elections.

✓ Example 11.1.3: Runoff Election for Condominium Association President

A condominium association elects a new president every two years by a two-round system of voting. If none of the candidates receive a majority, the association charter states that the top two candidates will be eligible to participate in a runoff election. In a particular year, five residents were nominated. The results of the first round are given in the table below.

Candidate	Votes in First Round
Abou	18
Baiocchi	10
Campana	5
Dali	11
Eugene	4

1. Is there a winner based on the first round? Why or why not?
2. If there is a winner, who won? If there should be a runoff, who will advance to the second round?

Answer

1. A majority of 48 total votes is required to win. Begin by finding 50 percent of 48, which is calculated as follows: $0.50(48) = 24$. A majority is 25 or more. No candidate has a majority, so there is no winner based on the first round.
2. Abou and Dali advance to the second round.

Your Turn 11.1.3

The student government bylaws of a particular college require that a new president is elected annually by plurality voting. In the event of a tie, the bylaws require the candidate(s) with the fewest votes to be removed from the ballot and a runoff election to be held with the remaining candidates. This process is repeated until a single candidate receives a plurality and wins the election. The results of two voting rounds are given in the table below.

Candidate	Votes in First Round	Votes in Second Round
Ferguson	158	168
Garcia	103	104
Hearn	157	180
Isaac	123	123
Jackman	58	Eliminated
Kelly	72	74
Lim	158	180

Which two candidates tied in the second round? Does this mean there will be a third election?

If there was a winner, who won? If there should be a runoff, who must be removed from the ballot?

Steps to Determine Winner by Plurality or Majority Election with Runoff

To determine the winner by plurality or when a majority election with runoff occurs, we take these three steps:

Step 1: If a majority is required to win the election, determine the number of votes needed to achieve a majority. This is the least whole number greater than 50 percent of the total votes. If a majority is not required, move to Step 2.

Step 2: Count the number of votes for each candidate in the current round of voting. If a single candidate has enough votes to win a plurality, or a majority as appropriate, then you are done! Otherwise, eliminate a predetermined number of candidates based on the rules of the election. Elimination conditions may vary. For example, the rules may state that the candidate(s) with the fewest votes will be eliminated (as in the Hare method), or that only the candidates meeting a certain threshold will move on (as in a two-round system). Once the appropriate candidates are eliminated, move on to Step 3.

Step 3: Hold a runoff election. If the runoff is simulated using a list of voter's preferences, renumber the preferences to reflect the remaining number of options in such a way that the original order of preference is retained. Then repeat Step 2.

Note: The second and third steps may be repeated as many times as necessary for voting procedures that allow multiple runoffs.

✓ Example 11.1.4: Family Dinner Night

The five members of the Chionilis family—Annette, Rene, Seema, Titus, and Galen—have decided to get takeout for dinner. They are trying to decide on a restaurant. The options are Rainbow China, Dough Boys Pizza, Taco City, or Caribbean Flavor. They will use majority election with runoffs where the restaurant with the fewest votes is eliminated in each round. The preferences of each family member are listed by first initial in the table below. An entry of 1 represents the person's first choice; 2, their second; and so on. For example, Annette's second choice is Dough Boys Pizza.

Options	A	R	S	T	G
Rainbow China	1	3	3	1	3

Options	A	R	S	T	G
Dough Boys Pizza	2	2	1	2	1
Taco City	3	4	2	4	2
Caribbean Flavor	4	1	4	3	4

Use the information in the table to answer the following questions.

1. Which common type of runoff voting method is this?
2. List the results of each round of voting based on this information and determine which restaurant was ultimately chosen.

Answer

1. the Hare Method
2. **Step 1:** Determine the number of votes necessary to have a majority. There are five family members, so 50 percent of 5 is $0.50(5) = 2.5$. A majority is three or more votes.
   Step 2: Count the number of votes for each restaurant in the first election. In a list of voter preferences, the 1s represent the top choice of each voter, which corresponds to their vote in the first round.
   Results of Round 1:
   - Rainbow China — 2 votes
   - Dough Boys — 2 votes
   - Taco City — 0 votes
   - Caribbean Flavor — 1 vote

   No restaurant received a majority. Eliminate Taco City, which has the fewest first place votes:

Options	A	R	S	T	G
Rainbow China	1	3	3	1	3
Dough Boys Pizza	2	2	1	2	1
Caribbean Flavor	4	1	4	3	4

   Step 3: Hold a runoff election. In other words, hold a second round. Since we have a list of the voters' preferences with the eliminated option removed, we will renumber the preferences as first, second, and third so that we keep the original order of preference. The result is that we will count the second-place vote of any voter whose first choice was eliminated.

Options	A	R	S	T	G
Rainbow China	1	3	2	1	2
Dough Boys Pizza	2	2	1	2	1
Caribbean Flavor	3	1	3	3	3

   Step 4: Repeat the process from Step 2. Count the number votes for each restaurant in the first-round election. Since we are using a list of preferences, we need to count the number of 1s received by each restaurant.
   Results of Round 2:
   - Rainbow China — 2 votes
   - Dough Boys — 2 votes
   - Caribbean Flavor — 1 vote

No single restaurant has three votes. Eliminate Caribbean Flavor, which has the fewest first place votes.

OPTIONS	A	R	S	T	G
Rainbow China	1	3	2	1	2
Dough Boys Pizza	2	2	1	2	1

Step 5: Repeat the process from Step 3. Hold another runoff election. This will be Round 3. Renumber the voters' preferences as first and second this time.

Options	A	R	S	T	G
Rainbow China	1	2	2	1	2
Dough Boys Pizza	2	1	1	2	1

Step 6: Repeat the process from Step 2 one last time. Count the number of first place votes for each remaining restaurant.

Results of Round 3:

- Rainbow China — 2 votes
- Dough Boys — 3 votes

Determine whether any one choice has a majority. Yes! Dough Boys has three votes, so it is the winner!

Your Turn 11.1.4

There are six members on the board of a Parent Teacher Association (PTA) at a local elementary school: the president (P), the vice president (V), the recording secretary (R), the liaison to the administration (L), the treasurer (T), and the chief fundraiser (C). The board must decide which equipment to purchase for the classrooms with moneys from their annual fundraisers. The preferences of the board members are shown in the table below.

	P	**V**	**R**	**L**	**T**	**C**
Option A	3	4	1	3	4	1
Option B	1	3	2	1	3	2
Option C	2	2	3	2	1	4
Option D	4	1	4	4	2	3

The board uses plurality method and a runoff in the event of a tie, such that the option(s) with the least votes will be eliminated in each round. Which option will be chosen?

Ranked-Choice Voting

In Example 11.1.4 and Your Turn 11.1.4, you were given a list that ordered each voter’s preferences. This ordering is called a preference ranking. A ballot in which a voter is required to give an ordering of their preferences is a ranked ballot, and any voting system in which a voter uses a ranked ballot is referred to as ranked voting.

The vote for the Academy Awards uses a ranked ballot. The table below provides an example of a ranked ballot for the 2020 Academy Award nominees for Best Director.

Candidate for Best Director	Rank top choice as 1, next choice as 2, and so on.				
Martin Scorsese, *The Irishman*	1	2	3	4	5
Todd Phillips, *Joker*	1	2	3	4	5
Sam Mendes, *1917*	1	2	3	4	5
Quentin Tarantino, *Once Upon a Time in Hollywood*	1	2	3	4	5
Bong Joon-ho, *Parasite*	1	2	3	4	5

As you decide on the voting methods that will be used in your new democracy, budget must be a consideration. You might consider a particular type of ranked voting called ranked-choice voting (RCV), which simulates a series of runoff elections without the usual time and expense involved when voters must repeatedly return to the polls, like we did in Example 11.1.4.

The method of ranked-choice voting (RCV), also called instant runoff voting (IRV), is a version of the Hare Method, using *preference ranking* so that, if no single candidate receives a majority, the least popular selections can be eliminated and the results can be recounted, without the need for more elections.

Checkpoint

Ranked voting can be confused with ranked-choice voting, but ranked voting is a more general category which includes ranked-choice voting and several other voting methods.

As we explore examples of ranked voting, we will summarize the voters' preference rankings using a table in which the top row shows the number of ballots that ranked the options in the same order. Let's practice interpreting the information in this type of table.

✓ Example 11.1.5: Interpreting the Sample Preference Summary

Refer to the table below containing voters' preference rankings to answer the following questions.

Number of Ballots	100	200	150	75
Option A	1	4	3	4
Option B	2	3	4	2
Option C	4	2	1	1
Option D	3	1	2	3

1. How many voters ranked the options in the following order: Option A in fourth place, Option B in second place, Option C in first place, and Option D in third place?
2. How many ballots in total were collected?
3. How many voters indicated that Option C was their first choice?

Answer

1. The column farthest to the right displays this ordering. The top entry in this column is 75; so, there were 75 voters who ranked the options in this way.

2. The sum of the top row gives the total number of ballots collected: $100 + 200 + 150 + 75 = 525$. So, there were 525 ballots collected.
3. In the Option C row, there are two entries of 1 which indicate a first choice for that option. These occur in the last two columns. The sum of the top entries in these columns is $150 + 75 = 225$. So, 225 voters indicated Option C as their first choice.

Your Turn 11.1.5

A kindergarten class votes on their favorite colors using a ranked ballot. Use the results in the following table to answer the questions.

Number of Ballots	**4**	**6**	**4**	**7**
Red	2	6	2	5
Blue	1	2	1	4
Green	6	5	5	2
Yellow	5	4	6	3
Purple	4	3	4	1
Pink	3	1	3	6

How many students voted in total?

How many students voted for yellow as their favorite color?

How many students in the class indicated that blue was their favorite color while green was their least favorite color?

Now that we’ve covered how to read a summary of preference rankings, let’s practice using the ranked-choice method to determine the winner of an election. Recall that ranked-choice voting is still the Hare Method where the candidate with the very least number of votes is eliminated each round until a majority is attained. The difference here is that the voters have completed a ranked ballot, so they don't have to visit the polls multiple times. Here are the steps for ranked-choice voting.

Steps to Determine Winner by Ranked-Choice Voting

To determine the winner when ranked-choice voting occurs, we take these three steps:

Step 1: Determine the number of votes needed to achieve a majority. This is the least whole number greater than 50 percent of the total votes.

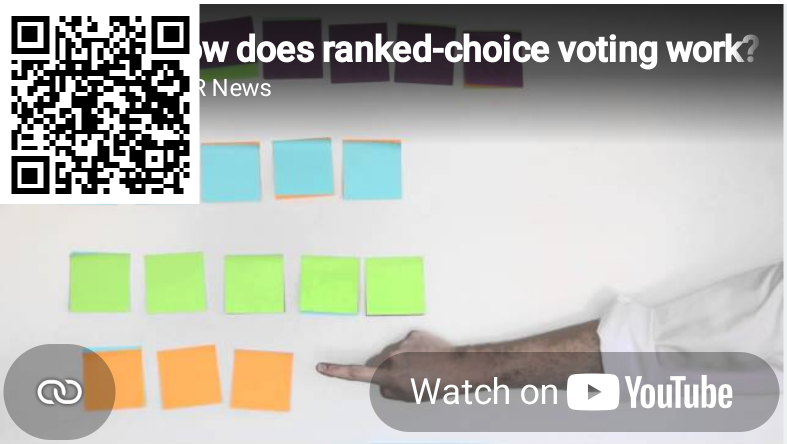
Step 2: Count the number of first place votes for each candidate. If a candidate has a majority, that candidate wins the election and we are done! Otherwise, eliminate the candidate(s) with the fewest votes and complete Step 3.

Step 3: Reallocate the votes to the remaining candidates, and repeat Step 2.

Checkpoint

Be methodical to avoid arithmetic errors. Make sure that each time you count the number of first place votes they sum to the number of ballots.

Video



How Does Ranked-Choice Voting Work?

Example 11.1.6: Most Popular Color in Kindergarten

Let's review the kindergarten class color preferences again, and this time determine which color would be selected based on these results using the ranked-choice method.

Number of Ballots	4	6	4	7
Red	2	6	2	5
Blue	1	2	1	4
Green	6	5	5	2
Yellow	5	4	6	3
Purple	4	3	4	1
Pink	3	1	3	6

Answer

Step 1: Determine whether any candidate received a majority. There were 21 ballots. Fifty percent of 21 is $0.50(21) = 10.5$. A majority is 11.

Step 2: Count the number of first place votes for each candidate. If a candidate has a majority, that candidate wins the election. Otherwise, eliminate the candidate(s) with the fewest votes.

- Red: 0
- Blue: $4 + 4 = 8$
- Green: 0
- Yellow: 0
- Purple: 7
- Pink: 6

Notice that $8 + 7 + 6 = 21$, which is the total number of ballots. Confirming this helps to catch any arithmetic or counting errors. No candidate has a majority with 11 or more votes. We must eliminate red, green, and yellow which had the fewest votes with 0 each. The remaining votes that must be counted for Round 2 are given in the table below.

Number of Ballots	4	6	4	7
Blue	1	2	1	4
Purple	4	3	4	1
Pink	3	1	3	6

Step 3: Reallocate the votes to the remaining candidates. We can do this by numbering the choices as 1, 2, and 3 in such a way that the order of preference is retained as seen in the table below:

Number of Ballots	4	6	4	7
Blue	1	2	1	2
Purple	3	3	3	1
Pink	2	1	2	3

Step 4: Repeat the process from Step 2. Count the number of first place votes for each candidate. If a candidate has a majority, that candidate wins the election. Otherwise, eliminate the candidate(s) with the fewest votes.

- Blue: $4+4=8$
- Purple: 7
- Pink: 6

Confirm that $8+7+6=21$. Great! No candidate has 11 or more votes. We must eliminate pink which had the fewest votes with 6. The remaining votes that must be counted for Round 2 are shown in the table below.

Number of Ballots	4	6	4	7
Blue	1	2	1	2
Purple	3	3	3	1

Step 5: Repeat the process from Step 3. Reallocate the votes to the remaining candidates. We can do this by numbering the choices as 1 and 2;

Number of Ballots	4	6	4	7
Blue	1	1	1	2
Purple	2	2	2	1

Step 6: Repeat the process from Step 2 one last time. Count the number of first place votes for each candidate. If a candidate has a majority, that candidate wins the election. Otherwise, eliminate the candidate(s) with the fewest votes.

- Blue: $4+6+4=14$
- Purple: 7

Blue has a majority and wins the election!

Video

Determine Winner of Election by Ranked-Choice Method (aka Instant Runoff)

Your Turn 11.1.6

Suppose that 58 *Star Wars* fans were asked to vote for their favorite *Star Wars* character. They were given a ranked ballot, and the results are shown in the following table. Use ranked-choice voting to determine the winner.

Number of Ballots	7	6	10	8	4	5	6	7	2	3
Han Solo	3	5	1	3	2	4	2	2	1	4
Princess Leia	2	1	3	6	4	3	3	1	3	2

Number of Ballots	7	6	10	8	4	5	6	7	2	3
Luke Skywalker	6	6	6	1	5	5	5	4	6	1
Chewbacca	4	4	5	4	6	1	6	5	4	5
Yoda	1	2	4	2	3	2	1	3	2	3
R2-D2	5	3	2	5	1	6	4	6	5	6

Borda Count Voting

Ranked-choice voting is one type of ranked voting that simulates multiple runoffs based on ranked ballots. Another type of ranked voting is the **Borda count method**, which uses ranked ballots that award candidates points corresponding to the number of candidates ranked lower on each ballot.

To understand how this works, let's review the favorite colors of our kindergarten class from the table below. Let's focus on the votes represented by the first column of the preference summary.

Number of Ballots	4	6	4	7
Red	2	6	2	5
Blue	1	2	1	4
Green	6	5	5	2
Yellow	5	4	6	3
Purple	4	3	4	1
Pink	3	1	3	6

Each student had six options. This first column tells us that four students ranked blue as their first choice, red as their second choice, pink as their third choice, purple as their fourth choice, yellow as their fifth choice, and green as their sixth choice. Blue was ranked higher than $6-1=5$ other colors. For each of the four students who completed their ballot in this way, blue would receive five points. Since there were four ballots with this ordering, blue would receive $5(4)=20$ points from the first column. To determine the total points for each candidate, we have to find the sum of the points they received in each column.

To determine the winner of a contest using the Borda count method, we must compare total number of points earned by each candidate. The candidate with the most points is the winner. Each row of the preference summary corresponds to a single candidate. To find the number of points received by a particular candidate in the preference summary, or their **Borda score**, we will need to focus on the row in which that candidate appears.

Before we practice determining the winner of a Borda count election, let's examine how to find the Borda score for a single candidate.

✓ Example 11.1.7: Most Popular Color in Kindergarten Revisited

Let's review the ballots from the kindergarten class again, as shown in the table below. This time, let's determine the Borda score received by the color purple.

Number of Ballots	4	6	4	7
Red	2	6	2	5
Blue	1	2	1	4

Number of Ballots	4	6	4	7
Green	6	5	5	2
Yellow	5	4	6	3
Purple	4	3	4	1
Pink	3	1	3	6

Answer

Step 1: Find the number of points received by the candidate in each column.

Column 1: $4 \times (6-4) = 4 \times 2 = 8$

Column 2: $6 \times (6-3) = 6 \times 3 = 18$

Column 3: $4 \times (6-4) = 4 \times 2 = 8$

Column 4: $7 \times (6-1) = 7 \times 5 = 35$

Step 2: Find the sum of the points received in each column. This is the total number of points received by this candidate: $8 + 18 + 8 + 35 = 69$

This process can also be combined into one step as shown here.

$$\begin{aligned} &4 \times (6-4) + 6 \times (6-3) + 4 \times (6-4) + 7 \times (6-1) \\ &= 4 \times 2 + 6 \times 3 + 4 \times 2 + 7 \times 5 \\ &= 8 + 18 + 8 + 35 \\ &= 69 \end{aligned}$$

Purple received 69 points in this election.

Checkpoint

When calculating a Borda score in one step, be careful to use the correct order of operations. Perform the subtraction inside each pair of parentheses first, then perform each multiplication, and then perform each addition.

Your Turn 11.1.7

Consider the color preferences of the kindergarten class once more. Using the Borda count method, determine the total number of points the color blue received.

Now let's determine the winner of an election by comparing the Borda scores for each of the candidates.

✓ Example 11.1.8: Determine the Winner by Two Ranked Voting Methods

Use the table below, which displays a sample preference summary, to answer the questions that follow.

Number of Ballots	95	90	110	115
Option A	4	4	1	1
Option B	2	2	2	2
Option C	3	1	3	4
Option D	1	3	4	3

1. Use the ranked-choice voting method to determine the winner of the election.
2. Use the Borda count method to determine the winner of the election.

Answer

1. **Step 1:** Determine the number of votes needed to achieve a majority. The number of ballots is $95+90+110+115=410$. Fifty percent of 410 is $0.50(410)=205$. So, 206 votes or more is a majority.

   Step 2: Count the number of first place votes for each candidate.

   - Option A has $110+115=225$
   - Option B has 0
   - Option C has 90
   - Option D has 95

   Since Option A has a majority, Option A is the winner by the ranked-choice method.

2. The Borda scores would be:

   - Option A: $95(4-4)+90(4-4)+110(4-1)+115(4-1)=675$
   - Option B: $95(4-2)+90(4-2)+110(4-2)+115(4-2)=820$
   - Option C: $95(4-3)+90(4-1)+110(4-3)+115(4-4)=475$
   - Option D: $95(4-1)+90(4-3)+110(4-4)+115(4-3)=490$

   Since Option B has a Borda score of 820 points, Option B is the winner by the Borda count method.

Your Turn 11.1.8

Answer the following questions using the table below, which summarizes Imaginarian voter preferences.

Number of Ballots	**12**	19	**27**	**29**	**31**	**21**
Candidate A	1	1	2	2	3	3
Candidate B	2	3	1	3	1	2
Candidate C	3	2	3	1	2	1

Use the ranked-choice voting method to determine the winner of the election.

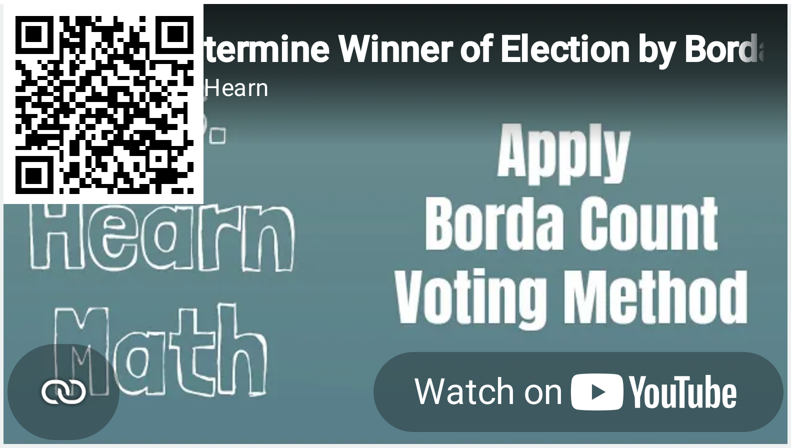
Use the Borda count method to determine the winner of the election.

Compare the results of the two methods. Did the same candidate win? What observations can you make about the results?

The Borda count method may seem too complicated to even consider using for Imaginaria, but each voting method has its own pros and cons. The Borda count method, for example, favors compromise candidates over divisive candidates. A **compromise candidate** is not the first choice of most of the voters, but is more acceptable to the population as a whole than the other candidates. A **divisive candidate** is simultaneously the first choice of a large portion of the voters and the last choice of another large portion of the voters.

In Example 11.8, Candidate A was ranked first by 225 voters, but was ranked last by 185 voters. No voters ranked Candidate A as second or third. It appears that, although Candidate A had the majority of first place votes, there was a significant minority who strongly disliked them. Candidate A was a divisive candidate. Candidate B, on the other hand, was the second choice of every voter, making Candidate B a good compromise. The Borda count method chose Candidate B, a compromise candidate, that was more acceptable to the population as a whole. This scenario is cited by both opponents and proponents of the Borda count method.

Video



Determine Winner of Election by Borda Count Method

Pairwise Comparison and Condorcet Voting

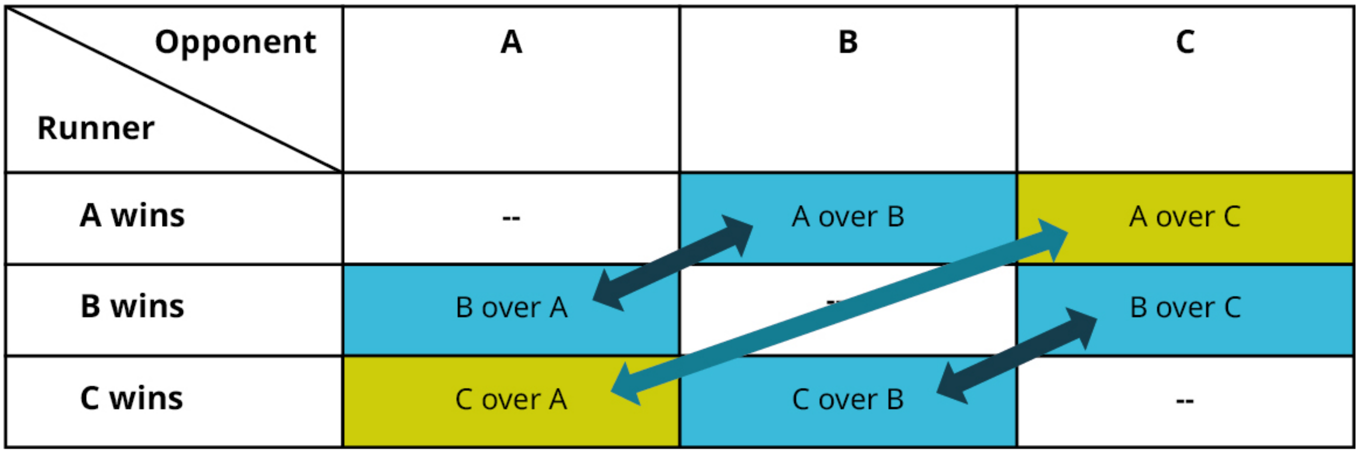
We have discussed two kinds of ranked voting methods so far: ranked-choice and Borda count. A third type of ranked voting is the **pairwise comparison method**, in which the candidates receive a point for each candidate they would beat in a one-on-one election and half a point for each candidate they would tie. If one candidate earns more points than the others, then that candidate wins. This method is one of several **Condorcet voting methods**, which are methods in which candidates are ranked and then compared pairwise to each other, a candidate having to beat all others in order to win. These methods vary in the way candidates are scored, and there is not always a clear winner. A candidate who wins each possible pairing is known as a **Condorcet candidate**. These terms are named after the Marquis de Condorcet, a French philosopher and mathematician who preferred the pairwise comparison method to the Hare method and made public arguments in its favor.

People in Mathematics: Marquis de Condorcet

Condorcet voting methods are named for the Marquis de Condorcet, a French philosopher and mathematician known for, among other accomplishments, writing "*Sur l'admission des femmes au droit de Cité*" ("On the Admission of Women to the Rights of Citizenship"), in 1789, the first published essay on the political rights of women.

For more details visit this Web site.

If you include a Condorcet voting method in the constitution of Imaginaria, the election supervisors may want to use a pairwise comparison matrix like the one in Figure 11.3. It's a tool used to list the number of wins associated with each pairing of two candidates. Each candidate will receive a point for each win and a half a point for each tie. Each pairing is listed twice, once for the number of wins of a candidate over a particular challenger and once for the number of wins of the challenger over that candidate.



Opponent Runner	A	B	C
A wins	--	A over B	A over C
B wins	B over A	--	B over C
C wins	C over A	C over B	--

Figure 11.1.2: Pairwise Comparison Matrix for Three Candidates

Steps to Determine a Winner by Pairwise Comparison Method Using a Matrix

To determine the winner when the pairwise comparison method is used, we take these three steps:

Step 1: On the matrix, indicate a losing matchup by crossing out a box, $\boxed{\times}$, and tie match ups by drawing a slash through the box, $\boxed{\backslash}$.

Step 2: Award each candidate 1 point for a win, half a point for a tie, and 0 points for a loss.

Step 3: Identify the winner, which is the candidate with the most points.

Video

Determine Winner of Election by Using the Pairwise Comparison Method

Before you decide on the pairwise comparison method for Imaginaria, review what’s involved in constructing a pairwise comparison matrix from a summary of ranked ballots. Then we can use the matrix to determine the winner of the election. Does the winner using the Borda method still win?

Example 11.1.9: Construct and Use a Pairwise Comparison Matrix

Consider the summary of ranked ballots shown in the table below. Determine the winner of an election using the pairwise comparison method.

Number of Ballots	95	90	110	115
Option A	4	4	1	1
Option B	2	2	2	2
Option C	3	1	3	4
Option D	1	3	4	3

1. Construct a pairwise comparison matrix for the sample summary of ranked ballots in the table above.
2. Use the pairwise comparison method to determine a winner.
3. Recall that in Example 11.8, Candidate A won by the ranked-ballot method, and Candidate B won by the Hare method. Did the same candidate win using the pairwise comparison method?
4. Is the winner a Condorcet candidate?

Answer

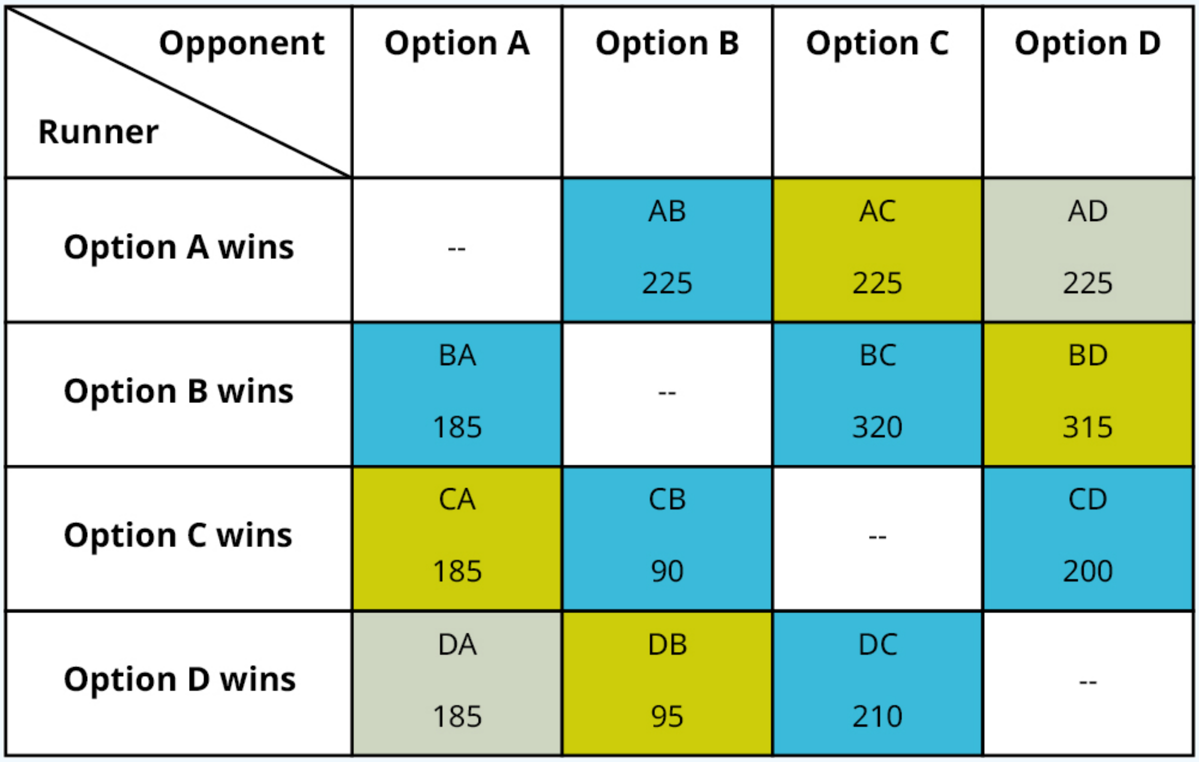
There are four candidates on the ballots. We will need a row and a column for each candidate in addition to the headings, so we will draw a five by five matrix.

Opponent Runner	Option A	Option B	Option C	Option D
Option A wins	--	A over B	A over C	A over D
Option B wins	B over A	--	B over C	B over D
Option C wins	C over A	C over B	--	C over D
Option D wins	D over A	D over B	D over C	--

Figure ✓ 11.1.3 Pairwise Comparison Matrix for Four Candidates

1. **Step 1:** Refer to Figure 11.4 to determine the values that belong in each cell.
    - A over B: A is preferred to B in columns 3 and 4. So, A scores $110 + 115 = 225$ points.
    - A over C: A is preferred to C in columns 3 and 4. So, A scores 225 points again.
    - A over D: Similarly, A scores 225 points.
    - B over A: B is preferred to A in columns 1 and 2. So, B scores $95 + 90 = 185$ points.

   Step 2: Continuing in this way, we complete the pairwise comparison matrix, as shown in Figure 11.5.

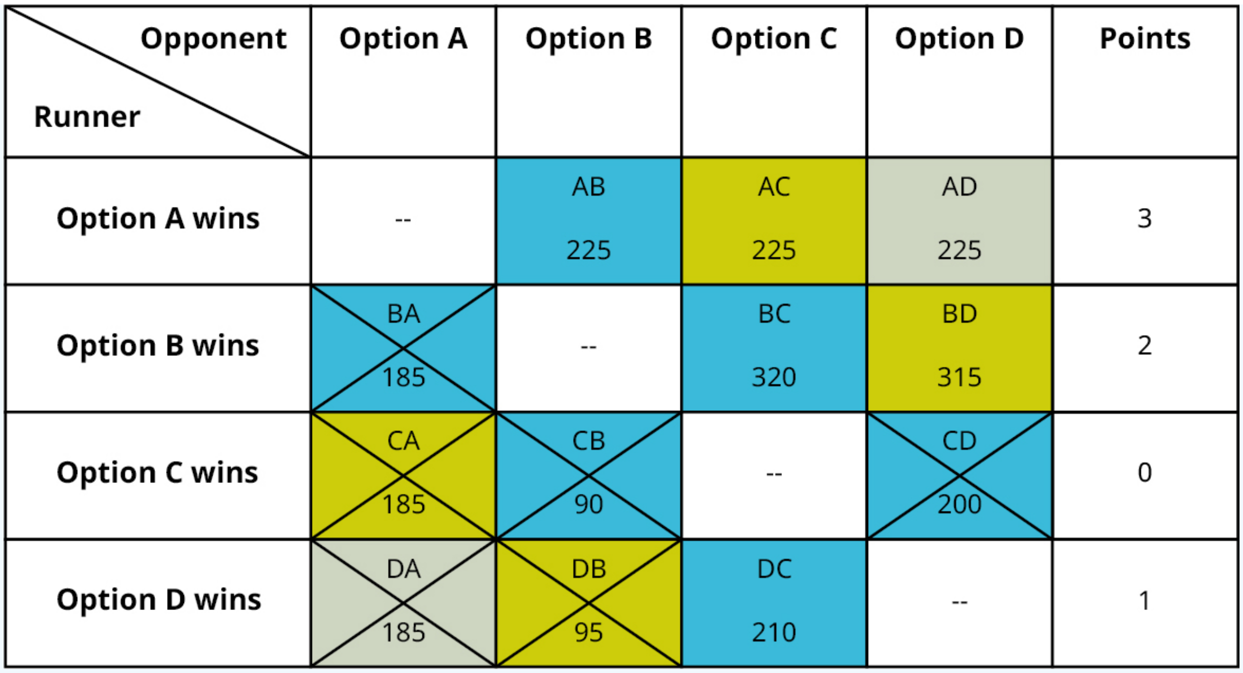


Opponent Runner	Option A	Option B	Option C	Option D
Option A wins	--	AB 225	AC 225	AD 225
Option B wins	BA 185	--	BC 320	BD 315
Option C wins	CA 185	CB 90	--	CD 200
Option D wins	DA 185	DB 95	DC 210	--

Figure ✓ 11.1.4: Pairwise Comparison Matrix for Four Candidates with Vote Counts

2. **Step 1:** Losing pairings are crossed off with an $\boxed{\times}$. In the event of a tie, we will draw a slash, $\boxed{\backslash}$.

   Step 2: Determine the number of points for each candidate by analyzing their row of wins. Each win is 1 point, each loss, $\boxed{\times}$, is 0 points, and each tie, $\boxed{\backslash}$, is half a point. Construct an additional column for each candidate's points.



Opponent / Runner	Option A	Option B	Option C	Option D	Points
Option A wins	--	AB 225	AC 225	AD 225	3
Option B wins	~~BA 185~~	--	BC 320	BD 315	2
Option C wins	~~CA 185~~	~~CB 90~~	--	~~CD 200~~	0
Option D wins	~~DA 185~~	~~DB 95~~	DC 210	--	1

Figure ✔ 11.1.5: Pairwise Comparison Matrix for Four Candidates with Pairwise Winners and Points Column Added

Step 3: The winner by the pairwise comparison method is Option A with 3 points.

3. Option A was not the winner by the Hare method.
4. The winner, Option A, is a Condorcet candidate because Option A won each pairwise comparison.

Checkpoint

Notice that the pairwise vote totals are not used to determine the points. Vote totals are only used to determine a win or a loss. Avoid the common error of adding the values in each row to get the points.

Your Turn 11.1.9

According to *Variety* magazine, there were 8,469 eligible Oscar voters in 2020. To make our matrix easier to work with, we've rounded this number up to 8,700. We'd never do this in a real voting situation. Suppose that the voter preferences for the ballot for Best Director had been as shown in the table below.

Number of Ballots	2,400	2,100	1,900	1,200	1,100
Martin Scorsese, *The Irishman*	3	1	4	5	2
Todd Phillips, *Joker*	1	4	5	2	4
Sam Mendes, *1917*	5	5	3	4	5

Number of Ballots	2,400	2,100	1,900	1,200	1,100
Quentin Tarantino, *Once Upon a Time in Hollywood*	4	2	1	3	3
Bong Joon-ho, *Parasite*	2	3	2	1	1

Refer to the table to answer each question.

Construct a pairwise comparison matrix for the Best Director ballots.

Who is the “Best Director” according to the pairwise comparison method?

Is the winner a Condorcet candidate?

Three Key Questions

Before you decide if you want to use the pairwise comparison method for Imaginarian elections, let’s consider three questions that might affect your decision.

I. Is there always a winner?
II. If there is a winner, is the winner always a Condorcet candidate?
III. If there is a Condorcet candidate, does that candidate always win?

Let’s think about why these questions might be important to you if you chose the pairwise comparison method. First, if no candidate meets the criteria to win an election, you will need a backup plan such as a runoff election. Second, if the winner is not a Condorcet candidate, then there is at least one candidate who beat the winner in a pairwise matchup and the supporters of that candidate might question the validity of the election. Finally, if there is a Condorcet candidate who beat every other candidate in a pairwise matchup, it is reasonable to conclude that it would be unfair for anyone else to win. The rest of the examples in this section should illustrate these key concepts.

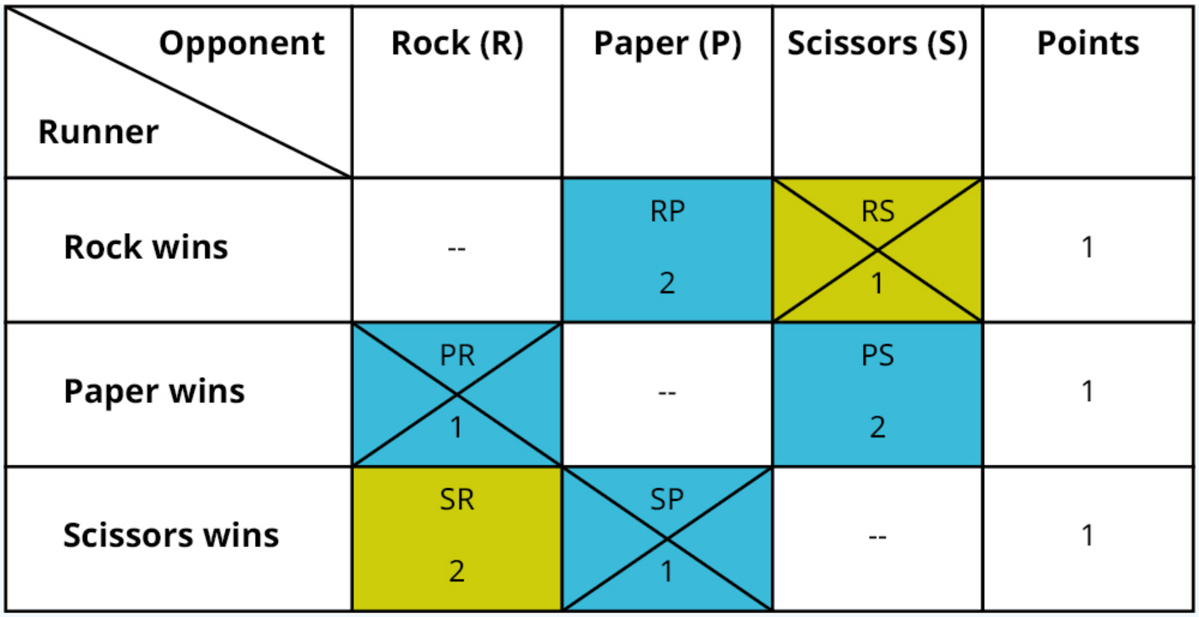
✓ Example $11.1.10$: Rock, Paper, Scissors by Pairwise Comparison

Suppose that three people are playing the game Rock, Paper, Scissors. On the count of three, each person shows a hand signal for rock, paper, or scissors. Each hand signal beats another hand signal. The group keeps having a tie because Person A always picks rock, Person B always picks paper which beats rock, and Person C always picks scissors which beats paper, and is beaten by rock! This leads to a disagreement about which choice is best. They decide to use the pairwise comparison method determine the winner. Their preference rankings are given in the following table.

Voters	A	B	C
Rock (R)	1	3	2
Paper (P)	2	1	3
Scissors (S)	3	2	1

Answer

Construct the comparison matrix:



Runner \ Opponent	Rock (R)	Paper (P)	Scissors (S)	Points
Rock wins	--	RP 2	RS 1	1
Paper wins	PR 1	--	PS 2	1
Scissors wins	SR 2	SP 1	--	1

Figure ✓ 11.1.6 Rock, Paper, Scissors Pairwise Comparison Matrix

There is a tie! There is no winner.

Example 11.1.10 illustrates the answer to the first key question. The pairwise comparison method does not always result in a winner. For example, much like the game of Rock, Paper, Scissors, it is possible for a cyclic pattern to emerge in which each candidate beats the next until the last candidate who beats the first.

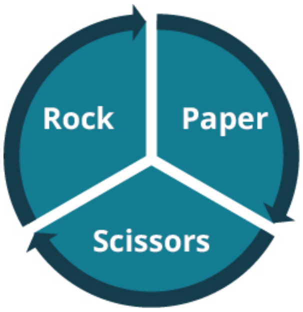
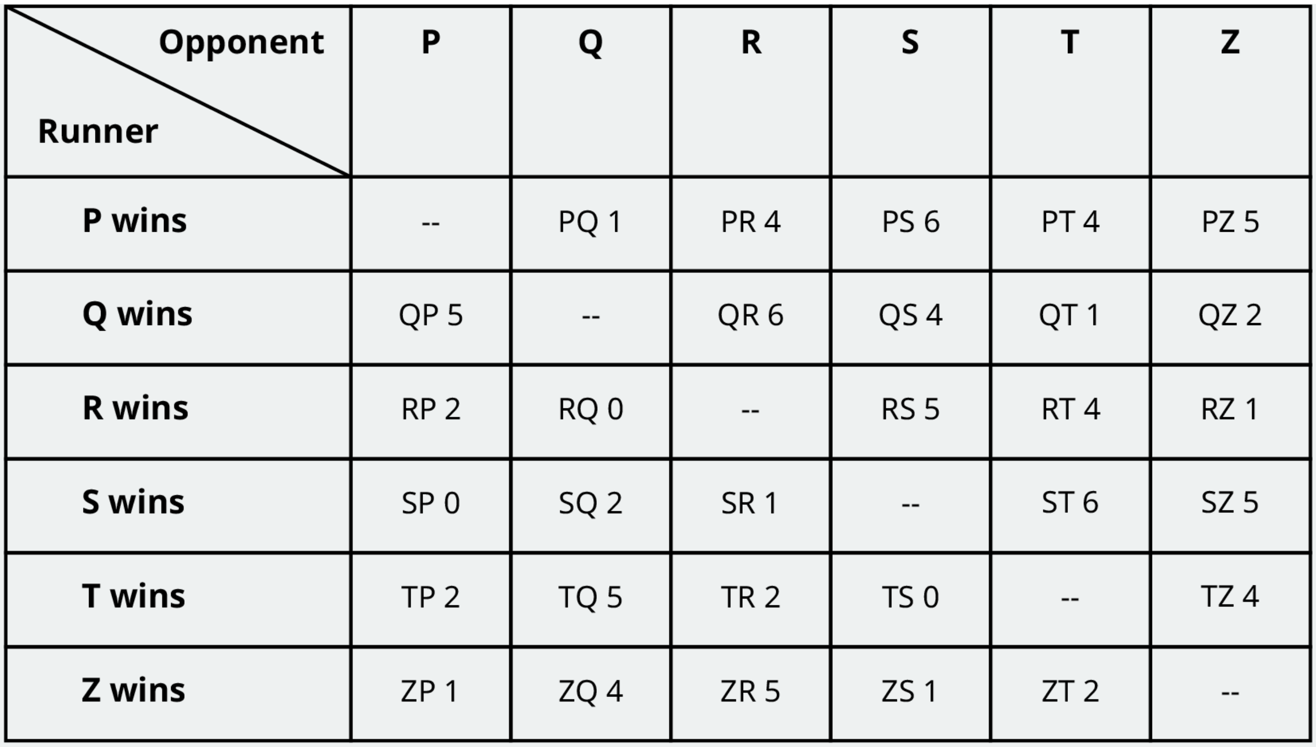


Figure 11.1.7: Rock, Paper, Scissors Cyclic Outcome

Your Turn 11.1.10

A pairwise comparison matrix is given. Determine the winner by the pairwise comparison method. If there is not a winner, explain why. If there is a winner, tell whether the winner is a Condorcet candidate.



Opponent Runner	P	Q	R	S	T	Z
P wins	--	PQ 1	PR 4	PS 6	PT 4	PZ 5
Q wins	QP 5	--	QR 6	QS 4	QT 1	QZ 2
R wins	RP 2	RQ 0	--	RS 5	RT 4	RZ 1
S wins	SP 0	SQ 2	SR 1	--	ST 6	SZ 5
T wins	TP 2	TQ 5	TR 2	TS 0	--	TZ 4
Z wins	ZP 1	ZQ 4	ZR 5	ZS 1	ZT 2	--

Figure 11.1.8: A Pairwise Comparison Matrix

Now, you have the answer to the second key question. The pairwise comparison matrix in YOUR TURN 11.10 is an example of a scenario where a winner is not a Condorcet candidate.

The answer to the third question is not as clear. If there is a Condorcet candidate, does that candidate win? So far, we have not come across a contradictory example where the Condorcet candidate didn't win, but we cannot know with certainty that it is not possible by looking at examples. Instead, we will need to use some reasoning. Let’s review some particular cases of elections with a certain number of candidates, and then we will try to generalize the scenario to an election with n candidates.

✓ Example 11.1.11: Does the Condorcet Candidate Win?

1. Suppose there is an election with five candidates—A, B, C, D, and E—and that Candidate C is a Condorcet candidate. How many points did Candidate C win?
2. What is the greatest number of points that any one of the other candidates could win?
3. Is it possible for Candidate C to lose or tie?

Answer

1. In any pairwise election with five candidates, each candidate must compete against four other candidates. It follows that the most points a single candidate can win is four points, which would occur if the candidate won every matchup. As a Condorcet candidate, Candidate C won all the pairwise matchups against Candidates A, B, D, and E, earning four points.
2. The rest of the candidates lost to Candidate C. The most points a particular candidate could win if they won matchups with each of the other three candidates is three points.
3. Since Candidate C has four points and the rest of the candidates have three points or less, Candidate C is the winner. Therefore, it is not possible for Candidate C to tie or lose.

Your Turn 11.1.11

Suppose there is an election with 26 candidates, A through Z, and that Candidate C is a Condorcet candidate.

How many points did Candidate C win?

What is the greatest number of points that any one of the other candidates could win?

Is it possible for Candidate C to lose or tie?

Let's consider a general case where there are n candidates. One of the candidates is a Condorcet candidate. Since the Condorcet candidate wins all matchups, the Condorcet candidate wins $n-1$ points. Since each of the other candidates lost to the Condorcet candidate, the most a single candidate could win is $n-2$. Since the Condorcet candidate won $n-1$ points and each other candidate won $n-2$ points or fewer, the Condorcet candidate is the winner. You have your answer to the third key question! If there is a Condorcet candidate, that candidate is always the winner.

Approval Voting

The last type of voting system you will consider for your budding democracy is an approval voting system. In this system, each voter may approve any number of candidates without rank or preference for one over another (among the approved candidates), and the candidate approved by the most voters wins. This voting system has aspects in common with plurality voting and Condorcet voting methods, but it has characteristics that distinguish it from both. An approval voting ballot lists the candidates and provides the option to approve or not approve each candidate.

The term "approval voting" was not used until the 1970s Brams, Steven J.; Fishburn, Peter C. (2007), *Approval Voting*, Springer-Verlag, p. xv, ISBN 978-0-387-49895-9, although its use has been documented as early as the 13th century (Brams, Steven J. (April 1, 2006). *The Normative Turn in Public Choice* (PDF) (Speech). Presidential Address to Public Choice Society. New Orleans, Louisiana.) Approval voting has the appeal of being simpler than ranked voting methods. It also allows an individual voter to support more than one candidate equally. This has appeal for those who do not want a split vote among a few mainstream candidates to lead to the election of a fringe candidate. It also has appeal for those who want an underdog to have a chance of success because voters will not worry about wasting their vote on a candidate who is not believed likely to win.

✓ Example 11.1.12: Rock, Paper, Scissors, Lizard, Spock

Suppose that Person A/B/C were just about to give up on their game of Rock, Paper, Scissors when they were joined by Person D who reminded them that their updated version, Rock, Paper, Scissors, Lizard, Spock was a far superior game with the added rules that Lizard eats Paper, Paper disproves Spock, Spock vaporizes Rock, Rock crushes Lizard, Lizard poisons Spock, Spock smashes Scissors, and Scissors decapitates Lizard.

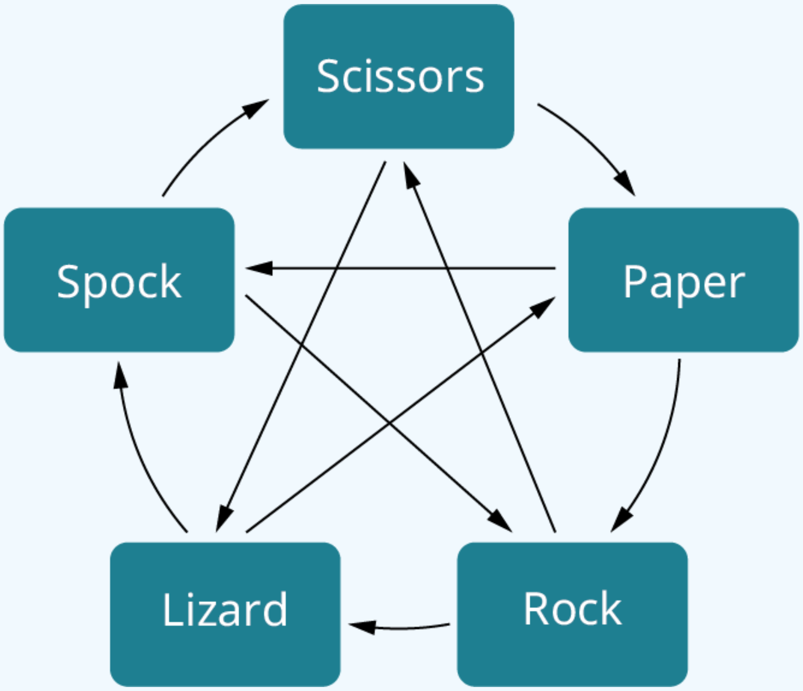


Figure ✓ 11.1.9 Rock, Paper, Scissors, Lizard, Spock Dominance

Person D encourages their friends to hold a new election. This time, for the sake of simplicity, the group decides to use approval voting to determine the best move in the game. The summary of approval ballots for Rock, Paper, Scissors, Lizard, Spock is given in the table below.

Voters	A	B	C	D
Rock	Yes	No	No	No
Paper	No	Yes	No	No
Scissors	No	No	Yes	No
Lizard	No	No	No	Yes
Spock	Yes	Yes	Yes	Yes

Answer

Count the number of approval votes for each candidate by counting the number of “Yes” votes in each row of the table.

- Rock: 1
- Paper: 1
- Scissors: 1
- Lizard: 1
- Spock: 4

Spock is the winning candidate, approved by four voters!

Your Turn 11.1.12

The Chionilis family is trying to decide on a restaurant again, but now they don’t want to deal with multiple runoffs or even ranking. They will use the approval voting method shown in the following table. Each family member will approve their top two choices. Rainbow China won when multiple runoffs were used. Find the winner for tonight’s dinner.

Options	**A**	**R**	**S**	**T**	**G**	**E**	**M**	**D**
Rainbow China	Yes	No	No	Yes	No	No	Yes	Yes
Dough Boys Pizza	Yes	Yes	Yes	Yes	No	Yes	No	Yes
Taco City	No	No	Yes	No	Yes	Yes	No	No
Caribbean Flavor	No	Yes	No	No	Yes	No	Yes	No

Example 11.1.13: The Chionilis Family Is Hungry Again!

The eight members of the Chionilis family—Annette, Rene, Seema, Titus, Galen, Elena, Max and Demitri—have another decision to make. Approval voting worked out nicely the last time. They are going to use it again, but this time, Annette, Rene, Seema, and Galen are feeling a little indecisive. They can't narrow their choice down to two. They will approve their three top choices, but the other family members will only approve two. These choices are reflected in the following table. Determine the restaurant that will be chosen.

Options	A	R	S	T	G	E	M	D
Rainbow China	Yes	Yes	Yes	Yes	Yes	No	Yes	Yes

Options	A	R	S	T	G	E	M	D
Dough Boys Pizza	Yes	Yes	Yes	Yes	No	Yes	No	Yes
Taco City	Yes	No	Yes	No	Yes	Yes	No	No
Caribbean Flavor	No	Yes	No	No	Yes	No	Yes	No

Answer

Count the number of approval votes for each restaurant by counting the number of “Yes” votes in each row.

- Rainbow China: 7
- Dough Boys Pizza: 6
- Taco City: 4
- Caribbean Flavor: 3

This time, Rainbow China won!

Your Turn 11.1.13

What would be the outcome of the election if every member of the Chionilis family approved their top three choices from the table below?

Options	A	R	S	T	G
Rainbow China	1	3	3	1	3
Dough Boys Pizza	2	2	1	2	1
Taco City	3	4	2	4	2
Caribbean Flavor	4	1	4	3	4

Compare and Contrast Voting Methods to Identify Flaws

Wow! We have covered a lot of options for the voting methods. Now, you need to decide which one is best for Imaginaria. Imaginarians might consider characteristics of certain voting systems desirable and others undesirable. In some cases, voters may consider these undesirable traits to be flaws in a voting system that are significant enough to motivate them to reject that system. If you are feeling a bit overwhelmed by this decision, maybe it would help to read about the experiences of others who have faced similar questions.

Consider the 2000 U.S. presidential election in which Green Party candidate Ralph Nader and Reform Party candidate Pat Buchanan were on the ballet running against the mainstream candidates, Democrat Al Gore and Republican George W. Bush. The voting results for Florida are given in Table 11.3.

Table 11.3 Florida Results in the 2000 U.S. Presidential Election (source: https://www.fec.gov/resources/cms-co...ltsbyState.pdf)

Candidate	Party	Votes	Percentage
(G) George W. Bush	Republican	2,912,790	48.85%
(A) Al Gore	Democrat	2,912,253	48.84%
(R) Ralph Nader	Green	97,488	1.63%
(P) Pat Buchanan	Reform	17,484	0.29%
(H) Harry Brown	Libertarian	16,415	0.28%

Candidate	Party	Votes	Percentage
(O) 7 Other Candidates	Other	6,680	0.11%
	Total	5,963,110	

In more than one state, Buchanan was able to split the Republican vote enough to allow Gore to win that state. Nader split the Democrat vote in Florida and New Hampshire by enough votes to prevent Gore from winning those states. Had Gore won either state, he would have had enough electoral votes to win the election. Instead, Bush won. This is an example of a flaw in the plurality system of voting: the spoiler.

A spoiler is a less popular candidate who takes votes from a more popular candidate with similar positions, swinging the race to another candidate with vastly different views that they would not support. This encourages voters not to vote for the candidate that they perceive to be the best, but instead for the candidate they can live with who they perceive to have a better chance of winning. Some voters may prefer a method such as approval voting, which does not have this trait in common with plurality voting.

✓ Example 11.1.14: The Spoiler Controversy

Because the vote counts for George W. Bush and Al Gore differed by only 537 votes, many Democrats blamed Ralph Nader and the Green Party for their loss. Let’s consider how the election results might have differed if the approval voting method had been used.

Use Table 11.3 and the following assumptions to extrapolate the results of an approval method election:

- 100 percent of Pat Buchanan supporters would approve George W. Bush.
- 100 percent of Ralph Nader supporters would approve Al Gore.
- 72 percent of Libertarians would approve George W. Bush.
- 28 percent of Libertarians would approve Al Gore (as was roughly the known percentage at the time according to the Cato Institute).
- 50 percent of the supporters of other candidates would approve George Bush while 50 percent would approve Al Gore.

Answer

Step 1: Create a summary of approval ballots based on the given assumptions. For the Libertarian candidate, 72 percent of 16,415 of the votes is $0.72(16,415) = 11,819$ and 28 percent is $0.28(16,415) = 4,596$. For the other candidates, 50 percent of the votes is $0.50(6,680) = 3,340$.

Number of Votes	2,912,790 (G)	2,912,253 (A)	97,488 (R)	17,484 (B)	11,819 (72% H)	4,596 (28% H)	3,340 (50% O)	3,340 (50% O)
(G) George W. Bush	Yes	No	No	Yes	Yes	No	No	Yes
(A) Al Gore	No	Yes	Yes	No	No	Yes	Yes	No
(R) Ralph Nader	No	No	Yes	No	No	No	No	No
(P) Pat Buchanan	No	No	No	Yes	No	No	No	No
(H) Harry Brown	No	No	No	No	Yes	Yes	No	No
(O) 7 Other Candidates	No	No	No	No	No	No	Yes	Yes

Step 2: Count the number of approval votes for each candidate.

- George W. Bush: $2,912,790+17,484+11,819+3340=2,945,433$
- Al Gore: $2,912,253+97,488+4,596+3,340=3,017,677$
- Ralph Nader: $97,488$
- Pat Buchanan: $17,484$
- Harry Brown: $11,819+4,596=16,415$
- Other Candidates: $6,680$

In this scenario, Al Gore is the winner.

Your Turn 11.1.14

Extrapolate the results of an approval method election using Table 11.3, the assumptions from Example 11.14, and the additional assumptions that the supporters of Al Gore would all approve Ralph Nader and that half of the supporters of George Bush would approve Pat Buchanan while half would approve Harry Brown.

The results in Example 11.14 and Your Turn 11.14 highlight one of the characteristics of approval voting. Ralph Nader moved up from a distant third place finish to a close second place finish when Al Gore's supporters approved him on their ballots. In this way, fringe candidates have a better chance of winning, which some voters consider a flaw but others consider a benefit.

Another aspect of approval voting systems that is a concern to many voters is that candidates in approval elections might encourage their loyal supporters to approve them and only them to avoid giving support to any other candidate. If this occurred, the election in effect becomes a traditional plurality election. This is a flaw that cannot occur in an instant runoff system since all candidates are ranked.

Example 11.1.15: Three Habitable Planets

In the future, humans have explored distant solar systems and found three habitable planets which could be colonized. Since it will take all available resources to colonize one planet, humans must agree on the planet. Planet A has the most comfortable climate and most plentiful resources, but it is the farthest from Earth making travel to the planet a challenge. Planet B is half the distance but will require more resources to make comfortable. Planet C is the least suitable of the three and terraforming will be required, but it is close enough to make travel between Earth and Planet C possible on a more regular basis. The table below provides the voter preferences for the colonization of each planet.

Percentage of Voters	45%	15%	40%
Planet A	1	3	3
Planet B	2	1	2
Planet C	3	2	1

If the entire population were able to vote, determine the winning planet using each of the methods listed below.

1. Plurality
2. Ranked-choice method
3. Borda count

Answer

1. The plurality method only considers the top choice of each voter. By this system, Planet A has 45 percent of the vote, Planet B has 15 percent of the vote, and Planet C has 40 percent of the vote. Planet A wins.
2. Using either instant runoff or a two-round system, Planet B with only 15 percent of the vote will be eliminated in the first round. In Round 2, the 15 percent that voted for Planet B would vote for their second choice, Planet C. This leaves Planet A with 45 percent and Planet C with 55 percent. Planet C has a majority and wins the election.
3. To find the Borda score for each candidate, imagine there are exactly 100 voters. Then the summary of ranked ballots looks like:

Out of 100 Voters	45	15	40
Planet A	1	3	3
Planet B	2	1	2
Planet C	3	2	1

The Borda score for each candidate is as follows:
Planet A: $(3-1)(45)+(3-3)(15)+(3-3)(40)=90$
Planet B: $(3-2)(45)+(3-1)(15)+(3-2)(40)=115$
Planet C: $(3-3)(45)+(3-2)(15)+(3-1)(40)=95$
Planet B wins.

Your Turn 11.1.15

The juniors at a high school in Central Florida are voting for a theme park to visit for an end of the year field trip. The options are Disney's four theme parks: Animal Kingdom, Magic Kingdom, EPCOT Center, or Hollywood Studios.

Percentage of Voters	35%	25%	30%
Animal Kingdom	1	3	4
Magic Kingdom	2	1	3
Epcot Center	3	4	1
Hollywood Studios	4	2	2

Determine the outcome of the vote by each of the given voting methods.

1. Plurality
2. Ranked-choice method
3. Borda count

The election in Example 11.1.15 involves a scenario in which there are two extreme candidates, Planet A and Planet B, and a moderate candidate, Planet C. The supporters of the extreme candidates prefer the moderate candidate to the other extremist ones. This makes Planet C a compromise candidate. In this case, both the plurality method and ranked-choice voting resulted in the election of one of the extreme candidates, but the Borda count method elected the compromise candidate in this scenario. Depending on a person's perspective, this may be perceived as a flaw in either ranked-choice and plurality systems, or the Borda count method.

In Fairness in Voting Methods, we will analyze the fairness of each voting system in greater detail using objective measures of fairness.

Tech Check: Voting Calculators

It is possible to create Excel spreadsheets that complete the calculations necessary to determine the winner of an election by various voting methods. In some cases, this work has already been done and posted online. As you practice applying the various voting methods that could be used in Imaginaria, quick Internet search will lead to sites such as Ms. Hearn Math with free specialty calculators.

These sites can be a great way to check your results!

Check Your Understanding

1. Name three voting methods that use a ranked ballot.
2. Determine whether the following statement is true or false: The same ranked ballots may result in a different winner depending on which voting method is used.
3. Determine whether the following statement is true or false. A majority candidate is always a Condorcet candidate.
4. The ________________ method is a system of voting using ranked ballots in which each candidate is awarded points corresponding to the number of candidates ranked lower on each ballot.
5. The _________________ method is a system of voting using ranked ballots (or multiple elections) in which each candidate receives a point for each candidate they would beat in a one-on-one election and half a point for each candidate they would tie.
6. The ___________________ method is a runoff voting system in which only the candidate(s) with the very least votes are eliminated.
7. Explain the differences between two-round voting and ranked-choice voting.

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11.1.0: Exercises

For the following exercises, identify the winning candidate based on the described voter profile, if possible. If it is not possible, state so. Explain your reasoning.

? Exercise 11.1.0.1

In a plurality election, the candidates have the following vote counts: A 125, B 132, C 149, D 112.

? Exercise 11.1.0.2

In the first round of a ranked-choice election with three candidates—A, B, and C. Candidate A received 55 first place rankings; Candidate B received 25; and Candidate C received 30.

? Exercise 11.1.0.3

The pairwise matchup points for each candidate were: A 1, B $1\frac{1}{2}$, D $\frac{1}{2}$.

? Exercise 11.1.0.4

In a Borda count election, the candidates have the following Borda scores: A 15, B 11, C 12, D 16.

? Exercise 11.1.0.5

There is a pairwise comparison election with candidates A, B, and C. Candidate A had the most first choice rankings, Candidate B has the highest Borda score, and Candidate C is a Condorcet candidate.

? Exercise 11.1.0.6

In the first round of a ranked-choice election with three candidates—A, B, and C—Candidate A received 20 first place rankings, Candidate B received 25, and Candidate C received 30.

For the following exercises, use the table.

O'Malley	De La Fuente	Clinton	Sanders	Other
110,227	67,331	17,174,432	13,245,671	322,276

Popular Vote in the 2016 U.S. Democratic Presidential Primary (source: Federal Election Commission, *Federal Elections 2016 Report*)

? Exercise 11.1.0.7

Calculate the number of votes required to have a majority of the popular vote in the 2016 U.S. Democratic Presidential Primary.

? Exercise 11.1.0.8

Which candidate had a plurality? Did this candidate have a majority?

For the following exercises, use the given table.

Candidate	Votes
Bush	281,189

Candidate	Votes
Trump	13,783,037
Cruz	7,455,780
Rubio	3,354,067
Carson	822,242
Kasich	4,198,498
Other	337,714

Popular Vote in the 2016 U.S. Republican Presidential Primary (source: Federal Election Commission, *Federal Elections 2016 Report*)

? Exercise 11.1.0.9

Calculate the number of votes required to have a majority of the popular vote in the 2016 U.S. Republican Presidential Primary.

? Exercise 11.1.0.10

Which candidate had a plurality? Did this candidate have a majority?

For the following exercises, use Table 11.4 and Table 11.5.

? Exercise 11.1.0.11

Suppose the Republican Primary in 2016 was a two-round system. Would there be a second round? Why or why not? If so, which candidates would advance to the second round?

? Exercise 11.1.0.12

Suppose the Democratic Primary in 2016 was a two-round system. Would there be a second round? Why or why not? If so, which candidates would advance to the second round?

? Exercise 11.1.0.13

Suppose the Democratic Primary in 2016 used the Hare method. Would there be a second round? Why or why not?

? Exercise 11.1.0.14

Suppose the Republican Primary in 2016 used the Hare method. Would there be a second round? Why or why not?

For the following exercises, use the following table and the Hare method.

Options	A	B	C	D	E
Candidate 1	1	3	3	1	3
Candidate 2	2	1	1	2	4
Candidate 3	3	4	2	4	1
Candidate 4	4	2	4	3	2

? Exercise 11.1.0.15

How many votes are needed to win by the Hare method?

? Exercise 11.1.0.16

How many votes does each candidate receive in Round 1?

? Exercise 11.1.0.17

Which candidates advance to Round 2?

? Exercise 11.1.0.18

How many votes does each remaining candidate receive in Round 2?

? Exercise 11.1.0.19

Will there be a third round? Why or why not?

? Exercise 11.1.0.20

Which candidate wins the election?

For the following exercises, use the sample summary of ranked ballots in the given table.

Number of Ballots	10	20	15	5
Option A	1	4	3	4
Option B	2	3	4	2
Option C	4	2	1	3
Option D	3	1	2	1

Sample Summary of Ranked Ballots

? Exercise 11.1.0.21

How many votes were recorded, and how many are required to have a majority?

? Exercise 11.1.0.22

How many voters indicated that Option A was their first choice?

? Exercise 11.1.0.23

How many voters indicated that Option B was their first choice?

? Exercise 11.1.0.24

How many voters indicated that Option A was their last choice?

Exercise 11.1.0.25

How many voters indicated that Option B was their last choice?

Exercise 11.1.0.26

Use ranked-choice voting to determine the two candidates in the final round and the number of votes they each receive in that round.

Exercise 11.1.0.27

Is there a winning candidate? If so, which candidate? Justify your answer.

Suppose that 55 Star Wars fans were asked to vote for their favorite new Star Wars character. They were given a ranked ballot, and the results are shown in the table. Use this table and ranked-choice voting for the following exercises.

Number Of Ballots	7	6	10	8	4	5	6	7	2
Finn	3	5	1	3	2	4	2	2	1
Rey	2	1	3	2	4	3	3	1	3
Poe	1	6	4	1	5	5	5	4	2
BB8	4	2	2	4	6	1	6	5	4
Rose	6	4	5	6	3	2	1	3	6
Kylo	5	3	6	5	1	6	4	6	5

Favorite New Heroes in Star Wars Sequels Ballot Preferences

Exercise 11.1.0.28

How many votes does each candidate get on the first round of voting?

Exercise 11.1.0.29

How many votes are required to get a majority?

Exercise 11.1.0.30

Which candidates remain in the final round, and how many votes do they have?

Exercise 11.1.0.31

Who is the winner of the election?

Refer to Table 11.6 for the following exercises.

Exercise 11.1.0.32

Find the Borda score for each candidate.

? Exercise 11.1.0.33

Compare your results from question 32 to those from question 26. Compare the winner and the second-place candidate using the Borda count method to those using the ranked-choice method. Are they the same?

Refer to Table 11.7 for the following exercises.

? Exercise 11.1.0.34

Find the Borda score for each candidate.

? Exercise 11.1.0.35

Compare your results from question 34 to those from question 30. Compare the winner and the second-place candidate using the Borda count method to those using the ranked-choice method. Are they the same?

For the following exercises, use the table below.

Number of Ballots	100	80	110	105	55
Option A	1	1	4	4	2
Option B	2	2	2	3	1
Option C	4	4	1	1	4
Option D	3	3	3	2	3

? Exercise 11.1.0.36

Do any candidates appear to be divisive candidates? Justify your answer.

? Exercise 11.1.0.37

Do any candidates appear to be compromise candidates? Justify your answer.

? Exercise 11.1.0.38

How many votes are required for a majority?

? Exercise 11.1.0.39

Which candidate is eliminated first by the ranked-choice method?

? Exercise 11.1.0.40

Which candidate is eliminated second by the ranked-choice method?

? Exercise 11.1.0.41

Which candidate is the winner by the ranked-choice method?

? Exercise 11.1.0.42

What are the Borda scores for each candidate?

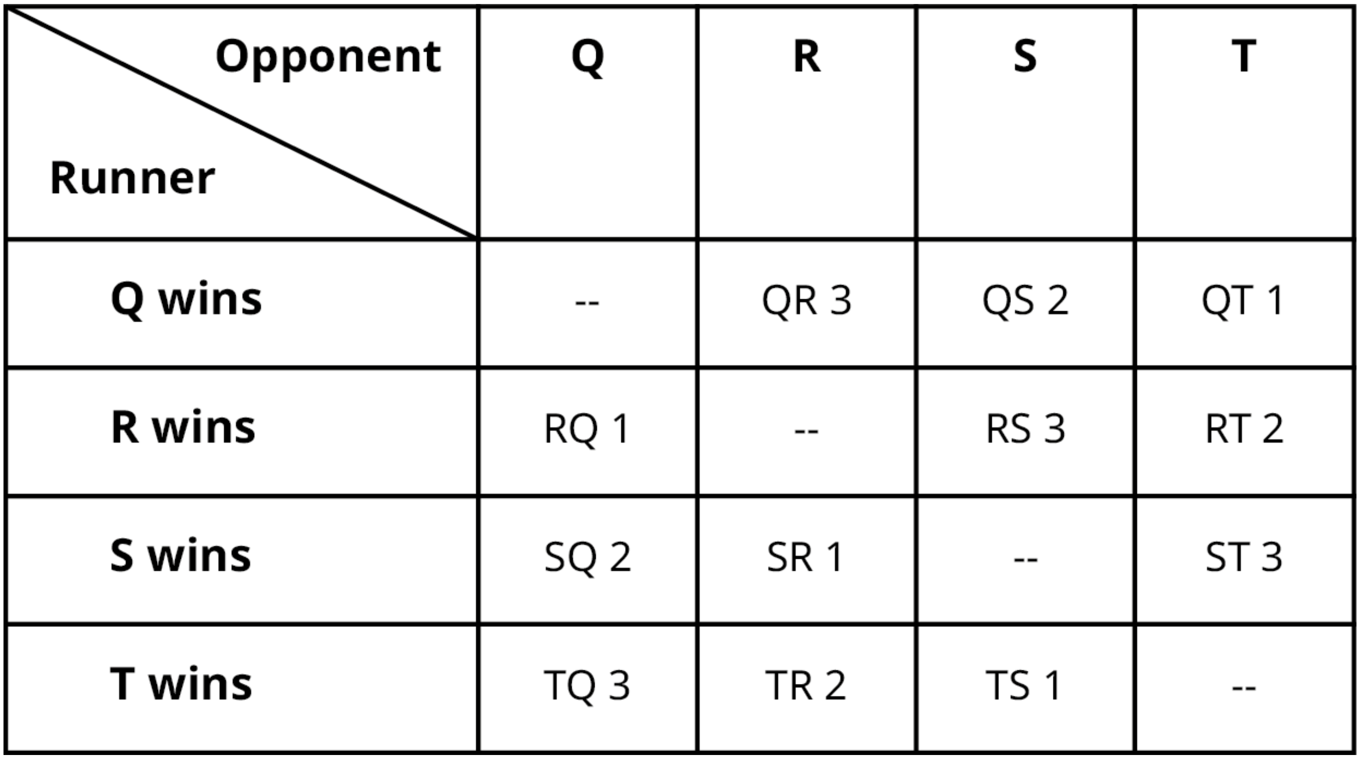
? Exercise 11.1.0.43

Which candidate is the winner by the Borda count method?

? Exercise 11.1.0.44

Which method resulted in a win for the compromise candidate: ranked-choice voting or the Borda count method or both?

Use the pairwise comparison matrix in the given figure for the following exercises.



Opponent Runner	Q	R	S	T
Q wins	--	QR 3	QS 2	QT 1
R wins	RQ 1	--	RS 3	RT 2
S wins	SQ 2	SR 1	--	ST 3
T wins	TQ 3	TR 2	TS 1	--

Pairwise Comparison Matrix for Candidates Q, R, S and T

? Exercise 11.1.0.45

Analyze the pairwise comparison matrix. Display the pairings in a table and indicate the winner of each matchup by marking an ☒ through the losing matchups and a single slash through the ties.

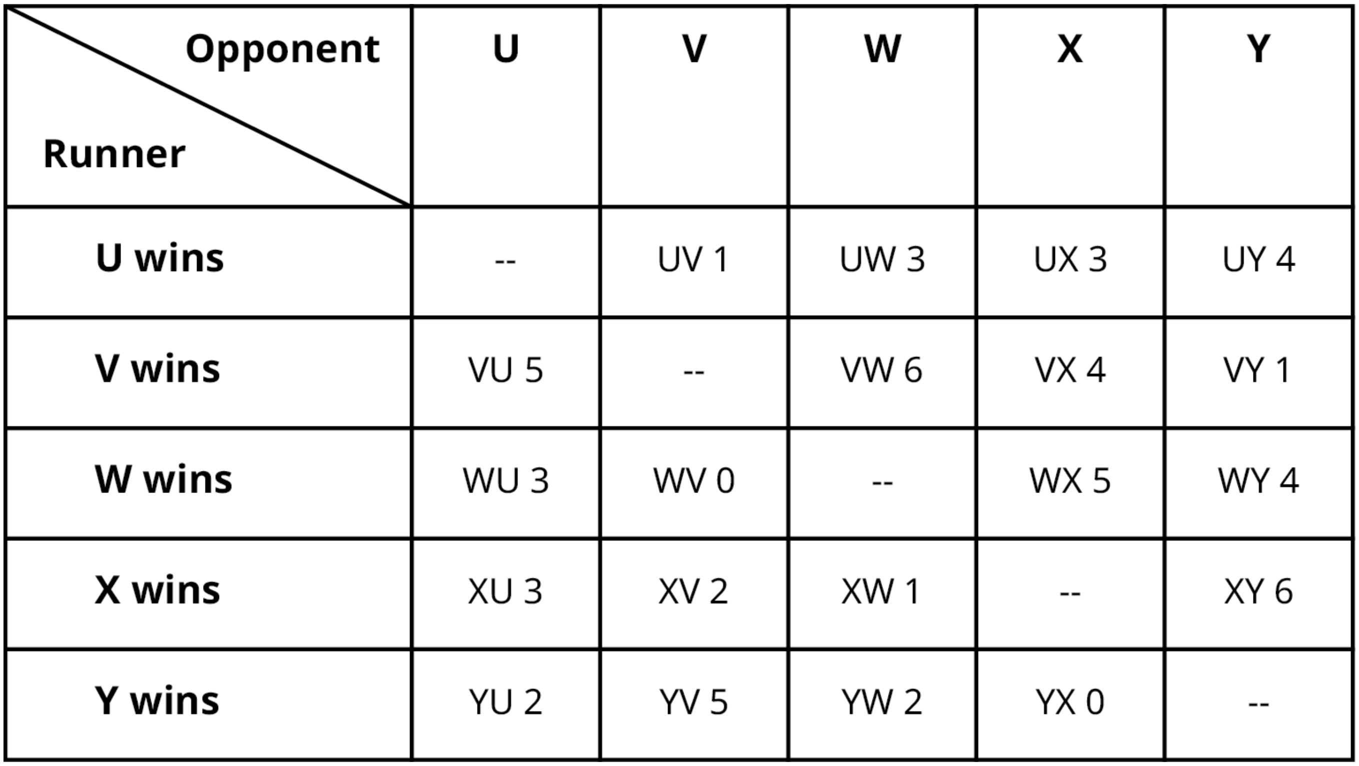
? Exercise 11.1.0.46

Calculate the points received by each candidate in the pairwise comparison matrix.

? Exercise 11.1.0.47

Determine whether there is a winner of the pairwise comparison election represented by the matrix. If there is a winner, determine whether the winner is a Condorcet candidate.

Use the pairwise comparison matrix in the given figure for the following exercises.



Runner \ Opponent	U	V	W	X	Y
U wins	--	UV 1	UW 3	UX 3	UY 4
V wins	VU 5	--	VW 6	VX 4	VY 1
W wins	WU 3	WV 0	--	WX 5	WY 4
X wins	XU 3	XV 2	XW 1	--	XY 6
Y wins	YU 2	YV 5	YW 2	YX 0	--

Pairwise Comparison Matrix for Candidates U, V, W, X, and Y

Exercise 11.1.0.48

Analyze the pairwise comparison matrix. Display the pairings in a table and indicate the winner of each matchup.

Exercise 11.1.0.49

Calculate the points received by each candidate in the pairwise comparison matrix.

Exercise 11.1.0.50

Determine whether there is a winner of the pairwise comparison election represented by the matrix. If there is a winner, determine whether the winner is a Condorcet candidate and explain your reasoning.

Exercise 11.1.0.51

In J.K. Rowling’s *Harry Potter* series, Albus Dumbledore was the headmaster of Hogwarts for many years. Imagine that an election is to be held to find his successor. Severus Snape, the head of Slytherin House, will be running against the heads of Gryffindor and Ravenclaw, Minerva McGonagall and Filius Flitwick. Use the preference rankings for each candidate in the following table to construct a pairwise comparison matrix.

Percentage of Vote	25%	40%	35%
(S) Snape	1	3	3
(M) McGonagall	3	1	2
(F) Flitwick	2	2	1

? Exercise 11.1.0.52

Analyze the pairwise comparison matrix you constructed for question 51. Display the pairings in a table and indicate the winner of each matchup.

? Exercise 11.1.0.53

Use the pairwise comparison matrix from questions 51 and 52 to find the number of points earned by each candidate. Who is the winner by the pairwise comparison method?

? Exercise 11.1.0.54

Is the winner of the Hogwarts headmaster election a Condorcet candidate? Explain how you know.

? Exercise 11.1.0.55

The women of *The Big Bang Theory* decide to hold their own approval voting election to determine the best option in Rock, Paper, Scissors, Lizard, Spock. Use the summary of their approval ballots in the table below to determine the number of votes for each candidate. Determine the winner, or state that there is none.

Voters	Penny	Bernadette	Amy
Rock	Yes	No	No
Paper	Yes	Yes	No
Scissors	Yes	Yes	Yes
Lizard	No	No	No
Spock	Yes	No	Yes

For the following exercises, use the table below.

Percentage of Vote	40%	35%	25%
Candidate A	1	3	2
Candidate B	2	1	3
Candidate C	3	2	1

? Exercise 11.1.0.56

Which candidate is the winner by the ranked-choice method?

? Exercise 11.1.0.57

Suppose that they used the approval method and each voter approved their top two choices. Which candidate is the winner by the approval method?

? Exercise 11.1.0.58

Which candidate is the winner by the Borda count method?

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11.2: Fairness in Voting Methods

Figure 11.2.1: Citizens strive to ensure their voting system is fair. (credit: “Governor Votes Early” by Maryland GovPics/Flickr, CC BY 2.0)

Learning Objectives

1. Compare and contrast fairness of voting using majority criterion.
2. Compare and contrast fairness of voting using head-to-head criterion.
3. Compare and contrast fairness of voting using monotonicity criterion.
4. Compare and contrast fairness of voting using irrelevant alternatives criterion.
5. Apply Arrow’s Impossibility Theorem when evaluating voting fairness.

Now that we’ve covered a variety of voting methods and discussed their differences and similarities, you might be leaning toward one method over another. You will need to convince the other founders of Imaginaria that your preference will be the best for the country. Before your collaborators approve the inclusion of a voting method in the constitution, they will want to know that the voting method is a fair method. In this section, we will formally define the characteristics of a fair system. We will analyze each voting previously discussed to determine which characteristics of fairness they have, and which they do not. In order to guarantee one ideal, we must often sacrifice others.

The Majority Criterion

One of the most fundamental concepts in voting is the idea that most voters should be in favor of a candidate for a candidate to win, and that a candidate should not win without majority support. This concept is known as the majority criterion.

With respect to the four main ranked voting methods we have discussed—plurality, ranked-choice, pairwise comparison, and the Borda count method—we will explore two important questions:

1. Which of these voting systems satisfy the majority criterion and which do not?
2. Is it always “fair” for a voting system to satisfy the majority criterion?

Keep in mind that this criterion only applies when one of the candidates has a majority. So, the examples we will analyze will be based on scenarios in which a single candidate has more than 50 percent of the vote.

✓ Example 11.2.1: Roommates Choose Fast Food

It's final exams week and seven college students are hungry. They must get food, but from which drive thru? Their preferences are listed in the table below. The majority have listed McDonald's as their top choice. Let's calculate what the results of the election will be using various voting methods.

Voters	A	B	C	D	E	F	G
(M) McDonald's	1	1	1	1	5	5	5
(B) Burger King	2	2	2	2	2	2	2
(T) Taco Bell	4	4	3	4	3	3	1
(O) Pollo Tropical	3	5	4	3	4	1	3
(I) Pizza Hut	5	3	5	5	1	4	4

1. Which restaurant is the winner using the plurality voting method?
2. Which restaurant is the winner using the ranked-choice voting method?
3. Does the majority criterion apply? If so, for which of voting method(s), if any, did the majority criterion fail?

Answer

1. For plurality voting, we only need to count the first-place votes for each candidate. In this case, McDonald's has four first place votes, which is a majority and wins the election automatically.
2. For ranked-choice voting, McDonald's also wins because it has a majority at the end of Round 1.
3. The majority criterion does apply because one candidate had a majority of the first-place votes. The majority criterion did not fail by either method, because in each case the majority candidate won.

Your Turn 11.2.1

Use the information in the table to answer the following questions.

Voters	L	M	N	O	P
Option A	2	2	3	2	1
Option B	1	4	1	4	2
Option C	3	1	4	1	3
Option D	4	1	2	3	4

Which option is the winner when using the plurality voting method?

Which option is the winner when using the ranked-choice voting method?

Does the majority criterion apply? If so, for which voting method(s), if any, did the majority criterion fail?

From Example 11.16, it appears that the plurality and ranked-choice voting methods satisfy the majority criterion. In general, the majority candidate always wins in a plurality election because the candidate that has more than half of the votes has more votes than any other candidate. The same is true for ranked-choice voting; and there will never be a need for a second round when there is a majority candidate. Let's examine how some of the other voting methods stand up to the majority criterion.

✓ Example 11.2.2: Roommates Choose Fast Food

Those seven college students are hungry again! Their preferences haven't changed, as shown below. Let's calculate if the results change when we use different voting methods.

VOTERS	A	B	C	D	E	F	G
(M) McDonald's	1	1	1	1	5	5	5
(B) Burger King	2	2	2	2	2	2	2
(T) Taco Bell	4	4	3	4	3	3	1
(O) Pollo Tropical	3	5	4	3	4	1	3
(I) Pizza Hut	5	3	5	5	1	4	4

1. Which restaurant is the winner using the pairwise comparison voting method?
2. Which restaurant is the winner using the Borda count voting method?
3. Does the majority criterion apply? If so, for which voting method(s), if any, did the majority criterion fail?

Answer

1. For pairwise comparison, notice that McDonald's is a Condorcet candidate because it wins every pairwise comparison. So, McDonald's is the winner.
2. For the Borda count, we must calculate the Borda score for each candidate: McDonald's is 16, Burger King is 21, Taco Bell is 13, Pollo Tropical is 12, Pizza Hut is 8. The winner is Burger King!
3. Yes, the majority criterion applies because McDonald's has the majority of first place votes. The majority criterion only fails using the Borda method.

Your Turn 11.2.2

Use the information in the following table to find the winner using each of the voting methods in parts 1 and 2, then answer the question in part 3.

Voters	L	M	N	O	P
Option A	2	2	3	2	2
Option B	1	4	1	4	1
Option C	3	1	4	1	3
Option D	4	3	2	3	4

Pairwise comparison

Borda count

Does the majority criterion apply? If so, for which voting method(s) did the majority criterion fail?

Example 11.17 demonstrates a concept that we also saw in Borda Count Voting—the Borda method frequently favors the compromise candidate over the divisive candidate. This can happen even when the divisive candidate has a majority, as it did in this example. Although a majority of the voters were in favor of McDonald's, a significant minority was strongly opposed to McDonald's, ranking it last. Since the Borda score includes all rankings, this strong opposition has an impact on the outcome of the election.

Pairwise comparison will always satisfy the majority criterion because the candidate with the majority of first-place votes wins each pairwise matchup. While it is possible for the majority candidate to win by the Borda count method, it is not guaranteed. So, the Borda method fails the majority criterion. A summary of each voting method as it relates to the majority criterion is found in the following table.

Voting Method	Majority Criterion
Plurality	Satisfies
Ranked-choice	Satisfies
Pairwise comparison	Satisfies
Borda count	Violates

If you prefer the Borda method, you might argue that its failure to satisfy the majority criterion is actually one of its strengths. As we saw in Example 11.16 and Example 11.17, the majority have the power to vote for their own benefit at the expense of the minority. While four students were very enthusiastic about McDonald’s, three students were strongly opposed to McDonald’s. It is reasonable to say that the better option would be Burger King, the compromise candidate, which everyone ranked highly and no one strongly opposed. The Borda method is designed to favor a candidate that is acceptable to the population as a whole. In this way, the Borda method avoids a downfall of strict majority rule known as the tyranny of the majority, which occurs when a minority of a population is treated unfairly because their situation is different from the situation of the majority.

The people of Imaginaria should know that the power of the majority to vote their will has serious implications for other groups. For example, according to the UCLA School of Law Williams Institute, the LGBTQ+ community in the United States makes up approximately 4.5 percent of the population. When elections occur that include issues that affect the LGBTQ+ community, members of the LGBTQ+ community depend on the 95.5 percent of the population who do not identify as LGBTQ+ to consider their perspectives when voting on issues such as same-sex marriage, the use of public restrooms by transgender people, and adoption by same-sex couples.

People in Mathematics: Robert Dahl

In his book, *Democracy and Its Critics*, Robert Dahl wrote, “If a majority is not entitled to do so, then it is thereby deprived of its rights; but if a majority is entitled to do so, then it can deprive the minority of its rights.” Dahl was a renowned political theorist, but he is also considered to be a mathematician since his work utilizes ideas from an area of mathematics known as Game Theory (Mathematics Genealogy Project, NDSU Department of Mathematics with the American Mathematical Society).

Who Knew?: Three Branches of Government

Concerns about the consequences of majority rule are not new. In 1788, John Adams warned of the consequences of majority rule and he argued for three branches of government as a way to temper them. In the early 1800s, a young French aristocrat named Alexis de Tocqueville toured the United States and wrote *Democracy in America*, which focused on the impact of democracy on political and civil societies. He observed, even then, the dominance of the white majority over the Indigenous people and enslaved people, which was perpetuated by majority rule.

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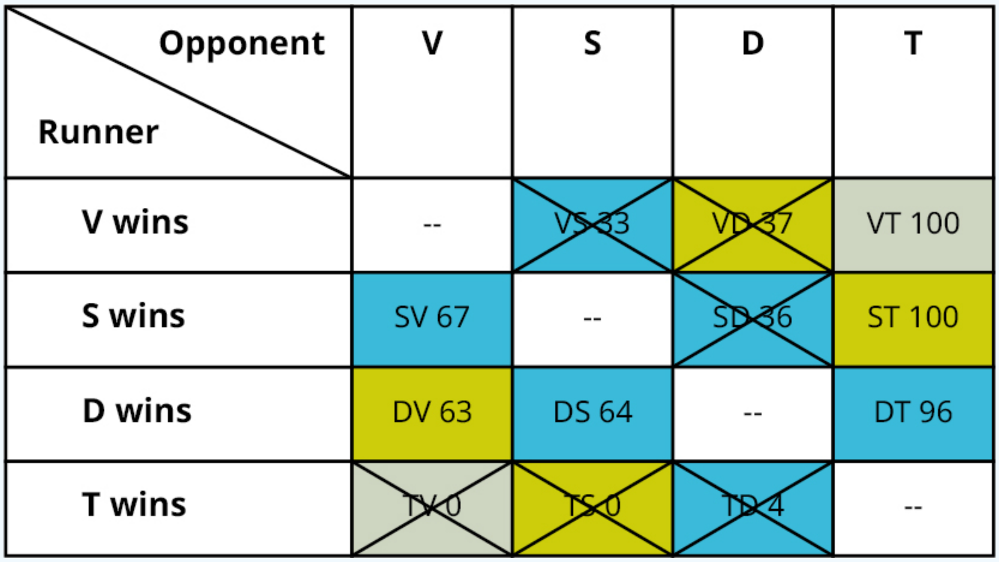
Head-to-Head Criterion

Another fairness criterion you must consider as you select a voting method for Imaginaria is the Condorcet criterion, also known as the head-to-head criterion. An election method satisfies the Condorcet criterion provided that the Condorcet candidate wins the election whenever a Condorcet candidate exists. A Condorcet method is any voting method that satisfies the Condorcet criterion.

Recall from Three Key Questions that not every election has a Condorcet candidate; the Condorcet criterion will not apply to every election. Also recall that a Condorcet candidate cannot lose an election by pairwise comparison. So, the pairwise comparison voting method is said to satisfy the Condorcet criterion.

✓ Example 11.2.3: Spending Tax Refund

A survey asked a random sample of 100 people in the United States to rank their priorities for spending their tax refund. The options were (V) go on vacation, (S) put into savings, (D) pay off debt, or (T) other. The pairwise comparison matrix for the results is in Figure 11.12. Determine whether the Condorcet criterion applies.



Opponent Runner	V	S	D	T
V wins	--	~~VS 33~~	~~VD 37~~	VT 100
S wins	SV 67	--	~~SD 36~~	ST 100
D wins	DV 63	DS 64	--	DT 96
T wins	~~TV 0~~	~~TS 0~~	~~TD 4~~	--

Figure ✓ 11.2.2 Pairwise Comparison Matrix for Tax Refund Spending

Answer

The Condorcet criterion only applies when there is a Condorcet candidate. “Pay off debt” (D) is a Condorcet candidate because D wins every matchup. Yes, the Condorcet criterion applies to this election.

Your Turn 11.2.3

Determine whether the Condorcet criterion applies based on the summary of ranked ballots given in the table below.

Votes	3	2
Option A	1	3
Option B	2	1
Option C	3	2

Example 11.2.4: Spending Tax Refund

Let’s return to the survey about tax refund spending from Example 11.18. We know that the Condorcet criterion applies because Option D, “Pay off debt,” is a Condorcet candidate, which wins every pairwise match up.

Use the information in the ballot summary from the table below to find the winner and determine whether the Condorcet criterion is satisfied in this election when each of the following voting methods are used.

Votes	33	32	31	4
On a vacation (V)	1	3	3	2
Put into savings (S)	3	1	2	1
Pay off debt (D)	2	2	1	4
Other (T)	4	4	4	3

1. Plurality
2. Ranked-choice voting
3. Borda count

Answer

1. V wins 33 first place votes; S, 36; D, 31; and T, 0. So candidate S, “Put into savings,” has a plurality and wins. Since the Condorcet candidate D didn’t win, the Condorcet criterion is violated.
2. Use the steps outlined in Ranked-Choice Voting for determining the winner of an election by ranked-choice voting, the application of the Hare method in which instant runoffs are used.

   Step 1: The number of votes needed to achieve a majority is 51.

   Step 2: As illustrated in part 1, no candidate has a majority of first-place votes; so the candidate with the fewest votes, T, must be eliminated.

   Step 3: Reallocate votes to the remaining candidates for the second round:

Votes	33	32	31	4
On a vacation (V)	1	3	3	2
Put into savings (S)	3	1	2	1
Pay off debt (D)	2	2	1	3

   Step 4: Repeat the process from Step 2. Count the first-place votes for each candidate: V has 33 votes, S has 36 votes, D has 31. votes. Eliminate candidate D, “Pay off debt,” for the third round.

   Step 5: Repeat the process from Step 3. Reallocate votes to the remaining candidates for the third round:

Votes	33	32	31	4

Votes	33	32	31	4
On a vacation (V)	1	2	2	2
Put into savings (S)	2	1	1	1

Step 6: Repeat the process from Step 2 one last time. Count the first-place votes for each candidate: V has 33, S has 67. Candidate S, “Put into savings,” has a majority and wins. Since the Condorcet candidate D candidate D, “Pay off debt,” didn’t win, the Condorcet criterion is violated.

3. Calculate the Borda score for each candidate.

V: $33(4-1)+32(4-3)+31(4-3)+4(4-2)=170$

S: $33(4-3)+32(4-1)+31(4-2)+4(4-1)=203$

D: $33(4-2)+32(4-2)+31(4-1)+4(4-4)=223$

T: $33(4-4)+32(4-4)+31(4-4)+4(4-3)=4$

Candidate D, “Pay off debt,” has the highest Borda score and wins. Since D was the Condorcet candidate, this election satisfies the Condorcet criterion.

Your Turn 11.2.41

Use the summary of ranked ballots below to find the winner and determine whether the Condorcet criterion is satisfied when each of the following voting methods are used. Recall that Option A is a Condorcet candidate.

Votes	3	2
Option A	1	3
Option B	2	1
Option C	3	2

1. Plurality
2. Ranked-choice voting
3. Borda count

As we have seen, the plurality method, ranked-choice voting, and the Borda count method each fail the Condorcet criterion in some circumstances. Of the four main ranked voting methods we have discussed, only the pairwise comparison method satisfies the Condorcet criterion every time. A summary of each voting method as it relates to the Condorcet criterion is found in the following table.

Voting Method	Condorcet Criterion
Plurality	Violates
Ranked-choice	Violates
Pairwise comparison	Satisfies
Borda count	Violates

Monotonicity Criterion

The citizens of Imaginaria might be surprised to learn that it is possible for a voter to cause a candidate to lose by ranking that candidate higher on their ballot. Is that fair? Most voters would say, “Absolutely not!!” This is an example of a violation of the fairness criterion called the **monotonicity criterion**, which is satisfied when no candidate is harmed by up-ranking nor helped by down-ranking, provided all other votes remain the same.

Consider a scenario in which voters are permitted a first round that is not binding, and then they may change their vote before the second round. Such a first round can be called a “straw poll.” Now, let’s suppose that a particular candidate won the straw poll. After that, several voters are convinced to increase their support, or up-rank, that winning candidate and no voters decrease that support. It is reasonable to expect that the winner of the first round will also win the second. Similarly, if some of the voters decide to decrease their support, or down-rank, a losing candidate, it is reasonable to expect that candidate will still lose in the second round.

You might be wondering why it’s called the monotonicity criterion. In mathematics, the term monotonicity refers to the quality of always increasing or always decreasing. For example, a person’s age is monotonic because it always increases, whereas a person’s weight is not monotonic because it can increase or decrease. If the only changes to the votes for a particular candidate after a straw poll are in one direction, this change is considered monotonic.

If you are going to make an informed decision about which voting method to use in Imaginaria, you need to know which of the four main ranked voting methods we have discussed—plurality, ranked-choice, pairwise comparison, and the Borda count method—satisfy the monotonicity criterion.

✓ Example 11.2.5: Favorite Dog Breed by Plurality

The local animal shelter is having a vote-by-donation charity event. For a $10 donation, an individual can complete a ranked ballot indicating their favorite large dog breed: standard poodle, golden retriever, Labrador retriever, or bulldog. Use the summary of ballots below to answer each question.

Votes	42	53	61	24
(S) Standard Poodle	1	3	2	1
(G) Golden Retriever	3	1	4	4
(L) Labrador Retriever	4	2	1	2
(B) Bulldog	2	4	3	3

1. Determine the winner of the election by plurality.
2. Suppose that the 53 voters in the second column increased their ranking of the winner by 1. Determine the winner by plurality with the new rankings.
3. Does this election violate the monotonicity criterion?
4. Do you think the result of part 3 is also true for plurality voting and the monotonicity criterion in general? Why or why not?

Answer

1. The number of votes for each candidate are: S 66, G 53, L 61, and B 0. The winner is the standard poodle.
2. If the 53 voters in the second column rank S as 2 and L as 3, then the number of votes for each candidate are: S with 66, G with 53, L with 61, and B with 0. The winner is still the standard poodle.
3. This election does not violate the monotonicity criterion because the winner was not hurt by up-ranking.
4. In general, increasing the ranking for a winner of a plurality election will either leave them with the same or more first place votes while leaving the other candidates with the same or fewer first place votes. So a plurality election will never violate the monotonicity criterion.

Your Turn 11.2.5

Use the Favorite Large Dog Breed Ballot Summary to answer each question.

Determine the winner of the election by Borda count.

Suppose that the 61 voters in the third column increased their ranking of the winner by 1. Determine the winner by Borda count with the new rankings.

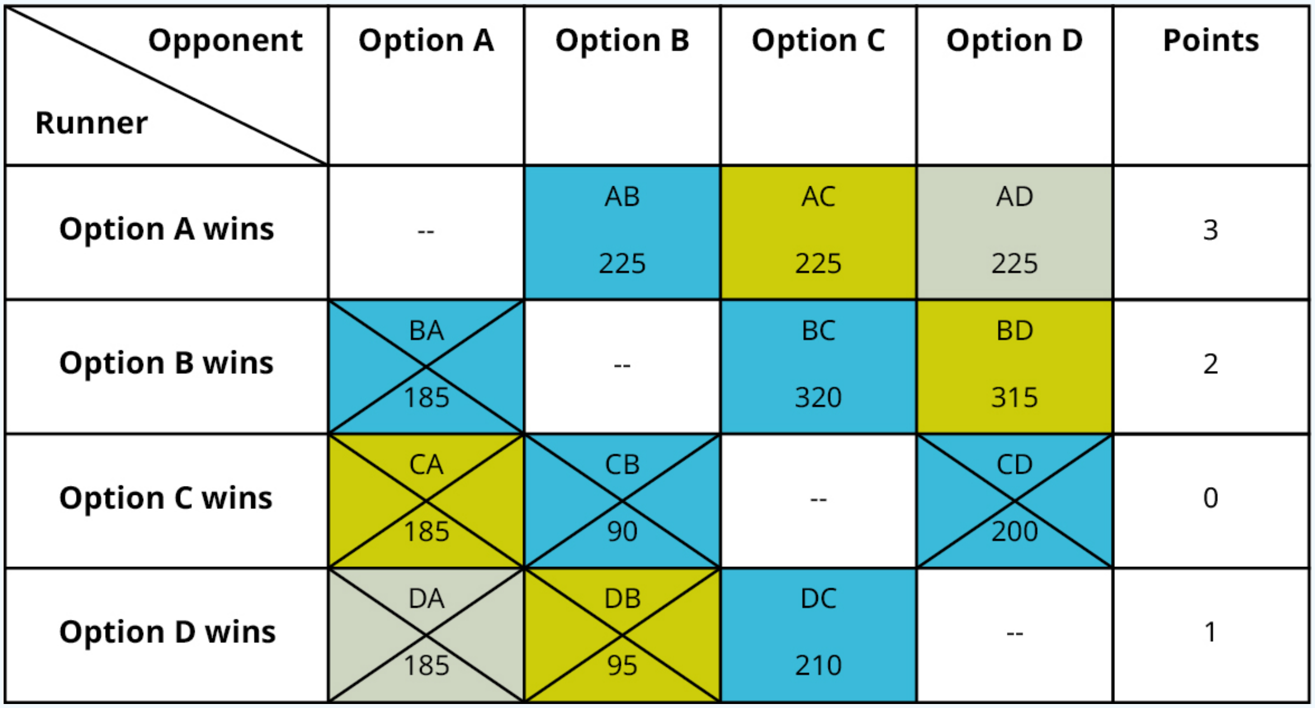
Does this election violate the monotonicity criterion?

Do you think the result of part 3 is true for Borda count and the monotonicity criterion in general? Why or why not?

✓ Example 11.2.6: Favorite Dog Breed by Pairwise Comparison

Earlier, we discovered that the summary of ranked ballots shown in the table below results in the pairwise comparison matrix in Figure 11.13. Use this information to answer the questions.

Number of Ballots	95	90	110	115
Option A	4	4	1	1
Option B	2	2	2	2
Option C	3	1	3	4
Option D	1	3	4	3



Opponent Runner	Option A	Option B	Option C	Option D	Points
Option A wins	--	AB 225	AC 225	AD 225	3
Option B wins	BA 185	--	BC 320	BD 315	2
Option C wins	CA 185	CB 90	--	CD 200	0
Option D wins	DA 185	DB 95	DC 210	--	1

Figure ✓ 11.2.3 Analyzed Pairwise Comparison Matrix for Sample Summary of Ranked Ballots

1. Determine the winner of the election by the pairwise comparison method.
2. Suppose that the 95 voters in the first column increased their ranking of the winner by 1. Determine the winner by the pairwise comparison method with the new rankings.
3. Does this election violate the monotonicity criterion?
4. Do you think the result of part 3 is true for the pairwise comparison method and the monotonicity criterion in general? Why or why not?

Answer

1. By the pairwise comparison method, Option A wins with three points.
2. If the 95 voters in the first column of Figure 11.13 increased their ranking of the winner by 1, then C would fall into fourth place and A would move up to third place on those ballots. This would only affect the matchup between A and C, and the result would be that A would gain 95 votes while C would lose 95 votes. This means A would have 320 votes and C would have 90. Since A already was ahead of C, this just puts A further ahead and causes no change to the election results.
3. Since the winner A is not hurt by an up-rank, and the loser C is not helped by a down-rank, this election is fair by the monotonicity criterion.

4. Yes, the monotonicity criterion would be satisfied by the pairwise comparison method, because an up-rank of the winner can never decrease the number of pairwise wins. Similarly, a down-rank can never increase the number of pairwise wins.

Your Turn 11.2.6

Use the Favorite Large Dog Breed Ballot Summary to answer each question.

Determine the winner of the election by the ranked-choice method.

Suppose that the 24 voters in the last column of the table increased their ranking of the winner by 1. Determine the winner by the ranked-choice method with the new rankings.

Does this election violate the monotonicity criterion?

Do you think the result in question 3 is true for the ranked-choice method and the monotonicity criterion in general? Why or why not?

The last few examples illustrate that the plurality method, pairwise comparison voting, and the Borda count method each satisfy the monotonicity criterion. Of the four main ranked voting methods we have discussed, only the ranked-choice method violates the monotonicity criterion. A summary of each voting method as it relates to the Condorcet criterion is found in the table below.

Voting Method	Monotonicity Criterion
Plurality	Satisfies
Ranked-choice	Violates
Pairwise comparison	Satisfies
Borda count	Satisfies

Irrelevant Alternatives Criterion

We have covered a lot about voting fairness, but there is one more fairness criterion that you and the other Imaginarians should know. Consider this well-known anecdote that is sometimes attributed to the American philosopher Sidney Morgenbesser:

A man is told by his waiter that the dessert options this evening are blueberry pie or apple pie. The man orders the apple pie. The waiter returns and tells him that there is also a third option, cherry pie. The man says, "In that case, I would like the blueberry pie." (*Gaming the Vote: Why Elections Aren't Fair (and What We Can Do About It)*, William Pound stone, p. 50, ISBN 0-8090-4893-0)

This story illustrates the concept of the Irrelevant Alternatives Criterion, also known as the **Independence of Irrelevant Alternatives Criterion (IIA),** which means that the introduction or removal of a third candidate should not change or reverse the rankings of the original two candidates relative to one another. In particular, if a losing candidate is removed from the race or if a new candidate is added, the winner of the race should not change.

Example 11.2.7: Apple, Blueberry, or Cherry?

Suppose that 30 students in a class are going to vote on whether to have apple, blueberry, or cherry pie. Use the summary of ranked ballots in below to answer each question.

Number of Ballots	14	12	4
(A) Apple Pie	1	3	3
(B) Blueberry Pie	2	1	2
(C) Cherry Pie	3	2	1

1. Determine the winner of the election by plurality.
2. Which candidate would win a plurality election if cherry pie were removed from the ballot?
3. Does this election violate the IIA?

Answer

1. The number of first place votes for each candidate is: A with 14, B with 12, and C with 4. Apple pie has the most first-place votes and wins the election.
2. If cherry pie is removed from the ballot, then the four voters in the third column now rank blueberry pie as their first choice. So the four votes for C now belong to B. This means that blueberry pie has 16 votes compared to the 14 votes for apple pie. Blueberry pie now wins the plurality election.
3. Yes, the election violates the IIA because the removal of a losing candidate from the ballot changed the winner of the election.

Your Turn 11.2.7

The local animal shelter is having another vote-by-donation charity event! This time, for a $10 donation, an individual can complete a ranked ballot indicating their favorite small dog breed, miniature, from the ones provided: miniature poodle, Yorkshire terrier, or Chihuahua. Use the summary of ballots below to answer each question.

Votes	53	42	24	11
(P) Miniature Poodle	1	3	3	2
(Y) Yorkshire Terrier	2	2	1	1
(C) Chihuahua	3	1	2	3

1. Determine the winner of the election by the ranked-choice method.
2. Determine the winner of the election if poodles are removed from the ballot.
3. Does this election violate IIA?

Example 11.2.8: Best Fourth Wall Breaking Stare on The Office

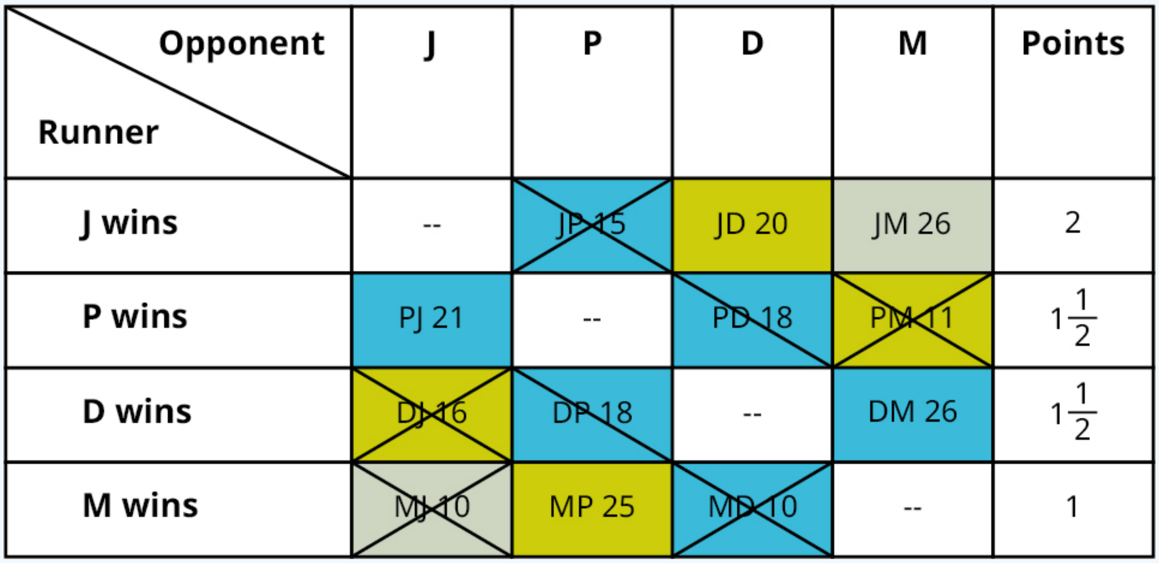
The NBC sitcom *The Office* ran for nine years and has been one of the most popular streamed television shows of all time. One of the trademarks of the show was that characters would often break the fourth wall to communicate with the audience just by staring directly into the camera. In fact, there is a website dedicated to "*The Office* stares" where you can watch over 700 of these stares! Suppose that 36 fans were asked which character had the best "*The Office* stare." Use the ballot summary in below to answer each question.

Number of Ballots	9	11	7	6	3
(J) Jim Halpert (John Krasinski)	1	2	4	2	4
(P) Pam Beesly-Halpert (Jenna Fischer)	4	1	2	4	3
(D) Dwight Schrute (Rainn Wilson)	2	3	3	1	2
(M) Michael Scott (Steve Carell)	3	4	1	3	1

1. Determine the winner of the election by the pairwise comparison method.
2. Determine the winner of the election by the pairwise comparison method if Michael Scott is removed from the ballot.
3. Does this election violate IIA?

Answer

1. Construct and analyze a pairwise comparison matrix:



Opponent / Runner	J	P	D	M	Points
J wins	--	~~JP 15~~	JD 20	JM 26	2
P wins	PJ 21	--	~~PD 18~~	~~PM 11~~	$1\frac{1}{2}$
D wins	~~DJ 16~~	~~DP 18~~	--	DM 26	$1\frac{1}{2}$
M wins	~~MJ 10~~	MP 25	~~MD 10~~	--	1

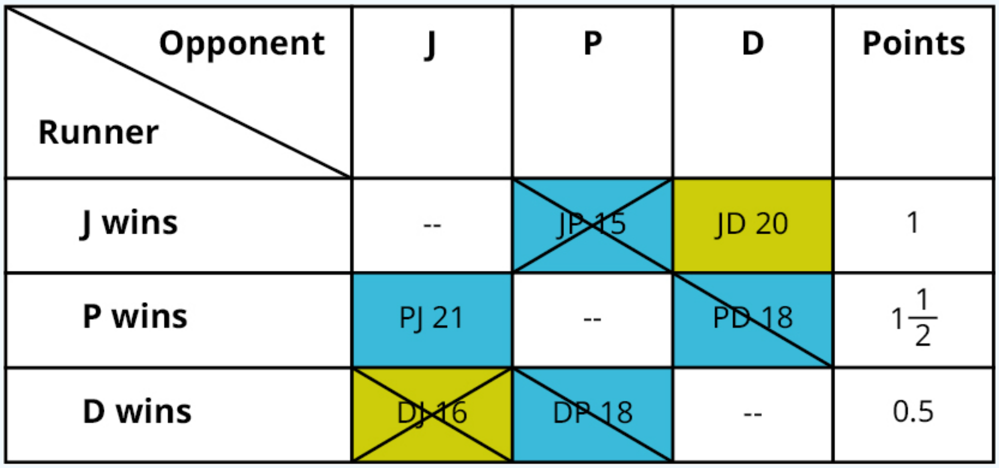
Figure ✓ 11.2.4 Pairwise Comparison Matrix for Jim, Pam, Dwight, and Michael

Jim Halpert wins with 2 points.

2. If Michael Scott is removed, the summary of ranked ballots becomes:

Number of Ballots	9	11	7	6	3
(J) Jim Halpert (John Krasinski)	1	2	3	2	3
(P) Pam Beesly-Halpert (Jenna Fischer)	3	1	1	3	2
(D) Dwight Schrute (Rainn Wilson)	2	3	2	1	1

Construct and analyze a pairwise comparison matrix:



Opponent / Runner	J	P	D	Points
J wins	--	~~JP 15~~	JD 20	1
P wins	PJ 21	--	~~PD 18~~	$1\frac{1}{2}$
D wins	~~DJ 16~~	~~DP 18~~	--	0.5

Figure ✓ 11.2.5: Pairwise Comparison Matrix for Jim, Pam, and Dwight

Pam wins with $1\frac{1}{2}$ points.

3. Yes, this violates the IIA, because the winning candidate was hurt by the elimination of a losing candidate.

Your Turn \(\PageIndex{8\)

Use the initial ballot summary from Example 11.23 to answer the questions.

Determine the winner of the election by the Borda count method.

Determine the winner of the election by the Borda count method if Michael Scott is removed from the ballot.

Does this election violate IIA?

We have seen that all four of the main voting systems we are working with fail the Irrelevant Alternatives Criterion (IIA). A summary of each voting method as it relates to the IIA criterion is found in the table below.

Voting Method	Irrelevant Alternatives Criterion
Plurality	Violates
Ranked-choice	Violates
Pairwise comparison	Violates
Borda count	Violates

Who Knew?

Electronic Voting: Does Your Vote Count?

In order for an election to be fair, voting must be accessible to everyone and every vote must be counted. When hundreds of thousands to millions of votes must be collected and counted in a short period of time, deciding it can be challenging to be on counting procedures that are accurate and secure. Electronic voting machines or even Internet voting can speed up the process, but how reliable are these methods? This has been a subject for debate for years.

In a press release on August 3, 2007, California Secretary of State Debra Bowen explained the results of an extensive review of electronic voting systems in her state. She said that transparency and auditability were key. She went on to say, “I think voters and counties are the victims of a federal certification process that hasn’t done an adequate job of ensuring that the systems made available to them are secure, accurate, reliable and accessible. Congress enacted the Help America Vote Act, which pushed many counties into buying electronic systems that—as we’ve seen for some time and we saw again in the independent UC review—were not properly reviewed or tested to ensure that they protected the integrity of the vote.” Secretary Bowden subsequently ordered that voting machines must have tighter security to be used in California. (DB07:042, Secretary of State Debra Bowen Moves to Strengthen Voter Confidence in Election Security Following Top-to-Bottom Review of Voting Systems, sos.ca.gov/elections.) In some instances, the use of electronic voting in parliamentary elections has been discontinued completely for security reasons. For example, according to the National Democratic Institute, the Netherlands returned to all paper ballots and hand counting in 2006. Will you use voting machines or Internet voting in Imaginaria?

So far, every one of the voting methods we have analyzed has failed one or more of the fairness criteria in one election or another.

Voting Method	Majority Criterion	Condorcet Criterion (Head-to-Head Criterion)	Monotonicity Criterion	(Independence of) Irrelevant Alternatives Criterion
Plurality	Satisfies	Violates	Satisfies	Violates
Ranked-choice	Satisfies	Violates	Satisfies	Violates
Pairwise comparison	Satisfies	Satisfies	Violates	Violates
Borda count	Violates	Violates	Satisfies	Violates

You might be wondering if there is a voting system you could recommend for Imaginaria that satisfies all the fairness criteria. If there is one, it remains to be discovered and it is not a voting system that is based solely on preference rankings. In 1972, Harvard Professor of Economics Kenneth J. Arrow received the Nobel Prize in Economics for proving Arrow's Impossibility Theorem, which states that any voting system, either existing or yet to be created, in which the only information available is the preference rankings of the candidates, will fail to satisfy at least one of the following fairness criteria: the majority criterion, the Condorcet criterion, the monotonicity criterion, and the independence of irrelevant alternatives criterion. This theorem only applies to a specific category of voting systems—those for which the preference ranking is the only information collected. There are other types of voting systems to which the Impossibility Theorem does not apply. For example, there is a class of voting systems called Cardinal voting systems that allow for rating the candidates in some way.

"Rating" is different from "ranking" because a voter can give different candidates the same rating. Consider the five-star rating systems used by various industries, or the thumbs up/thumbs down rating system used on YouTube. Could there be a Cardinal voting system that does not violate any of the fairness criteria we have discussed? It's possible, but more research must be done in order to prove it.

People in Mathematics: Kenneth J. Arrow

In 1972, Kenneth J. Arrow, a Harvard Professor of Economics, received the Nobel Prize in Economics jointly with Sir John Hicks, another world renowned economist, for their contributions to economic theory. In particular, Professor Arrow proved mathematically that no ranked voting system meets all four of the fairness criteria discussed in this section. The statement of this fact is known as Arrow's Impossibility Theorem. Visit this site for more details on Professor Arrow

Check Your Understanding

1. Which fairness criterion is violated by all four of the main ranked voting methods presented in this chapter?
2. Which of the four main ranked voting methods presented in this chapter satisfies the Condorcet criterion?
3. Which of the four main ranked voting methods presented in this section violates the majority criterion?
4. Which of the four main ranked voting methods presented in this section violates the monotonicity criterion?
5. According to Arrow's Impossibility Theorem, which of the four main ranked voting methods presented in this chapter violate at least one of the fairness criteria?
6. Determine whether the following statement is true or false and explain your reasoning: Any ranked election that violates the majority criterion also violates the Condorcet criterion.
7. Determine whether the following statement is true or false and explain your reasoning: Any ranked election that violates the Condorcet criterion also violates the majority criterion.
8. Does Arrow's Impossibility Theorem apply to approval voting? Why or why not?
9. Determine whether each statement is true or false. Explain your reasoning.
10. Arrow's Impossibility Theorem guarantees that ranked voting systems always lead to unfair elections.
11. Approval voting is in the class of voting systems called Cardinal Voting systems.

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11.2.0: Exercises

For the following exercises, identify which fairness criteria, if any, are violated by characteristics of the described voter profile. Explain your reasoning.

? Exercise 11.2.0.1

In a plurality election, the candidates have the following vote counts: A 125, B 132, C 149, D 112. The pairwise matchup points for each candidate would have been: A 1, B 3, C 1, D 1.

? Exercise 11.2.0.2

In a Borda count election, the candidates have the following Borda scores: A 1245, B 1360, C 787. Candidate A received 55 percent of the first-place rankings.

? Exercise 11.2.0.3

In a pairwise comparison election, the four candidates initially received the following points for winning matchups: A 2, B $1\frac{1}{2}$, $1\frac{1}{2}$, C 1, D $1\frac{1}{2}$. When candidate C dropped out of the election, the remaining candidates received: A 1, B $1\frac{1}{2}$, D $\frac{1}{2}$.

? Exercise 11.2.0.4

In a Borda count election, the candidates have the following Borda scores: A 15, B 11, C 12, D 16. The pairwise matchup points for the same voter profiles would have been: A 2, B 0, C 1, D 3.

? Exercise 11.2.0.5

In a Borda count election, the candidates have the following Borda scores: A 15, B 11, C 12, D 16. When Candidate E was added to the ballot, the Borda scores became: A 25, B 21, C 15, D 24, E 18.

? Exercise 11.2.0.6

In a pairwise comparison election, Candidate C was a Condorcet candidate in a straw poll. When the actual election took place, several voters up-ranked Candidate C on their ballots, but no other changes were made to the voter preferences, and Candidate B won the election.

For the following exercises, use the table below.

Votes	49	51
Candidate A	3	1
Candidate B	1	2
Candidate C	2	3

? Exercise 11.2.0.7

In a pairwise comparison election, Candidate A was in first place, Candidate B was in second place, and Candidate C was in third place. When the actual election tool place, the only changes were that several voters down-ranked Candidate B on their ballots, but the outcome remained the same.

? Exercise 11.2.0.8

Determine the Borda score for each candidate and the winner of the election using the Borda count method.

? Exercise 11.2.0.9

Is there a majority candidate? If so, which candidate?

? Exercise 11.2.0.10

Does this Borda method election violate the majority criterion? Justify your answer.

? Exercise 11.2.0.11

Is there a Condorcet candidate? If so, which candidate?

? Exercise 11.2.0.12

Does this Borda method election violate the Condorcet criterion? Justify your answer.

? Exercise 11.2.0.13

If Candidate C is removed from the ballot, which candidate wins by the Borda count method?

? Exercise 11.2.0.14

Does this Borda count method election violate IIA? Justify your answer.

Use the table below for the following exercises.

Votes	7	9	12	15	5
Candidate A	4	4	1	1	4
Candidate B	1	1	2	2	3
Candidate C	3	2	3	4	1
Candidate D	2	3	4	3	2

? Exercise 11.2.0.15

Determine Borda score for each candidate and the winner of the election using the Borda count method.

? Exercise 11.2.0.16

Is there a majority candidate? If so, which candidate?

? Exercise 11.2.0.17

Does this Borda method election violate the majority criterion? Justify your answer.

Exercise 11.2.0.18

Is there a Condorcet candidate? If so, which candidate?

Exercise 11.2.0.19

Does the Borda method election violate the Condorcet criterion? Justify your answer.

Exercise 11.2.0.20

Can an election that fails the majority criterion satisfy the Condorcet criterion? Why or why not?

For the following exercises, use the table below.

Number of Ballots	10	7	5	5	4
Candidate A	1	3	3	3	4
Candidate B	3	2	1	4	1
Candidate C	2	4	2	1	2
Candidate D	4	1	4	2	3

Exercise 11.2.0.21

Determine the Borda score for each candidate and the winner of the election using the Borda count method.

Exercise 11.2.0.22

Is there a majority candidate?

Exercise 11.2.0.23

Does the election violate the majority criterion? Justify your answer.

Exercise 11.2.0.24

Determine the winner by pairwise comparison.

Exercise 11.2.0.25

Is there a Condorcet candidate?

Exercise 11.2.0.26

Does the Borda election violate the Condorcet criterion? Justify your answer.

Exercise 11.2.0.27

Determine the winner by the ranked-choice method.

? Exercise 11.2.0.28

Does the ranked-choice election violate the majority criterion? Justify your answer.

? Exercise 11.2.0.29

Does the ranked-choice election violate the Condorcet criterion? Justify your answer.

? Exercise 11.2.0.30

Can an election that fails the Condorcet criterion satisfy the majority criterion? Why or why not?

Use the table below for the following exercises.

Number of Ballots	49	48	3
Candidate A	1	3	3
Candidate B	2	1	2
Candidate C	3	2	1

? Exercise 11.2.0.31

Determine the winner of the election using the plurality method.

? Exercise 11.2.0.32

Determine the winner by pairwise comparison.

? Exercise 11.2.0.33

Is there a Condorcet candidate?

? Exercise 11.2.0.34

Does this plurality election violate the Condorcet criterion? Justify your answer.

? Exercise 11.2.0.35

If Candidate C is removed from the ballot, which candidate wins by plurality?

? Exercise 11.2.0.36

Does this plurality violate the IIA? Explain your reasoning.

Use the sample ballot summary below for the following exercises.

Number of Ballots	16	20	24	8
Candidate A	1	3	2	1
Candidate B	3	1	4	4
Candidate C	4	2	1	2
Candidate D	2	4	3	3

? Exercise 11.2.0.37

Determine the winner of the election using the ranked-choice method.

? Exercise 11.2.0.38

Determine the winner by pairwise comparison.

? Exercise 11.2.0.39

Is there a Condorcet candidate?

? Exercise 11.2.0.40

Does this ranked-choice election violate the Condorcet criterion? Justify your answer.

? Exercise 11.2.0.41

If the four voters in the last column rank Candidate C ahead of A, which candidate wins by the ranked-choice method?

? Exercise 11.2.0.42

Does this ranked-choice election violate the monotonicity criterion? Explain your reasoning.

Use the sample ballot summary below for the following exercises.

Number of Ballots	15	12	9	3
Candidate A	1	3	3	2
Candidate B	2	2	1	1
Candidate C	3	1	2	3

? Exercise 11.2.0.43

Determine the winner of the election using the ranked-choice method.

? Exercise 11.2.0.44

How could it be demonstrated that this ranked-choice election violates IIA?

? Exercise 11.2.0.45

Determine the winner of the election by the Borda method.

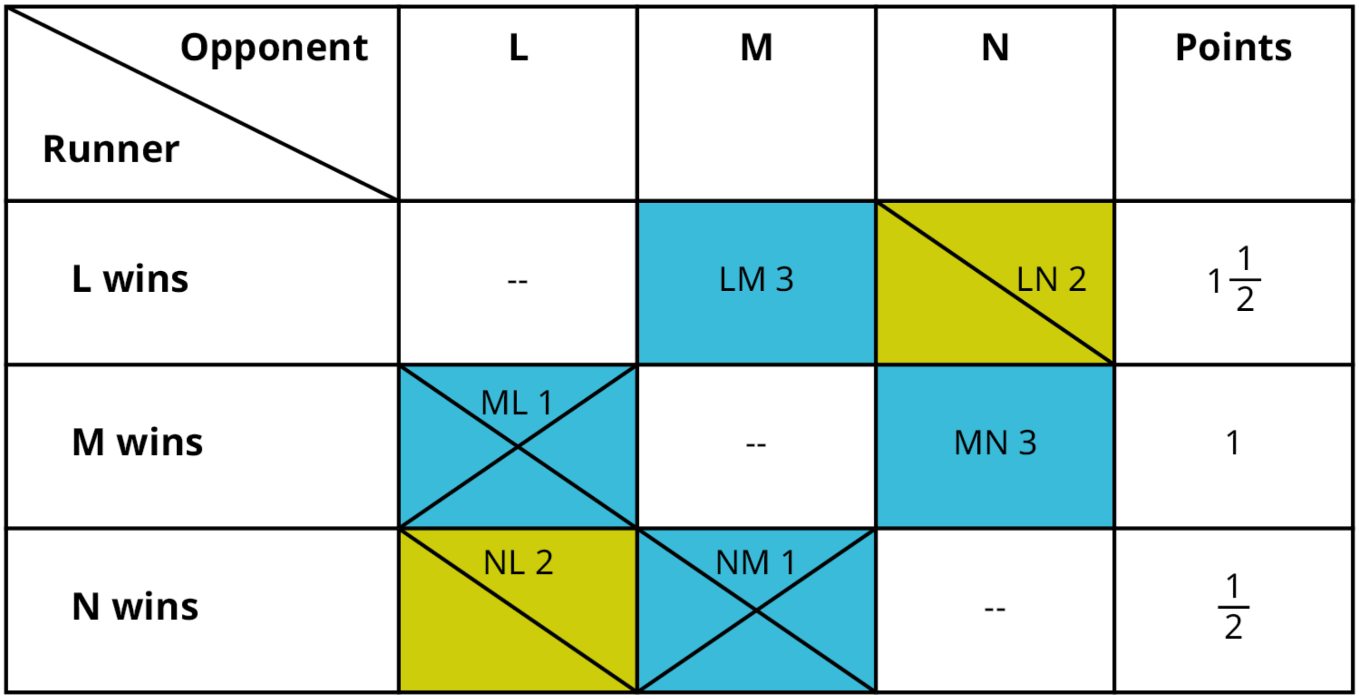
? Exercise 11.2.0.46

Does this Borda method election violate the IIA? Why or why not?

? Exercise 11.2.0.47

Does this Borda method election violate the monotonicity criterion? Why or why not?

Use the pairwise comparison matrix in the given figure for the following exercises.



Opponent Runner	L	M	N	Points
L wins	--	LM 3	LN 2	$1\frac{1}{2}$
M wins	ML 1	--	MN 3	1
N wins	NL 2	NM 1	--	$\frac{1}{2}$

Exercise 11.2.0.48

Which candidate wins the pairwise election?

Exercise 11.2.0.49

Determine the winner by pairwise comparison if N were removed from the ballot.

Exercise 11.2.0.50

Determine the winner by pairwise comparison if M were removed from the ballot.

Exercise 11.2.0.51

Does this pairwise election satisfy the IIA?

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11.3: Standard Divisors, Standard Quotas, and the Apportionment Problem

Figure 11.3.1: Every person at a party gets their fair slice of the cake. (credit: “apple spice cake” by Mark Bonica/Flickr, CC BY 2.0)

Learning Objectives

1. Analyze the apportionment problem and applications to representation.
2. Evaluate applications of standard divisors.
3. Evaluate applications of standard quotas.

The Apportionment Problem

In the new democracy of Imaginaria, there are four states: Fictionville, Pretendstead, Illusionham, and Mythbury. Each state will have representatives in the Imaginarian Legislature. You might now have an agreement on which voting method your citizens will use to elect representatives. However, before that process can even begin, you must decided on how many representatives each state will receive. This decision will present its own challenges.

When sharing your birthday cake, it’s only fair that everyone gets the same portion size, right? You were portioning the cake by dividing it up equally and giving everyone a slice. A great thing about cake is that you can slice it any way you want, but how do you apportion, or divide and distribute, items that can't be sliced? Suppose that you have a box of 16 Ring Pops™, gem-shaped lollipops on a plastic ring. You are going to share the box with four other kids. Dividing the 16 Ring Pops™ among the group of five leads to a problem; after each person in the group gets three Ring Pops™, there is still one left! Who gets the last one? The apportionment problem is how to fairly divide, or apportion, available resources that must be distributed to the recipients in whole, not fractional, parts.

The apportionment problem applies to many aspects of life, including the representatives in the Imaginarian legislature. The table below provides a short list of examples of resources that must be apportioned in whole parts, and the recipients of those resources.

Resource	Recipients
Covid-19 Vaccines	Nations around the world
Airport Terminals	Airlines
Faculty Positions at a University	Departments
Public Schools	Communities

Resource	Recipients
U.S. House of Representatives Seats	States
Parliamentary Seats	Political Parties

Fair division of a resource is not necessarily equal division of the resource like when distributing cake slices. When distributing airport terminals amongst airlines, there are many factors to consider such as the size of the airline, the number and types of aircraft they have, and the demand for the service. In most cases, fairness is defined as being proportional; two quantities are proportional if they have the same relative size. In the case of the Covid-19 vaccine, the expectation would be that countries with larger populations get more doses of the vaccine. In the Imaginarian legislature, the expectation may be that the states with larger populations will receive the larger number of representatives. This concept is referred to as a part-to-part ratio.

Suppose that a supermarket has a special on pies, two for $5. The first customer purchases four pies for $10, and the second customer purchases eight pies for $20. The dollar to pie ratio for the first customer is mfrac10dollars4pies=2.5dollars per pie10dollars4pies=2.5dollars per pie and the dollar to pie ratio for the second customer is mfrac20dollars5pies=2.5dollars per pie20dollars5pies=2.5dollars per pie. So, the dollar to pie ratio is constant. Although the customers do not spend the same amount of money, the amount each spent was proportional to the number of pies purchased.

Now suppose that the supermarket changed the special to $5 for the first pie, and $2 for each additional pie. In that case, four pies would cost $\$5+3(\$2)=\$11$, while 8 pies would cost $\$5+7(\$2)=\$19$. The dollar to pie ratios would be mfrac11dollars4pies=2.75dollars per pie11dollars4pies=2.75dollars per pie and mfrac19dollars8pies=2.375dollars per pie19dollars8pies=2.375dollars per pie, respectively. This special does not result in a constant part to part ratio. The dollars spent are not proportional to the number of pies purchased.

Video

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What Are the Different Types of Ratios?

✓ Example ✓ 11.3.1: Ratio of Faculty to Students at a College

The following table provides a comparison of the number of faculty members in each department at a particular college to the student head count in that department and the number of class sections in that department in the Spring semester. Use this information to answer the questions.

Department	Mathematics	English	History	Science
(S) Student Head Count	4800	2376	1536	2880
(C) Class Sections	120	108	48	96
(T) Total Faculty	30	27	12	24
(F) Full-Time Faculty	10	9	4	8
(P) Part-Time Faculty	20	18	8	16

1. Determine the ratios for each department: S to C, C to T, S to T, F to P
2. What are the units of the ratios that you found?
3. Which of these pairs, if any, has a constant part to part ratio? State the ratio.
4. Does it appear that the total number of faculty positions were allocated to each department based on student head count, the number of class sections, or neither? Justify your answer.

Answer

1. Divide the first quantity by the second, for each department, as shown in the table below.
2. Answers are provided in last column of the table.

Department	eMathematics	English	History	Science	Units of Ratios Found
S to C	$\frac{4800}{120}=40$	$\frac{2376}{108}=22$	$\frac{1536}{48}=32$	$\frac{2880}{96}=30$	Students per class section
C to T	$\frac{120}{30}=4$	$\frac{108}{27}=4$	$\frac{48}{12}=4$	$\frac{96}{24}=4$	Class sections per faculty member
S to T	$\frac{4800}{30}=160$	$\frac{2376}{27}=88$	$\frac{1536}{12}=128$	$\frac{2880}{24}=120$	Students per faculty member
F to P	$\frac{10}{20}=\frac{1}{2}$	$\frac{9}{18}=\frac{1}{2}$	$\frac{4}{8}=\frac{1}{2}$	$\frac{8}{16}=\frac{1}{2}$	Full-time faculty member per part-time faculty member

3. The ratio of class sections to faculty members is a constant ratio of four. The ratio of full-time faculty to part-time faculty is a constant ratio of $\frac{1}{2}$.
4. It appears that the faculty positions were allocated based on the number of class sections because there is a constant ratio of four class sections per faculty member.

Your Turn 11.3.1

The SAT is to be administered at a high school. In preparation, pencils have been distributed to each of the classrooms based on the room capacity. Use the information in the following table to answer each question.

Room Number	B	C	D	E
Room Capacity (number of student desks)	24	18	32	22
Number of Pencils	36	27	48	33

Find the part to part ratio of desks to pencils for each room. Represent it as both a reduced fraction and a decimal rounded to the nearest hundredth as needed.

Find the part to part ratio of pencils to desks for each room. Represent it as both a reduced fraction and a decimal rounded to the nearest hundredth as needed.

Have the pencils been distributed proportionally? If so, what is the constant ratio of pencils to desks? Give the units.

There are some useful relationships between quantities that are proportional to each other. When there is a constant ratio between two quantities, the one quantity can be found by multiplying the other by that ratio. Remember the supermarket special on pies, 2 pies for $5? The ratio of dollars to pies is mfracdollars2pies=2.5dollars per pie5 dollars2pies=2.5dollars per pie and the ratio of pies to dollars is$\frac{2 \text{ pies}}{5 \text{ dollars}} = 0.4 \text{ pies per}$dollar2pies5dollars=0.4pies per dollar. These two values are reciprocals of each other, $\frac{1}{2.5} = 0.4$ and $\frac{1}{0.4} = 2.5$. This means that multiplying by one has the same effect as dividing by the other. This also means that knowing either constant ratio allows us to calculate the price given the number of pies. To find the cost of 20 pies, multiply by the ratio of dollars to pies or divide by the ratio of pies to dollars.

- $20 \text{ pies} \times 2.50 \text{ dollars per pie} = 50 \text{ dollars}$50dollars
- $20 \text{ pies} \div 0.4 \text{ pies per dollar} = 50 \text{ dollars}$50dollars

These patterns are true in general.

FORMULA

Let A be a particular item and B another such that there is a constant ratio of A to B

- r- ratio of B' s to A's $= \frac{1}{\text{ratio of } A' \text{ s to } B'\text{s}}$ and ratio of A' 's to B's $= \frac{1}{\text{ratio of } B'\text{s to } A'\text{s}}$
- units of $A = ($ units of $B) \times ($ ratio of A 's to $B's) = \frac{\text{units of } B}{\text{ratio of } B' \text{ sto } A' \text{ s}}$
- units of $B = ($ units of $A) \times ($ ratio of B' 's to A' s $) = \frac{\text{units of } A}{\text{ratio of } A' \text{ 's to } B' \text{ s}}$

✓ Example ✓ 11.3.2: Ratio of Faculty to Students at a College

Refer to the information given in Example ✓ 11.3.1 .

1. If there are 32 class sections each semester in the Fine Art department, and the same ratio is used to determine the number of faculty members, how many faculty members would you expect to see in the Fine Art department?
2. If the Health Sciences department has 6 full-time faculty members, how many part-time faculty members are in the department?

Answer

1. Multiply the number of class sections by the ratio of faculty members per class section to find the number of faculty. Since there are 4 faculty members per class, the number of faculty members in 32 classes should be $32.4 = 128$ faculty members.
2. Multiply the number of full-time faculty by the ratio of part-time to full-time to find the number of part-time. Since ratio of full-time faculty to part-time faculty at the college is $\frac{1}{2}$ or 1 full-time per 2 part time, the ratio of part-time to full-time is $\frac{2}{1} = 2$ part-time to 1 full-time; so the number of part-time faculty in a department with 6 full-time faculty should be $6.4 = 24$ part-time faculty.

Your Turn 11.3.2

Refer again to the table providing information on classroom capacities and pencil distribution.

Room Number	B	C	D	E
Room Capacity (number of student desks)	24	18	32	22
Number of Pencils	36	27	48	33

1. Determine the number of pencils that would be allocated to a classroom F with 28 desks by multiplying the number of desks by the ratio of pencils to desks.
2. Determine the number of pencils that would be allocated to a classroom F with 28 desks by dividing the number of desks by the ratio of desks to pencils.
3. Assuming pencils continued to be distributed in the same manner, if there were another classroom, G, that received 51 pencils, how many desks would we expect to find in the room?

The apportionment application that will be important to the founders of Imaginaria occurs in representative democracies in which elected persons represent a group. The United Kingdom, France, and India each have a parliament, and the United States has a Congress, just as Imaginaria will have a legislature! The citizens of a country must decide what portion of the representatives each group, such as a state or province or even a political party, will have. A larger portion of representatives means greater influence over policy.

Example 11.3.3: Ratio of U.S. Representatives to State Population

The table below contains a list of the five U.S. states with the greatest number of representatives in the U.S. House of Representatives, along with the population of that state in 2021. Use the information in the table to answer the questions.

Table https://www.britannica.com/topic/Uni...-State-1787120 [representative seats])

State	Representative Seats	State Population
(CA) California	53	39,613,000
(TX) Texas	36	29,730,300
(NY) New York	27	19,300,000
(FL) Florida	27	21,944,600
(PA) Pennsylvania	18	12,804,100

1. What is the ratio of State Population to Representative Seats for each state to the nearest hundred thousand?
2. What is the ratio of Representative Seats to State Population for each state rounded to seven decimal places?
3. What is the ratio of Representative Seats to State Population for each state rounded to six decimal places?
4. Does there appear to be a constant ratio? Justify your answer.

Answer

1. CA 700,000; TX 800,000; NY 700,000; FL 800,000; PA 700,000
2. CA 0.0000013; TX 0.0000012; NY 0.0000014; FL 0.0000012; PA 0.0000014
3. CA 0.000001; TX 0.000001; NY 0.000001; FL 0.000001; PA 0.000001
4. The ratio of State Population to Representative Seats seems to be either 700,000 or 800,000. There does appear to be a constant ratio of about 0.000001 of Representative Seats to State Population if we round off to the sixth decimal place.

Your Turn 11.3.3

Table 11.3.2 contains a list of the five U.S. states ranked sixth through tenth in the number of representatives in the U.S. House of Representatives, along with the population of that state in 2021. Use the information in the table to answer the questions.

Table https://www.britannica.com/topic/Uni...-State-1787120 [representative seats])

State	Representative Seats	State Population
(IL) Illinois	18	12,804,100
(OH) Ohio	16	11,714,600
(MI) Michigan	14	9,992,430
(GA) Georgia	14	10,830,000
(NC) North Carolina	13	10,701,000

1. What is the ratio of State Population to Representative Seats for each state to the nearest hundred thousand?
2. What is the ratio of Representative Seats to State Population for each state rounded to seven decimal places?
3. What is the ratio of Representative Seats to State Population for each state rounded to six decimal places?
4. Does there appear to be a constant ratio? Are the results similar to the top five states?

Video

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You might be wondering why the ratio doesn't appear to be quite the same depending on the rounding of the values. We will see that the key to this variation lies in the fractions. Just like the five children sharing 16 Ring Pops™, there are going to be leftovers and there are many methods for deciding what to do with those leftovers.

The Standard Divisor

There are two houses of congress in the United States: the Senate and the House of Representatives. Each state has two senators, but the number of representatives depends on the population of the state. The number of representative seats in the U.S. House of Representatives is currently set by law to be 435. In order to distribute the seats fairly to each state, the ratio of the population of the U.S. to the number of representative seats must be calculated. The ratio of the total population to the house size is called the standard divisor, and it is the number of members of the total population represented by one seat.

Although apportionment applies to many other scenarios, such as the pencil distribution during the SAT, the terminology of apportionment is based on the House of Representatives scenario. Thus, several government-related terms take on a more general meaning. The **states** are the recipients of the apportioned resource, the **seats** are the units of the resource being apportioned, the

house size is the total number of seats to be apportioned, the **state population** is the measurement of the state's size, and the **total population** is the sum of the state populations.

FORMULA

Standard Divisor $= \frac{\text{Total Population}}{\text{House Size}}$

Example 11.3.4: The Standard Divisor of the U.S. House of Representatives 2021

As of this writing, the Census.gov website U.S. Population clock showed a population of 332,693,997. There are 435 seats in the U.S. House of Representatives. Find the standard divisor rounded to the nearest tenth.

Answer

Dividing $330{,}147{,}881$ people by 435 seats, there are 758,960.6 people per representative.

Your Turn 11.3.4

By the end of the first U.S. Congress in 1791, there were 13 states, 65 representative seats, and approximately 3,929,214 citizens. Find the standard divisor rounded to the nearest tenth.

Whether the standard divisor is less than, equal to, or greater than 1 depends on the ratio of the population to the number of seats.

- The standard divisor will be equal to 1 if the total population is equal to the number of seats. This would mean that each member of the population is allocated their own personal seat.
- The standard divisor will be a number between 0 and 1 when the total population is less than the number of seats. This means that each member of the population is allocated more than one seat.
- The standard divisor will be a number greater than 1when the total population is greater than the number of seats. This means that a certain number of members of the population will share 1 seat.

If the population is five children and the house consists of five pieces of candy, the standard divisor is $\frac{\text{5 children}}{\text{5 candies}} = 1$ child per candy meaning each child gets one candy. If the population is five children and ten pieces of candy, the standard divisor is $\frac{\text{5 children}}{\text{10 candies}} = 0.5$ child per candy meaning that each child gets more than one candy. If the population is five children and four pieces of candy, the standard divisor is $\frac{\text{5 children}}{\text{4 candies}} = 1.25$ child per candy meaning that each child gets less than one candy.

If the seats in the Imaginarian legislature are distributed to the states based on population, then the house size will be less than the population and we should expect the standard divisor to be a number greater than 1.

Example 11.3.5: School Resource Officers in Brevard County, Florida

The public schools in a certain county have been allotted 349 school resource officers to be distributed among 327 public schools attended by approximately 271,500 students.

1. Identify the states, seats, house size, state population, and total population in this apportionment scenario.
2. Describe the ratio the standard divisor represents in this scenario and calculate the standard divisor to the nearest tenth.

Answer

1. The states are the schools in that county. The seats are the school resource officers. The house size is the number of school resource officers, which is 349. The state population is the number of students in a particular school, which was not given. The total population consists of the sum of the school populations, which is 271,500.

2. The standard divisor is the ratio of the total population to the house size, which is the number of students served by each resource officer. Divide 271,500 students $\div 349$ officers $= 777.9$ students per officer.

Your Turn 11.3.5

The Hernandez family and the Higgins family went trick-or-treating together for Halloween last year. They returned with 313 pieces of candy, which they will now apportion to the families. The Hernandez family has three children and the Higgins family has four children.

1. Identify the states, seats, house size, state population, and total population in this apportionment scenario.
2. Describe the ratio the standard divisor represents in this scenario and calculate the standard divisor to four decimal places.

The Standard Quota

Once the standard divisor for the Imaginarian legislature is calculated, the next task is to determine the number of seats that each state should receive, which is referred to as the state's standard quota. Unless all the states have the same population, each state will receive a different number of seats because the quantities will be proportionate to the state populations. To determine those amounts, we will use an idea we learned earlier. Recall that, when the number of units of item A is proportionate to the number of units of item B, we have: unitsofA=units ofBratio ofB's toA'sunits ofA=units ofBratio ofB's toA's

In this case, we are trying to calculate the number of seats a state should be apportioned, the state's standard quota. So A So A would refer to seats allocated to a particular state, while B would refer to the state population. This means that the ratio of B to A is the ratio of the total population to house size, which is the standard divisor. So in apportionment terms, we have the following formula.

FORMULA

$$\text{State's Standard Quota} = \frac{\text{State Population}}{\text{Standard Divisor}} \text{ seats}$$

Example ✓ 11.3.4 outlined that the Census.gov website U.S. Population clock showed a population of 330,147,881, there are 435 seats in the U.S. House of Representatives, and the standard divisor was 758,960.6 people per representative. The state of California has a population of approximately 39,613,000. Use these values to determine the standard quota for California to two decimal places.

Answer

$$\text{California's standard Quota} = \frac{\text{California's population}}{\text{Standard Divisor}} = \frac{39{,}613{,}000}{758{,}960.6} = 52.19 \text{ representatives}$$

Your Turn 11.3.6

By the end of the first U.S. Congress in 1791, there were 13 states, 65 representative seats, and approximately 3,929,214 citizens. In that year, the state of Delaware had a population of approximately 59,000. people. Use this information and the standard divisor you found in Your Turn 11.27 to find Delaware's standard quota rounded to two decimal places.

✓ Example 11.3.7: Apportionment of Laptops in a Science Department

The science department of a high school has received a grant for 34 laptops. They plan to apportion them among their six classrooms based on each classroom's student capacity. Use the values in the table below to find the standard quota for each classroom.

Room	Students
A	30
B	25
C	28
D	32
E	24

Room	Students
F	27

Answer

Step 1: Identify the state population, total population, and the house size. The states are the classrooms, and the state populations are listed in the table. The total population is the sum of the state populations, which is 166. The house size is the number of seats, or laptops, to be allocated, which is 34.

Step 2: Calculate the standard divisor by dividing the total population by the house size.

$$\text{Standard Divisor } = \frac{\text{Total of Room Capacities}}{\text{Number of Laptops}} = \frac{166}{34} \approx 4.88$$

students per laptop.

Step 3: Calculate the standard quota by dividing the state population by the standard divisors, as shown in the table below. Room's Standard Quota $= \frac{\text{Room Capacity}}{\text{Standard Divisor}}$

Room	Room Capacity	Room’s Standard Quota
A	30	$30 \div 4.88 \approx 6.15$ laptops
B	25	$25 \div 4.88 \approx 5.12$ laptops
C	28	$28 \div 4.88 \approx 5.74$ laptops
D	32	$32 \div 4.88 \approx 6.56$ laptops
E	24	$24 \div 4.88 \approx 4.92$ laptops
F	27	$27 \div 4.88 \approx 5.53$ laptops

Step 4: Find the sum of the standard quotas. $6.15 + 5.12 + 5.74 + 6.56 + 4.91 + 5.53 = 34.01$. This is only slightly off from the number of laptops -34 - which can be caused by rounding off in previous steps. This is a good indication that the calculations were correct. If you find that the value of the sum of the standard quotas is significantly different from the house size (number of seats), it is possible that the standard divisor was calculated using too few decimal places. Calculate the standard divisor and standard quotas again but round off to a greater number of decimal places.

Your Turn 11.3.7

This year the Hernandez family and the Higgins family were joined by the Ho family for Halloween trick-or-treating. The Hernandez family has three children, the Higgins family has four children, and the Ho family has two children. This time, they collected 527 pieces of candy, which they are going to apportion based on the number of children. Find the standard quota for each family. Round all values to four decimal places. If traditional rounding methods are applied to determine the actual number whole number values of pieces of candy received by each family, do the values sum to 527?

Check Your Understanding

In the following questions, assume there is a constant ratio between units of A and units of B. Two students are having a discussion. Determine who is correct: Student 1, Student 2, both, or neither.

1. Student 1 says that the number of units of A is the product of the number of units of B times the ratio of A to B.

Student 2 says that the number of units of A is the quotient of the number of units of B divided by the ratio of A to B.

2. Student 1 says that the number of units of A is the quotient of the number of units of B divided by the ratio of B to A.

Student 2 says that the number of units of B is the quotient of the number of units of A divided by the ratio of A to B.

3.Suppose there are 110 of item A and 55 of item B.

Student 1 says the ratio of A to B is $\frac{1}{2}$ Student 2 says the ratio of B to A is $\frac{1}{0.5}$.

4. Suppose the constant ratio of A to B is 0.75 . Student 1 says the ratio of A to B is $\frac{4}{3}$ Student 2 says the ratio of B to A is $\frac{3}{4}$.

5. Student 1 says that Standard Divisor $= \frac{\text{House Size}}{\text{Total Population}}$.

Student 2 says that Standard Divisor $= \frac{\text{Sum of the State Populations}}{\text{Total Number of Seats Available}}$.

6. Student 1 says that State's Standard Divisor $= \frac{\text{Total Population}}{\text{State's Standard Quota}}$.

Student 2 says that State's Standard Quota $= \frac{\text{State Population}}{\text{Standard Divisor}}$.

7. Student 1 says that State's Standard Quota $=$ State Population $\div \frac{\text{Total Population}}{\text{House Size}}$.

Student 2 says that State's Standard Quota $= \frac{\text{State Population}}{\text{Standard Divisor}}$

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11.3.0: Exercises

For the following exercises, identify the states, the seats, and the state population (the basis for the apportionment) in the given scenarios.

? Exercise 11.3.0.1

A parent has 25 pieces of candy to split among their four children. They will earn the candy based on how many minutes of chores they children did this week.

? Exercise 11.3.0.2

The board of trustees of a college has recently approved the installation of 70 new emergency blue lights in five parking lots. The number of lights in each lot will be proportionate to the size of the parking lot, which is to be measured in acres.

? Exercise 11.3.0.3

The reading coach at an elementary school has 52 prizes to distribute to students as a reward for time spent reading.

(The Coronavirus Crisis, by Pien Huang, Shots Health News From NPR, npr.org, November 24, 2020)

For the following exercises, use the given information to find the standard divisor to the nearest hundredth. Include the units.

? Exercise 11.3.0.6

The total population is 2,235 automobiles, and the number of seats is 14 warehouses.

? Exercise 11.3.0.7

The total population is 135 hospitals, and the number of seats is 200 respirators.

For the following exercises, use the given information to find the standard quota. Include the units.

? Exercise 11.3.0.8

The state population is eight residents in a unit, and the standard divisor is 1.75 residents per parking space.

? Exercise 11.3.0.9

The state population is 52 ICU patients each week, and the number of seats is 6.5 patients per respirator.

? Exercise 11.3.0.10

The total population is 145 basketball players, the number of seats is 62 trophies, and the state population is 14 basketball players on Team Tigers.

? Exercise 11.3.0.11

The total population is 12 giraffes, the number of seats is nine water troughs, and the state population is three giraffes in Enclosure C.

For the following exercises, use the table below, which shows student head count, class section, and total faculty in each of four college departments.

Department	(M) Math	(E) English	(H) History	(S) Science	(O) College Overall
(T) Student Head Count	4800	2376	1536	2880	87118
(C) Class Sections	120	108	48	96	3712
(F) Faculty Members	30	27	12	24	928

? Exercise 11.3.0.12

Determine the F to T ratios for each department rounded to four decimal places as needed. What are the units?

? Exercise 11.3.0.13

Determine the C to F ratios for each department rounded to four decimal places as needed. What are the units?

? Exercise 11.3.0.14

What is the F to T ratio for the college overall? Include units. How does it compare to the F to T ratios for individual departments?

? Exercise 11.3.0.15

What is the overall C to F ratio? Include units. How does it compare to the C to F ratios for individual departments?

? Exercise 11.3.0.16

Does there appear to be a constant F to T ratio? If so, what is the ratio? If not, what implications does this have about the different departments?

? Exercise 11.3.0.17

Does there appear to be a constant C to F ratio? If so, what is the ratio? If not, what implications does this have about the different departments?

? Exercise 11.3.0.18

If the departments are the states, the students are the population, and the faculty members are the seats, use the College Overall column to determine the standard divisor for the apportionment of the faculty rounded to two decimal places as needed. Include the units.

? Exercise 11.3.0.19

If the departments are the states, the classes are the population, and the faculty members are the seats, use the Overall College column to determine the standard divisor rounded to two decimal places as needed. Include the units.

? Exercise 11.3.0.20

Use the standard divisor from question 18 to find the standard quota for each department rounded to two decimal places as needed. What are the units?

Exercise 11.3.0.21

Use the standard divisor from question 19 to find the standard quota for each department rounded to two decimal places as needed.

For the following exercises, use this information: Wakanda, the domain of the Black Panther, King T'Challa has six fortress cities. In Wakandan, the word "birnin" means "fortress city." King T'Challa has found 111 Vibranium artifacts that must be distributed among the fortress cities of Wakanda. He has decided to apportion the artifacts based on the number of residents of each birnin. The table below displays the populations of major Wakandan cities.

Fortress Cities	Birnin Djata (D)	Birnin T'Chaka (T)	Birnin Zana (Z)	Birnin S'Yan (S)	Birnin Bashenga (B)	Birnin Azzaria (A)	Total Population
Residents	26,000	57,000	27,000	18,000	64,000	45,000	237,000

Exercise 11.3.0.22

Identify the states, the seats, and the state population (the basis for the apportionment) in this scenario.

Exercise 11.3.0.23

Find the standard divisor for the apportionment of the Vibranium artifacts. Round to the nearest tenth as needed. Include the units.

Exercise 11.3.0.24

Find each birnin's standard quota for the apportionment of the Vibranium artifacts. Round to the nearest hundredth as needed. What are the units?

Exercise 11.3.0.25

Find the sum of the standard quotas. Is it reasonably close to the number of artifacts available for distribution?

For the following exercises, suppose that 6.4 million doses of COVID-19 vaccine are to be distributed among U.S. states. The vaccines will either be distributed based on the total state population or based on the number of people over 65 years old, as shown in the table below.

State	State Population	State Population Age 65+	Percentage of State Population 65+
(CA) California	39,613,000	5,669,000	14.3%
(TX) Texas	29,730,300	3,602,000	12.6%
(NY) New York	19,300,000	3,214,000	16.4%
(FL) Florida	21,944,600	4,358,000	20.5%
(PA) Pennsylvania	12,804,100	2,336,000	18.2%
(US) United States	330,151,000	52,345,000	15.8%

Exercise 11.3.0.26

Find the standard divisor for the apportionment of the vaccine doses by population using the estimate for the total U.S. population. Round to the nearest tenth as needed. Include the units.

? Exercise 11.3.0.27

Find each state's standard quota for the apportionment of the vaccine doses. Round to the nearest tenth as needed. What are the units?

? Exercise 11.3.0.28

Find the standard divisor for the apportionment of the vaccine doses by population age 65 and older using the estimate for the total U.S. population of people aged 65 and older. Round to the nearest tenth as needed. Include the units.

? Exercise 11.3.0.29

Find each state's standard quota for the apportionment of the vaccine doses by total state population. Round to the nearest tenth as needed. What are the units?

? Exercise 11.3.0.30

Compare the standard quota for each state based on the entire state population to the standard quota for each state based on the portion of the population age 65 and older. Which states would receive more doses of vaccine if the apportionment were based on the population of people age 65 and older?

? Exercise 11.3.0.31

Approximately 15.8 percent of U.S. residents are age 65 and older.

a. Which of the five states listed have a percentage of residents age 65 and older greater than 15.8 percent?
b. Which of the five states listed have a percentage of residents age 65 and older less than 15.8 percent?
c. Explain the correlation.

For the following exercises, use this information: Children from five families—the Chorro family, the Eswaran family, the Javernick family, the Lahde family, and the Stolly family—joined a town-wide Easter egg hunt. When they returned with their baskets, they had 827 eggs! They decided to share their eggs among the families based on the number of children in each family, as shown in the table below.

Family	Number of Children
(C) Chorro	3
(E) Eswaran	2
(J) Javernick	4
(L) Lahde	1
(S) Stolly	5

? Exercise 11.3.0.32

Identify the states, the seats, and the state population (the basis for the apportionment) in this scenario.

? Exercise 11.3.0.33

Find the standard divisor for the apportionment of the Easter eggs. Round to five decimal places as needed. Include the units.

? Exercise 11.3.0.34

Find each family’s standard quota for the apportionment of the Easter eggs. Round to the nearest hundredth as needed. What are the units?

? Exercise 11.3.0.35

Find the sum of the standard quotas from exercise 34. Is the sum reasonably close to the number of Easter eggs available for distribution?

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11.4: Apportionment Methods

Learning Objectives

1. Describe and interpret the apportionment problem.
2. Apply Hamilton’s Method.
3. Describe and interpret the quota rule.
4. Apply Jefferson’s Method.
5. Apply Adams’s Method.
6. Apply Webster’s Method.
7. Compare and contrast apportionment methods.
8. Identify and contrast flaws in various apportionment methods.

A Closer Look at the Apportionment Problem

In Standard Divisors, Standard Quotas, and the Apportionment Problem we calculated the standard divisor and the standard quotas in various apportionment scenarios. The results of those calculations routinely led to fractions and decimals of units. However, the seats in the House of Representatives, laptops in a classroom, or a variety of other resources, are indivisible, meaning they cannot be divided up into fractional parts. This leaves a decision to be made. For example, if the standard quota for the number of laptops to be distributed to a classroom is 12.44 units, how do we deal with the fractional part of 0.44? It is unclear if the classroom should receive 12 units, 13 units, or some other value. Let’s try traditional rounding to the nearest whole number value.

Figure $11.4.1$: Schoolchildren depend on apportionment of resources like laptops among schools and classroom. (credit: “Richmond Public Schools” by Virginia Department of Education/Flickr, CC BY 2.0)

Example $11.4.1$: Installing Emergency Lights

The board of trustees of a college has recently approved the installation of 70 new emergency blue lights in three parking lots. The number of lights in each lot will be proportionate to the size of the parking lot, which isto be measured in acres. The total number of acres is 34; so the standard divisor is $\frac{34}{70} \approx 0.4857$. The standard quota for each lot is listed in the table below. Use this information to answer each question.

Lot	Acres	Lot’s Standard Quota
A	15	$15 \div 0.4857 \approx 30.88$ emergency blue lights
B	9	$9 \div 0.4857 \approx 18.53$ emergency blue lights
C	10	$10 \div 0.4857 = 20.59$ emergency blue lights

1. Use traditional rounding to determine the number of lights assigned to each lot.
2. Find the sum of the values from part 1.
3. Does the sum found in part 2 equal the number of lights available?

Answer

1. If traditional rounding is used, there will be 31, 19, and 21 lights distributed to each lot, respectively.
2. The total of these values is 71.
3. No, the total from part 2 is one more than the number of lights available. In other words, one of the parking lots must get 1 fewer light than apportioned.

Your Turn 11.4.1

The science department of a high school has received a grant for 34 laptops. They plan to apportion them among their six classrooms based on each classroom's student capacity. Use the standard quotas in the table below to answer each question.

Room	Room Capacity	Room's Standard Quota
A	30	$30 \div 4.88 \approx 6.15$ laptops
B	25	$25 \div 4.88 \approx 5.12$ laptops
C	28	$28 \div 4.88 \approx 5.74$ laptops
D	32	$32 \div 4.88 \approx 6.56$ laptops
E	24	$24 \div 4.88 \approx 4.92$ laptops
F	27	$27 \div 4.88 \approx 5.53$ laptops

Use traditional rounding to determine the number of laptops assigned to each classroom.

Find the sum of the values from part 1.

Does the sum found in part 2 equal the number of laptops available?

Example 11.4.1 demonstrates that we cannot successfully apportion indivisible resources by rounding off each standard quota using traditional rounding. This leaves us with a problem. What is a fair way to distribute the fractional parts of the standard quotas? We will refer to this as the **apportionment problem**. Several methods for making this decision will be discussed.

In 2015, the U.S. Air Force had a fleet of approximately 281 A-10C Thunderbolt II aircraft. Suppose that the Air Force administration wanted to distribute 27 aircrafts across six bases based on the number of qualified pilots stationed at those bases. Use the information in the table below to answer each question.

Base Pilots (A) Alpha 13 (B) Bravo 12 (C) Charlie 5 (D) Delta 16 (E) Echo 7 (F) Foxtrot 9

1. Identify the states, the seats, and the state population (the basis for the apportionment) in this scenario.
2. Find the standard divisor for the apportionment of the aircraft. Round to four decimal places as needed. Include the units.
3. Find each Air Force base's standard quota for the apportionment of the aircraft. Round to the nearest hundredth as needed. What are the units?
4. How does this example demonstrate the apportionment problem? Will traditional rounding solve the problem?

Answer

1. The states are the bases, the seats are the aircraft, and the state populations are the pilots at a given base.

2. Standard Divisor $= \frac{\text{Total Population}}{\text{House Size}} = \frac{13+12+5+16+7+9}{27} = \frac{62}{27} \approx 2.2963$ pilots per aircraft.

3. $\mathrm{A}\frac{13}{2.2963} \approx 5.66,\ \mathrm{B}\frac{12}{2.2963} \approx 5.23, \mathrm{C}\frac{5}{2.2963} \approx 2.18, \mathrm{D}\frac{16}{2.2963} \approx 6.97, \mathrm{E}\frac{7}{2.2963} \approx 3.05,\ \mathrm{F}\ \frac{9}{2.2963} \approx 3.92$. The units are aircraft.

4. This example demonstrates the apportionment problem because it is not possible to send a fractional number of aircraft to an Air Force base. On the other hand, if we use traditional rounding methods to get whole numbers, the results are $5+5+2+7+3+4=26$ aircraft will be apportioned, which is one less than the number of aircraft that were supposed to be apportioned.

Your Turn 11.4.2

The reading coach at an elementary school has 13 gift cards to distribute to their three students as a reward for time spent reading. When they calculated the standard quota for each student based on the number of minutes they student had read, the results were: 4.49 gift cards, 4.03 gift cards, and 4.48 gift cards. How does this demonstrate the apportionment problem?

Hamilton's Method of Apportionment

One of the problems encountered when standard quotas are transformed into whole numbers using traditional rounding is that it is possible for the sum of the values to be greater than the number of seats available. A reasonable way to avoid this is to always round down, even when the first decimal place is five or greater. For example, a standard quota of 12.33 and a standard quota of 12.99 would both round down to 12. This is called the lower quota.

✓ Example 11.4.3: Lower Quota for Apportionment of Aircraft

The Air Force administration wants to distribute 27 aircrafts across six bases based on the number of qualified pilots stationed at those bases. The standard quotas for each base are listed in the table below. Use this information to answer the questions.

Base	Standard Quota
(A) Alpha	$\frac{13}{2.2963} \approx 5.66$ aircraft
(B) Bravo	$\frac{12}{2.2963} \approx 5.23$ aircraft
(C) Charlie	$\frac{5}{2.2963} \approx 2.18$ aircraft
(D) Delta	$\frac{16}{2.2963} \approx 6.97$ aircraft
(E) Echo	$\frac{7}{2.2963} \approx 3.05$ aircraft
(F) Foxtrot	$\frac{9}{2.2963} \approx 3.92$ aircraft

1. Give the lower quota for each Air Force base.
2. Find the sum of the lower quotas. By how much does this sum fall short of the actual number of aircraft?

Answer

1. Round down. The lower quota for each Air Force base is 5, 5, 2, 6, 3, 3, respectively.
2. The sum is 24. This is 3 fewer than the actual number of aircraft.

Your Turn 11.4.3

The apportionment of 70 new emergency blue lights in three parking lots is based on acreage. The standard quota for each lot is listed in the table below. Use this information to answer each question.

Lot	Acres	Lot’s Standard Quota
A	15	$15 \div 0.4857 \approx 30.88$ emergency blue lights
B	9	$9 \div 0.4857 \approx 18.53$ emergency blue lights
C	10	$10 \div 0.4857 = 20.59$ emergency blue lights

Give the lower quota for each parking lot.

Find the sum of the lower quotas.

By how much does this sum fall short of the actual number of emergency lights?

If the standard quotas are all rounded down, their sum will always be less than or equal to the house size. Then, it would only remain to find a fair way to distribute any remaining seats. Alexander Hamilton, who was a general in the American Revolution, author of the Federalist Papers, and the first secretary of the treasury, took this approach to apportionment.

Steps for Hamilton's Method of Apportionment

There are five steps we follow when applying Hamilton's Method of apportionment:

1. Find the standard divisor.
2. Find each state's standard quota.
3. Give each state the state's lower quota (with each state receiving at least 1 seat).
4. Give each remaining seat one at a time to the states with the largest fractional parts of their standard quotas until no seats remain.
5. Check the solution by confirming that the sum of the modified quotas equals the house size.

Video

milton's Method of Apportionment
Hearn
Hamilton Method of Apportionment Step By Step & Using Free Online Calculator
Hearn Math
Watch on YouTube

Hamilton Method of Apportionment

Example 11.4.4: Hawaiian School Districts

Suppose that the Hawaii State Department of Education has a budget for 616 schools and is doing a research study to determine the equitable number of schools to have in each of the five counties based on the residents under 19 years old, This data is provided in the table below. Using the Hamilton method, calculate how many schools would be funded in each state.

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Residents under age 19	46,310	224,230	20	16,560	38,450	325,570

Answer

Step 1: Calculate the standard divisor. Divide the total population, 325,570, by the house size, 616 seats. The standard divisor is 528.52.

Step 2: Find each state's standard quota:

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Standard Quota	$\frac{46,310}{528.52} \approx 87.62$	$\frac{224,230}{528.52} \approx 424.26$	$\frac{20}{528.52} \approx 0.04$	$\frac{16,560}{528.52} \approx 31.33$	$\frac{38,450}{528.52} \approx 72.75$	616

Step 3: Find each state's lower quota and their sum:

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Lower Quota	87	424	1	31	72	615

Step 4: Compare the sum of the states' lower quotas, 615, to the house size, 616. One seat remains to be apportioned and must be given to the state with the largest fractional part: Maui with 0.75. So, the final Hamilton quotas are as follows: Hawaii 87, Honolulu 424, Kalawao 1, Kauai 31, and Maui 73.

Step 5: Find the total to confirm the sum of the quotas equals the house size, 616. Then $87+424+1+31+73=616$. The apportionment is complete.

Your Turn 11.4.4

In the country of Imaginaria, there will be four states: Fictionville, Pretendstead, Illusionham, and Mythbury. Suppose there will be 35 seats in the legislature of Imaginaria. Use Hamilton's method of apportionment to determine the number of seats in each state based on the populations in the following table.

	Fictionville	Pretendstead	Illusionham	Mythbury	Total
Population	71,000	117,000	211,000	1,194,000	1,593,000

Tech Check: Apportionment Calculators

Check out websites such as Ms. Hearn Math for a free Hamilton apportionment calculator.

This can be a useful tool to confirm your results!

The Quota Rule

A characteristic of an apportionment that is considered favorable is when the final quota values all either result from rounding down or rounding up from the standard quotas. The value that results from rounding down is called the lower quota, and the value that results from rounding up is called the upper quota.

As we explore more methods of apportionment, we will consider whether they satisfy the quota rule. If a scenario exists in which a particular apportionment allocates a value greater than the upper quota or less than the lower quota, then that apportionment violates the quota rule and the apportionment method that was used violates the quota rule.

Example 11.4.5: Which Apportionment Method Satisfies the Quota Rule?

Several apportionment methods have been used to allocate 125 seats to ten states and the results are shown in the table below. Determine which apportionments do not satisfy the quota rule and justify your answer.

	State A	State B	State C	State D	State E	State F	State G
Standard Quota	41.26	16.00	5.77	2.64	7.82	10.47	0.21
Lower Quota	41	16	5	2	7	10	0
Upper Quota	42	17	6	3	8	11	1
Method X	43	16	5	2	7	10	1
Method Y	41	16	6	2	8	10	1
Method Z	42	16	7	3	7	9	1

Answer

Look for states such that the number of seats allocated differs from the lower or upper quota. Method X violates the quota rule because State A receives 43 seats instead of 41 or 42. Method Z violates the quota rule because State C receives 7 seats instead of 5 or 6 and State F receives 9 instead of 10 or 11.

Your Turn 11.4.5

Apportionment Method V has been used to allocate 125 seats to ten states as shown in the table below. Determine whether the apportionment satisfies the quota rule and justify your answer.

	State A	State B	State C	State D	State E	State F	State G
Standard Quota	41.26	10.70	16.00	13.11	17.00	5.77	2.64
Lower Quota	41	10	16	13	17	5	2
Upper Quota	42	11	17	14	18	6	3
Method V	42	11	17	13	18	4	2

It is possible for an apportionment method to satisfy the quota rule in some scenarios but violate it in others. However, because the Hamilton method always begins with the lower quota and either adds one to it or keeps it the same, the final Hamilton quota will always consist of values that are either lower quota values or upper quota values. When an apportionment method has this characteristic, it is said to satisfy the quota rule. So, we can say:

The Hamilton method of apportionment satisfies the quota rule.

Although the Hamilton method of apportionment satisfies the quota rule, it can result in some unexpected outcomes, which has caused it to pass in and out of favor of the U.S. government over the years. There are several apportionment methods that have been popular alternatives, such as Jefferson's method of apportionment that the founders of Imaginaria should consider.

Jefferson's Method of Apportionment

Another approach to dealing with the fractional parts of the standard quotas is to modify the standard divisor so that the total of the resulting modified lower quotas is the necessary number of seats. This is the approach used by Jefferson.

In Jefferson's method, the change to the standard divisor is made so that the total of the modified lower quotas equals the house size. The change in the standard divisor to get the modified divisor is relatively small. There is not a formula for this. The modified divisor is found by "guess and check." It is important to remember that *increasing* the divisor *decreases* the quotas, but *decreasing* the divisor *increases* the quotas. So, if you need a larger quota, try reducing the divisor, and if you need a smaller quota, try increasing the divisor.

Example 11.4.6: Modifying a Standard Divisor

Suppose the population of a state is 50 and the standard divisor is 12.5.

1. Find the state's standard quota.
2. Increase the standard divisor by 2 units and use the modified divisor to determine the modified quota for the state.
3. Decrease the modified divisor from part 2 by 1.5 units and use the new modified divisor to determine the modified quota for the state.
4. Choose any value of divisor between the value of the modified divisor from part 2 and the value of the modified divisor from part 3 and use it to determine the modified quota for the state.
5. Which modified quota was the largest, the modified quota from part 2, from part 3, or from part 4? Explain why.

Answer

1. The state's standard quota is $\frac{50}{12.5} = 4$.
2. The modified divisor is 14.5 . The modified quota is $\frac{50}{14.5} \approx 3.45$.

3. The modified divisor is 13 . The modified quota is $\frac{50}{13} \approx 3.85$.

4. One value between 13 and 14.5 is 13.5 . With a modified divisor of 13.5 , the modified quota is $\frac{50}{13.5} \approx 3.70$.

5. The modified quota from part 3 was the largest because the divisor was the smallest of the three. Dividing the same number by a smaller value gives a larger result.

Your Turn 11.4.6

Suppose the population of a state is 12 and the standard divisor is 0.225.

1. Find the state's standard quota.
2. Decrease the standard divisor by 0.200 units and use the modified divisor to determine the modified quota for the state.
3. Increase the modified divisor from part 2 by 0.100 units and use the new modified divisor to determine the modified quota for the state.
4. Choose any value of divisor between the value of the modified divisor from part 2 and the value of the modified divisor from part 3 and use it to determine the modified quota for the state.
5. Which modified quota was the smallest, the modified quota from part 2, from part 3, or from part 4? Explain why.

When you use Jefferson's method, you might have to adjust the divisor several times find modified lower quotas that sum to the house size. First, guess what the divisor should be based on the sum of the lower quotas and then increase or decrease it from there based on whether the sum needs to be smaller or larger respectively. If the result still does not produce lower quotas that sum to the house size, adjust again. Keep a record of the values that didn't work to help you narrow your search.

Steps for Jefferson's Method of Apportionment

We take four steps to apply Jefferson's Method of apportionment:

- **Step 1:** Find the standard divisor.
- **Step 2:** Find each state's quota. This will be the standard quota the first time Step 2 is completed and the standard divisor is used, but Step 2 may be repeated as needed using a modified divisor and resulting in modified quotas.
- **Step 3:** Find the states' lower quotas (with each state receiving at least one seat), and their sum.
- **Step 4:** If the sum from Step 3 equals the number of seats, the apportionment is complete. If the sum of the lower quotas is less than the number of seats, reduce the standard divisor. If the sum of the lower quotas is greater than the number of seats, increase the standard divisor. Return to Step 2 using the modified divisor.

Example 11.4.7: Hawaiian State Representative Districts

Suppose that the Hawaii State Department of Education has a budget for 616 schools and is doing a research study to determine the equitable number of schools to have in each of five counties based on the residents under the age of 19. With the data in the table below, apply Jefferson's method to apportion the schools to the counties.

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Residents under Age 19	46,310	224,230	20	16,560	38,450	325,570

Answer

Step 1: The process for finding the standard divisor, standard quotas, and lower quotas is the same in the Hamilton and Jefferson methods of apportionment. We walked through the Hamilton Method in Example 11.34, and following these steps resulted in lower quotas as shown in the table below.

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Standard Quota	$\frac{46,310}{528.52} \approx 87.62$	$\frac{224,230}{528.52} \approx 424.26$	$\frac{20}{528.52} \approx 0.04$	$\frac{16,560}{528.52} \approx 31.33$	$\frac{38,450}{528.52} \approx 72.75$	616

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Lower Quota	87	424	1	31	72	615

Step 2: Compare the sum of the states' lower quotas, 615, to the house size, 616. Since 615 is less than 616, use a modified divisor that is less than the standard divisor of 528.52. Try 526.00.

Step 3: Find each state's modified quota, lower quota, and the sum of the lower quotas based on the modified divisor of 526:

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Modified Quota	$\frac{46{,}310}{526.00} \approx 88.04$	$\frac{224{,}230}{526.00} \approx 426.29$	$\frac{20}{526.00} \approx 0.04$	$\frac{16{,}560}{526.00} \approx 31.48$	$\frac{38{,}450}{526.00} \approx 72.75$	616
Lower Quota	88	426	1	31	72	618

Step 4: The new sum of the lower quotas is 2 units greater than 616. We have overshot the goal. So, increase the divisor to a value between 526.00 and 528.52. Try 527.00.

Step 5: Repeat the process of finding the quotas. Find each state's modified quota, lower quota, and the sum of the lower quotas based on the modified divisor of 526.00:

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Modified Quota	$\frac{46{,}310}{527.00} \approx 87.87$	$\frac{224{,}230}{527.00} \approx 425.48$	$\frac{20}{527.00} \approx 0.04$	$\frac{16{,}560}{527.00} \approx 31.42$	$\frac{38{,}450}{527.00} \approx 72.96$	616
Lower Quota	87	425	1	31	72	616

Step 6: The new sum of the lower quotas equals the house size. The apportionment is complete.

The apportionment is: Hawaii County 87, Honolulu County 425, Kalawao County 1, Kauai 31, and Maui 72 schools.

When using Jefferson's method, the modified divisors you use may be different from what another person chooses, but final apportionment values will be the same.

Your Turn 11.4.71

Let's return to the Imaginarian states of Fictionville, Pretendstead, Illusionham, and Mythbury. Suppose that there are going to be 35 seats in the legislature. This time use Jefferson's method of apportionment to determine the number of seats in each state based on the populations in the table below. How many seats would each state receive?

	Fictionville	Pretendstead	Illusionham	Mythbury	Total
Population	71,000	117,000	211,000	1,194,000	1,593,000

Notice that, in this apportionment, Mythbury received more than the upper quota. Since this apportionment of representatives to Imaginarian states by Jefferson's method does not satisfy the quota rule, we say that:

Jefferson's method violates the quota rule.

We have discussed two apportionment methods: one that satisfies the quota rule and one that does not. Before you decide which method to use in Imaginaria, there are a couple more options to consider.

Video

Jefferson Apportionment Method
Hearn
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Math
Jefferson Method of Apportionment Step By Step & Using Free Online Calculator
Watch on YouTube

Jefferson Apportionment Method

Tech Check: Apportionment Calculators

It is possible to create Excel spreadsheets that complete the calculations necessary to complete a Jefferson Apportionment. In some cases, this work has already been done and posted online. Check out websites such Ms. Hearn Math for a free Jefferson apportionment calculator.

This can be a useful tool to confirm your results!

Adams's Method of Apportionment

Adams's method of apportionment is another method of apportionment that is based on a modified divisor. However, instead of basing the changes on the sum of the lower quotas, as Jefferson did, Adams used the upper quotas.

To apply Adams's Method of apportionment, there are four steps we follow:

1. Find the standard divisor.
2. Find each state's quota. This will be the standard quota the first time Step 2 is completed, and the standard divisor is used, but Step 2 may be repeated as needed using a modified divisor and resulting in modified quotas.
3. Find the states' upper quotas and their sum.
4. If the sum from Step 3 equals the number of seats, the apportionment is complete. If the sum of the upper quotas is less than the number of seats, reduce the standard divisor. If the sum of the upper quotas is greater than the number of seats, increase the standard divisor. Return to Step 2 using the modified divisor.

Example 11.4.8: Hawaiian School Districts

As in earlier examples, suppose that the Hawaii State Department of Education has a budget for 616 schools and is doing a research study to determine the equitable number of schools to have in each of the five counties based on the residents under the age of 19. Use the data in the following table and the Adams method to apportion the schools to the counties.

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Residents under Age 19	46,310	224,230	20	16,560	38,450	325,570

Answer

Step 1: The steps of finding the standard divisor and each state's quota are the same in the Jefferson and Adams methods. As in Example 11.37, the standard divisor is 528.52.

Step 2: Find each state's upper quota and their sum:

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Standard Quota	$\frac{46,310}{528.52} \approx 87.62$	$\frac{224,230}{528.52} \approx 424.26$	$\frac{20}{528.52} \approx 0.04$	$\frac{16,560}{528.52} \approx 31.33$	$\frac{38,450}{528.52} \approx 72.75$	616
Upper Quota	88	425	1	32	73	619

Step 3: Compare the sum of the states' upper quotas, 619, to the house size, 616. Since 619 is greater than 616, we need to reduce the size of the quotas. Use a modified divisor that is greater than the standard divisor of 528.52. Try 534.00.

Step 4: Find each state's modified quota, upper quota, and the sum of the upper quotas based on the modified divisor of 534:

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Modified Quota	$\frac{46,310}{534.00} \approx 86.72$	$\frac{224,230}{534.00} \approx 419.91$	$\frac{20}{534.00} \approx 0.04$	$\frac{16,560}{534.00} \approx 31.01$	$\frac{38,450}{534.00} \approx 72.00$	616
Upper Quota	88	420	1	32	72	613

Step 5: The new sum of the upper quotas is 3 units less than 616. Larger quotas are needed. So, decrease the divisor to a value between 534.00 and 528.52. Try 532.00.

Step 6: Find each state's modified quota, upper quota, and the sum of the upper quotas based on the modified divisor of 532.00:

County	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Modified Quota	$\frac{46,310}{532.00} \approx 87.05$	$\frac{224,230}{532.00} \approx 421.48$	$\frac{20}{532.00} \approx 0.04$	$\frac{16,560}{532.00} \approx 31.13$	$\frac{38,450}{532.00} \approx 72.27$	616
Upper Quota	88	422	1	32	73	616

Step 7: The new sum of the upper quotas equals the house size. The apportionment is complete.

The apportionment is Hawaii County 88, Honolulu County 422, Kalawao County 1, Kauai 32, and Maui 73 schools.

When using Adams's method, just as with Jefferson's method, the modified divisors you use may be different from what another person chooses, but final apportionment values will be the same.

Your Turn 11.4.8

There are four states in Imaginaria: Fictionville, Pretendstead, Illusionham, and Mythbury. Assume there will be 35 seats in the legislature of Imaginaria. Use Adams's method of apportionment to determine the number of seats in each state based on the populations in the table below. How many seats would each state receive?

	Fictionville	Pretendstead	Illusionham	Mythbury	Total
Population	71,000	117,000	211,000	1,194,000	1,593,000

In this apportionment, Mythbury received less than the state's lower quota. So, this apportionment is an example of a scenario in which the Adams's method violates the quota rule.

Adams's method of apportionment violates the quota rule.

So far, only Hamilton's method satisfies the quota rule, but there is one more apportionment method you should consider for Imaginaria.

Video

ams Method Apportionment Calcul
Hearn
Adams Method of Apportionment Step by Step, and the FREE Online Calculator!
Hearn Math
Watch on YouTube

Adams Method Apportionment Calculator

Tech Check: Apportionment Calculators

Check out websites such as Ms. Hearn Math for a free Adams Method apportionment calculator.

This can be a useful tool to confirm your results!

Webster's Method of Apportionment

Webster's method of apportionment is another method of apportionment that is based on a modified divisor. However, instead of basing the changes on the sum of the lower quotas, as Jefferson did or the sum of the upper quotas as Adams did, Webster used traditional rounding.

To apply Webster's method of apportionment, there are four steps we take:

1. Find the standard divisor.
2. Find each state's quota. This will be the standard quota the first time Step 2 is completed, and the standard divisor is used, but Step 2 may be repeated as needed using a modified divisor and resulting in modified quotas.
3. Round each state's quota to the nearest whole number and find the sum of these values.
4. If the sum of the rounded quotas equals the number of seats, the apportionment is complete. If the sum of the rounded quotas is less than the number of seats, reduce the divisor. If the sum of the rounded quotas is greater than the number of seats, increase the divisor. Return to Step 2 using the modified divisor.

When using Webster's method, just as with Jefferson's method, the modified divisors you use may be different from what another person chooses, but final apportionment values will be the same.

Example 11.4.9: Hawaiian School Districts

Use the data in the table below to apportion 616 schools to Hawaiian counties. This time, use Webster's method.

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Residents under Age 19	46,310	224,230	20	16,560	38,450	325,570

Answer

To apply Webster's method of apportionment, there are four steps we take:

Step 1: The processes of finding the standard divisor and standard quota are the same in the Jefferson, Adams, and Webster's methods. As in the previous examples, the standard divisor is 528.52.

Step 2: Find each state's rounded quota and their sum:

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Standard Quota	$\frac{46{,}310}{528.52} \approx 87.62$	$\frac{224{,}230}{528.52} \approx 424.26$	$\frac{20}{528.52} \approx 0.04$	$\frac{16{,}560}{528.52} \approx 31.33$	$\frac{38{,}450}{528.52} \approx 72.75$	616
Rounded Quota	88	424	1	31	73	617

Step 3: Compare the sum of the states' rounded quotas, 617, to the house size, 616. Since 617 is greater than 616, we need to reduce the size of the quotas. Use a modified divisor that is greater than the standard divisor of 528.52. Try 534.00.

Step 4: Find each state's modified quota, rounded quota, and the sum of the rounded quotas based on the modified divisor of 534:

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Modified Quota	$\frac{46{,}310}{534.00} \approx 86.72$	$\frac{224{,}230}{534.00} \approx 419.91$	$\frac{20}{534.00} \approx 0.04$	$\frac{16{,}560}{534.00} \approx 31.01$	$\frac{38{,}450}{534.00} \approx 72.00$	616
Upper Quota	87	420	1	31	72	612

Step 5: The new sum of the rounded quotas is 4 units less than 616. Larger quotas are needed. So, decrease the divisor to a value between 534.00 and 528.52. Try 530.00.

Step 6: Find each state's modified quota, rounded quota, and the sum of the rounded quotas based on the modified divisor of 530.00:

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Modified Quota	$\frac{46{,}310}{530.00} \approx 87.38$	$\frac{224{,}230}{530.00} \approx 423.08$	$\frac{20}{530.00} \approx 0.04$	$\frac{16{,}560}{530.00} \approx 31.25$	$\frac{38{,}450}{530.00} \approx 72.55$	616
Upper Quota	87	423	1	31	73	615

Step 7: The new sum of the rounded quotas is 1 unit less than 616. Larger quotas are needed. So, decrease the divisor to a value between 528.52 and 530.00. Try 529.50.

Step 8: Find each state's modified quota, rounded quota, and the sum of the rounded quotas based on the modified divisor of 529.50:

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Modified Quota	$\frac{46{,}310}{529.50} \approx 87.46$	$\frac{224{,}230}{529.50} \approx 423.48$	$\frac{20}{529.50} \approx 0.04$	$\frac{16{,}560}{529.50} \approx 31.27$	$\frac{38{,}450}{529.50} \approx 72.62$	616
Upper Quota	87	423	1	31	73	615

Step 9: The new sum is still only 1 unit less than 616. Larger quotas are needed, but not much larger. So, decrease the divisor to a value between 528.52 and 529.50. Try 529.30.

Step 10: Find each state's modified quota, rounded quota, and the sum of the rounded quotas based on the modified divisor of 529.30:

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Modified Quota	$\frac{46{,}310}{529.30} \approx 87.49$	$\frac{224{,}230}{529.30} \approx 423.63$	$\frac{20}{529.30} \approx 0.04$	$\frac{16{,}560}{529.30} \approx 31.29$	$\frac{38{,}450}{529.30} \approx 72.64$	616
Upper Quota	87	424	1	31	73	616

Step 11: The new sum of the rounded quotas equals the house size. The apportionment is complete.

The apportionment is Hawaii County 87, Honolulu County 424, Kalawao County 1, Kauai 31, and Maui 73 schools.

Your Turn 11.4.9

If you use Webster's method to apportion 35 legislative seats to the 4 states of Imaginaria, Fictionville, Pretendstead, Illusionham, and Mythbury, with the populations given in the table below, what is the resulting apportionment?

	Fictionville	Pretendstead	Illusionham	Mythbury	Total
Population	71,000	117,000	211,000	1,194,000	1,593,000

So far, we know that the Hamilton method satisfies the quota rule, while the Jefferson and Adams methods do not. The apportionments in the Example and Your Turn above are both scenarios in which the Webster method satisfies the quota rule. Does it always? We have a little more work to do to find out. However, one thing is clear. Not all apportionment methods have the same results. Before you make such an important decision for Imaginaria, it's important to think about the differences in the apportionments that result from these four methods. How will the differences affect the citizens of Imaginaria?

Tech Check: Apportionment Calculators

Check out websites such as Ms. Hearn Math for a free Webster Method apportionment calculator.

This can be a useful tool to confirm your results!

Comparing Apportionment Methods

Recall that the four apportionment methods discussed in this chapter differ in two main ways:

- Whether or not a modified divisor is used
- The type of rounding of the quotas that is used

How might these differences affect Imaginarians? In the next two examples, we will compare the results when different apportionment methods are applied to the same scenario.

✓ Example 11.4.11: Hawaiian School Districts with Different Apportionment Methods

Let's use the results from Example ✓ 11.4.5, Example ✓ 11.4.8, Example ✓ 11.4.9, and Example ✓ 11.4.10 to compare the four apportionment methods we have discussed. The following table summarizes the results of the results of the Hamilton, Jefferson, Adams and Webster methods when applied to the apportionment of 616 schools to Hawaiian counties.

	Hawaii	Honolulu	Kalawao	Kauai	Maui
Under 19 years old	46,310	224,230	20	16,560	38,450
Hamilton	87	424	1	31	73
Jefferson	87	425	1	31	72
Adams	88	422	1	32	73
Webster	87	424	1	31	73

1. Do any of the apportionment methods result in the same apportionment? If so, which ones?
2. Which apportionment method would the citizens of the largest county likely favor most and least? Justify your answer.
3. As a group, which apportionment method would the citizens of the other four counties likely favor most and least? Justify your answer.

Answer

1. Yes, the Hamilton and Webster methods result in the same apportionment.
2. The largest county is Honolulu. The citizens would likely favor the Jefferson method of apportionment most since they received the most seats by that method. They would likely favor the Adams method of apportionment least because they received the least number of seats by that method.
3. As a group, the other four counties received 192 seats by either the Hamilton or Webster method, 194 seats by the Adams method, and 191 seats by the Jefferson method. They would likely favor the Adams method the most and favor the Jefferson methods the least.

🖈 Your Turn 11.4.11

In Your Turn 🖈 11.4.5, Example 🖈 11.4.8, Example 🖈 11.4.9, and Example 🖈 11.4.10 , you apportioned 35 legislative seats among the four states of Imaginaria using the Hamilton, Jefferson, Adams, and Webster methods of apportionment. To understand how the differences in the apportionments might affect Imaginarians, answer these questions.

Which apportionment method would the citizens of the largest state likely favor most and least? Justify your answer.

As a group, which apportionment method would the citizens of the other three states likely favor most and least? Justify your answer.

The Adams method favored the smaller states and the Jefferson method favored the larger states in the previous example, but is this the case in general?

Since the Jefferson method begins with the lower quotas, any adjustment to the quotas will be an increase. As you have seen, this is accomplished by using a modified divisor that is smaller than the standard divisor. The next example compares the impact of a decreasing divisor on the modified quotas of large states to the impact of the same size decrease on small states.

✓ Example 11.4.12: Effect of Decreasing Divisors on Modified Quotas

The following table displays the effect of reducing the size of the divisor. Observe the effect this has on the modified quotas of smaller states versus larger states and use the table answer each question.

		Modified Quotas		
State	Population	Divisor: 10,500	Divisor: 10,000	Divisor: 9,500
A	10,000	0.95	1	1.05
B	100,000	9.52	10	10.53
C	1,000,000	95.24	100	105.26

1. When the divisor decreases from 10,500 to 10,000, how many representatives are gained by each state based on the lower quota?
2. When the divisor decreases from 10,000 to 9,500, how many representatives are gained by each state based on the lower quota?
3. Which state gains the most representatives each time the divisor is decreased?

Answer

1. Since a state must have at least one seat, State A begins with 1 seat and still has one seat. State B begins with 9 seats and increases to 10 seats. State C begins with 95 seats and increases to 100 seats. So, State A gains 0, B gains 1, and C gains 5 seats.
2. State A begins with 1 and still has 1. State B begins with 10 and still has 10. State C begins with 100 and increases to 105. So, State A gains 0, State B gains 0, and State C gains 5.
3. State C, the largest state, gains the most representatives each time the divisor is decreased.

This example demonstrates that the Jefferson method is biased toward states with larger populations because the modified divisor is smaller than the standard divisor. On the other hand, the Adams's method, which begins with the upper quotas, must increase the standard divisor in order to reduce the quotas. Once again, the effect on the number of seats is greater for the larger states, but this time they are decreased. This means that the Adams's method favors states with smaller populations.

Your Turn 11.4.12

The following table displays the effect of increasing the size of the divisor. Observe the effect this has on the modified quotas of smaller states versus larger states and use the table to answer each question.

		Modified Quotas		
State	Population	Divisor: 11,500	Divisor: 12,000	Divisor: 12,500
A	10,000	0.87	0.83	0.8
B	100,000	8.7	8.33	8
C	1,000,000	86.96	83.33	80

When the divisor increases from 11,500 to 12,000, how many representatives are lost by each state based on the upper quota?

When the divisor increases from 12,000 to 12,500, how many representatives are lost by each state based on the upper quota?

Which state loses the most representatives each time the divisor is increased?

Flaws in Apportionment Methods

As we have seen, different apportionment methods can have the same results in some scenarios but different results in others. Citizens of states which receive fewer seats with a particular apportionment method will view the apportionment method as flawed and argue in favor of a different method. This inevitably creates debates regarding the use of one method over another. Methods that favor larger states are likely to be challenged by smaller states, methods that favor smaller states are likely to be challenged by larger states, and methods that violate the quota rule are likely to be challenged by states of any size depending on the circumstances.

Suppose that the State of Hawaii House of Representatives had 51 representatives, each with their own district. Imagine that redistricting were underway, and the representative districts were to be apportioned to each of five counties based on population. The following table shows the apportionment that would result from the use of the Jefferson, Adams, and Webster methods of apportionment.

	Hawaii	Honolulu	Kalawao	Kauai	Maui
Population	201,500	974,600	100	72,300	167,400
Lower Quota	7	35	0	2	6
Upper Quota	8	36	1	3	7
Jefferson	7	35	1	2	6
Adams	7	34	1	3	6
Webster	7	34	1	3	6

From the table, you can see that Hawaii, Kalawao, and Maui receive the same number of seats regardless of the method used. However, citizens of Honolulu would likely reject the Adams and Webster methods arguing that they violate the quota rule. Similarly, citizens of Kauai would probably reject the Jefferson method based on the argument that it unfairly favors the larger states. This scenario demonstrates that the Adams and Webster methods violate the quota rule, but the Jefferson method also violates the quota rule at times. **The Hamilton method is the only method that satisfies the quota rule in all scenarios.** It also consistently favors neither larger nor smaller states. Unfortunately, it can have some strange and results in certain circumstances, which you will see in the next section.

Who Knew?: Gerrymandering: A Subtle Way to Impact Apportionment

In addition to your choice of voting method and your choice of apportionment method, there is another important decision to make which could potentially have a huge impact on the fairness of elections in Imaginaria—the creation of electoral districts. In example above, we imagined that there were 51 state legislators in Hawaii, each representing their own district. But how did the legislators decide on the boundaries for these districts? Typically, boundaries are drawn so that each district has approximately the same number of residents, but the percentage of residents in each district with a particular political affiliation can swing the power from one group to another. When the districts are drawn to impact the power of a political party, ethnic or racial group, or other group, this is called gerrymandering. For example, districts can be drawn so that a particular group is spread thinly across districts, increasing the likelihood that they will not have strong representation.

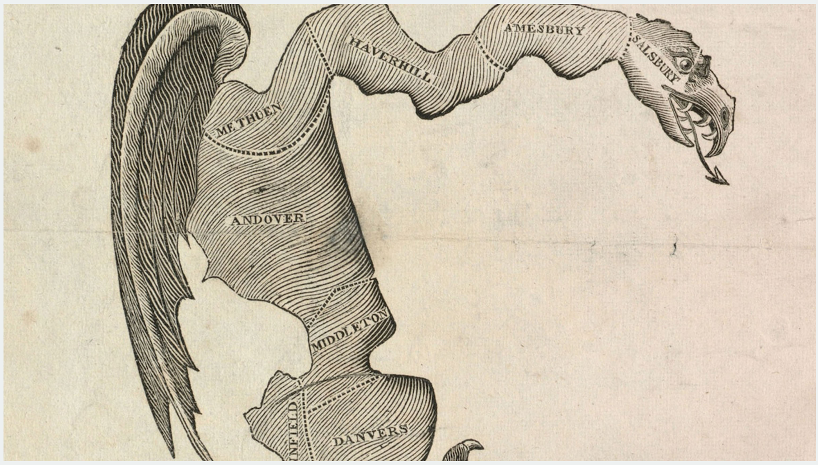

Figure 11.4.2: This cartoon map conveys the idea that the drawing of the map may impact election outcomes. (credit: “The Gerry Mander”/Wikimedia Commons, Public Domain)

"The Gerry-mander" first appeared in this cartoon-map in the Boston Gazette, March 26, 1812, and was soon reproduced in several other Massachusetts newspapers in response to election district changes initiated by governor Eldridge Gerry. Note that while the practice is named after him, Gerry was not the first to employ it.

People in Mathematics: Jonathan Mattingly

Jonathan Mattingly is a mathematician who was featured in a *Nature* article titled “The Mathematicians Who Want to Save Democracy”. Mattingly is a mathematician at Duke University in North Carolina and he runs election simulations based on alternate versions of electoral districts in order to analyze the effects of gerrymandering. He has even been asked to testify as an expert witness in court. Mattingly and other mathematicians who are working on the problem will potentially have an impact on the redistricting that will occur as a result of the 2020 census. (Carrie Arnold, “The Mathematicians Who Want to Save Democracy,” *Nature 546*, 200–202, 2017.)

Check Your Understanding

1. Which of the four apportionment methods discussed in this section does not use a modified divisor?
2. Which of the four apportionment methods discussed in this section satisfies the quota rule?
3. Which of the four apportionment methods discussed in this section is biased toward states with larger populations?
4. Which of the four apportionment methods discussed in this section is biased toward states with smaller populations?
5. Which of the four apportionment methods discussed in this section begin the apportionment with a state’s upper quota and adjust down?
6. Which of the four apportionment methods discussed in this section begin the apportionment with a state’s lower quota and adjust up?
7. Which of the four apportionment methods discussed in this section use traditional rounding?
8. Does the change from a standard divisor to a modified divisor tend to change the number of seats for larger or smaller states more?

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11.4.0: Exercises

For the following exercises, use the standard quotas given in the table below.

	State A	State B	State C	State D	State E	State F	Total Seats
Scenario X	17.63	26.62	10.81	16.01	13.69	15.24	100
Scenario Y	12.37	7.59	71.71	6.75	5.76	20.81	125
Scenario Z	3.53	31.56	2.95	5.12	9.84	NA	53

? Exercise 11.4.0.1

Round the standard quota for each state in Scenario X using traditional rounding. Find the sum of the modified quotas. What is the difference between the sum and the house size?

? Exercise 11.4.0.2

Round the standard quota for each state in Scenario Y using traditional rounding. Find the sum of the modified quotas. What is the difference between the sum and the house size?

? Exercise 11.4.0.3

Round the standard quota for each state in Scenario Z using traditional rounding. Find the sum of the modified quotas. What is the difference between the sum and the house size?

? Exercise 11.4.0.4

Find the lower quota for each state in Scenario Y. If each state is allocated its lower quota, how many seats remain to be allocated?

? Exercise 11.4.0.5

Find the lower quota for each state in Scenario X. If each state is allocated its lower quota, how many seats remain to be allocated?

? Exercise 11.4.0.6

Find the lower quota for each state in Scenario Z. If each state is allocated its lower quota, how many seats remain to be allocated?

? Exercise 11.4.0.7

Find the upper quota for each state in Scenario X and determine how much the sum of the upper quotas exceeds the house size.

? Exercise 11.4.0.8

Find the upper quota for each state in Scenario Y and determine how much the sum of the upper quotas exceeds the house size.

? Exercise 11.4.0.9

Find the upper quota for each state in Scenario Z and determine how much the sum of the upper quotas exceeds the house size.

Exercise 11.4.0.10

Determine the Hamilton apportionment for Scenario Y.

Exercise 11.4.0.11

Determine the Hamilton apportionment for Scenario X.

Exercise 11.4.0.12

Determine the Hamilton apportionment for Scenario Z.

For the following exercises, use the information in the table below, which gives standard and final quotas for Methods X, Y, and Z.

	State A	**State B**	**State C**	**State D**	**State E**
Standard Quota	1.67	3.33	5.00	6.67	8.33
Method X	2	2	5	7	9
Method Y	1	3	5	7	9
Method Z	1	3	5	6	10

Exercise 11.4.0.13

Does the apportionment resulting from method X satisfy the quota rule? Why or why not?

Exercise 11.4.0.14

Does the apportionment resulting from method Z satisfy the quota rule? Why or why not?

Exercise 11.4.0.15

Does the apportionment resulting from method Y satisfy the quota rule? Why or why not?

In the movie *Black Panther*, the hero lives in the fictional country of Wakanda. Imagine that 111 Vibranium artifacts must be distributed among the fortress cities, or birnin, of Wakanda based on the population of each birnin. Use the population and standard quota information in the table below for the following exercises.

	Birnin Djata (D)	Birnin T'Chaka (T)	Birnin Zana (Z)	Birnin S'Yan (S)	Birnin Bashenga (B)	Birnin Azzaria (A)	Total
Residents	26,000	57,000	27,000	18,000	64,000	45,000	237,000
Standard Quota	12.18	26.70	12.65	8.43	29.98	21.08	111

Exercise 11.4.0.16

Modify the standard quota for each state using traditional rounding. Find the sum of the modified quotas. What is the difference between the sum and the house size?

? Exercise 11.4.0.17

Find the standard lower quota for each state. If each state is allocated its lower quota, how many seats remain to be allocated?

? Exercise 11.4.0.18

Find the standard upper quota for each state and determine how much the sum of the upper quotas exceeds the house size.

? Exercise 11.4.0.19

Use the Hamilton method to apportion the artifacts.

? Exercise 11.4.0.20

Find the modified lower quota for each state using a modified divisor of 2,000. Is the sum of the modified quotas too high, too low, or equal to the house size?

? Exercise 11.4.0.21

Find the modified lower quota for each state using a modified divisor of 2,100. Is the sum of the modified quotas too high, too low, or equal to the house size?

? Exercise 11.4.0.22

Use the Jefferson method to apportion the artifacts. Determine whether it is necessary to modify the divisor. If so, indicate the value of the modified divisor.

? Exercise 11.4.0.23

Does the Jefferson method result in an apportionment that satisfies or violates the quota rule in this scenario?

? Exercise 11.4.0.24

Find the modified upper quota for each state using a modified divisor of 2,250. Is the sum of the modified quotas too high, too low, or equal to the house size?

? Exercise 11.4.0.25

Find the modified upper quota for each state using a modified divisor of 2,150. Is the sum of the modified quotas too high, too low, or equal to the house size?

? Exercise 11.4.0.26

Use the Adams method to apportion the artifacts. Determine whether it is necessary to modify the divisor. If so, indicate the value of the modified divisor.

? Exercise 11.4.0.27

Does the Adams method result in an apportionment that satisfies or violates the quota rule in this scenario?

Exercise 11.4.0.28

Which method of apportionment, Jefferson or Adams, is a resident of Birnin T'Chaka likely to prefer? Justify your answer.

Exercise 11.4.0.29

Use the Webster method to apportion the artifacts. Determine whether it is necessary to modify the divisor. If so,indicate the value of the modified divisor.

Exercise 11.4.0.30

Does the Webster method result in an apportionment that satisfies or violates the quota rule in this scenario?

Exercise 11.4.0.31

Which of the four methods of apportionment from this section (Hamilton, Jefferson, Adams, or Webster) are the residents of Birnin S'Yan likely to prefer? Justify your answer.

Children from five families—the Chorro family, the Eswaran family, the Javernick family, the Lahde family, and the Stolly family—joined a town Easter egg hunt. When they returned with their baskets, they had 827 eggs! They decided to share their eggs amongst the families based on the number of children in each family. Use the population and standard quota information in the table below for the following exercises.

	(C) Chorro	(E) Eswaran	(J) Javernick	(L) Lahde	(S) Stolly	Total
Children	3	2	4	1	5	15
Standard Quota	155.04	103.36	206.72	103.36	258.40	827

Exercise 11.4.0.32

Modify the standard quota for each state using traditional rounding. Find the sum of the modified quotas. What is the difference between the sum and the house size?

Exercise 11.4.0.33

Find the standard lower quota for each state. If each state is allocated its lower quota, how many seats remain to be allocated?

Exercise 11.4.0.34

Find the standard upper quota for each state, and determine how much the sum of the upper quotas exceeds the house size.

Exercise 11.4.0.35

Use the Hamilton method to apportion the Easter eggs.

Exercise 11.4.0.36

Find the modified lower quota for each state using a modified divisor of 0.01800. Is the sum of the modified quotas too high, too low, or equal to the house size?

Exercise 11.4.0.37

Find the modified lower quota for each state using a modified divisor of 0.01810. Is the sum of the modified quotas too high, too low, or equal to the house size?

Exercise 11.4.0.38

Use the Jefferson method to apportion the Easter eggs. Determine whether it is necessary to modify the divisor. If so, indicate the value of the modified divisor.

Exercise 11.4.0.39

Does the Jefferson method result in an apportionment that satisfies or violates the quota rule in this scenario?

Exercise 11.4.0.40

Find the modified upper quota for each state using a modified divisor of 0.0182. Is the sum of the modified quotas too high, too low, or equal to the house size?

Exercise 11.4.0.41

Find the modified upper quota for each state using a modified divisor of 0.01816. Is the sum of the modified quotas too high, too low, or equal to the house size?

Exercise 11.4.0.42

Use the Adams method to apportion the Easter eggs. Determine whether it is necessary to modify the divisor. If so, indicate the value of the modified divisor.

Exercise 11.4.0.43

Does the Adams method result in an apportionment that satisfies or violates the quota rule in this scenario?

Exercise 11.4.0.44

Use the Webster method to apportion the Easter eggs. Determine whether it is necessary to modify the divisor. If so, indicate the value of the modified divisor.

Exercise 11.4.0.45

Does the Webster method result in an apportionment that satisfies or violates the quota rule in this scenario?

For the following exercises, use this information: Suppose that the State of Delaware received 2,000 packs of COVID-19 vaccines, with ten doses per pack. These (unopened) packs must be distributed to the three counties based on total population. Use the population information in the table below to determine how many packs of vaccine will be distributed to each county based on the given apportionment method.

	(N) New Castle	(K) Kent	(S) Sussex
Residents	558,753	180,786	234,225

? Exercise 11.4.0.46

Hamilton’s Method

? Exercise 11.4.0.47

Jefferson’s Method

? Exercise 11.4.0.48

Adams's Method

? Exercise 11.4.0.49

Webster’s Method

? Exercise 11.4.0.50

Notice that the apportionments found in questions 46, 47, 48, and 49 all satisfy the quota rule. Does this contradict the statement, “The Jefferson, Adams, and Webster methods of apportionment all violate the quota rule”? Why or why not?

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11.5: Fairness in Apportionment Methods

Learning Objectives

1. Describe and illustrate the Alabama paradox.
2. Describe and illustrate the population paradox.
3. Describe and illustrate the new-states paradox.
4. Identify ways to promote fairness in apportionment methods.

Apportionment Paradoxes

The citizens of Imaginaria will want the apportionment method to be as fair as possible. There are certain characteristics that they would reasonably expect from a fair apportionment.

- If the house size is increased, the state quotas should all increase or remain the same but never decrease.
- If one state's population is growing more rapidly than another state's population, the faster growing state should not lose a seat while a slower growing state maintains or gains a seat.
- If there is a fixed number of seats, adding a new state should not cause an existing state to gain seats while others lose them.

However, apportionment methods are known to contradict these expectations. Before you decide on the right apportionment for Imaginarians, let's explore the apportionment paradox, a situation that occurs when an apportionment method produces results that seem to contradict reasonable expectations of fairness.

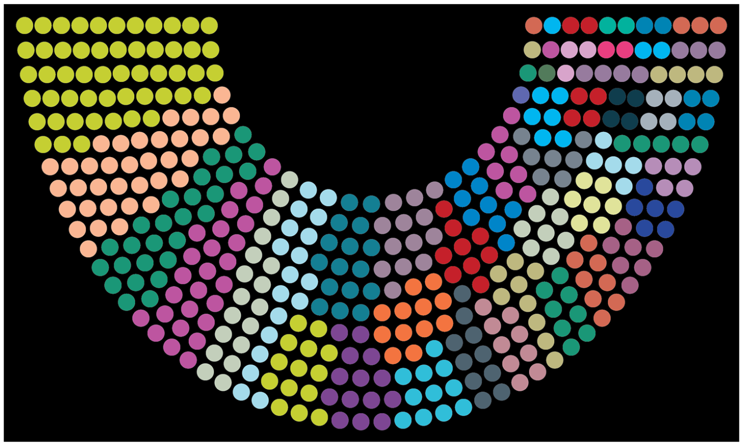

Figure 11.5.1: In this seating chart for the House of Representatives, each color indicates representatives from a particular state.

There is a lot that the founders of Imaginaria can learn from U.S. history. The constitution of the United States requires that the seats in the House of Representatives be apportioned according to the results of the census that occurs every decade, but the number of seats and the apportionment method is not stipulated. Over the years, several different apportionment methods and house sizes have been used and scrutinized for fairness. This scrutiny has led to the discovery of several of these apportionment paradoxes.

The Alabama Paradox

At the time of the 1880 U.S. Census, the Hamilton method of apportionment had replaced the Jefferson method. The number of seats in the House of Representatives was not fixed. To achieve the fairest apportionment possible, the house sizes were chosen so that the Hamilton and Webster methods would result in the same apportionment. The chief clerk of the Census Bureau calculated the apportionments for house sizes between 275 and 350. There was a surprising result that became known as the Alabama paradox, which is said to occur when an increase in house size reduces a state's quota. Alabama would receive eight seats with a house size of 299, but only receive seven seats if the house size increased to 300. (Michael J. Caulfield (Gannon University), "Apportioning Representatives in the United States Congress - Paradoxes of Apportionment," *Convergence* (November 2010), DOI:10.4169/loci003163)

Example $11.5.1$: The 1880 Alabama Quota

The 1880 census recorded the population of Alabama as 1,513,401 and that of the U.S. as 62,979,766.

1. Calculate the standard divisor and standard quota for the State of Alabama based on a house size of 299.
2. Calculate the standard divisor and standard quota for the State of Alabama based on a house size of 300.
3. Did the standard quota increase or decrease when the house size increased?
4. Consider the Hamilton method of apportionment. Explain how Alabama's final quota could be smaller with a larger standard quota.

Answer

1. The standard divisor is $62{,}979{,}766 \div 299 = 210, \text{resulting in } 634.6689$ citizens per seat. The standard quota for Alabama is $1{,}513{,}401 \div 210{,}634.7 = 7.1850$ seats.
2. The standard divisor is $62{,}979{,}766 \div 300 = 209, \text{resulting in } 932.5533$ citizens per seat. The standard quota for Alabama is $1{,}513{,}401 \div 209{,}932.6 = 7.2090$ seats.
3. The standard quota increased.
4. In each case, the state would receive the lower quota of 7 and then be awarded one more seat if the fractional part of the standard quota were high enough relative to the fractional parts of the other states' standard quotas. When the house size was 299, Alabama received one of the remaining seats after the lower quotas were distributed. When the house size was 300, Alabama did not receive one of the remaining seats after the lower quotas were distributed. It must have been the case that either the fractional part 0.2090 ranked lower amongst the other fractional parts of the state quotas than the fractional part 0.1850 did, or there were fewer remaining seats, or both.

After the 1900 census, the Census Bureau again calculated the apportionment based on various house sizes. It was determined that Colorado would receive three seats with a house size of 356, but only two seats with a house size of 357.

Your Turn $11.5.1$

The 1900 census recorded the population of Colorado as 539,700 and that of the U.S. as 76,212,168.

a. Calculate the standard divisor and standard quota for the State of Colorado based on a house size of 356.
b. Calculate the standard divisor and standard quota for the State of Colorado based on a house size of 357.
c. Did the standard quota increase or decrease when the house size increased?
d. Consider the Hamilton method of apportionment. Explain how Colorado's final quota could be smaller with a larger standard quota.

Example $11.5.2$: Hamilton's Method and the Alabama Paradox

Suppose that States A and B each have a population of 6, while State C has a population of 2.

1. Use the Hamilton method to apportion 10 seats.
2. Use the Hamilton method to apportion 11 seats.
3. Does this example demonstrate the Alabama paradox? If so, how?

Answer

1. **Step 1:** The total population is 14. The standard divisor is $14 \div 10 = 1.4$ individuals per seat.

   Step 2: The states' standard quotas are as follows: A $6 \div 1.4 \approx 4.29$, B $6 \div 1.4 \approx 4.29$, and C $2 \div 1.4 \approx 1.43$.

   Step 3: The states' lower quotas are as follows: A 4, B 4, and C 1.

   Step 4: The sum of the lower quotas is 9, which means there is one seat remaining to be apportioned. State C has the highest fractional part and receives the additional seat.

   Step 5: The final apportionment is as follows: A 4, B 4, and C 2, which sums to 10.

2. **Step 1:** The total population is 14. The standard divisor is $14 \div 11 \approx 1.2727$ individuals per seat.

   Step 2: The states' standard quotas are: A $6 \div 1.2727 \approx 4.71$, B $6 \div 1.2727 \approx 4.71$, and C $2 \div 1.2727 \approx 1.57$.

Step 3: The states’ lower quotas are: A 4, B 4, and C 1.

Step 4: The sum of the lower quotas is 9, which means there are two seats remaining to be apportioned. A and B have the highest fractional parts and receive the additional seats.

Step 5: The final apportionment is: A 5, B 5, and C 1.

3. Yes, this demonstrates the Alabama paradox because State C receives two seats if the house size is 10, but only one seat if the house size is 11.

Your Turn 11.5.2

Suppose that the founders of Imaginaria decide to have a parliament that apportions seats to four political parties based on the portion of the vote each party has earned. Also, suppose that Party A has 56.7 thousand votes, Party B has 38.5 thousand votes, Party C has 4.2 thousand votes, and Party D has 0.6 thousand votes.

a. Use the Hamilton method to apportion 323 seats.
b. Use the Hamilton method to apportion 324 seats.
c. Does this example demonstrate the Alabama paradox? If so, how?

The Population Paradox

It is important for the founders of Imaginaria to keep in mind that the populations of states change as time passes. Some populations grow and some shrink. Some populations increase by a large amount while others increase by a small amount. These changes may necessitate a reapportionment of seats, or the recalculation of state quotas due to a change in population. It would be reasonable for Imaginarians to expect that the state with a population that has grown more than others will gain a seat before the other states. Once again, this is not always the case with the Hamilton method of apportionment. The population growth rate of a state is the ratio of the change in the population to the original size of the population, often expressed as a percentage. This value is positive if the population is increasing and negative if the population is decreasing. The population paradox occurs when a state with an increasing population loses a seat while a state with a decreasing population either retains or gains seats. More generally, the population paradox occurs when a state with a higher population growth rate loses seats while a state with a lower population growth rate retains or gains seats.

Notice that the population paradox definitions has two parts. If either part applies, then the population paradox has occurred. The first part of the definition only applies when one state has a decreasing population. The second part of the definition applies in all situations, whether there is a state with a decreasing population or not. It will be easier to identify situations that involve a decreasing population. The other situations requires the calculation of a growth rate. The reason that we don't have to calculate a growth rate when one state has a decreasing population and the other has an increasing population is that increasing population has a positive growth rate which is always greater than the negative growth rate of a decreasing population.

Checkpoint

A state must lose a seat in order for the population paradox to apply. It is not enough for a state with a lower growth rate to gain a seat while a state with a higher growth rate retains the same number of seats.

Example 11.5.3: Apportionment of Respirators to Hospitals

Suppose that 18 respirators are to be apportioned to three hospitals based on their capacities. The Hamilton method is used to allocate the respirators in 2020, then to reallocate based on new capacities in 2021. The results are shown in the table below. How does this demonstrate the population paradox?

Hospital	Capacity in 2020	Respirators in 2020	Change in Capacity	Growth Rate = $\frac{\text{Change in Capacity}}{\text{Capacity in}}$20 20Change in Capacity Capacity in 2020	Capacity in 2021	Respirators in 2021

Hospital	Capacity in 2020	Respirators in 2020	Change in Capacity	Growth Rate = $\frac{\text{Change in Capacity}}{\text{Capacity in}}$20 20Change in Capacity Capacity in 2020	Capacity in 2021	Respirators in 2021
A	825	9	57	$\frac{57}{825} \approx 0.0691 = 6.91\%$	882	9
B	613	7	13	$\frac{13}{613} \approx 0.0212 = 2.12\%$	626	6
C	239	2	3	$\frac{3}{239} = 0.0126 = 1.26\%$	242	3

Answer

Hospital B lost a respirator while hospital C gained one, even though hospital B had a higher growth rate than hospital C.

Your Turn 11.5.3

Suppose that 18 respirators are to be apportioned to three hospitals based on their capacities. The respirators are allocated based on the Hamilton method in 2020, then reallocated based on new capacities in 2021. The results are shown in the table below. How does this demonstrate the population paradox?

Hospital	Capacity in 2020	Respirators in 2020	Change in Capacity	Growth Rate \ (\frac\)	Capacity in 2021	Respirators in 2021
A	237	5	6	$6 \div 237 = 0.0253$ $= 2.53\%$	243	6
B	889	21	69	$69 \div 889 \approx 0.0776$ $= 7.76\%$	958	21
C	674	16	18	$18 \div 674 \approx 0.0267$ $= 2.67\%$	692	15

FORMULA

The growth rate of a population can be calculated by subtracting the previous population size from the current population size, and then dividing the difference by the previous population size.

$$\text{population growth rate} = \text{mfrac}$$

sizeprevious population sizepopulation growth rate=current population size−previous population sizeprevious population size

Checkpoint

Make sure to calculate the subtraction before the division. If you are entering the values in a calculator, it is helpful to put parentheses around the subtracted terms.

Example 11.5.4: The Congress of Costaguana

The country of Costaguana has three states: Azuera with a population of 894,000; Punta Mala with a population of 696,000; and Esmeralda with a population of 215,000. There are 38 seats in the Congress of Costaguana. The apportionment of the seats is determined by Hamilton's method to be: 19 for Azuera, 15 for Punta Mala, and 4 for Esmerelda. A census reveals that the population has grown and the seats must be reapportioned. If Azuera now has 953,000 residents, Punta Mala now has 706,000 residents, and Esmerelda now has 218,000 residents, how many seats will each state receive upon reapportionment? How is this an example of the population paradox?

Answer

The Hamilton reapportionment is: 19 for Azuera, 14 for Punta Mala, and 5 for Esmerelda. This is an example of the population paradox because Punta Mala lost a seat to Esmerelda, even though Punta Mala's population grew by 1.44 percent while Esmerelda's only grew by 1.40 percent.

Your Turn 11.5.4

The country of Elbonia has three states: Mudston with a population of 866,000; WallaWalla with a population of 626,000; and Dilberta with a population of 256,000. There are 38 seats in the Congress of Elbonia. The apportionment of the seats is determined by Hamilton's method to be: 19 for Mudston, 14 for WallaWalla, and 5 for Dilberta. A census reveals that the population has grown and the seats must be reapportioned. If Mudston now has 921,000 residents, WallaWalla now has 640,000 residents, and Dilberta now has 260,000 residents, how many seats will each state receive upon reapportionment? How is this an example of the population paradox?

The New-States Paradox

As a founder of Imaginaria, you might also consider the possibility that Imaginaria could annex nearby lands and increase the number of states. This occurred several times in the United States such as when Oklahoma became a state in 1907. The House size was increased from 386 to 391 to accommodate Oklahoma's quota of five seats. When the seats were reapportioned using Hamilton's method, New York lost a seat to Maine despite the fact that their populations had not changed. This is an example of the new-state paradox, which occurs when the addition of a new state is accompanied by an increase in seats to maintain the standard ratio of population to seats, but one of the existing states loses a seat in the resulting reapportionment.

✓ Example 11.5.51: New State of Oscuridad

The country of San Lorenzo has grown from two states to three. The house size of the congress has been increased by eight and the seats have been reapportioned to accommodate the new state of Oscuridad. The constitution mandates the use of the Hamilton method of apportionment. Use this information and the following table to answer the questions.

State	Population (in hundreds)	Original Apportionment	Reapportionment
Clara	7,100	39	40
Velasco	9,080	51	50
Oscuridad	1,500	Not Applicable	8

1. What was the original house size?
2. What is the new house size?
3. How is this reapportionment an example of the new-states paradox?

Answer

1. $39+51=90$
2. $40+50+8=98$
3. The original state of Velasco lost a seat to the original state of Clara when the new state of Oscuridad was added.

Your Turn 11.5.5

The country of Narnia has grown from two states to three. The house size of the congress has been increased by five and the seats have been reapportioned to accommodate the new state of Chippingford. The constitution mandates the use of the Hamilton method of apportionment. Use this information and the following table to answer the questions.

State	Population (in hundreds)	Original Apportionment	Reapportionment
Beruna	7600	39	40

State	Population (in hundreds)	Original Apportionment	Reapportionment
Beaversdam	9720	51	50
Chippingford	1000	Not Applicable	5

a. What was the original house size?
b. What is the new house size?
c. How is this reapportionment an example of the new-states paradox?

Example 11.5.6: The Growing Country of Gulliversia

The country of Gulliversia has two states: Lilliput with a population of 700,000 and Brobdingnag with a population of 937,000. The constitution of Gulliversia requires that the 90 congressional seats be apportioned by Hamilton’s method. Lilliput has received 38 seats while Brobdingnag has received 52 seats. Recently, the island of Houyhnhnmsland with a population of 170,000 has joined the union, becoming a state of Gulliversia. When Houyhnhnmsland is included, nine additional seats must be apportioned to maintain the same ratio of seats to citizens. Use Hamilton’s method to reapportion the 99 seats to the three states. How is the resulting apportionment an example of the new-states paradox?

Answer

The reapportionment gives 39 seats to Lilliput, 51 seats to Brobdingnag, and 9 seats to Houyhnhnmsland. This is an example of the new-states paradox because the original state of Brobdingnag lost a seat to the original state of Lilliput when the new state was added to the union.

Your Turn 11.5.7

The country of Neverland has two states: Neverwood with a population of 760,000 and Mermaids Lagoon with a population of 943,000. The constitution of Neverland requires that the 84 congressional seats be apportioned by Hamilton’s method. Neverwood has received 37 seats while Mermaids Lagoon has received 47 seats. Recently, the island of Marooners Rock with a population of 190,000 has joined the union, becoming a state of Neverland. When Marooners Rock is included, nine additional seats must be apportioned to maintain the same ratio of seats to citizens. Use Hamilton’s method to reapportion the 93 seats to the three states. How is the resulting apportionment an example of the new-states paradox?

When a new state is added, it is necessary to determine the amount that the house size must be increased to retain the original ratio of population to seats, in other words to keep the original standard divisor. To calculate the new house size, divide the new population by the original standard divisor, and round to the nearest whole number.

FORMULA

$\text{New House Size} =$ mfracPopulationOriginal Standard Divisorrounded to the nearest whole number.New House Size=New PopulationOriginal Standard Divisorrounded to the nearest whole number.

Example 11.5.8: Oklahoma Joins the Union

Oklahoma was admitted as the 46th state on November 16, 1907. Before Oklahoma joined the union, the U.S. population was approximately 75,030,000 and the House of Representatives had 386 seats. The new state had a population of approximately 970,000. Use this information to estimate the original standard divisor to the nearest hundred, the new population, the new house size, and the number of seats Oklahoma should receive.

Answer

Step 1: $\text{Original Standard Divisor} \approx \frac{75{,}030{,}000}{386} \approx 194{,}400$Original Standard Divisor≈75,030,000386≈194,400

Step 2: New Population $\approx 75,030,000 + 970,000 = 76,000,000$

Step 3: $\text{New House Size} \approx \frac{76{,}000{,}000}{194{,}400} \approx 391$

Step 4: There are $391 - 386 = 5$ new seats to be apportioned to Oklahoma.

Your Turn $11.5.8$

New Mexico was admitted as the 47th state on January 6, 1912. Before New Mexico joined the union, the U.S. population was approximately 76,000,000 and the House of Representatives had 391 seats. The new state had a population of approximately 300,000. Use this information to estimate the original standard divisor to the nearest hundred, the new population, the new house size, and the number of seats New Mexico should receive.

The Search for the Perfect Apportionment Method

The ideal apportionment method would simultaneously satisfy the following four fairness criteria.

- Satisfy the quota rule
- Avoid the Alabama paradox
- Avoid the population paradox
- Avoid the new-states paradox

We have seen that the Hamilton method allows the Alabama paradox, the population paradox, and the new-states paradox in some apportionment scenarios. Let's explore the results of the other methods of apportionment we have discussed in some of the same scenarios.

✓ Example $11.5.9$: Orange Grove and the New-States Paradox

The incorporated town of Orange Grove consists of two subdivisions: The Oaks with 1,254 residents and The Villages with 10,746 residents. A council with 100 members supervises the municipality's operations. The council votes to annex an unincorporated subdivision called The Lakes with a population of 630. They plan to increase the size of the council to maintain the ratio of seats to residents such that the new council will have 100 seats plus the number of seats given to The Lakes. Use each of the following apportionment methods and indicated number of additional seats to find the original and new apportionment and determine whether the new-state paradox occurs.

1. Jefferson's method with five additional seats.
2. Adams's method with six additional seats.
3. Webster's method with five additional seats

Answer

1. Using a modified divisor of 119, the original apportionment would have been: The Oaks 10 and The Villages 90. Using a modified divisor of 119, the new apportionment would be: The Oaks 10, The Villages 90, and The Lakes 5. The new-state paradox does not occur.
2. Using a modified divisor of 121, the original apportionment would have been: The Oaks 11 and The Villages 89. Using a modified divisor of 121, the new apportionment would be: The Oaks 11, The Villages 89, and The Lakes 6. The new-state paradox does not occur.
3. Using the standard divisor of 120, the original apportionment would have been: The Oaks 10 and The Villages 90. Using a modified divisor of 119.5, the new apportionment would be: The Oaks 10, The Villages 90, and The Lakes 5. The new-state paradox does not occur.

Your Turn $11.5.9$

Suppose that in 2016, States A, B, and C had populations of 13 million, 12 million, and 112 million, respectively. In 2020, State A has grown by 1 million residents, State B has lost 1 million residents, and State C has gained 2 million residents. Compare the apportionments in 2016 to 2020 using each method given below. Which of the four methods violate(s) the population paradox in this scenario?

We have seen in our examples that neither the population paradox nor the new-states paradox occurred when using the Jefferson, Adams, and Webster methods. It turns out that, although all three of these divisor methods violate the quota rule, none of them ever causes the population paradox, new-states paradox, or even the Alabama paradox. On the other hand, the Hamilton method satisfies the quota rule, but will cause the population paradox, the new-states paradox, and the Alabama paradox in some scenarios.

In 1983, mathematicians Michel Balinski and Peyton Young proved that no method of apportionment can simultaneously satisfy all four fairness criteria.

There are other apportionment methods that satisfy different subsets of these fairness criteria. For example, the mathematicians, Balinski and Young who proved the Balinski-Young Impossibility Theorem created a method that both satisfies the quota rule and is free of the Alabama paradox. (Balinski, Michel L.; Young, H. Peyton (November 1974). “A New Method for Congressional Apportionment.” *Proceedings of the National Academy of Sciences*. 71 (11): 4602–4606.) However, no method may always follow the quota rule and simultaneously be free of the population paradox. (Balinski, Michel L.; Young, H. Peyton (September 1980). "The Theory of Apportionment" (PDF). Working Papers. International Institute for Applied Systems Analysis. WP-80-131.)

So, as you and your fellow founders of Imaginaria make the important decision about the right apportionment method for Imaginaria, do not look for a perfect apportionment method. Instead, look for an apportionment method that best meets the needs and concerns of Imaginarians.

Check Your Understanding

1. In the following questions, a scenario is given. Determine whether the scenario is an example of a quota rule violation, the Alabama paradox, the population paradox, the new-states paradox, or none of these.
2. A city purchased five new firetrucks and apportioned them among the existing fire stations. Although your neighborhood fire station has the same proportion of the city’s firetrucks as before the new ones were purchased, it now has one fewer.
3. The school resources officers in a county were reapportioned based on the most recent census. The number of students at Chapel Run Elementary went up while the number of students at Panther Trail Elementary went down. However, Chapel Run now has one fewer resources officer while Panther Trail has one more than it did previously.
4. The standard quota for States A, B, and C are 1.19, 2.73, and 5.71 respectively. State A received 1 seat, State B received 3 seats, and State C received 4 seats.
5. A corporation that owns several hospitals purchased an additional hospital, causing the doctor to patient ratio to decrease.
6. When the city of Cocoa annexed an adjacent unincorporated community, the number of seats on the city council was increased to maintain the standard ratio of citizens to seats, but one existing community of Cocoa still lost a seat on the city council to another existing community of Cocoa when the new community was added.
7. In the following questions, a type of scenario is described. Indicate which paradox could arise in a scenario of this kind.
8. A reapportionment occurs because the populations of the states change, and the house size remains the same.
9. A reapportionment occurs because the house size increases, and the populations of the states remain the same.
10. A reapportionment occurs because an additional state is added to the union, the populations of the original states remain the same, and the house size is increased to correspond to the population of the new state.

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11.5.0: Exercises

In the following exercises, determine whether the scenario violates the Alabama paradox. Justify your answer.

Exercise 11.5.0.1

A company with an office in each of four cities must distribute 145 new Chromebooks to the four offices. It is determined that Office A will receive 42, Office B will receive 17, Office C will receive 35, and Office D will receive 51. At the last minute, it is discovered that there are 146 Chromebooks. When they are reapportioned, Office A receives 42, Office B receives 16, Office C receives 36, and Office D receives 52.

Exercise 11.5.0.2

A county with three towns has 30 garbage trucks to apportion. Attenborough receives 6 trucks, Breckenridge receives 8 trucks, and Cabbotsville receives 16 trucks, in proportion to their populations. When one additional truck is purchased, the reallocation results in 5 trucks for Attenborough, 9 for Breckenridge, and 8 for Cabbotsville though there has been no change in populations.

Exercise 11.5.0.3

A group of 60 dentists work for a company that runs five offices. The dentists have been apportioned to the offices by the number of patients. Office A receives 14 dentists, Office B receives 11, Office C receives 11, Office D receives 12, and Office E receives 12. When a new dentist joins the group, the new apportionment gives the following: A 14, B 11, C 11, D 12, and E 13.

In the following exercises, determine whether the scenario violates the population paradox. Justify your answer.

Exercise 11.5.0.4

A company with locations in three cities plans to give 200 achievement awards that shall be apportioned to the three cities by population. City A with 9,150 employees is allotted 61 awards, City B with 6,040 employees is allotted 40 awards, and City C with 14,810 employees is allotted 99 awards to distribute. Then it is discovered that the number of employees is out of date. The awards are reallocated based on the new populations: City A with 9,180; City B with 6,040; and City C with 14,930. It turns out that the apportionment remains the same.

Exercise 11.5.0.5

A soccer club must apportion soccer balls to the teams among four age brackets based on the number of teams in each bracket. Bracket U8 receives 32 balls, U12 receives 46 balls, U15 receives 29 balls, and U18 receives 25 balls. Mid-season, the balls are reapportioned. U8 has decreased by 10 percent, U12 has increased by 20 percent, U15 remains the same, and U18 has increased by 10 percent. The reapportionment gives 33 balls to U8, 46 balls to U12, 29 balls to U15, and 14 balls to U18.

Exercise 11.5.0.6

READ (Reading Education Assistance Dogs) trains and certifies therapy animals to serve in classrooms and help children develop a love of reading in a low-stress environment. Suppose that 20 therapy teams are apportioned to three schools based on their populations. When the population of School A decreases by 10 percent, the population of School B increases by 10 percent, and the population of School C remains constant, the apportionment remains the same.

In the following exercises, determine whether the scenario violates the new-states paradox. Justify your answer.

? Exercise 11.5.0.7

A charity organization has 851 volunteers in Country A and 3449 volunteers in Country B. There are 43 lead organizers that must be apportioned to the two locations. Country A receives 9 while Country B receives 34. When the operations are expanded to Country C with 725 volunteers, 7 new lead organizers are added to the team. When the lead organizers are reapportioned, Country A receives 9, Country B receives 34, and Country C receives 7 lead organizers.

? Exercise 11.5.0.8

A country has two states. There are 22 seats in the legislature. State A has 4 seats, and State B has 18 seats. When State C joins the union, the number of representatives is increased by five. Under the new apportionment, State A receives 3 seats, State B receives 19 seats, and State C receives 5 seats.

? Exercise 11.5.0.9

In the Garunga Solar System, there are four inhabited planets. Three of the planets are members of the United Association of Garungan Planets (UAGP). The UAGP has 67 seats. Planet Angluertka has 40 seats, planet Bangluertka has 27 seats, and planet Clangluertka has 13 seats. When planet Danggluertka decides to join the UAGP, 14 seats are added to accommodate them proportionately. The new allocation of seats gives 40 seats to Angluertka, 26 seats to Bangluertka, 14 seats to Clangluertka, and 14 seats to Danggluertka.

In the following exercises, use the Hamilton method of apportionment to answer the questions.

? Exercise 11.5.0.10

When the number of seats changed from 147 to 148, the standard quotas changed from A 44.24, B 17.35, C 37.12, and D 48.29 to A 45.54, B 17.47, C 37.37, and D 48.62.

a. How did the increase in seats impact the apportionment?
b. Is this apportionment an example of a paradox? Justify your answer.

? Exercise 11.5.0.11

When the number of seats changed from 126 to 127, the standard quotas changed from A 9.57, B 29.49, C 33.89, D 28.43, and E 24.61 to A 9.65, B 29.72, C 34.16, D 38.66, and E 24.81.

a. How did the increase in seats impact the apportionment?
b. Is this apportionment an example of a paradox? Justify your answer.

? Exercise 11.5.0.12

When the number of seats changed from 25 to 26, the standard quotas changed from A 2.21, B 5.25, C 11.27, and D 6.27 to A 2.30, B 5.46, C 11.72, and D 6.52.

a. How did the increase in seats impact the apportionment?
b. Is this apportionment an example of a paradox? Justify your answer.

? Exercise 11.5.0.13

When the number of seats changed from 25 to 26, the standard quotas changed from A 2.43, B 5.42, and C 8.15 to A 2.46, B 5.47, and C 8.07.

a. How did the increase in seats impact the apportionment?
b. Is this apportionment an example of a paradox? Justify your answer.

? Exercise 11.5.0.14

The house size is 18. When the population of State A increases by 11.76 percent, State B increases by 16.22 percent, and State C increases by 12.18 percent, the standard quotas change from A 2.46, B 6.00, and C 9.53 to A 2.41, B 6.25, and C 9.34.

a. How did the change in populations impact the apportionment?
b. Is this apportionment an example of a paradox? Justify your answer.

? Exercise 11.5.0.15

The house size is 100. When the population of State A increases by 20 percent, State B increases by 10 percent, State C increases by 30 percent, and the populations of States D, E, and F remain the same. The standard quotas change from A 12.50, B 25.00, C 9.38, D 18.75, E 12.50 and F 21.88 to A 13.91, B 25.51, C 11.30, D 17.39, E 11.59, and F 20.29.

a. How did the change in populations impact the apportionment?
b. Is this apportionment an example of a paradox? Justify your answer.

? Exercise 11.5.0.16

The house size is 17. When the population of State A increases by 12.73 percent, State B increases by 15.63 percent, and State C increases by 12.90 percent, the standard quotas change from A 2.53, B 5.90, and C 8.57 to A 2.51, B 5.99, and C 8.50.

a. How did the change in populations impact the apportionment?
b. Is this apportionment an example of a paradox? Justify your answer.

? Exercise 11.5.0.17

The house size is 38 seats. When the population of State A increases by 6.60 percent, State B increases by 1.44 percent, and State C increases by 1.40 percent, the standard quotas change from A 18.82, B 14.65, and C 4.53 to A 19.59, B 14.29, and C 4.41.

a. How did the change in populations impact the apportionment?
b. Is this apportionment an example of a paradox? Justify your answer.

? Exercise 11.5.0.18

The house size is 24 seats. When the population of State A increases by 28 percent, State B increases by 26 percent, and State C increases by 15 percent, the standard quotas change from A 3.38, B 6.32, and C 14.30 to A 3.63, B 6.67, and C 13.71.

a. How did the change in populations impact the apportionment?
b. Is this apportionment an example of a paradox? Justify your answer.

? Exercise 11.5.0.19

The house size was 60. There were three states with standard quotas of A 4.18, B 15.38, and C 40.44. A fourth state was annexed, and the house size was increased to 65. The new standard quotas are A 4.17, B 15.33, C 40.32, and D 5.18.

a. How did the additional state impact the apportionment?
b. Is this apportionment an example of a paradox? Justify your answer.

? Exercise 11.5.0.20

The house size was 50. There were three states with standard quotas of A 9.41, B 24.42, and C 16.17. A fourth state was annexed, and the house size was increased to 66. The new standard quotas are A 9.36, B 24.30, C 16.09, and D 16.25.

a. How did the additional state impact the apportionment?
b. Is this apportionment an example of a paradox? Justify your answer.

Exercise 11.5.0.21

The house size was 27. There were three states with standard quotas of A 6.39, B 11.40, and C 9.21. A fourth state was annexed, and the house size was increased to 35. The new standard quotas are A 6.38, B 11.37, C 9.19, and D 8.06.

a. How did the additional state impact the apportionment?
b. Is this apportionment an example of a paradox? Justify your answer.

Exercise 11.5.0.22

The house size was 100. There were three states with standard quotas of A 26.09, B 30.43, and C 43.48. A fourth state was annexed, and the house size was increased to 122. The new standard quotas are A 26.14, B 30.50, C 43.57, and D 21.78.

a. How did the additional state impact the apportionment?
b. Is this apportionment an example of a paradox? Justify your answer.

In the following exercises, use the information in the table below.

State	A	B	C	D	E	F	G	H	I	J	K	L	P	Q	R
Population	624	1,219	979	**3,462**	**7,470**	**4,264**	**5,300**	263	809	931	781	676	**150**	**250**	**350**
Original House Size	38			**204**				126					**50**		
Updated House Size	39			**205**				127					**51**		

Exercise 11.5.0.23

Consider States A, B, and C.

a. Determine the apportionment for States A, B, and C with the original house size using the Hamilton method.
b. Determine the apportionment for States A, B, and C with the updated house size using the Hamilton method.
c. Does the change in the house size and use of the Hamilton method cause the Alabama paradox? Explain your reasoning.

Exercise 11.5.0.24

Consider States D, E, F, and G.

a. Determine the apportionment for States D, E, F, and G with the original house size using the Hamilton method.
b. Determine the apportionment for States D, E, F, and G with the updated house size using the Hamilton method.
c. Does the change in the house size and use of the Hamilton method cause the Alabama paradox? Explain your reasoning.

Exercise 11.5.0.25

Consider States H, I, J, K, and L.

a. Determine the apportionment for States H, I, J, K, and L with the original house size using the Hamilton method.
b. Determine the apportionment for States H, I, J, K, and L with the updated house size using the Hamilton method.
c. Does the change in the house size and use of the Hamilton method cause the Alabama paradox? Explain your reasoning.

Exercise 11.5.0.26

Consider States P, Q, and R.

a. Determine the apportionment for States P, Q, and R with the original house size using the Hamilton method.
b. Determine the apportionment for States P, Q, and R with the updated house size using the Hamilton method.
c. Does the change in the house size and use of the Hamilton method cause the Alabama paradox? Explain your reasoning.

In the following exercises, use the information in the table below.

State	Original Population	Updated Population	Population Growth Rate	House Size
A	889	958	7.76%	
B	674	692	2.67%	42
C	237	243	2.53%	
D	**12,032**	**14,124**	$d\%$	
E	**10,789**	**9,726**	$e\%$	**135**
F	**995**	**2,304**	$f\%$	
G	901	1,156	28.3%	
H	1,683	2,125	26.3%	24
I	3,808	4,369	14.7%	
X	**56**	**63**	$x\%$	
Y	**125**	**141**	$y\%$	**16**
Z	**182**	**213**	$z\%$	
P	6,534	6,534	$p\%$	
Q	7,832	7,810	$q\%$	40
R	13,959	13,992	$r\%$	
S	20,515	21,164	$s\%$	

Exercise 11.5.0.27

Calculate the population growth rates d, e, and f for States D, E, and F. Give answer as a percentage rounded to one decimal place.

Exercise 11.5.0.28

Calculate the population growth rates p, q, r, and s for States P, Q, R, and S. Give answer as a percentage rounded to one decimal place.

Exercise 11.5.0.29

Calculate the population growth rates x, y, and z for States X, Y, and Z. Give answer as a percentage rounded to one decimal place.

Exercise 11.5.0.30

Consider States A, B, and C.

a. Determine the Hamilton apportionment for States A, B, and C with the original population.
b. Determine the Hamilton apportionment for States A, B, and C with the updated population.
c. Does the increase in population of States A, B, and C from the original population to the updated population and the use of the Hamilton method cause the population paradox? Explain your reasoning.

Exercise 11.5.0.31

Consider States D, E, and F.

a. Determine the Hamilton apportionment for States D, E, and F with the original population.
b. Determine the Hamilton apportionment for States D, E, and F with the updated population.
c. Does the increase in population and the use of the Hamilton method cause the population paradox? Explain your reasoning.

Exercise 11.5.0.32

Consider States G, H, and I.

a. Determine the Hamilton apportionment for States G, H, and I with the original population.
b. Determine the Hamilton apportionment for States G, H, and I with the updated population.
c. Does the increase in population and the use of the Hamilton method cause the population paradox? Explain your reasoning.

Exercise 11.5.0.33

Consider States X, Y, and Z.

a. Determine the Hamilton apportionment for States X, Y, and Z with the original population.
b. Determine the Hamilton apportionment for States X, Y, and Z with the updated population.
c. Does the increase in population and the use of the Hamilton method cause the population paradox? Explain your reasoning.

Exercise 11.5.0.34

Consider States P, Q, R, and S.

a. Determine the Hamilton apportionment for States P, Q, R, and S with the original population.
b. Determine the Hamilton apportionment for States P, Q, R, and S with the updated population.
c. Does the increase in population and the use of the Hamilton method cause the population paradox? Explain your reasoning.

In the following exercises, use the information in the table below.

State	Population	Original House Size	New House Size
A	627		
B	1,287	25	
C	973		**32**
D	**815**		
E	520		
F	1,510	50	
G	1,060		h
H	**950**		

State	Population	Original House Size	New House Size
P	1,222	100	
Q	473		
R	**225**		r
K	1,688		
L	7,912	48	
M	**1,448**		m
T	150		
U	250		
V	350	50	w
W	**450**		

Exercise 11.5.0.35

Consider States A, B, and C.

a. Calculate the standard divisor based on the original house size.
b. Use the Hamilton method to apportion the seats.

Exercise 11.5.0.36

Consider States E, F, and G.

a. Calculate the standard divisor based on the original house size.
b. Use the Hamilton method to apportion the seats.

Exercise 11.5.0.37

Consider States P and Q.

a. Calculate the standard divisor based on the original house size.
b. Use the Hamilton method to apportion the seats.

Exercise 11.5.0.38

Consider States K and L.

a. Calculate the standard divisor based on the original house size.
b. Use the Hamilton method to apportion the seats.

Exercise 11.5.0.39

Suppose that States A, B, and C annex State D and increase the house size proportionately.

a. Calculate the standard divisor based on the new house size.
b. Use the Hamilton method to reapportion the seats.
c. Does the new-states paradox occur?

? Exercise 11.5.0.40

Suppose that States E, F, and G annex State H and increase the house size proportionately.

a. Determine the new house size, h, that is necessary.
b. Calculate the standard divisor based on the new house size.
c. Use the Hamilton method to reapportion the seats.
d. Does the new-states paradox occur?

? Exercise 11.5.0.41

Suppose that States P and Q annex State R and increase the house size proportionately.

a. Determine the new house size, r, that is necessary.
b. Calculate the standard divisor based on the new house size.
c. Use the Hamilton method to reapportion the seats.
d. Does the new-states paradox occur?

? Exercise 11.5.0.42

Suppose that States K and L annex State M and increase the house size proportionately.

a. Determine the new house size, m, that is necessary.
b. Calculate the standard divisor based on the new house size.
c. Use the Hamilton method to reapportion the seats.
d. Does the new-states paradox occur?

? Exercise 11.5.0.43

Suppose that States T, U, and V annex State W and increase the house size proportionately.

a. Calculate the standard divisor based on the original house size and only States T, U, and V.
b. Use the Hamilton method to apportion the seats to T, U, and V.
c. Determine the new house size when State W is annexed.
d. Calculate the standard divisor based on the new house size.
e. Use the Hamilton method to reapportion the seats.
f. Does the new-states paradox occur? (Refer to part b.)

? Exercise 11.5.0.44

Suppose 24 seats are apportioned to States A, B, and C with populations of 16, 15, and 125 respectively. Then the populations of States A, B, and C change to 17, 15, and 126 respectively.

a. Demonstrate that the population paradox occurs when the Hamilton method is used.
b. Determine whether the population paradox occurs when the Webster method is used. Justify your answer.

? Exercise 11.5.0.45

Suppose that 10 seats are apportioned to States A, B, and C with populations 6, 6, and 2 respectively. Then the number of seats is increased to 11. Demonstrate that the Alabama paradox occurs when the Hamilton method is used.

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SECTION OVERVIEW

11.6: Chapter Summary

11.6.0: Key Terms

11.6.1: Key Concepts

11.6.2: Videos

11.6.3: Formula Review

11.6.4: Projects

11.6.5: Chapter Review

11.6.6: Chapter Test

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11.6.0: Key Terms

11.1 Voting Methods

- majority
- plurality
- runoff election
- runoff voting system
- two-round system
- Hare Method
- preference ranking
- ranked ballot
- ranked-choice voting (RCV)
- instant runoff voting (IRV)
- Borda count method
- Borda score
- Compromise candidate
- divisive candidate
- pairwise comparison method
- Condorcet voting methods
- Condorcet candidate
- approval voting system
- approval voting ballot
- spoiler

11.2 Fairness in Voting Methods

- majority criterion
- tyranny of the majority
- Condorcet criterion
- Condorcet method
- monotonicity criterion
- up-rank
- down-rank
- independence of irrelevant alternatives criterion (IIA)
- Arrow's Impossibility Theorem
- cardinal voting system

11.3 Standard Divisors, Standard Quotas, and the Apportionment Problem

- apportion
- apportionment problem
- proportional
- part-to-part ratio
- representative democracies
- standard divisor
- states
- seats
- house size
- state population
- total population
- standard quota

11.4 Apportionment Methods

- apportionment problem
- lower quota
- upper quota

11.5 Fairness in Apportionment Methods

- apportionment paradox
- Alabama paradox
- reapportionment
- population growth rate
- population paradox
- new-state paradox

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11.6.1: Key Concepts

11.1 Voting Methods

- In plurality voting, the candidate with the most votes wins.
- When a voting method does not result in a winner, runoff voting can be used to do so.
- Ranked-choice voting, also known as instant runoff voting, is one type of ranked voting system.
- The Borda count method is a type of ranked voting system in which each candidate is given a Borda score based on the number of candidates ranked lower than them on each ballot.
- When pairwise comparison is used, the winner will be the Condorcet candidate if one exists.
- Approval voting allows voters to give equally weighted votes to multiple candidates.
- When a voter finds a characteristic of a particular voting method unappealing, they may consider that characteristic a flaw in the voting method and look for an alternative method that does not have that characteristic.

11.2 Fairness in Voting Methods

- There are several common measures of voting fairness, including the majority criterion, the head-to head criterion, the monotonicity criterion, and the irrelevant alternatives criterion.
- According to Arrow’s Impossibility Theorem, each voting method in which the only information is the order of preference of the voters will violate one of the fairness criteria.

11.3 Standard Divisors, Standard Quotas, and the Apportionment Problem

- The apportionment problem is how to fairly divide and distribute available resources to recipients in whole, not fractional, parts.
- To distribute the seats in the U.S. House of Representatives fairly to each state, calculations are based on state population, total population, and house size, or the total number of seats to be apportioned.
- The standard divisor is the ratio of the total population to the house size, and the standard quota is the number of seats that each state should receive.

11.4 Apportionment Methods

- Hamilton’s method of apportionment uses the standard divisor and standard lower quotas, and it distributes any remaining seats based on the size of the fractional parts of the standard lower quota. Hamilton’s method satisfies the quota rule and favors neither larger nor smaller states.
- Jefferson’s method of apportionment uses a modified divisor that is adjusted so that the modified lower quotas sum to the house size. Jefferson’s method violates the quota rule and favors larger states.
- Adams’s method of apportionment uses a modified divisor that is adjusted so that the modified upper quotas, sum to the house size. Adams’s method violates the quota rule and favors smaller states.
- Webster’s method of apportionment uses a modified divisor that is adjusted so that the modified state quotas, rounded using traditional rounding, sum to the house size. Webster’s method violates the quota rule but favors neither larger nor smaller states.

11.5 Fairness in Apportionment Methods

- Several surprising outcomes can occur when apportioning seats that voters may find unfair: Alabama paradox, population paradox, and new-state paradox.
- Apportionment methods are susceptible to apportionment paradoxes and may violate the quota rule.
- The Balinsky-Young Impossibility Theorem indicates that no apportionment can satisfy all fairness criteria.

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11.6.2: Videos

11.1 Voting Methods

- How Does Ranked-Choice Voting Work?
- Determine Winner of Election by Ranked-Choice Method (aka Instant Runoff)
- Determine Winner of Election by Borda Count Method
- Determine Winner of Election by Pairwise Comparison Method

11.2 Fairness in Voting Methods

- Separation of Powers and Checks and Balances

11.3 Standard Divisors, Standard Quotas, and the Apportionment Problem

- What Is a Ratio?
- What Are the Different Types of Ratios?
- Math Antics – Rounding

11.4 Apportionment Methods

- Hamilton Method of Apportionment
- Jefferson Apportionment Method
- Adams Method Apportionment Calculator

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11.6.3: Formula Review

11.3 Standard Divisors, Standard Quotas, and the Apportionment Problem

Let A be a particular item and B another such that there is a constant ratio of A to B.

- ratio of $B'\text{s}$ to $A'\text{s} = \frac{1}{\text{ratio of } A'\text{s to } B'\text{s}}$ and ratio of $A'\text{s}$ to $B'\text{s} = \frac{1}{\text{ratio of } B'\text{s to } A'\text{s}}$
- units of $A = (\text{ units of } B) \times (\text{ ratio of } A \text{ 's to } B's) = \frac{\text{units of } B}{\text{ratio of } B' \text{ sto } A'\text{s}}$
- units of $B = (\text{ units of } A) \times (\text{ ratio of } B's \text{ to } A's) = \frac{\text{units of } A}{\text{ratio of } A \text{ 's to } B'\text{s}}$

Standard Divisor $= \frac{\text{Total Population}}{\text{House Size}}$

State's Standard Quota $= \frac{\text{State Population}}{\text{Standard Divisor}}$ seats

11.5 Fairness in Apportionment Methods

$$\text{population growth rate } = \frac{\text{current population size } - \text{ previous population size}}{\text{previous population size}}$$

$$\text{New House Size } = \frac{\text{New Population}}{\text{Original Standard Divisor}} \text{ rounded to the nearest whole number.}$$

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11.6.4: Projects

The First Census: Challenges of Collecting Population Data

As you and your fellow founders write the Imaginarian constitution, you must also design systems to collect accurate information about the population of Imaginaria. Why is this important and what are the challenges you will face? Let’s find out! Research and answer each question. (Questions adapted from “Authorizing the First Census–The Significance of Population Data,” Census.gov, United States Census Bureau)

1. The First Congress laid out a plan for collecting this data in Chapter II, *An Act Providing for the Enumeration of the Inhabitants of the United States*, which was approved March 1, 1790. What group did Congress select to carry out the first enumeration? Why did they choose this group? What might be the advantages and disadvantages to this approach?
2. Choose at least two challenges faced by the U.S. government during the first enumeration and explain how the information gathered might have helped address them.
3. President James Madison was a U.S. Representative who participated in the census debate in the first congress. What were the main points of his remarks and how were they relevant to the overall debate about the first enumeration?
4. What issues do you think the U.S. Census Bureau encounters today as it continues to collect and process data about the U.S. population that might be significant to you and the other founders of Imaginaria?

The Party System: How Many Political Parties Are Enough?

The electoral system of Imaginaria will likely involve multiple political parties. The way these parties interact with the system may be determined by the founders of the new democracy. Let’s explore the ways in which political parties interact with the governments of various democracies by researching and answering each of the following questions.

1. What concerns did the founders of the United States have about political parties? Have any of their concerns become a reality? Were political parties addressed in the U.S. Constitution? How did they become such a large part of politics in the U.S.? As a founder of Imaginaria, would you address political parties in your constitution?
2. What is frontloading? Does our current system of frontloading impact fair representation? Why do two small, racially homogeneous states hold their primaries first? Do you think this impacts the final results?
3. How does the interaction political parties and the electoral system in the United States differ from that of other countries? Give at least three specific examples. Of these three, which would you be most likely to use as a model for Imaginaria?

Technology and Voting: How Has the Digital Age Impacted Elections?

Unlike most other countries, Imaginaria will be founded in the digital age. Let’s explore the impact this might have on how you choose to set up your electoral system. Research and answer each question.

1. Participation in an electoral system if very important. In what ways does the Internet positively affect participation? In what ways does it negatively affect participation. What roll, if any, should government of Imaginaria play in tempering or promoting the Impact of the Internet?
2. Find at least three examples of governments that utilize Internet voting around the world. What concerns have slowed the spread of this technology? What are the advantages and disadvantages of Internet voting? Would you be in favor of Internet voting for Imaginaria? Why or why not?

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11.6.5: Chapter Review

Voting Methods

Exercise 11.6.5.1

In a plurality election, the candidates have the following vote counts: A 125, B 132, C 149, and D 112. Which candidate has the plurality and wins the election?

For the following exercises, use the table below.

Options	A	B	C	D	E
Candidate 1	1	3	3	1	3
Candidate 2	2	1	1	2	4
Candidate 3	3	4	2	4	1
Candidate 4	4	2	4	3	2

Exercise 11.6.5.2

Which candidate has a plurality?

Exercise 11.6.5.3

Does the plurality candidate have a majority?

Exercise 11.6.5.4

Determine the winner of the election by the Hare method based on the sample preference summary in the table.

For the following exercises, use the table below.

Number of Ballots	10	20	15	5
Option A	1	4	3	4
Option B	2	3	4	2
Option C	4	2	1	3
Option D	3	1	2	1

Exercise 11.6.5.5

Use ranked-choice voting to determine the two options in the final round and the number of votes they each receive in that round.

Exercise 11.6.5.6

Is there a winning option? If so, which option? Justify your answer.

For the following exercises, use the table below.

Number of Ballots	100	80	110	105	55

Number of Ballots	100	80	110	105	55
Candidate A	1	1	4	4	2
Candidate B	2	2	2	3	1
Candidate C	4	4	1	1	4
Candidate D	3	3	3	2	3

? Exercise 11.6.5.7

What are the Borda scores for each candidate?

? Exercise 11.6.5.8

Which candidate is the winner by the Borda count method?

Use Pairwise Comparison Matrix for Candidates U, V, W, X, and Y to answer Questions 9 and 10.

? Exercise 11.6.5.9

Calculate the points received by each candidate in pairwise comparison matrix.

? Exercise 11.6.5.10

Determine the winner of the pairwise comparison election represented by matrix. If there is a winner, determine whether the winner is a Condorcet candidate and explain your reasoning. If there is no winner, indicate this.

? Exercise 11.6.5.11

The ladies of *The Big Bang Theory* decide to hold their own approval voting election to determine the best option in Rock, Paper, Scissors, Lizard, Spock. Use the summary of their approval ballots in the table below to determine the number of votes for each candidate. Determine the winner, or state that there is none.

VOTERS	Penny	Bernadette	Amy
Rock	Yes	No	No
Paper	Yes	Yes	No
Scissors	Yes	Yes	Yes
Lizard	No	No	No
Spock	Yes	No	Yes

For the following exercises, use the table below.

Percentage of Vote	40%	35%	25%
Candidate A	1	3	2
Candidate B	2	1	3
Candidate C	3	2	1

Exercise 11.6.5.12

Which candidate is the winner by the ranked-choice method?

Exercise 11.6.5.13

Suppose that you used the approval method and each voter approved their top two choices. Which candidate is the winner by the approval method?

Exercise 11.6.5.14

Which candidate is the winner by the Borda count method?

Fairness in Voting Methods

Exercise 11.6.5.15

In a Borda count election, the candidates have the following Borda scores: A 1245, B 1360, and C 787. Candidate A received 55% of the first place rankings. Identify which fairness criteria, if any, are violated by characteristics of the described voter profile in this Borda count election. Explain your reasoning.

For the following exercises, use the table below.

Number of Ballots	8	10	12	4
Option A	1	3	2	1
Option B	3	1	4	4
Option C	4	2	1	2
Option D	2	4	3	3

Exercise 11.6.5.16

Determine Borda score for each candidate, and the winner of the election using the Borda count method.

Exercise 11.6.5.17

Is there a majority candidate? If so, which candidate?

Exercise 11.6.5.18

Does the Borda method election violate the majority criterion? Justify your answer.

Exercise 11.6.5.19

In a Borda count election, the candidates have the following Borda scores: A 15, B 11, C 12, and D 16. The pairwise match up points for the same voter profiles would have been A 2, B 0, C 1, and D 3. Identify which fairness criteria, if any, are violated by characteristics of the described voter profile in this Borda election. Explain your reasoning.

Exercise 11.6.5.20

Determine the winner of the election using the ranked-choice method.

Exercise 11.6.5.21

If the four voters in the last column rank C ahead of A, which candidate wins by the ranked-choice method?

Exercise 11.6.5.22

Does this ranked-choice election violate the monotonicity criterion? Explain your reasoning.

For the following exercises, use the table below.

Number of Ballots	15	12	9	3
Option A	1	3	3	2
Option B	2	2	1	1
Option C	3	1	2	3

Exercise 11.6.5.23

Determine the winner of the election by the Borda method.

Exercise 11.6.5.24

Does this Borda method election violate the IIA? Why or why not?

Exercise 11.6.5.25

Which of the ranked voting methods in this chapter, if any, meets the majority criterion, the head-to-head criterion, the monotonicity criterion, and the irrelevant alternatives criterion?

Standard Divisors, Standard Quotas, and the Apportionment Problem

Exercise 11.6.5.26

Identify the states, the seats, and the state population (the basis for the apportionment) in the given scenario: The reading coach at an elementary school has 52 prizes to distribute to their students as a reward for time spent reading.

Exercise 11.6.5.27

Use the given information to find the standard divisor to the nearest hundredth. Include the units. The total population is 2,235 automobiles, and the number of seats is 14 warehouses.

Exercise 11.6.5.28

Use the given information to find the standard quota. Include the units. The state population is eight residents in a unit, and the standard divisor is 1.75 residents per parking space.

Apportionment Methods

Exercise 11.6.5.29

Which of the four apportionment methods discussed in this section does not use a modified divisor?

? Exercise 11.6.5.30

Determine the Hamilton apportionment for Scenario X in the table below.

	State A	State B	State C	State D	State E	State F	Total Seats
Scenario X	17.63	26.62	10.81	16.01	13.69	15.24	100

? Exercise 11.6.5.31

Does the apportionment resulting from Method X in the table below satisfy the quota rule? Why or why not?

	State A	State B	State C	State D	State E
Standard Quota	1.67	3.33	5.00	6.67	8.33
Apportionment Method X	2	2	5	7	9

For the following exercises, use the table below and the following information: In Wakanda, the domain of the Black Panther, King T’Challa, has six fortress cities. In Wakandan, the word “birnin” means “fortress city.” King T’Challa has found 111 Vibranium artifacts that must be distributed among the fortress cities of Wakanda. He has decided to apportion the artifacts based on the number of residents of each birnin.

Fortress Cities	Birnin Djata (D)	Birnin T'Chaka (T)	Birnin Zana (Z)	Birnin S'Yan (S)	Birnin Bashenga (B)	Birnin Azzaria (A)	Total
Residents	26,000	57,000	27,000	18,000	64,000	45,000	237,000
Standard Quota	12.18	26.70	12.65	8.43	29.98	21.08	111

? Exercise 11.6.5.32

Does the Jefferson method result in an apportionment that satisfies or violates the quota rule in this scenario?

? Exercise 11.6.5.33

Find the modified upper quota for each state using a modified divisor of 2,250. Is the sum of the modified quotas too high, too low, or equal to the house size?

? Exercise 11.6.5.34

Use the Adams method to apportion the artifacts. Determine whether it is necessary to modify the divisor. If so, indicate the value of the modified divisor.

? Exercise 11.6.5.35

Does the Adams method result in an apportionment that satisfies or violates the quota rule in this scenario?

? Exercise 11.6.5.36

Use the Webster method to apportion the artifacts. Determine whether it is necessary to modify the divisor. If so, indicate the value of the modified divisor.

Exercise 11.6.5.37

Does the Webster method result in an apportionment that satisfies or violates the quota rule in this scenario?

Exercise 11.6.5.38

Which of the four methods of apportionment from this section are the residents of Birnin S'Yan likely to prefer? Justify your answer.

Exercise 11.6.5.39

Does the change from a standard divisor to a modified divisor tend to change the number of seats for larger or smaller states more?

Exercise 11.6.5.40

Which of the four apportionment methods—Jefferson, Adams, Hamilton, or Webster—satisfies the quota rule?

Fairness in Apportionment Methods

Exercise 11.6.5.41

A city purchased five new firetrucks and apportioned them among the existing fire stations. Although your neighborhood fire station has the same proportion of the city's firetrucks as before the new ones were purchased, it now has one fewer. Is this scenario an example of a quota rule violation, the Alabama paradox, the population paradox, the new-states paradox, or none of these?

Exercise 11.6.5.42

When the number of seats changed from 25 to 26, the standard quotas changed from A 2.21, B 5.25, C 11.27, and D 6.27 to A 2.30, B 5.46, C 11.72, and D 6.52.

a. How did the increase in seats impact the apportionment?
b. Is this apportionment an example of a paradox? Justify your answer.

Exercise 11.6.5.43

The school resources officers in a county were reapportioned based on the most recent census. The number of students at Chapel Run Elementary went up while the number of students at Panther Trail Elementary went down, but Chapel Run now has 1 fewer resources officers while Panther Trail has one more than it did previously. Is this scenario an example of a quota rule violation, the Alabama paradox, the population paradox, the new-states paradox, or none of these?

For the following exercises, the house size is 24 seats. When the population of A increases by 28 percent, B increases by 26 percent, and C increases by 15 percent, the standard quotas change from A 3.38, B 6.32, and C 14.30 to A 3.63, B 6.67, and C 13.71.

Exercise 11.6.5.44

How did the change in populations impact the apportionment?

Exercise 11.6.5.45

Is this apportionment an example of a paradox? Justify your answer.

? Exercise 11.6.5.46

When the city of Cocoa annexed an adjacent unincorporated community, the number of seats on the city council was increased to maintain the standard ratio of citizens to seats, but one existing community of Cocoa still lost a seat on the city council to another existing community of Cocoa when the new community was added. Is this scenario an example of a quota rule violation, the Alabama paradox, the population paradox, the new-states paradox, or none of these?

For the following exercises, the house size was 27. There were three states with standard quotas of A 6.39, B 11.40, and C 9.21. A fourth state was annexed, and the house size was increased to 35. The new standard quotas are A 6.38, B 11.37, C 9.19, and D 8.06.

? Exercise 11.6.5.47

How did the additional state impact the apportionment?

? Exercise 11.6.5.48

Is this apportionment an example of a paradox? Justify your answer.

For the following exercises, suppose 11 seats are apportioned to States A, B, and C with populations of 50, 129, and 181 people, respectively. Then the populations of States A, B, and C change to 57, 151, and 208, respectively.

? Exercise 11.6.5.49

Demonstrate that the population paradox occurs when the Hamilton method is used.

? Exercise 11.6.5.50

Demonstrate that the population paradox does not occur when the Jefferson method is used. Justify your answer.

? Exercise 11.6.5.51

Demonstrate that the population paradox does not occur when the Adams method is used. Justify your answer.

? Exercise 11.6.5.52

Demonstrate that the population paradox does not occur when the Webster method is used. Justify your answer.

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11.6.6: Chapter Test

For the following exercises, use the table below.

Number of Ballots	12	17	15	13
Option A	1	2	4	3
Option B	3	1	3	2
Option C	4	3	1	4
Option D	2	4	2	1

? Exercise 11.6.6.1

Determine the winner of the election by plurality.

? Exercise 11.6.6.2

Determine the Borda scores for each candidate to determine the winner by Borda count method.

? Exercise 11.6.6.3

Create and analyze a pairwise comparison matrix based on the preference summary to determine the winner of the election by pairwise comparison.

? Exercise 11.6.6.4

From the table below use ranked-choice voting to determine the winner of the election.

Number of Ballots	28	5	30	5	16	16
Option L	3	2	1	1	2	3
Option R	1	1	3	2	3	2
Option E	2	3	2	3	1	1

For the following exercises, identify which fairness criteria, if any, are violated by characteristics of the described voter profile. Explain your reasoning

? Exercise 11.6.6.5

In a Borda count election, the candidates have the following Borda scores: A 1345, B 1260, C 685. Candidate B received 51% of the first-place rankings.

? Exercise 11.6.6.6

In a plurality election, the candidates have the following percentages of first place votes: A 25, B 21, C 30, D 24. The pairwise matchup points for the same voter profiles would have been A 3, B 0, C 2, D 2.

For the following exercises, use the table below.

Number of Ballots	13	14	11	12
Option A	2	1	3	3

Number of Ballots	13	14	11	12
Option B	3	2	4	1
Option C	4	4	1	2
Option D	1	3	3	4

? Exercise 11.6.6.7

Determine the winner by ranked-choice voting if two of the voters in the second column up-rank the original winner. Refer to Question 4. Which fairness criterion, if any, is violated?

? Exercise 11.6.6.8

Determine the winner by ranked-choice voting if candidate R is removed from the election. Refer to Question 4. Which fairness criterion, if any, is violated?

For the following exercises, use this information: The incorporated town of Orange Grove consists of two subdivisions: The Oaks with 1,254 residents, and The Villages with 10,746 residents. A council with 100 members supervises the municipality's operations with representation proportionate to the number of residents.

? Exercise 11.6.6.9

Identify the states, the seats, and the state population (the basis for apportionment) in the given scenario.

? Exercise 11.6.6.10

Determine the standard divisor for the apportionment

? Exercise 11.6.6.11

Determine each state's standard quota rounded to two decimal places.

For the following exercises, use this information: Air Force administration wanted to distribute 27 aircraft across 6 bases based on the number of qualified pilots stationed at those bases. The standard quota is 2.2963. The standard quotas for each base are listed in the table below.

Air Force Base	(A) Alpha	(B) Bravo	(C) Charlie	(D) Delta	(E) Echo	(F) Foxtrot
Pilots	13	12	5	16	7	9
Standard Quota	5.66	5.23	2.18	6.97	3.05	3.92

? Exercise 11.6.6.12

Determine the states' lower quotas and the states' upper quotas.

? Exercise 11.6.6.13

Use Adams's method to apportion the aircraft.

Exercise 11.6.6.14

Use Jefferson's method to apportion the aircraft.

Exercise 11.6.6.15

The apportionment of 616 schools to 5 Hawaiian counties by various methods is displayed in the table below.

County	Hawaii	Honolulu	Kalawao	Kauai	Maui
Lower Quota	87	424	0	31	72
Upper Quota	88	425	1	32	73
Jefferson	87	425	1	31	72
Adams	88	422	1	32	73
Webster	87	424	1	31	73

Apportionment by which methods, if any, fail to satisfy the quota rule? Explain your reasoning. For the following exercises, use this information: The incorporated town of Orange Grove consists of two subdivisions: The Oaks with 1,254 residents, and The Villages with 10,746 residents. A council with 100 members supervises the municipality's operations. The Hamilton method was used to apportion the council seats. The Oaks has 10 seats on the council, while The Villages has 90 seats. The council votes to annex an unincorporated subdivision called The Lakes with a population of 630. They plan to increase the size of the council to maintain the ratio of seats to residents such that the new council will have 100 seats plus the number of seats given to The Lakes.

Exercise 11.6.6.16

What is the standard divisor from the original apportionment?

Exercise 11.6.6.17

What is the new house size?

Exercise 11.6.6.18

Use the Hamilton method to reapportion the seats.

Exercise 11.6.6.19

Is the reapportionment an example of the new-states paradox? If so, how?

For the following exercises, use this information: determine whether the reapportionment violates the Alabama paradox, the population paradox, or neither. Justify your answer.

Exercise 11.6.6.20

States A, B, C, and D received 21, 25, 26, and 28 seats respectively. When the population remains the same, but house size is increased, the reapportionment is A 20, B 26, C 27, and D 29.

Exercise 11.6.6.21

States A, B, C, and D received 21, 25, 26, and 28 seats respectively. When the house size remains the same, the population of state A increased, the population of state B decreased, and the populations of states C and D remained the same, the reapportionment is A 20, B 26, C 26, and D 28.

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CHAPTER OVERVIEW

12: Graph Theory

- 12.0: Introduction
- 12.1: Graph Basics
  - 12.1.0: Exercises
- 12.2: Graph Structures
  - 12.2.0: Exercises
- 12.3: Comparing Graphs
  - 12.3.0: Exercises
- 12.4: Navigating Graphs
  - 12.4.0: Exercises
- 12.5: Euler Circuits
  - 12.5.0: Exercises
- 12.6: Euler Trails
  - 12.6.0: Exercises
- 12.7: Hamilton Cycles
  - 12.7.0: Exercises
- 12.8: Hamilton Paths
  - 12.8.1: Exercises
- 12.9: Traveling Salesperson Problem
  - 12.9.1: Exercises
- 12.10: Trees
  - 12.10.1: Exercises
- 12.11: Chapter Summary
  - 12.11.0: Key Terms
  - 12.11.1: Key Concepts
  - 12.11.2: Videos
  - 12.11.3: Formula Review
  - 12.11.4: Projects
  - 12.11.5: Chapter Review
  - 12.11.6: Chapter Test

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12.0: Introduction

In this chapter, you will learn the fundamental skills needed to work with graphs used in an area of mathematics known as graph theory. You can think of these graphs as a kind of map. Maps have served many purposes over the course of history. You probably use GPS maps to navigate to various destinations. A scientist from ancient Greece named Ptolemy wanted an accurate map of the world to make more accurate astrological predications. In recent years, neurobiologists have mapped the cerebral cortex to better understand the human brain. Social network analysts map online interactions to assist advertisers in reaching target audiences. Like other maps, the graphs you will study in this chapter can serve many purposes, but they do not have a lot of the details you might expect in a map such as size, shape, and distance between objects. All of that is stripped away so that we can focus on one element of maps, the connections between objects.

Figure 12.0.1 Networks connect cities around the globe. (credit: "Globalization" by faith.e.murphy/Flickr, Public Domain)

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12.1: Graph Basics

Figure 12.1.1: Cell phone networks connect individuals. (credit: "Business people using their phones" by Rawpixel Ltd./Flickr, CC BY 2.0)

Learning Objectives

1. Identify parts of a graph.
2. Model applications of graph basics.

When you hear the word, *graph*, what comes to mind? You might think of the semanticsxyxy-coordinate system you learned about earlier in this course, or you might think of the line graphs and bar charts that are used to display data in news reports. The graphs we discuss in this chapter are probably very different from what you think of as a graph. They look like a bunch of dots connected by short line segments. The dots represent a group of objects and the line segments represent the connections, or relationships, between them. The objects might be bus stops, computers, Snapchat accounts, family members, or any other objects that have direct connections to each other. The connections can be physical or virtual, formal or casual, scientific or social. Regardless of the kind of connections, the purpose of the graph is to help us visualize the structure of the entire network to better understand the interactions of the objects within it.

Parts of a Graph

In a graph, the objects are represented with dots and their connections are represented with lines like those in Figure 12.3. Figure 12.3 displays a simple graph labeled G and a multigraph labeled H. The dots are called vertices; an individual dot is a **vertex**, which is one object of a set of objects, some of which may be connected. We often label vertices with letters. For example, Graph G has vertices a, b, c, and d, and Multigraph H has vertices, e, f, g, and h. Each line segment or connection joining two vertices is referred to as an edge. H is considered a multigraph because it has a double edge between f and h, and a double edge between h and g. Another reason H is called a multigraph is that it has a loop connecting vertex e to itself; a loop is an edge that joins a vertex to itself. Loops and double edges are not allowed in a simple graph.

To sum up, a simple graph is a collection of vertices and any edges that may connect them, such that every edge connects two vertices with no loops and no two vertices are joined by more than one edge. A multigraph is a graph in which there may be loops or pairs of vertices that are joined by more than one edge. In this chapter, most of our work will be with simple graphs, which we will call graphs for convenience.

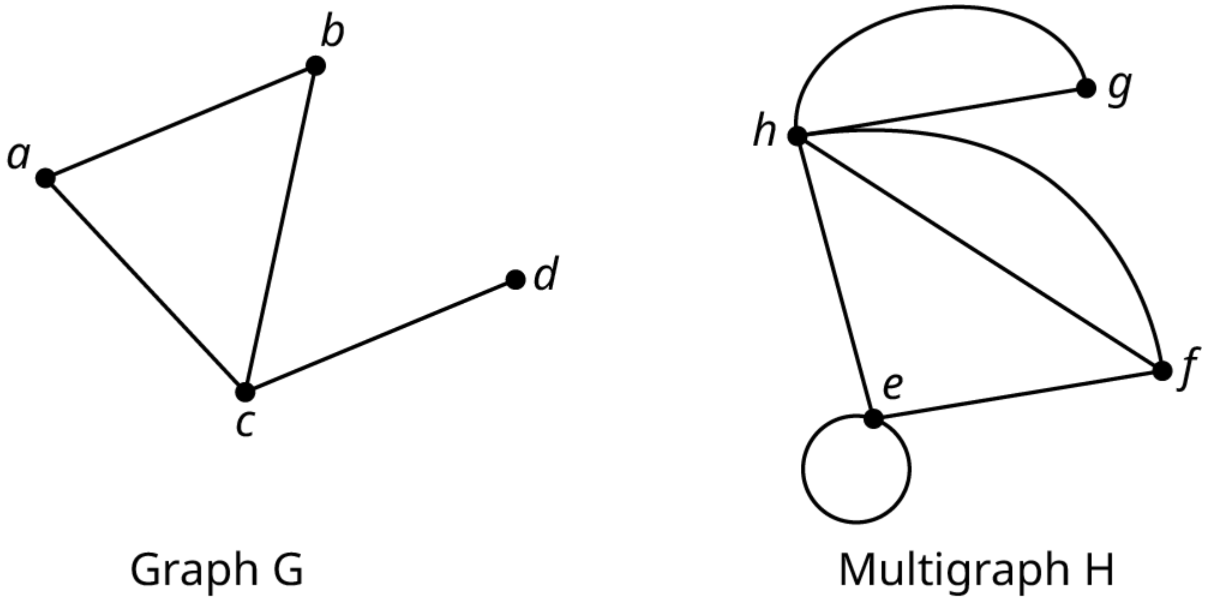


Figure 12.1.2: A Graph and a Multigraph

It is not necessary for the edges in a graph to be straight. In fact, you can draw an edge any way you want. In graph theory, the focus is on which vertices are connected, not how the connections are drawn (see Figure 12.4). In a graph, each edge can be named by the two letters of the associated vertices. The four edges in Graph *X* in Figure 12.4 are *ab*, *ac*, *ad*, and *ae*. The order of the letters is not important when you name the edge of a graph. For example, *ab* refers to the same edge as *ba*.

Checkpoint

A graph may have vertices that are not joined to other vertices by edges, such as vertex f in Graph X in Figure 12.4, but any edge must have a vertex at each end.

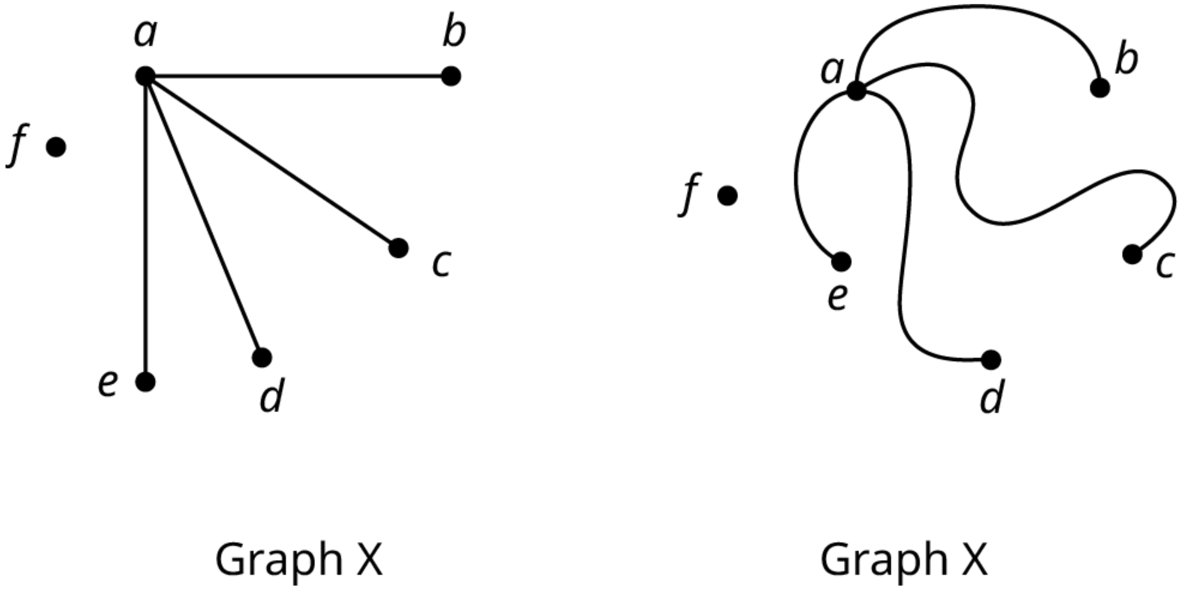


Figure 12.1.3: Different Representations of the Same Graph

Example 12.1.1: Identifying Edges and Vertices

Name all the vertices and edges of graph *F* in Figure 12.1.4.

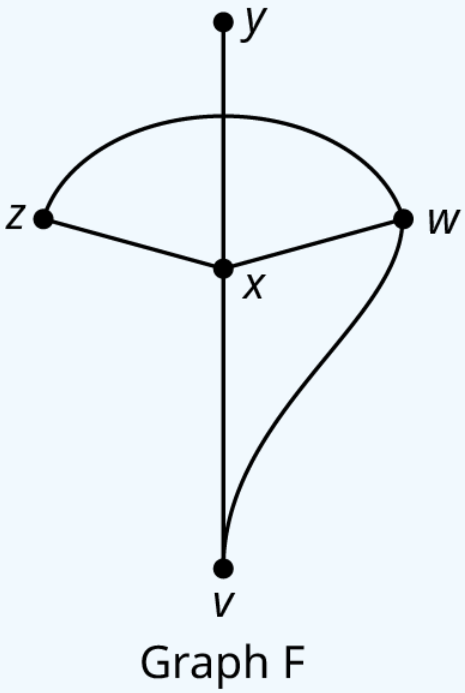


Figure 12.1.4 Graph F

Answer

The vertices are v, w, x, y, and z. The edges are vw, vx, wx, wz, xy, and xz.

Checkpoint

When listing the vertices and edges in a graph, work in alphabetical order to avoid accidentally listing the same item twice. When you are finished, count the number of vertices or edges you listed and compare that to the number of vertices or edges on the graph to ensure you didn't miss any.

Your Turn 12.1.1

Name all the vertices and edges of Graph A.

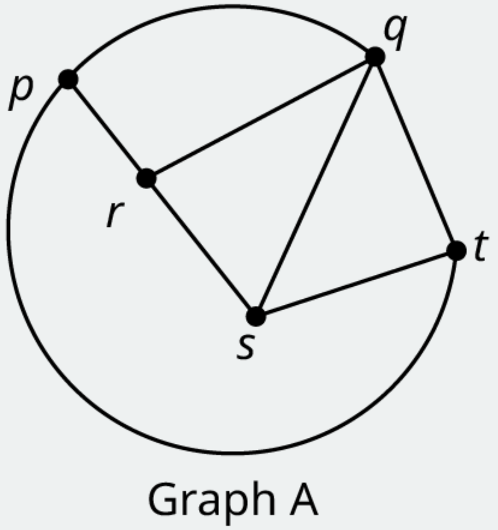


Figure 12.1.5: Graph A

Since the purpose of a graph is to represent the connections between objects, it is very important to know if two vertices share a common edge. The two vertices at either end of a given edge are referred to as neighboring, or adjacent. For example, in Figure 12.5, vertices x and w are adjacent, but vertices y and w are not.

✓ Example 12.1.2: Identifying Vertices That Are Not Adjacent

Name all the pairs of vertices of graph F in Figure 12.5 that are *not* adjacent.

Answer

The pairs of vertices that are not adjacent in graph F are v and y, v and z, w and y, and y and z.

Your Turn 12.1.2

Name all the pairs of vertices of graph A in Figure 12.6 that are *not* adjacent.

People in Mathematics: Sergey Brin and Laurence Page

The “Google boys,” Sergey Brin and Laurence Page, transformed the World Wide Web in 1998 when they used the mathematics of graph theory to create an algorithm called Page Rank, which is known as the Google Search Engine today. The two computer scientists identified webpages as vertices and hyperlinks on those pages as edges because hyperlinks connect one website to the next. The number of edges influences the ranking of a website on the Google Search Engine because the websites with more links to “credible sources” are ranked higher. ("Page Rank: The Graph Theory-based Backbone of Google," September 20, 2011, Cornell University, Networks Blog.

Analyzing Geographical Maps with Graphs

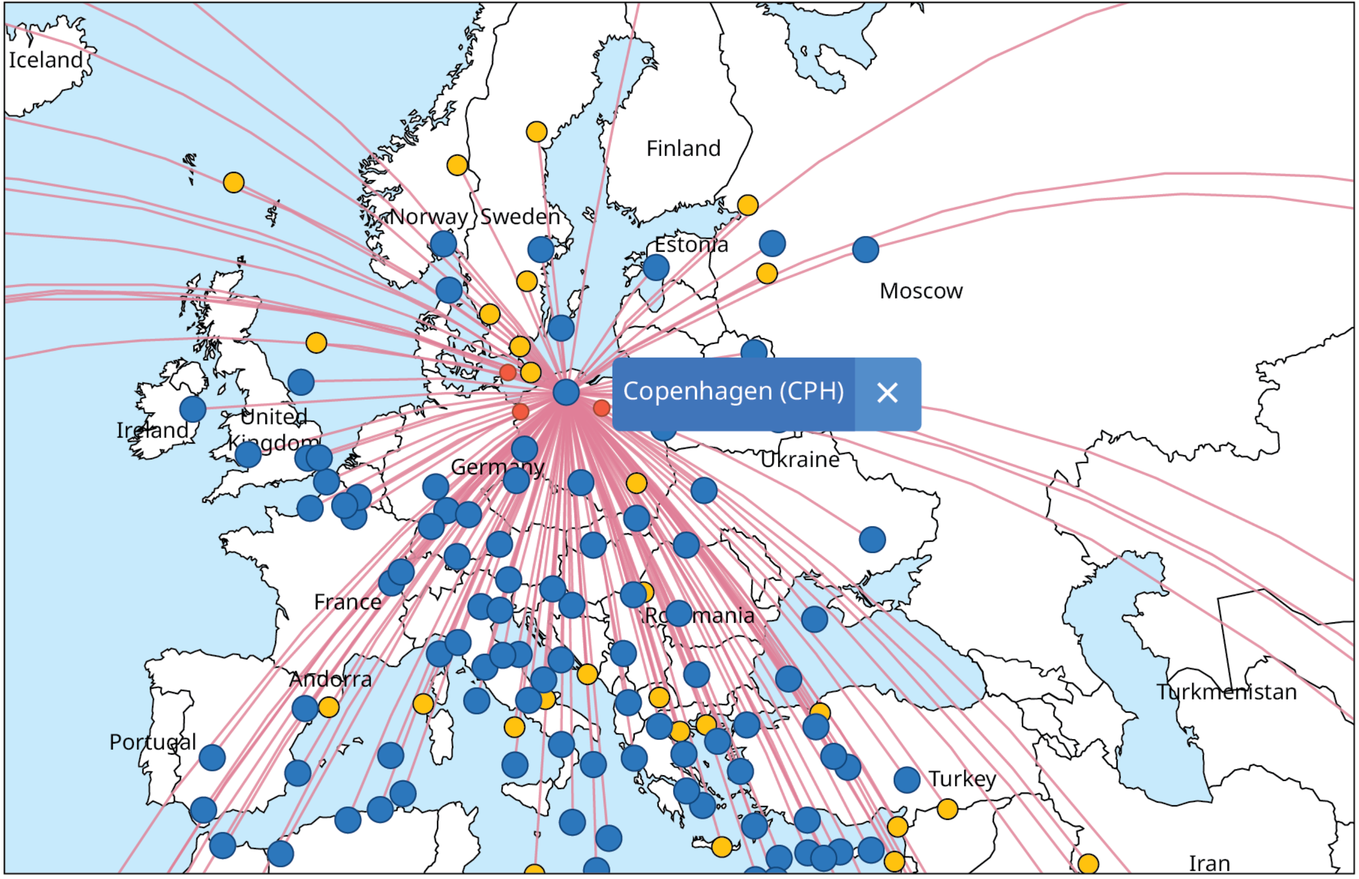


Figure 12.1.6: Commercial airlines' route systems create a global network.

When graphs are used to model and analyze real-world applications, the number of edges that meet at a particular vertex is important. For example, a graph may represent the direct flight connections for a particular airport as in Figure $12.1.7$. Representing the connections with a graph rather than a map shifts the focus away from the relative positions and toward which airports are connected. In Figure $12.1.7$, the vertices are the airports, and the edges are the direct flight paths. The number of flight connections between a particular airport and other South Florida airports is the number of edges meeting at a particular vertex. For example, Key West has direct flights to three of the five airports on the graph. In graph theory terms, we would say that vertex FYW has degree 3. The degree of a vertex is the number of edges that connect to that vertex.

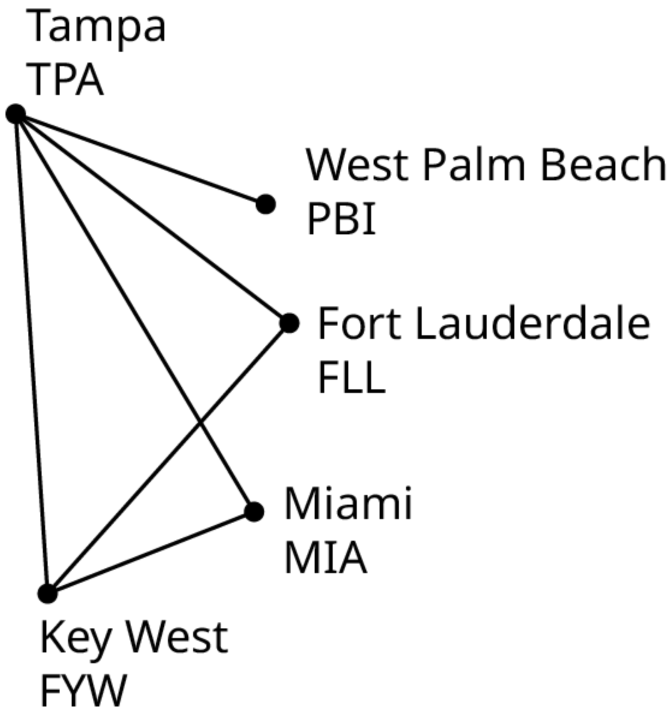


Figure $12.1.7$: Direct Flights between South Florida Airports

✓ Example $12.1.3$: Determining the Degree of a Vertex

Determine the degree of each vertex of Graph *J* in Figure $12.1.8$. If graph *J* represents direct flights between a set of airports, do any of the other airports have direct flights to two or more of the other cities on the graph?

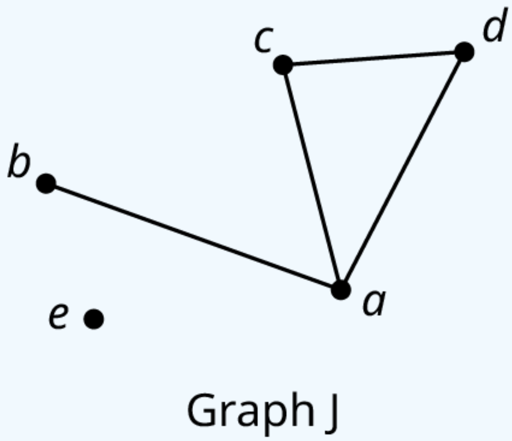
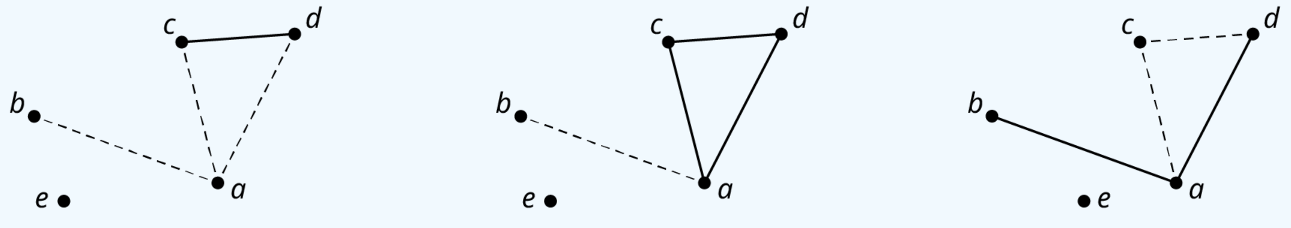


Figure $112.1.8$ Graph *J*

Answer

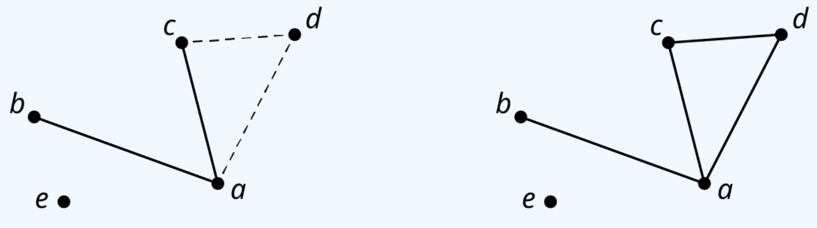
For each vertex, count the number of edges that meet at that vertex. This value is the degree of the vertex. In Figure 12.10, the dashed edges indicate the edges that meet at the marked vertex.



Graph J
Vertex a has degree 3.

Graph J
Vertex b has degree 1.

Graph J
Vertex c has degree 2.



Graph J
Vertex d has degree 2.

Graph J
Vertex e has degree 0.

Figure 12.1.9 Degrees of Vertices of Graph *J*

Vertex *a* has degree 3, vertex *b* has degree 1, vertices *c* and *d* each have degree 2, and vertex *e* has degree 0. Airports *a*, *c*, and *d* have direct flights to two or more of the other airports.

Your Turn 12.1.3

Name a vertx of Graph *A* in Figure 12.1.6 with degree 4.

Graphs are also used to analyze regional boundaries. The states of Utah, Colorado, Arizona, and New Mexico all meet at a single point known as the “Four Corners,” which is shown in the map in Figure 12.11.

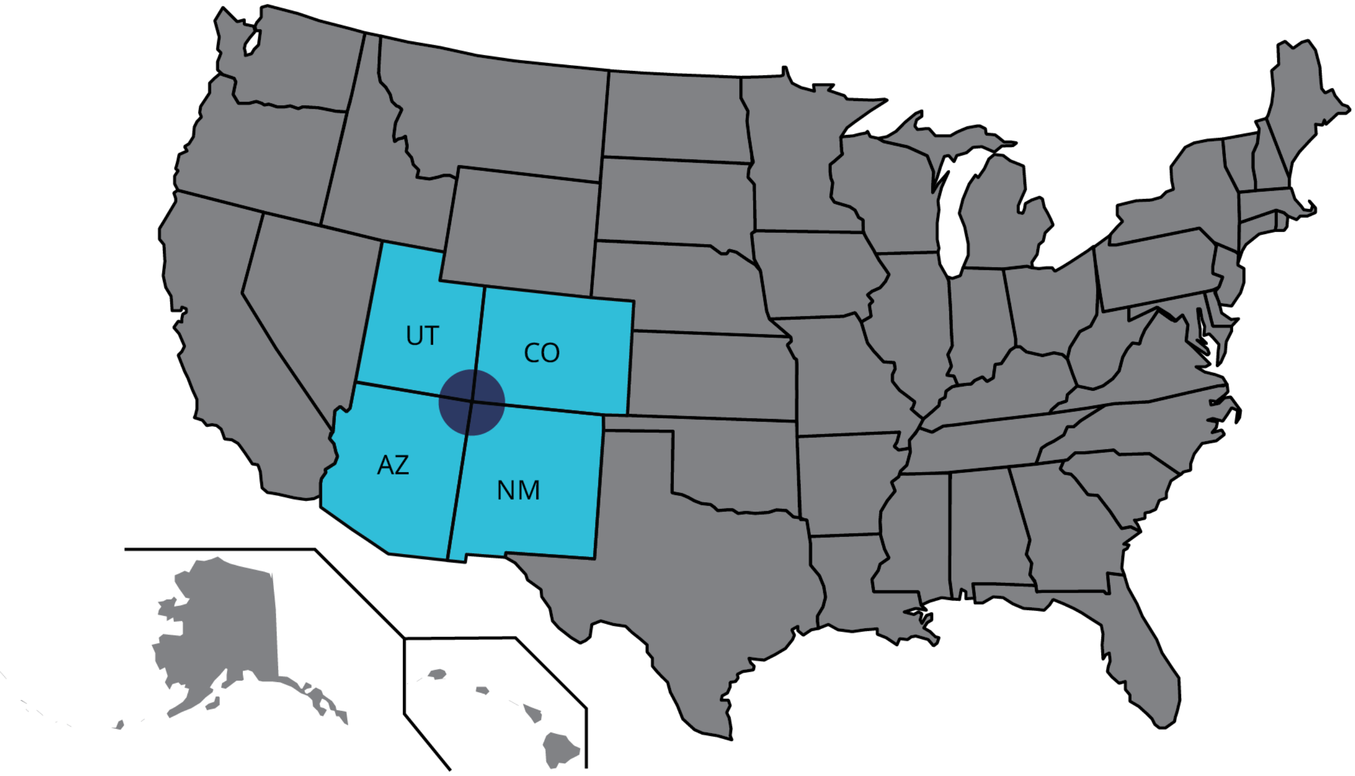


Figure 12.1.10: Map of the Four Corners

In Figure 12.1.11, each vertex represents one of these states, and each edge represents a shared border. States like Utah and New Mexico that meet at only a single point are *not* considered to have a shared border. By representing this map as a graph, where the connections are shared borders, we shift our perspective from physical attributes such as shape, size and distance, toward the existence of the relationship of having a shared boundary.

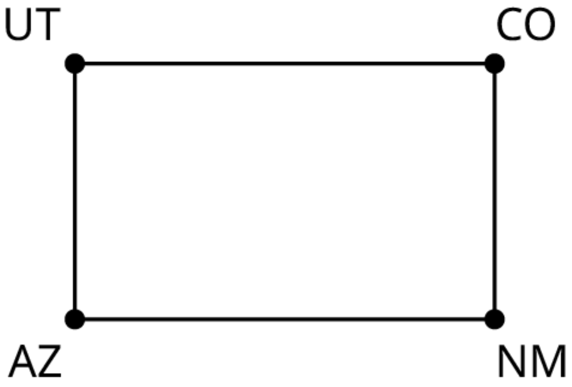


Figure 12.1.11: Graph of the Shared Boundaries of Four Corners States

✓ Example 12.1.4: Graphing the Midwestern States

A map of the Midwest is given in Figure 12.1.12. Create a graph of the region in which each vertex represents a state and each edge represents a shared border.

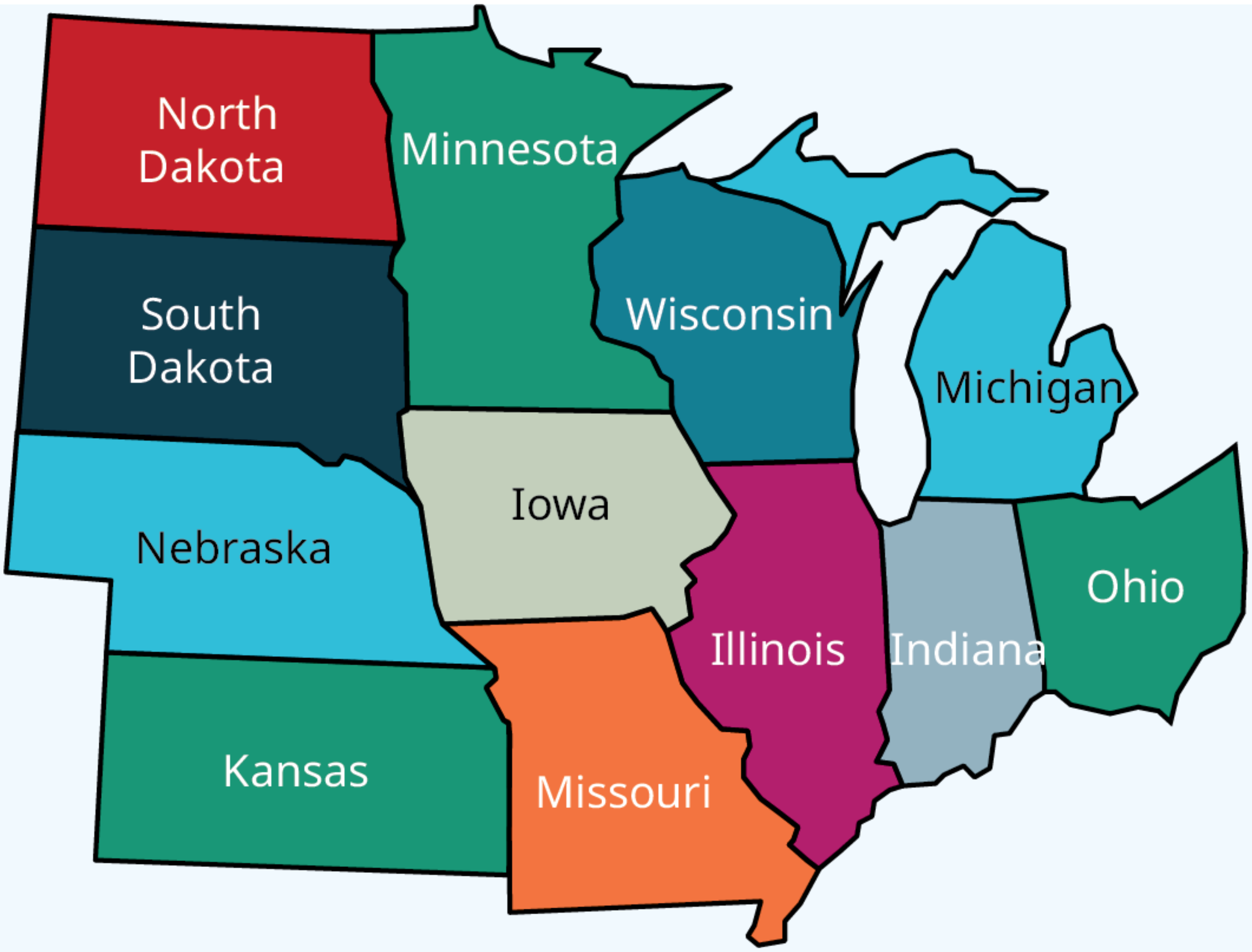

North Dakota
Minnesota
South Dakota
Wisconsin
Michigan
Iowa
Nebraska
Ohio
Illinois
Indiana
Kansas
Missouri

Figure 12.1.12 Map of Midwestern States

Answer

Step 1: For each state, draw and label a vertex as in Figure 12.1.13

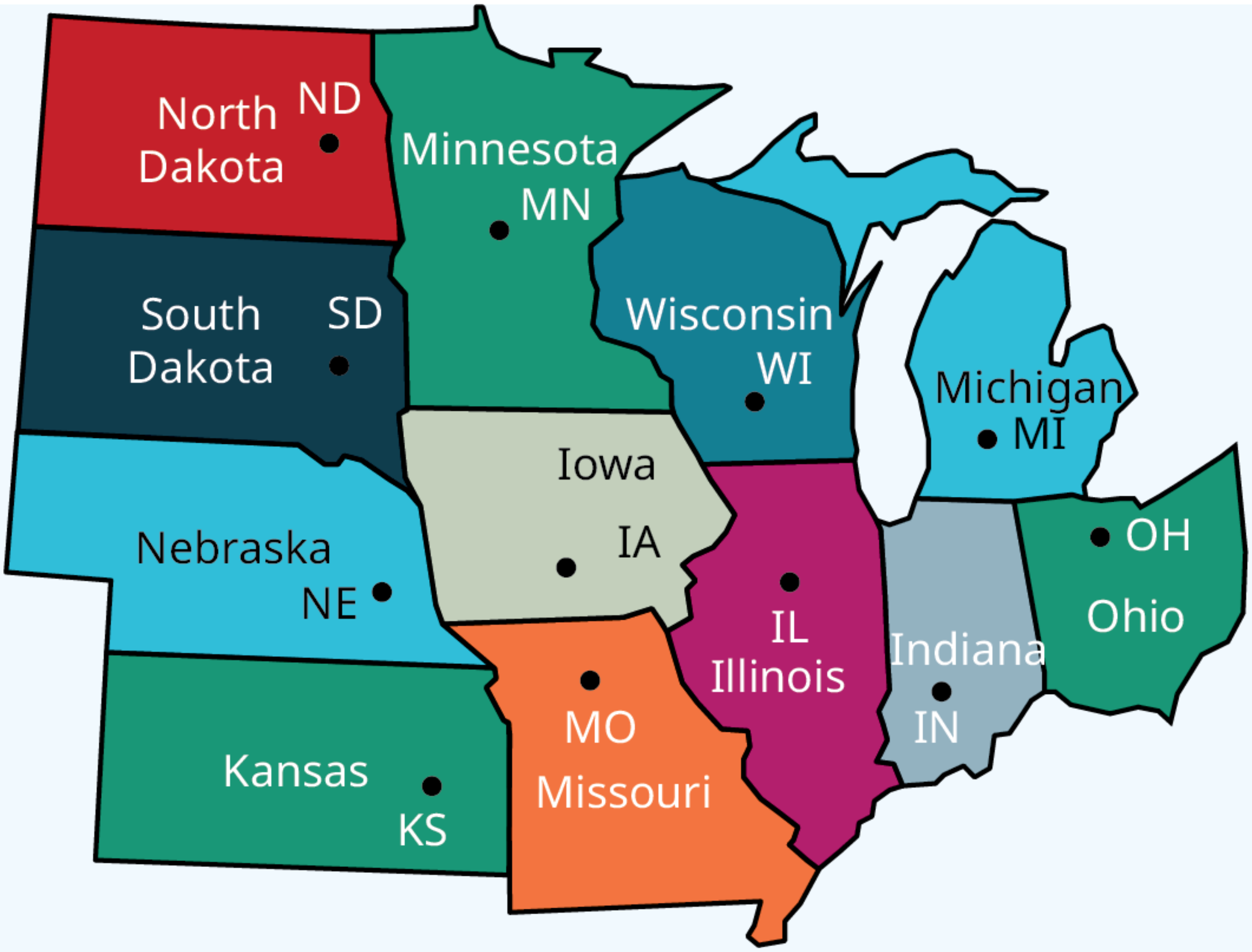
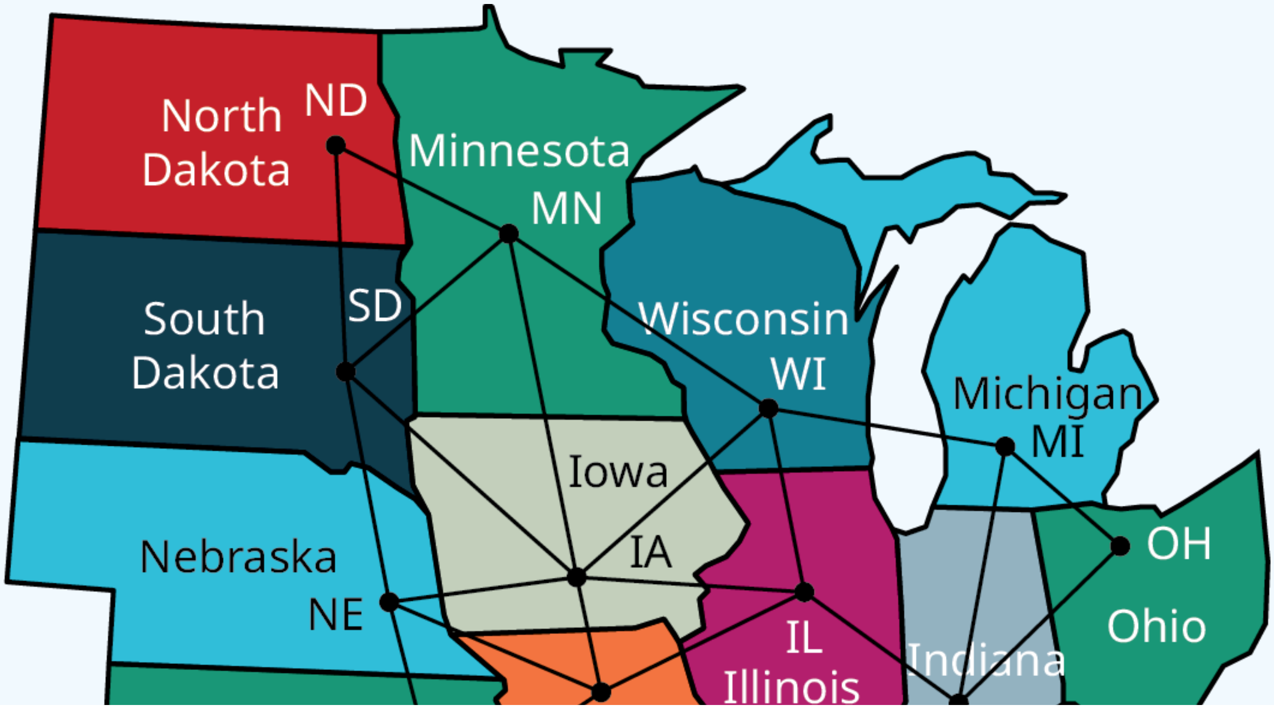

North Dakota
ND
Minnesota
MN
South Dakota
SD
Wisconsin
WI
Michigan
MI
Iowa
IA
Nebraska
NE
IL
Illinois
Indiana
IN
OH
Ohio
Kansas
KS
MO
Missouri

Figure 12.1.13 Vertex Assigned to Each Midwestern State

Step 2:

Draw edges between any two states that share a common land border as in Figure 12.1.14


North Dakota
ND
Minnesota
MN
South Dakota
SD
Wisconsin
WI
Michigan
MI
Iowa
IA
Nebraska
NE
OH
Ohio
IL
Illinois
Indiana

MO
IN
Kansas
KS
Missouri

Figure 12.1.14: Edge Assigned to Each Pair of Midwestern States with Common Border

The graph is given in Figure 12.1.15.

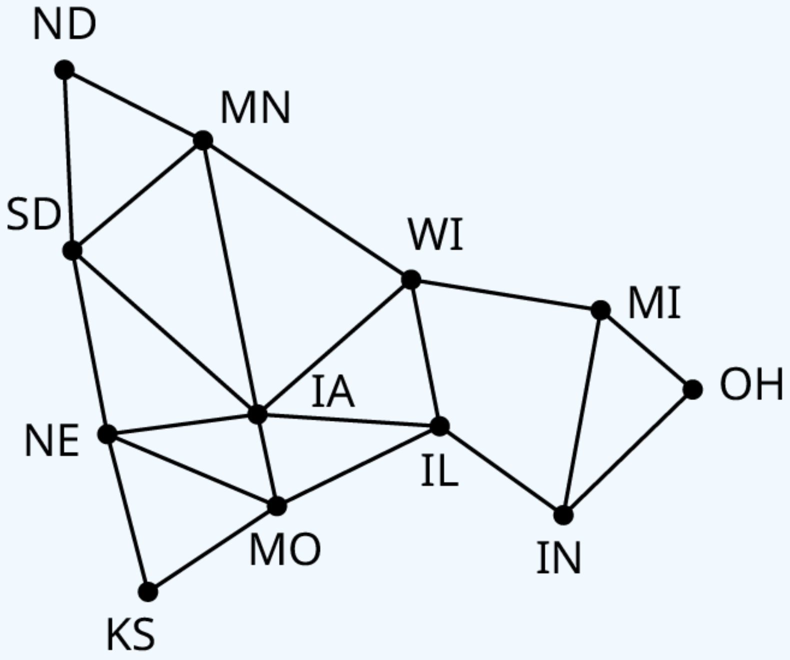

ND
MN
SD
WI
MI
OH
IA
NE
IL
MO
IN
KS

Figure 12.1.15: Final Graph Representing Common Boundaries between Midwestern States

Your Turn 12.1.4

The figure shows a map of the Island of Oahu in the State of Hawaii divided into regions. Draw a graph in which each vertex represents one of the regions and each edge represents a shared land border.

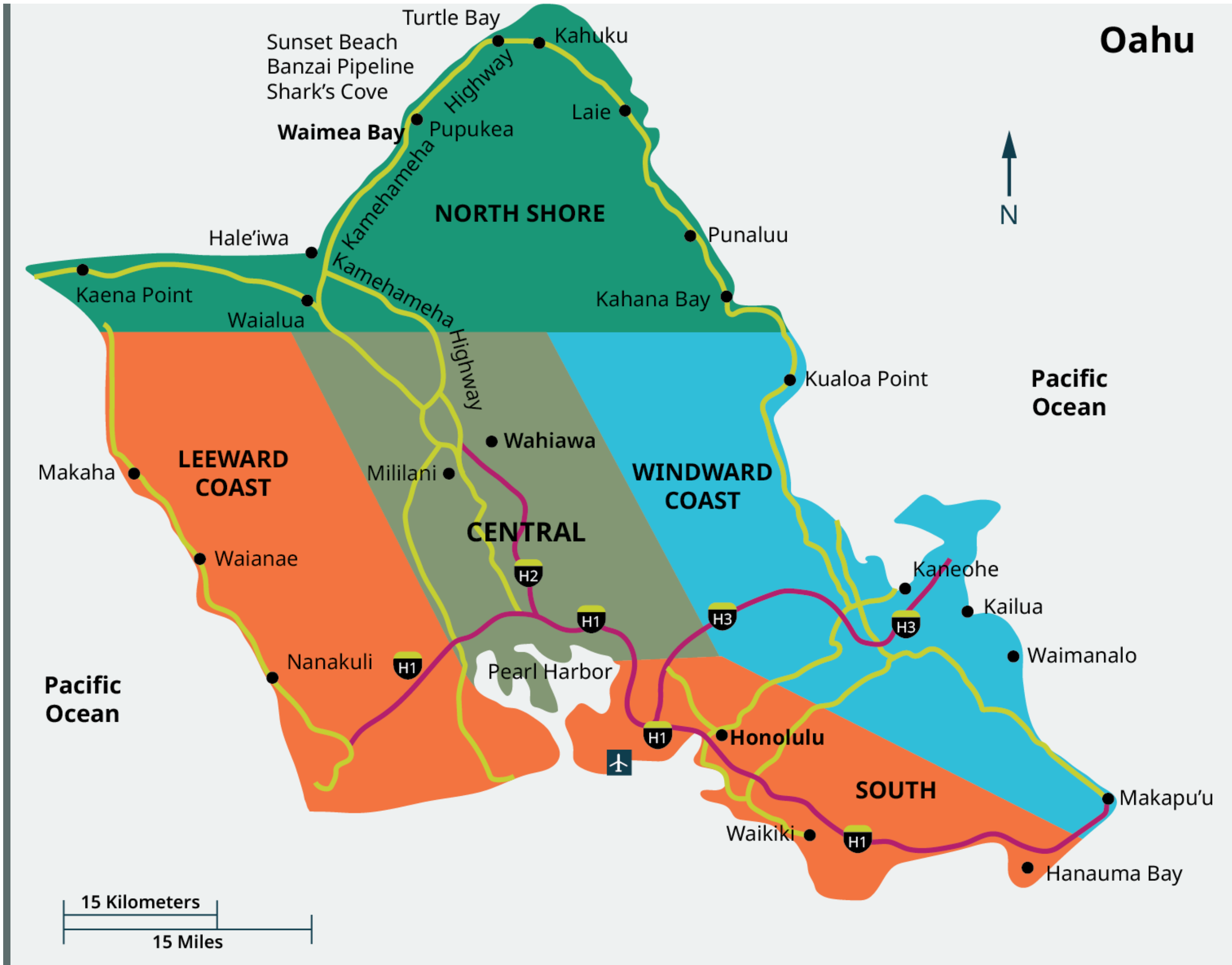
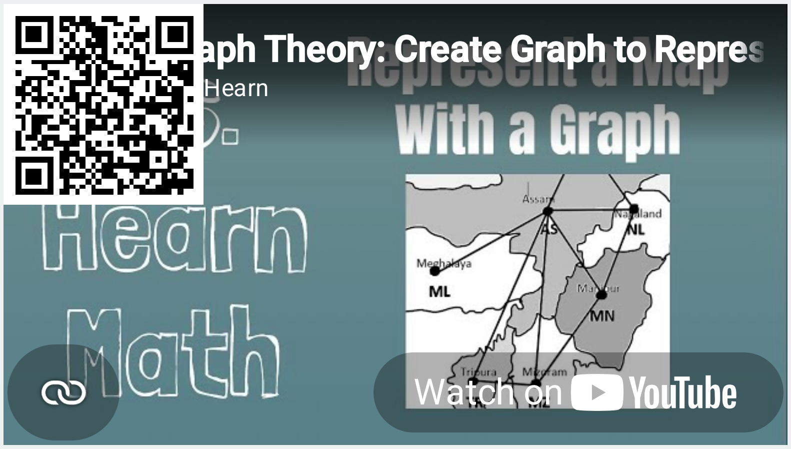

Oahu
Turtle Bay
Sunset Beach
Banzai Pipeline
Shark's Cove
Kahuku
Kamehameha Highway
Laie
Waimea Bay
Pupukea
NORTH SHORE
N
Hale'iwa
Punaluu
Kaena Point
Waialua
Kahana Bay
Kamehameha Highway
Kualoa Point
Pacific Ocean
Wahiawa
Makaha
LEEWARD COAST
Mililani
WINDWARD COAST
CENTRAL
Waianae
Kaneohe
H2
H1
H3
Kailua
H3
Waimanalo
Nanakuli
H1
Pearl Harbor
Pacific Ocean
H1
Honolulu
SOUTH
Makapu'u
Waikiki
H1
Hanauma Bay
15 Kilometers
15 Miles

Figure 12.1.16: Map of Oahu

Video


aph Theory: Create Graph to Repres
Hearn
Represent a Map
With a Graph
Hearn
Math
Assam
AS
Nagaland
NL
Meghalaya
ML
Manipur
MN
Tripura
Mizoram
Watch on YouTube

Graph Theory: Create a Graph to Represent Common Boundaries on a Map

Graphs of Social Interactions

Geographical maps are just one of many real-world scenarios which graphs can depict. Any scenario in which objects are connected to each other can be represented with a graph, and the connections don't have to be physical. Just think about all the connections you have to people around the world through social media! Who is in your network of Twitter followers? Whose Snapchat network are you connected to?

✓ Example 12.1.5: Graphing Chloe's *Roblox* Friends

Roblox is an online gaming platform. Chloe is interested to know how many people in her network of *Roblox* friends are also friends with each other so she polls them. Explain how a graph or multigraph might be drawn to model this scenario by identifying the objects that could be represented by vertices and the connections that could be represented by edges. Indicate whether a graph or a multigraph would be a better model.

Answer

> The objects that are represented with vertices are *Roblox* friends. A *Roblox* friendship between two friends will be represented as an edge between a pair of vertices. There will be no double edges because it is not possible for two friends to be linked twice in *Roblox*; they are either friends or they are not. Also, a player cannot be a friend to themself, so there is no need for a loop. Since there are no double edges or loops, this is best represented as a graph.

Your Turn 12.1.5

In a particular poker tournament, five groups of five players will play at a table until one player wins, then the five winning players will play each other at a table in a final round. Explain how a graph or multigraph might be drawn to model this scenario by identifying the objects that could be represented by vertices and the connections that could be represented by edges. Indicate whether a graph or a multigraph would be a better model.

Who Knew?: Using Graph Theory to Reduce Internet Fraud

Could graphs be used to reduce Internet fraud? At least one researcher thinks so. Graph theory is used every day to analyze our behavior, particularly on social network sites. Alex Buetel, a computer scientist from Carnegie Mellon University in Pittsburgh, Pennsylvania, published a research paper in 2016 that discussed the possibilities of distinguishing the normal interactions from those that might be fraudulent using graph theory. Buetel wrote, "To more effectively model and detect abnormal behavior, we model *how* fraudsters work, catching previously undetected fraud on Facebook, Twitter, and Tencent Weibo and improving classification accuracy by up to 68%." In the same paper, the researcher discusses how similar techniques can be used to model many other applications and even, "predict *why* you like a particular movie." (Alex Beutel, "User Behavior Modeling with Large-Scale Graph Analysis," http://reports-archive.adm.cs.cmu.ed...-CS-16-105.pdf, May 2016, CMU-CS-16-105, Computer Science Department, School of Computer Science, Carnegie Mellon University, Pittsburgh, PA)

Check Your Understanding

1. A simple graph has no loops.
   a. True
   b. False

2. It is not possible to have a vertex of degree 0.
   a. True
   b. False

3. A graph with three vertices has at most three edges.
   a. True
   b. False

4. A multigraph with three vertices has at most three edges.
   a. True
   b. False

5. If vertex a is adjacent to vertex b, and vertex b is adjacent to vertex c, then vertex a must be adjacent to vertex c.

a. True
b. False

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12.1.0: Exercises

For the following exercises, use the figure showing Diagrams A, B, C, and D.

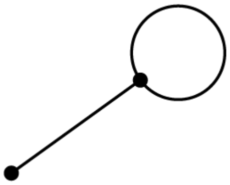


Diagram A

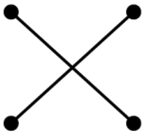


Diagram B

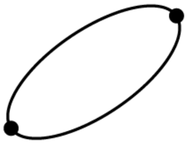


Diagram C

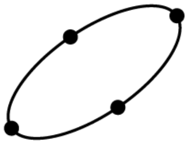


Diagram D

Exercise 12.1.0.1

Identify any multigraphs.

Exercise 12.1.0.2

Identify any graphs.

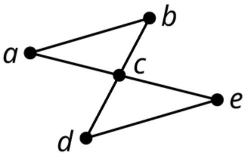
Exercise 12.1.0.3

Identify any multigraph that has a loop.

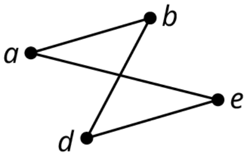
Exercise 12.1.0.4

Identify any multigraph that has a double edge.

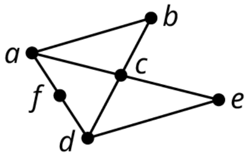
For the following exercises, use the figure showing Graphs S, T, U, V, and W.



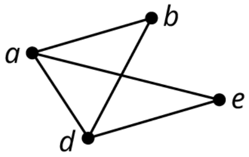
Graph S



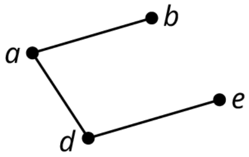
Graph T



Graph U



Graph V



Graph W

Exercise 12.1.0.5

Determine the number of vertices in Graph T.

Exercise 12.1.0.6

Determine the number of vertices in Graph U.

Exercise 12.1.0.7

Identify the graph with the most vertices.

Exercise 12.1.0.8

Identify the graphs with four vertices.

Exercise 12.1.0.9

Identify the graph with the most edges.

Exercise 12.1.0.10

Identify the graph with the fewest edges.

Exercise 12.1.0.11

Name the vertices in Graph S.

Exercise 12.1.0.12

Name the vertices in Graph V.

Exercise 12.1.0.13

Determine the number of edges in Graph U.

Exercise 12.1.0.14

Determine the number of edges in Graph T.

Exercise 12.1.0.15

Name the edges in Graph V.

Exercise 12.1.0.16

Name the edges in Graph W.

Exercise 12.1.0.17

Identify any pairs of vertices in Graph T that are not adjacent.

Exercise 12.1.0.18

Identify any vertices in Graph V that are not adjacent.

Exercise 12.1.0.19

Find the degree of vertex a in graph U.

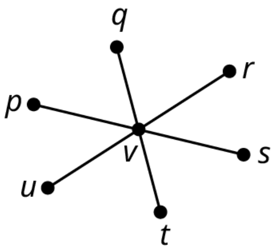
Exercise 12.1.0.20

Find the degree of vertex a in graph S.

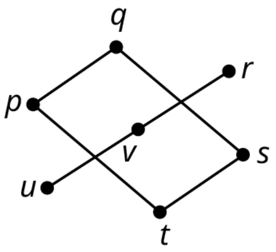
Exercise 12.1.0.21

Which two graphs have a vertex of degree 4?

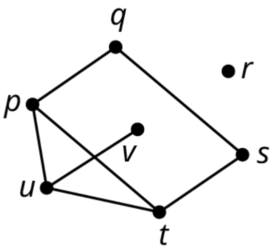
For the following exercises, use the figure showing Graphs A, B, C, and D.



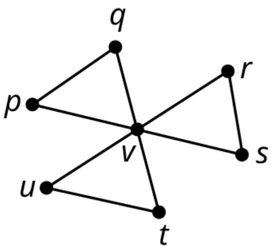
Graph A



Graph B



Graph C



Graph D

? Exercise 12.1.0.22

Identify the graph with 1 vertex of degree 6 and 6 vertices of degree 1.

? Exercise 12.1.0.23

Identify the graph with 1 vertex of degree 6 and 6 vertices of degree 2.

? Exercise 12.1.0.24

Identify the graph with exactly 1 vertex of degree 0.

? Exercise 12.1.0.25

Identify the graph with exactly 2 vertices of degree 1.

? Exercise 12.1.0.26

Name all the pairs of adjacent vertices in Graph *B*.

? Exercise 12.1.0.27

Name all the pairs of adjacent vertices in Graph *A*.

? Exercise 12.1.0.28

Identify the graph in which the sum of the degrees of the vertices is 14.

For each of the following exercises, a scenario is given. Explain how a graph or multigraph might be drawn to model the scenario by identifying the objects that could be represented by vertices and the connections that could be represented by edges. Indicate whether a graph or a multigraph would be a better model.

? Exercise 12.1.0.29

The Centers for Disease Control and Prevention (CDC) is tracking the spread of a virus. The CDC is attempting to determine where the virus began by identifying where each known carrier contracted the virus.

? Exercise 12.1.0.30

The faculty members in a college mathematics department have a “telephone tree” that assigns each faculty member two other faculty members to call with information in the event of an emergency. The chairperson of the department contacts two faculty members. Each of these faculty members contacts two faculty members, and so on, until all faculty members have been notified.

? Exercise 12.1.0.31

There is a theory that any two people on earth are no more than six social connections apart, which is known as “six degrees of separation” or the “six handshake rule.” It means that there is a chain of a “friend of a friend” that connects any two people through at most six steps.

? Exercise 12.1.0.32

The U.S. Postal Service has a network of post offices that move mail around the United States. Mail trucks from one post office follow routes deliver mail to other locations. These mail trucks also pick up mail to bring back to their home post office.

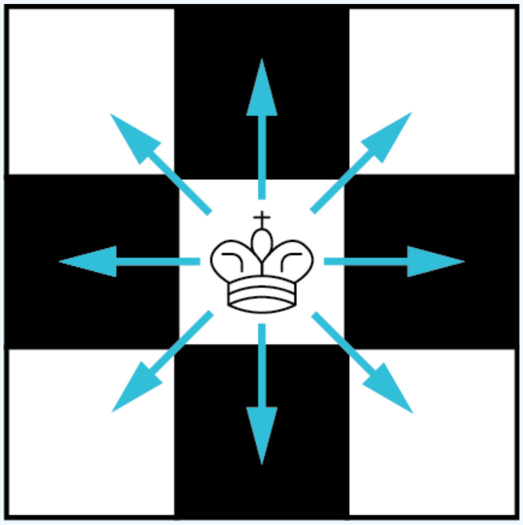
In the following exercises , draw a graph to represent the given scenario.

? Exercise 12.1.0.33

Draw a graph to model the scenario described in Your Turn 12.5.

? Exercise 12.1.0.34

In the game of chess, a king can move one space in any direction, vertically, horizontally, or diagonally, as indicated in the figure. Draw a graph to represent the possible movements of a king on a three-by-three section of a chess board such that each edge represents one move, and each vertex represents a position where the king might be at the beginning or end of a turn. Assume that the king can have unlimited turns.



? Exercise 12.1.0.35

Chloe is interested to know how many in her network of *Roblox* friends are also friends with each other, so she polls them. The following is a list of each of Chloe’s friends and their common friends. Create a graph of the *Roblox* friends in Chloe’s network, including Chloe.

- Aria is friends with no one else in Chloe’s network.
- Benicio is friends with Dakshayani, Eun-ah, and Fukashi.
- Dakshayani is friends with Benicio.
- Eun-ah is friends with Benicio and Fukashi.
- Fukashi is friends with Benicio and Eun-ah.

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12.2: Graph Structures

Learning Objectives

1. Describe and interpret relationships in graphs.
2. Model relationships with graphs.

Graph theory is used in neuroscience to study how different parts of the brain connect. Neurobiologists use functional magnetic resonance imaging (fMRI) to measure levels of blood in different parts of the brain, called nodes. When nodes are active at the same time, it suggests there is a functional connection between them so they form a network. This network can be represented as a graph where the vertices are the nodes and the functional connections are the edges between them. (Mikey Taylor, "Graph Theory & Machine Learning in Neuroscience," Medium.com, June 24, 2020.

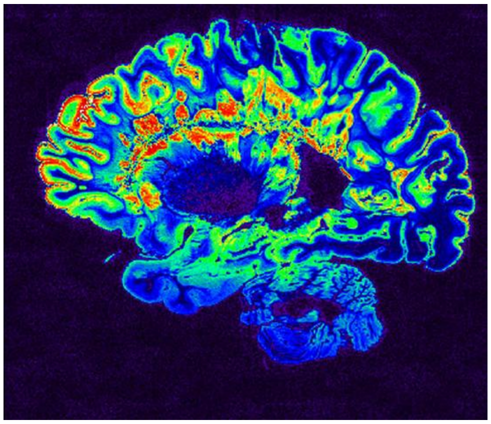


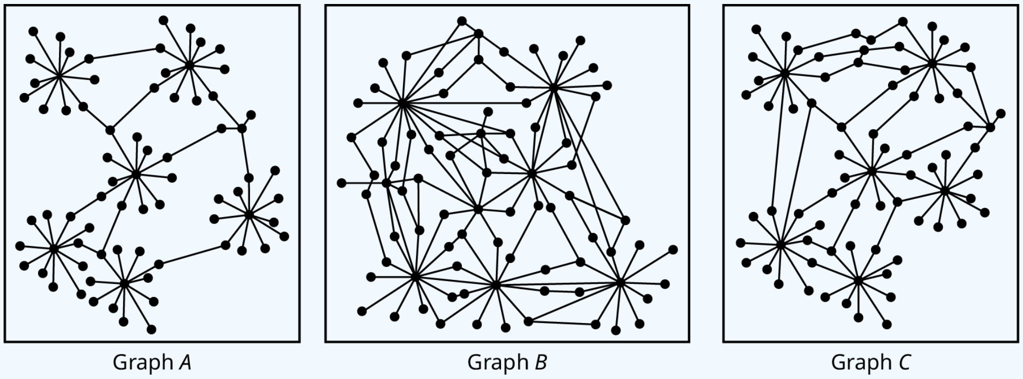
Figure 12.2.1: Neuroimaging shows brain activity. (credit: "MRI Scan" by NIH Image Gallery/Flickr, Public Domain)

Importance of the Degrees of Vertices

One reason scientists study these networks is to determine how successful the communication within a network continues to be when it experiences failures in nodes and connections. Graphs can be used to study the resilience of these networks. (Mikey Taylor, "Graph Theory & Machine Learning in Neuroscience," Medium.com, June 24, 2020)

✓ Example ✓ 12.2.1: Using Graphs to Understand Relationships

The graphs in Figure ✓ 12.2.2 represent neural networks, where the vertices are the nodes, and the edges represent functional connections between them. Which graph do you think would represent a network with the highest resistance to failure? Which graph would probably be the most vulnerable to failure? How might this relate to the degrees of the vertices?



Graph *A* Graph *B* Graph *C*

Figure ✓ 12.2.2 Models of Neural Networks

Answer

Graph B represents a network that would have the highest resistance to failure because there are more edges connecting the nodes indicating there are more connections between nodes than on either of the other graphs. This would make the network more resistant to failure because there are more options to work around a faulty edge or node. Graph A would probably be the most susceptible to failure because the network depends on a few particular edges and vertices for connections. There are more vertices of higher degree in Graph B than in Graph A because there are more edges connecting the nodes.

Your Turn 12.2.1

Sociologists use graphs to study connections between people and to identify characteristics that make communities more resilient, more likely to stay connected over time. In the graphs shown, the edges represent social connections and the vertices are individuals.

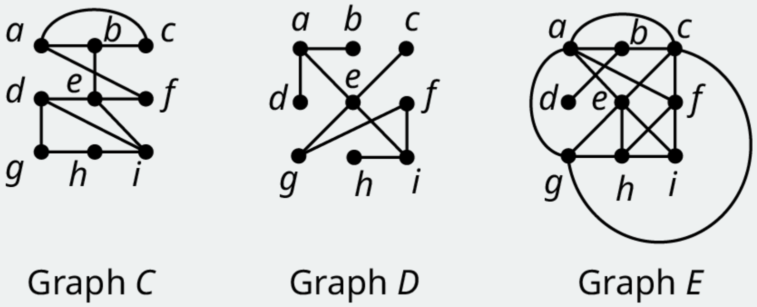


Figure $12.2.3$: Models of Communities

Which graph do you think represents the most resilient community? What is the sum of the degrees of all the vertices in this graph?

Which graph do you think represents the community that is least likely to stay connected over time? What is the sum of the degrees of all the vertices in this graph?

Is the sum of the degrees higher or lower in a more resilient community?

Relating the Number of Edges to the Degrees of Vertices

In the applications of graph theory to neuroscience and sociology in Example $12.2.1$ and YOUR TURN $12.2.1$, there was a correlation between the degrees of vertices and the resilience of a network. Researchers in many fields have also observed a direct relationship between the number of edges in a graph and the degrees of the vertices. To begin to understand this relationship, consider a graph with five vertices and zero edges as in Figure $12.2.4$.

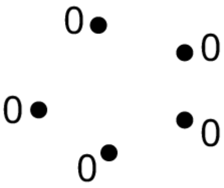


Figure $12.2.4$: Graph with Five Vertices and zero Edges

Instead of being marked with a name, each vertex in Figure 12.21 is marked with its degree. In this case, all of the degrees are 0 so the sum of the degrees is also zero. Suppose that we add an edge between any two existing vertices and indicate the degrees of the vertices. This gives us a graph with five vertices and one edge like the graph in Figure $12.2.5$.

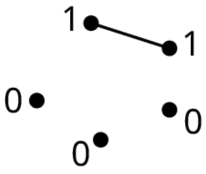


Figure $12.2.5$: Graph with Five Vertices and One Edge

Note that the degrees of two vertices increased, each by 1. So, the sum of the degrees is now 2. Suppose that we continue in this way, adding one edge at a time and making note of the number of edges and the sum of the degrees of the vertices as in Figure 12.23.

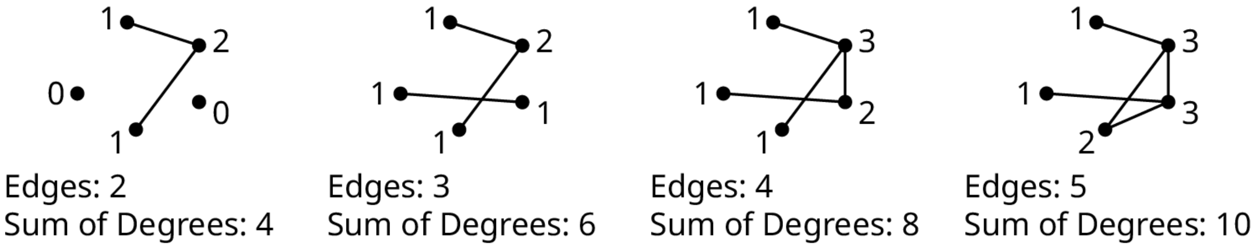


Figure 12.2.6: Comparing Number of Edges to Sum of Degrees

Figure 12.2.7 demonstrates a characteristic that is true of all graphs of any shape or size. When the number of edges is increased by one, the sum of the degrees increases by two. This happens because each edge has two ends and each end increases the degree of one vertex by one unit. As a result, the sum of the degrees of the vertices on any graph is always twice the number of edges. This relationship is known as the **Sum of Degrees Theorem**.

FORMULA

For the Sum of Degrees Theorem,

$$sumofthedegrees\backslash(= 2\times \backslash)numberofedges$$

or

$$\text{number of edges } = \frac{\text{sum of degrees}}{2}$$

mfrac

Example 12.2.2: Finding the Sum of Degrees

Suppose that a graph has five edges.

1. Find the sum of the degrees of the vertices.
2. Draw two different graphs that demonstrate this conclusion.

Answer

1. The sum of the degrees is twice the number of edges: 2 × 5=10. The sum of the degrees is 10.
2. See Figure ✓ 12.2.8.

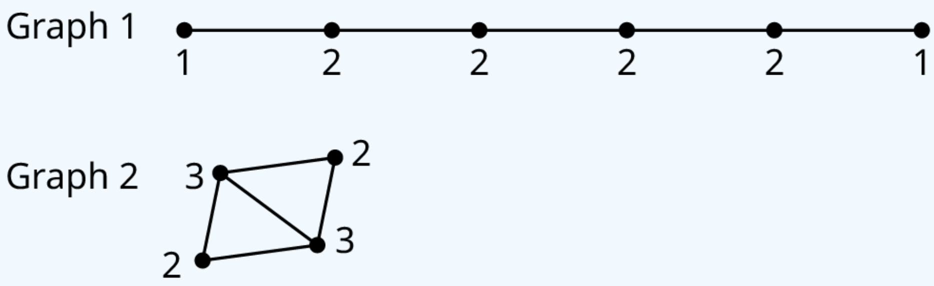


Figure ✓ 12.2.8

Your Turn 12.2.2

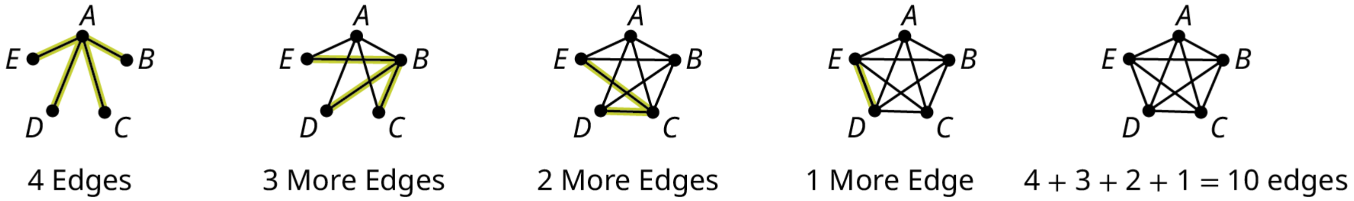
Suppose that the sum of the degrees of a graph is six.

Find the number of edges.

Draw two graphs that demonstrate your conclusion.

Completeness

Suppose that there were five strangers in a room, A, B, C, D, and E, and each one would be introduced to each of the others. How many introductions are necessary? One way to begin to answer this question is to draw a graph with each vertex representing an individual in the room and each edge representing an introduction as in Figure $12.2.9$.



4 Edges   3 More Edges   2 More Edges   1 More Edge   4 + 3 + 2 + 1 = 10 edges

Figure $12.2.9$: Model of Introductions between Five Strangers

Let's approach the problem by thinking about how many new people Person A would meet, then Person B, and so on, making sure not to repeat any introductions. The first graph in Figure $12.2.9$ shows Person A meeting Persons B, C, D, and E, for a total of 4 introductions. The next graph shows that Person B still has to meet Persons C, D, and E, for a total of 3 more introductions. The next graph shows that Person C still has to meet Persons D and E, which is 2 more introductions. The next graph shows that Person D only remains to meet Person E, which is 1 more introduction. The final graph has $4+3+2+1\ =10$ edges representing 10 introductions. The last graph is an example of a complete graph because each pair of vertices is joined by an edge. Another way of saying this is that the graph is complete because each vertex is adjacent to every other vertex. Figure $12.2.10$ shows complete graphs with three, four, five, and six vertices. Math input error

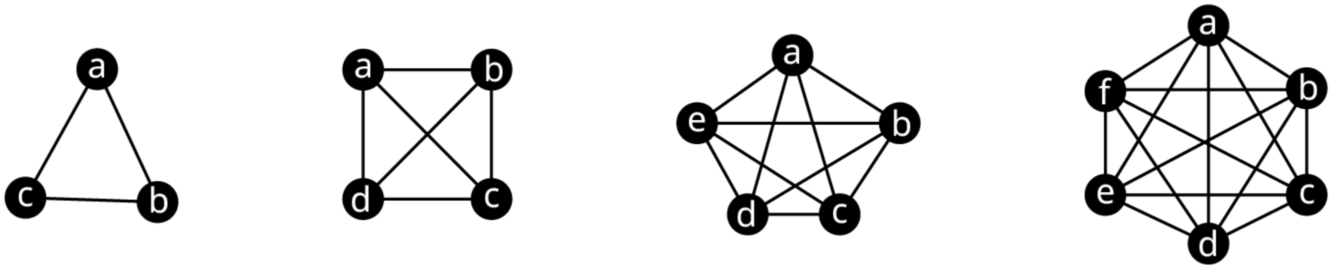


Figure $12.2.10$6: Complete Graphs with Up to Six Vertices

Suppose we want to know the number of introductions necessary in a room with six people. This would be represented by a complete graph with six vertices, and the total number of introductions would be $5+4+3+2+1=15$, the number of edges in the graph. In fact, you can always find the number of introductions in a room with n people by adding all the whole numbers from 1 to $n-1$.

FORMULA

The number of edges in a complete graph with n vertices is the sum of the whole numbers from 1 to $n-1$, $1+2+3+\cdots+(n-1)$.

Suppose that we want to determine how many introductions are necessary in a room with 500 strangers. In other words, suppose that we want to determine the number of edges in a complete graph with 500 vertices. Adding up all the numbers from 1 to 499 could take a long time! In the next example, we use the Sum of Degrees Theorem to make the problem more manageable.

✓ Example $12.2.3$: Using the Sum of Degrees Theorem

Use the Sum of Degrees Theorem to determine the number of introductions required in a room with

1. 6 strangers.
2. 10 strangers.
3. n strangers.

Answer

1. Since there are 6 strangers, there are 6 vertices. Since each individual must meet 5 other individuals, there are 5 edges meeting at each vertex which means each vertex has degree 5. Since there are 6 vertices of degree 5 , the sum of degrees is $6 \cdot 5 = 30$. By the Sum of Degrees Theorem, the number of edges is half the sum of the degrees, which is $\frac{30}{2} = 15$. So, the total number of introductions is 15.
2. Since there are 10 strangers, there are 10 vertices. Since each individual must meet 9 other individuals, there are 9 edges meeting at each vertex which means each vertex has degree 9. Since there are 10 vertices of degree 9 , the sum of degrees is $10 \cdot 9 = 90$. By the Sum of Degrees Theorem, the number of edges is half the sum of the degrees, which is $\frac{90}{2} = 45$. So, the total number of introductions is 45.
3. Since there are n strangers, there are n vertices. Since each individual must meet $n-1$ other individuals, there are $n-1$ edges meeting at each vertex which means each vertex has degree $n-1$. Since there are n vertices of degree $n-1$, the sum of degrees is $n(n-1)$. By the Sum of Degrees Theorem, the number of edges is half the sum of the degrees, which is $\frac{n(n-1)}{2}$. So, the total number of introductions is $\frac{n(n-1)}{2}$.

Your Turn 12.2.3

Determine the number of introductions necessary in a room with 500 strangers using the Sum of Degrees Theorem.

Now we have a shorter way to calculate the number of introductions in a room with n strangers, and the number of edges on a complete graph with n vertices. Let's update our formula.

FORMULA

The number of edges in a complete graph with n vertices is $1+2+3+\cdots+(n-1) = \frac{n(n-1)}{2}$.

Subgraphs

Sometimes a graph is a part of a larger graph. For example, the graph of South Florida Airports from Figure 12.8 is part of a larger graph that includes Orlando International Airport in Central Florida, which is shown in Figure 12.2.11

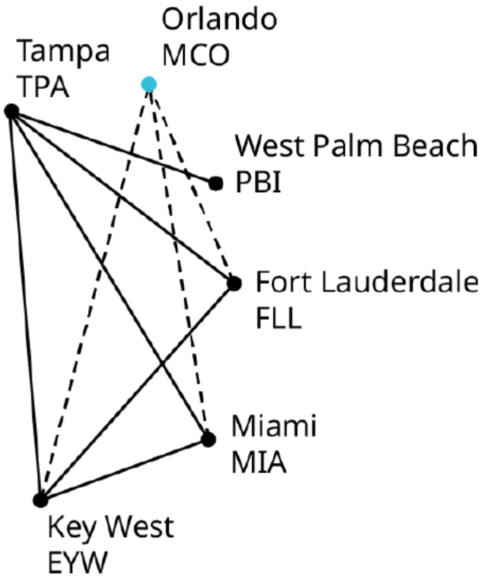


Figure 12.2.11: Orlando and South Florida Airports

The graph in Figure 12.2.11 includes an additional vertex, MCO, and additional edges shown with dashed lines. The graph of direct flights between South Florida airports from Figure 12.1.7 is called a **subgraph** of the graph that also includes direct flights between Orlando and the same South Florida airports in Figure 12.2.11. In general terms, if Graph B consists entirely of a set of edges and vertices from a larger Graph A, then B is called a subgraph of A.

✓ Example 12.2.4: Identifying a Subgraph

In Figure ✓ 12.2.12, Graph G is given, along with four diagrams. Determine whether each diagram is or is not a subgraph of Graph G and explain why.

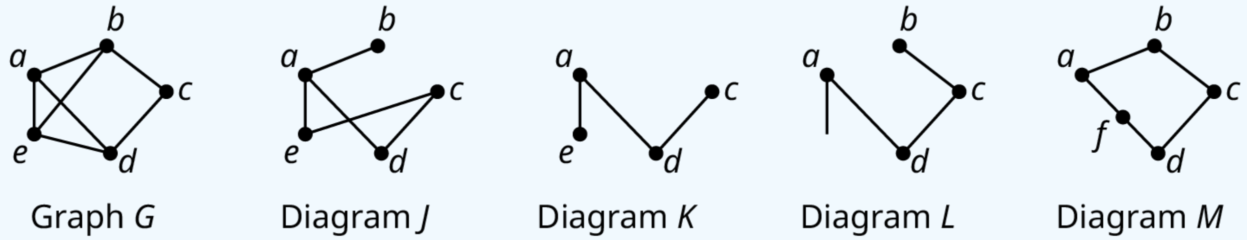

a
b
c
d
e
Graph G
a
b
c
d
e
Diagram J
a
c
d
e
Diagram K
a
b
c
d
Diagram L
a
b
c
d
f
Diagram M

Figure ✓ 12.2.12 Graph G and Potential Subgraphs

Answer

- Diagram J is not a subgraph of Graph G because edge ec is not in Graph G.
- Diagram K is a subgraph of Graph G because all of its vertices and edges were part of Graph G.
- Diagram L is not a graph at all, because there is a line extending from vertex a that does not connect a to another vertex. So, Diagram L cannot be a subgraph.
- Diagram M is not a subgraph of Graph G because it contains a vertex f, which is not part of G.

Your Turn 12.2.4

Draw a subgraph of Graph F from Figure 12.1.3 that has exactly four vertices and five edges.

Identifying and Naming Cycles

When you think of a cycle in everyday life, you probably think of something that begins and ends the same way. For example, the water cycle (Figure 12.29) begins with water in a lake or ocean, which evaporates into water vapor, condenses into clouds, and then returns to earth as rain or some other form of precipitation that settles into lakes or oceans. A cycle in graph theory is similar in that it begins and ends in the same way: It is a series of connected edges that begin and end at the same vertex but otherwise never repeat any vertices.

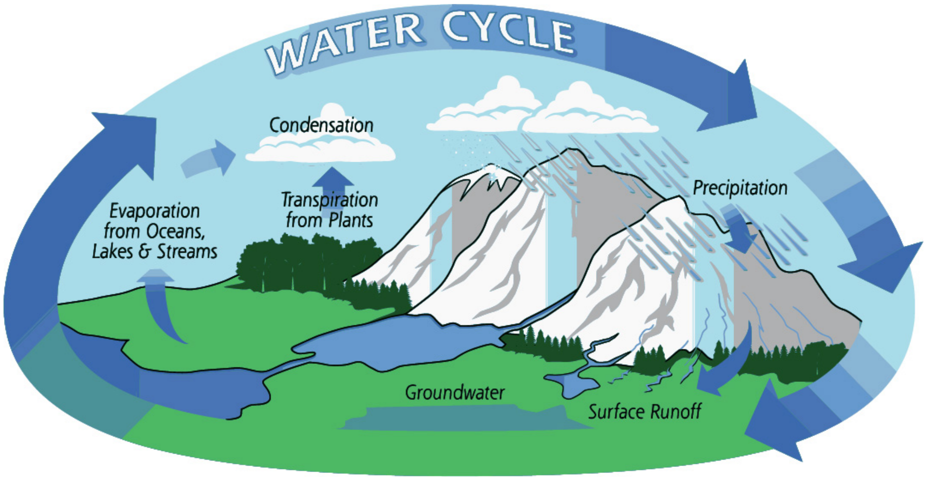
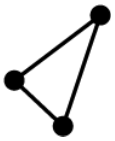
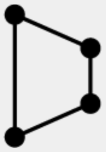
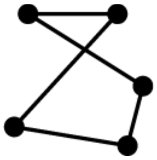
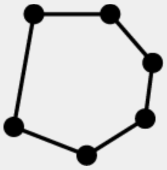
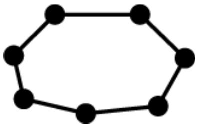
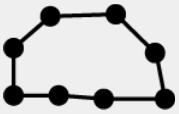
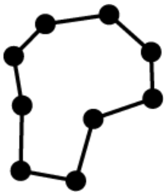
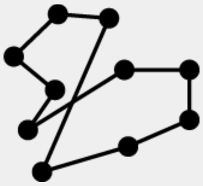

WATER CYCLE
Condensation
Transpiration from Plants
Evaporation from Oceans, Lakes & Streams
Precipitation
Groundwater
Surface Runoff

Figure 12.2.13: The water cycle begins and ends with water. (credit: "Diagram of the water cycle" by NASA, Public Domain)

In a cycle, there are always the same number of vertices as edges, and all vertices must be of degree 2. Cycles are often referred to by the number of vertices. For example, a cycle with 5 vertices can be called a 5-cycle. Cycles can also be named after polygons based on the number of edges. For example a 5-cycle is also called a pentagon. Table 12.2.1 lists these names for cycles of size 3 through size 10.

Table 12.2.1 Categories of Cycles

Cycle Category	Number of Edges	Example
triangle	3	
quadrilateral	4	
pentagon	5	
pentagon	6	
heptagon	7	
octagon	8	
octagon	9	
octagon	10	

There are many more polygon names, including a megagon that has a million edges and a googolgon that has 10^{100} edges, but usually we just say n-gon when the number n is past 10 . For example, a cycle with 11 edges could be called an 11-gon.

Notice that the 10-cycle, or decagon, appears to cross over itself in Table 12.2.1. Remember, graphs can be drawn differently but represent the same connections. In Figure 12.2.14, the same decagon is transformed into a graph that does not appear to overlap itself. We have done this without changing any of the connections so both diagrams represent the same relationships, and both diagrams are considered decagons.

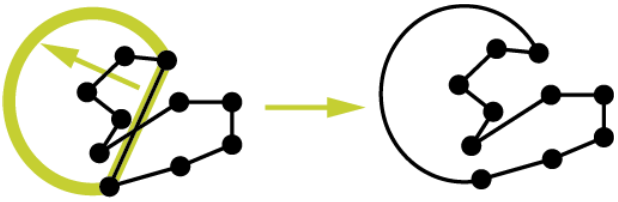


Figure 12.2.14: Transformation of a Decagon

Who Knew?: Googol to Google

Did the name "googolgon" ring a bell for you? If so, there is a good reason for it. The creators of Google.com renamed their search engine Google, a misspelled reference to the number "googol" alluding to the enormous number of calculations their algorithm can tackle. A googol is a 1 followed by 100 zeroes, or 10^{100}. William L. Hosch, "Google, American Company," https://www.britannica.com/topic/Google-Inc, Encyclopedia Britannica online)Math input error

Cyclic Subgraphs and Cliques

When cycles appear as subgraphs within a larger graph, they are called **cyclic subgraphs**. Cyclic subgraphs are named by listing their vertices sequentially. The vertex where you begin is not important. Graph K in Figure 12.2.15 has two triangle cycles (g, h, j) and (h, i, j), and one quadrilateral cycle (g, h, i, j).

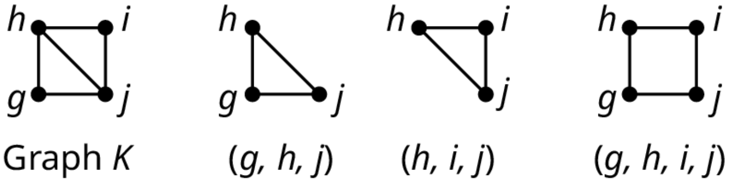


Figure 12.2.15: Cycles in Graph K

Checkpoint

The same cycle can be named in many ways, but we must keep in mind that listing two vertices consecutively indicates an edge exists between them. For example, (g, h, i, j) can also be named (h, i, j, g) or (j, i, h, g). However, you cannot name it (g, i, h, j,) which doesn't reflect the correct order of the vertices. We can see that the order (g, i, h, j) is incorrect because the cycle has no edges gi or hj.

✓ Example 12.2.5: Identifying Cyclic Subgraphs

Identify the types of cyclic subgraphs in Graph H in Figure ✓ 12.2.16 and name them.

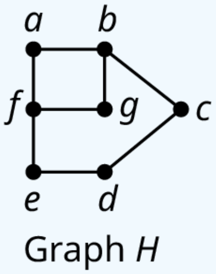


Figure ✓ 12.2.16 Graph H

Answer

> In order to avoid missing a cycle, begin by analyzing the vertex a, then proceed in alphabetical order. Vertex a is part of two cycles, the quadrilateral cycle (a, b, f, g,) and the hexagonal cycle (a, b, c, d, e, f). Next look at vertex b. It is part of the hexagonal cycle (b, c, d, e, f, g). After checking each of the remaining vertices, we see there are no other cycles.

Checkpoint

In order to avoid naming the same cycle twice, consider naming cycles beginning with the letter closest to the beginning of the alphabet and then progressing clockwise.

Your Turn 12.2.5

Which types of cycles are in Graph G in Figure 12.2.16? Use the vertices of the graph to give a name for one of each type that you find.

We have seen that sociologists use graphs to study the structures of social networks. In sociology, there is a principle known as Triadic Closure. It says that if two individuals in a social network have a friend in common, then it is more likely those two individuals will become friends too. Sociologists refer to this as an open triad becoming a closed triad. This concept can be visualized as graphs in Figure a and person b each has a friendship with person c. In the closed triad, person a and person a have also developed a friendship. Notice that the graph representing the closed triad is a three-cycle, or a triangle, in graph theory terms.

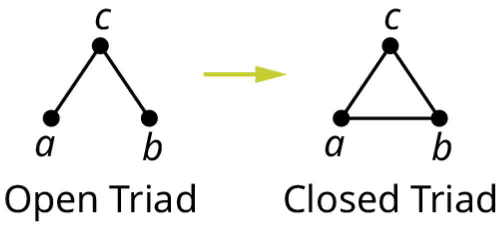


Figure 12.2.17: Graphs of Open Triad and Closed Triad

Another topic of interest to sociologists, as well as computer scientists and scientists in many fields, is the concept of a **clique**. In a social group, a clique is a subgroup who are all friends. A triad is an example of a clique with three people, but there can be cliques of any size. In graph theory, a clique is a complete subgraph.

Example 12.2.6: Identifying Triads and Cliques

The graphs in YOUR TURN 12.2.1 represent social communities. The vertices are individuals and the edges are social connections. Use the graphs to answer the questions.

1. Which graph has the most triads? Name the triangles.
2. Which graph has the fewest triads? Name the triangles.
3. How might an increase in the number of triads in a graph affect the resilience of a community? Explain your reasoning.
4. Identify a clique with more than three vertices in Graph E by naming its vertices.

Answer

1. Graph E has the most triads. The triangles are (a, b, c), (a, e, g), (c, f, i), (e, g, h), (e, h, i), and (f, h, i).
2. Graph D has the fewest triads. There are no triangles in the graph.
3. More triads means vertices of greater degree, which indicates greater resilience. Specifically, if an edge is removed from a closed triad, there the two individuals who are no longer adjacent are still connected by way of the third member of the triad.
4. The vertices a, e, c, and g form a clique with four vertices.

Your Turn 12.2.6

From Exercise 35 in Section 12.1.0 Exercises, Chloe is interested to know how many in her network of Roblox friends are also friends with each other, so she polls them. The following is a list of each of Chloe’s friends and their common friends. Find a pentagon cycle in the graph of the Roblox friends in Chloe’s network that includes Chloe.

- Aria is friends with no one else in Chloe’s network.
- Benicio is friends with Dakshayani, Eun-ah, and Fukashi.

> - Dakshayani is friends with Benicio.
> - Eun-ah is friends with Benicio and Fukashi.
> - Fukashi is friends with Benicio and Eun-ah.

Three-Cycles in Complete Graphs

Just as complete graphs have a predictable number of edges, complete graphs have a predictable number of cyclic subgraphs. Let's look at the three-cycles within complete graphs with up to six vertices, which are shown in Figure 12.2.18.

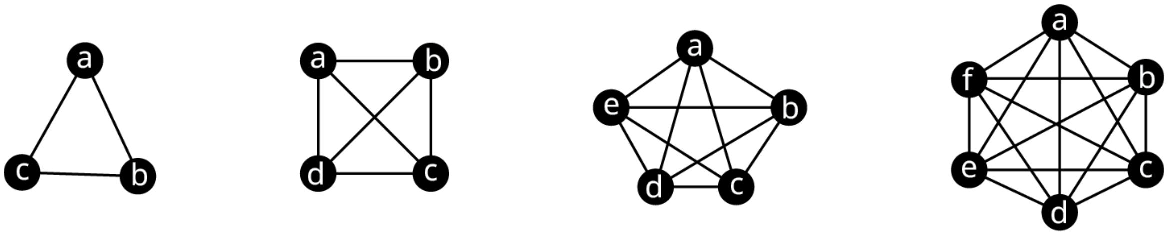


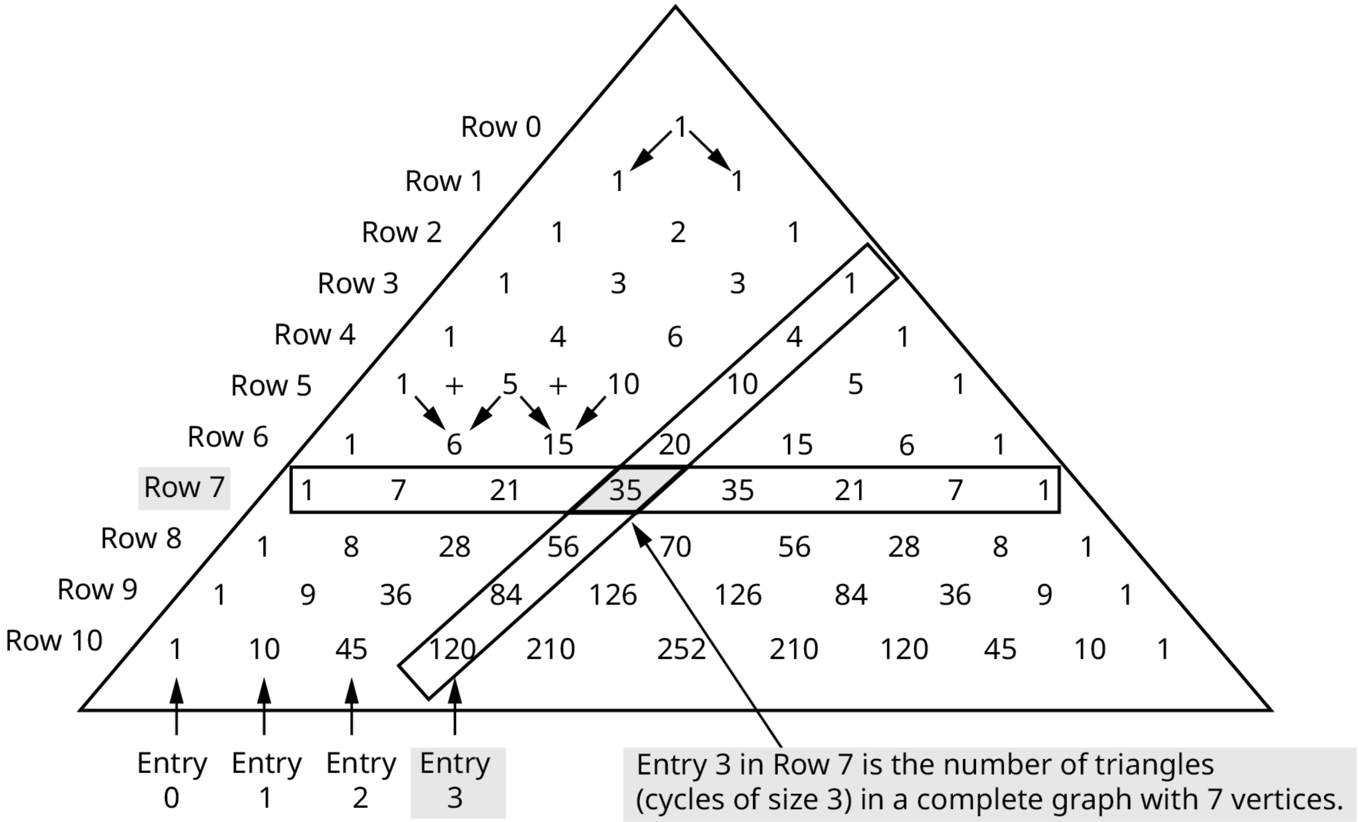
Figure 12.2.18: Complete Graphs with Up to Six Vertices

Let's list the names of all the triangles in each graph. ***Since every pair of vertices is adjacent, any three vertices on a complete graph form a triangle***. There is only one triangle in the complete graph with three vertices, (a, b, c). For the rest of the graphs, it is important that we take an organized approach. Start with the vertex that is first alphabetically, listing any triangles that include that vertex also in alphabetical order. Then, proceed to the next vertex in the alphabet, and list any triangles that include that vertex, except those that are already listed. Keep going in this way as shown in Table 12.2.

Table 12.2.2 Listing Triangles in Complete Graphs

Complete Graph With:	3 Vertices	4 Vertices	5 Vertices	6 Vertices
All *a* triangles	(a, b, c)	(a, b, c), (a, b, d), (a, c, d)	(a, b, c), (a, b, d), (a, b, e) (a, c, d), (a, c, e) (a, d, e)	(a, b, c), (a, b, d), (a, b, e), (a, b, f) (a, c, d), (a, c, e), (a, c, f) (a, d, e), (a, d, f) (a, e, f)
Other *b* triangles	None	(b, c, d)	(b, c, d), (b, c, e), (b, d, e)	(b, c, d), (b, c, e), (b, c, f) (b, d, e), (b, d, f) (b, e, f)
Other *c* triangles	None	None	(c, d, e)	(c, d, e), (c, d, f) (c, e, f)
Other *d* triangles	None	None	None	(d, e, f)
Total	1	$(2+1)+1$ $=3+1$ $=4$	$(3+2+1)+(2+1)+1$ $=6+3+1$ $=10$	(4+3+2+1)+(3+2+1)+(2+1)+1 =10+6+3+1 =20

Look at the last row in Table 12.2.2. Do you see a pattern emerge for counting triangles in a complete graph? Without drawing a complete graph with 7 vertices, we can predict that it will have $(5+4+3+2+1)+(4+3+2+1)+(3+2+1)+(2+1)+1=35$ triangles inside it. This pattern also appears in a famous diagram known to Western mathematicians as "Pascal's Triangle." Figure 12.35 displays the first 11 rows of Pascal's Triangle. Row 0 of Pascal's Triangle only has the number 1 in it. The first and last entries in each of the other rows are also 1s. Otherwise, all the entries are formed by adding two entries from the previous row. For example, in row 6, entry 1 is 6, which was found by adding 1 and the 5 from the previous row, and entry 2 is 15, which was found by adding the 5 and the 10 from the previous row, as shown in Table 12.2.2.


Row 0
1
Row 1
1 1
Row 2
1 2 1
Row 3
1 3 3 1
Row 4
1 4 6 4 1
Row 5
1 + 5 + 10 10 5 1
Row 6
1 6 15 20 15 6 1
Row 7
1 7 21 35 35 21 7 1
Row 8
1 8 28 56 70 56 28 8 1
Row 9
1 9 36 84 126 126 84 36 9 1
Row 10
1 10 45 120 210 252 210 120 45 10 1
Entry 0
Entry 1
Entry 2
Entry 3

Entry 3 in Row 7 is the number of triangles (cycles of size 3) in a complete graph with 7 vertices.

Figure 12.2.19: Pascal’s Triangle

Checkpoint

IMPORTANT! When you count the rows and entries of Pascal’s Triangle begin with row 0 and entry 0, not row 1 and entry 1.

Video

e mathematical secrets of Pascal's
-Ed
MATHEMATICAL SECRETS OF PASCAL'S TRIANGLE
Watch on YouTube

The Mathematical Secrets of Pascal's Triangle by Wajdi Mohamed Ratemi

If we begin to list the third entries in each row of Pascal's triangle from the top down, we see 1, 4, 10, 20, 35, and so on. Notice that these values are exactly the number of triangles in a complete graph of sizes 3, 4, 5, 6, and 7, respectively. Specifically, entry 3 in row 7 is 35, the number of triangles in a complete graph of size 7. Let's practice using Pascal’s triangle to find the number of triangles in a complete graph of a particular size.

STEPS TO FIND THE NUMBER OF TRIANGLES IN A COMPLETE GRAPH OF SIZE *n*

- **Step 1** Calculate the entries in row *n* of Pascal's Triangle using sums of pairs of entries in previous rows as needed.
- **Step 2** The number of triangles is entry 3 in row *n*. Remember to count the entries 0, 1, 2, and 3, from left to right.

✓ Example 12.2.7: Using Pascal's Triangle to Find the Number of Triangles in a Complete Graph

Use Pascal's Triangle in Figure 12.35 to find the number of triangles in a complete graph with 11 vertices.

Answer

Step 1: Identify the row number and calculate its entries. Since the complete graph has 11 vertices, we will need to look in row 11 of Pascal's Triangle. Use row 10 in Figure 12.35 to find the entries in row 11 as shown in Figure ✓ 12.2.20.

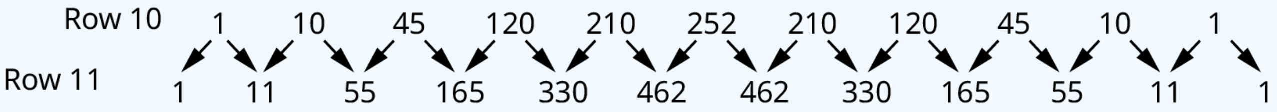


Figure ✓ 12.2.20 Rows 10 and 11 of Pascal's Triangle

Step 2: Find entry number 3. The number of triangles is in entry 3. Remember to count the entries beginning with entry 0 as shown in Figure ✓ 12.2.20.

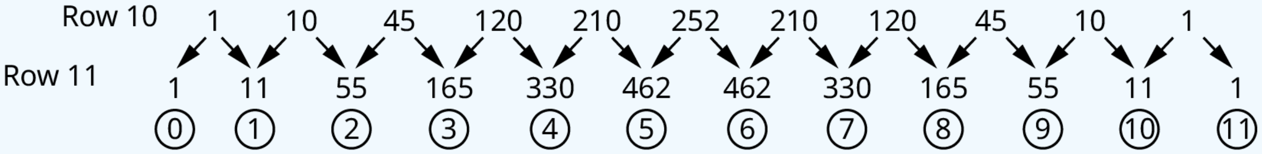


Figure ✓ 12.2.21: Identifying an Entry in Pascal's Triangle

Entry 3 in row 11 is 165. So, there are 165 triangles in a complete graph with 11 vertices.

Your Turn 12.2.7

A sociologist is studying a community of 13 individuals in which every person has a social connection to every other person. How many closed triads are there in the community?

Tech Check

Since the entries in Pascal's Triangle are useful in many applications, many resources such as online and traditional calculator functions have been developed to assist in calculating the values. The website dCode.xyz has a tool that automatically calculates entries in Pascal's Triangle using these steps:

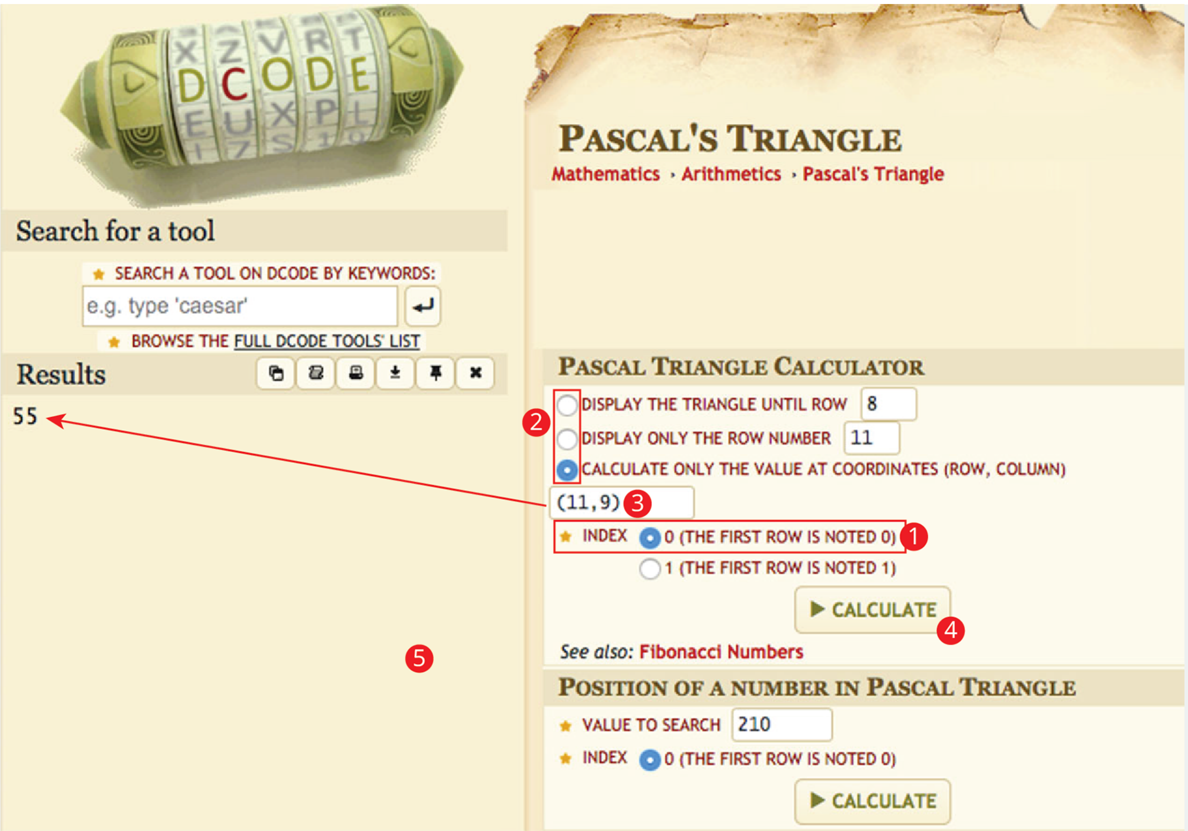
Step 1: Make sure to select Index 0.

Step 2: Select your preference, display the triangle until a particular row (line), display only one row (line), or calculate the value at coordinates (which means give the value of a single entry).

Step 3: Enter row number (line number) you are interested, or enter the row number and entry number in parentheses: (row number, entry number).

Step 4: Select Calculate.

Step 5: Retrieve the answer from the results box.



PASCAL'S TRIANGLE

Mathematics › Arithmetics › Pascal's Triangle

★ SEARCH A TOOL ON DCODE BY KEYWORDS: e.g. type 'caesar'

★ BROWSE THE FULL DCODE TOOLS' LIST

55

PASCAL TRIANGLE CALCULATOR

○ DISPLAY THE TRIANGLE UNTIL ROW 8
○ DISPLAY ONLY THE ROW NUMBER 11
● CALCULATE ONLY THE VALUE AT COORDINATES (ROW, COLUMN)
(11,9)
★ INDEX ● 0 (THE FIRST ROW IS NOTED 0)
○ 1 (THE FIRST ROW IS NOTED 1)

▶ CALCULATE

See also: Fibonacci Numbers

POSITION OF A NUMBER IN PASCAL TRIANGLE

★ VALUE TO SEARCH 210
★ INDEX ● 0 (THE FIRST ROW IS NOTED 0)

▶ CALCULATE

Figure 12.38: Pascal's Triangle Tool on dCode.xyz

Check Your Understanding

1. A complete graph with four vertices must have exactly four edges.

   a. True
   b. False

2. In a cycle, every vertex has degree two.

   a. True
   b. False

3. The number of vertices in a graph is twice the number of edges.

   a. True
   b. False

4 . The cycle (a, b, c, d) can also be called (c, b, a, d).

   a. True
   b. False

5. The cycle (a, b, c, d) can also be called (a, b, d, c).

   a. True
   b. False

6. The only cliques that are cycles are cliques of three vertices.

   a. True
   b. False

7. A student found that the sum of the degrees of the vertices in a graph was 13. Why is that impossible?

8. A student finds the number of edges in a complete graph by taking half the number of vertices, then subtracting one from the number of vertices, then multiplying these two values together. Will the student get the correct result?

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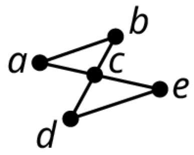
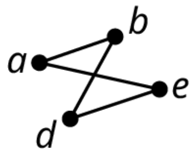
12.2.0: Exercises

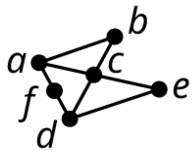
Use the figure to answer the following exercises.



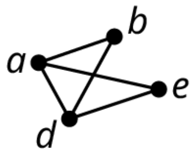
Graph S



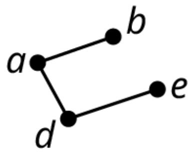
Graph T



Graph U



Graph V



Graph W

? Exercise 12.2.0.1

List any graphs that are subgraphs of Graph U.

? Exercise 12.2.0.2

List any graphs that are subgraphs of Graph V.

? Exercise 12.2.0.3

Explain why Graph T is not a subgraph of Graph S.

? Exercise 12.2.0.4

List any graphs that have a quadrilateral cyclic subgraph and name the quadrilaterals.

Use the Sum of Degrees Theorem as needed to answer the following exercises.

? Exercise 12.2.0.5

How many edges are in a graph if the sum of the degrees of its vertices is 18.

? Exercise 12.2.0.6

Draw two possible graphs to demonstrate that the sum of the degrees of the vertices in a graph with 7 edges is 14. Label the degrees of the vertices.

? Exercise 12.2.0.7

What is the sum of the degrees of the vertices of a graph that has 122 edges?

? Exercise 12.2.0.8

A graph has 3 vertices of degree 2, 4 vertices of degree 1, and 2 vertices of degree 3. How many edges are in the graph?

? Exercise 12.2.0.9

There are 6 vertices and 11 edges in a graph, 2 of degree 5, 1 of degree 4, and 2 of degree 3. What is the degree of the remaining vertex?

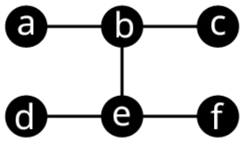
Exercise 12.2.0.10

A complete graph has 5 vertices. What is the sum of the degrees of the vertices?

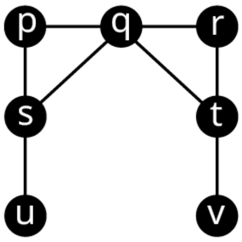
Exercise 12.2.0.11

A complete graph has 120 edges. How many vertices does it have?

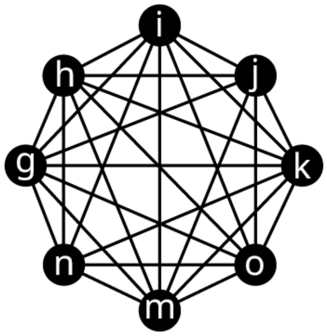
Use the figure to answer the following exercises. Identify the graph or graphs with the given characteristic.



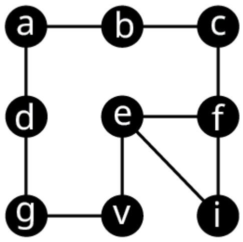
Graph 1



Graph 2



Graph 3



Graph 4

Exercise 12.2.0.12

The sum of the degrees of the vertices is 16.

Exercise 12.2.0.13

The graph is complete.

Exercise 12.2.0.14

The graph has no cyclic subgraphs

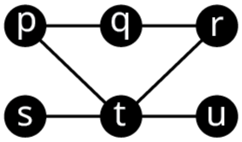
Exercise 12.2.0.15

The graph contains at least one octagon.

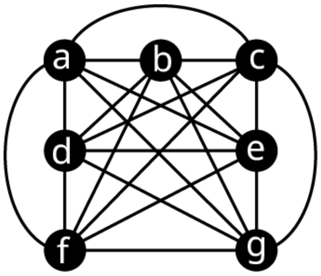
Exercise 12.2.0.16

The graph contains exactly two 3-cycles.

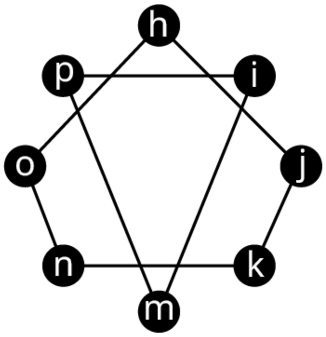
Use the figure to answer the following exercises. Identify the graph or graphs with the given characteristic.



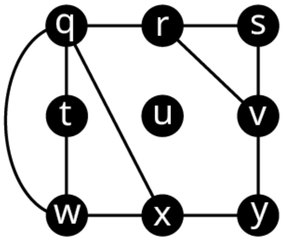
Graph 5



Graph 6



Graph 7



Graph 8

? Exercise 12.2.0.17

The graph is a subgraph of Graph 6.

? Exercise 12.2.0.18

The sum of the degrees of the vertices is 16.

? Exercise 12.2.0.19

The graph is complete.

? Exercise 12.2.0.20

The graph has no pentagons.

? Exercise 12.2.0.21

The graph contains at least one octagon.

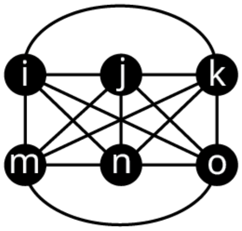
? Exercise 12.2.0.22

The graph contains exactly two cyclic subgraphs.

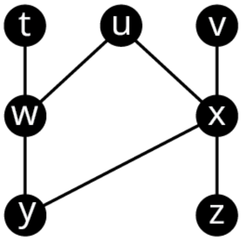
? Exercise 12.2.0.23

Use Table 12.1 to name a heptagon in Graph 8.

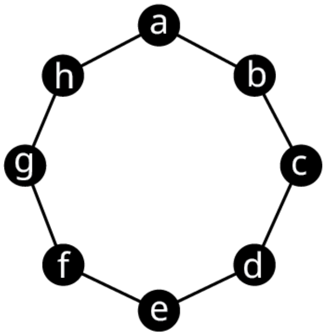
Use the figure to answer the following exercises. Identify the graph or graphs with the given characteristic.



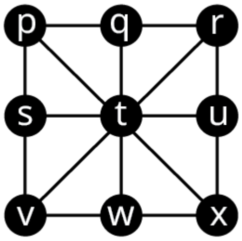
Graph 9



Graph 10



Graph 11



Graph 12

Exercise 12.2.0.24

The sum of the degrees of the vertices is 16.

Exercise 12.2.0.25

The graph is complete.

Exercise 12.2.0.26

The graph contains exactly one quadrilateral.

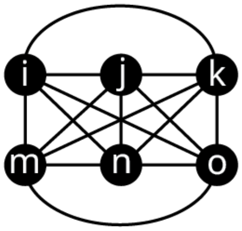
Exercise 12.2.0.27

The graph contains at least one octagon.

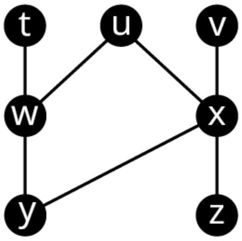
Exercise 12.2.0.28

The graph contains exactly one cyclic subgraph.

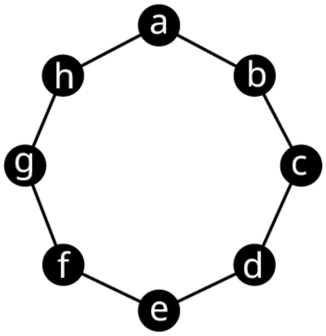
Use Table 12.1 to answer the following exercises about the below figure.



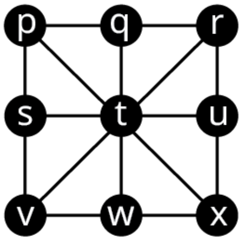
Graph 9



Graph 10



Graph 11



Graph 12

Exercise 12.2.0.29

List every quadrilateral in Graph 12.

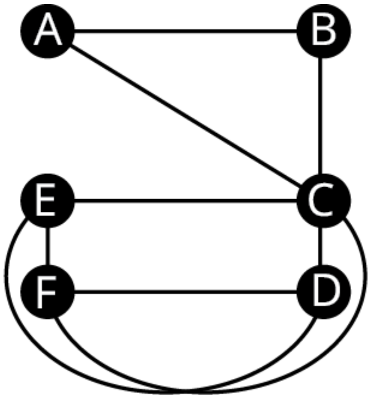
Exercise 12.2.0.30

List every quadrilateral in Graph 10.

Exercise 12.2.0.31

Determine the number of quadrilaterals in Graph 9.

Use the figure to answer the following exercises.



Exercise 12.2.0.32

Name five triangles in the graph.

Exercise 12.2.0.33

Identify a clique with more than three vertices in the graph by listing its vertices.

For the following exercises, draw a graph with the given characteristics.

Exercise 12.2.0.34

4 vertices, 6 edges, a subgraph that is a 4-cycle.

Exercise 12.2.0.35

11 vertices, the only cyclic subgraphs are triangles.

Exercise 12.2.0.36

Complete graph, no quadrilateral subgraph

Exercise 12.2.0.37

4 vertices, 4 edges, no cyclic subgraphs

Exercise 12.2.0.38

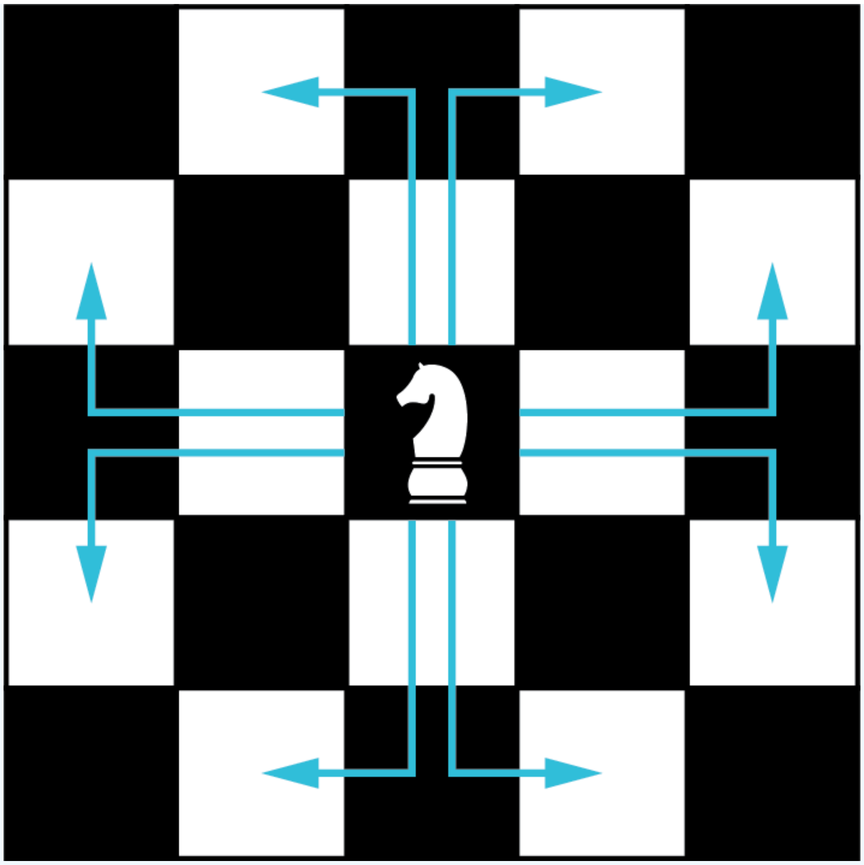
Complete graph, sum of the degrees of the vertices is 20.

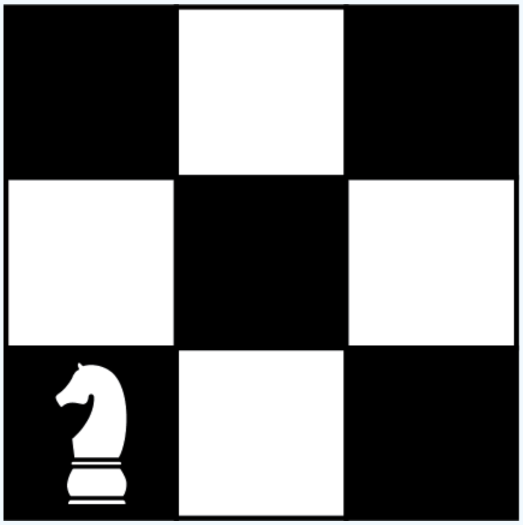
Exercise 12.2.0.39

7 vertices, largest clique has 5 vertices.

Exercise 12.2.0.40

In chess, a knight can move in any direction, but it must move two spaces then turn and move one more space. The eight possible moves a knight can make from a space in the center of a five-by-five grid are shown in the first figure. Draw a graph that represents all the legal moves of a knight on a three-by-three grid starting from the lower left corner as shown in the second figure where the vertices will represent the spaces occupied by the knight and the edges will indicate a move from one space to the next.





Exercise 12.2.0.41

What kind of cycle is the resulting graph you drew for Exercise 40?

Exercise 12.2.0.42

Use Pascal’s triangle to find number of triangles in a complete graph with 14 vertices.

Exercise 12.2.0.43

Do you think that a graph representing network of friends on Facebook is likely to be complete or not? Explain your reasoning.

Exercise 12.2.0.44

Would the graph of a family tree, in which edges represent parent-child lineage, contain any cycles? Why or why not?

Exercise 12.2.0.45

We have seen that the number of triangles in a complete graph with 7 vertices can be calculated using the pattern $(5+4+3+2+1)+(4+3+2+1)+(3+2+1)+(2+1)+1=35$. This pattern gets very long for complete graphs with more vertices, but we have seen sums from 1 to a number before, and we had a shortcut! Recall from Example 12.8 that $1+2+3+\cdots+(n-1)=\frac{n(n-1)}{2}$. This makes finding the sum from 1 up to $n-1$ shorter. We can write each of the sums we need in the form $\frac{n(n-1)}{2}$. To find the sum from 1 to 5 , since $n-1=5$, use $n=6: 1+2+3+4+5=\frac{6(5)}{2}$ To find the sum from 1 to 4 , since $n-1=4$, use $n=5: 1+2+3+4=\frac{5(4)}{2}$
To find the sum from 1 to 3 , since $n-1=3$, use $n=4: 1+2+3=\frac{4(3)}{2}$
To find the sum from 1 to 2 , since $n-1=2$, use $n=3: 1+2=\frac{3(2)}{2}$
To find the sum from 1 to 1 , (Sounds silly, but it helps to see the pattern!) since $n-1=1$, use $n=2$:

$$1=\frac{2(1)}{2}$$

Use these to rewrite the calculation for the number of triangles in a graph with 7 vertices.

$$\begin{aligned}
&(5+4+3+2+1)+(4+3+2+1)+(3+2+1)+(2+1)+1\\
=&\underbrace{1}_{\frac{2(1)}{2}}+\underbrace{(1+2)}_{\frac{3(2)}{2}}+\underbrace{(1+2+3)}_{\frac{4(3)}{2}}+\underbrace{(1+2+3+4)}_{\frac{5(4)}{2}}+\underbrace{(1+2+3+4+5)}_{\frac{6(5)}{2}}\\
=&\frac{2(1)}{2}+\frac{3(2)}{2}+\frac{4(3)}{2}+\frac{5(4)}{2}+\frac{6(5)}{2}\\
=&\underbrace{\frac{1(2)+2(3)+3(4)+4(5)+5(6)}{2}}=35
\end{aligned}$$

The number of triangles in complete graph with 7 vertices.
Similarly, for the number of triangles in a complete graph with 8 vertices, instead of

$$(6+5+4+3+2+1)+(5+4+3+2+1)+(4+3+2+1)+(3+2+1)+(2+1)+1=56$$

use the shorter formula $\frac{1(2)+2(3)+3(4)+4(5)+5(6)+6(7)}{2}=56$.
Use this pattern to find the number of triangles in a complete graph with 10 vertices. In which row and entry of Pascal's triangle can you also find this result?

Exercise 12.2.0.46

Use the fact that the sum from 1 to $n-1$ is $\frac{n(n-1)}{2}$ to write a formula for the number of triangles in a complete graph with n vertices.

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12.3: Comparing Graphs

Learning Objectives

1. Identify the characteristics used to compare graphs.
2. Determine when two graphs represent the same relationships
3. Explore real-world examples of graph isomorphisms
4. Find the complement of a graph

Maps of the same region may not always look the same. For example, a map of Earth on a flat surface looks distorted at the poles. When the same regions are mapped on a spherical globe, the countries that are closer to the polls appear smaller without the distortion. Despite these differences, the two maps still communicate the same relationships between nations such as shared boundaries, and relative position on the earth. In essence, they are the same map. This means that for every point on one map, there is a corresponding point on the other map in the same relative location. In this section, we will determine exactly what characteristics need to be the shared by two graphs in order for us to consider them "the same."

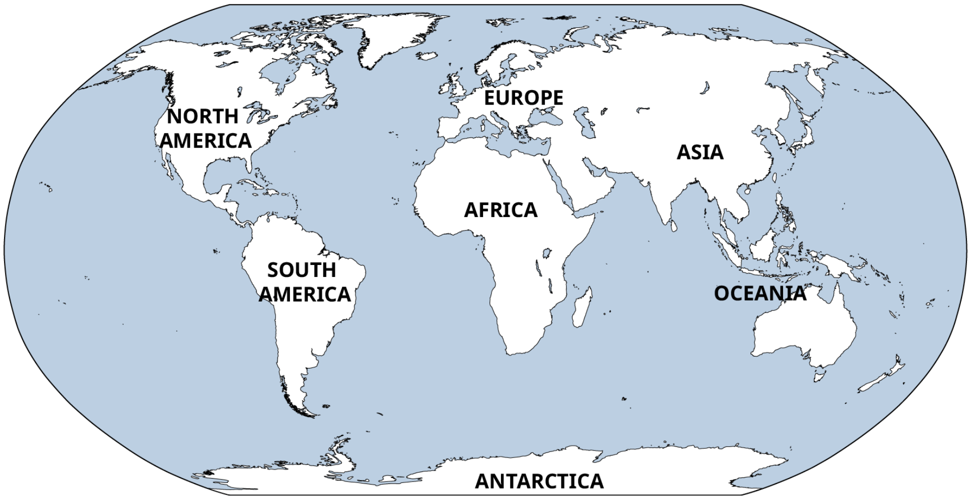


Figure 12.3.1 A flat map represents the surface of Earth in two dimensions. (credit: modification of work "World Map" by Scratchinghead/Wikimedia Commons, Public Domain)

Figure 12.3.2 A globe represents the surface of Earth in three dimensions. (credit: "Globe" by Wendy Cope/Flickr, CC BY 2.0)

When Are Two Graphs Really the Same Graph?

In arithmetic, when two numbers have the same value, we say they are equal, like ½ = 0.5. Although ½ and 0.5 look different, they have the same value, because they are assigned the same position on the real number line. When do we say that two graphs are equal?

Figure 12.3.3 shows Graphs *A* and *F*, which are identical except for the labels. Graphs are visual representations of connections. As long as two graphs indicate the same pattern of connections, like Graph *A* and Graph *F*, they are considered to be equal, or in graph theory terms, isomorphic.

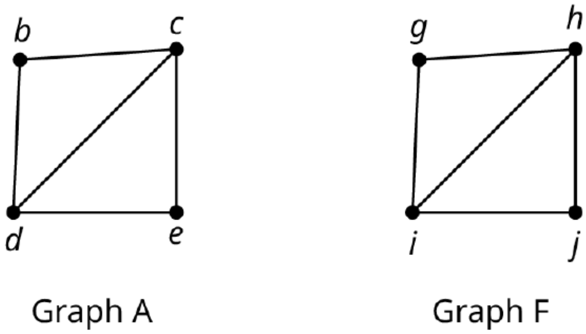


Figure 12.3.3: Identical Graphs with Different Labels

Two graphs are isomorphic if either one of these conditions holds:

1. One graph can be transformed into the other without breaking existing connections or adding new ones.
2. There is a correspondence between their vertices in such a way that any adjacent pair in one graph corresponds to an adjacent pair in the other graph.

It is important to note that if either one of the isomorphic conditions holds, then both of them do. When we need to decide if two graphs are isomorphic, we will need to make sure that one of them holds. For example, Figure 12.3.4 shows how Graph *T* can be bent and flipped to look like Graph *Z*, which means that Graphs *T* and *Z* satisfy condition 1 and are isomorphic.

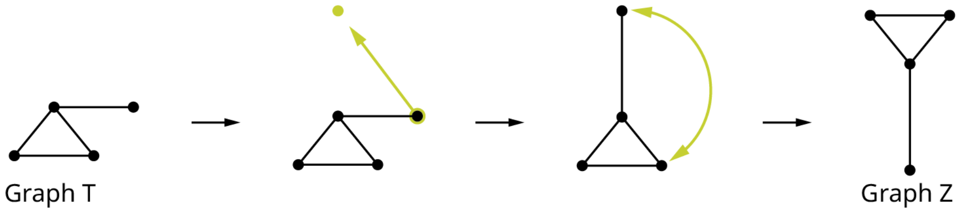


Figure 12.3.4 Graph *T* Transformed Into Graph *Z*

Also, notice that the vertices that were adjacent in the first graph are still adjacent in the transformed graph as shown in Figure 12.3.5. For example, vertex 3 is still adjacent to vertex 4, which means they are still neighboring vertices joined by a single edge.

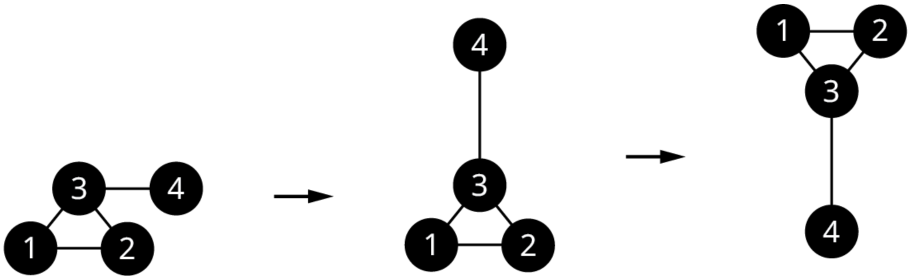
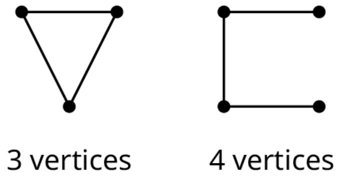
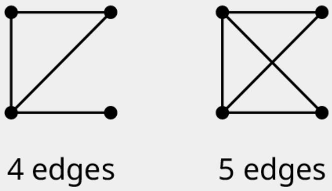
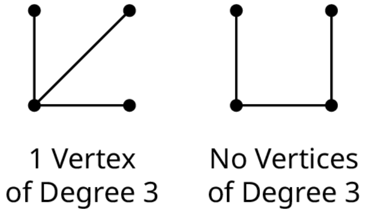
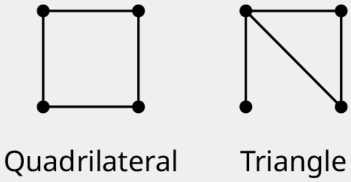


Figure 12.3.5: Adjacent Vertices Are Still Adjacent

When Are Two Graphs Really Different?

Verifying that two graphs are isomorphic can be a challenging process, especially for larger graphs. It makes sense to check for any obvious ways in which the graphs might differ so that we don't spend time trying to verify that graphs are isomorphic when they are not. If two graphs have any of the differences shown in Table 12.3, then they cannot be isomorphic.

Table 12.3.1 Characteristics of Graphs That Are Not Isomorphic

Unequal number of vertices	 3 vertices 4 vertices
Unequal number of edges	 4 edges 5 edges
Unequal number of vertices of a particular degree	 1 Vertex of Degree 3 No Vertices of Degree 3
Different cyclic subgraphs	 Quadrilateral Triangle

Recognizing Isomorphic Graphs

Isomorphic graphs that represent the same pattern of connections can look very different despite having the same underlying structure. The edges can be stretched and twisted. The graph can be rotated or flipped. For example, in Figure 12.3.6, each of the diagrams represents the same pattern of connections.

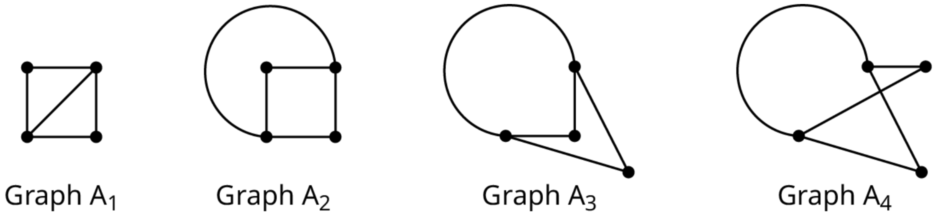

Graph A_1
Graph A_2
Graph A_3
Graph A_4

Figure 12.3.64: Four Representations of the Same Graph

Looking at Figure 12.44, how can we know that these graphs are isomorphic? We will start by checking for any obvious differences. Each of the graphs in Figure 12.3.6 has four vertices and five edges; so, there are no differences there. Next, we will focus on the degrees of the vertices, which have been labeled in Figure 12.3.7

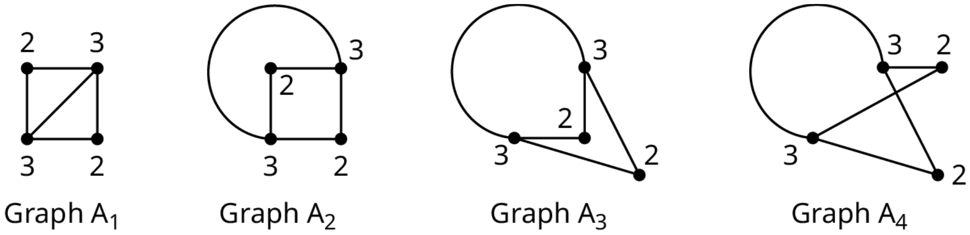

2
3
3
2
Graph A_1
3
2
3
2
Graph A_2
3
2
3
2
Graph A_3
3
2
3
2
Graph A_4

Figure 12.3.7: Graphs with Vertices of the Same Degrees

As shown in Figure 12.3.7, each graph has two vertices of degree 2 and two vertices of degree 3; so, there are no differences there. Now, let's check for cyclic subgraphs. These are highlighted in Figure 12.3.86.

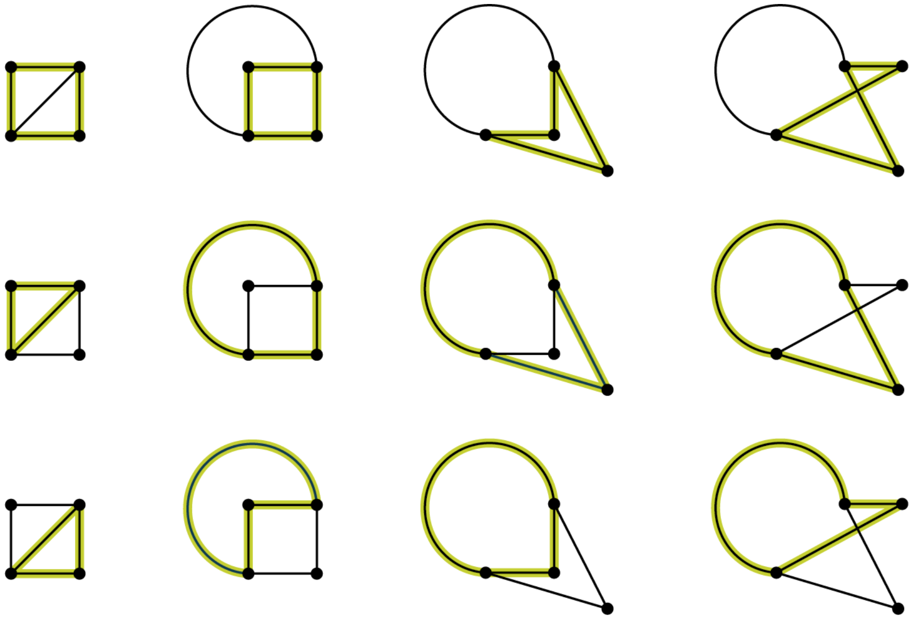


Figure 12.3.8: Graphs with the Same Cyclic Subgraphs

As shown in Figure 12.3.8, each graph has two triangles and one quadrilateral; so, no differences there either. It is beginning to look likely that these graphs are isomorphic, but we will have to look further to be sure.

To know with certainty that these graphs are isomorphic, we need to confirm one of the two conditions from the definition of isomorphic graphs. With smaller graphs, you may be able to visualize how to stretch and twist one graph to get the other to see if condition 1 holds. Imagine the edges are stretchy and picture how to pull and twist one graph to form the other. If you can do this without breaking or adding any connections, then the graphs are really the same. Figure 12.3.9 demonstrates how to change graph A_4 to get A_3, graph A_3 to get A_2, and graph A_2 to get A_1.

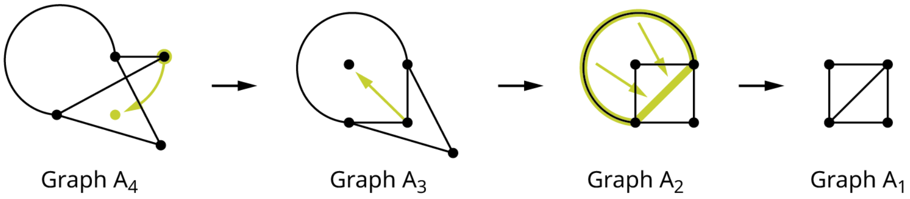


Figure 12.3.9: Transforming Graphs

Now that we have used visual analysis to see that condition 1 holds for graphs A_1, A_2, A_3, and A_4 in Figure 12.46, we know that they are isomorphic. In Figure 12.47, one of the edges of graph A_4 crossed another edge of the graph. By transforming it into graph A_3, we have “untangled” it. Graphs that can be untangled are called planar graphs. The complete graph with five vertices is an example of a nonplanar graph-that means that, no matter how hard you try, you can’t untangle it. But, when you try to figure out if two graphs are the same, it can be helpful to untangle them as much as possible to make the similarities and differences more obvious.

✓ Example 12.3.1: Identifying Isomorphic Graphs

Which of the three graphs in Figure ✓ 12.3.10 are isomorphic, if any? Justify your answer.

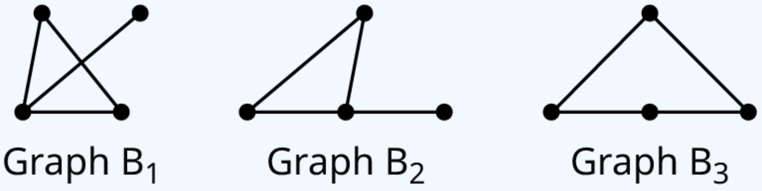


Figure ✓ 12.3.10: Three Similar Graphs

Answer

Step 1: Check for differences in number of vertices, number of edges, degrees of vertices, and types of cycles to see if an isomorphism is possible.

- Vertices: They all have the same number of vertices, 4.
- Edges: They all have the same number of edges, 4.
- Degrees: Graph B_1 and Graph B_2 each have a vertex of degree 3, while Graph B_3 does not. So, Graph B_3 is not isomorphic to either of the other two graphs, but Graph B_1 and Graph B_2 could possibly be isomorphic.
- Cycles: Focus on any cycles in Graph B_1 and Graph B_2. Each graph has a triangle as shown in Figure ✓ 12.3.11.

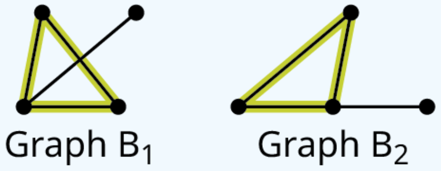


Figure ✓ 12.3.11 Cycles of Graph B_1 and Graph B_2

Step 2: If no differences were found and an isomorphism is possible, verify one of the conditions in the definition of isomorphic.

Since we were able to determine that Graph B_1 and Graph B_2 have no obvious differences, and they are relatively small graphs, we will attempt to transform one graph into the other, which would verify condition 1. Graph B_1 can be transformed into Graph B_2 without breaking or adding connections as shown in Figure ✓ 12.3.12. Begin by untangling graph B_1. Then rotate or flip as needed to see that the graphs match.

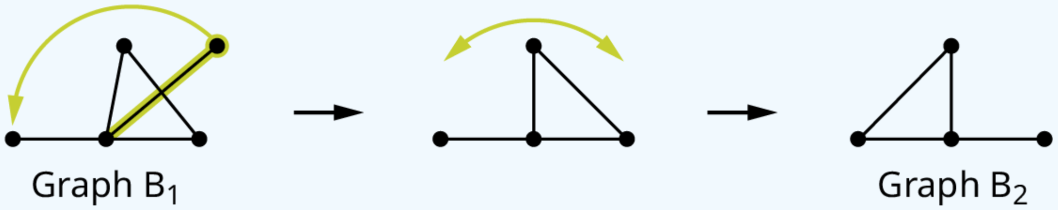


Figure ✓ 12.3.12: Transformation of Graph B_1

So, Graph B_1 and Graph B_2 have the same structure and are isomorphic.

📌 Your Turn 12.3.1

Name four differences between Graph C_1 and Graph C_2 that show they cannot be isomorphic.

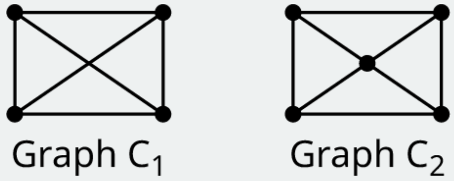
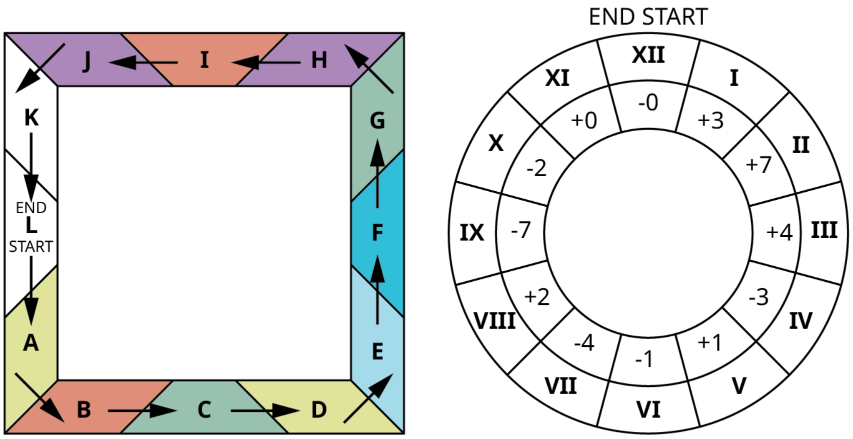


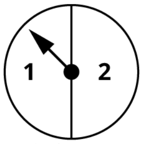
Figure 📌 12.3.13: Two Distinct Graphs

Have you ever noticed that many popular board games may look different but are really the same game? A good example is the many variations of the board game *Monopoly®*, which was submitted to the U.S. Patent Office in 1935. Although the rules have been revised a bit, a very similar game board is still in use today. There have been many versions of Monopoly over the years. Many have been stylized to reflect a popular theme, such as a show or sports team, while retaining the same game board structure. If we were to represent these different versions of the game using a graph, we would find that the graphs are isomorphic. Let's analyze some game boards using graph theory to determine if they have the same structure despite having different appearances.



Colors Game Figure A

Clock Game Figure B



Spinner

Figure 12.3.14: Two Game Boards

✓ Example 12.3.2: Deciding If Graphs are Isomorphic

A teacher uses games to teach her students about colors and numbers as shown in Figure ✓ 12.3.14.

In the Colors Game, shown in Figure *A*, each player begins in the space marked START and proceeds in a counterclockwise direction. On each turn, the player spins a spinner marked 1 and 2 and moves forward the number of spaces shown on the spinner. If the player lands on a space marked with any color other than white, the player must move forward or back to the other space of the same color. The first player to land in or pass the space marked END wins.

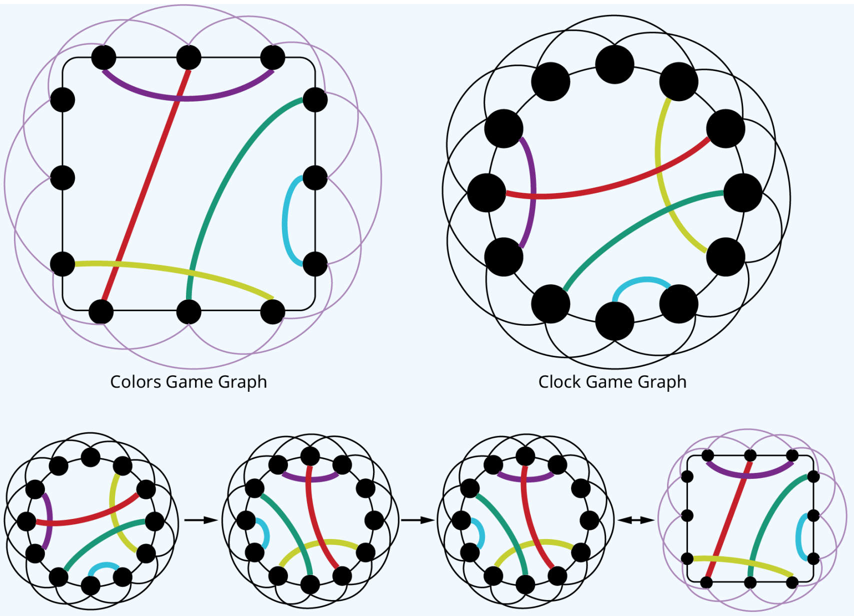
In the Clock Game, shown in Figure *B*, each player begins in the space marked START and proceeds in a clockwise direction. On each turn, the player spins a spinner marked 1 and 2 and moves forward the number of spaces shown on the spinner. In the same turn, the player must read the number in the space and move forward the number of spaces indicated by a positive value or backward the number of spaces indicated by a negative value. Then the turn ends. The first player to land in or pass the space marked END wins.

Draw a graph or multigraph to represent each game in which the vertices are the spaces and the edges represent the ability of a player to move between the spaces either by a spin or as dictated by a marked color or number. (We will ignore the direction of motion for simplicity.) Transform one of the graphs to show that it is isomorphic to the other and explain what this tells you about the games.

This tells us that the games are essentially the same game with the same moves even though the games appear to be different.

Answer

> The graph representing the Clock Game can be transformed as shown in Figure ✓ 12.3.15 so that we can see that the graphs are isomorphic. There is a correspondence between their vertices in such a way that any adjacent pair in one graph corresponds to an adjacent pair in the other graph.



Colors Game Graph

Clock Game Graph

Rotated clockwise

Flipped horizontally

Matches of colour game

Figure ✓ 12.3.15 Graphs for Clock Game and Colors Game

Your Turn 12.3.2

Let's compare the games in Figures *A* and *B*. In the game Diamonds, shown in Figure *A*, each player begins in the space marked START and proceeds in a counterclockwise direction. On each turn, the player rolls a six-sided die and moves forward the number of spaces shown on the die. If the player lands on a space marked with a diamond, the player jumps to the next space marked with a diamond and stops there. The first player to land in or pass the space marked END wins. In the Dots game, shown in Figure *B*, each player begins in the space marked 1 and proceeds in numerical order through the numbered spaces. On each turn, the player rolls a six-sided die and moves forward the number of spaces shown on the die. When a player lands on a space marked with a green dot, they immediately move to the space indicated by the arrow. Construct a graph to represent each gameboard such that each vertex represented a space on the game board and each edge represented the ability of a player to move directly from one space to the next. For example, in the Dots game, the vertex representing space 3 would be adjacent to 2, 4, and 6. Identify at least three differences between a graph based on Diamonds and a graph based on Dots to show the graphs are not isomorphic.

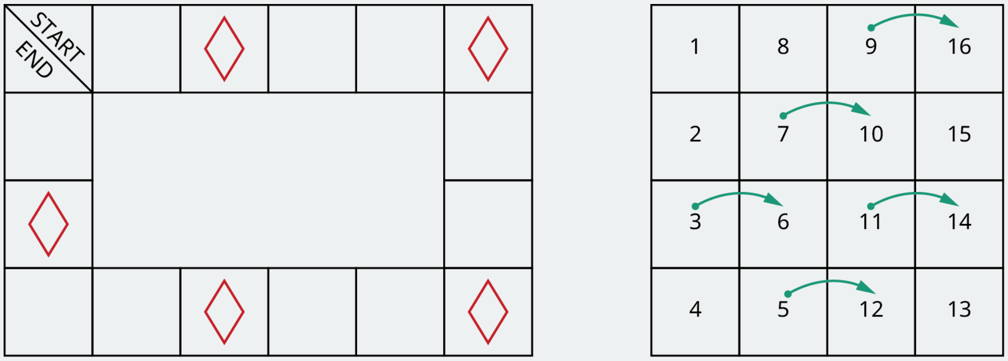


Figure *A*
Diamond game

Figure *B*
Dots game

Figure 12.3.16: Two Game Boards

People in Mathematics: Elizabeth Magie

Game designer, engineer, comedian, and political activist, Elizabeth Magie designed a game called “Landlord’s Game” to educate fellow citizens about the dangers of monopolies and the benefits of wealth redistribution through a land tax. Magie, whose father had campaigned with Abraham Lincoln, was a proponent of a land tax, an idea popularized by Henry George’s 1879 book, *Progress and Poverty*. She designed the game to be played by two sets of rules for comparison. In one version, the goal was to dominate opponents by creating monopolies, leaving one wealthy player standing in the end. In the other version, all the players were rewarded when a monopoly was created through a simulated land tax. She patented this game for the first time in 1904. Does the game board in Figure 12.3.16 look familiar?

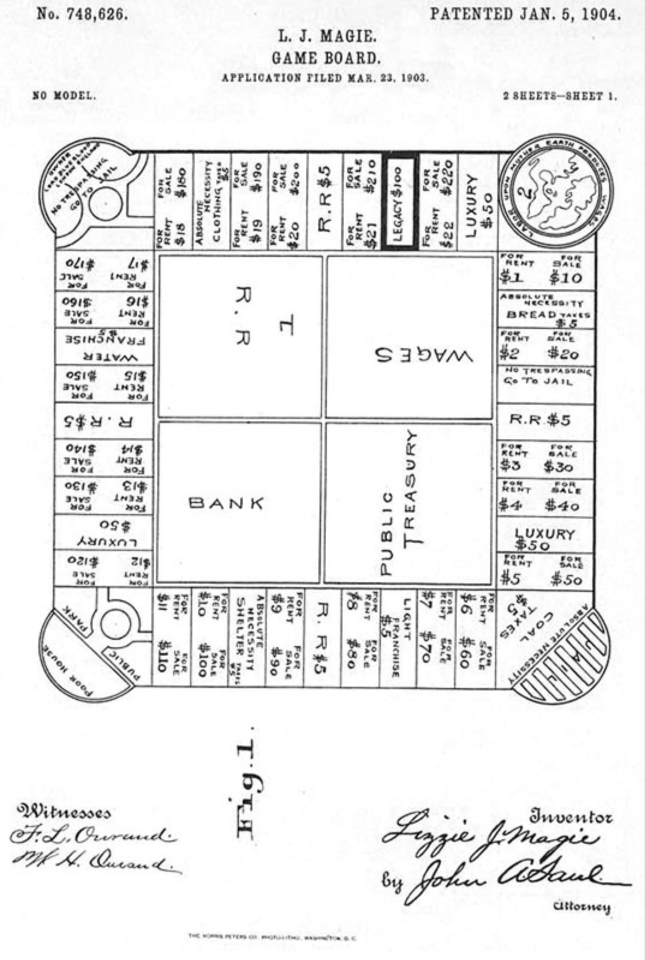

No. 748,626.
PATENTED JAN. 5, 1904.
L. J. MAGIE.
GAME BOARD.
APPLICATION FILED MAR. 23, 1903.
NO MODEL.
2 SHEETS—SHEET 1.
GO TO JAIL
LEGACY $100
LUXURY $50
R.R $5
WAGES
BANK
PUBLIC TREASURY
R.R
T
FRANCHISE
WATER
ABSOLUTE NECESSITY BREAD
NO TRESPASSING Go To JAIL
LIGHT FRANCHISE
ABSOLUTE NECESSITY SHELTER
ABSOLUTE NECESSITY CLOTHING
COAL
PUBLIC PARK
POOR HOUSE
LABOR UPON MOTHER EARTH PRODUCES WAGES
Fig. 1.
Witnesses
F. L. Ourand
W. H. Ourand
Inventor
Lizzie J. Magie
By John A. Saul
Attorney

Figure 12.3.16 *Landlord’s Game* Patent (credit: "Landlord's Game Patent" Wikimedia Commons, Public Domain)

That’s right! A modified version of the game was obtained from friends by a man named Charles Darrow who renamed it Monopoly and sold to Parker Brothers. Parker Brothers later paid Magie $500 for the right to her patent on it. The property tax version was left behind, and the modern game of Monopoly was born. (Mary Pilon, *Monopoly’s Lost Female Inventor*, September 1, 2018, National Women’s History Museum, “*Monopoly’s* Lost Female Inventor,”

Identifying and Naming Isomorphisms

When two graphs are isomorphic, meaning they have the same structure, there is a correspondence between their vertices, which can be named by listing corresponding pairs of vertices. This list of corresponding pairs of vertices in such a way that any adjacent pair in one graph corresponds to an adjacent pair in the other graph is called an **isomorphism**. Consider the isomorphic graphs in Figure 12.3.8. In Figure 12.3.17, we could replace the labels Graph F with the labels from Graph A and have an identical graph, as in Figure 12.3.18.

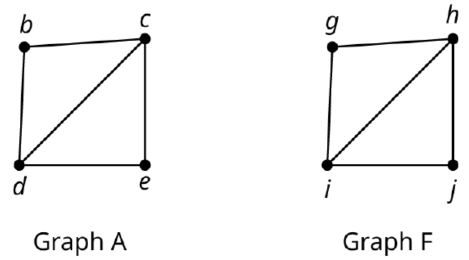

b
c
d
e
Graph A
g
h
i
j
Graph F

Figure 12.3.17: Identical Graphs with Different Labels

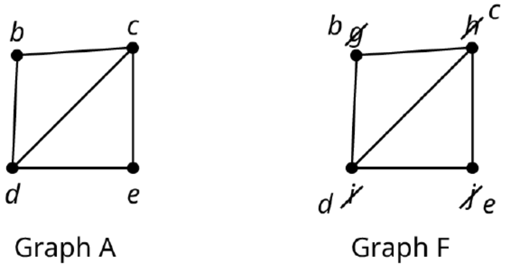

b
c
d
e
Graph A
b ~~g~~
~~h~~ c
d ~~i~~
~~j~~ e
Graph F

Figure 12.3.18: Corresponding Vertices

So, we can identify an isomorphism between Graph *A* and Graph *F* by listing the corresponding pairs of vertices: *b*-*g*, *c*-*h*, *d*-*i*, and *e*-*j*. Notice that *b* is adjacent to *c* and *g* is adjacent to *h*. This must be the case since *b* corresponds to *g* and *c* corresponds to *h*. The same is true for other pairs of adjacent vertices.

An isomorphism between graphs is not necessarily unique. There can be more than one isomorphism between two graphs. We can see how to form a different isomorphism between Graph *A* and Graph *F* from Figure 12.36 by rotating Graph *F* clockwise and comparing the rotated version of *F* to Graph *A* as in YOUR TURN 12.3.13. Now, we can see that a second isomorphism exists, which has the correspondence: *b*-*j*, *d*-*h*, *c*-*i*, and *e*-*g* as shown in Figure 12.3.19.

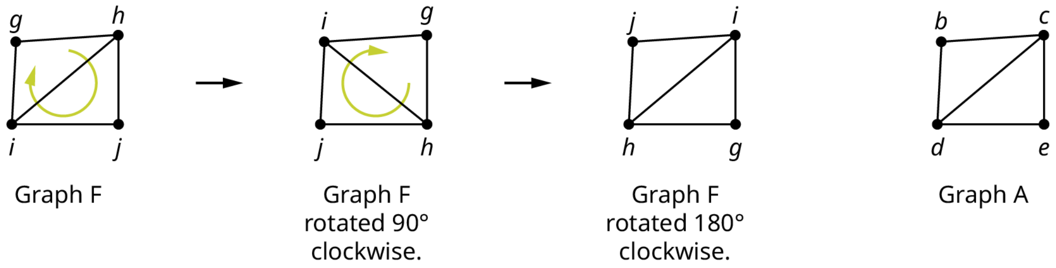
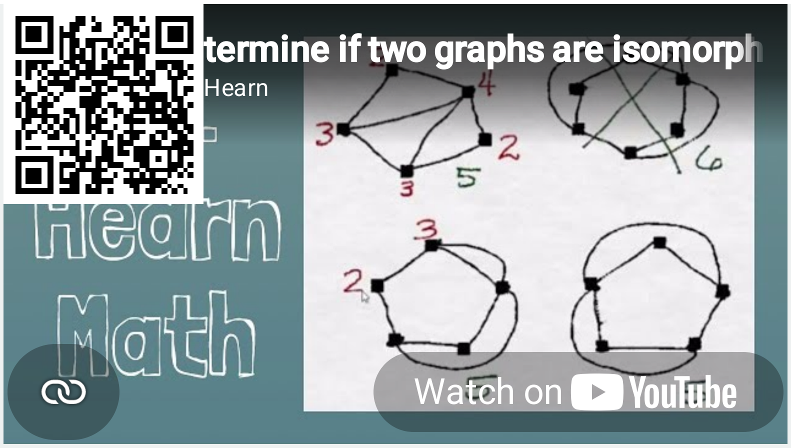

g
h
i
j
Graph F
i
g
j
h
Graph F
rotated 90°
clockwise.
j
i
h
g
Graph F
rotated 180°
clockwise.
b
c
d
e
Graph A

Figure 12.3.19 Transforming Graph *F* into Graph *A*

Checkpoint

When you name isomorphisms, one way to check that your answer is reasonable is to make sure that the degrees of corresponding vertices are equal.

Video


termine if two graphs are isomorph
Hearn
Hearn
Math
Watch on YouTube

Determine If Two Graphs Are Isomorphic and Identify the Isomorphism

✓ Example 12.3.31: Identifying Isomorphisms

We showed that the Graphs B_1 and B_2 in Figure ✓ 12.3.8 are isomorphic. In Figure ✓ 12.3.20, labels have been assigned to the vertices of Graphs B_1 and B_2. Identify an isomorphism between them by listing corresponding pairs of vertices.

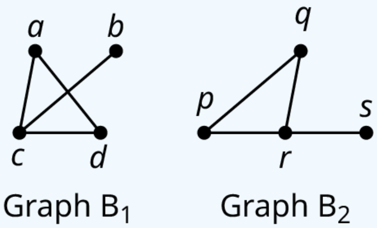

a
b
c
d
q
p
s
r
Graph B₁
Graph B₂

Figure ✓ 12.3.20: Two Isomorphic Graphs

Answer

Figure ✓ 12.3.8 showed how to transform Graph B_1 to get Graph B_2. In Figure ✓ 12.3.21, we will do the same, but this time we will include the labels.

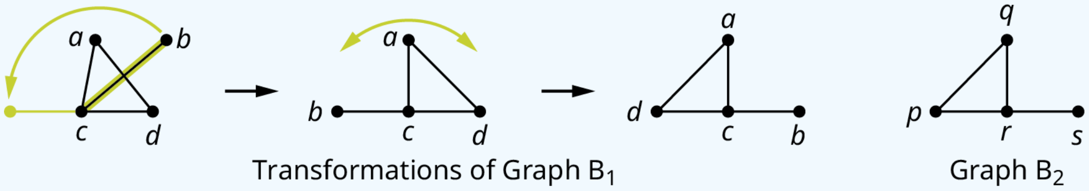

a
b
c
d
a
b
c
d
a
d
c
b
Transformations of Graph B₁
q
p
r
s
Graph B₂

Figure ✓ 12.3.21 Transform Graph B_1 into Graph B_2

From YOUR TURN ✓ 12.3.8, we can see the corresponding vertices: *a*-*q*, *d*-*p*, *c*-*r*, and *b*-*s*, which is an isomorphism of the two graphs.

Your Turn 12.3.3

Graphs *E* and *T* are isomorphic. Name two isomorphisms.

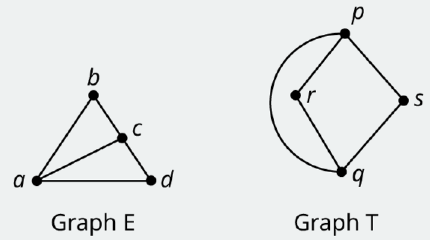


Figure 12.3.22: Graph E and Graph T

✓ Example 12.3.4: Recognizing an Isomorphism

Determine whether Graphs G and S in Figure ✓ 12.3.23 are isomorphic. If not, explain how they are different. If so, name the isomorphism.

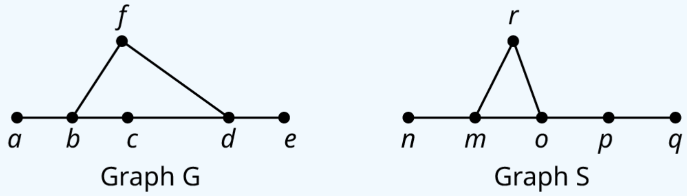


Figure ✓ 12.3.23 Graph G and Graph S

Step 2: This step is not necessary because we now know Graphs G and S are not isomorphic.

Answer

Step 1: Check for any differences.

- Vertices: Graphs G and S each have six vertices.
- Edges: Graphs G and S each have six edges.
- Degrees: Each graph also has two vertices of degree 1, two vertices of degree 2, and two vertices of degree 3.
- Cycles: From Figure ✓ 12.3.24, we can see that Graph G contains a quadrilateral cycle (b, f, d, c) but Graph S has no quadrilaterals. Also, Graph S contains a triangle cycle (m, r, o) but Graph G has no triangles. This means that the graphs are not isomorphic.

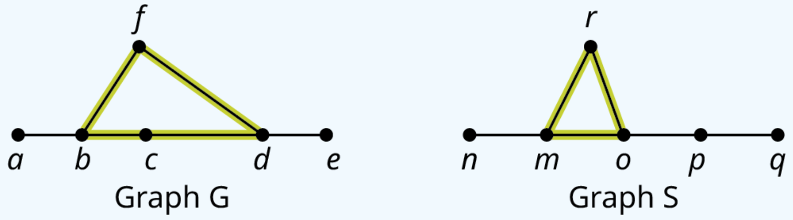


Figure ✓ 12.3.24 Comparing Subgraphs

Your Turn 12.3.4

Three students have been asked to determine if Graphs E and T are isomorphic and justify their answers.

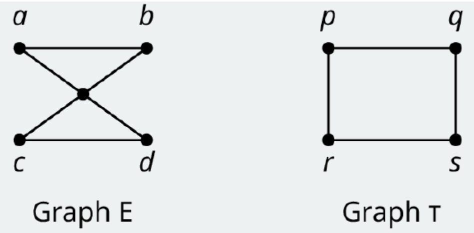


Figure 12.3.25: Graph E and Graph T

Javier believes that Graphs E and T are not isomorphic because Graph E contains a triangle cycle and Graph T does not. He supports his conclusion with the graphs shown.

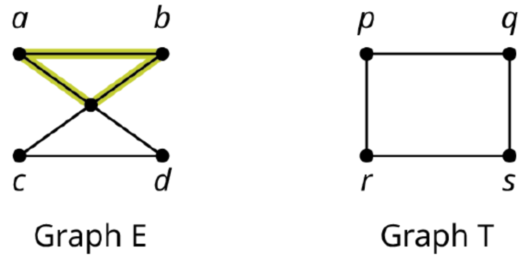


Figure 12.3.26: Javier's Highlighted Cycle

Maubi also believes that Graphs E and T are not isomorphic, but says that it is because Graph T has a quadrilateral cycle (p, q, r, s) while Graph E does not. Caden believes that Graphs E and T are isomorphic. She said one isomorphism is a-p, b-q, c-s, and d-r. Determine who is correct, who is incorrect, and explain how you know.

Complementary Graphs

Suppose that you are a camp counselor at Camp Woebegone and you are holding a camp Olympics with four events. The campers have signed up for the events. You drew a graph in Figure 12.3.27 to help you visualize which events have campers in common.

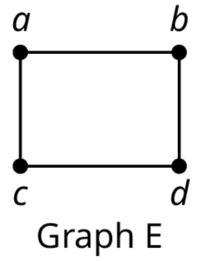


Figure 12.3.27: Graph of Events with Campers in Common

Graph E in Figure 12.3.27 shows that some of the same campers will be in events a and b, as well as b and d, c and d, and a and c. What do you think the graph would look like that represented the events that do not have campers in common? It would have the same vertices, but any pair of adjacent edges in Graph E, would not be adjacent in the new graph, and vice versa. This is called a **complementary** graph, as shown in Figure 12.3.28. Two graphs are complementary if they have the same set of vertices, but any vertices that are adjacent in one, are not adjacent in the other. In this case, we can say that one graph is the **complement** of the other.

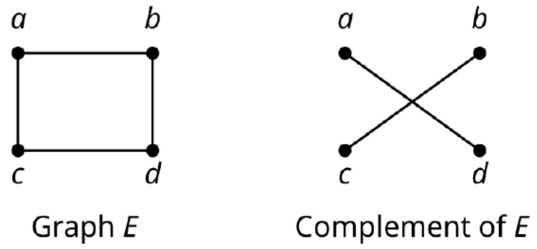


Figure 12.3.28: Graphs of Camp Olympics

One way to find the complement of a graph is to draw the complete graph with the same number of vertices and remove all the edges that were in the original graph. Let's say you wanted to find the complement of Graph *E* from Figure 12.3.28, and you didn't already know it was Graph *F*. You could start with the complete graph with four vertices and remove the edges that are in Graph *E* as shown in Figure 12.3.29.

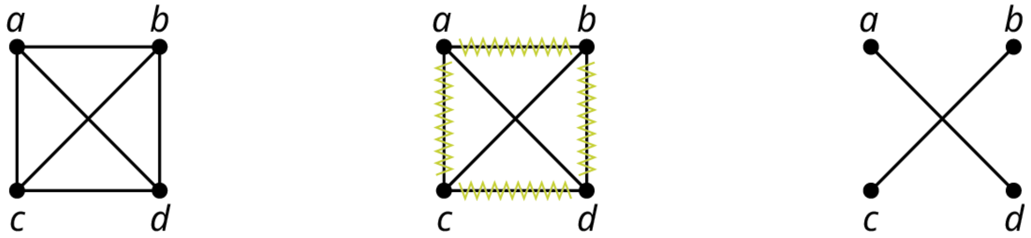


Figure 12.3.29: Use Complete Graph to Find Complement

✓ Example 12.3.5: Finding a Complement

A particular high school has end-of-course exams in (E3) English 3, (E4) English 4, (M) Advanced Math, (C) Calculus, (W) World History, (U) U.S. History, (B) Biology, and (P) Physics. No English 3 students are taking English 4, World History, or Biology; no English 4 students are also in Calculus, Advanced Math, U.S. History, or Physics; no Physics students are also taking Advanced Math; No World History students are also taking U.S. History; and no Advanced Math students are also taking Calculus.

1. Create a graph in which the vertices represent the exams, and an edge between a pair of vertices indicates that there are no students taking both exams.
2. Find the complement of the graph in part 1.
3. Explain what the graph in part 2 represents.

Answer

In Figure ✓ 12.3.30, we have drawn a vertex for each exam and edges between any vertices that have no students in common.

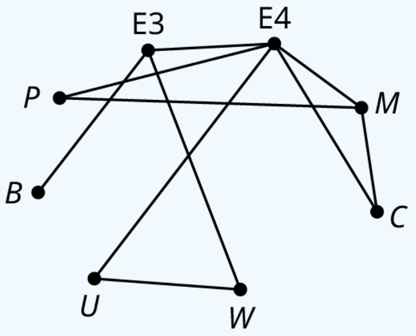


Figure ✓ 12.3.30 Graph of Exams with No Students in Common

One way to get the complement of the graph in Figure ✓ 12.3.30 is to draw a complete graph with the same number of vertices and remove the edges they have in common as shown in Figure ✓ 12.3.31.

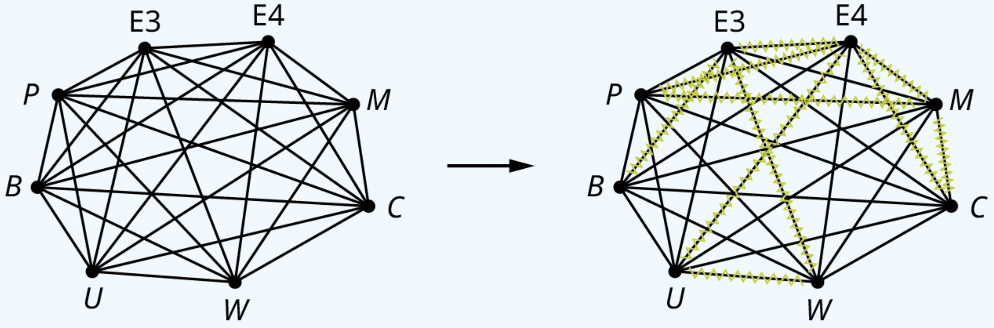


Figure ✓ 12.3.31: Remove Unwanted Edges from Complete Graph

The final graph of the complement is in Figure ✓ 12.3.32.

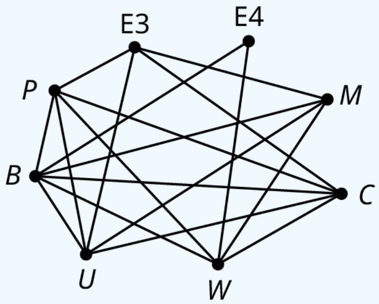


Figure ✓ 12.3.32: Graph of Exams with Students in Common

In the graph in Figure 12.71, the vertices are still the exams, and a pair of adjacent vertices represents a pair of exams that have students in common.

Your Turn 12.3.5

Find the complement of the Graph *H*.

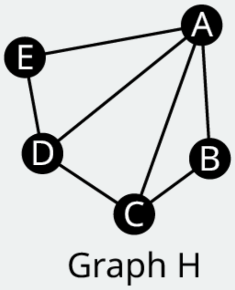


Figure 12.3.33

When two graphs are really the same graph, they have the same missing edges. So, when two graphs have a lot of edges, it may actually be easier to determine if they are isomorphic by looking at which edges are missing rather than which edges are included. In other words, we can determine if two graphs are isomorphic by checking if their complements are isomorphic.

✓ Example 12.3.6: Using a Complement to Find an Isomorphism

Use Figure ✓ 12.3.34 to answer each question.

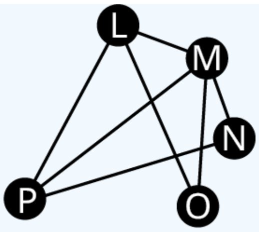


Figure ✓ 12.3.34 Graph *K*

1. Find the complement of Graph *K*.
2. Identify an isomorphism between the complement of Graph *K* from part 1, and the complement of Graph *H* in YOUR TURN 12.17.
3. Confirm that the correspondence between the vertices you found in part 2 also gives an isomorphism between Graph *H* from YOUR TURN 12.17, and Graph *K* from Figure ✓ 12.3.34.

Answer

1. The complement of Graph *K* can be found by removing the edges of Graph *K* from a complete graph with the same vertices as shown in Figure ✓ 12.3.35.

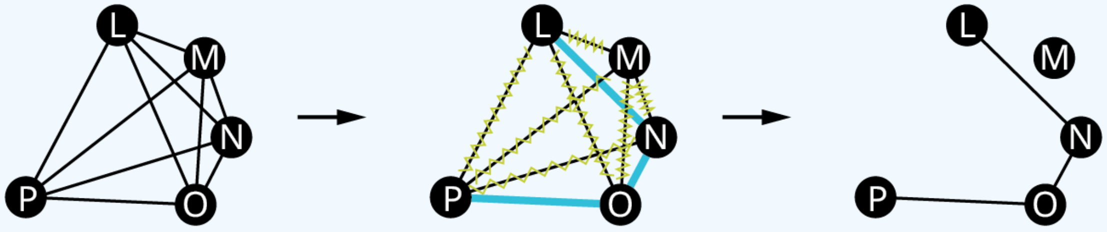

L
M
N
P
O
L
M
N
P
O
L
M
N
P
O

Figure ✓ 12.3.35 Find Complement of Graph *K*

2. An isomorphism between the complement of Graph *K* and the complement of Graph *H* is *A*-*M*, *C*-*L*, *E*-*N*, *B*-*O*, and *D*-*P*, which is confirmed by transforming the complement of Graph *K* in Figure ✓ 12.3.36.

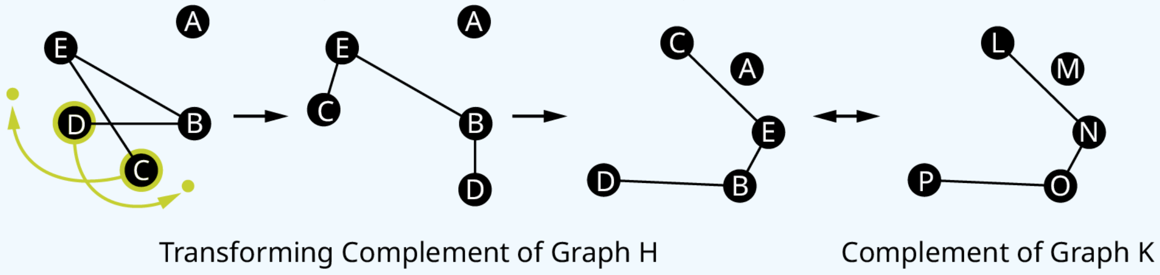

A
E
D
B
C
A
E
C
B
D
C
A
E
D
B
L
M
N
P
O
Transforming Complement of Graph H
Complement of Graph K

Figure ✓ 12.3.36: Isomorphism between Complements of *K* and *H*

3. Figure ✓ 12.3.37 shows how Graph *K* can be transformed into Graph *H* to confirm that the correspondence is *A*-*M*, *C*-*L*, *E*-*N*, *B*-*O*, and *D*-*P* also gives an isomorphism between Graph *K* and Graph *H*.

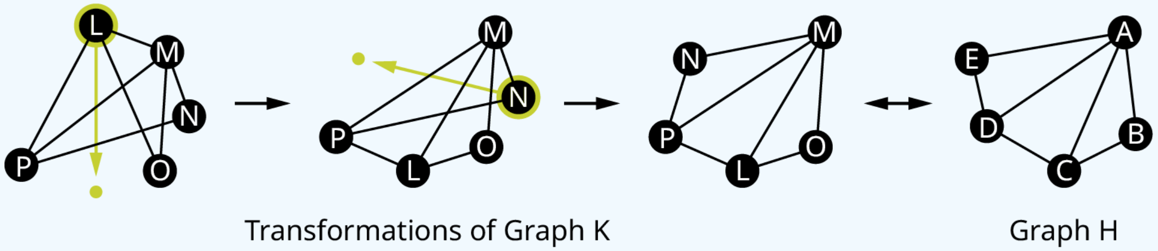

L
M
N
P
O
M
N
P
O
L
M
N
P
O
L
A
E
D
B
C
Transformations of Graph K
Graph H

Figure ✓ 12.3.37 Isomorphism between *K* and *H*

This means we now have three conditions that guarantee two graphs are isomorphic.

First Way: One graph can be transformed into the other without breaking existing connections or adding new ones.

Second Way: There is a correspondence between their vertices in such a way that any adjacent pair in one graph corresponds to an adjacent pair in the other graph.

Third Way: Their complements are isomorphic.

If any one of these statements is true, then they are all true. If any one of these statements is false, then they are all false.

Your Turn 12.3.6

Suppose that a Graph M is a complete graph and Graph N is the complement of M. What are the degrees of the vertices of Graph N? How do you know?

WORK IT OUT

Here is an activity you can do with a few of your classmates that will build your graph comparison skills.

Step 1: Draw a planar graph with the following characteristics: exactly five vertices, one vertex of degree four, at least two vertices of degree three, and exactly eight edges. Give names to the vertices. Make sure you do not use the same letters or numbers to label your vertices as your classmates do.

Step 2: Analyze your graph. What is the degree of each vertex? Does your graph have any cyclic subgraphs? If so, list them and indicate their sizes.

Step 3: Draw and analyze the complement of your graph. How many edges and vertices does it have? What is the degree of each vertex? Does the complementary graph have any cyclic subgraphs? If so, list them and indicate their sizes.

Step 4: Compare your graphs to each of your classmates' graphs. Does your graph have the same number of edges and vertices as the graph of your classmate? Does your graph have the same size cyclic subgraphs as the graph of your classmate? How does the complement of your graph compare to the complement of the graph of your classmate? Determine if your graph is isomorphic to your classmates' graph. If so, give a correspondence that demonstrates the isomorphism. If not, explain how you know.

Check Your Understanding

1. For the following exercises, determine whether each statement is always true, sometimes true, or never true.
2. Two graphs are isomorphic, and the graphs have the same structure.
3. A graph with four vertices is isomorphic to a graph with five vertices.
4. The sums of the degrees of the vertices of two graphs are equal, but the two graphs are not isomorphic.
5. Two graphs are isomorphic, but the graphs have a different number of edges.
6. One graph can be transformed to look like a second graph without removing or adding any connections, and the two graphs are isomorphic.
7. Two graphs are isomorphic, and there is more than one isomorphism between the two graphs.
8. Two graphs have the same number of vertices, but there is no isomorphism between them.
9. There is a correspondence between the vertices of Graph A and the vertices of Graph B such that the adjacent vertices in Graph A always correspond to vertices of Graph B, but the two graphs are not isomorphic
10. Two graphs have the same number of edges, the same number of vertices, vertices of the same degree, and have all the same subgraphs, but they are not isomorphic
11. Two graphs are isomorphic, and the sum of the degrees of the vertices of one equals the sum of the degrees of the other graph.
12. If two graphs are isomorphic, then their complements are isomorphic.

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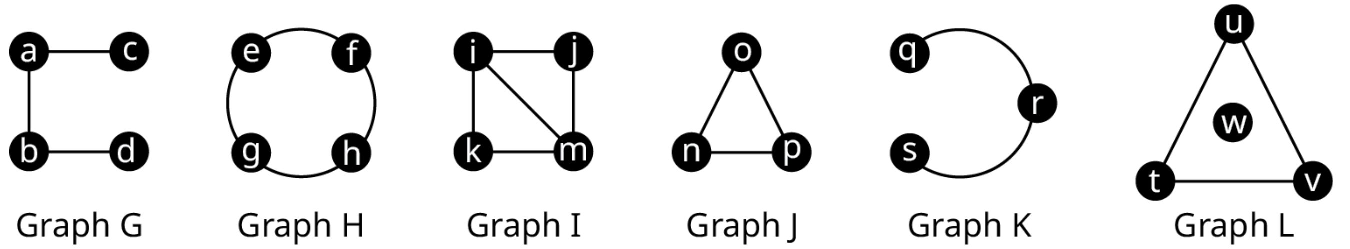
12.3.0: Exercises

Use the figure to answer the following exercises. A pair of graphs is given. Identify three differences between them that demonstrate the graphs are not isomorphic.


a
c
b
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Graph G
e
f
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Graph H
i
j
k
m
Graph I
o
n
p
Graph J
q
r
s
Graph K
u
w
t
v
Graph L

? Exercise 12.3.0.1

G and H

? Exercise 12.3.0.2

G and I

? Exercise 12.3.0.3

G and J

? Exercise 12.3.0.4

G and K

? Exercise 12.3.0.5

G and L

? Exercise 12.3.0.6

H and I

? Exercise 12.3.0.7

H and J

? Exercise 12.3.0.8

H and K

? Exercise 12.3.0.9

H and L

? Exercise 12.3.0.10

I and J

? Exercise 12.3.0.11

I and K

? Exercise 12.3.0.12

I and L

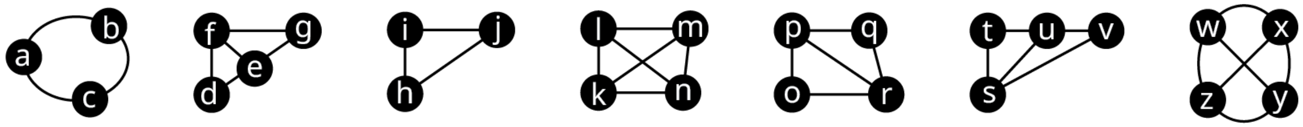
? Exercise 12.3.0.13

J and K

? Exercise 12.3.0.14

K and L

Use the figure to answer the following exercises. In each exercise, a pair of graphs is given. Determine if one graph is a subgraph of the other graph, or the two graphs are isomorphic.



Graph P Graph Q Graph R Graph S Graph T Graph U Graph V

? Exercise 12.3.0.15

P and Q

? Exercise 12.3.0.16

P and R

? Exercise 12.3.0.17

P and U

? Exercise 12.3.0.18

Q and R

? Exercise 12.3.0.19

Q and S

? Exercise 12.3.0.20

Q and T

? Exercise 12.3.0.21

Q and V

Exercise 12.3.0.22

S and V

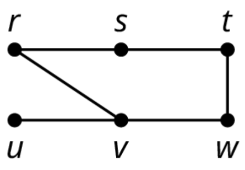
Exercise 12.3.0.23

T and U

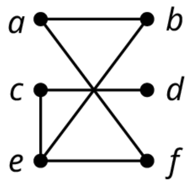
Exercise 12.3.0.24

U and V

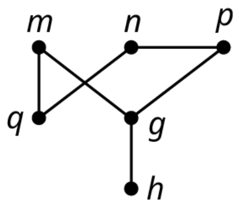
Use the figure to answer the following exercises. In each exercise, a pair of graphs is given. Either give a reason that the graphs are not isomorphic, or show how one of the graphs can be transformed to look like the other.



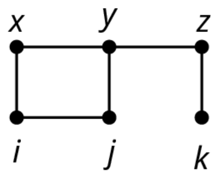
Graph A



Graph B



Graph C



Graph D

Exercise 12.3.0.25

A and B

Exercise 12.3.0.26

A and C

Exercise 12.3.0.27

A and D

Exercise 12.3.0.28

B and C

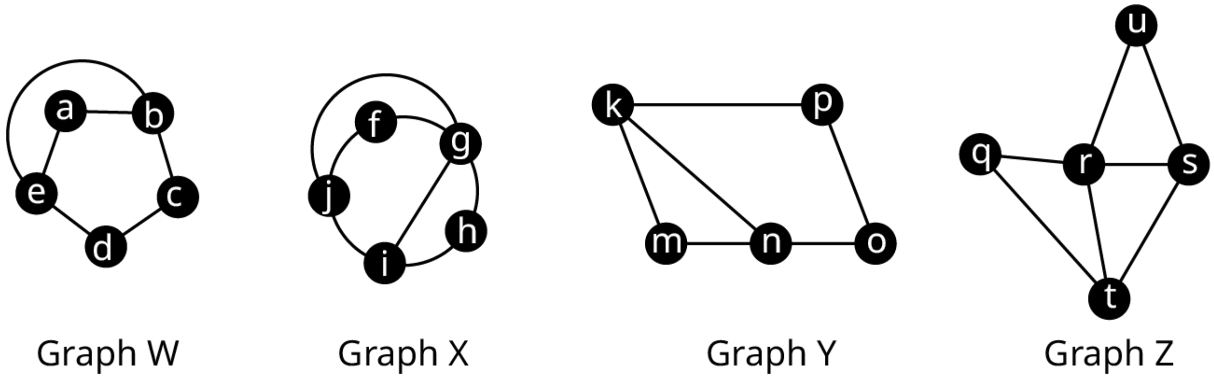
Exercise 12.3.0.29

B and D

Exercise 12.3.0.30

C and D

Use the figure to answer the following exercises. Determine if one graph is a subgraph of the other graph, or the two graphs are isomorphic. If they are isomorphic, name an isomorphism. If one is a subgraph of the other, indicate a correspondence between the vertices that demonstrates the relationship.


a
b
c
d
e
Graph W
f
g
h
i
j
Graph X
k
p
m
n
o
Graph Y
u
q
r
s
t
Graph Z

? Exercise 12.3.0.31

W and Z

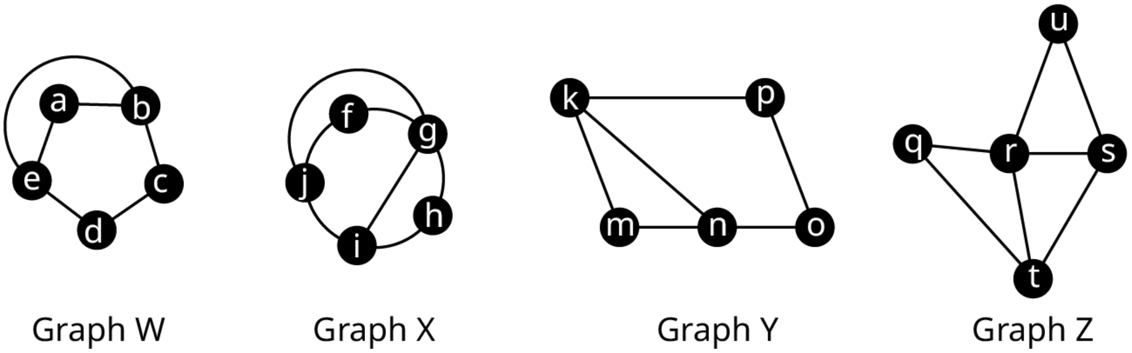
? Exercise 12.3.0.32

X and Y

? Exercise 12.3.0.33

X and Z

Use the figure to answer the following exercises.


a
b
c
d
e
Graph W
f
g
h
i
j
Graph X
k
p
m
n
o
Graph Y
u
q
r
s
t
Graph Z

? Exercise 12.3.0.34

Find the complement of Graph W.

? Exercise 12.3.0.35

Find the complement of Graph Y.

? Exercise 12.3.0.36

Find the complement of Graph X.

? Exercise 12.3.0.37

Find an isomorphism between the complement of W and the complement of Y if one exists. If not, explain how you know.

Exercise 12.3.0.38

Are W and Y isomorphic? Explain how you know.

Exercise 12.3.0.39

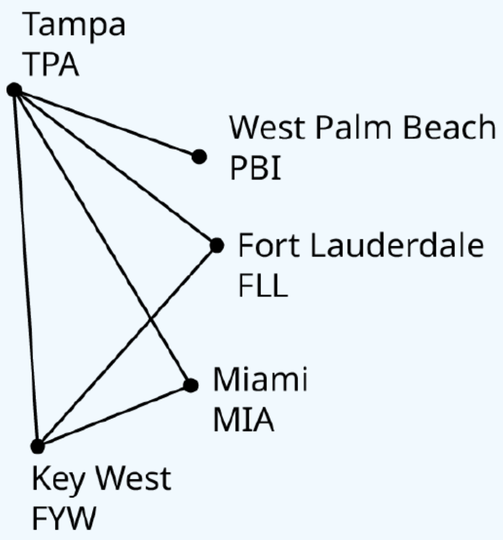
Find an isomorphism between the complement of W and the Complement of X if one Exists. If not, explain how you know.

Exercise 12.3.0.40

Are W and X isomorphic? Explain how you know.

Exercise 12.3.0.41

Find the complement of the graph in the given figure representing direct flights between south Florida airports. Explain what the graph represents.


Tampa
TPA
West Palm Beach
PBI
Fort Lauderdale
FLL
Miami
MIA
Key West
FYW

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12.4: Navigating Graphs

Learning Objectives

1. Describe and identify walks, trails, paths, and circuits.
2. Solve application problems using walks, trails, paths, and circuits.
3. Identify the chromatic number of a graph.
4. Describe the Four-Color Problem.
5. Solve applications using graph colorings.

Now that we know the basic parts of graphs and we can distinguish one graph from another, it is time to really put our graphs to work for us. Many applications of graph theory involve navigating through a graph very much like you would navigate through a maze. Imagine that you are at the entrance to a maze. Your goal is to get from one point to another as efficiently as possible. Maybe there are treasures hidden along the way that make straying from the shortest path worthwhile, or maybe you just need to get to the end fast. Either way, you definitely want to avoid any wrong turns that would cause unnecessary backtracking. Luckily, graph theory is here to help!

Figure $12.4.1$: Visitors navigate a garden maze. (credit: “Longleat Maze” by Niki Odolphie/Wikimedia, CC BY 2.0)

Walks

Suppose Figure $12.4.2$ is a maze you want to solve. You want to get from the start to the end.

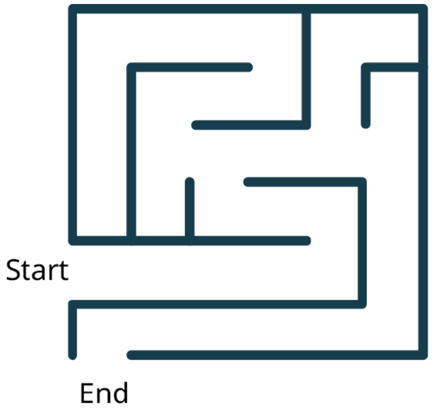


Figure $12.4.28$: Your Maze

You can approach this task any way you want. The only rule is that you can't climb over the wall. To put this in the context of graph theory, let's imagine that at every intersection and every turn, there is a vertex. The edges that join the vertices must stay within the walls. The graph within the maze would look like Figure 12.4.3.

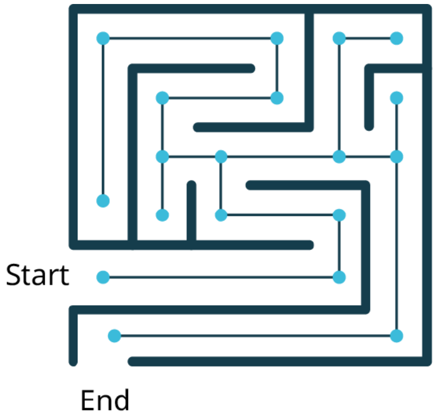


Figure 12.4.3: The Graph in Your Maze

One approach to solving a maze is to just start walking. It is not the most efficient approach. You might cross through the same intersection twice. You might backtrack a bit. It's okay. We are just out for a walk. It might look something like the black sequence of vertices and edges in Figure 12.4.4.

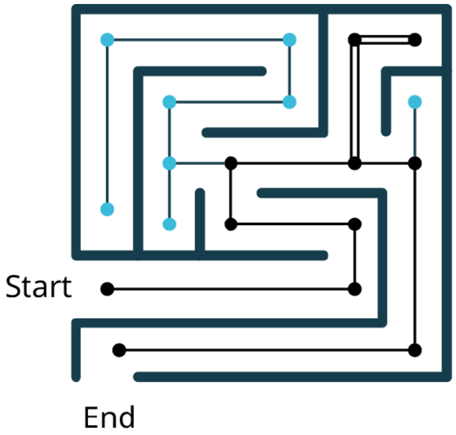


Figure 12.4.4: The Walk Through the Graph in Your Maze

This type of sequence of adjacent vertices and edges is actually called a walk (or directed walk) in graph theory too!

A walk can be identified by naming the sequence of its vertices (or by naming the sequence of its edges if those are labeled). Let's take the graph out of the context of the maze and give each vertex a name and each edge of the walk a direction as in Figure 12.4.5.

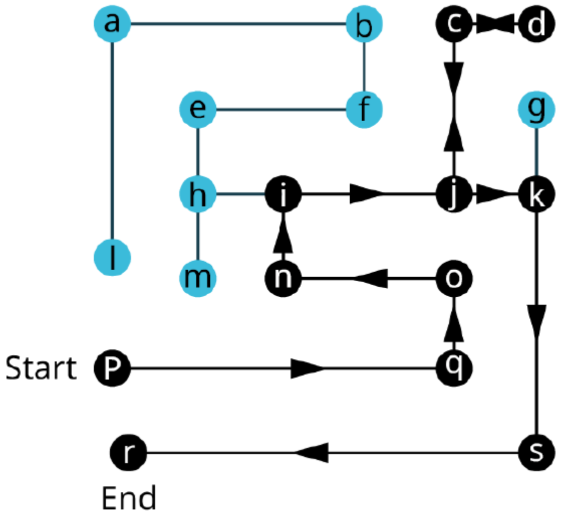


Figure 12.4.5: The Graph without Your Maze

The name of this walk from p to r is $p \rightarrow q \rightarrow o \rightarrow n \rightarrow i \rightarrow j \rightarrow c \rightarrow d \rightarrow c \rightarrow j \rightarrow k \rightarrow s \rightarrow r$. When a particular edge on our graph was traveled in both directions, it had arrows in both directions and the letters of vertices that were visited more than once had to be repeated in the name of the walk.

Checkpoint

Not every list of vertices or list of edges makes a path. The order must take into account the way in which the edges and vertices are connected. The list must be a sequence, which means they are in order. No edges or vertices can be skipped and you cannot go off the graph.

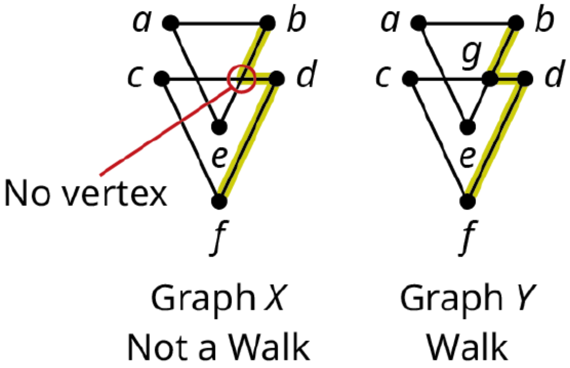


Figure 12.4.6: A Walk or Not a Walk?

The highlighted edges Graph Y in Figure 12.82 represent a walk between f and b. The highlighted edges in Graph X do **not** represent a walk between f and b, because there is a turn at a point that is not a vertex. This is like climbing over a wall when you are walking through a maze. Another way of saying this is that $b \rightarrow d \rightarrow f$ is not a walk, because there is no edge between b and d.

Checkpoint

A point on a graph where two edges cross is not a vertex.

✓ Example 12.4.1: Naming a Walk Through A House

Figure ✓ 12.4.7 shows the floor plan of a house. Use the floor plan to answer each question.

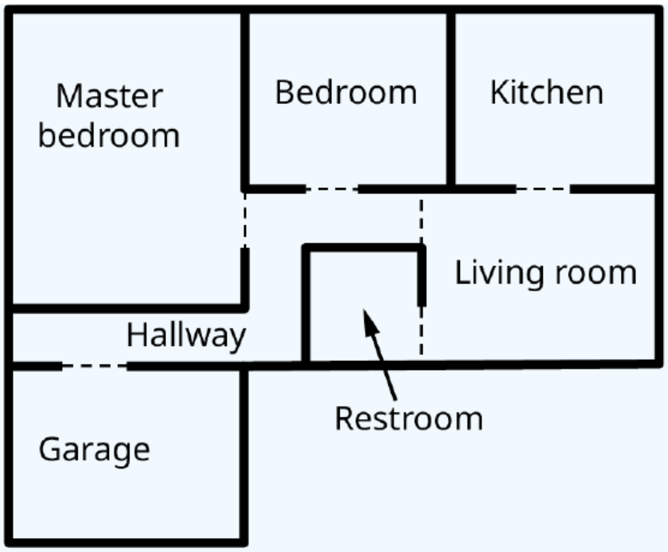

Master bedroom
Bedroom
Kitchen
Living room
Hallway
Restroom
Garage

Figure ✓ 12.4.7 Floor Plan of a House

1. Draw a graph to represent the floor plan in which each vertex represents a different room (or hallway) and edges represent doorways between rooms.
2. Name a walk through the house that begins in the living room, ends in the garage and visits each room (or hallway) at least once.

Answer

1. **Step 1:** We will need a vertex for each room and it is convenient to label them according to the names of the rooms. Visualize the scenario in your head as shown in Figure ✓ 12.4.8. You don't have to write this step on your paper.

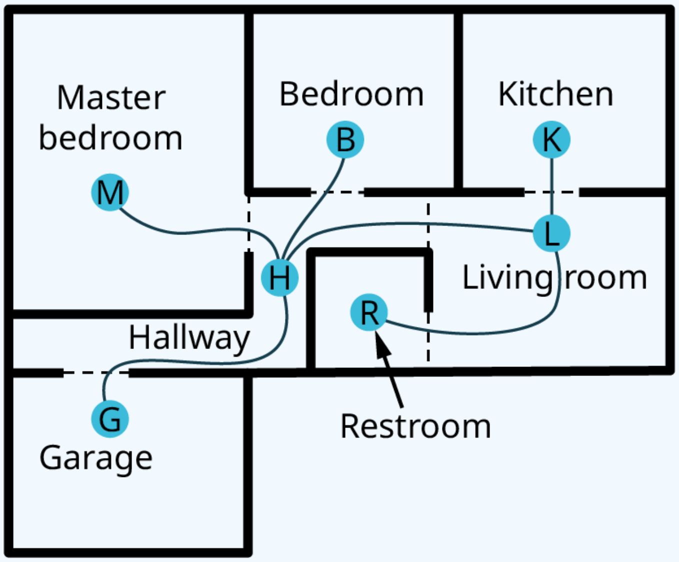

Master bedroom
Bedroom
Kitchen
M
B
K
L
H
R
Living room
Hallway
G
Garage
Restroom

Figure ✓ 12.4.8 Assigning Vertices to Rooms

Step 2: Draw a graph to represent the scenario. Start with the vertices. Then connect those vertices that share a doorway in the floorplan as shown in Figure ✓ 12.4.9.

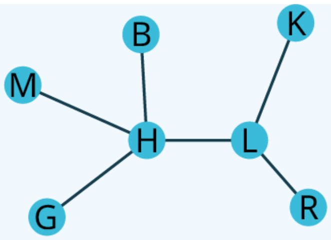


Figure ✓ 12.4.9: Graph of the Floor plan

2. **Step 1:** Draw a path that begins at vertex L, representing the living room, and ending at vertex G, representing the garage making sure to visit every room at least once. There are many ways this can be done. You may want to number the edges to keep track of their order. One example is shown in Figure ✓ 12.4.10.

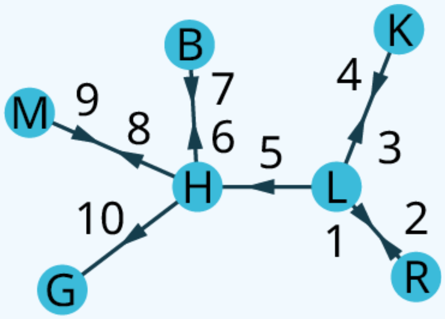

B
K
M
9
7
4
8
6
5
3
H
L
10
2
1
G
R

Figure ✓ 12.4.10: Draw the Path from L to G

Step 2: Name the path that you followed by listing the vertices in the order you visited them.

$L \to R \to L \to K \to L \to H \to B \to H \to M \to H \to G$

Your Turn 12.4.1

Recall from the Graph Basics section that a graph in which there are multiple edges between the same pair of vertices is called a multigraph. The figure shows a map of the four bridges that link Staten Island to Brooklyn and New Jersey and a multigraph representing the map. In the multigraph, the edges are labeled instead of the vertices. The edges represent the bridges, G (Goethals Bridge), B (Bayonne Bridge), C (Outerbridge Crossing), and V (Verrazzano-Narrows Bridge). The vertices represent New Jersey, Staten Island, and Brooklyn.

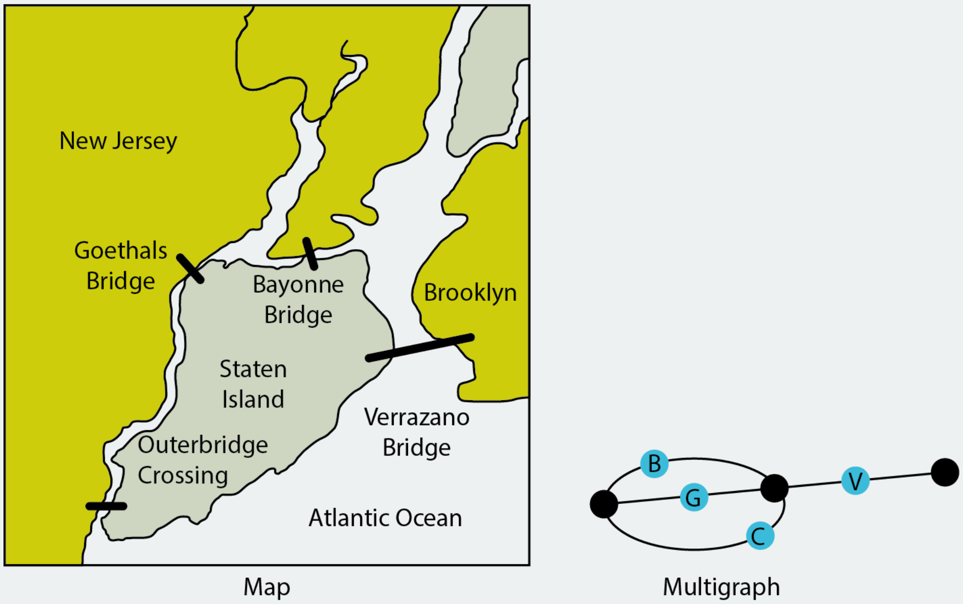

New Jersey
Goethals Bridge
Bayonne Bridge
Brooklyn
Staten Island
Verrazano Bridge
Outerbridge Crossing
Atlantic Ocean
Map
B
G
V
C
Multigraph

Figure 12.4.11: Map and Multigraph of Staten Island Bridges

> A path can be named by a sequences of edges instead of a sequence of vertices. Determine which of the following sequence of edges is a walk.
>
> a. $B \rightarrow V \rightarrow C \rightarrow G \rightarrow C$
> b. $V \rightarrow C \rightarrow B \rightarrow G \rightarrow C$
> c. $C \rightarrow V \rightarrow G \rightarrow B \rightarrow B$
> d. $G \rightarrow V \rightarrow B \rightarrow V \rightarrow C$

Paths and Trails

A walk is the most basic way of navigating a graph because it has no restrictions except staying on the graph. When there are restrictions on which vertices or edges we can visit, we will call the walk by a different name. For example, if we want to find a walk that avoids travelling the same edge twice, we will say we want to find a trail (or directed trail). If we want to find a walk that avoids visiting the same vertex twice, we will say, we want to find a path (or directed path).

Walks, trails, and paths are all related.

1. All paths are trails, but trails that visit the same vertex twice are not paths.
2. All trails are walks, but walks in which an edge is visited twice would not be trails.

We can visualize the relationship as in Figure $12.4.12$.

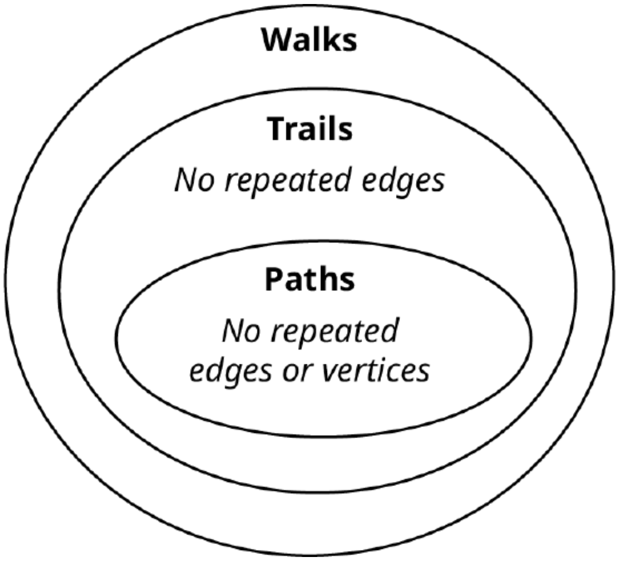


Figure $12.4.12$: Walks, Trails, and Paths

Let’s practice identifying walks, trails, and paths using the graphs in Figure $12.4.13$.

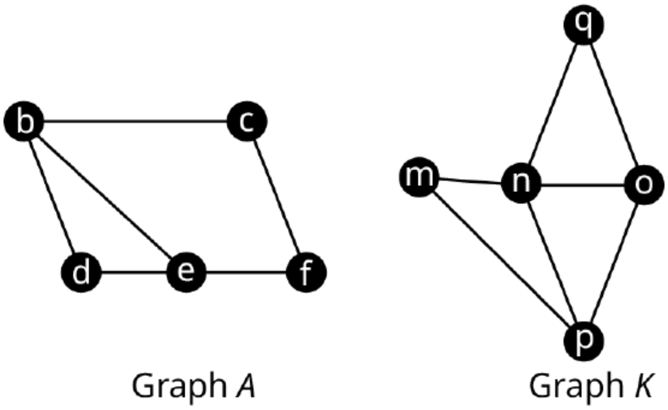


Figure $12.4.13$: Graphs A and K

✓ Example 12.4.2: Identifying Walks, Paths, and Trails

Consider each sequence of vertices from Graph *A* in Figure 12.89. Determine if it is only a walk, both a walk and a path, both a walk and a trail, all three, or none of these.

1. $b \rightarrow c \rightarrow d \rightarrow e \rightarrow f$
2. $c \rightarrow b \rightarrow d \rightarrow b \rightarrow e$
3. $c \rightarrow f \rightarrow e \rightarrow d \rightarrow b \rightarrow c$
4. $b \rightarrow e \rightarrow f \rightarrow c \rightarrow b \rightarrow d$

Answer

1. First, check to see if the sequence of vertices is a walk by making sure that the vertices are consecutive. As you can see in Figure ✓ 12.4.14, there is no edge between vertex *c* and vertex *d*.

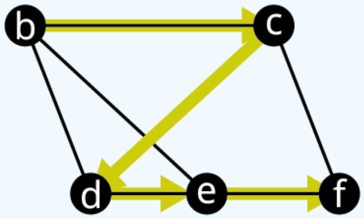


Figure ✓ 12.4.14

This means that the sequence is not a walk. If it is not a walk, then it can't be a path and it cannot be a trail, so, it is none of these.

- First, check to see if the sequence is a walk. As you can see in Figure ✓ 12.4.15, the vertices are consecutive.

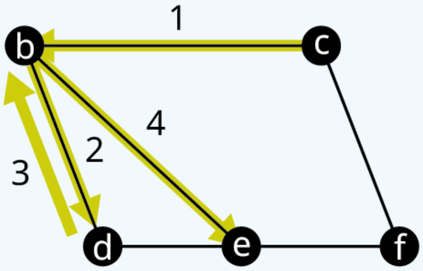


Figure ✓ 12.4.5:

This means that the sequence is a walk. Since the vertex *b* is visited twice, this walk is not a path. Since edge semanticsbdbd is traveled twice, this walk is not a trail. So, the sequence is only a walk.

- First, check to see if the sequence is a walk. We can see in Figure ✓ 12.4.16 that the vertices are consecutive, which means it is a walk.

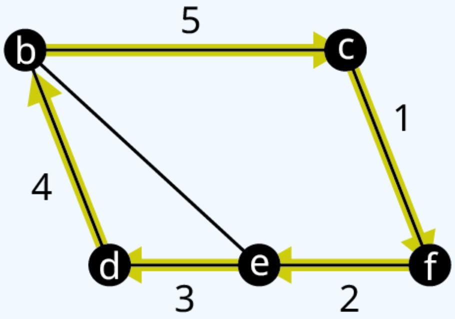


Figure ✓ 12.4.16:

Next, check to see if any vertex is visited twice. Remember, we do **not** consider beginning and ending at the same vertex to be visiting a vertex twice. So, no vertex was visited twice. This means we have a walk that is also a path. Next check to see if any edge was visited twice; none were. So, the sequence is a walk, a path, and a trail.

- First, check to see if the sequence is a walk. We can see in Figure ✓ 12.4.17 that the vertices are consecutive, which means it is a walk.

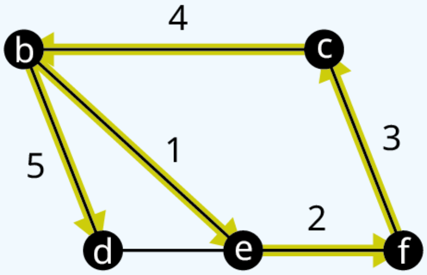


Figure ✓ 12.4.17:

Next check to see if any vertex is visited twice. Since vertex b is visited twice, this is not a path. Finally, check to see if any edges are traveled twice. Since no edges are traveled twice, this is a trail. So, the sequence of vertices is a walk and a trail.

Your Turn 12.4.2

Consider each sequence of vertices from Graph K in Figure 12.106. Determine if it is only a walk, both a walk and a path, both a walk and a trail, all three, or none of these.

$n \to q \to o \to p$

$p \to n \to q \to o \to n \to m$

$m \to n \to o \to p \to q$

Video

alks Trails and Paths in Graph Theo
Hearn
Hearn
Math
Identifying
Walks
Trails
& Paths
in Graph Theory
Watch on YouTube

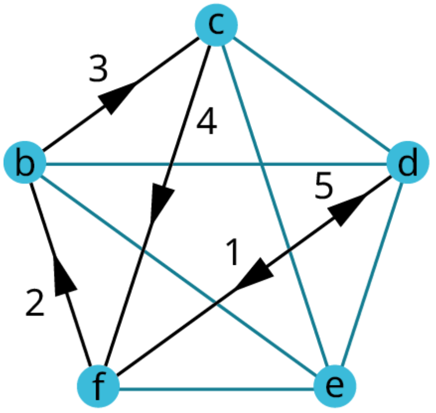
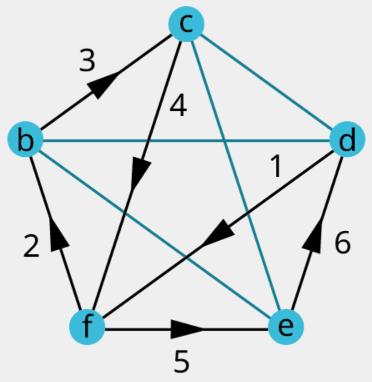
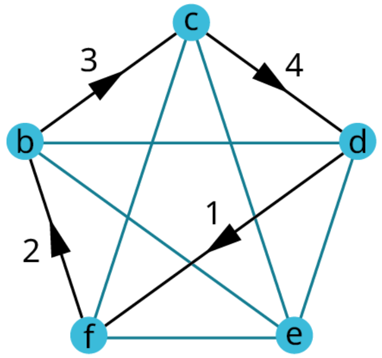
Walks, Trails, and Paths in Graph Theory

Circuits

In many applications of graph theory, such as creating efficient delivery routes, beginning and ending at the same location is a requirement. When a walk, path, or trail end at the same location or vertex they began, we call it closed. Otherwise, we call it open (does not begin and end at the same location or vertex). Some examples of closed walks, closed trails, and closed paths are given in in the following table.

Table 12.4.1: Closed Walks, Trails, and Paths

DESCRIPTION	EXAMPLE	CHARACTERISTICS

DESCRIPTION	EXAMPLE	CHARACTERISTICS
A closed walk is a walk that begins and ends at the same vertex.	 $d \to f \to b \to c \to f \to d$	Alternating sequence of vertices and edges Begins and ends at the same vertex
A closed trail is a trail that begins and ends at the same vertex. It is commonly called a circuit.	 $d \to f \to b \to c \to f \to e \to d$	No repeated edges Begins and ends at the same vertex
A closed path is a path that begins and ends at the same vertex. It is also referred to as a directed cycle because it travels through a cyclic subgraph.	 $d \to f \to b \to c \to d$	No repeated edges or vertices Begins and ends at the same vertex

Since walks, trails, and paths are all related, closed walks, circuits, and directed cycles are related too.

1. All circuits are closed walks, but closed walks that visit the same edge twice are not circuits.
2. All directed cycles are circuits, but circuits in which a vertex is visited twice are not directed cycles.

We can visualize the relationship as in Figure 12.4.18.

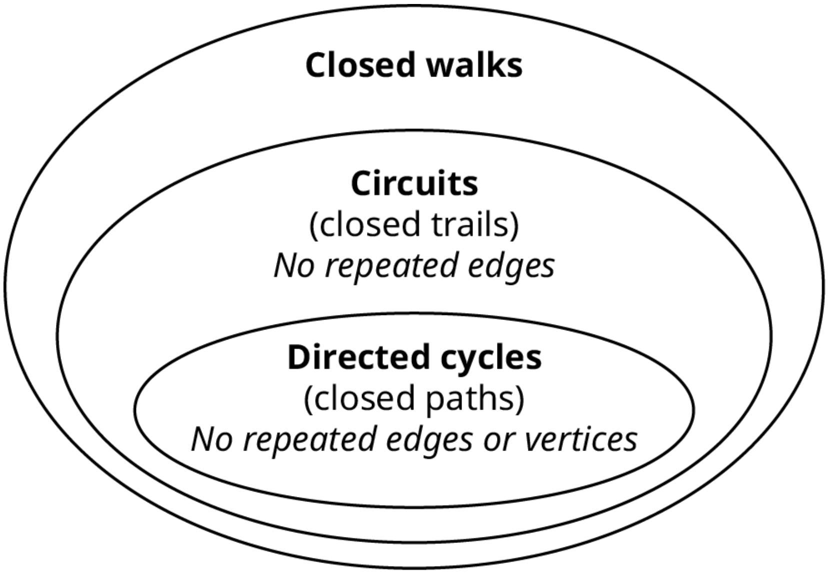

Closed walks
Circuits
(closed trails)
No repeated edges
Directed cycles
(closed paths)
No repeated edges or vertices

Figure 12.4.18: Closed Walks, Circuits, and Directed Cycles

The same circuit can be named using any of its vertices as a starting point. For example, the circuit $d \to f \to b \to c \to d$ can also be referred to in the following ways.

$a \to b \to c \to d \to a$ is the same as

$$\begin{cases} b \to c \to d \to a \to b \\ c \to d \to a \to b \to c \\ d \to a \to b \to c \to d \end{cases}$$

Let's practice working with closed walks, circuits (closed trails), and directed cycles (closed paths). In the graph in Figure 12.4.19, the vertices are major central and south Florida airports. The edges are direct flights between them.

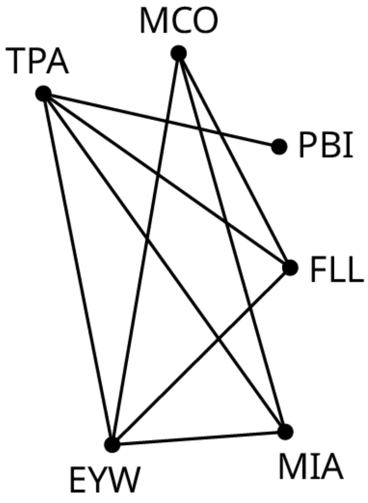

MCO
TPA
PBI
FLL
EYW
MIA

Figure 12.4.19: Major Central and South Florida Airports

✓ Example 12.4.3: Determining a Closed Walk, Circuit, or Directed Cycle

Suppose that you need to travel by air from Miami (MIA) to Orlando (MCO) and you were restricted to flights represented on the graph. For the trip to Orlando, you decide to purchase tickets with a layover in Key West (EYW) as shown in Figure ✓ 12.4.20, but you still have to decide on the return trip. Determine if your roundtrip itinerary is a closed walk, a circuit, and/or a directed cycle, based on the return trip described in each part.

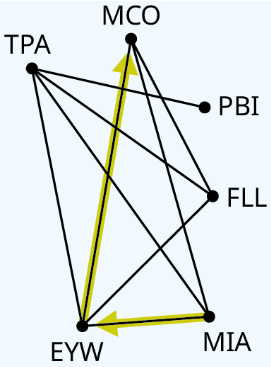

MCO
TPA
PBI
FLL
EYW
MIA

Figure ✓ 12.4.20 MIA to EYW to MCO

1. You returned to Miami (MIA) by reversing your route.
2. Your direct flight back left Orlando (MCO) but was diverted to Fort Lauderdale (FLL)! From there you flew to Tampa (TPA) before returning to Miami (MIA).

Answer

1. The whole trip was $MIA \rightarrow EYW \rightarrow MCO \rightarrow EYW \rightarrow MIA$. This is a closed walk, because it is a walk that begins and ends at the same vertex. It is not a circuit, because it repeats edges. If it is not a circuit, then it cannot be a directed cycle.
2. The whole trip was $MIA \rightarrow EYW \rightarrow MCO \rightarrow FLL \rightarrow TPA \rightarrow MIA$. This is a closed walk, because it is a walk that begins and ends at the same vertex. It is a circuit because no edges were repeated. It is also a directed cycle because no vertices were repeated either. So, it is all three!

Your Turn 12.4.3

Suppose that you want to travel from Palm Beach (PBI) to any other city listed in Figure 12.113 and then return to Palm Beach.

Why is it impossible to find an itinerary that is a circuit?

How is this related to the degree of the vertex PBI?

Video

osed Walks, Closed Trails (Circuits)
Hearn
Closed Walks
Closed Trails
Circuits
Closed Paths
Directed Cycles
Hearn
Math
in
Graph Theory
Watch on YouTube

Closed Walks, Closed Trails (Circuits), and Closed Paths (Directed Cycles) in Graph Theory

Graph Colorings

In this section so far, we have looked at how to navigate graphs by proceeding from one vertex to another in a sequence that does not skip any vertices, but in some applications we may want to skip vertices. Remember the camp Olympics at Camp Woebegone in Comparing Graphs? You were planning a camp Olympics with four events. The campers signed up for the events. You drew a graph to help you visualize which events have campers in common. The vertices of Graph E in Figure 12.4.21 represent the events and adjacent vertices indicate that there are campers who are participating in both.

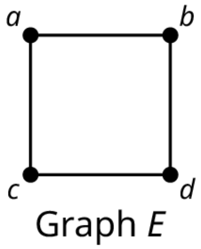


Figure 12.4.21: Graphs of Camp Olympics

In this case, we do NOT want events represented by two adjacent vertices to occur in the same timeslot, because that would prevent the campers who wanted to participate in both from doing so. We can use the graph in Figure 12.4.21 to count the timeslots we need so there are no conflicts. Let's assign each timeslot a different color. We could categorize events that happen at 1 pm as Red; 2 pm, Purple; 3 pm, Blue; and 4 pm, Green. Then assign different colors to any pair of adjacent vertices to ensure that the events they represent do not end up in the same timeslot. Figure 12.4.22 shows several of the ways to do this while obeying the rule that no pair of adjacent vertices can be the same color.

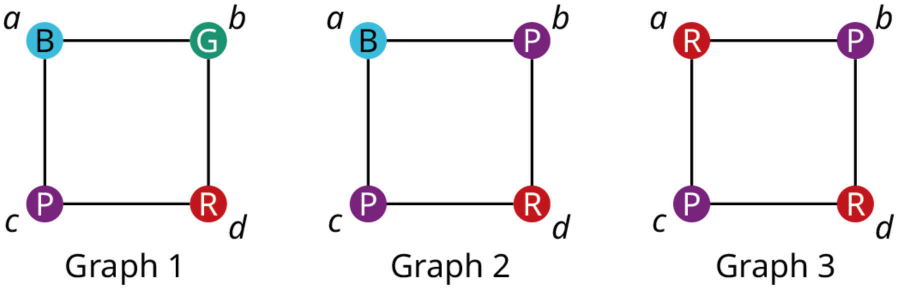


Figure 12.4.22: Graph Colorings

In Figure 12.4.22, the graphs with vertices colored so that no adjacent vertices are the same color are called graph colorings. Notice that Graph 3 has the fewest colors, which means it shows us how to have the fewest number of timeslots. The events marked in red, a and d, can be held at the same time because they are not adjacent and do not have conflicts. Also, the events marked in purple, b and c, can be held at the same time. We would not need green or blue timeslots at all!

A graph that uses n colors is called an n-coloring. The smallest number of colors needed to color a particular graph is called its chromatic number.

Graph colorings can be used in many applications like the scheduling scenario at camp Woebegone. Let's look at how they work in more detail. Figure 12.4.23 shows two different colorings of a particular graph. Coloring A is called a four-coloring, because it uses four colors, red (R), green (G), blue (B), and purple (P). Coloring B is called a three-coloring because it uses three. The colors allow us to visually subdivide the graphs into groups. The only rule is that adjacent vertices are different colors so that they are in different groups.

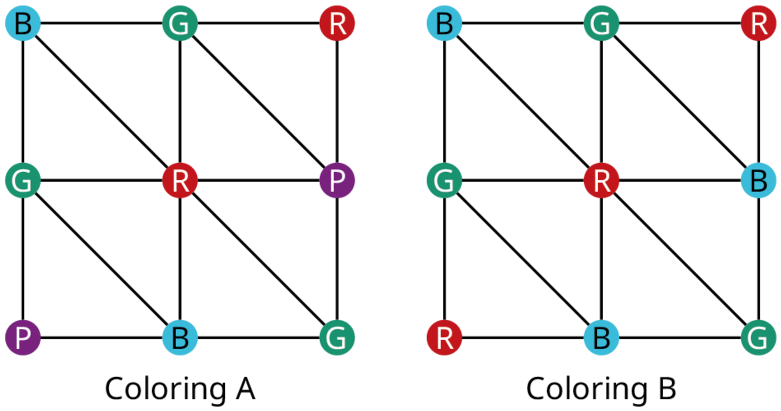
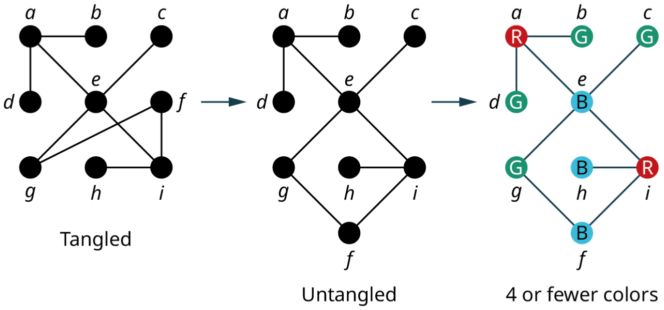
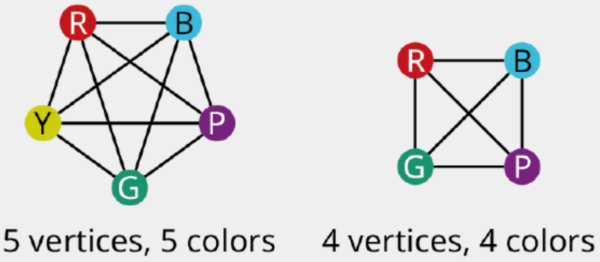
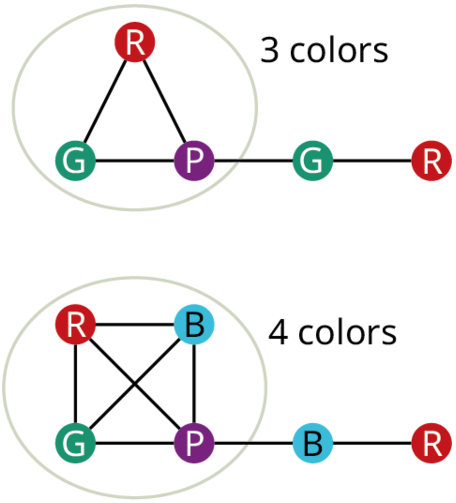


Figure 12.4.23: Two Colorings of the Same Graph

It turns out that a three-coloring is the best we can do with the graph in Figure 12.4.23. No matter how many different patterns you try with only two colors, you will never find one in which the adjacent vertices are always different colors. In other words, the graph has a chromatic number of three. For large graphs, computer assistance is usually required to find the chromatic numbers. There is no formula for finding the chromatic number of a graph, but there are some facts that are helpful in Table 12.4.2.

Table 12.4.2: Facts about Graph Colorings

Fact	Example
Recall that planar graphs are untangled; that is, they can be drawn on a flat surface so that no two edges are crossing. **If a graph is planar, it can be colored with four colors or possibly fewer.**	
Each vertex in a complete graph is adjacent to every other vertex forcing us to color them all different colors.**The chromatic number of a complete graph is the same as the number of vertices.**	

Fact	Example
If a graph has a clique, a complete subgraph, then each vertex in the clique must have a different color and the vertices outside the clique may or may not have even more colors. **The chromatic number of a graph is *at least* the number of vertices in its biggest clique.**	 R 3 colors G P G R R B 4 colors G P B R

Creating Colorings to Solve Problems

Let’s see how these facts can help us color the graph in Figure $12.4.24$.

- Since the graph is planar, the chromatic number is *no more than* four.
- The graph is not complete, but it has complete subgraphs of three vertices. In other words, it has triangles like the one shown with blue vertices in Figure 12.100. This means that the chromatic number is *at least* three.

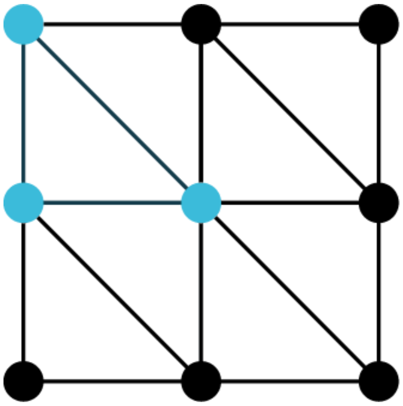
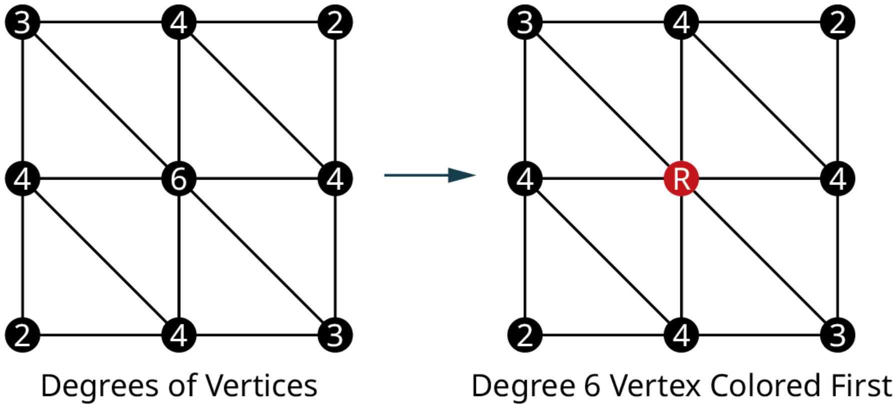


Figure $12.4.4$: A Graph with Triangles

We know we can color this graph in three or four colors. It is usually best to start by coloring the vertex of highest degree as shown in Figure $12.4.25$. In this case, we used red (R). The color is not important.



Degrees of Vertices

Degree 6 Vertex Colored First

Figure $12.4.25$: Color Highest Degree Vertex First

We want to color as many of the vertices the same color as possible; so, we look at all the vertices that are not adjacent to the red vertex and begin to color them, red starting with the one among them of highest degree. Since the only vertices that are not adjacent are both degree 2, choose either one and color it red as shown in Figure 12.4.26.

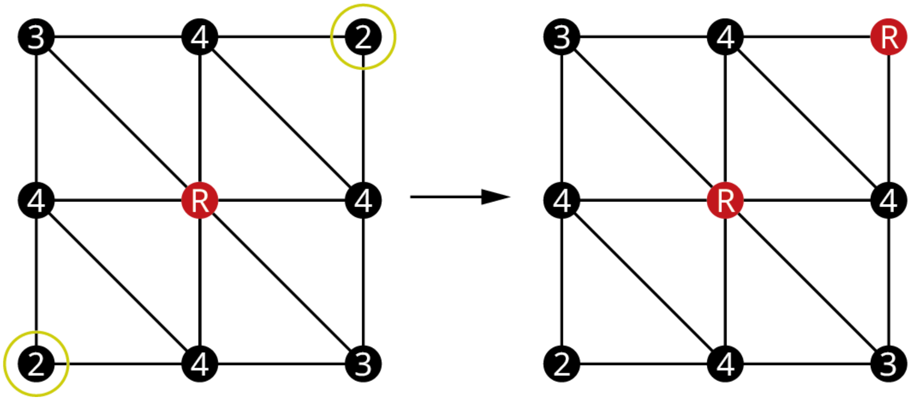


Figure 12.4.26: Color More Red from Highest to Lowest Degree

Now, there is only one vertex left that is not adjacent to a red; so, color it red. Of the remaining vertices, the highest degree is four; so, color one of the vertices of degree four in a different color. These two steps are shown in Figure 12.4.27.

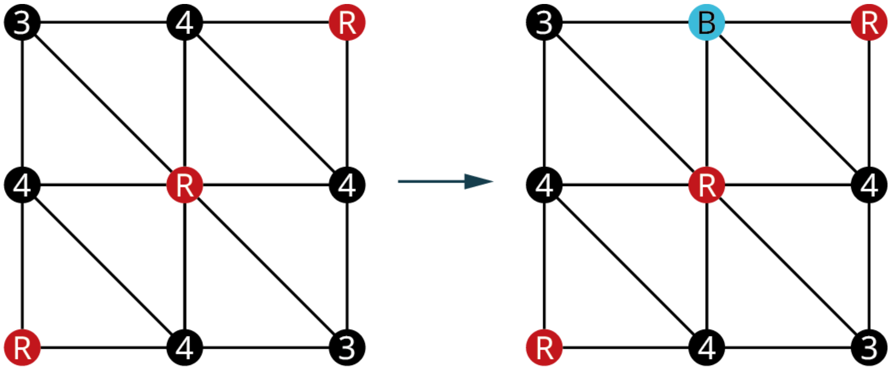


Figure 12.4.27: Complete the Reds and Start Another Color

Repeat the same procedure. There are three remaining vertices that are not adjacent to a blue. Color as many blue as possible, with priority going to vertices of higher degree as shown in Figure 12.4.28.

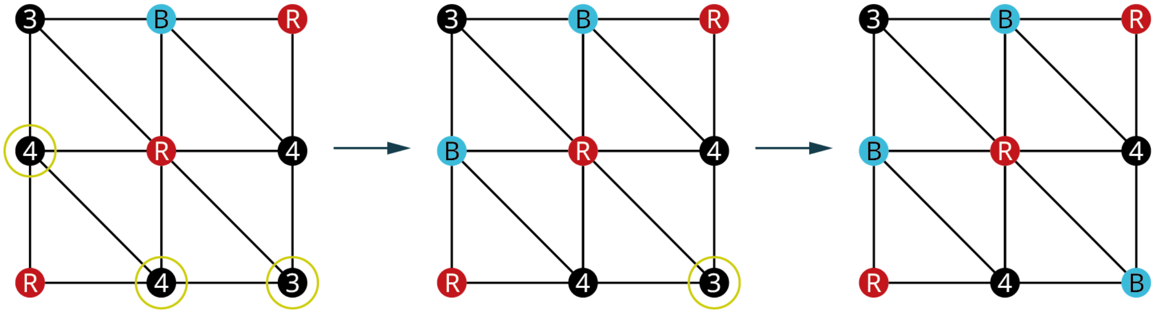


Figure 12.4.28: Repeat Procedure for Color Blue

All the remaining vertices are adjacent to blue. So, it is time to repeat the procedure with another color as shown in Figure 12.4.29.

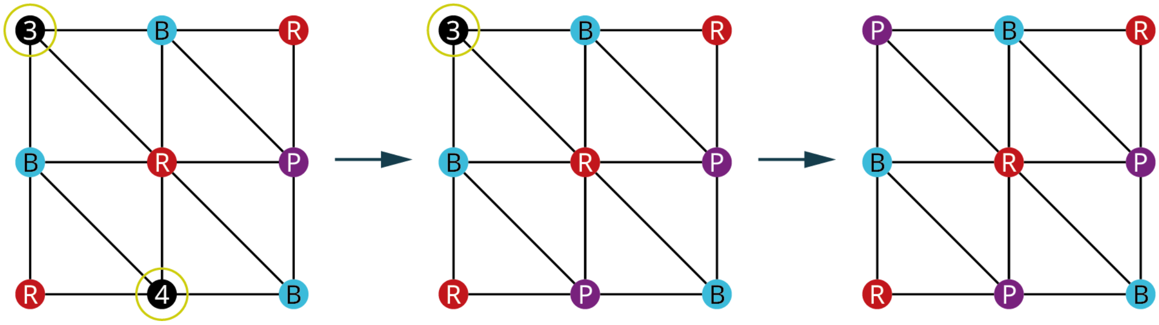
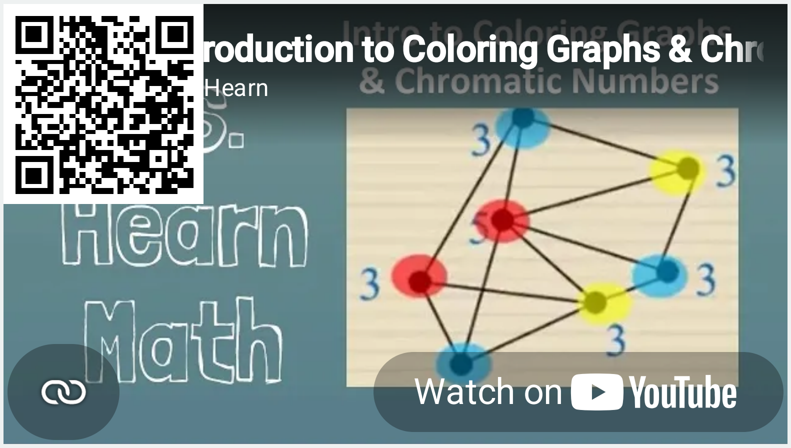

3
B
R
B
R
P
R
4
B
3
B
R
B
R
P
R
P
B
P
B
R
B
R
P
R
P
B

Figure 12.4.29: Repeat Procedure for Color Purple

All the vertices are now colored with a three-coloring so we know the chromatic number is *at most* three, but we knew the chromatic number was *at least* three because the graph has a triangle. So, we are now certain it is *exactly* three.

Video


roduction to Coloring Graphs & Chr
Hearn
Intro to Coloring Graphs & Chromatic Numbers
Hearn
Math
3
3
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3
3
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Watch on YouTube

Coloring Graphs Part 1: Coloring and Identifying Chromatic Number

Suppose the vertices of the graph in Figure 12.4.30 represented the nine events in the Camp Woebegone Olympics, and edges join any events that have campers in common, but nonadjacent vertices do not.

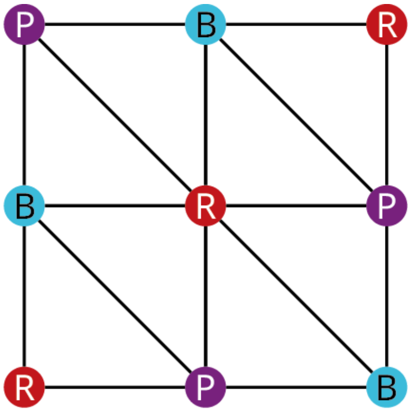


Figure 12.4.30: Coloring for Nine Event Camp Woebegone Olympics

Any vertices of the same color are nonadjacent; so, they have no conflicts. Since this graph is a three-coloring, all nine events could be scheduled in just three timeslots!

✓ Example 12.4.4: Understanding Chromatic Numbers

We previously discussed a high school, which holds end-of-course exams in (E3) English 3, (E4) English 4, (M) Advanced Math, (C) Calculus, (W) World History, (U) U.S. History, (B) Biology, and (P) Physics. We were given a list of courses that had no students in common. We used that information to find the graph in Figure ✓ 12.4.31, which shows edges between exams with students in common. Use the graph we found to answer each question.

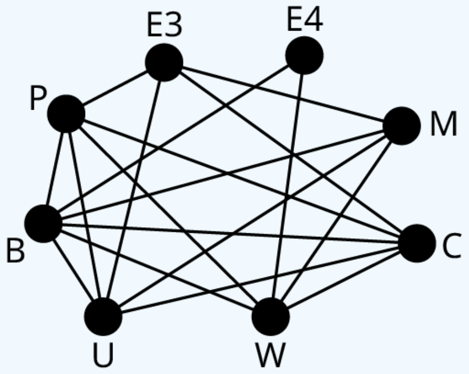


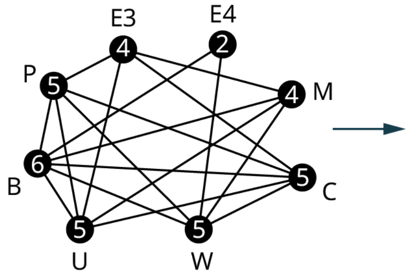
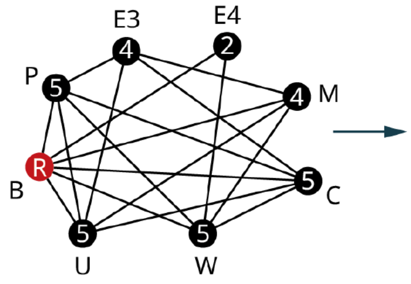
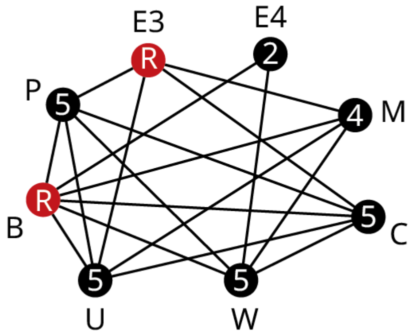
Figure ✓ 12.4.31 Graph of Exams with Students in Common

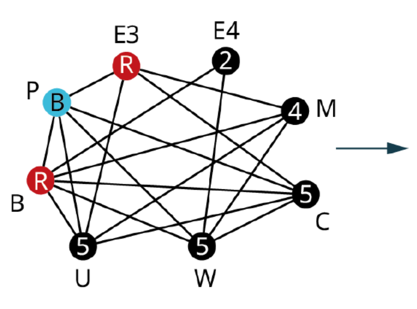
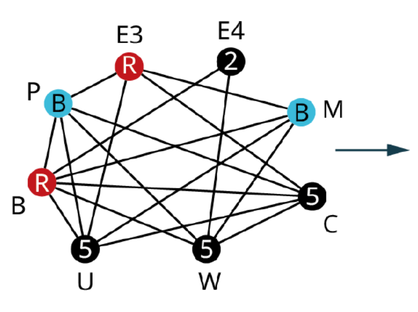
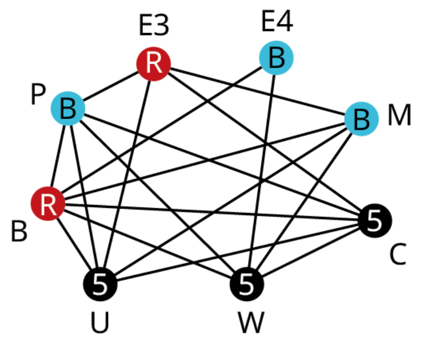
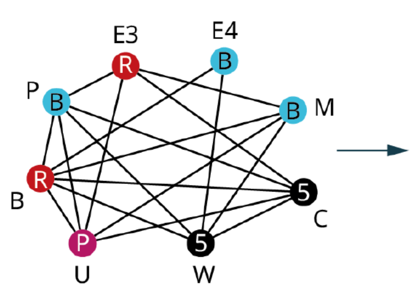
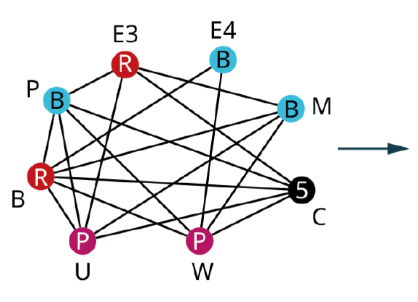
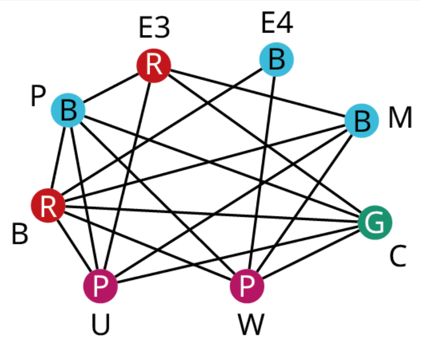
1. The graph contains a clique of size 4 formed by the vertices P, E3, C, and U. What does this tell you about the chromatic number?
2. The graph is not planar, meaning that you cannot untangle it. What does this tell you about the chromatic number?
3. Create a coloring by coloring vertex of highest degree first, coloring as many other vertices as possible each color from highest to lowest degree, then repeating this process for the remaining vertices.
4. Do you know what the minimum number of timeslots is? If so, what is it and how do you know? If not, what are the possibilities?

Answer

1. We would need four different colors just for the clique with four vertices; so, the chromatic number is *at least* four.
2. It is possible for the chromatic number to be greater than four.
3. The process is shown in Table ✓ 12.4.3.

Table ✓ 12.4.3 Coloring the Graph

 E3 E4 4 2 P 5 4 M 6 B 5 C 5 5 U W	 E3 E4 4 2 P 5 4 M R B 5 C 5 5 U W	 E3 E4 R 2 P 5 4 M R B 5 C 5 5 U W
This is the original graph. Vertex B has highest degree.	Color vertex B. Vertex E3 is the only remaining vertex that is NOT adjacent to B.	Color vertex E3 the same color. Vertices P, U, W, and C are the remaining vertices with highest degree. Pick one to color.

 E3 R E4 2 P B 4 M R B 5 C 5 U 5 W	 E3 R E4 2 P B B M R B 5 C 5 U 5 W	 E3 R E4 B P B B M R B 5 C 5 U 5 W
Color vertex P a new color. Vertices E4 and M are NOT adjacent to P, and M has higher degree.	Color vertex M the same color. Vertex E4 is the only remaining vertex NOT adjacent to P or M.	Color vertex E4 the same color. Vertices U, W, and C are the remaining vertices with highest degree. Pick one to color.
 E3 R E4 B P B B M R B 5 C P U 5 W	 E3 R E4 B P B B M R B 5 C P U P W	 E3 R E4 B P B B M R B G C P U P W
Color U a new color. Vertex W is the only remaining vertex NOT adjacent to U.	Color vertex W the same color. Vertex C is the only remaining vertex that has NOT been colored.	Color vertex C a new color. The coloring is final. We used four colors.

The last graph in Table ✓ 12.4.3 is the final coloring.

4. Yes, the minimum number of times slots is the chromatic number. We knew the chromatic number had to be at least four because there was a clique with four vertices. Now we have found a four-coloring of the graph which tells us that the chromatic number is at most four. So, we know four must be the chromatic number.

Checkpoint

The procedure used in Example 22 to color the graph is not guaranteed to result in a graph that has the minimum number of colors possible, but it is usually results in a coloring that is close to the chromatic number.

Your Turn 12.4.4

The given figure shows the same graph colored in several ways. The largest clique of the graph has three vertices. Use this information to answer each question.

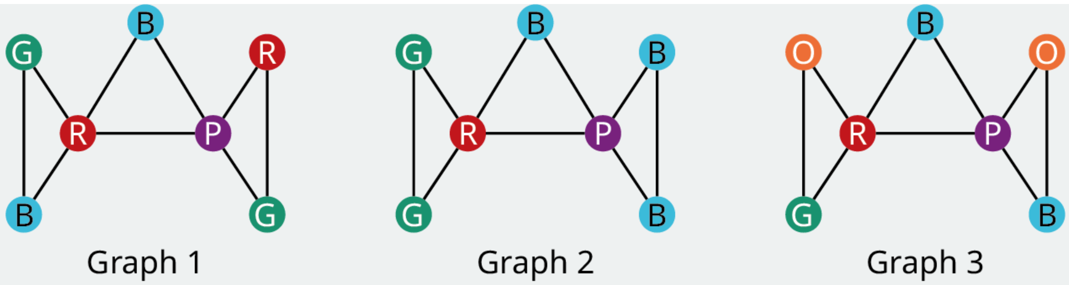


Figure 12.4.32: Possible Colorings of the Same Graph

Is the graph planar? What does your answer tell you about the chromatic number?

What is the lowest possible chromatic number of this graph based on the size of any cliques?

What are the possible chromatic numbers?

Which graph or graphs meet the definition of a valid graph coloring?

Draw an n-coloring for the graph where n is the answer to Exercise 1.

The Four Color Problem

The idea of coloring graphs to solve problems was inspired by one of the most famous problems in mathematics, the "four color problem." The idea was that, no matter how complicated a map might be, only four colors were needed to color the map so that no two regions that shared a boundary would be the same color. For many years, everyone suspected this to be true, because no one could create a map that needed more than four colors, but they couldn't prove it was true in general. Finally, graphs were used to solve the problem!

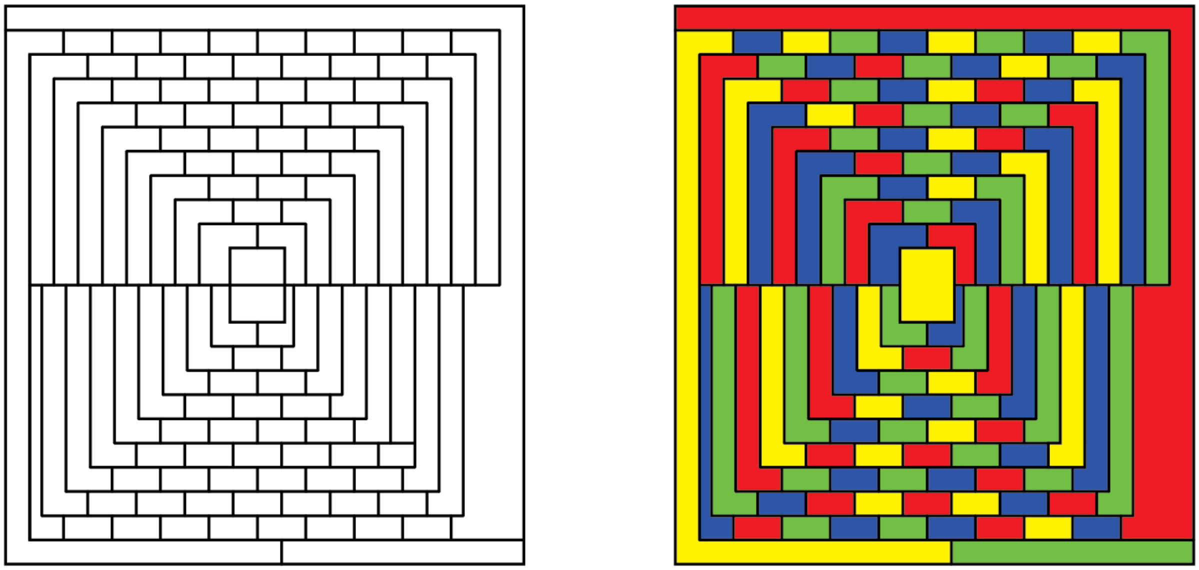
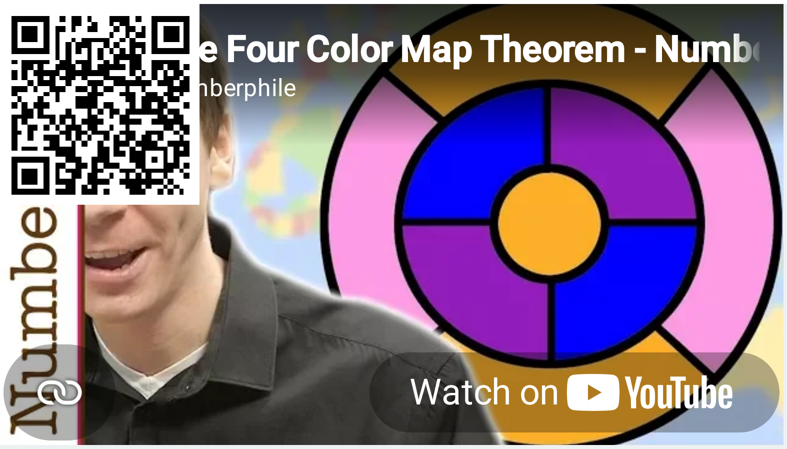


Figure 12.4.33: Using only four colors, no two adjacent regions have the same color.

Video


e Four Color Map Theorem - Numb
nberphile
Numbe
Watch on YouTube

The Four Color Map Theorem – Numberphile

We saw how maps can be represented as graphs in Graph Basics. Figure $12.4.34$ shows a map of the midwestern region of the United States.

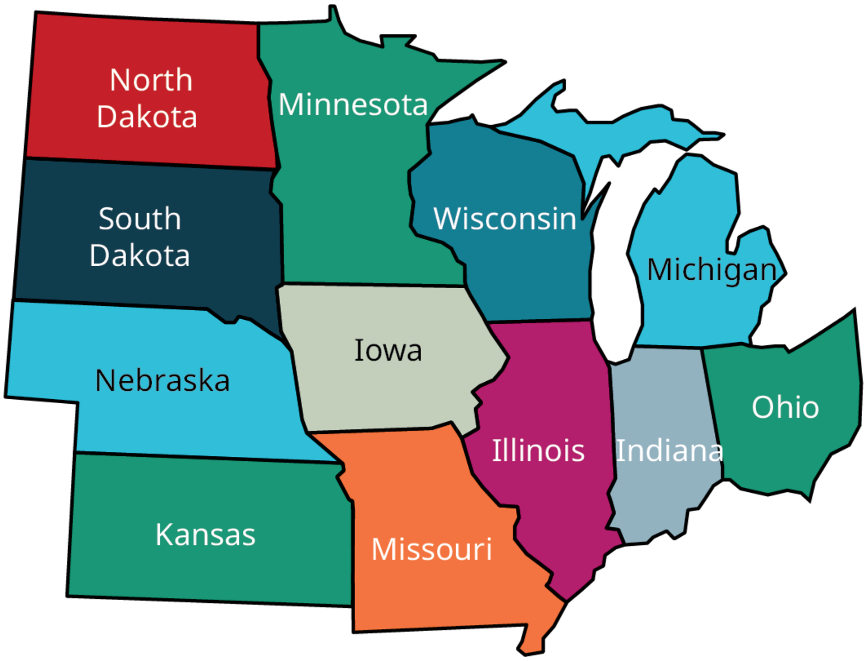

North Dakota
Minnesota
South Dakota
Wisconsin
Michigan
Iowa
Nebraska
Ohio
Illinois
Indiana
Kansas
Missouri

Figure $12.4.34$: Map of Midwestern States

Figure $12.4.34$ shows how this map can be associated with a graph in which each vertex represents a state and each edge indicates the states that share a common boundary. Figure$12.4.35$ shows the final graph.

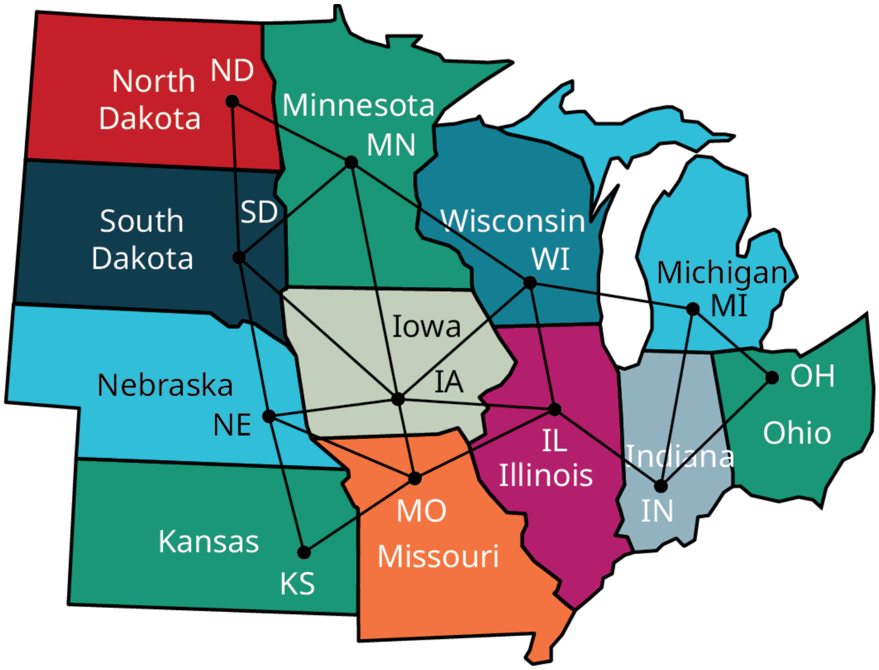

North Dakota
ND
Minnesota
MN
South Dakota
SD
Wisconsin
WI
Michigan
MI
Iowa
IA
Nebraska
NE
OH
Ohio
IL
Illinois
Indiana
IN
MO
Missouri
Kansas
KS

Figure 12.4.35: Edge Assigned to Each Pair of Midwestern States with Common Border

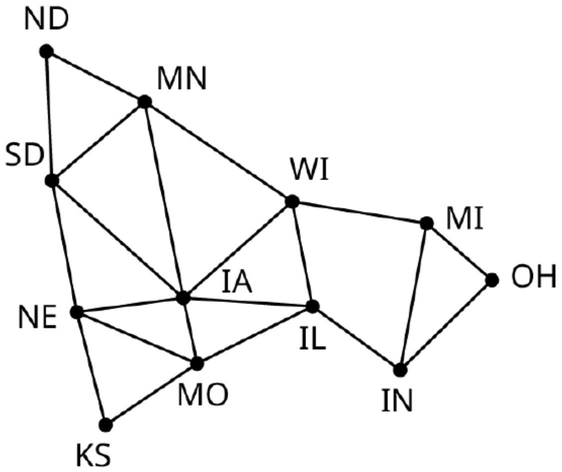
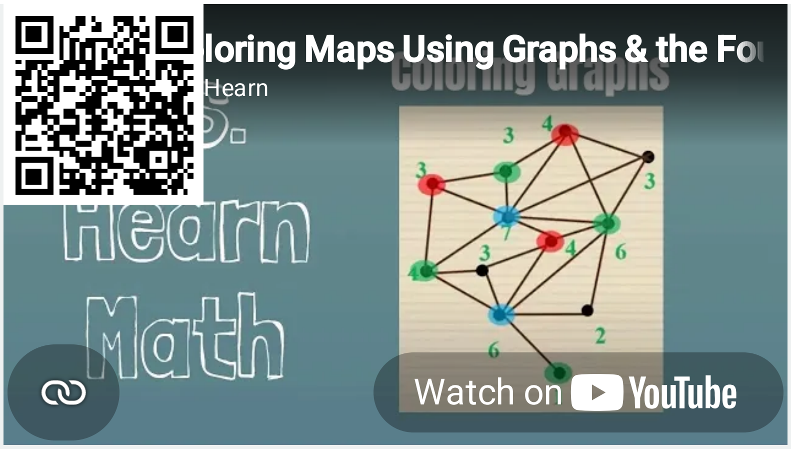

ND
MN
SD
WI
MI
OH
IA
NE
IL
MO
IN
KS

Figure 12.4.36: Final Graph Representing Common Boundaries between Midwestern States

Notice that the graph representing the common boundaries between midwestern states is planar, meaning that it can be drawn on a flat surface without edges crossing. As we have seen, any planar graph has a chromatic number of four or less. This very well-known fact is called the Four-Color Theorem, or Four-Color Map Theorem.

Video


loring Maps Using Graphs & the Fo
Hearn
Hearn
Math
Coloring Graphs
Watch on YouTube

Coloring Graphs Part 2: Coloring Maps and the Four Color Problem

People in Mathematics: Four-Color Theorem

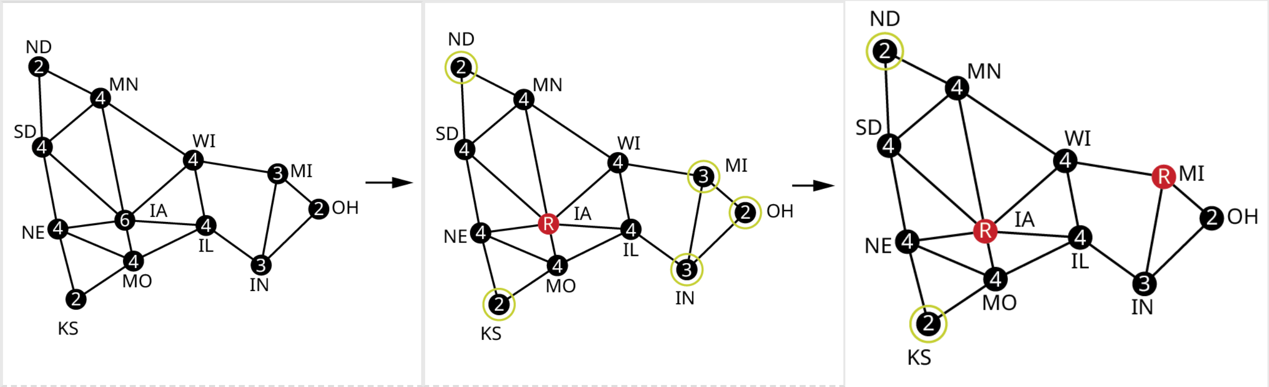
In the 19th century, Francis Guthrie was coloring a map of British counties and noticed that he could color so that no two adjacent counties were the same color using only four colors. This seemed to be the case for other maps as well, but he could not figure out why. He shared this with his brother, Frederick, who subsequently took the problem to his professor, August De Morgan, one of the most famous mathematicians of all time. De Morgan never was able to solve the problem, but he shared it with his colleagues. The problem continued to intrigue and perplex mathematicians for over a hundred years, inspiring discoveries in several areas of mathematics. It was finally proven by Kenneth Appel and Wolfgang Haken from the University of Illinois in 1976 using a combination of computer-aided calculations and graph theory. This was the first time in history that a mathematical theorem was proven using a computer! (Jesus Najera, “The Four-Color Theorem,” *Cantor’s Paradise*, a Medium publication, October 22, 2019)

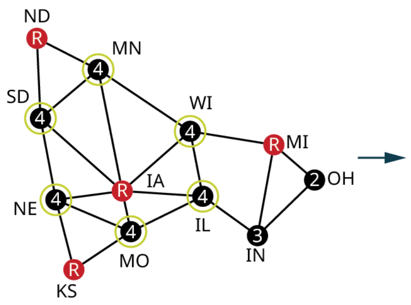
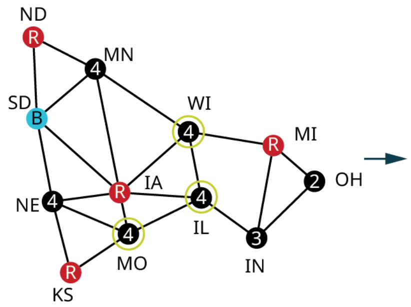
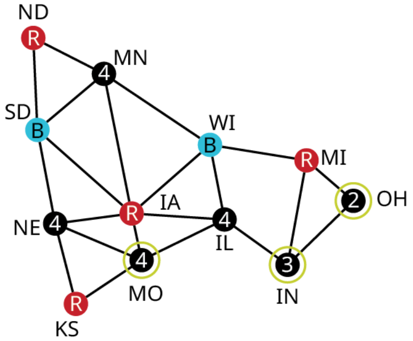
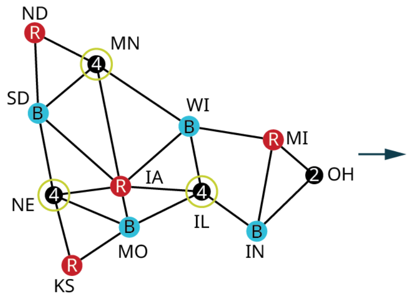
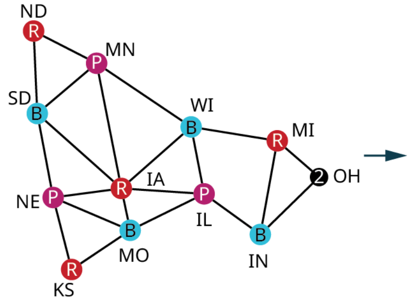
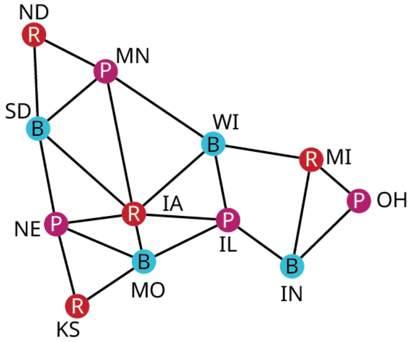
? Example $12.4.5$: Using Four Colors or Fewer to Color a Graph

Find a coloring of the graph in Figure ? $12.4.34$, which uses four colors or fewer. Use the resulting coloring as a guide to recolor the map in Figure ? $12.4.34$. How many colors did you use? Does this support the conclusion of the Four-Color Theorem? If so, how?

Answer

Table ? $12.4.4$ Steps to Color Graph of Midwestern States


ND
2
MN
4
SD
4
WI
4
MI
3
OH
2
NE
4
IA
6
IL
4
MO
4
IN
3
KS
2
ND
2
MN
4
SD
4
WI
4
MI
3
OH
2
NE
4
R
IA
IL
4
MO
4
IN
3
KS
2
ND
2
MN
4
SD
4
WI
4
R
MI
OH
2
NE
4
R
IA
IL
4
MO
4
IN
3
KS
2

Step 1: Graph with degrees of vertices labeled. Vertex IA has highest degree.	**Step 2:** Color vertex IA any color. Vertices ND, KS, MI, OH, and IN are NOT adjacent to IA. MI and IN have highest degree, 3.	**Step 3:** Color either MI or IN the same color as IA. Vertex ND and KS are the only remaining vertices not adjacent to a red vertex. They both have degree 2.
		
Step 4: Since KS and ND are not adjacent to each other, we can save a step and color both red. The highest degree of remaining vertices is four.	**Step 5:** Choose one of the vertices of degree 4, SD, to color with a new color, blue. The vertices WI, IL, MO, IN,and OH are NOT adjacent to blue. WI, IL, and MO have the highest degree.	**Step 6:** Choose one of WI, IL, or MO to color. We color WI blue. MO, IN and OH are NOT adjacent to blue. MO has the highest degree of these.
		
Step 7: Color MO blue. All remaining vertices are adjacent to blue. Choose a new color. Four vertices remain, MN, NE, IL, and OH. MN, NE, and IL have the highest degree.	**Step 8:** Since MN, IL, and NE are not adjacent, save steps and color all three the new color, purple. Vertex OH is the only remaining vertex that has NOT been colored.	**Step 9:** Since vertex OH is not adjacent to purple, color it purple. This is the final graph. We used three colors.

The final graph in ? 12.4.4 shows how we would color the map. In Figure ? 12.4.37 we have colored the map to correspond to the colors on the graph.

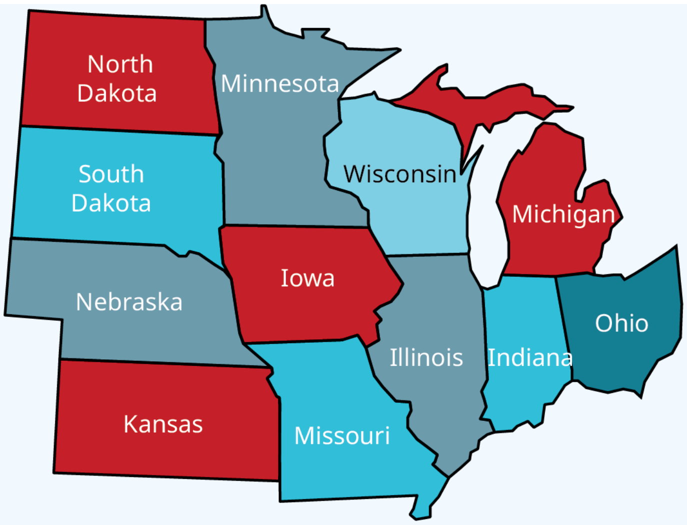

North Dakota
Minnesota
South Dakota
Wisconsin
Michigan
Nebraska
Iowa
Ohio
Illinois
Indiana
Kansas
Missouri

Figure ❓ 12.4.37: Midwestern States in Three Colors

We used three colors to color the graph. This supports the Four-Color Theorem, because the graph is planar and its chromatic number is less than four.

Your Turn 12.4.5

The first figure shows a map of the Island of Oahu in the State of Hawaii divided into regions, and the second figure shows a graph of the map. What is the chromatic number? Find a graph coloring that reflects the chromatic number of the graph.

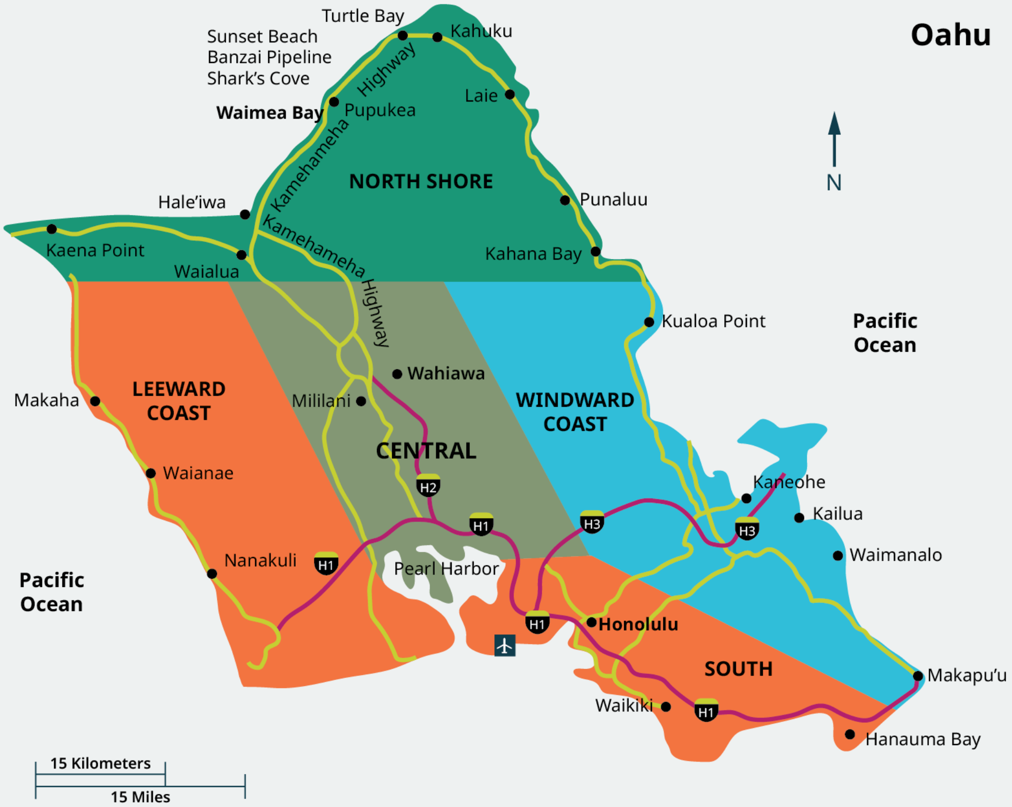

Oahu
Turtle Bay
Sunset Beach
Banzai Pipeline
Shark's Cove
Kahuku
Waimea Bay
Pupukea
Laie
Kamehameha Highway
NORTH SHORE
Hale'iwa
Punaluu
Kaena Point
Waialua
Kahana Bay
Kualoa Point
Pacific Ocean
N
Wahiawa
Makaha
LEEWARD COAST
Mililani
WINDWARD COAST
CENTRAL
Waianae
H2
Kaneohe
Kailua
H1
H3
Waimanalo
Nanakuli
Pearl Harbor
Pacific Ocean
Honolulu
SOUTH
Makapu'u
Waikiki
Hanauma Bay
15 Kilometers
15 Miles

Figure 📌 12.4.38: Map of Oahu

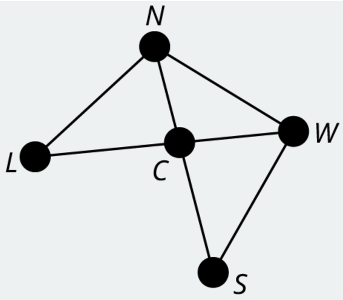


Figure 12.4.39: Graph Representing Common Boundaries between Regions of Oahu

Who Knew?: Coloring A Möbius Strip

Have you ever heard of a möbius strip? It is a flat object with only one side. This might sound theoretical, but you can make one for yourself. Take a narrow strip of paper, make a half twist in it, and tape the ends together. To prove to yourself that it has only one side, pick a point anywhere on the paper and start drawing a line. You can draw the line continuously without lifting your pen from the paper and eventually, you will get back to the point at which you started. Since you never had to flip over the paper, you have just proved it has one side!

Now, you might be wondering what this has to do with Graph Theory. It turns out that a map drawn on a möbius strip cannot necessarily be drawn using four colors. This doesn't contradict the Four-Color Theorem, which only applies to graphs that are planar. A möbius strip does not live on a "plane," or flat surface, which means the maps are not always planar like those drawn on a flat piece of paper. In the case of a möbius strip, it is the Six-Color Map Theorem! Figure 12.4.40 is an example of a möbius strip with a map that requires five colors.

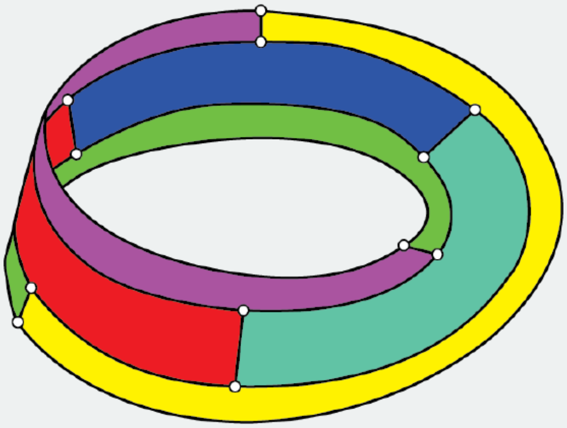
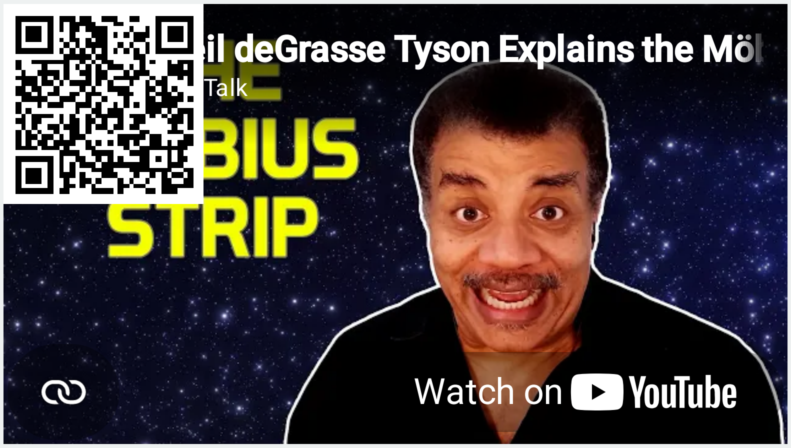


Figure 12.4.40: A Five-Coloring on a Möbius Strip

Video



> Neil deGrasse Tyson Explains the Möbius Strip

Check Your Understanding

For the following exercises, determine whether each statement is always true or sometimes true.

1. A trail is a path.
2. A trail is a walk.
3. A walk is a path.
4. A circuit is a trail.
5. A directed cycle is a path.
6. A circuit is a directed cycle.
7. A directed cycle is a circuit.
8. If a graph has an n-coloring, then its chromatic number is n.
9. If the chromatic number of a graph is n, then it has an n-coloring.
10. If a graph is planar, then it has a chromatic number of at most four.

For the following exercise, fill in the blanks to make the statement true.

11. A walk that _____________ is a trail.
12. A trail that _____________ is a circuit.
13. A circuit that ____________ is a directed cycle.
14. A closed walk that _____________ is a circuit.
15. A complete graph with n vertices has a chromatic number of_____.
16. A graph with a clique with n-vertices has a chromatic number of __________ n.

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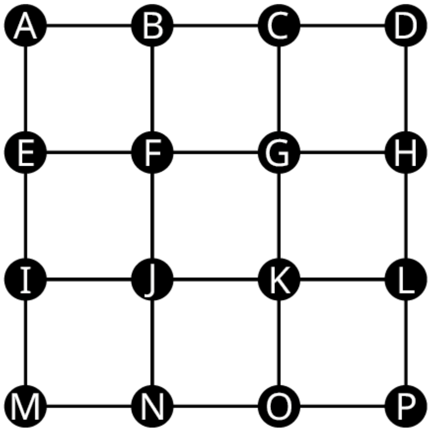
12.4.0: Exercises

For the following exercises, identify each sequence of vertices from the figure as a walk, trail, and/or path. Select all that apply.



Exercise 12.4.0.1

$A \to B \to F \to G \to K \to J \to F \to B$

Exercise 12.4.0.2

$G \to K \to O \to N \to J \to K \to L$

Exercise 12.4.0.3

$F \to J \to K \to G \to B \to A$

Exercise 12.4.0.4

$I \to J \to K \to L \to K \to J \to N$

Exercise 12.4.0.5

$M \to N \to O \to K \to L \to H$

Exercise 12.4.0.6

$A \to F \to K \to P$

Exercise 12.4.0.7

$N \to J \to F \to B \to C \to G \to F \to E$

Exercise 12.4.0.8

$E \to F \to J \to I \to E$

For the following exercises, identify each sequence of vertices from Figure 12.134 as a closed walk, circuit (closed trail), and/or directed cycle (closed path). Select all that apply.

Exercise 12.4.0.9

$A \rightarrow B \rightarrow F \rightarrow G \rightarrow K \rightarrow J \rightarrow F \rightarrow B \rightarrow A$

Exercise 12.4.0.10

$G \rightarrow K \rightarrow O \rightarrow N \rightarrow J \rightarrow K \rightarrow L$

Exercise 12.4.0.11

$F \rightarrow J \rightarrow N \rightarrow O \rightarrow K \rightarrow J \rightarrow I \rightarrow E \rightarrow F$

Exercise 12.4.0.12

$I \rightarrow J \rightarrow K \rightarrow G \rightarrow F \rightarrow E \rightarrow I$

Exercise 12.4.0.13

$M \rightarrow N \rightarrow O \rightarrow K \rightarrow J \rightarrow I \rightarrow M$

Exercise 12.4.0.14

$N \rightarrow J \rightarrow F \rightarrow B \rightarrow C \rightarrow G \rightarrow F \rightarrow J \rightarrow N$

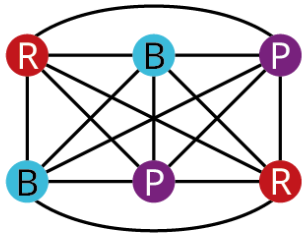
Exercise 12.4.0.15

$A \rightarrow B \rightarrow G \rightarrow F \rightarrow E \rightarrow A$

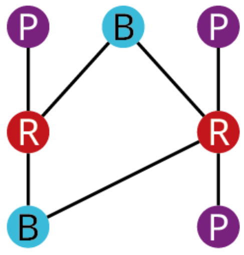
Exercise 12.4.0.16

$E \rightarrow F \rightarrow G \rightarrow K \rightarrow J \rightarrow F \rightarrow B \rightarrow A \rightarrow E$

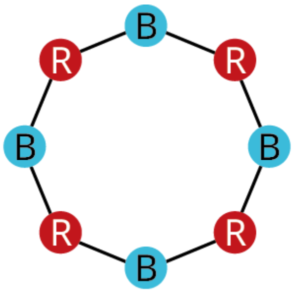
For the following exercises, use the graphs shown. Identify the graph or graphs with the given characteristics.



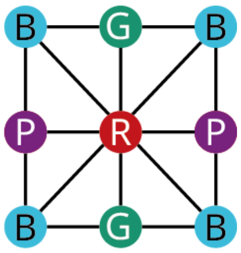
Graph 1



Graph 2



Graph 3



Graph 4

Exercise 12.4.0.17

The colors do not follow the definition of a graph coloring.

Exercise 12.4.0.18

The graph is a two-coloring.

Exercise 12.4.0.19

The graph is a three-coloring.

Exercise 12.4.0.20

The graph is a four-coloring.

Exercise 12.4.0.21

The graph is planar.

Exercise 12.4.0.22

The graph has a chromatic number of two.

Exercise 12.4.0.23

Fewer colors could be used to color the graph.

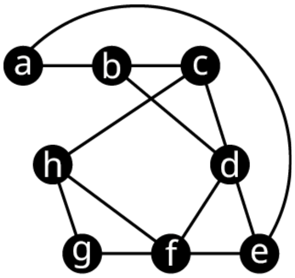
Exercise 12.4.0.24

The chromatic number is more than four.

Exercise 12.4.0.25

The Four-Color Theorem applies to the graph.

For the following exercises, complete the sequence of vertices from the graph to create the indicated kind of closed walk.



Exercise 12.4.0.26

Closed path: $a \rightarrow b \rightarrow \square \rightarrow \square \rightarrow \square \rightarrow a$

Exercise 12.4.0.27

Closed path that visits edge ed twice: $e \rightarrow d \rightarrow \square \rightarrow \square \rightarrow \square \rightarrow e$

Exercise 12.4.0.28

Directed cycle: $e \rightarrow a \rightarrow \square \rightarrow \square \rightarrow \square \rightarrow e$

Exercise 12.4.0.29

Directed cycle: $b \to d \to \square \to \square \to \square \to b$

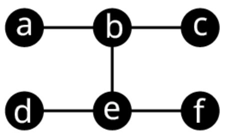
Exercise 12.4.0.30

Circuit that visits vertex d twice: $b \to c \to d \to \square \to \square \to \square \to b$

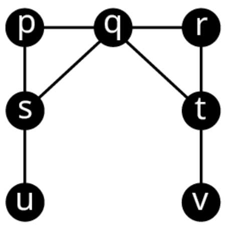
Exercise 12.4.0.31

Circuit that visits vertex f twice: $g \to h \to f \to \square \to \square \to \square \to g$

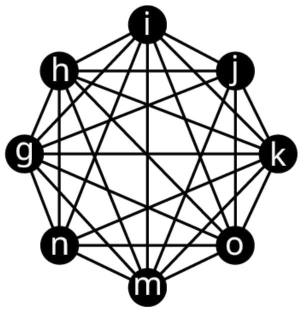
For the following exercises, use the graphs shown.



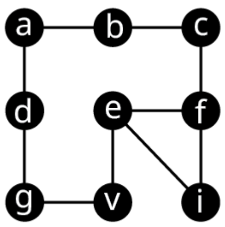
Graph 1



Graph 2



Graph 3



Graph 4

Exercise 12.4.0.32

Find the chromatic number n for graph 1 and give an n-coloring.

Exercise 12.4.0.33

Find the chromatic number n for graph 2 and give an n-coloring.

Exercise 12.4.0.34

Find the chromatic number n for graph 3 and give an n-coloring.

Exercise 12.4.0.35

Find the chromatic number n for graph 4 and give an n-coloring.

For the following exercises, indicate the smallest and largest possible chromatic number of the graph described.

Exercise 12.4.0.36

The graph has 15 vertices and contains a clique with 9 vertices.

Exercise 12.4.0.37

A planar graph with 100 vertices and a clique with 3 vertices.

Exercise 12.4.0.38

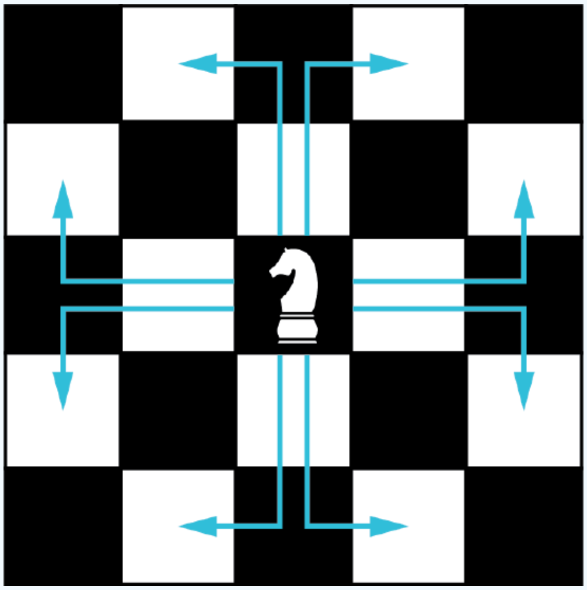
A planar graph with more than 2 vertices and no cliques.

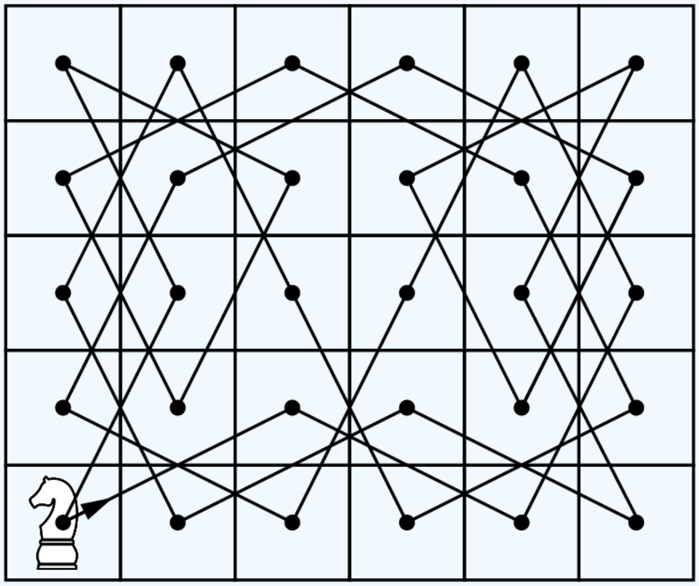
Exercise 12.4.0.39

A complete graph with 2,123 vertices.

Exercise 12.4.0.40

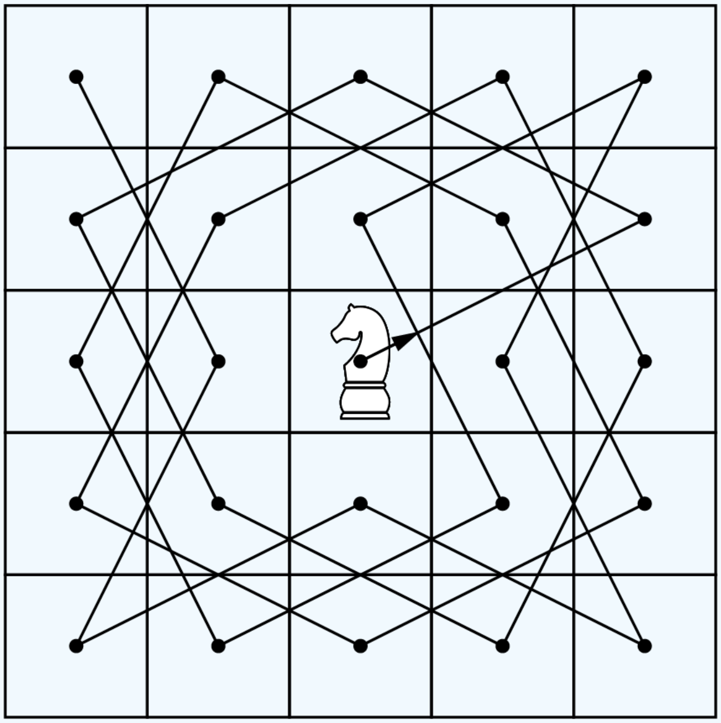
In chess, a knight can move in any direction, but it must move two spaces then turn and move one more space. The 8 possible moves a knight can make from a space in the center of a five-by-five grid are shown in the first figure. A knight’s tour is a sequence of moves by a knight on a chessboard (of any size) such that the knight visits every square exactly once. If the knight’s tour brings the knight back to its starting position on the board, it is called a closed knight’s tour. Otherwise, it is called an open knight’s tour. Determine if the closed knight’s tour in the second figure is most accurately described as a closed walk, a circuit, or a directed cycle. Explain your reasoning.





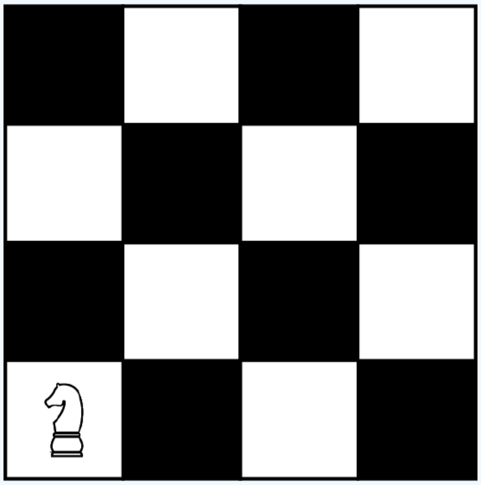
Exercise 12.4.0.41

An open knight’s tour on a five-by-five board is shown in the figure. Is it most accurately described as a walk, a trail, or a path?



Exercise 12.4.0.42

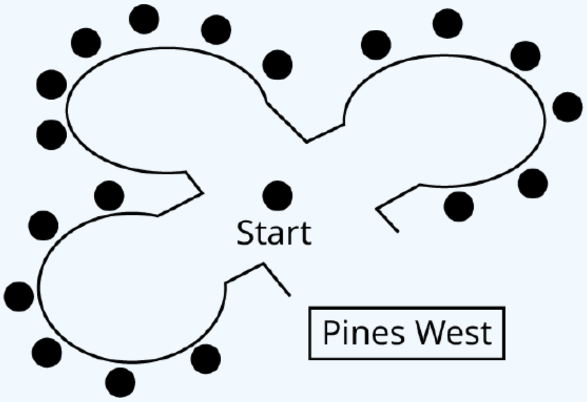
A knight’s tour is not possible on a four-by-four board like the one shown in the figure. Find an open tour of the board in which the knight is permitted to travel through a given space more than once to achieve the goal of visiting every space. Is your tour most accurately described as a walk, trail, or path? Explain why.



The neighborhood of Pines West has three cul-de-sacs that meet at an intersection as shown in the figure. A postal delivery person starts at the intersection and visits each house in a cul-de-sac once, returns to the intersection, visits each house in the next cul-de-sac, and so on, returning to the intersection when finished.

? Exercise 12.4.0.43

Describe how the route can be represented as a graph. If there is no backtracking, in other words, the person never reverses direction, is the route followed by the postal delivery person best described as a walk, trail, path, closed walk, circuit, or directed cycle? Explain your reasoning.

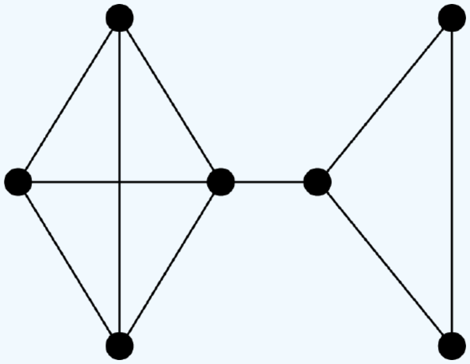


? Exercise 12.4.0.44

Draw the graph for the route described in the neighborhood. Find its chromatic number. Find a map coloring that shows the chromatic number.

? Exercise 12.4.0.45

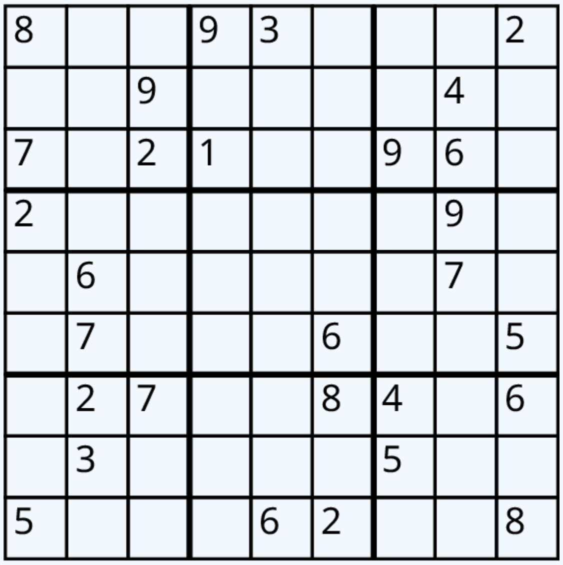
When radio towers are within range of each other, they must use different frequencies. In the graph, the vertices represent the towers, and the edges indicate the towers are in range of each other. Use graph colorings to determine the number of frequencies needed to avoid two towers in the same range being assigned the same frequency. Give an example of a coloring to support your conclusion.



? Exercise 12.4.0.46

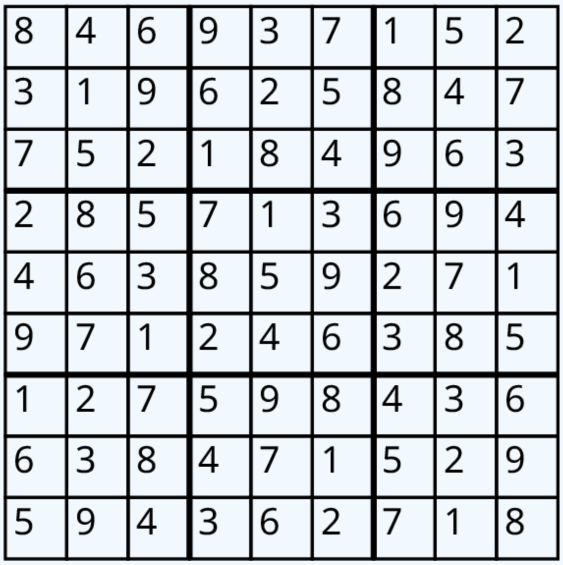
A Sudoku puzzle consists of a nine-by-nine grid which is subdivided into nine three-by-three smaller grids. Given an incomplete grid, the goal is to complete it in such a way that each row contains the numbers 1 through 9, each column contains the numbers 1 through 9, and each three by three grid contains the numbers 1 through 9 as shown in the figure. If a graph were drawn with a vertex to represent each entry in the grid, what should the edges represent so that a nine-coloring of the graph would be a solution to the puzzle?

Sudoku Puzzle



8			9	3				2
		9					4	
7		2	1			9	6	
2							9	
	6						7	
	7				6			5
	2	7			8	4		6
	3					5		
5				6	2			8

Solution to Puzzle



8	4	6	9	3	7	1	5	2
3	1	9	6	2	5	8	4	7
7	5	2	1	8	4	9	6	3
2	8	5	7	1	3	6	9	4
4	6	3	8	5	9	2	7	1
9	7	1	2	4	6	3	8	5
1	2	7	5	9	8	4	3	6
6	3	8	4	7	1	5	2	9
5	9	4	3	6	2	7	1	8

At a wedding reception, the bride and groom would like their family and friends to get to know each other better. They have decided that if two people know each other, they will not seat them at the same table. The following is a list of guests and who they know. Guests A, B, C, and D all know each other; guests E, F, G, and H all know each other; guests I, J, and K all know each other; guests P, Q and R all know each other, guests L, M, and O all know each other. Additionally, I knows D and I knows G, J knows M, K knows P, N knows L, and N knows O.

Exercise 12.4.0.47

Draw a graph to show the relationships.

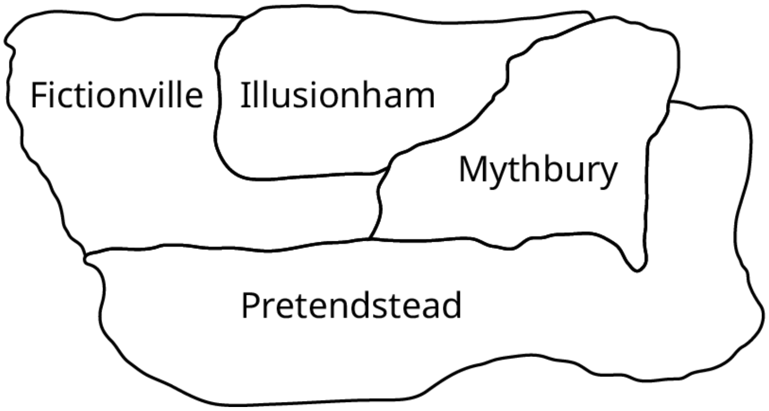
Exercise 12.4.0.48

Determine the minimum number of tables needed to seat the guests so that they will sit with people they don't know.

? Exercise 12.4.0.49

Give a coloring to support your conclusion.

The map of the states of Imaginaria is given. Use the figure to answer the following exercises.



? Exercise 12.4.0.50

Draw a graph to represent the map.

? Exercise 12.4.0.51

Determine the chromatic number of the graph you found.

? Exercise 12.4.0.52

Give a graph coloring that supports your answer in the previous question.

? Exercise 12.4.0.53

Color the provided map to correspond with the coloring you found in the previous question.

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12.5: Euler Circuits

Figure $12.5.1$: Delivery trucks move goods from place to place. (credit: “Mack Midliner” by Jason Lawrence/Flickr, CC BY 2.0)

Learning Objectives

1. Determine if a graph is connected.
2. State the Chinese postman problem.
3. Describe and identify Euler Circuits.
4. Apply the Euler Circuits Theorem.
5. Evaluate Euler Circuits in real-world applications.

The delivery of goods is a huge part of our daily lives. From the factory to the distribution center, to the local vendor, or to your front door, nearly every product that you buy has been shipped multiple times to get to you. If the cost and time of that delivery is too great, you will not be able to afford the product. Delivery personnel have to leave from one location, deliver the goods to various places, and then return to their original location and do all of this in an organized way without losing money. How do delivery services find the most efficient delivery route? The answer lies in graph theory.

Connectedness

Before we can talk about finding the best delivery route, we have to make sure there is a route at all. For example, suppose that you were tasked with visiting every airport on the graph in Figure $12.5.2$ by plane. Could you accomplish that task, only taking direct flight paths between airports listed on this graph? In other words, are all the airports connected by paths? Or are some of the airports disconnected from the others?

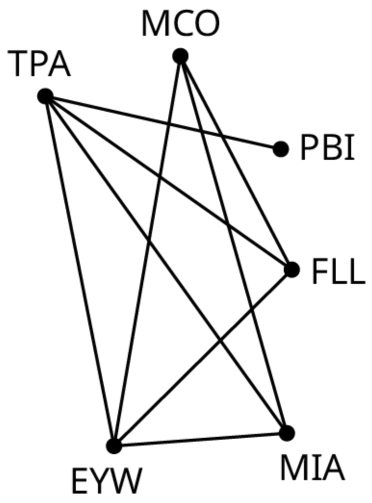


Figure $12.5.2$: Major Central and South Florida Airports

In Figure 12.118, we can see TPA is adjacent to PBI, FLL, MIA, and EYW. Also, there is a path between TPA and MCO through FLL. This indicates there is a path between each pair of vertices. So, it is possible to travel to each of these airports only taking direct flight paths and visiting no other airports. In other words, the graph is **connected** because there is a path joining every pair of vertices on the graph. Notice that if one vertex is connected to each of the others in a graph, then all the vertices are connected to each other. So, one way to determine if a graph is connected is to focus on a single vertex and determine if there is a path between that vertex and each of the others. If so, the graph is connected. If not, the graph is **disconnected,** which means at least one pair of vertices in a graph is not joined by a path. Let's take a closer look at graph X in Figure $12.5.3$. Focus on vertex a. There is a path between vertices a and b, but there is no path between vertex a and vertex c. So, Graph X is disconnected.

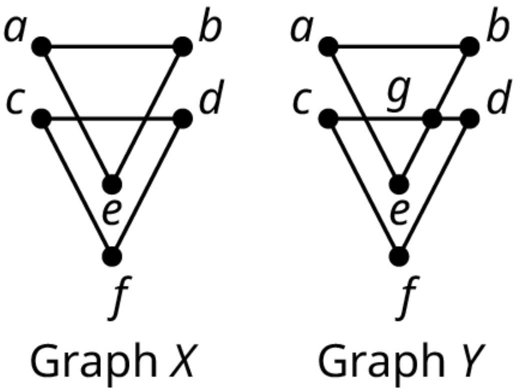


Figure $12.5.3$: Connected vs. Disconnected

When you are working with a planar graph, you can also determine if a graph is connected by untangling it. If you draw it so that so that none of the edges are overlapping, like we did with Graph X in Figure $12.5.4$, it is easier to see that the graph is disconnected.

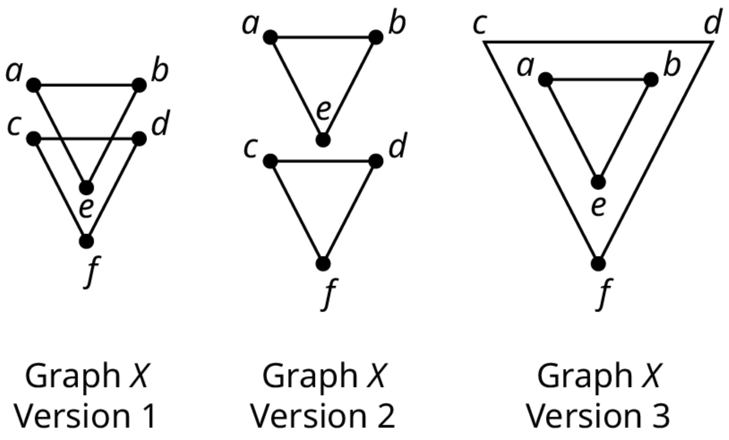


Figure 12.5.4: Untangling Graph X

Versions 2 and 3 of Graph *X* in Figure 12.5.4 each have the same number of vertices, number of edges, degrees of the vertices, and pairs of adjacent vertices as version 1. In other words, each version retains the same structure as the original graph. Since versions 2 and 3 of Graph *X*, do not have overlapping edges, it is easier to identify pairs of vertices that do not have paths between them, and it is more obvious that Graph *X* is disconnected. In fact, there are two completely separate, disconnected subgraphs, one with the vertices in $\{a, b, e\}$, and the other with the vertices $\{c, d, f\}$

These sets of vertices together with all of their edges are called **components** of Graph *X*. A component of a graph is a subgraph in which there is a path between each pair of vertices in the subgraph, but no edges between any of the vertices in the subgraph and a vertex that is not in the subgraph.

Now let's focus on Graph *Y* from Figure 12.5.3 . As in Graph *X*, there is a path between vertices *a* and *b*, as well as between vertices *a* and *e*, but Graph *Y* is different from Graph *X* because of vertex *g*. Not only is there a path between vertices *a* and *g*, but vertex *g* bridges the gap between *a* and *c* with the path $a \to b \to g \to c$. Similarly, there is a path between vertices *a* and *d* and vertices *a* and *f*: $a \to b \to g \to d$, $a \to b \to g \to d \to f$. Since there is a path between vertex *a* and every other vertex, Graph *Y* is connected. You can also see this a bit more clearly by untangling Graph *Y* as in Figure 12.121. Even when *Y* is drawn so that the edges do not overlap, the graph cannot be drawn as two separate, unconnected pieces. In other words, Graph *Y* has only one component with the vertices $\{a, b, c, d, e, f\}$.

We can give an alternate definition of **connected** and **disconnected** using the idea of components. A graph is connected if it has only one component. A graph is disconnected if it has more than one component. These alternate definitions are equivalent to the previous definitions. This means that you can confirm a graph is connected or disconnected either by checking to see if there is a path between each vertex and each other vertex, or by identifying the number of components. A graph is connected if it has only one component.

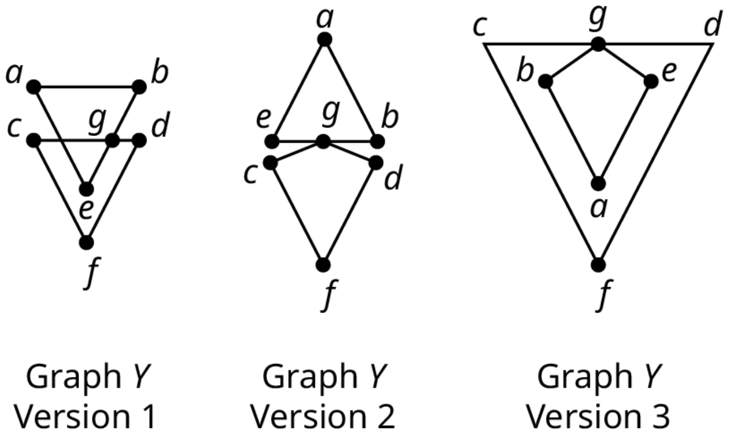


Figure 12.5.5: Untangling Graph Y

✓ Example 12.5.1: Determining If a Graph Is Connected or Disconnected

Use Figure ✓ 12.5.6 to answer each question.

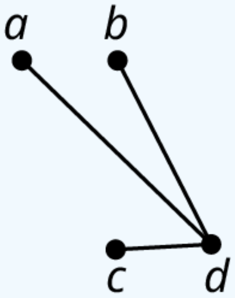


Figure ✓ 12.5.6 Graph *E*

1. Find a path between vertex *a* and every other vertex on the graph, if possible.
2. Identify all the components of Graph *E*.
3. Determine whether the graph is connected or disconnected and explain how you know.

Answer

1. The paths are $a \to d \to b$, $a \to d \to c$, and $a \to d$.
2. There is only one component in Graph *E*. It has the vertices $\{a, b, c, d\}$.
3. The graph is connected, because there is a path between vertex *a* and every other vertex. We can also so that Graph *E* is connected because it has only one component.

Your Turn 12.5.1

Determine whether each graph is connected or disconnected and identify the set of vertices that make up each of its components.

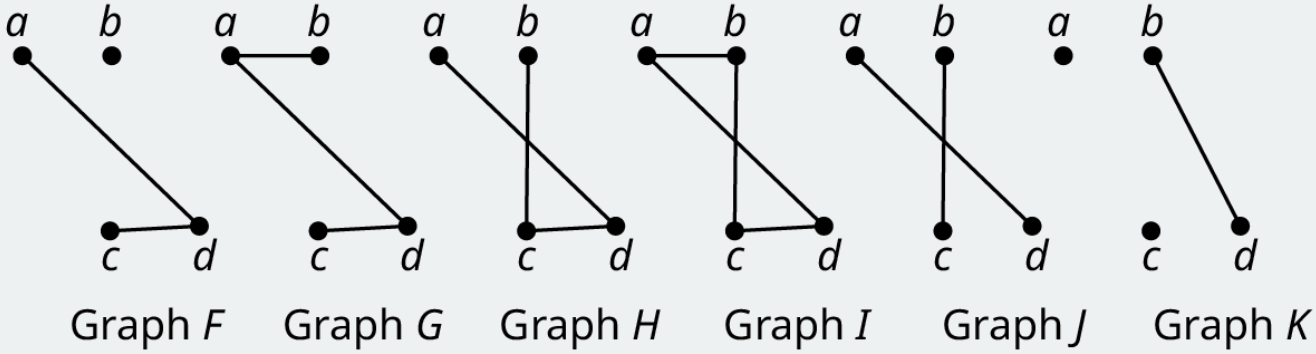
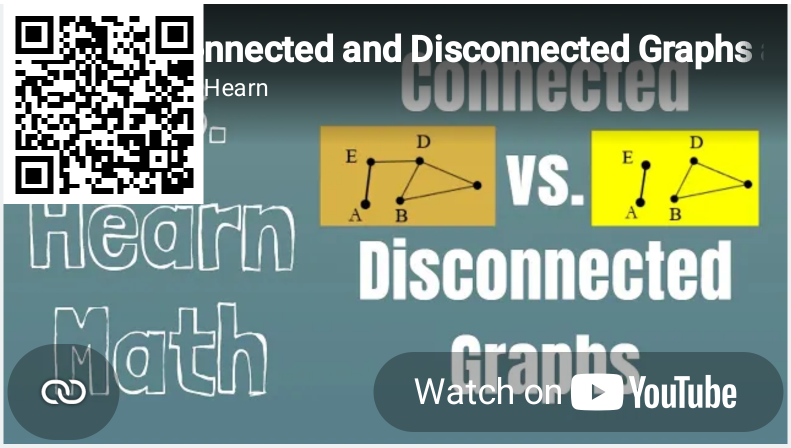


Figure 12.5.7

Video



Connected and Disconnected Graphs in Graph Theory

Example $12.5.2$: Applying Connectedness

The U.S. Interstate Highway System extends throughout the 48 contiguous states. It also has routes in the states of Hawaii and Alaska, and the commonwealth of Puerto Rico. Consider a graph representing the U.S. Interstate Highway System, in which there is a vertex for each of the 50 states and Puerto Rico, and an edge is drawn between any two vertices representing states that are connected by a highway in that system. Would this graph be connected or disconnected? Explain your reasoning.

Answer

Hawaii, Alaska, and Puerto Rico are geographically separate from the 48 contiguous states, and each from each other. This means that there are vertices on the graph with no path joining them to the other vertices. So the graph is disconnected.

Your Turn $12.5.2$

“Six degrees of separation” is the idea that any two people on Earth are connected by at most six social connections. Assume that this is true. Consider a graph in which each vertex is a person on Earth, and each edge is a social connection. Would this graph be connected or disconnected? Explain your reasoning.

Origin of Euler Circuits

The city of Konigsberg, modern day Kaliningrad, Russia, has waterways that divide up the city. In the 1700s, the city had seven bridges over the various waterways. The map of those bridges is shown in Figure $12.5.8$. The question as to whether it was possible to find a route that crossed each bridge exactly once and return to the starting point was known as the Konigsberg Bridge Problem. In 1735, one of the most influential mathematicians of all time, Leonard Euler, solved the problem using an area of mathematics that he created himself, graph theory!

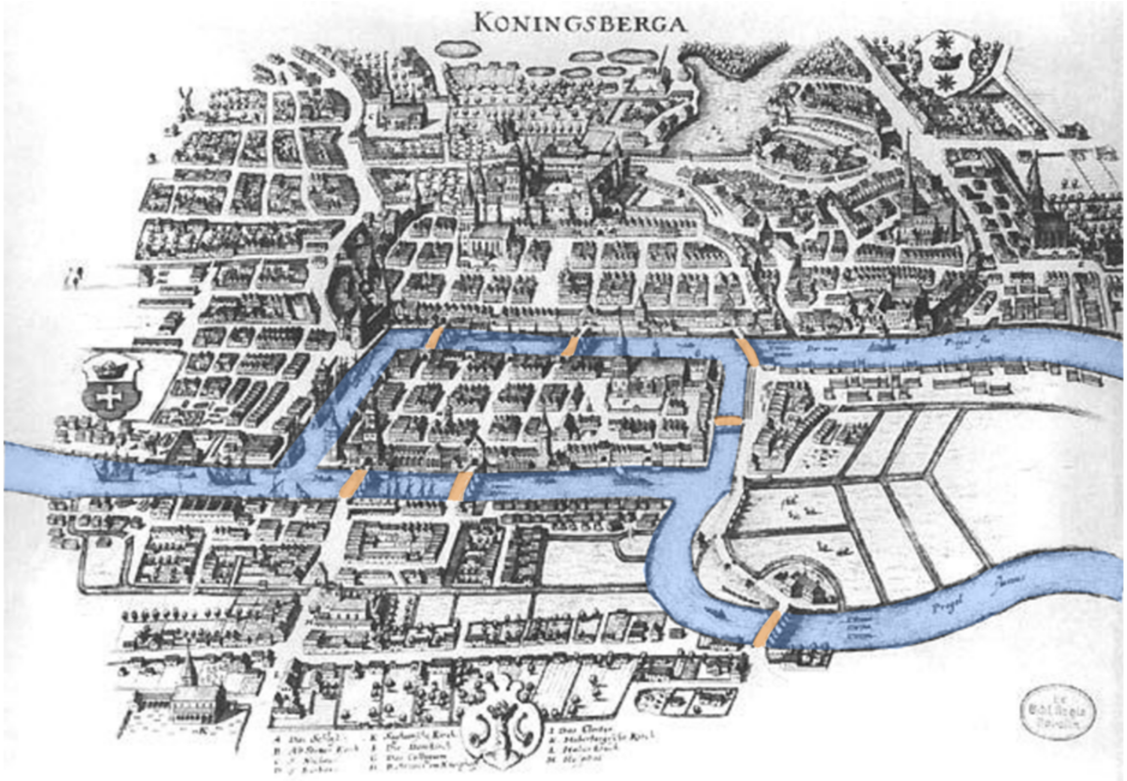

KONINGSBERGA

Figure $12.5.8$: Map of the Bridges of Konigsberg in 1700s (credit: “Konigsberg Bridge” by Merian Erben/Wikimedia Commons, Public Domain)

Euler drew a multigraph in which each vertex represented a land mass, and each edge represented a bridge connecting them, as shown in Figure $12.5.9$. Remember from Navigating Graphs that a circuit is a trail, so it never repeats an edge, and it is closed, so it begins and ends at the same vertex. Euler pointed out that the Konigsberg Bridge Problem was the same as asking this graph theory question: Is it possible to find a circuit that crosses every edge? Since then, circuits (or closed trails) that visit every edge in a graph exactly once have come to be known as Euler circuits in honor of Leonard Euler.

Video

cognizing Euler Trails and Euler Cir
Hearn
Recognizing
Euler Trails
& Euler Circuits
in Graph Theory
Hearn
Math
Watch on YouTube

Recognizing Euler Trails and Euler Circuits

Euler was able to prove that, in order to have an Euler circuit, the degrees of all the vertices of a graph have to be even. He also proved that any graph with that characteristic must have an Euler circuit. So, saying that a connected graph is Eulerian is the same as saying it has vertices with all even degrees, known as the **Eulerian circuit theorem**.

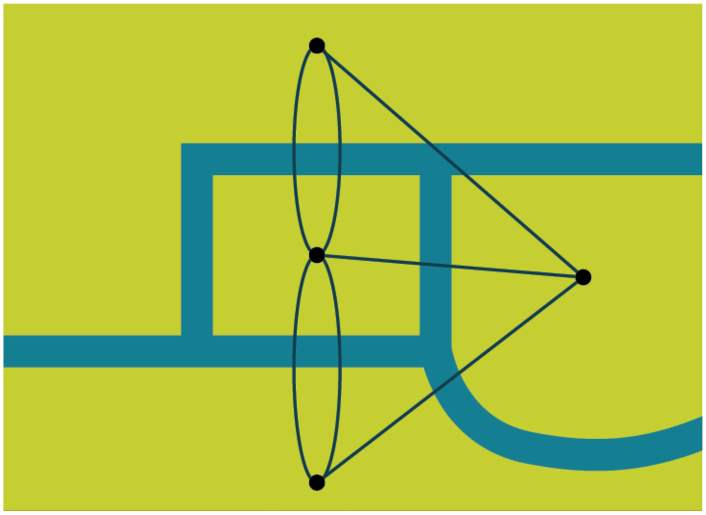


Figure 12.5.9: Graph of Konigsberg Bridges

To understand why the Euler circuit theorem is true, think about a vertex of degree 3 on any graph, as shown in Figure 12.5.10 .

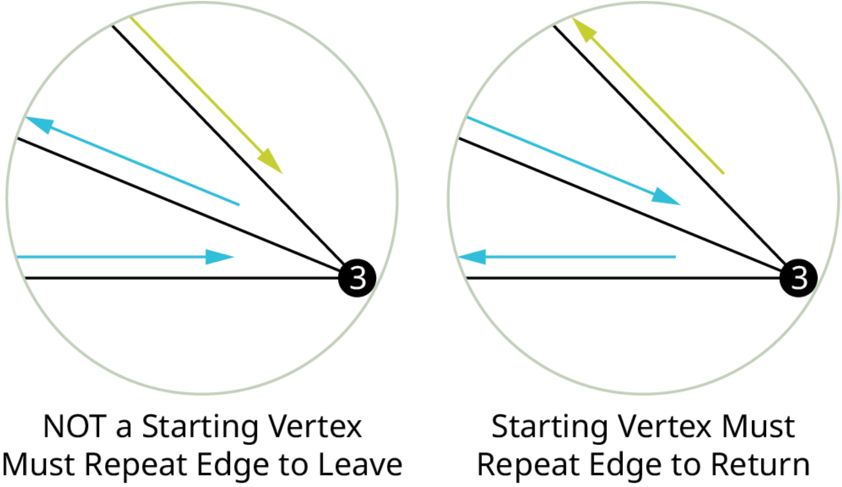


Figure 12.5.10: A Vertex of Degree 3

First imagine the vertex of degree 3 shown in Figure 12.5.10 is not the starting vertex. At some point, each edge must be traveled. The first time one of the three edges is traveled, the direction will be toward the vertex, and the second time it will be away from the vertex. Then, at some point, the third edge must be traveled coming in toward the vertex again. This is a problem, because the only way to get back to the starting vertex is to then visit one of the three edges a second time. So, this vertex cannot be part of an Euler circuit.

Next imagine the vertex of degree 3 shown in Figure 12.5.10 is the starting vertex. The first time one of the edges is traveled, the direction is away from the vertex. At some point, the second edge will be traveled coming in toward the vertex, and the third edge will be the way back out, but the starting vertex is also the ending vertex in a circuit. The only way to return to the vertex is now to travel one of the edges a second time. So, again, this vertex cannot be part of an Euler circuit.

For the same reason that a vertex of degree 3 can never be part of an Euler circuit, a vertex of any odd degree cannot either. We can use this fact and the graph in Figure 12.5.11 to solve the Konigsberg Bridge Problem. Since the degrees of the vertices of the graph in Figure 12.5.10 are not even, the graph is not Eulerian and it cannot have an Euler circuit. This means it is not possible to travel through the city of Konigsberg, crossing every bridge exactly once, and returning to your starting position.

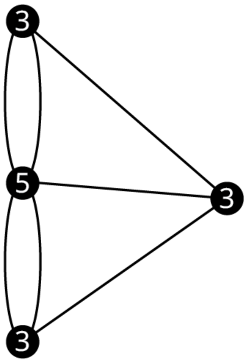


Figure 12.5.11: Degrees of Vertices in Konigsberg Bridge Multigraph

Video

Existence of Euler Circuits in Graph Theory

Chinese Postman Problem

At Camp Woebegone, campers travel the waterways in canoes. As part of the Camp Woebegone Olympics, you will hold a canoeing race. You have placed a checkpoint on each of the 11 different streams. The competition requires each team to travel each stream, pass through the checkpoints in any order, and return to the starting line, as shown in the Figure 12.5.12 .

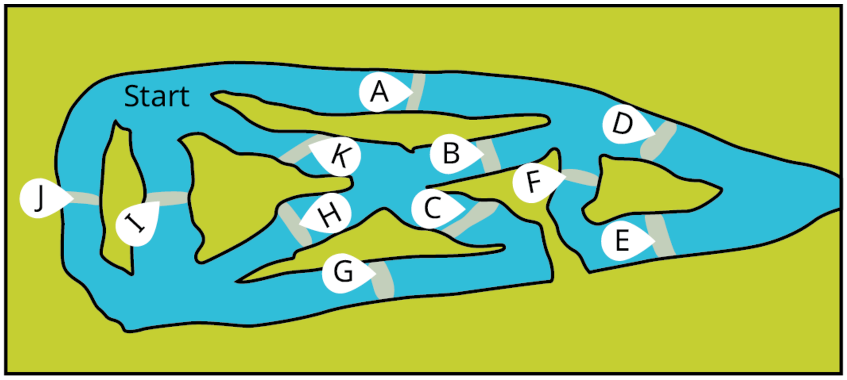


Figure 12.5.12: Map of Canoe Event Checkpoints

Since the teams want to go as fast as possible, they would like to find the shortest route through the course that visits each checkpoint and returns to the starting line. If possible, they would also like to avoid backtracking. Let’s visualize the course as a multigraph in which the vertices represent turns and the edges represent checkpoints as in Figure 12.5.13 .

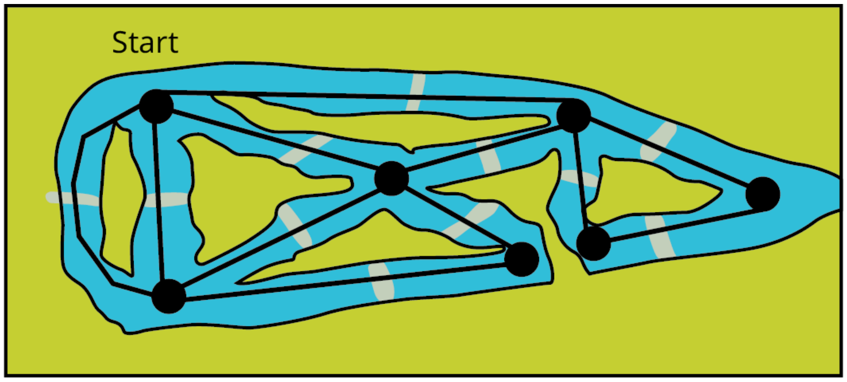


Figure 12.5.13: Multigraph of Canoe Event

The teams would like to find a closed walk that repeats as few edges as possible while still visiting every edge. If they never repeat an edge, then they have found a closed trail, which is a circuit. That circuit must cover all edges; so, it would be an Euler circuit. The task of finding a shortest circuit that visits every edge of a connected graph is often referred to as the **Chinese postman**

problem. The name Chinese postman problem was coined in honor of the Chinese mathematician named Kwan Mei-Ko in 1960 who first studied the problem.

If a graph has an Euler circuit, that will always be the best solution to a Chinese postman problem. Let’s determine if the multigraph of the course has an Euler circuit by looking at the degrees of the vertices in Figure 12.5.14 . Since the degrees of the vertices are all even, and the graph is connected, the graph is Eulerian. It is possible for a team to complete the canoe course in such a way that they pass through each checkpoint exactly once and return to the starting line.

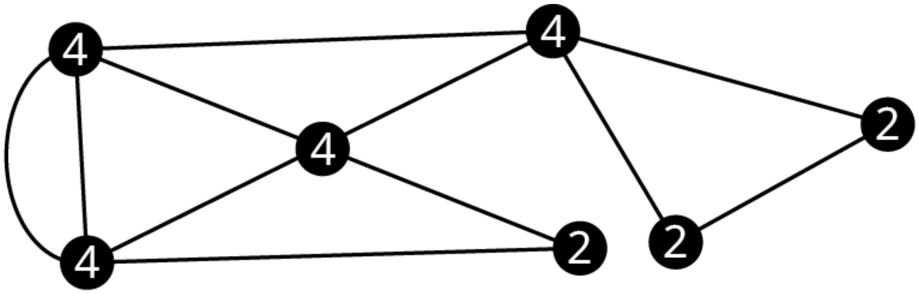


Figure 12.5.14: Degrees of Vertices in Graph of Canoe Event

✓ Example 12.5.3: Understanding Eulerian Graphs

A postal delivery person must deliver mail to every block on every street in a local subdivision. Figure ✓ 12.5.15 is a map of the subdivision. Use the map to answer each question.

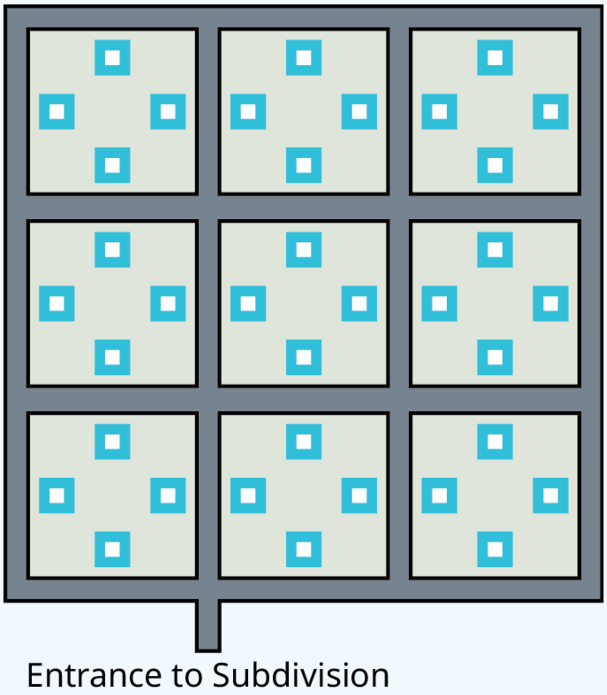


Figure ✓ 12.5.15 Map of Subdivision

1. Draw a graph or multigraph to represent the subdivision in which the vertices represent the intersections, and the edges represent streets.
2. Is your graph connected? Explain how you know.
3. Determine the degrees of the vertices in the graph.
4. Is your graph an Eulerian graph?
5. Is it possible for the postal delivery person to visit each block on each street exactly once? Justify your answer.

Answer

1. The graph is in Figure ✓ 12.5.16 .

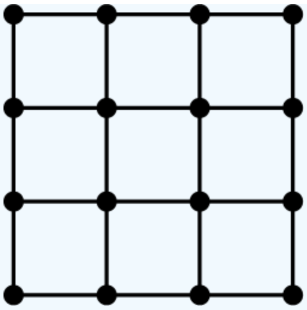


Figure ✓ 12.5.16 Graph of Subdivision

2. The graph is connected. It has only one component and there is a path between each pair of vertices.
3. There are four corner vertices of degree 2, there are eight exterior vertices of degree 3, and there are four interior vertices of degree 4.
4. The graph is not Eulerian because it has vertices of odd degree.
5. No. Since the graph is not Eulerian, there is no Euler circuit, which means that there is no route that would pass through every edge exactly once.

Your Turn 12.5.3

A pest control service has at least one customer on every block of every street or cul-de-sac in a neighborhood. Use the map of the neighborhood to answer each question.

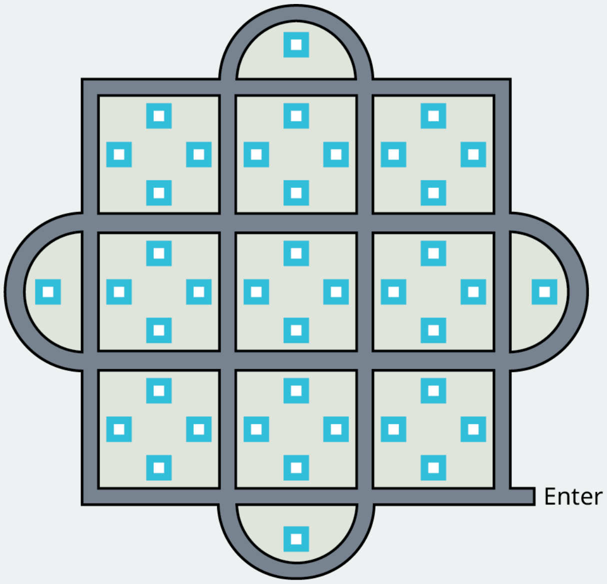


Figure 12.5.17: Map of Neighborhood

Draw a graph or multigraph of the neighborhood in which the vertices represent intersections, and the edges represent the streets between them.

1. Is your graph connected? Explain how you know.
2. Determine the degrees of the vertices in the graph.
3. Is your graph an Eulerian graph?
4. Is it possible for the postal delivery person to visit each block on each street exactly once and start and end at the same position? Justify your answer.

Identifying Euler Circuits

Solving the Chinese postman problem requires finding a shortest circuit through any graph or multigraph that visits every edge. In the case of Eulerian graphs, this means finding an Euler circuit. The method we will use is to find any circuit in the graph, then find a second circuit starting at a vertex from the first circuit that uses only edges that were not in the first circuit, then find a third circuit starting at a vertex from either of the first two circuits that uses only edges that were not in the first two circuits, and then continue this process until all edges have been used. In the end, you will be able to link all the circuits together into one large Euler circuit.

Let’s find an Euler circuit in the map of the Camp Woebegone canoe race. In Figure 12.5.18, we have labeled the edges of the multigraph so that the circuits can be named. In a multigraph it is necessary to name circuits using edges and vertices because there can be more than one edge between adjacent vertices.

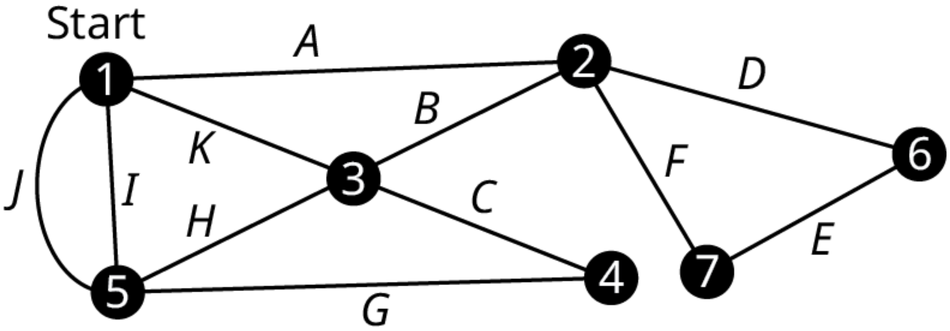


Figure 12.5.18: Multigraph of Canoe Race with Vertices and Edges Labeled

We will begin with vertex 1 because it represents the starting line in this application. In general, you can start at any vertex you want. Find any circuit beginning and ending with vertex 1. Remember, a circuit is a trail, so it doesn’t pass through any edge more than once. Figure 12.5.19 shows one possible circuit that we could use as the first circuit, $1 \to A \to 2 \to B \to 3 \to C \to 4 \to G \to 5 \to J \to 1$.

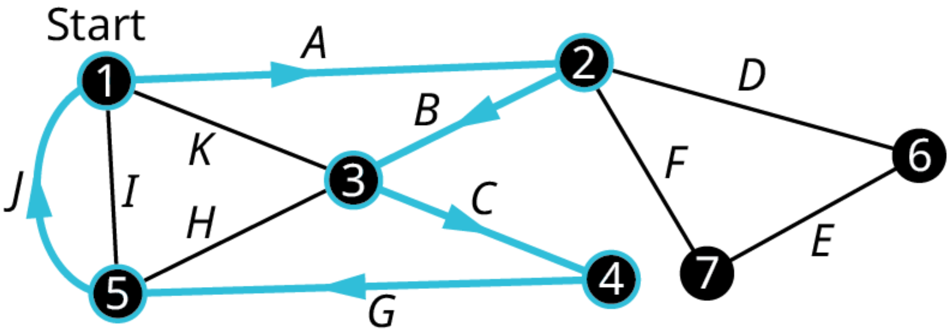


Figure 12.5.19: First Circuit

From the edges that remain, we can form two more circuits that each start at one of the vertices along the first circuit. Starting at vertex 3 we can use $3 \to H \to 5 \to I \to 1 \to K \to 3$ and starting at vertex 2 we can use $2 \to D \to 6 \to E \to 7 \to F \to 2$, as shown in Figure 12.5.20.

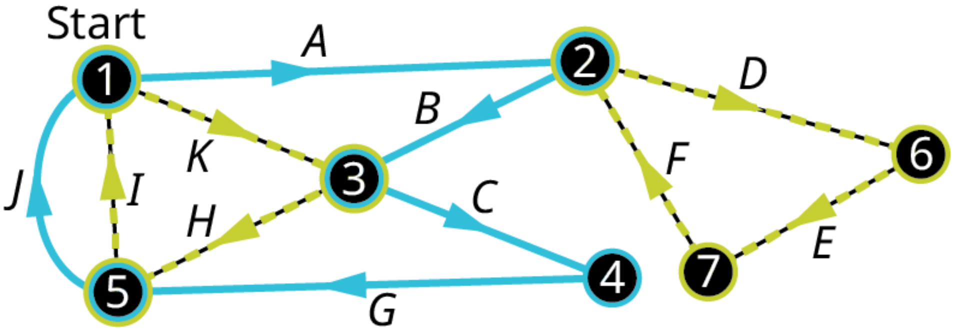


Figure 12.5.20: Second and Third Circuits

Now that each of the edges is included in one and only once circuit, we can create one large circuit by inserting the second and third circuits into the first. We will insert them at their starting vertices 2 and 3

$$1 \rightarrow A \rightarrow \boxed{2} \rightarrow B \rightarrow \boxed{3} \rightarrow C \rightarrow 4 \rightarrow G \rightarrow 5 \rightarrow J \rightarrow 1$$

becomes

$$1 \rightarrow A \rightarrow \boxed{2 \rightarrow D \rightarrow 6 \rightarrow E \rightarrow 7 \rightarrow F \rightarrow 2} \rightarrow B \rightarrow$$
$$\boxed{3 \rightarrow H \rightarrow 5 \rightarrow I \rightarrow 1 \rightarrow K \rightarrow 3} \rightarrow C \rightarrow 4 \rightarrow G \rightarrow 5 \rightarrow J \rightarrow 1$$

Finally, we can name the circuit using vertices, 1 → 2 → 6 → 7 → 2 → 3 → 5 → 1 → 3 → 4 → 5 → 1, or edges, A → D → E → F → B → H → I → K → C → G → J.

Let's review the steps we used to find this Eulerian Circuit.

Steps to Find an Euler Circuit in an Eulerian Graph

Step 1 - Find a circuit beginning and ending at any point on the graph. If the circuit crosses every edges of the graph, the circuit you found is an Euler circuit. If not, move on to step 2.

Step 2 - Beginning at a vertex on a circuit you already found, find a circuit that only includes edges that have not previously been crossed. If every edge has been crossed by one of the circuits you have found, move on to Step 3. Otherwise, repeat Step 2.

Step 3 - Now that you have found circuits that cover all of the edges in the graph, combine them into an Euler circuit. You can do this by inserting any of the circuits into another circuit with a common vertex repeatedly until there is one long circuit.

✓ Example 12.5.4: Finding an Euler Circuit

Use Figure ✓ 12.5.21 to answer each question.

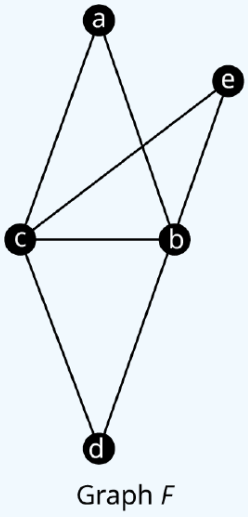


Figure ✓ 12.5.21 Graph F

1. Verify the Graph F is Eulerian.
2. Find an Euler circuit that begins and ends at vertex c.

Answer

1. The graph is connected because it has only one component. Also, each of the vertices in graph F has even degree as shown in Figure ✓ 12.5.22. So, the graph is Eulerian.

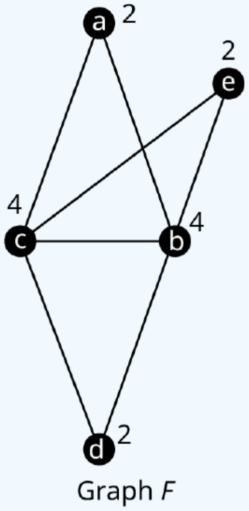


Figure ✓ 12.5.22 Degrees of Vertices in Graph F

2. **Step 1:** Beginning at vertex c, identify circuit $c \rightarrow e \rightarrow b \rightarrow c$ as shown in Figure ✓ 12.5.23. Since this circuit does not cover every edge in the graph, move on to Step 2.

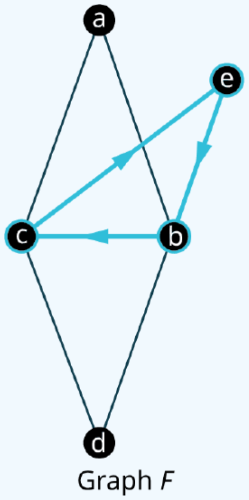


Figure ✓ 12.5.23: First Circuit in Graph F

Step 2: Find another circuit beginning at one of the vertices in the first circuit, using only edges that have not been used in the first circuit. It is possible to do this using either vertex c or vertex b. In Figure ✓ 12.5.24, we have used vertex b as the starting and ending point for a second circuit, $b \rightarrow d \rightarrow c \rightarrow a \rightarrow b$. Since all edges have been crossed by one of the two circuits, move on to Step 3.

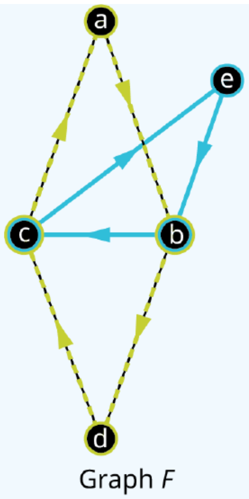


Figure ✓ 12.5.24: Second Circuit in Graph *F*

Step 3: Combine the two circuits into one. Replace vertex *b* in the first circuit with the whole second circuit that begins at vertex *b*.

$$c \to e \to \boxed{b} \to$$

cbecomesc→e→b→d→c→a→b→cc→e→b→cbecomesc→e→b→d→c→a→b→c.
An Euler circuit that begins at vertex *c* is $c \to e \to b \to d \to c \to a \to b \to c$.

Your Turn 12.5.4

Use Graphs *X* and *Y* to answer each question.

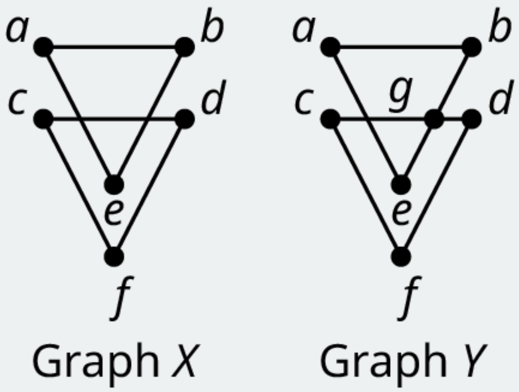


Figure 12.5.25

Which graph does not have an Euler circuit? Explain how you know.

Find an Euler circuit in the other graph.

Eulerization

The Chinese postman problem doesn't only apply to Eulerian graphs. Recall the postal delivery person who needed to deliver mail to every block of every street in a subdivision. We used the map in Figure 12.5.26 to create the graph in Figure 12.5.27.

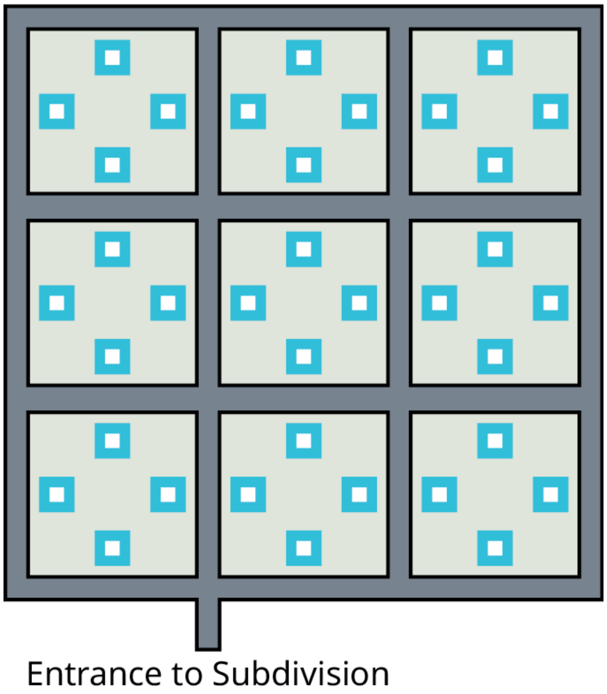


Figure 12.5.26: Map of Subdivision

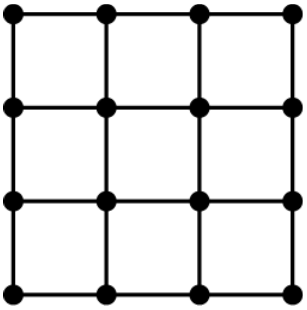


Figure 12.5.27: Graph of Subdivision

Since the graph of the subdivision has vertices of odd degree, there is no Euler circuit. This means that there is no route through the subdivision that visits every block of every street without repeating a block. What should our delivery person do? They need to repeat as few blocks as possible. The technique we will use to find a closed walk that repeats as few edges as possible is called **eulerization**. This method adds duplicate edges to a graph to create vertices of even degree so that the graph will have an Euler circuit.

In Figure 12.5.28, the eight vertices of odd degree in the graph of the subdivision are circled in green. We have added duplicate edges between the pairs of vertices, which changes the degrees of the vertices to even degrees so the resulting multigraph has an Euler circuit. In other words, we have eulerized the graph.

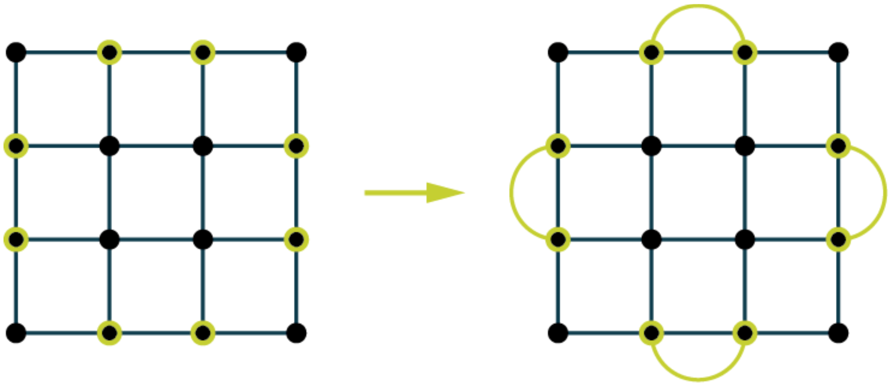


Figure 12.5.28: An Eulerized Graph

The duplicate edges in the eulerized graph correspond to blocks that our delivery person would have to travel twice. By keeping these duplicate edges to a minimum, we ensure the shortest possible route. It can be challenging to determine the fewest duplicate edges needed to eulerize a graph, but you can never do better than half the number of odd vertices. In the graph in Figure 12.5.28, we have found a way to fix the eight vertices of odd degree with only four duplicate edges. Since four is half of eight, we will never do better.

Checkpoint

IMPORTANT! The duplicate edges that we add indicate places where the route will pass twice. An entirely new edge between two vertices that were not previously adjacent would indicate that our postal delivery person created a new road through someone's property! So, ***we can duplicate existing edges, but we cannot create new ones****.*

✓ Example 12.5.5: Eulerizing Graphs

Use Graph A and multigraphs B, C, D, and E given in Figure ✓ 12.5.29 to answer the questions.

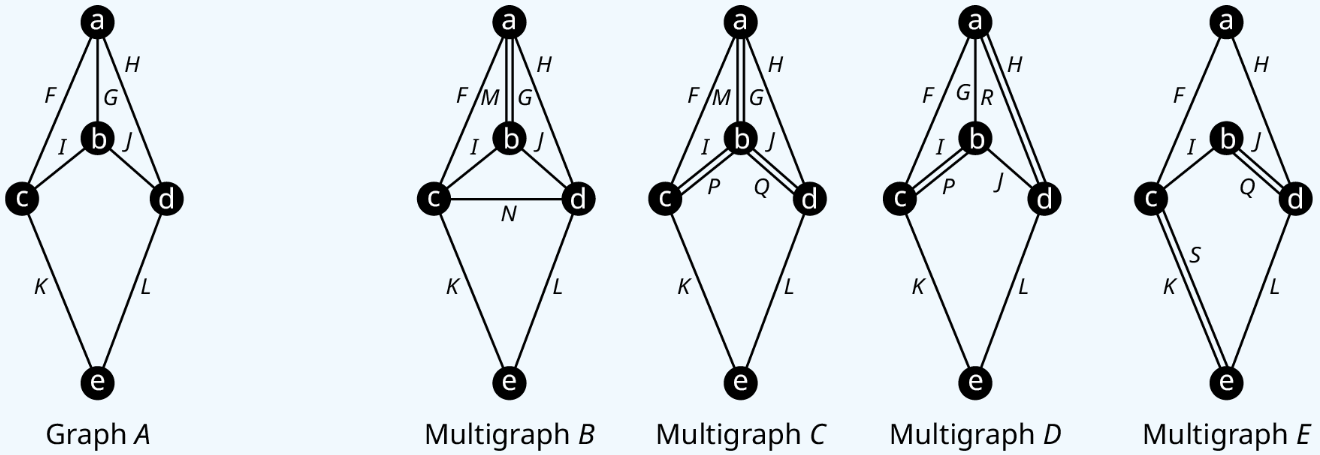


Figure ✓ 12.5.29 Graph A and Multigraphs B through E

1. Which of the multigraphs are not eulerizations of Graph A? Explain your answer.
2. Which eulerization of Graph A uses the fewest duplicate edges? How many does it use?
3. Is it possible to eulerize Graph A using fewer duplicate edges than your answer to part 2? If so, give an example. If not, explain why not.

Answer

1. Multigraph B is not an eulerization of A because the edge N between vertices c and d is not a duplicate of an existing edge. Multigraph E is not an eulerization of A because vertices b and e have odd degree.
2. Multigraph C uses 3 duplicate edges while multigraph D only uses 2. So, D uses the fewest.
3. Since there were four vertices in Graph A, the fewest number of edges that could possibly eulerize the graph is half of four, which is two. So, it is not possible to eulerize Graph A using fewer edges.

Your Turn 12.5.5

Create an eulerization using the fewest duplicate edges possible for Graph Z.

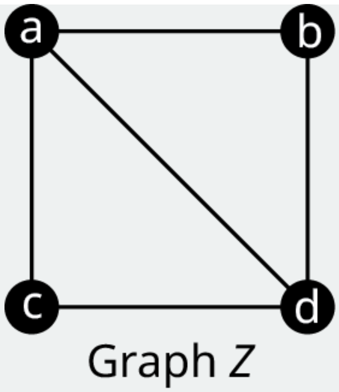

a
b
c
d
Graph Z

Figure 12.5.30: Graph Z

Checkpoint

IMPORTANT! Since only duplicate edges can be added to eulerize a graph, it is not possible to eulerize a disconnected graph.

WORK IT OUT

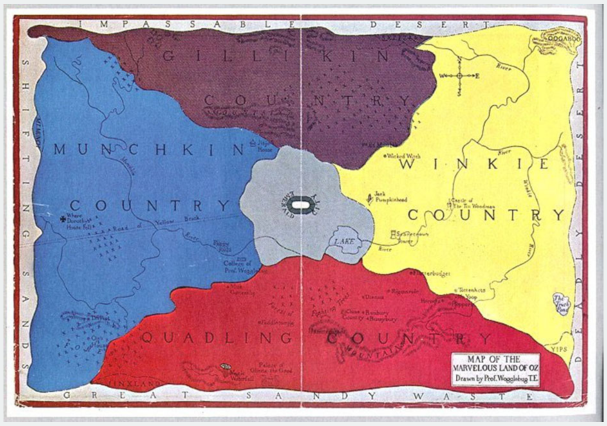

IMPASSABLE DESERT
GILLIKIN COUNTRY
MUNCHKIN COUNTRY
WINKIE COUNTRY
QUADLING COUNTRY
SHIFTING SANDS
DEADLY DESERT
GREAT SANDY WASTE
OZ
LAKE
MAP OF THE MARVELOUS LAND OF OZ
Drawn by Prof. Wogglebug T.E.

Figure 12.5.31: Map of the Magical Land of Oz (credit: "Map of Oz within the surrounding deserts" by L. Frank Baum/Wikimedia Commons, Public Domain)

In *The Wonderful Wizard of Oz*, written by L. Frank Baum and illustrated by W. W. Denslow, the region of the Emerald City lies at the center of the magical land of Oz, with Gillikin Country to the north, Winkie Country to the east, Munchkin Country to the west, and Quadling Country to the south. Munchkin Country and Winkie Country each shares a border with Gillikin Country and Quadling Country. Let's apply graph theory to Dorothy's famous journey through Oz. Draw a graph in which each vertex is one of the regions of Oz. Then answer each question:

1. Is there an Euler trail circuit that Dorothy could follow, instead of the yellow brick road, to lead her from the land of the Munchkins, through all the regions of Oz and back, passing over each border exactly once? If not, how could the graph be Eulerized most efficiently?
2. What is the chromatic number of the graph? Find a graph coloring that demonstrates your answer and use it to draw and color a graph of Oz.

Check Your Understanding

For the following exercises, determine whether each statement is always true, sometimes true, or never true.

? Exercise 12.5.1

A disconnected graph has only one component.

? Exercise 12.5.2

A graph that has all vertices of even degree is connected.

? Exercise 12.5.3

There is a route through the city of Konigsberg that a person may pass over each bridge exactly once and return to the starting point.

? Exercise 12.5.4

A graph with vertices of all even degree is Eulerian.

? Exercise 12.5.5

An Eulerian graph has all vertices of even degree.

? Exercise 12.5.6

An Euler circuit is a closed trail.

? Exercise 12.5.7

An Euler circuit is a closed path

? Exercise 12.5.8

To eulerize a graph, add new edges between previously nonadjacent vertices until no vertices have odd degree.

? Exercise 12.5.9

To eulerize a graph, add duplicate edges between adjacent vertices until no vertices have odd degree.

? Exercise 12.5.10

The number of duplicate edges required to eulerize a graph is half the number of vertices of odd degree.

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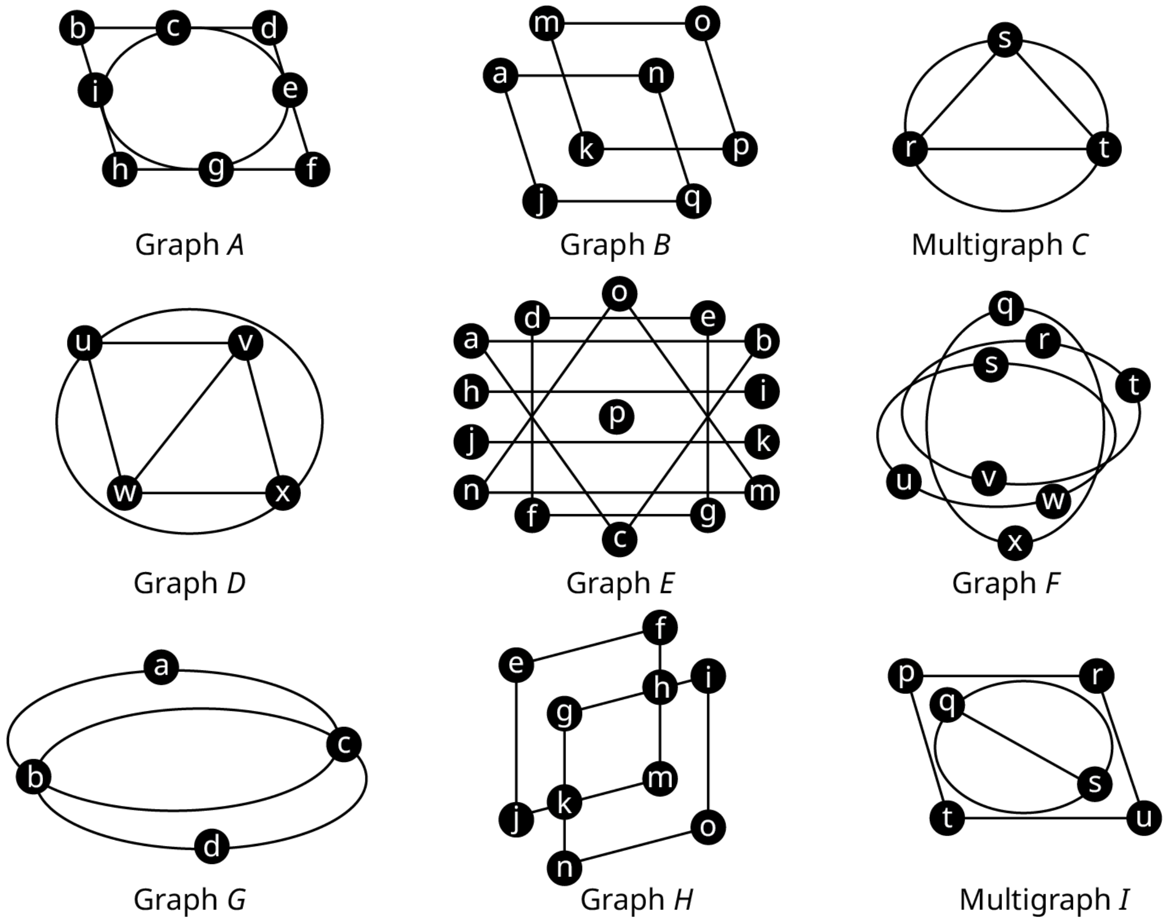
12.5.0: Exercises

Use Graphs *A*, *B*, *C*, *D*, *E*, *F*, *G*, *H*, and *I* to answer the following exercises. Identify any graph or graphs with the given characteristics or indicate that none do.


b
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Graph *A* Graph *B* Multigraph *C*

Graph *D* Graph *E* Graph *F*

Graph *G* Graph *H* Multigraph *I*

Figure 12.148

Exercise 12.5.0.1

Connected

Exercise 12.5.0.2

Disconnected

Exercise 12.5.0.3

Exactly two components

Exercise 12.5.0.4

Exactly three components

? Exercise 12.5.0.5

Exactly four components

? Exercise 12.5.0.6

Exactly five components

? Exercise 12.5.0.7

At least one vertex of odd degree

? Exercise 12.5.0.8

All vertices of even degree

? Exercise 12.5.0.9

An Euler circuit

? Exercise 12.5.0.10

A path between vertex j and each other vertex on the same graph

Use the graphs in the previous exercise to answer the following exercises. For each exercise, list the set of vertices for each component in the given graph.

? Exercise 12.5.0.11

Graph B

? Exercise 12.5.0.12

Graph E

? Exercise 12.5.0.13

Graph F

? Exercise 12.5.0.14

Multigraph I

Use the graphs in the initial exercise to answer the following exercises. For each exercise, a graph and a vertex on the graph are given. Find a path between the given vertex and each other vertex on the graph. If this is not possible, indicate that it is not.

? Exercise 12.5.0.15

Graph A, vertex c.

? Exercise 12.5.0.16

Graph B, vertex m.

? Exercise 12.5.0.17

Graph D, vertex x.

? Exercise 12.5.0.18

Graph F, vertex w.

? Exercise 12.5.0.19

Graph G, vertex a.

? Exercise 12.5.0.20

Graph H, vertex e.

Use the graphs in the initial exercise to answer the following exercises. For each exercise, a graph and a vertex on the graph are given. Find an Euler circuit beginning and ending at the given vertex if one exists. If no Euler circuits exist, explain how you know that they do not.

? Exercise 12.5.0.21

Graph A, vertex c.

? Exercise 12.5.0.22

Graph B, vertex k.

? Exercise 12.5.0.23

Graph D, vertex w.

? Exercise 12.5.0.24

Graph G, vertex b.

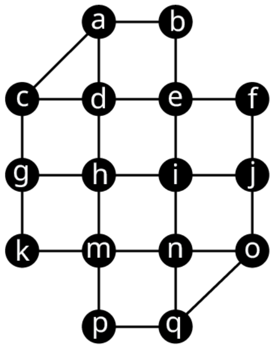
? Exercise 12.5.0.25

Graph H, vertex o.

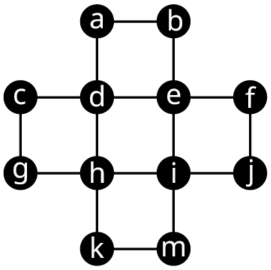
? Exercise 12.5.0.26

Multigraph I, vertex p.

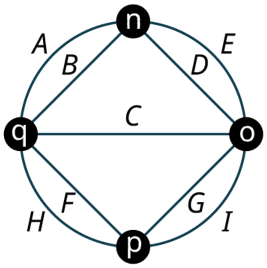
For the following exercises, use the connected graphs. In each exercise, a graph is indicated. Determine if the graph is Eulerian or not and explain how you know. If it is Eulerian, give an example of an Euler circuit. If it is not, state which edge or edges you would duplicate to eulerize the graph.



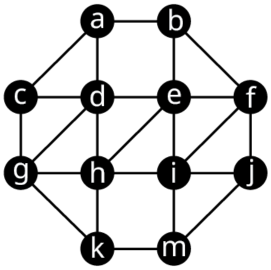
Graph *J*



Graph *K*



Multigraph *L*



Graph *M*

Exercise 12.5.0.27

Graph J

Exercise 12.5.0.28

Graph K

Exercise 12.5.0.29

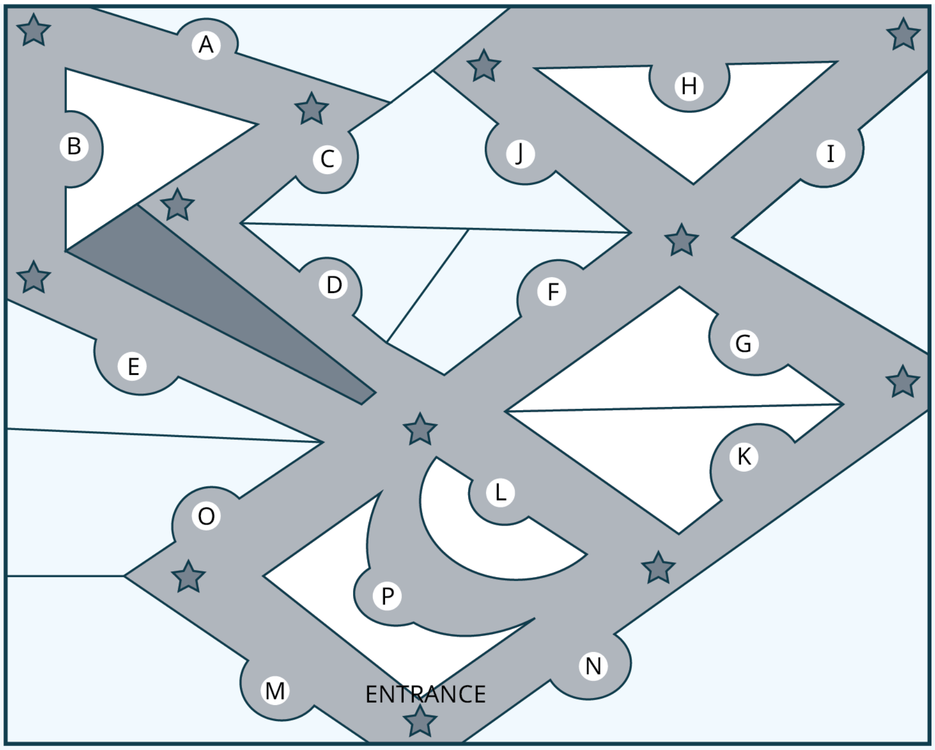
Multigraph L

Exercise 12.5.0.30

Graph M

Exercise 12.5.0.31

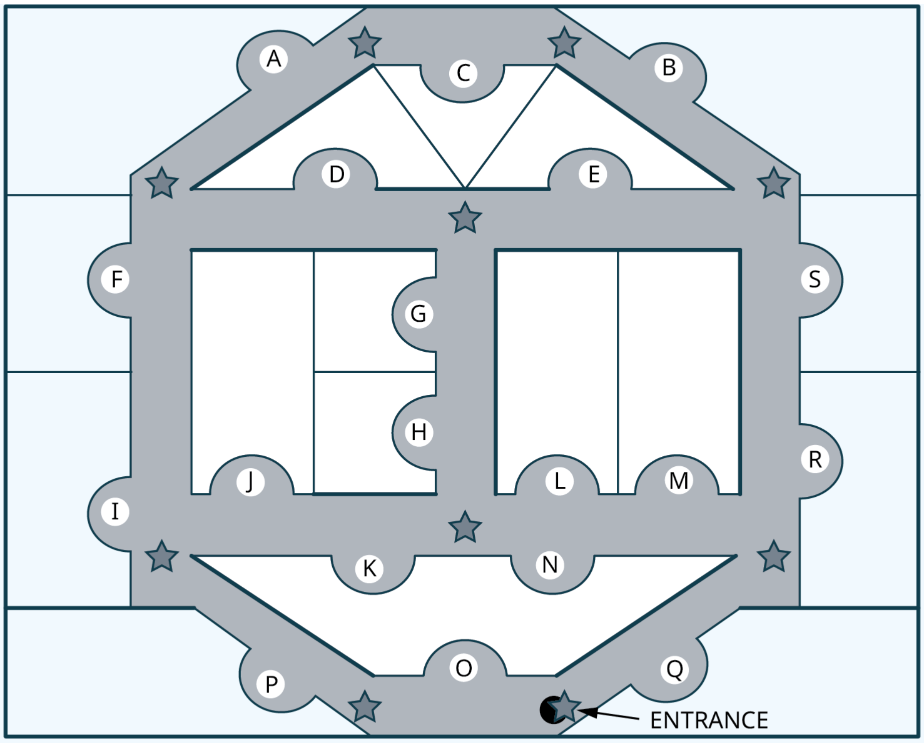
The figure shows a map of zoo exhibits A through P with walkways shown in gray. Draw a graph, or multigraph, to represent the routes through the zoo in which the edges represent walkways and the vertices represent turns and intersections, which are each marked with a star. Notice that there is exactly one exhibit between each pair of adjacent vertices. Label the edges with the corresponding exhibit. Use it to determine if it is possible for a visitor to begin at the entrance, view each exhibit exactly once, and end back at the entrance. If it is possible, give an example of a circuit on the graph that would represent a route the visitor could take. If it is not possible, explain why.



The figure shows the map of the exhibits at an indoor aquarium with hallways shown in gray. Turns and intersections of hallways are marked with stars.

Exercise 12.5.0.32

Use the Euler circuit theorem and a graph in which the edges represent hallways and the vertices represent turns and intersections to explain why a visitor to the aquarium cannot start at the entrance, visit every exhibit exactly once, and return to the entrance.



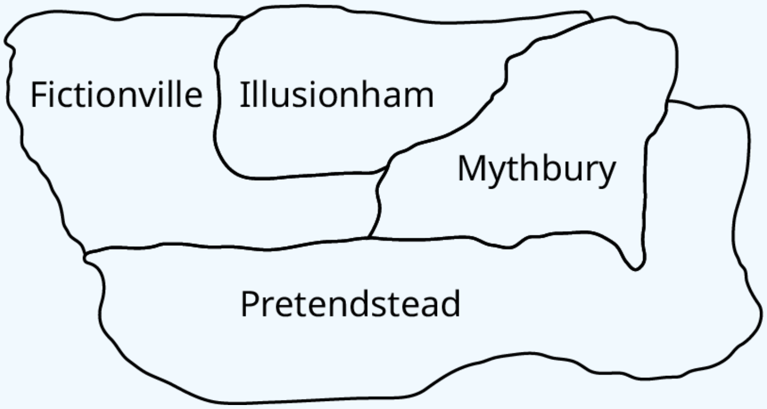
Exercise 12.5.0.33

Use an eulerization of the graph you found to determine the least amount of backtracking necessary to allow a visitor to begin at the entrance, see every exhibit at least once, and return to the entrance. How many hallways must be covered twice? Explain your reasoning.

The map of the states of Imaginaria is given.

Exercise 12.5.0.34

Use a graph to determine if it is possible to travel through Imaginaria crossing each the border between each pair of states exactly once, and returning to the state in which you started.



Exercise 12.5.0.35

Use an eulerization of the graph you found to determine the fewest borders that must be covered twice in order to cross each border at least once and return to the state in which you started. Explain your reasoning.

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12.6: Euler Trails

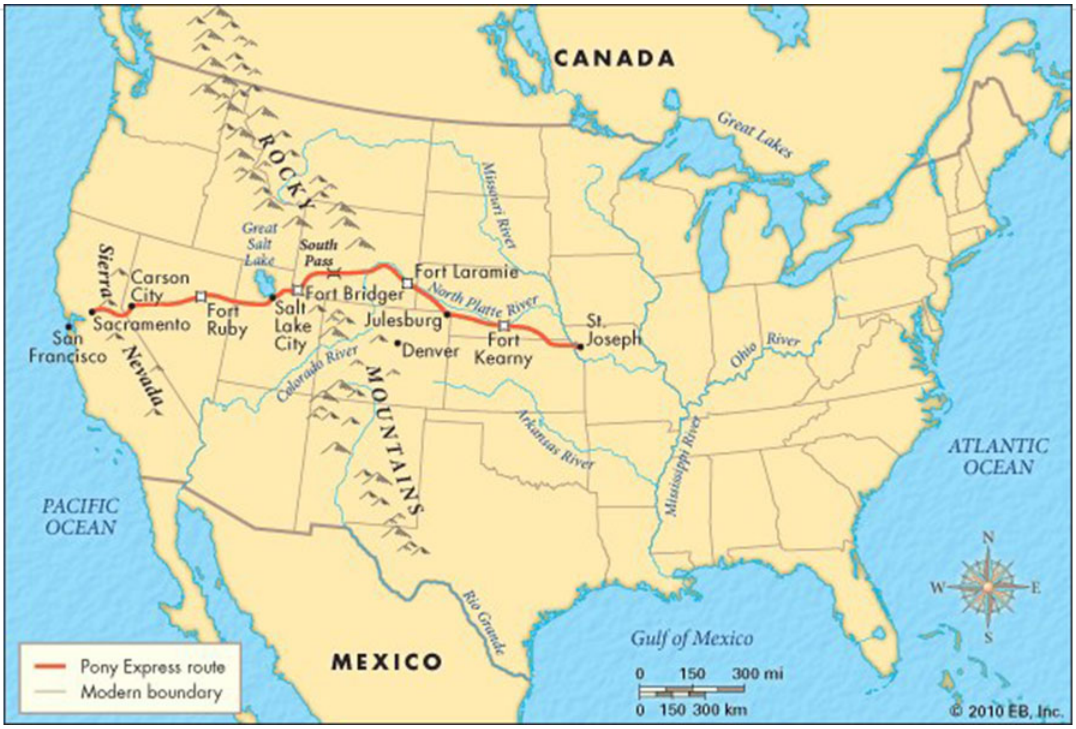

CANADA
Great Lakes
Missouri River
ROCKY
Great Salt Lake
South Pass
Fort Laramie
North Platte River
Sierra
Carson City
Fort Bridger
Fort Ruby
Salt Lake City
Julesburg
St. Joseph
San Francisco
Sacramento
Denver
Fort Kearny
Nevada
Colorado River
MOUNTAINS
Ohio River
Arkansas River
Mississippi River
ATLANTIC OCEAN
PACIFIC OCEAN
N
W
E
S
Rio Grande
Gulf of Mexico
MEXICO
Pony Express route
Modern boundary
0 150 300 mi
0 150 300 km
© 2010 EB, Inc.

Figure 12.6.1The Pony Express mail route spanned from California to Missouri. (credit: “Map of Pony Express” by Nathan Hughes Hamiltonh/Flickr, CC BY 2.0)

Learning Objectives

1. Describe and identify Euler trails.
2. Solve applications using Euler trails theorem.
3. Identify bridges in a graph.
4. Apply Fleury’s algorithm.
5. Evaluate Euler trails in real-world applications.

We used Euler circuits to help us solve problems in which we needed a route that started and ended at the same place. In many applications, it is not necessary for the route to end where it began. In other words, we may not be looking at circuits, but trails, like the old Pony Express trail that led from Sacramento, California in the west to St. Joseph, Missouri in the east, never backtracking.

Euler Trails

If we need a trail that visits every edge in a graph, this would be called an Euler trail. Since trails are walks that do not repeat edges, an Euler trail visits every edge exactly once.

✓ Example ✓ 12.6.1: Recognizing Euler Trails

Use Figure ✓ 12.6.2 to determine if each series of vertices represents a trail, an Euler trail, both, or neither. Explain your reasoning.

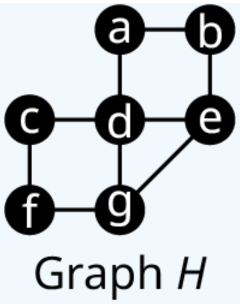


Figure
✓ 12.6.2
Graph H

1. $a \rightarrow b \rightarrow e \rightarrow g \rightarrow f \rightarrow c \rightarrow d \rightarrow e$
2. $a \rightarrow b \rightarrow e \rightarrow g \rightarrow f \rightarrow c \rightarrow d \rightarrow e \rightarrow b \rightarrow a \rightarrow d \rightarrow g$
3. $g \rightarrow d \rightarrow a \rightarrow b \rightarrow e \rightarrow d \rightarrow c \rightarrow f \rightarrow g \rightarrow e$

Answer

1. It is a trail only. It is a trail because it is a walk that doesn’t cover any edges twice, but it is not an Euler trail because it didn’t cover edges *ad* or *dg*.
2. It is neither. It is not a trail because it visits *ab* and *be* twice. Since it is not a trail, it cannot be an Euler trail.
3. It is both. It is a trail because it is a walk that doesn’t cover any edges twice, and it is an Euler trail because it visits all the edges.

Your Turn 12.6.1

Use the figure to determine if each sequence of vertices represents an Euler trail or not. If not, explain why.

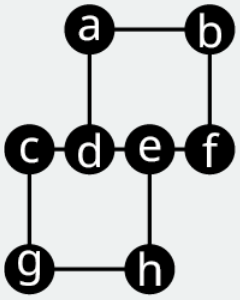


Figure 1 12.6.3: Graph *I*

1. $e \rightarrow f \rightarrow b \rightarrow a \rightarrow d \rightarrow c \rightarrow g \rightarrow h \rightarrow e \rightarrow d$
2. $d \rightarrow a \rightarrow b \rightarrow f \rightarrow e \rightarrow h \rightarrow g \rightarrow c$
3. $d \rightarrow a \rightarrow b \rightarrow e \rightarrow f \rightarrow b \rightarrow e \rightarrow d \rightarrow c \rightarrow g \rightarrow h \rightarrow e$

The Five Rooms Puzzle

Just as Euler determined that only graphs with vertices of even degree have Euler circuits, he also realized that the only vertices of odd degree in a graph with an Euler trail are the starting and ending vertices. For example, in Figure 12.6.3, Graph *H* has exactly two vertices of odd degree, vertex *g* and vertex *e*. Notice the Euler trail we saw in Exercise 3 of Example 12.6.1 began at vertex *g* and ended at vertex *e*.

This is consistent with what we learned about vertices off odd degree when we were studying Euler circuits. We saw that a vertex of odd degree couldn't exist in an Euler circuit as depicted in Figure 12.6.3. If it was a starting vertex, at some point we would leave the vertex and not be able to return without repeating an edge. If it was not a starting vertex, at some point we would return and not be able to leave without repeating an edge. Since the starting and ending vertices in an Euler trail are not the same, the start is a vertex we want to leave without returning, and the end is a vertex we want to return to and never leave. Those two vertices must have odd degree, but the others cannot.

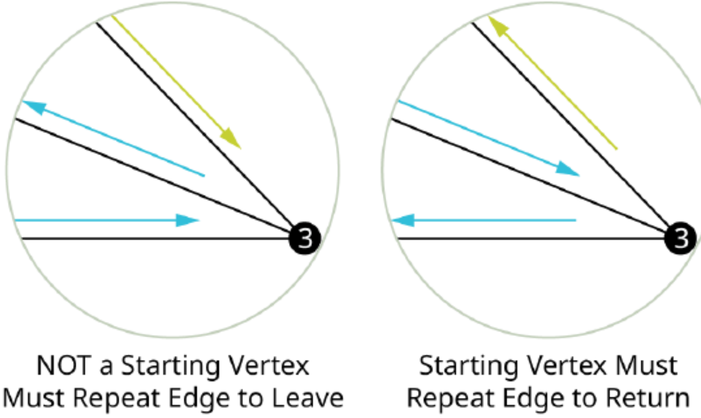


Figure 12.6.4: A Vertex of Degree 3

Let's use the Euler trail theorem to solve a puzzle so you can amaze your friends! This puzzle is called the "Five Rooms Puzzle." Suppose that you were in a house with five rooms and the exterior. There is a doorway in every shared wall between any two rooms and between any room and the exterior as shown in Figure 12.6.5. Could you find a route through the house that passes through each doorway exactly once?

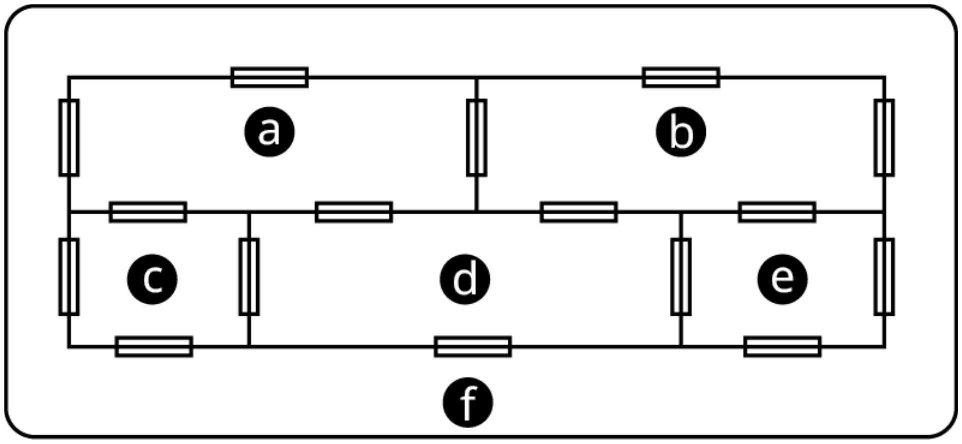


Figure 12.6.5: Five Rooms Puzzle

Let's represent the puzzle with a graph in which vertices are rooms (or the exterior) and an edge indicates a door between two rooms as shown in Figure 12.6.6.

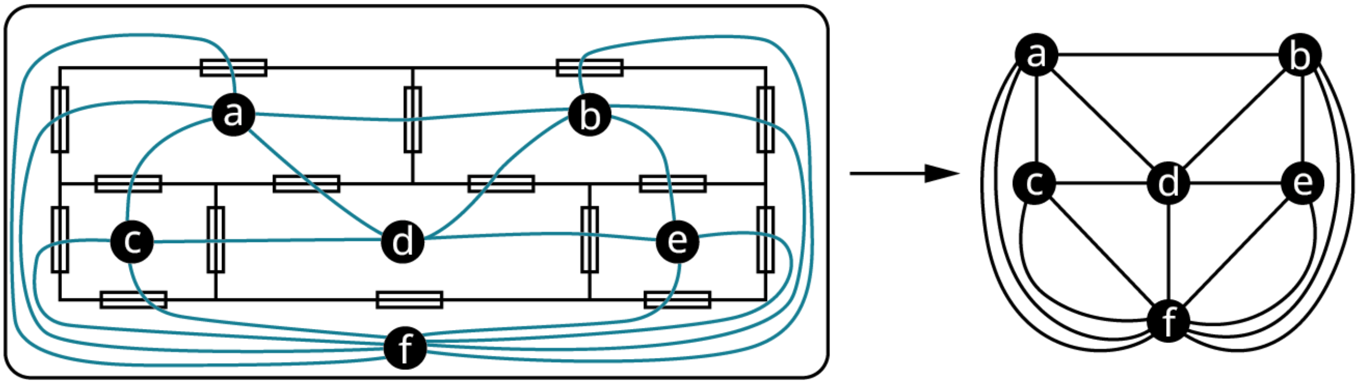


Figure 12.6.6: Graph of Five Rooms Puzzle

To pass through each doorway exactly once means that we cross every edge in the graph exactly once. Since we have not been asked to start and end at the same position, but to visit each edge exactly once, we are looking for an Euler trail. Let's check the degrees of the vertices.

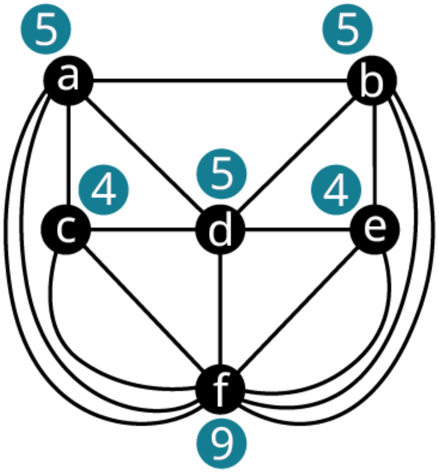


Figure 12.6.7: Degrees of Vertices in Five Rooms Puzzle

Since there are more than two vertices of odd degree as shown in Figure 12.6.7, the graph of the five rooms puzzle contains no Euler path. Now you can amaze and astonish your friends!

Bridges and Local Bridges

Now that we know which graphs have Euler trails, let's work on a method to find them. The method we will use involves identifying bridges in our graphs. A bridge is an edge which, if removed, increases the number of components in a graph. Bridges are often referred to as cut-edges. In Figure 12.6.8, there are several examples of bridges. Notice that an edge that is not part of a cycle is always a bridge, and an edge that is part of a cycle is never a bridge.

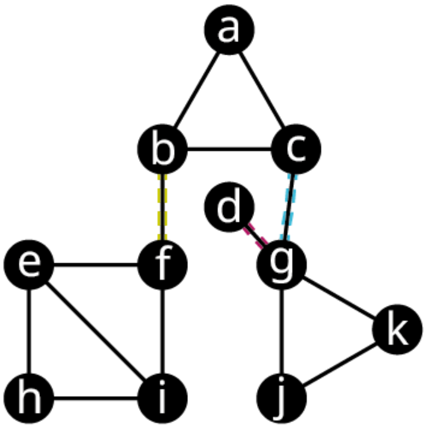


Figure 12.6.8: Graph with Bridges

Edges *bf*, *cg*, and *dg* are "bridges"

The graph in Figure 12.6.8 is connected, which means it has exactly one component. Each time we remove one of the bridges from the graph the number of components increases by one as shown in Figure 12.6.9. If we remove all three, the resulting graph in Figure 12.157 has four components.

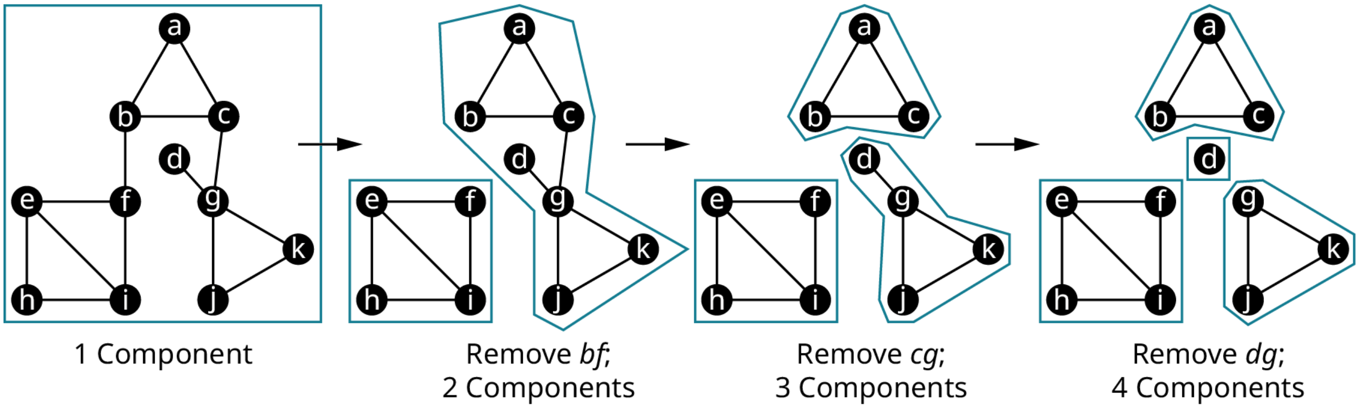


Figure 12.6.9 Removing a Bridge Increases Number of Components

In sociology, bridges are a key part of social network analysis. Sociologists study two kinds of bridges: local bridges and regular bridges. Regular bridges are defined the same in sociology as in graph theory, but they are unusual when studying a large social network because it is very unlikely a group of individuals in a large social network has only one link to the rest of the network. On the other hand, a local bridge occurs much more frequently. A local bridge is a friendship between two individuals who have no other friends in common. If they lose touch, there is no single individual who can pass information between them. In graph theory, a local bridge is an edge between two vertices, which, when removed, increases the length of the shortest path between its vertices to more than two edges. In Figure 12.158, a local bridge between vertices b and e has been removed. As a result, the shortest path between b and e is $b \rightarrow i \rightarrow j \rightarrow k \rightarrow e$, which is four edges. On the other hand, if edge ab were removed, then there are still paths between a and b that cover only two edges, like $a \rightarrow i \rightarrow b$.

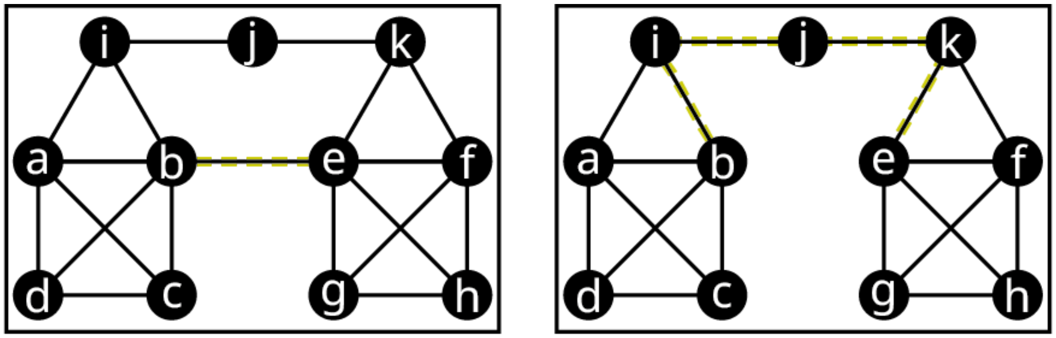


Figure 12.6.10: Removing a Local Bridge

The significance of a local bridge in sociology is that it is the shortest communication route between two groups of people. If the local bridge is removed, the flow of information from one group to another becomes more difficult. Let's say that vertex b is Brielle and vertex e is Ella. Now, Brielle is less likely to hear about things like job opportunities, that Ella many know about. This is likely to impact Brielle as well as the friends of Brielle.

✓ Example 12.6.2: Identifying Bridges and Local Bridges

Use the graph of a social network in Figure ✓ 12.6.11 to answer each question.

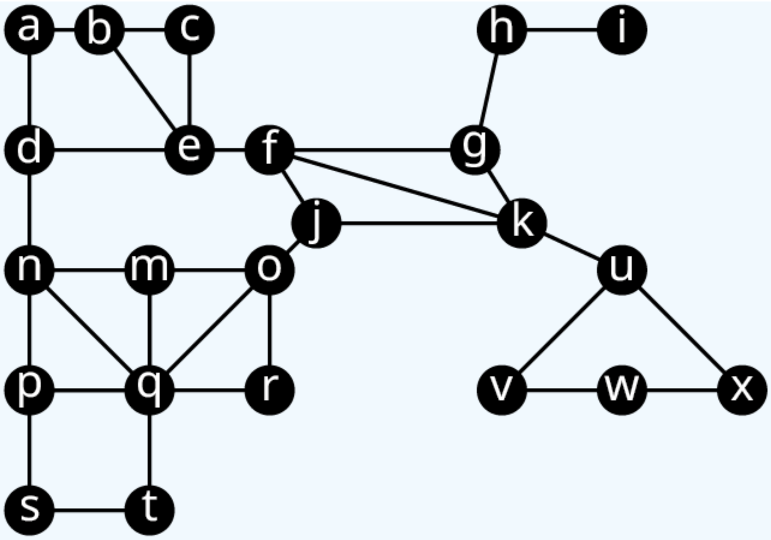


Figure ✓ 12.6.11 Graph of a Social Network

1. Identify any bridges.
2. If all bridges were removed, how many components would there be in the resulting graph?
3. Identify one local bridge.
4. For the local bridge you identified in part 3, identify the shortest path between the vertices of the local bridge if the local bridge were removed.

Answer

1. The edges *ku*, *gh*, and *hi* are bridges.
2. If the bridges were all removed, there would be four components in the resulting graph, {*i*}, {*h*}, {*u*, *v*, *w*, *x*}, and {*a*, *b*, *c*, *d*, *e*, *f*, *g*, *j*, *k*, *m*, *n*, *o*, *p*, *q*, *r*, *s*, *t*} as shown in Figure 12.160.

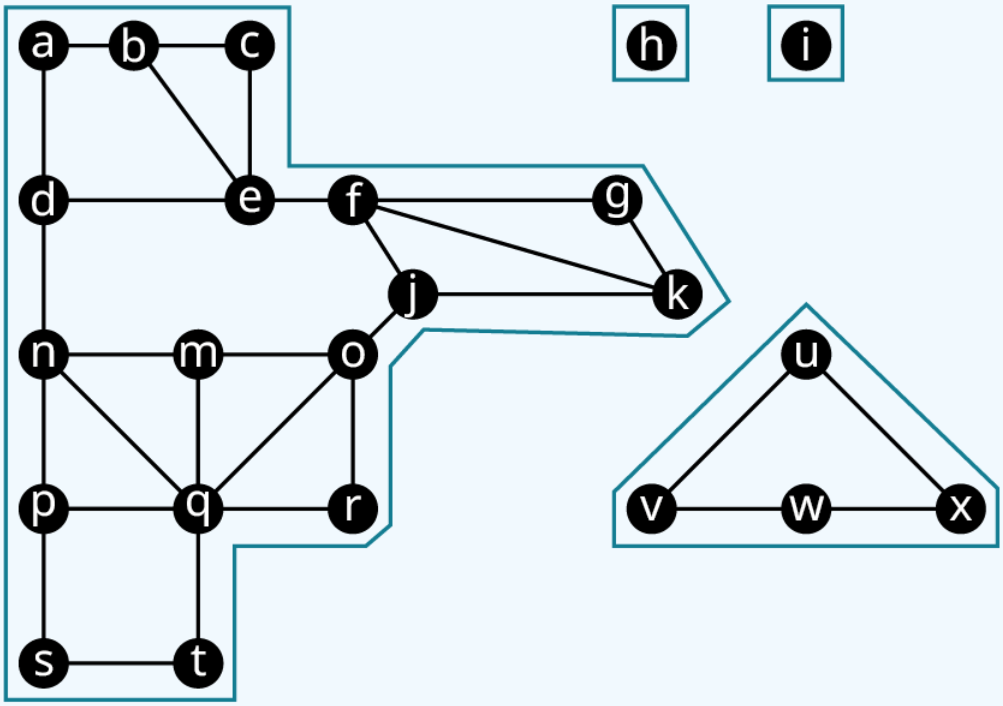


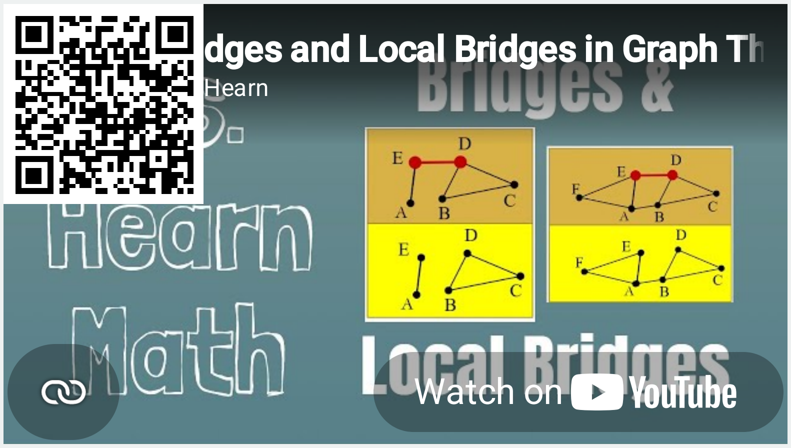
Figure ✓ 12.6.12 Graph of Social Network without Bridges

3. Three local bridges are *dn*, *ef*, and *qt*, among others.
4. If *dn* were removed, the shortest path between *d* and *n* would be *d* → *e* → *f* → *j* → *o* → *m* → *n*.

Your Turn 12.6.2

How many bridges and local bridges are in a complete graph with three or more vertices? Explain your reasoning.

Video



Bridges and Local Bridges in Graph Theory

Finding an Euler Trail with Fleury’s Algorithm

Now that we are familiar with bridges, we can use a technique called Fleury’s algorithm, which is a series of steps, or algorithm, used to find an Euler trail in any graph that has exactly two vertices of odd degree. Here are the steps involved in applying Fleury’s algorithm.

Here are the steps involved in applying Fleury’s algorithm.

Step 1: Begin at either of the two vertices of odd degree.

Step 2: Remove an edge between the vertex and any adjacent vertex that is NOT a bridge, unless there is no other choice, making a note of the edge you removed. Repeat this step until all edges are removed.

Step 3: Write out the Euler trail using the sequence of vertices and edges that you found. For example, if you removed *ab*, *bc*, *cd*, *de*, and *ef*, in that order, then the Euler trail is $a \rightarrow b \rightarrow c \rightarrow d \rightarrow e \rightarrow f$.

Figure 12.6.13 shows the steps to find an Euler trail in a graph using Fleury’s algorithm.

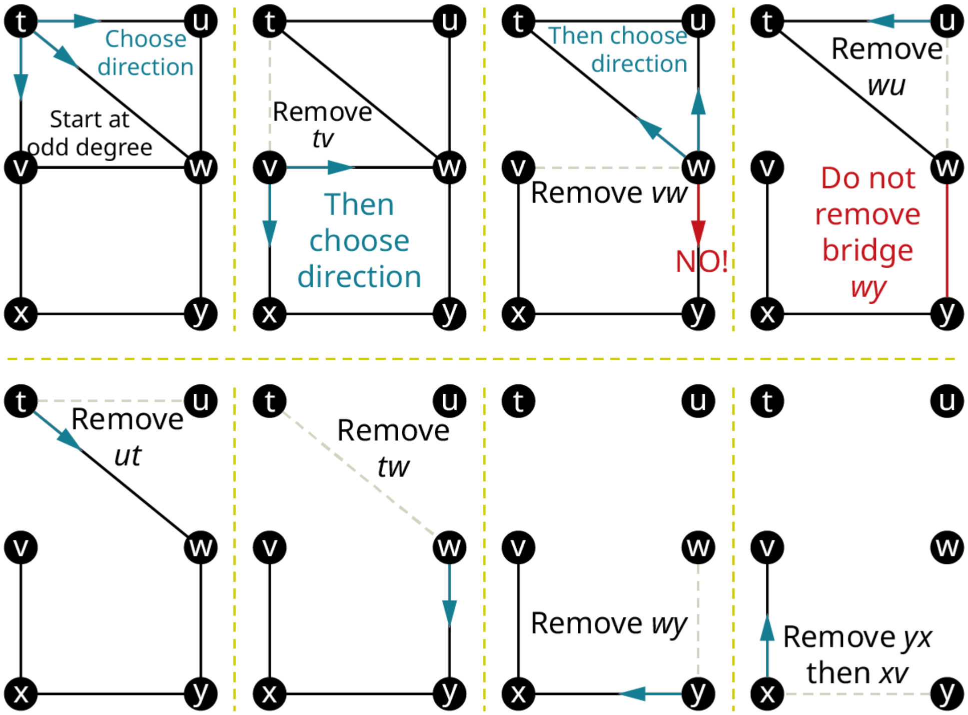

t
u
v
w
x
y
Choose direction
Start at odd degree
Remove tv
Then choose direction
Then choose direction
Remove vw
NO!
Remove wu
Do not remove bridge wy
Remove ut
Remove tw
Remove wy
Remove yx then xv

Figure 12.6.13 Using Fleury’s Algorithm To Find Euler Trail

The Euler trail that was found in Figure 12.6.13 is $t \rightarrow v \rightarrow w \rightarrow u \rightarrow t \rightarrow w \rightarrow y \rightarrow x \rightarrow v$.

✓ Example 12.6.3: Finding an Euler Trail with Fleury’s Algorithm

Use Fleury’s Algorithm to find an Euler trail for Graph J in Figure ✓ 12.6.14

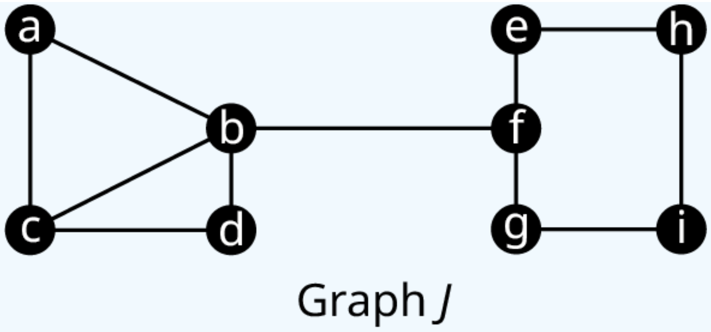


Figure ✓ 12.6.14: Graph *J*

Answer

Step 1: Choose one of the two vertices of odd degree, *c* or *f*, as your starting vertex. We will choose *c*.

Step 2: Remove edge *ca*, *cb*, or *cd*. None of these are cut edges so we can select any of the three. We will choose *cb* as shown in Figure ✓ 12.6.15 to be the first edge removed.

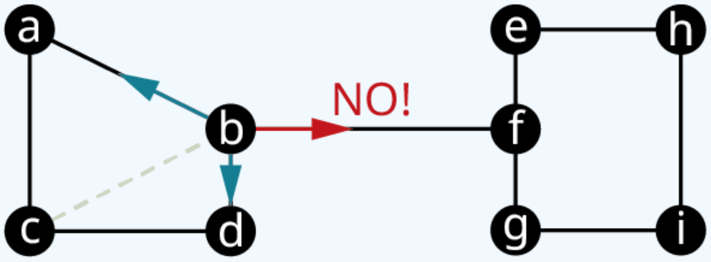


Figure ✓ 12.6.15 Graph *J* with *cb* Removed

Repeat Step 2 The next choice is to remove edge *ba*, *bd*, or *bf* as shown in Figure ✓ 12.6.16, but *bf* is not an option since it is a bridge. We will choose *ba* as shown in Figure 12.164 to be the second edge removed.

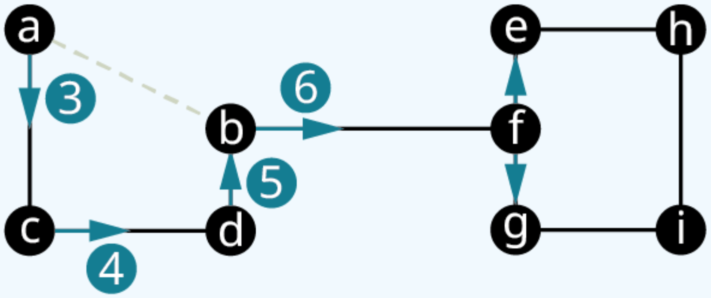


Figure ✓ 12.6.16 Graph *J* with *cb* and *ba* Removed

Repeat Step 2 for the third, fourth, fifth, sixth, and seventh edges. As shown in Figure ✓ 12.6.16, until we get to the seventh edge there is only one option each time, *ac*, *cd*, *db*, and *bf* in that order. For the seventh edge, we must choose between *fe* and *fg*. Neither of these are bridges. We choose *fe*. Figure ✓ 12.6.17 shows that *ac*, *cd*, *db*, *bf*, and *fe* have been removed.

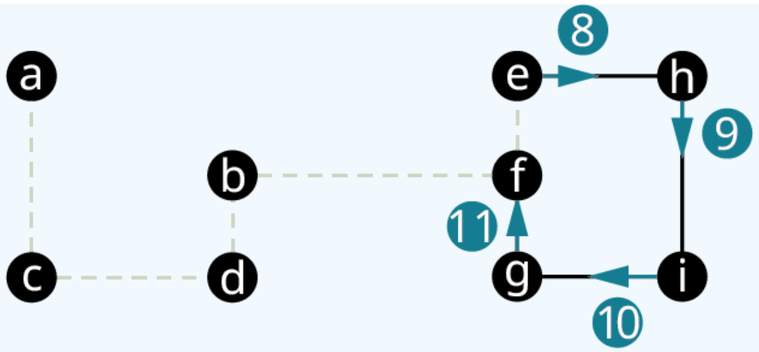


Figure ✓ 12.6.17 Graph *J* with Seven Edges Removed

Step 3: Write out the Euler trail using the vertices in the sequence that the edges were removed. We removed *cb*, *ba*, *ac*, *cd*, *db*, *bf*, *fe*, *eh*, *hi*, *ig*, and *gf*, in that order. The Euler trail is *c* → *b* → *a* → *c* → *d* → *b* → *f* → *e* → *h* → *i* → *g* → *f*.

Checkpoint

TIP! To avoid errors, count the number of edges in your graph and make sure thatyour Euler trail represents that number of edges.

Your Turn 12.6.3

Use Graph *L* to fill in the blanks to complete the steps in Fleury's algorithm.

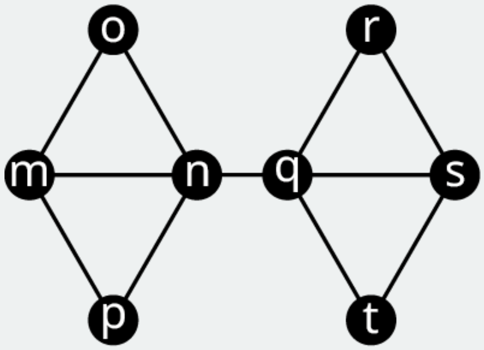


Figure 12.6.18: Graph *L*

1. The two vertices that can be used as the starting vertex are ____ and *s*.
2. If *sq* is the first edge removed, the three options for the second edge to be removed are *qr*, ___, and ___; however, ___ cannot be chosen because it is a ________________.
3. If *qr* is the second edge removed, the next four edges to be removed must be ___, ___, ___, and ___, in that order.
4. After *qn* is removed, the three options for the next edge to be removed are *no*, ___, and ___, however, ___ cannot be chosen because it is a ______________
5. If *no* is the next edge removed, the last four edges removed will be ___, ___, ___, and ___, in that order.
6. The final Euler trail using the answers to parts 1 through 5 is ___________________________.

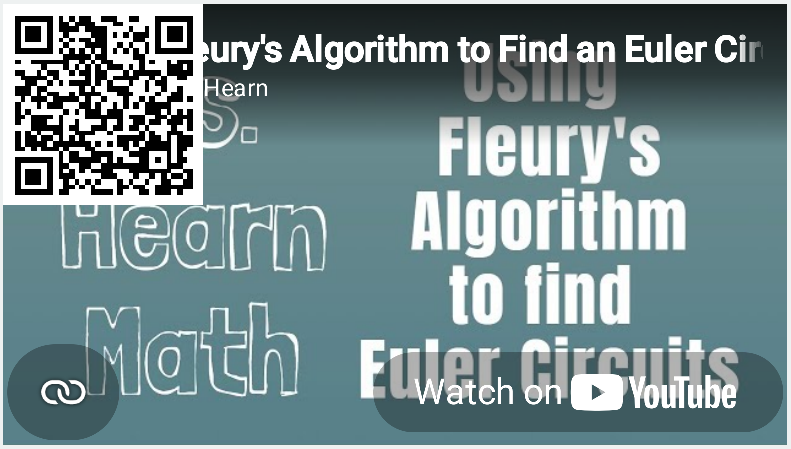
In the previous section, we found Euler circuits using an algorithm that involved joining circuits together into one large circuit. You can also use Fleury's algorithm to find Euler circuits in any graph with vertices of all even degree. In that case, you can start at any vertex that you would like to use.

Step 1: Begin at any vertex.

Step 2: Remove an edge between the vertex and any adjacent vertex that is NOT a bridge, unless there is no other choice, making a note of the edge you removed. Repeat this step until all edges are removed.

Step 3: Write out the Euler circuit using the sequence of vertices and edges that you found. For example, if you removed *ab*, *bc*, *cd*, *de*, and *ea*, in that order, then the Euler circuit is *a* → *b* → *c* → *d* → *e* → *a*.

Video


…ury's Algorithm to Find an Euler Cir
Hearn
Using Fleury's Algorithm to find Euler Circuits
Hearn Math
Watch on YouTube

Fleury's Algorithm to Find an Euler Circuit

Checkpoint

IMPORTANT! Since a circuit is a closed trail, every Euler circuit is also an Euler trail, but when we say Euler trail in this chapter, we are referring to an open Euler trail that begins and ends at different vertices.

✓ Example 12.6.4: Finding an Euler Circuit or Euler Trail Using Fleury's Algorithm

Use Fleury's algorithm to find either an Euler circuit or Euler trail in Graph G in Figure ✓ 12.6.19.

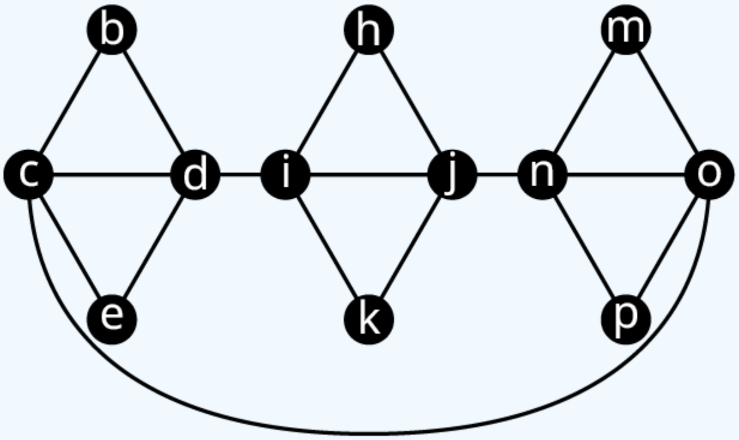

b
h
m
c
d
i
j
n
o
e
k
p

Figure ✓ 12.6.19: Graph G

Answer

Graph G has all vertices of even degree so it has an Euler circuit.

Step 1: Choose any vertex. We will choose vertex j.

Step 2: Remove one of the four edges that meet at vertex j. Since jn is a bridge, we must remove either jh, ji, or jk. We remove ji as shown in Figure 12.6.20.

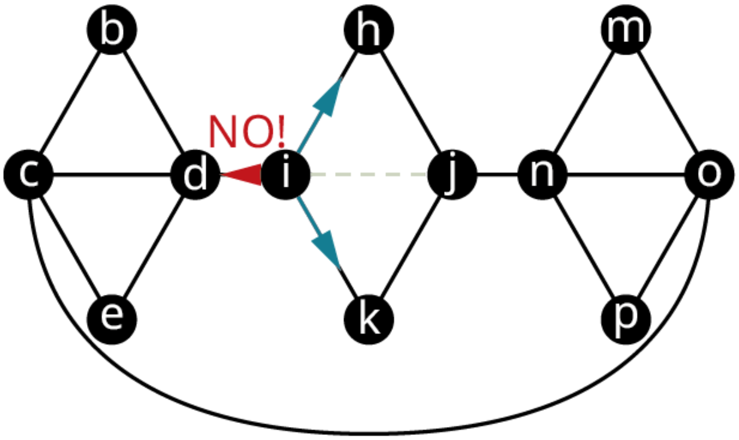


Figure 12.6.20 Graph *G* with 1 Edge Removed

Repeat Step 2: Since *id* is a bridge, we can remove either *ih* or *ik* next. We remove *ih*, and then the only option is to remove *hj* as shown in Figure 12.6.21.

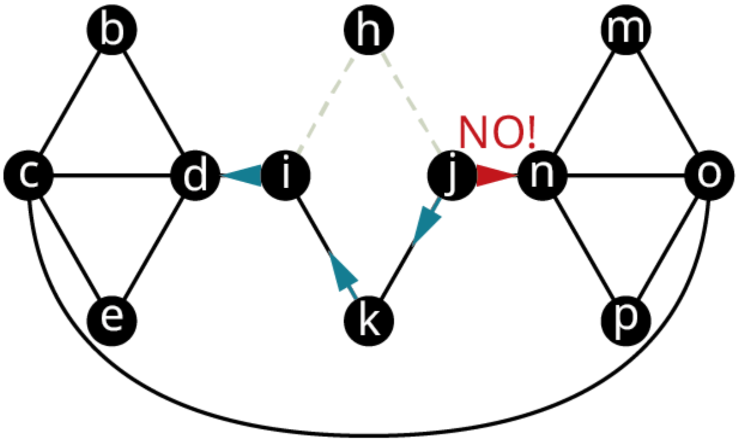


Figure 12.6.21 Graph *G* with 3 Edges Removed

Repeat Step 2: Since *jn* is a bridge, the next edge removed must be *jk*, and then the only option is to remove *ki* followed by *id* as shown in Figure 12.6.21. Even though *id* is a bridge, it can be removed because it is the only option at this point. Figure 12.6.22 shows Graph *G* with these additional edges removed.

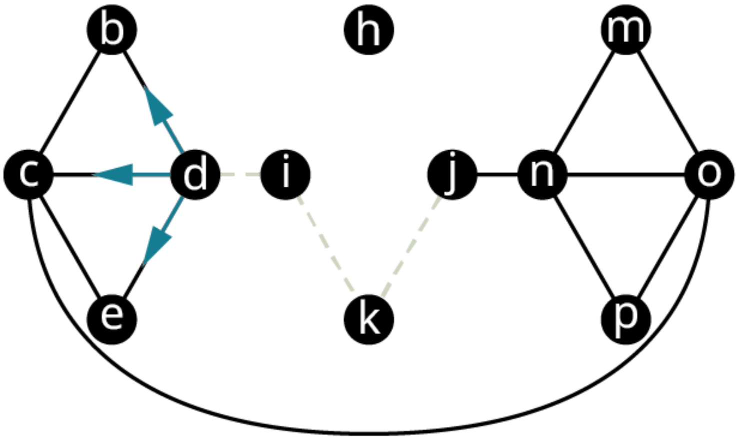


Figure 12.6.22 Graph *G* with 6 Edges Removed

Repeat Step 2: Choose any one of the edges *db*, *dc*, or *de*. We remove *dc* as shown in Figure 12.6.23.

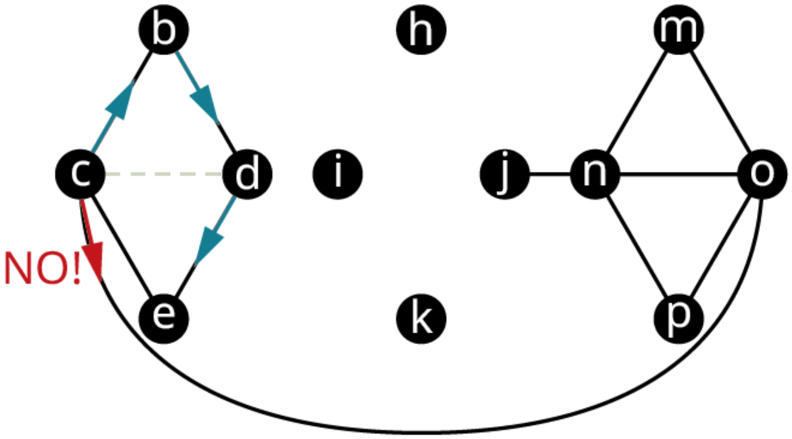


Figure 12.6.23 Graph G with 7 Edges Removed

Repeat Step 2: Since *co* is a bridge, choose *cb* next. We remove *cb*, then *bd*, and then *de* as shown in Figure 12.6.24.

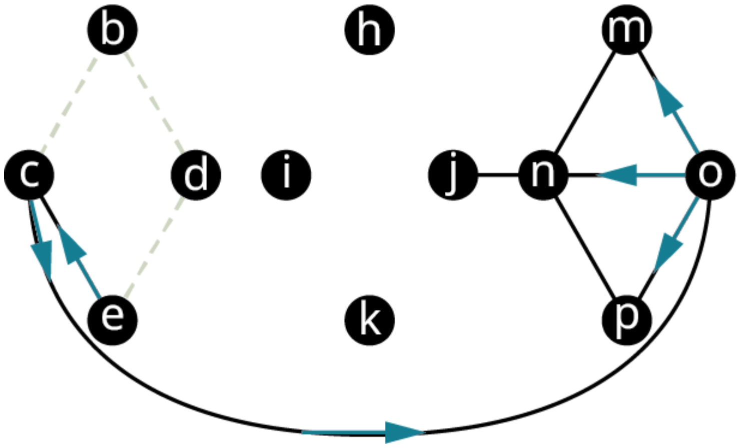


Figure 12.6.24 Graph G with 10 Edges Removed

Repeat Step 2: Next, remove *ec* and *co*. Then choose any of *op*, *pn*, or *om*. We remove on as shown in Figure 12.6.25.

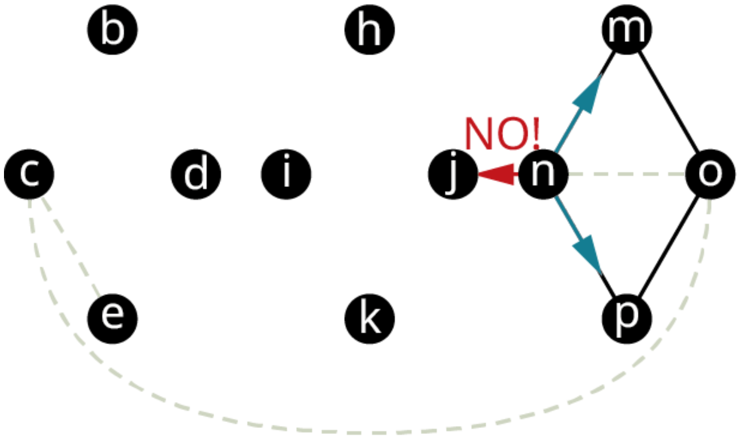


Figure 12.6.26 Graph G with 13 Edges Removed

Repeat Step 2: Next, remove either *nm*, *np*, or *nj*, but *nj* is a So, we remove *nm* as shown in Figure 12.6.27.

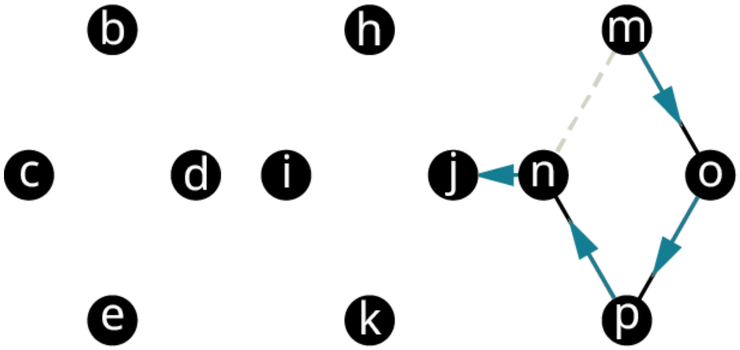


Figure 12.6.27 Graph *G* with 14 Edges Removed

Repeat Step 2: Next, remove *mo*, *op*, *pn*, and *nj*. And we are done!

Step 3: Notice that the algorithm brought us back to the vertex where we started, forming an Euler circuit. Write out the Euler circuit:

$j \to i \to h \to j \to k \to i \to d \to c \to b \to d \to e \to c \to o \to n \to m \to o \to p \to n \to j$

Your Turn 12.6.4

Find an Euler circuit or trail through the graph using Fleury's algorithm.

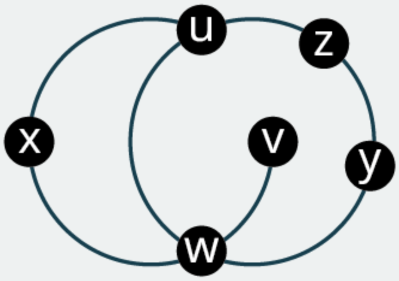


Figure 12.6.28: Graph *T*

WORK IT OUT

We have discussed a lot of subtle concepts in this section. Let's make sure we are all on the same page. Work with a partner to explain why each of the following facts about bridges are true. Support your explanations with definitions and graphs.

1. When a bridge is removed from a graph, the number of components increases.
2. A bridge is never part of a circuit.
3. An edge that is part of a triangle is never a local bridge.

Check Your Understanding

Fill in the blank to make the statement true.

1. An Euler trail is a trail that visits each ___________ exactly once.

2. __________ algorithm is a procedure for finding an Euler trail or circuit.

3. An Euler _____ always begins and ends at the same vertex, but an Euler _____ does not.

4. When a bridge is removed from a graph, the number of ________ is increased by one.

5. When a __________ is removed from a graph, the shortest path between its vertices will be greater than two.

6. When using Fleury's algorithm to find an Euler trail, never remove a _________ unless it is the only option.

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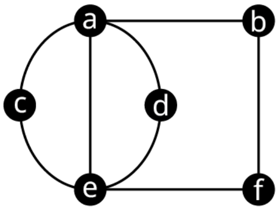
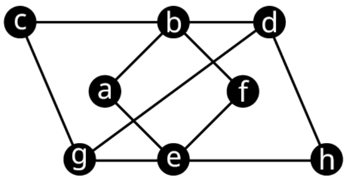
12.6.0: Exercises

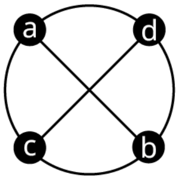
Use the figure to answer the following exercises. Identify the graph(s) with the given characteristics, if any.



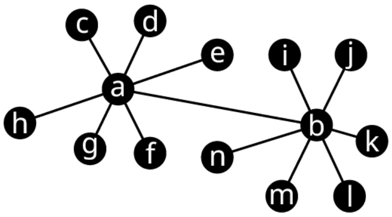
Graph *T*



Graph *U*



Graph *V*



Graph *W*

Exercise 12.6.0.1

Connected

Exercise 12.6.0.2

All vertices of even degree

Exercise 12.6.0.3

Exactly two vertices of odd degree

Exercise 12.6.0.4

Has an Euler trail

Exercise 12.6.0.5

Has an Euler circuit

Exercise 12.6.0.6

Has neither an Euler trail nor an Euler circuit

Exercise 12.6.0.7

ab is a bridge

Exercise 12.6.0.8

ef is a bridge

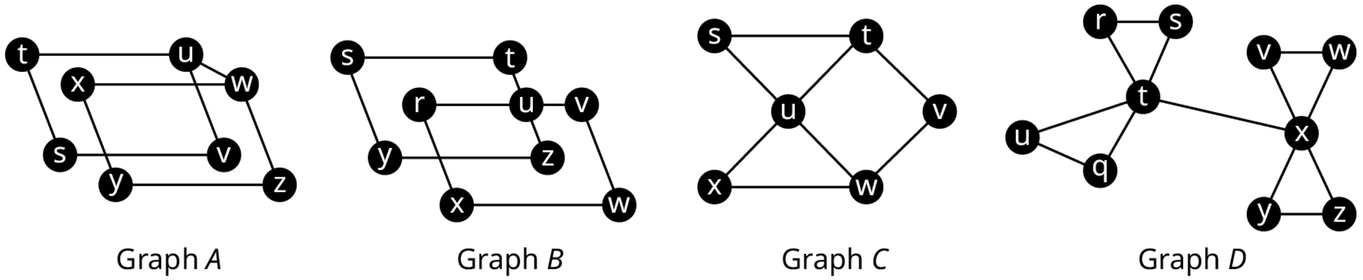
Exercise 12.6.0.9

ab is a local bridge

Exercise 12.6.0.10

ef is a local bridge

Use the figure to answer the following exercises. In each exercise, a graph and a sequence of vertices are given. Determine whether each sequence of vertices is an Euler trail, an Euler circuit, or neither for the graph. If it is neither, explain why.



Graph A Graph B Graph C Graph D

Figure 12.176

Exercise 12.6.0.11

Graph A, $w \to x \to y \to z \to w \to u \to t \to s \to v \to u$

Exercise 12.6.0.12

Graph A, $u \to v \to s \to t \to u \to w \to z \to y \to x \to w$

Exercise 12.6.0.13

Graph A, $s \to t \to u \to v \to u \to w \to z \to y \to x \to w$

Exercise 12.6.0.14

Graph A, $w \to x \to y \to z \to w \to v \to u \to t \to s \to v$

Exercise 12.6.0.15

Graph B, $u \to v \to w \to x \to r \to u \to t \to s \to y \to z \to u$

Exercise 12.6.0.16

Graph B, $v \to w \to x \to r \to u \to z \to y \to s \to t \to u$

Exercise 12.6.0.17

Graph C, $s \to t \to u \to v \to w \to x \to s$

Exercise 12.6.0.18

Graph C, $t \to u \to x \to w \to u \to s \to t \to v \to w$

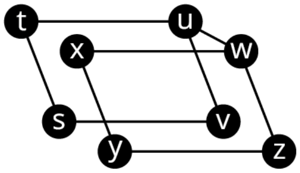
Exercise 12.6.0.19

Graph D, $t \to r \to s \to t \to u \to v \to t \to x \to v \to w \to x \to y \to z \to x$

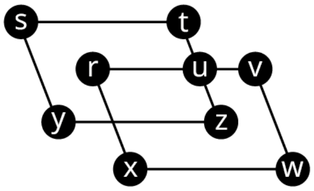
Exercise 12.6.0.20

Graph D, $x \to v \to w \to x \to y \to z \to x \to t \to r \to s \to t \to u \to v \to t$

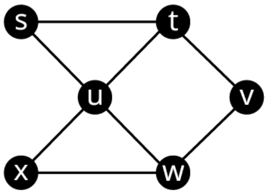
Use the figure to answer the following exercises. For each graph, identify a bridge if one exists. If it does not, state so. If it does, identify any components that are created when the bridge is removed.



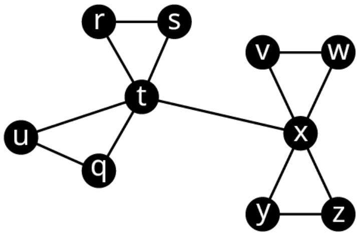
Graph *A*



Graph *B*



Graph *C*


r
s
v
w
t
u
x
q
y
z

Graph *D*

Exercise 12.6.0.21

Graph *A*

Exercise 12.6.0.22

Graph *B*

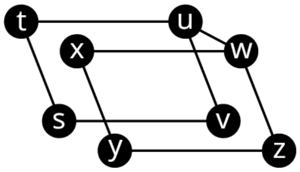
Exercise 12.6.0.23

Graph *C*

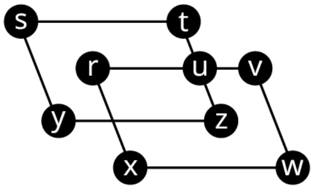
Exercise 12.6.0.24

Graph *D*

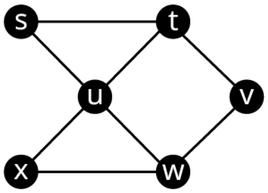
Use the figuree to answer the following exercises. For each graph, identify a local bridge if one exists. If it does not, state so. If it does, find a shortest path between the vertices of the local bridge if the local bridge is removed.



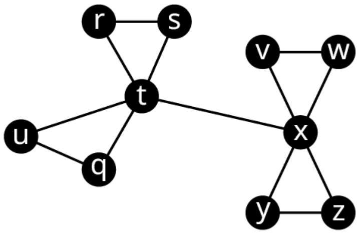
Graph *A*



Graph *B*



Graph *C*


r
s
v
w
t
u
x
q
y
z

Graph *D*

Exercise 12.6.0.25

Graph *A*

Exercise 12.6.0.26

Graph *B*

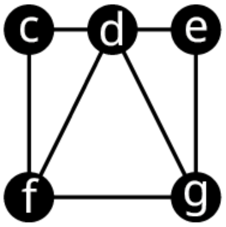
Exercise 12.6.0.27

Graph *C*

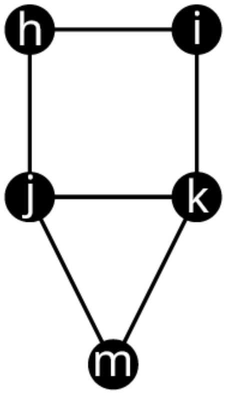
Exercise 12.6.0.28

Graph D

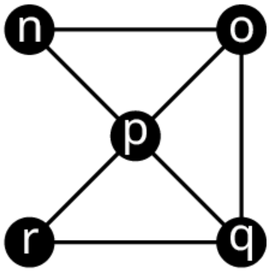
Use the graphs to answer the following exercises. In each exercise, a graph is given. Find two Euler trails in each graph using Fleury’s algorithm.



Graph Q



Graph R



Graph S

Exercise 12.6.0.29

Graph Q

Exercise 12.6.0.30

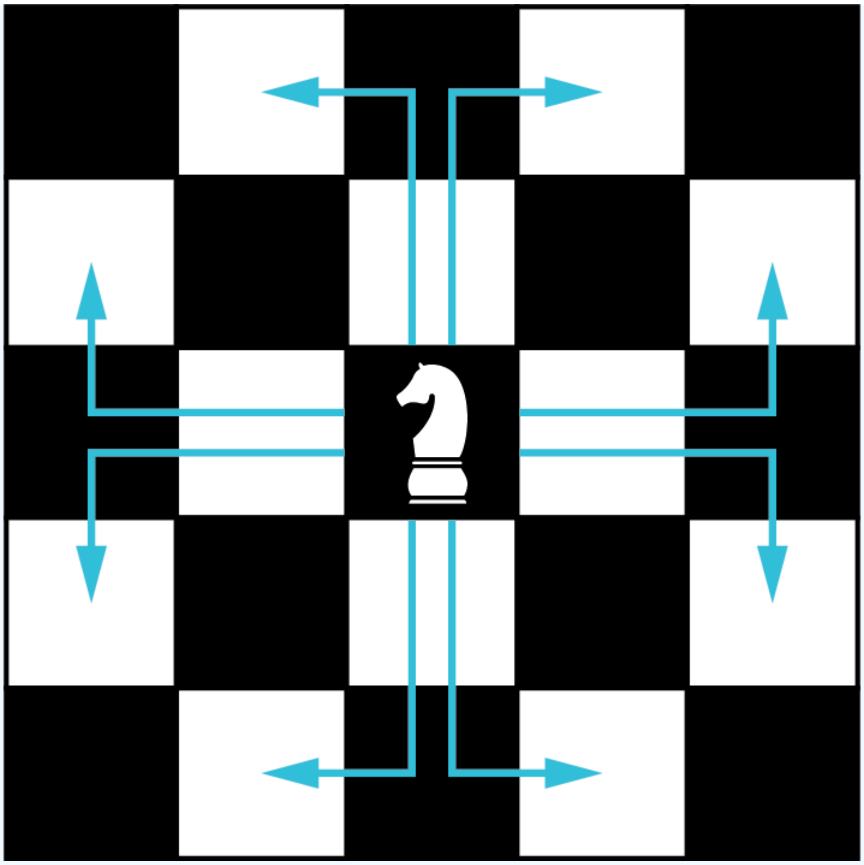
Graph R

Exercise 12.6.0.31

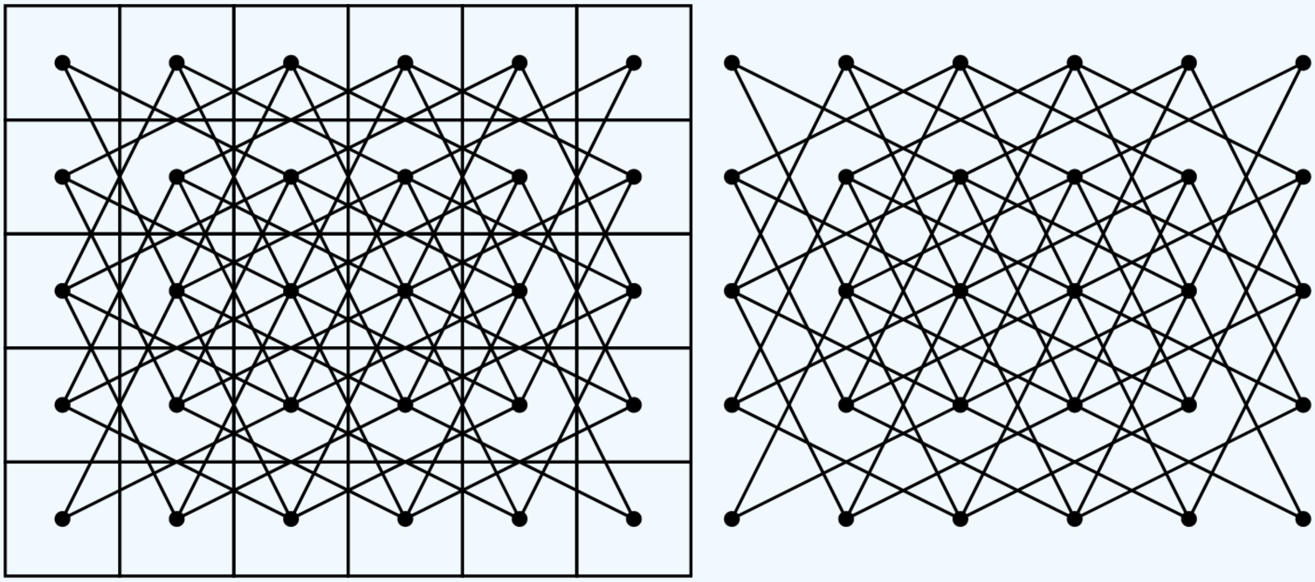
Graph S

Exercise 12.6.0.32

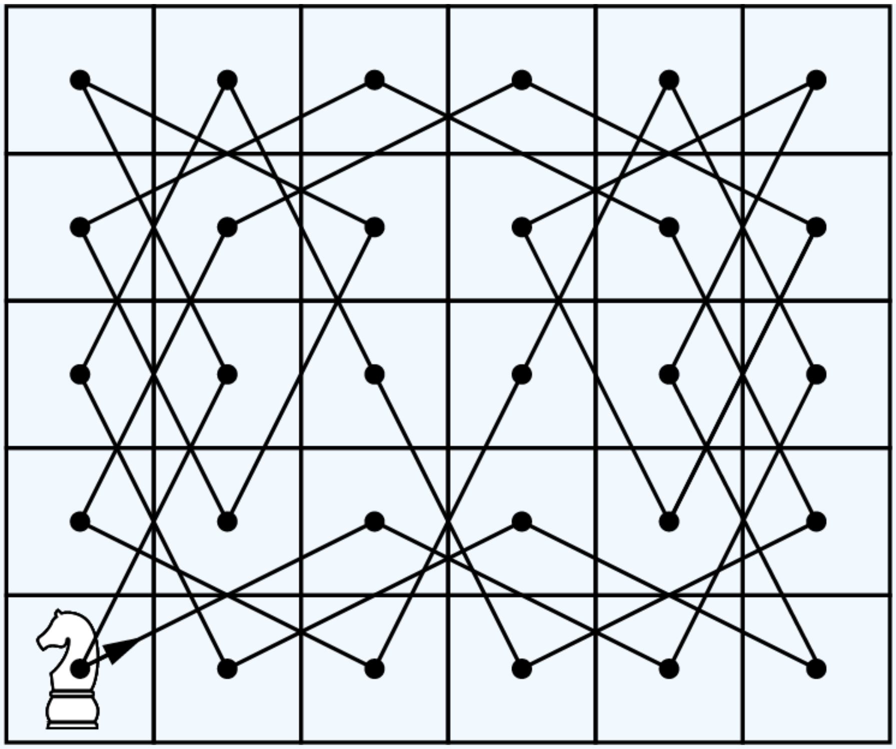
In chess, a knight can move in any direction, but it must move two spaces then turn and move one more space. The eight possible moves a knight can make from a given space are shown in the figure.



A graph in which each vertex represents a space on a five-by-six game board and each edge represents a move a knight could make is shown in the figure.

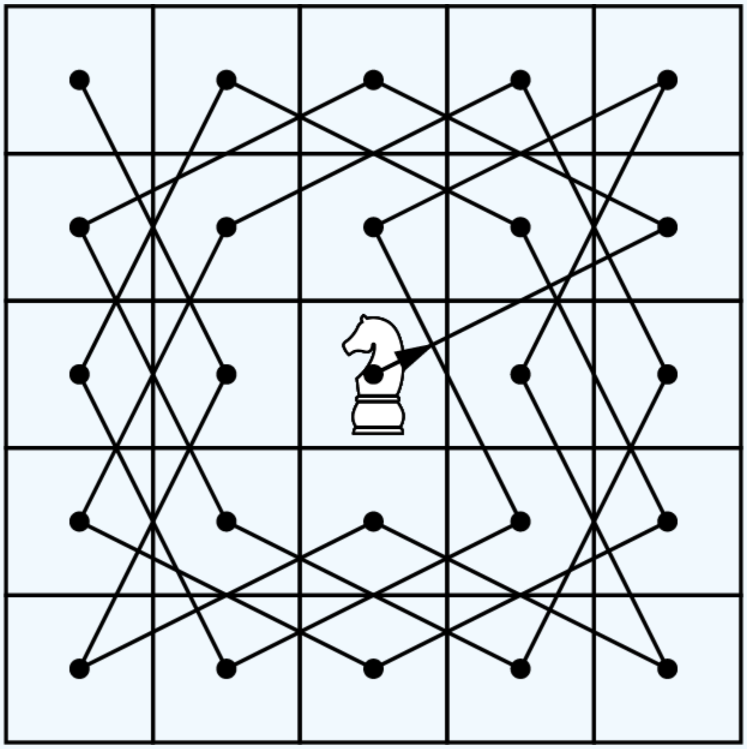


A knight’s tour is a sequence of moves by a knight on a chessboard (of any size) such that the knight visits every square exactly once. If the knight’s tour brings the knight back to its starting position on the board, it is called a closed knight’s tour. Otherwise, it is called an open knight’s tour. Determine if the closed knight’s tour in the figure is most accurately described as a trail, a circuit, an Euler trail, or an Euler circuit of the graph of all possible knight moves. Explain your reasoning.



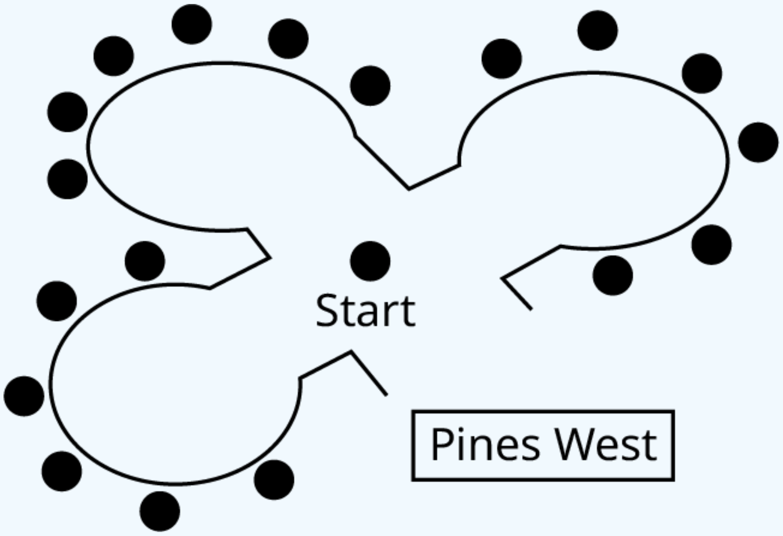
Exercise 12.6.0.33

Determine if the open knight’s tour in the figure is most accurately described as a trail, a circuit, an Euler trail, or an Euler circuit of the graph of all possible knight moves on a five-by-five game board. Explain your reasoning.



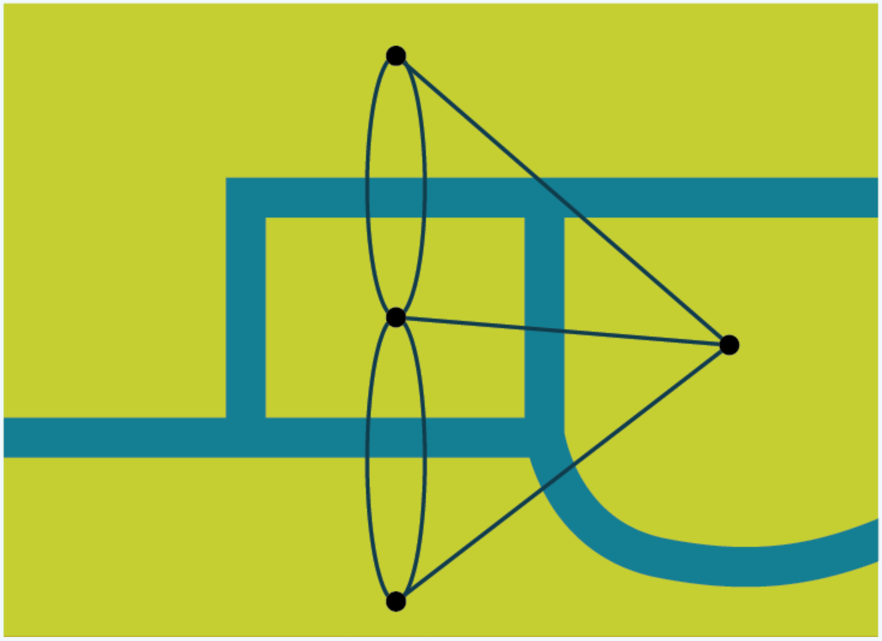
Exercise 12.6.0.34

The neighborhood of Pines West has three cul-de-sacs that meet at an intersection as shown in the figure. A postal delivery person starts at the intersection and visits each house in a cul-de-sac once, returns to the intersection, visits each house in the next cul-de-sac, and so on, returning to the intersection when finished. Describe how the route can be represented as a graph. If there is no backtracking, in other words, the person never reverses direction, is the route followed by the postal delivery person best described as a trail, a circuit, an Euler trail, or an Euler circuit? Explain your reasoning.



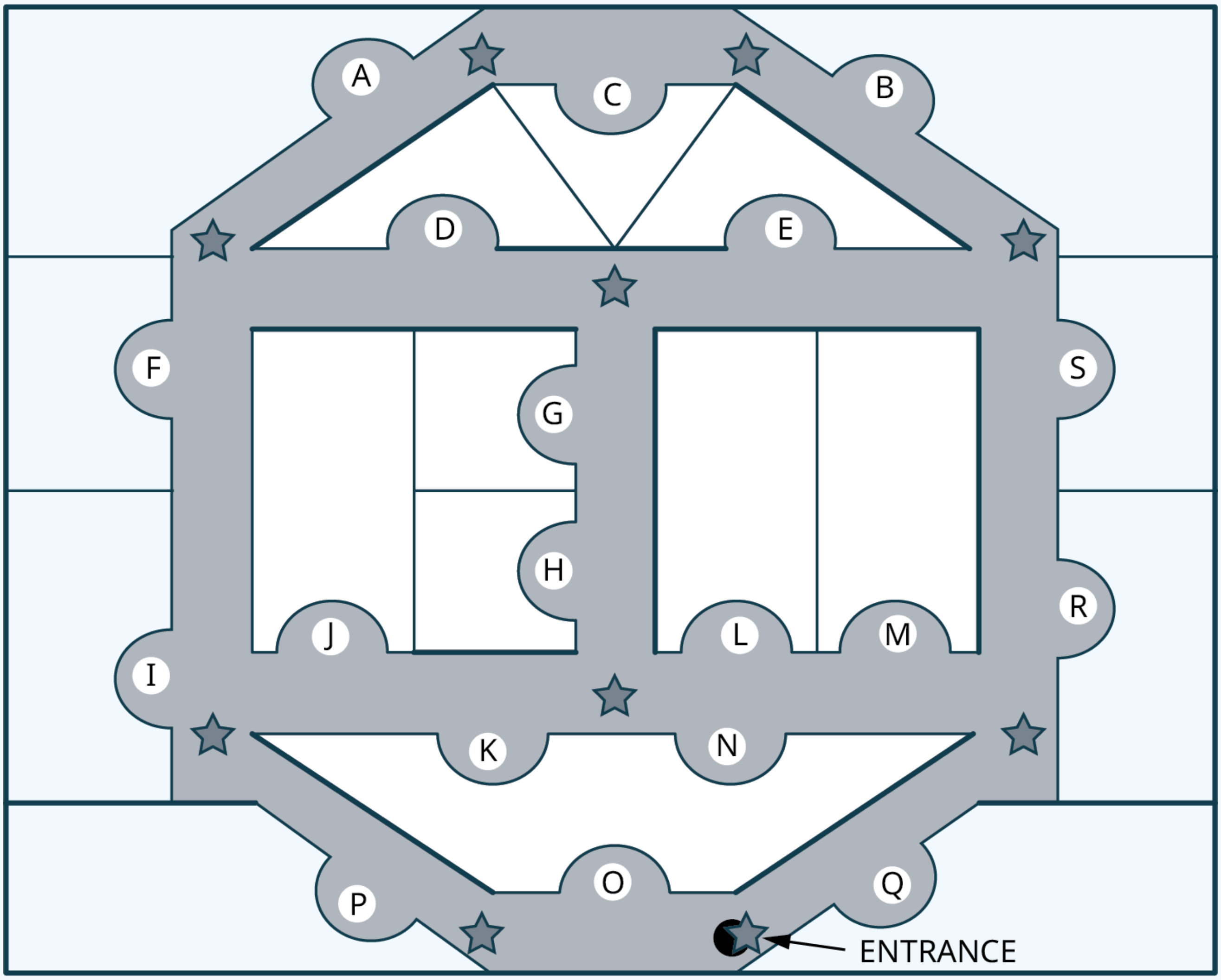
Exercise 12.6.0.35

Recall that the bridges of Konigsberg can be represented as a multigraph as shown in the figure. We have seen that no route through Konigsberg passes over each bridge exactly once and returns to the starting point. Is there a route that passes over each bridge exactly once but does not begin and end at the same point? Explain your reasoning.



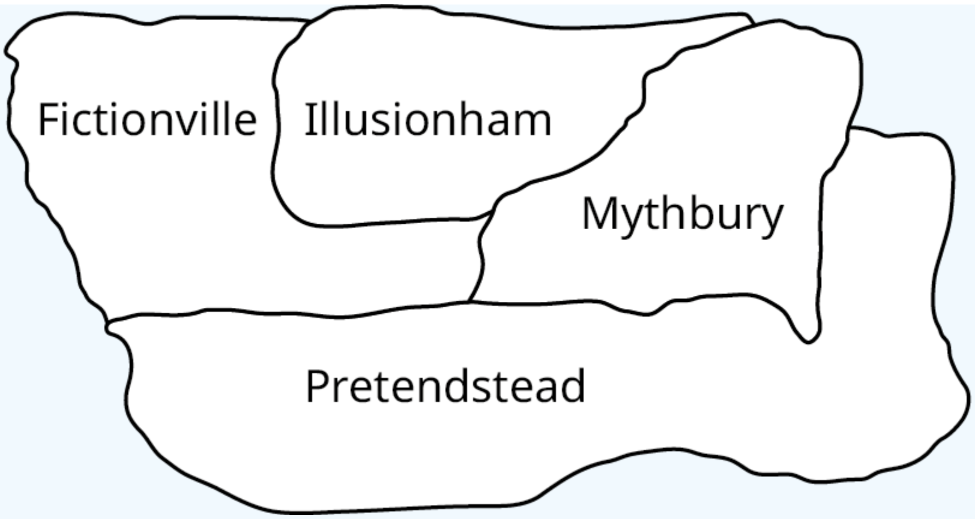
Exercise 12.6.0.36

The figure shows the map of the exhibits at an indoor aquarium. Use a graph in which the edges represent hallways and the vertices represent turns and intersections to explain why a visitor to the aquarium cannot start at one of the turns or intersections, passes by every exhibit exactly once, and end at one of the turns or intersections.



Exercise 12.6.0.37

The map of the states of Imaginaria is given. Use a graph to determine if it is possible to begin in one state, travel through Imaginaria crossing the border between each pair of states exactly once, and end in a different state. If it is possible, find such a route. If it is not, explain why.


Fictionville
Illusionham
Mythbury
Pretendstead

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12.7: Hamilton Cycles

Learning Objectives

1. Describe and identify Hamilton cycles.
2. Compute the number of Hamilton cycles in a complete graph.
3. Apply and evaluate weighted graphs.

In Euler Circuits and Euler Trails, we looked for circuits and paths that visited each *edge* of a graph exactly once. In this section, we will look for circuits that visit each *vertex* exactly once. Like many concepts in graph theory, the idea of a circuit that visits each vertex once was inspired by games and puzzles. As early as the 9th century, Indian and Islamic intellectuals wondered whether it was possible for a knight to visit every space on a chess board of a given size, which is equivalent to visiting every vertex of a graph that represents the chess board.

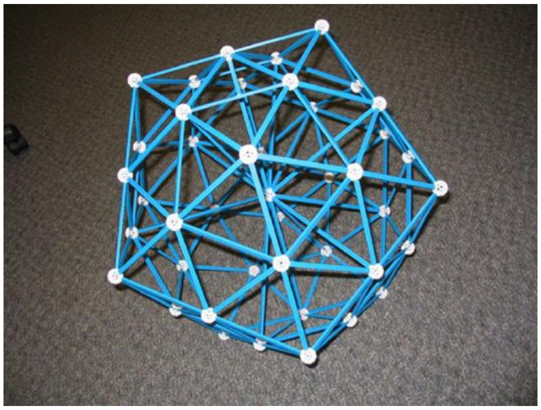


Figure $12.7.1$: The symmetries of an icosahedron, with 30 edges, 20 faces, and 12 vertices, can be analyzed using graph theory. (credit: "Really big icosahedron" by Clayton Shonkwiler/Wikimedia, CC BY 2.0)

In 1857, a mathematician named William Rowan Hamilton invented a puzzle in which players were asked to find a route along the edges of a dodecahedron (see Figure 12.178), which visited every vertex exactly once. Let’s explore how graph theory provides insight into these games as well as practical applications such as the Traveling Salesperson Problem.

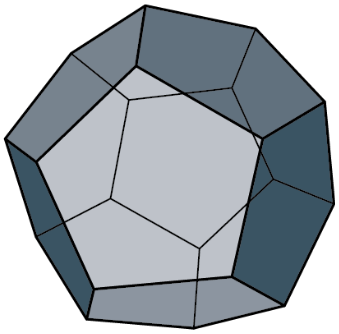


Figure $12.7.2$: A Dodecahedron

Hamilton’s Puzzle

Before we look at the solution to Hamilton's puzzle, let’s review some vocabulary we used in Figure $12.7.3$. It will be helpful to remember that directed cycle is a type of circuit that doesn’t repeat any edges or vertices.

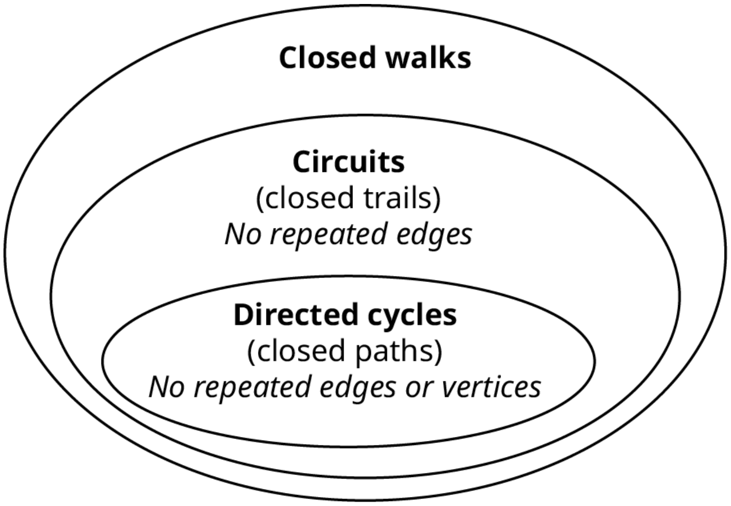

Closed walks
Circuits
(closed trails)
No repeated edges
Directed cycles
(closed paths)
No repeated edges or vertices

Figure 12.7.3: Closed Walks, Circuits, and Directed Cycles

The goal of Hamilton's puzzle was to find a route along the edges of the dodecahedron, which visits each vertex exactly once. A dodecahedron is a three-dimensional space figure with faces that are all pentagons as we saw in Figure 12.178.

Since it is easier to visualize two dimensions rather than three, we will flatten out the dodecahedron and look at the edges and vertices on a flat surface. Graph *A* in Figure 12.180 shows a two-dimensional graph of the edges and vertices, and Graph *B* shows an untangled version of Graph *A* in which no edges are crossing. Graph *B* in Figure 12.180 is very similar to the design of the game board that Hamilton invented for his puzzle.

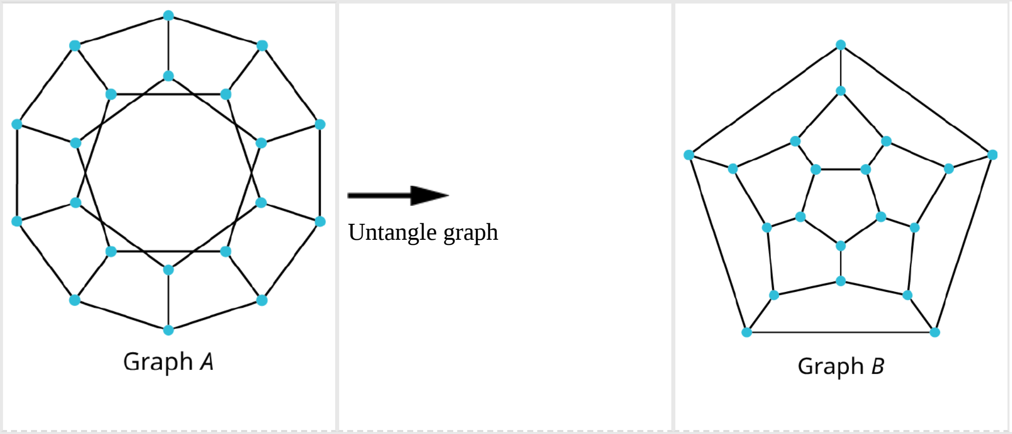

Untangle graph
Graph A
Graph B

Figure 12.7.4: Graph of Edges and Vertices of Dodecahedron

We can see that this is a planar graph because it can be “untangled.” In order to solve Hamilton’s puzzle, we need to find a circuit that visits every vertex once. A solution is shown in Figure 12.181.

.

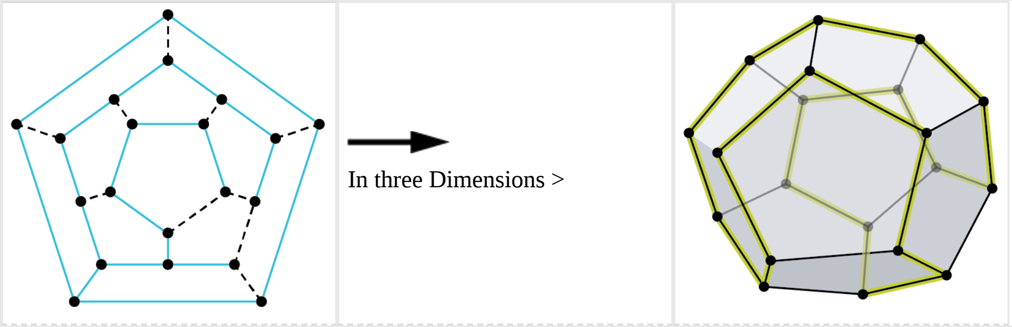

In three Dimensions >

Figure 12.7.5: A Solution to Hamilton’s Puzzle

A circuit that doesn’t repeat any vertices, like the one in Figure 12.181, is called a directed cycle. So, we can most accurately say that Hamilton’s puzzle asks us to find a directed cycle that visits every vertex in a graph exactly once. Because Hamilton created and solved this puzzle, these special circuits were named Hamilton cycles, or Hamilton circuits.

Who Knew?: Icosian Game

When Hamilton invented his puzzle in the 19th century, it was originally called the Icosian game. This was a reference to an icosahedron, not a dodecahedron. Why? Hamilton was actually interested in the symmetries of icosahedrons. It turns out that these two space figures are related to each other. The dodecahedron has 30 edges, 20 faces, and 12 vertices, while the icosahedron has 30 edges, 12 faces, and 20 vertices. By relating the vertices of the dodecahedron to the faces of the icosahedron, Hamilton was able to make the mathematical connections necessary to use graph theory and dodecahedrons to make discoveries about the symmetries of icosahedrons.

Hamilton Cycles vs. Euler Circuits

Let’s practice naming and identifying Hamilton cycles, as well as distinguishing them from Euler circuits. It is important to remember that Euler circuits visit all edges without repetition, while Hamilton cycles visit all vertices without repetition. Hamilton cycles are named by their vertices just like all circuits. An example is given in Figure 12.7.6.

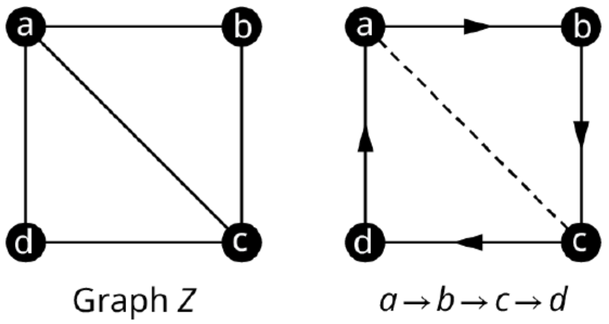


Figure 12.7.6: Hamilton Cycle in Graph Z

Notice that the Hamilton cycle $a \rightarrow b \rightarrow c \rightarrow d$ for Graph Z in Figure 12.182 is NOT an Euler circuit, because it does not visit edge semanticsacac. Some Hamilton cycles are also Euler circuits while some are not, and some Euler circuits are Hamilton cycles while some are not.

Checkpoint

TIP! Sometimes students confuse Euler circuits with Hamilton cycles. To help you remember, think of the E in Euler as standing for Edge.

✓ Example 12.7.1: Differentiating between Hamilton Cycles or Euler Circuits

Use Figure ✓ 12.7.7 to determine whether the given circuit is a Hamilton cycle, an Euler circuit, both, or neither.

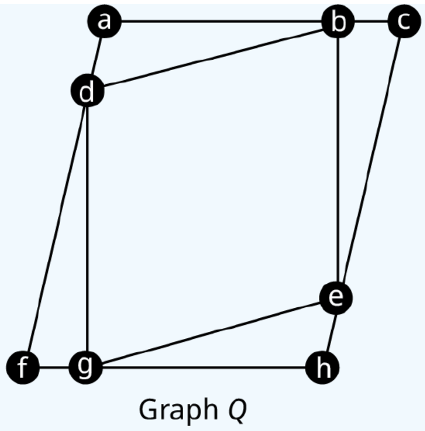


Figure 12.183 Graph Q

1. $a \to b \to c \to e \to h \to g \to f \to d \to a$
2. $g \to e \to h \to g \to f \to d \to a \to b \to d \to g$
3. $a \to b \to c \to e \to h \to g \to f \to d \to b \to e \to g \to d \to a$

Answer

1. This circuit is a Hamilton cycle only. It visits each vertex exactly once, so, it is a Hamilton cycle. It is not an Euler circuit because it doesn't visit all of the edges.
2. This circuit is neither Hamilton cycle nor an Euler circuit. It doesn't visit vertex c, so, it is not a Hamilton cycle. It also doesn't visit edges semanticsbebe and semanticscece, so, it is not an Euler circuit.
3. This circuit is an Euler circuit only. It visits several vertices more than once; so, it is not a Hamilton cycle. It visits every edge exactly once, so, it is an Euler circuit.

Checkpoint

TIP! Euler circuits never skip a vertex, so, they only fail to be Hamilton cycles when they visit a vertex more than once. Hamilton cycles never visit any edges twice, so, they only fail to be Euler circuits when they skip edges.

Your Turn 12.7.1

Consider the figure shown. Is the circuit $a \to b \to f \to e \to h \to i \to g \to d \to c \to a$ an Euler circuit, a Hamilton cycle, or both?

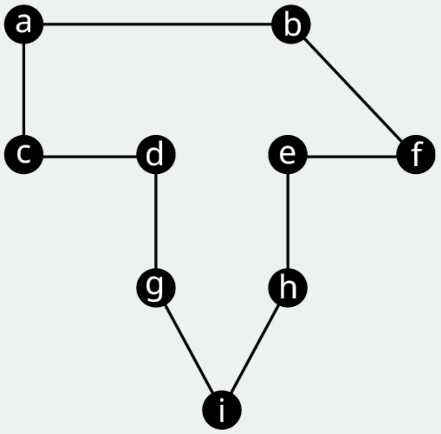


Figure 12.7.8: Graph T

Notice that the graph is a cycle. A cycle will always be Eulerian because all vertices are degree 2. Moreover, any circuit in the graph will always be both an Euler circuit and a Hamilton cycle. It is not always as easy to determine if a graph has a Hamilton cycle as it is to see that it has an Euler circuit, but there is a large group of graphs that we know will always have Hamilton cycles, the complete graphs. Since all vertices in a complete graph are adjacent, we can always find a directed cycle that visits all the vertices. For example, look at the directed six-cycle, $n \to o \to p \to q \to r \to s$, in the complete graph with six vertices in Figure 12.7.9.

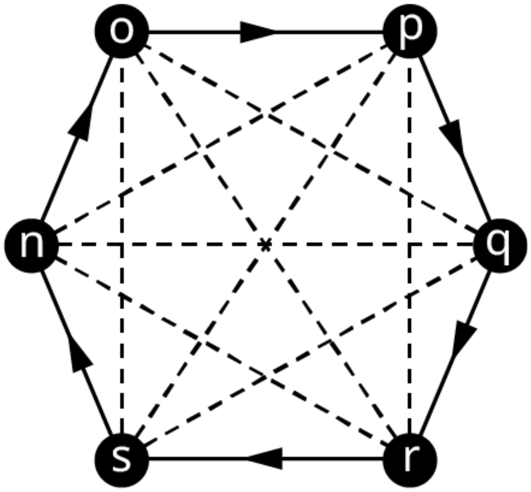


Figure 12.7.9: Directed Cycle in Complete Graph

That is not the only directed six-cycle in the graph though. We could find another just be reversing the direction, and we could find even more by using different edges. So, how many Hamilton cycles are in a complete graph with n vertices? Before we tackle this problem, let's look at a shorthand notation that we use in mathematics which will be helpful to us.

Factorials

In many areas of mathematics, we must calculate products like $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ or $11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$, products that involve multiplying all the counting numbers from a particular number down to 1. Imagine that the product happened to be all the numbers from 100 down to 1. That's a lot of writing! Instead of writing all of that out, mathematicians came up with a shorthand notation. For example, instead of $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$, we write $7!$, which is read "7 factorial." In other words, the product of all the counting numbers from n down to 1 is called n factorial and it is written $n!$

✓ Example 12.7.2: Evaluating Factorials

Evaluate $n!$ and $(n-1)!$ for $n = 4$.

Answer

$n! = 4! = 4 \cdot 3 \cdot 2 \cdot 1 =$24n!=4!=4·3·2·1=24 and $(n-1)\,! = (4-1)! = 3! = 3 \cdot 2 \cdot 1 = 6$

Your Turn 12.7.12

Evaluate $n!$ and $(n-1)!$ for $n = 6$.

A common use for factorials is counting the number of ways to arrange objects. Suppose that there were three students, Aryana, Byron, and Carlos, who wanted to line up in a row. How many arrangements are possible? There are six possibilities: ABC, ACB, BAC, BCA, CAB, or CBA. Notice that there were three students being arranged, and the number of possible arrangements is three.

FORMULA

The number of ways to arrange n distinct objects is $n!$.

Example 12.7.3: Counting Arrangements of Letters

Find the number of ways to arrange the letters a, b, c, and d.

Answer

$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

Your Turn 12.7.3

Find the number of ways to arrange the letters v, w, x, y, and z.

Counting Hamilton Cycles in Complete Graphs

Now, let's get back to answering the question of how many Hamilton cycles are in a complete graph. In Table 12.8, we have drawn all the four cycles in a complete graph with four vertices. Remember, cycles can be named starting with any vertex in the cycle, but we will name them starting with vertex a.

Table 12.7.1 Four-Cycles in Complete Graph with Four Vertices

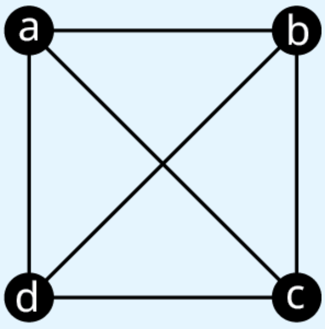
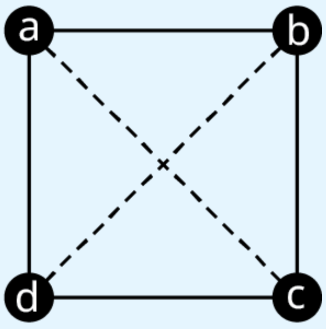
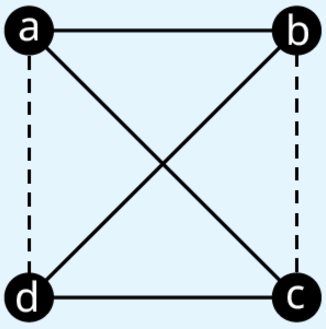
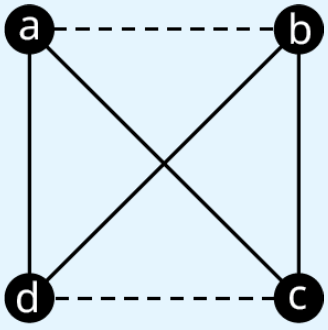
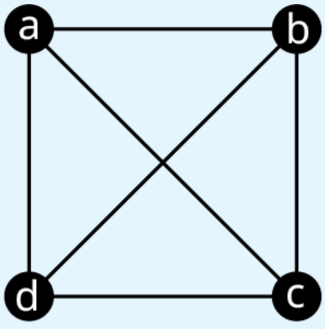
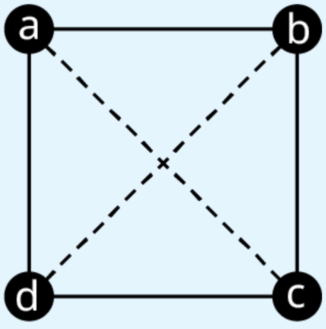
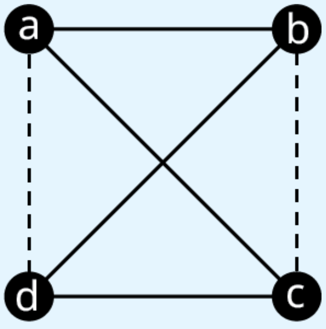
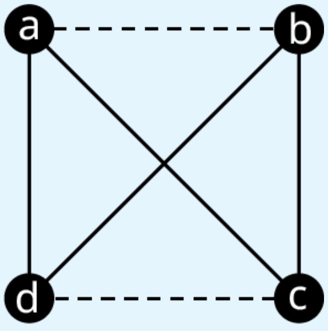
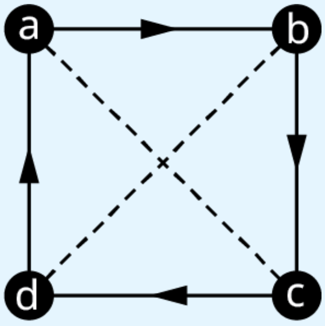
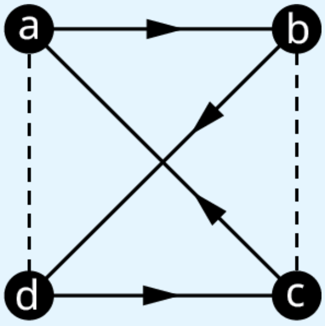
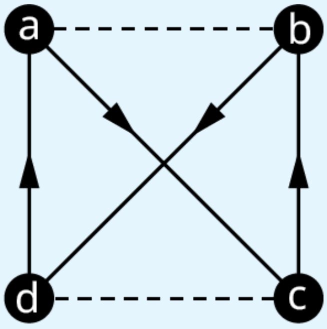
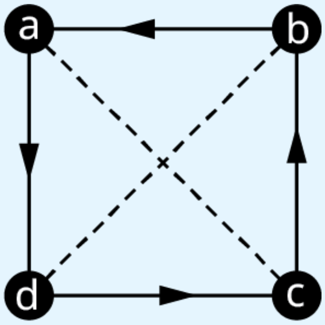
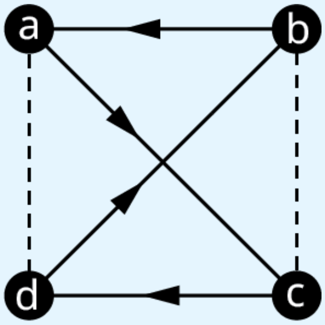
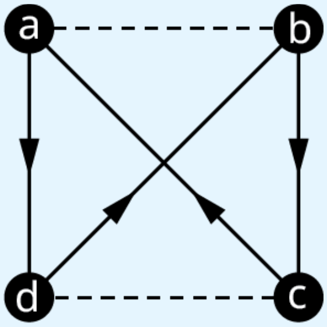
Complete Graph	Cycle	Cycle	Cycle
			
Cycle Name Clockwise	a, b, c, d	a, b, d, c	a, c, b, d
Cycle Name Counterclockwise	a, d, c, b	a, c, d, b	a, d, b, c

Table 12.7.1 shows that there are three unique four-cycles in a complete graph with four vertices. Notice that there were two ways to name each cycle, one reading the vertices in a clockwise direction and one reading the vertices in a counterclockwise direction. This is important to us because we are interested in Hamilton cycles, which are *directed* cycles. Although the cycles *(a, b, c, d)* and *(a, d, c, b)* are the same cycle, the directed cycles, $a \to b \to c \to d \to a$ and $a \to d \to c \to b \to a$, which travel the same route in reverse order are considered different directed cycles, as shown in Table 12.7.2.

Table 12.7.2 Hamilton Cycles in a Complete Graph with Four Vertices

Complete Graph	Cycle	Cycle	Cycle
			

Clockwise Hamilton Cycle	 $a \to b \to c \to d \to a$	 $a \to b \to d \to c \to a$	 $a \to c \to b \to d \to a$
Counter-clockwise Hamilton Cycle	 $a \to d \to c \to b \to a$	 $a \to c \to d \to b \to a$	 $a \to d \to b \to c \to a$

The six directed four-cycles in Table $12.7.2$ are the only distinct Hamilton cycles in a complete graph with four vertices. Six is also the number of ways to arrange the three letters b, c, and d. (Do you see why?) The number of ways to arrange three letters is $3! = 3 \cdot 2 \cdot 1 = 6$. Similarly, the number of Hamilton cycles in a graph with five vertices is the number of ways to arrange four letters, which is $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. In general, to find the number of Hamilton cycles in a graph, we take one less than the number of vertices and find its factorial.

FORMULA

The number of distinct Hamilton cycles in a complete graph with n vertices is $(n-1)!$

✓ Example $12.7.4$: Counting Hamilton Cycles in a Complete Graph

How many Hamilton cycles are in the complete graph in Figure ✓ $12.7.10$?

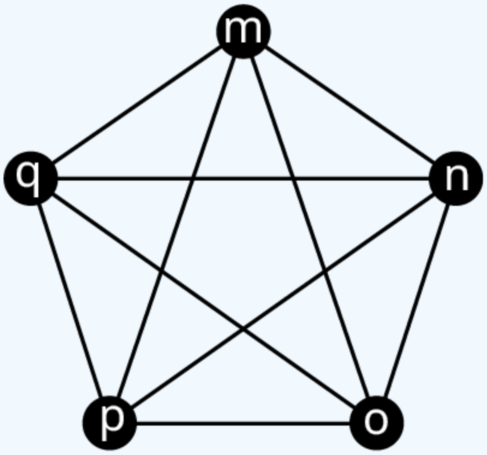


Figure ✓ $12.7.10$ Complete Graph L

Answer

There are five vertices in the graph. Using $n = 5$, we have $(n-1)! = (5-1)! = 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ Hamilton cycles.

Your Turn 12.7.4

How many Hamilton cycles are in the graph?

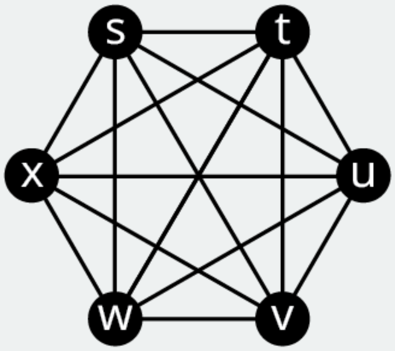


Figure 12.7.11: Complete Graph *R*

Weighted Graphs

Suppose that an officer in the U.S. Air Force who is stationed at Vandenberg Air Force base must drive to visit three other California Air Force bases before returning to Vandenberg. The officer needs to visit each base once. The vertices in the graph in Figure 12.7.12 represent the four U.S. Air Force bases, Vandenberg, Edwards, Los Angeles, and Beale. The edges are labeled to with the driving distance between each pair of cities.

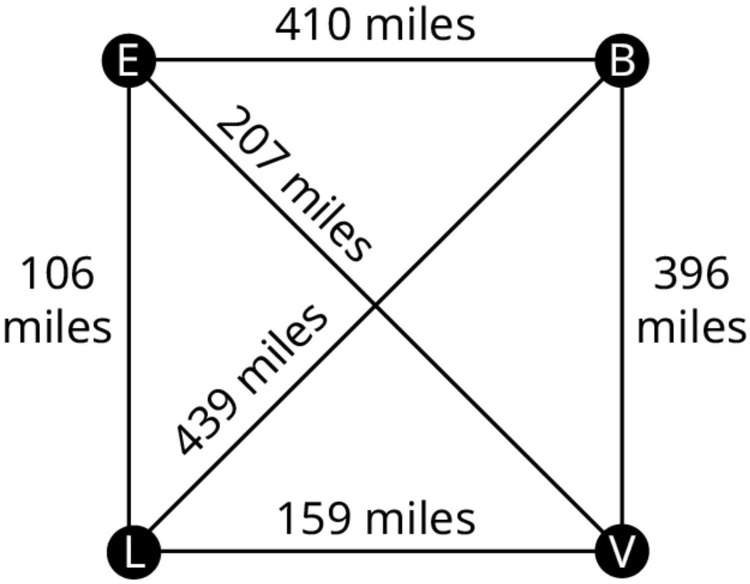


Figure 12.7.12: Graph of Four California Air Force Bases

The graph in Figure 12.7.12 is called a weighted graph, because each edge has been assigned a value or weight. The weights can represent quantities such as time, distance, money, or any quantity associated with the adjacent vertices joined by the edges. The total weight of any walk, trail, or path is the sum of the weights of the edges it visits.

Notice that the officer's trip can be represented as a Hamilton cycle, because each of the four vertices in the graph is visited exactly once.

✓ Example 12.7.5: Finding Hamilton Cycles of Lowest Weight

Use Figure ✓ 12.7.12 and the given Hamilton cycles to answer the following questions.

$V \to L \to E \to B \to V$

$V \to L \to B \to E \to V$

$V \to E \to L \to B \to V$

$V \to B \to E \to L \to V$

1. Which of the Hamilton cycles (directed cycles) lie on the same cycle (undirected cycle) in the graph?
2. Find the total weight of each cycle.
3. Of the four, which of the Hamilton cycles describes the shortest trip for the officer? Describe the route.

Answer

1. $V \to L \to E \to B \to V$ and $V \to B \to E \to L \to V$ follow the same edges in reverse order.
2. Any Hamilton cycles that lie on the same cycle will have the same edges and the same total weight.

   $V \to L \to E \to B \to V$ and $V \to B \to E \to L \to V$ each have total weight $159 + 106 + 410 + 396 = 1071$.

   $V \to L \to B \to E \to V$ has a total weight $159 + 439 + 410 + 207 = 1215$.

   $V \to E \to L \to B \to V$ has a total weight $396 + 439 + 106 + 207 = 1148$.
3. Hamilton cycles $V \to L \to E \to B \to V$ and $V \to B \to E \to L \to V$ each have the lowest total weight. The officer would take the route from Vandenberg, to Los Angeles, to Edwards, to Beale, and back to Vandenberg, or reverse that route.

Your Turn 12.7.5

Find the weight of the Hamilton cycle $m \to o \to p \to n \to q \to m$ in the given figure.

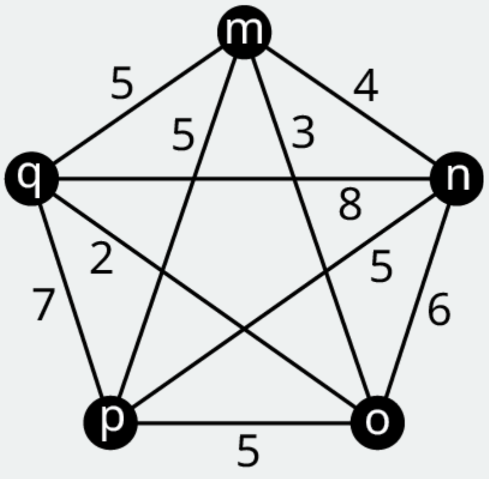


Figure 12.7.13: Weighted Complete Graph with Five Vertices

Check Your Understanding

For the following exercises, fill in the blank to make the statement true.

1. A Hamilton cycle is a circuit that visits each ___________ exactly once.
2. A ___________ graph with $n \geq 3$ vertices has $(n-1)!$ Hamilton cycles.
3. For the following exercises, fill in the blank with **is** or **is not** to make the statement true.
4. A Hamilton cycle ________ a circuit.
5. A Hamilton cycle that visits every edge ________ an Euler circuit.
6. A Hamilton cycle ________ different from a Hamilton circuit.

7. An Euler circuit that visits every vertex _________ a Hamilton cycle.
8. The total weight of a trail ________ the sum of the weights of the edges visited by the trail.
9. A weighted graph ________ always a complete graph.
10. The number of ways to arrange n objects ________ $(n-1)!$
11. Every cycle _____ a circuit.

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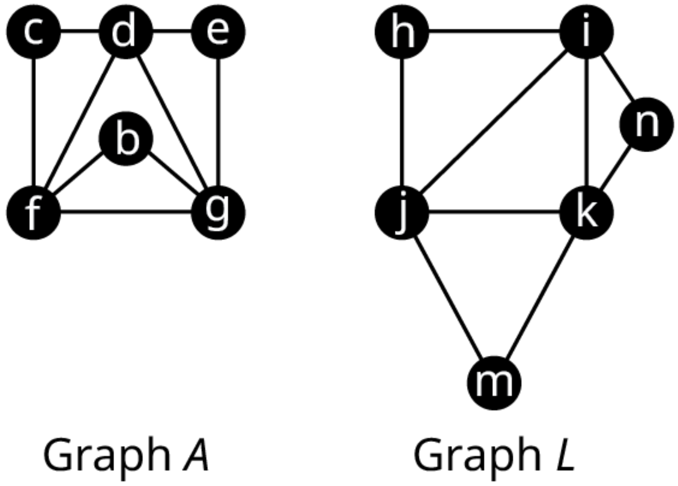
12.7.0: Exercises

For the following exercises, use the figure to determine whether the sequence of vertices in the given graph is a Hamilton cycle, an Euler circuit, both, or neither.



? Exercise 12.7.0.1

Graph A: $f \rightarrow b \rightarrow g \rightarrow e \rightarrow d \rightarrow c \rightarrow f$

? Exercise 12.7.0.2

Graph A: $g \rightarrow b \rightarrow f \rightarrow c \rightarrow d \rightarrow e \rightarrow g$

? Exercise 12.7.0.3

Graph A: $d \rightarrow e \rightarrow g \rightarrow d \rightarrow f \rightarrow b \rightarrow g \rightarrow f \rightarrow c \rightarrow d$

? Exercise 12.7.0.4

Graph L: $h \rightarrow i \rightarrow k \rightarrow n \rightarrow j \rightarrow h$

? Exercise 12.7.0.5

Graph L: $n \rightarrow i \rightarrow h \rightarrow j \rightarrow m \rightarrow k \rightarrow n$

? Exercise 12.7.0.6

Graph L: $j \rightarrow i \rightarrow n \rightarrow k \rightarrow i \rightarrow j \rightarrow k \rightarrow m \rightarrow j$

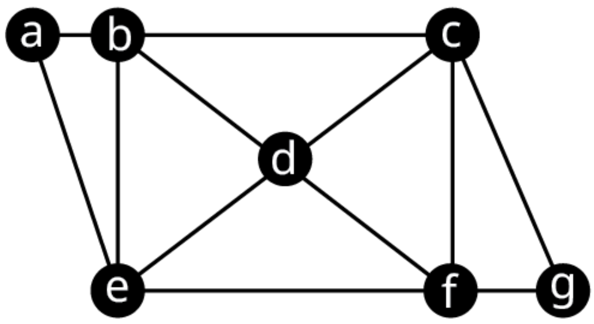
? Exercise 12.7.0.7

Graph U: $v \rightarrow w \rightarrow r \rightarrow s \rightarrow t \rightarrow o \rightarrow q \rightarrow v$

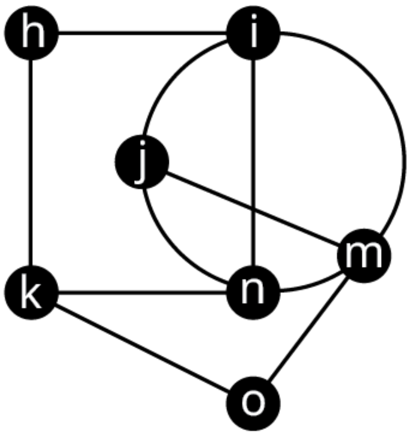
? Exercise 12.7.0.8

Graph U: $w \rightarrow q \rightarrow r \rightarrow s \rightarrow t \rightarrow o \rightarrow v \rightarrow w$

For the following exercises, use the figure to find a circuit that fits the description.



Graph P



Graph Q

? Exercise 12.7.0.9

A Hamilton cycle in Graph P that begins at vertex c.

? Exercise 12.7.0.10

An Euler circuit in Graph P that begins at vertex c.

? Exercise 12.7.0.11

A directed cycle in Graph P that is NOT a Hamilton cycle, and explain why it is not a Hamilton cycle.

? Exercise 12.7.0.12

A directed circuit in Graph P that is NOT an Euler circuit, and explain why it is not an Euler circuit.

? Exercise 12.7.0.13

A Hamilton cycle in Graph Q that begins at vertex n.

? Exercise 12.7.0.14

An Euler circuit in Graph Q that begins at vertex n.

? Exercise 12.7.0.15

A directed cycle in Graph Q that is NOT a Hamilton cycle, and explain why it is not a Hamilton cycle.

? Exercise 12.7.0.16

A directed circuit in Graph Q that is NOT an Euler circuit, and explain why it is not an Euler circuit.

? Exercise 12.7.0.17

Use the results of Exercises 9–16 to make an observation about which tends to involve a longer sequence of vertices, Hamilton cycles or Euler circuits. Explain why you think this is.

For the following exercises, evaluate the factorial expression for the given value of n.

? Exercise 12.7.0.18

$n!$, $n = 10$

? Exercise 12.7.0.19

$n!$, $n = 11$

? Exercise 12.7.0.20

$(n-1)!$, $n = 10$

? Exercise 12.7.0.21

$(n-1)!$, $n = 11$

For the following exercises, find the number of arrangements letters in the given word.

? Exercise 12.7.0.22

have

? Exercise 12.7.0.23

teamwork

? Exercise 12.7.0.24

anime

? Exercise 12.7.0.25

making

For the following exercises, find the number of Hamilton cycles in a complete graph with the given number of vertices.

? Exercise 12.7.0.26

12 vertices

? Exercise 12.7.0.27

13 vertices

? Exercise 12.7.0.28

9 vertices

? Exercise 12.7.0.29

8 vertices

Exercise 12.7.0.30

x vertices

For the following exercises, given the number of Hamilton cycles in a complete graph, determine the number of vertices.

Exercise 12.7.0.31

$5! = 120$ Hamilton cycles

Exercise 12.7.0.32

$6! = 720$ Hamilton cycles

Exercise 12.7.0.33

$7! = 5040$ Hamilton cycles

Exercise 12.7.0.34

$x!$ Hamilton cycles

For the following exercises, all the distinct Hamilton cycles for a complete graph are given. Indicate which pairs of Hamilton cycles (directed cycles) lie on the same cycle (undirected cycle) in the graph.

Exercise 12.7.0.35

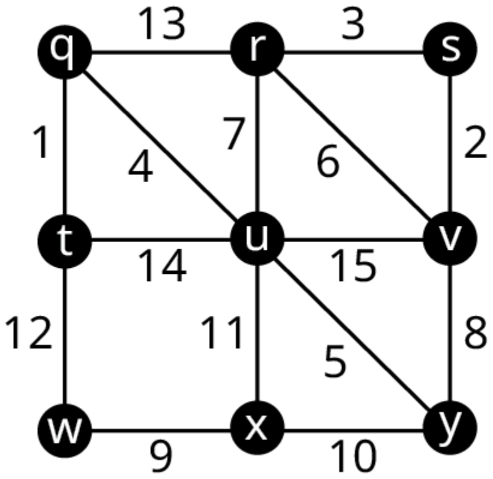
1. $b \to a \to c \to d \to b$
2. $b \to a \to d \to c \to b$
3. $b \to c \to a \to d \to b$
4. $b \to c \to d \to a \to b$
5. $b \to d \to a \to c \to b$
6. $b \to d \to c \to a \to b$

Exercise 12.7.0.36

1. $i \to f \to g \to h \to e \to i$
2. $i \to f \to g \to e \to h \to i$
3. $i \to f \to h \to g \to e \to i$
4. $i \to f \to h \to e \to g \to i$
5. $i \to f \to e \to g \to h \to i$
6. $i \to f \to e \to h \to g \to i$
7. $i \to g \to f \to h \to e \to i$
8. $i \to g \to f \to e \to h \to i$
9. $i \to g \to h \to f \to e \to i$
10. $i \to g \to h \to e \to f \to i$
11. $i \to g \to e \to f \to h \to i$
12. $i \to g \to e \to h \to f \to i$
13. $i \to h \to g \to f \to e \to i$
14. $i \to h \to g \to e \to f \to i$
15. $i \to h \to f \to g \to e \to i$
16. $i \to h \to f \to e \to g \to i$
17. $i \to h \to e \to g \to f \to i$
18. $i \to h \to e \to f \to g \to i$

19. $i \to e \to g \to h \to f \to i$
20. $i \to e \to g \to f \to h \to i$
21. $i \to e \to h \to g \to f \to i$
22. $i \to e \to h \to f \to g \to i$
23. $i \to e \to f \to g \to h \to i$
24. $i \to e \to f \to h \to g \to i$

For the following exercises, use the figure to find the weight of the given Hamilton cycle.



Exercise 12.7.0.37

$q \to r \to s \to v \to y \to x \to w \to t \to u \to q$

Exercise 12.7.0.38

$u \to y \to x \to w \to t \to q \to r \to s \to v \to u$

Exercise 12.7.0.39

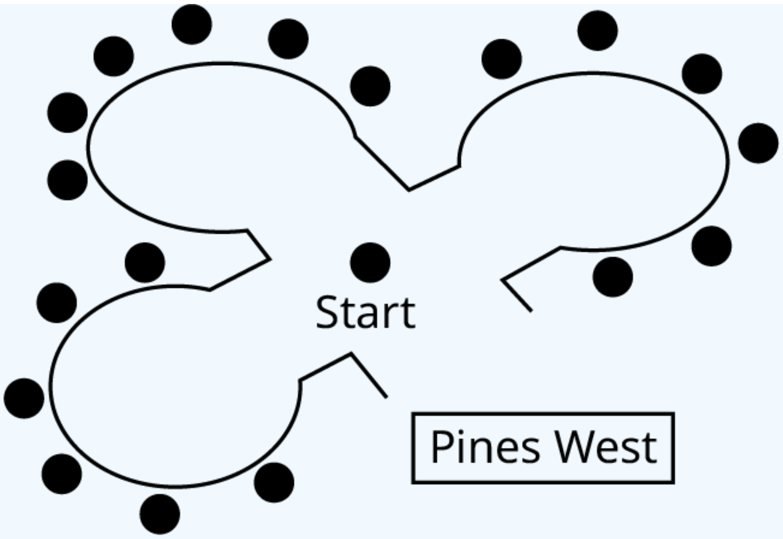
$y \to v \to s \to r \to u \to q \to t \to w \to x \to y$

Exercise 12.7.0.40

$u \to v \to s \to r \to q \to t \to w \to x \to y \to u$

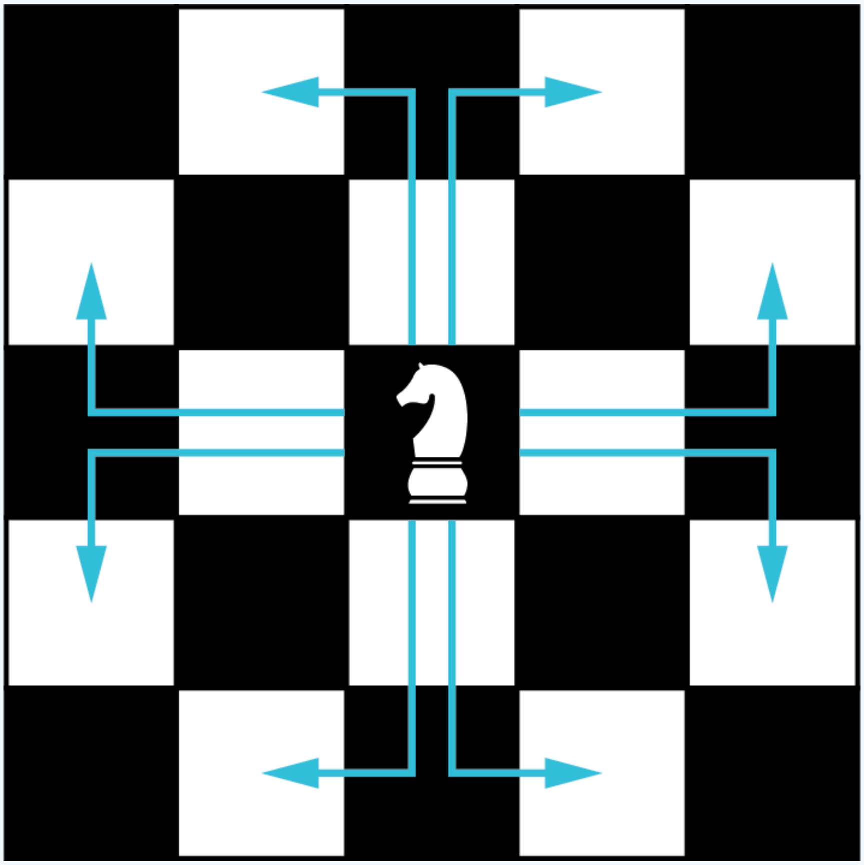
Exercise 12.7.0.41

The neighborhood of Pines West has three cul-de-sacs that meet at an intersection as shown. A postal delivery person starts at the intersection and visits each house in a cul-de-sac once, returns to the intersection, visits each house in the next cul-de-sac, and so on, returning to the intersection when finished. Describe how the route can be represented as a graph. If there is no backtracking, in other words, the person never reverses direction, is the route followed by the postal delivery person best described as a trail, an Euler circuit, a Hamilton cycle, or neither? Explain your reasoning.

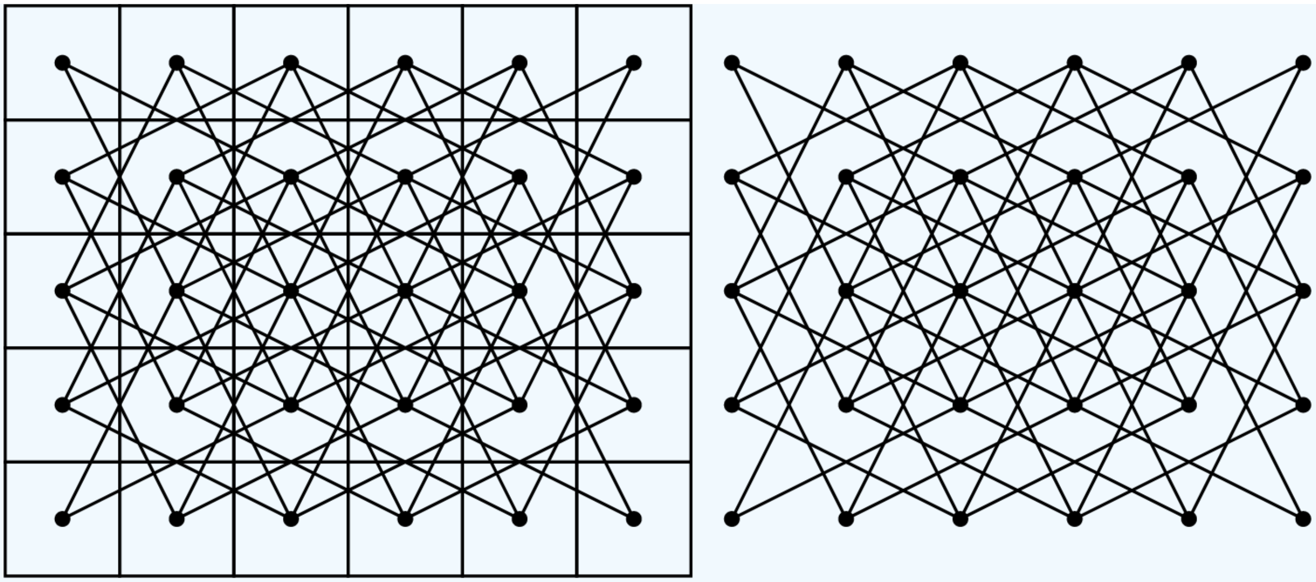


Exercise 12.7.0.42

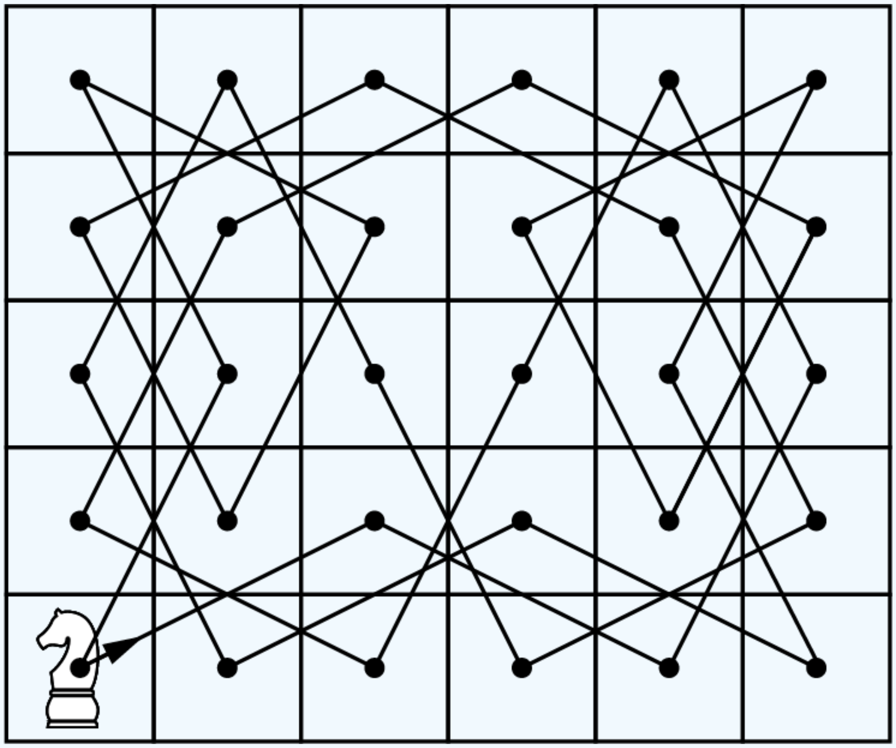
In chess, a knight can move in any direction, but it must move two spaces then turn and move one more space. The 8 possible moves a knight can make from a given space are shown in the figure.



A graph in which each vertex represents a space on a five-by-six game board and each edge represents a move a knight could make is shown in the figure.



A knight's tour is a sequence of moves by a knight on a chessboard (of any size) such that the knight visits every square exactly once. If the knight's tour brings the knight back to its starting position on the board, it is called a closed knight's tour. Otherwise, it is called an open knight's tour.



Determine if the closed knight's tour in Figure 12.239 is most accurately described as an Euler circuit or a Hamilton cycle, or both, of the graph of all possible knight moves. Explain your reasoning.

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12.8: Hamilton Paths

Figure 12.8.1 A school bus picks up children along a planned route. (credit: “Kids at School Bus Stop” by Ty Hatch/Flickr, CC BY 2.0)

Learning Objectives

1. Describe and identify Hamilton paths.
2. Evaluate Hamilton paths in real-world applications.
3. Distinguish between Hamilton paths and Euler trails.

In the United States, school buses carry 25 million children between school and home every day. The total distance they travel is around 6 billion kilometers per year. In the city of Boston, Massachusetts, the 2016 budget for running those buses was $120 million dollars. In 2017, the city held a competition to find ways to cut costs and the Quantum Team from the MIT Operations Research Center came to the rescue, using a computer algorithm to identify the most efficient and least costly routes, which saved the city of Boston $5 million each year and even reduced daily CO_2 emissions by 9,000 kilograms! (*This* U.S. *city put an algorithm in charge of its school bus routes and saved $5 million*, Sean Fleming, World Economic Forum)

The problem the Quantum Team tackled involves graph theory. Imagine a graph in which vertices are the bus depot, the school, and the bus stops along a particular route. The bus must start at the depot, visit every stop exactly once, and end at the school. The route is a special kind of path that visits every vertex exactly once. Can you guess what those paths are called?

Hamilton Paths

Just as circuits that visit each vertex in a graph exactly once are called Hamilton cycles (or Hamilton circuits), paths that visit each vertex on a graph exactly once are called Hamilton paths. As we explore Hamilton paths, you might find it helpful to refresh your memory about the relationships between walks, trails, and paths by looking at Figure 12.8.1. We know that paths are walks that don’t repeat any vertices or edges. So, a Hamilton path visits every vertex without repeating any vertices or edges. Figure 12.8.2 shows a path from vertex A to vertex E and a Hamilton path from vertex A to vertex E.

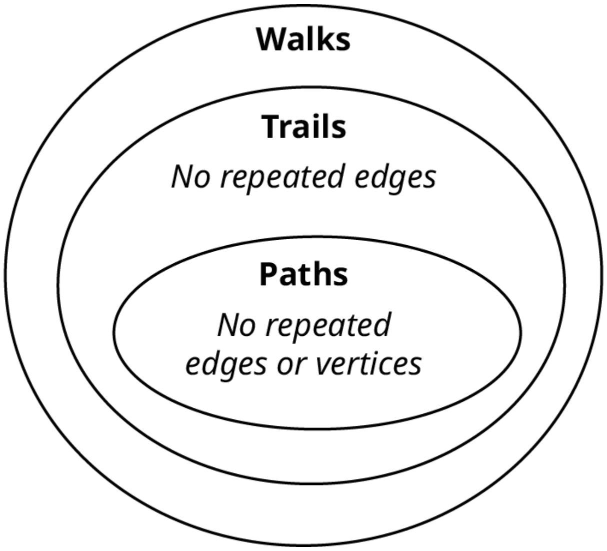

Walks
Trails
No repeated edges
Paths
No repeated edges or vertices

Figure 12.8.1: Walks, Trails, and Paths

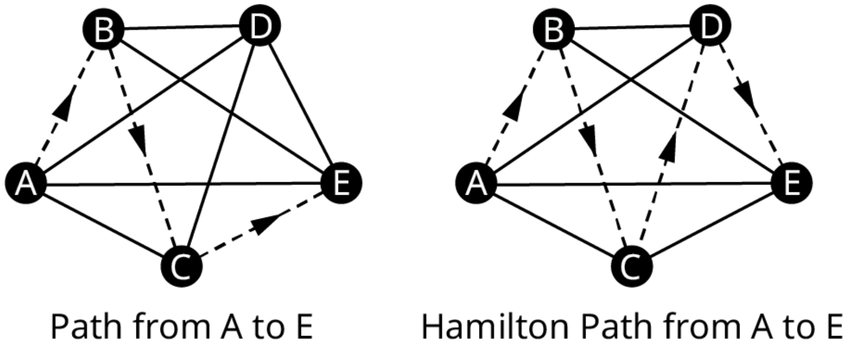

B
D
A
E
C
Path from A to E
B
D
A
E
C
Hamilton Path from A to E

Figure 12.8.2: Path or Hamilton Path?

✓ Example 12.8.1: Identifying Hamilton Paths

Which of the following sequences of vertices is a Hamilton path for Graph Q in Figure ✓ 12.8.3?

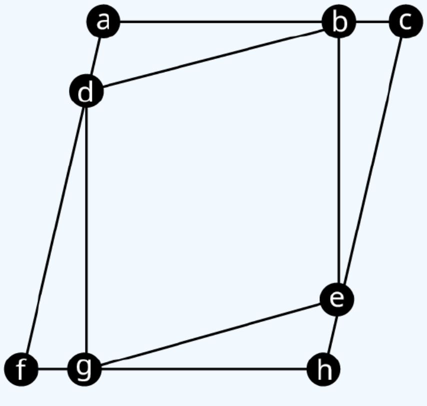


Figure ✓ 12.8.3 Graph Q

1. $a \rightarrow d \rightarrow b \rightarrow c \rightarrow e \rightarrow g \rightarrow f$
2. $c \rightarrow b \rightarrow e \rightarrow h \rightarrow g \rightarrow f \rightarrow d \rightarrow a$
3. $h \rightarrow e \rightarrow g \rightarrow d \rightarrow b \rightarrow e \rightarrow g \rightarrow f \rightarrow d \rightarrow a \rightarrow b \rightarrow c$

Answer

Sequence 1 is a path, because it is a walk that doesn't repeat any vertices or edges, but not a Hamilton path because it skips vertex h. Sequence 2 is a path that visits each vertex exactly once; so, it is a Hamilton path. Sequence 3 is a walk, but it is not a path because it visits vertices g, e, and b each more than once; so, it cannot be a Hamilton path. So, we can see that only sequence 2 is a Hamilton path.

Your Turn 12.8.1

Use the graphs to determine if the given sequence of vertices represents a Hamilton path or not.

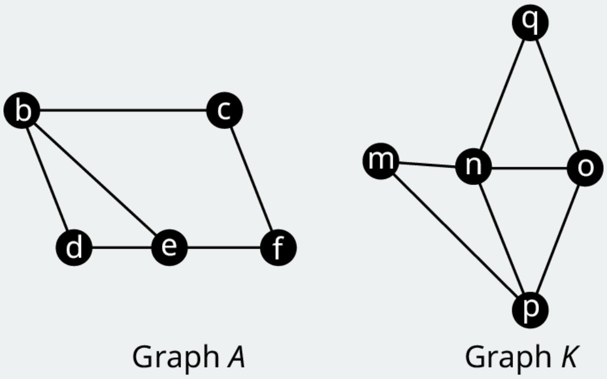


Figure 12.8.4: Graphs A and K

Graph A, $b \to c \to d \to e \to f$

Graph A, $b \to e \to f \to c \to b \to d$

Graph K, $n \to q \to o \to p \to m$

Checkpoint

TIP! Since a Hamilton path visits each vertex exactly once, it must have the same number of vertices listed as appear in the graph.

Finding Hamilton Paths

Suppose you were visiting an aquarium with some friends. The map of the aquarium is given in Figure 12.195. The letters represent the exhibits.

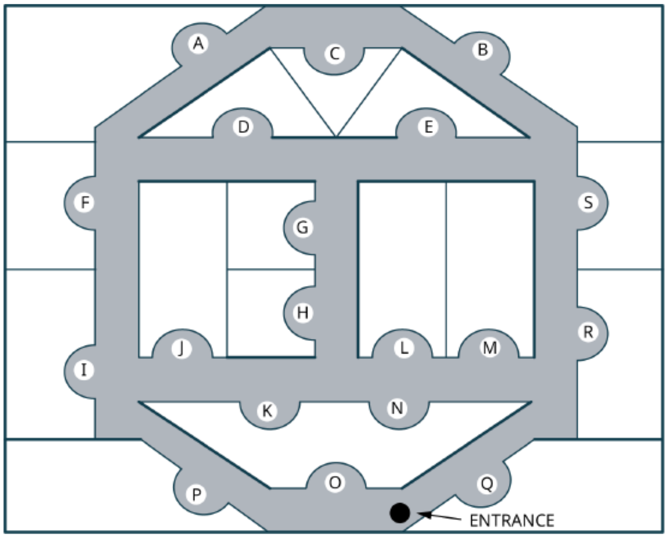


Figure 12.8.5: Map of Aquarium Exhibits

Figure 12.8.6 shows a graph of the aquarium in which each vertex represents an exhibit and each edge is a route between the pair of exhibits that doesn’t bypass another exhibit.

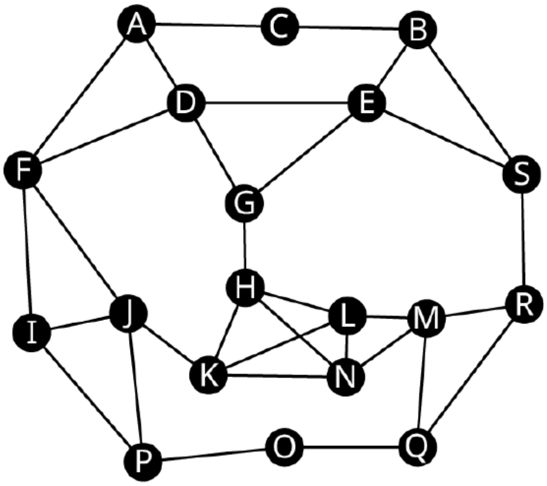


Figure 12.8.6: Graph of Aquarium

Let’s see if we can plan a tour of the exhibits that visits each exhibit exactly once, beginning at exhibit *O* and ending at exhibit *C*. Suppose that, after exhibit *O*, we plan to visit exhibit *Q* and then exhibit *M*. After *M*, should we plan to visit *N*, *L*, or *R*? Take a look at Figure 12.8.7. If *R* is not chosen next, that will cause a problem later on. Do you see what it is?

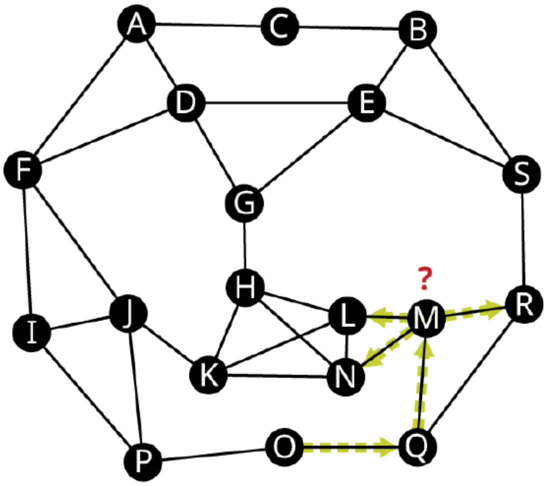


Figure 12.8.7 Choosing Vertex *L*, *N*, or *R*

If *L* or *N* is chosen next, the only way to get to *R* later will be to go from *S* to *R*, and then we will not be able to continue without repeating a vertex. So, we will pick *R* next, and then the only option is *S*. After *S* we have another choice to make. As shown in Figure 12.8.8, the next choice is between *B* and *E*. Keeping in mind that the goal is to end at *C*, which would be the better choice?

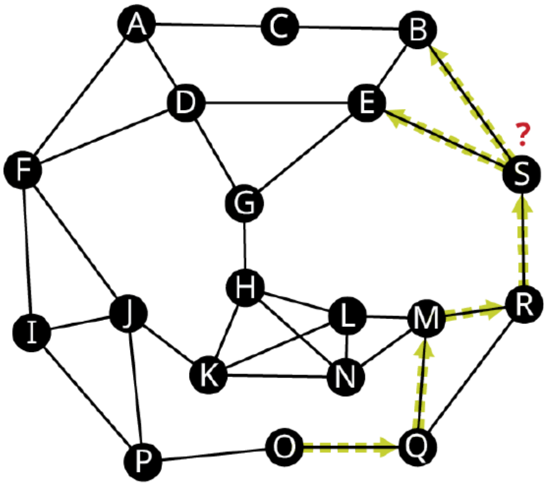


Figure 12.8.8 Choosing Vertex *B* or *E*

If you said vertex *B*, you are right! Otherwise, we will not be able to visit *B* later. After *B*, the only option is *E*. Then we can choose either *D* or *G*. Either will work fine. Let's choose *G* as shown in Figure 12.8.9. After *G*, you must visit *H*, but should you visit *K* or *L* after that?

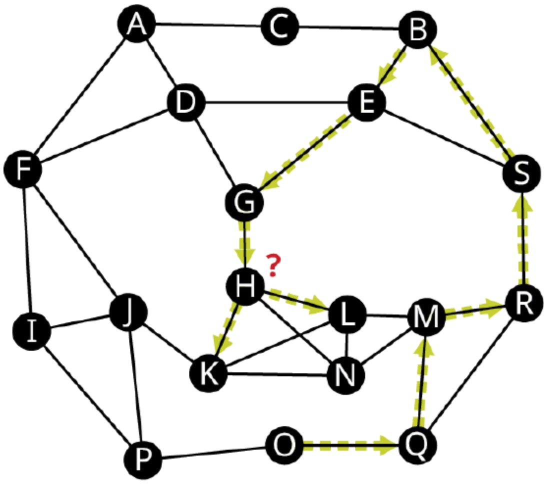


Figure
12.8.9
Choose Vertex *L* or *K*

If you said to go to vertex *L* next, you are right! Otherwise, it will be impossible to visit *N* without repeating a vertex. So, next is *L*, then *N*, then *K*, and then at *J* you have another decision to make we can see in Figure 12.8.10. Should you choose *F*, *I*, or *P* next?

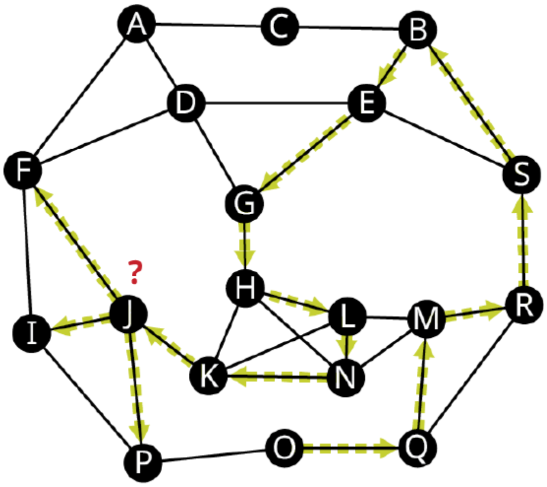


Figure 12.8.10 Choose Vertex *F*, *I*, or *P*

If you said *P*, you are right! If you choose either of the other two vertices, you will not be able to visit *P* later without passing through another vertex twice. Once *P* is chosen, vertex *I* must be next followed by *F*. Then you have to choose between *A* and *D* as shown in Figure 12.8.11.

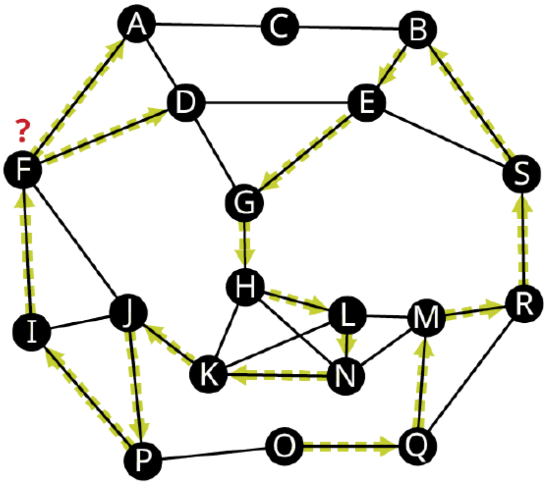


Figure
12.8.11
Choose Vertex *A* or *D*

In this case, we must go to *D* then to *A* so that we can visit *C* without backtracking. The complete Hamilton path is shown in Figure 12.8.12.

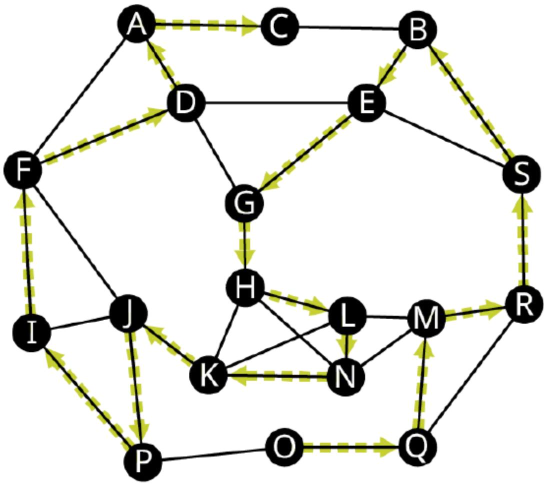


Figure 12.8.12: Complete Hamilton Path from O to C

So, one Hamilton path that begins at O and ends at C is $O \to Q \to M \to R \to S \to B \to E \to G \to H \to L \to N \to K \to J \to P \to I \to F \to D \to A \to C$.

There is no set sequence of steps that can be used to find a Hamilton path if it exists, but it does help to keep in mind where we are headed and avoid choices that will make returning to a particular vertex impossible without repeating vertices. Let's practice finding Hamilton paths.

✓ Example 12.8.2: Finding a Hamilton Path

Use Figure ✓ 12.8.13 to find a Hamilton path between vertices C and D.

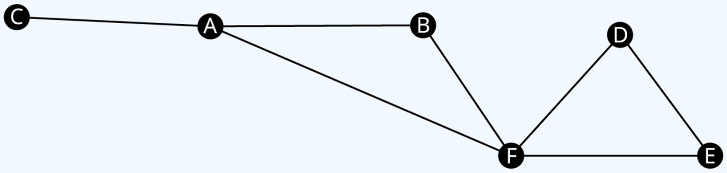


Figure ✓ 12.8.13: Graph G

Answer

If we start at vertex C, A must be next. Then we must choose between B and F. If we choose F, we will have to backtrack to get to include B; so, we must choose B. Once we choose B, we must choose F next. After F, we choose E, because we want to end at D. So, a Hamilton path between C and D is $C \to A \to B \to F \to E \to D$

Your Turn 12.8.2

Use Figure 12.8.13 to find a Hamilton path between vertices C and E.

Existence of a Hamilton Path

It turns out that there is no Hamilton path between vertices A and E in Graph G in Figure 12.8.13. To understand why, let's imagine there is a red apple tree on one side of a bridge and a green apple tree on the other side of the bridge. Now suppose someone asked you to pick up all the fallen apples under each tree without crossing the bridge more than once, and making sure that the first apple you pick up and the last apple you pick up are both red. You would say, that is impossible! To have the first and last apple be red would either require leaving the green apples on the ground or crossing the bridge twice.

Let's see how this relates to finding a Hamilton path between A and E in Graph G. The edge AC is a bridge because, if it were removed, the graph would become disconnected with two components, the component $\{C\}$ and the component $\{A, B, D, E, F\}$. So, we can think of the vertices A, B, D, E, and F as the red apples, vertex C as the green apple, and the edge AC is the bridge between them as in Figure 12.8.14.

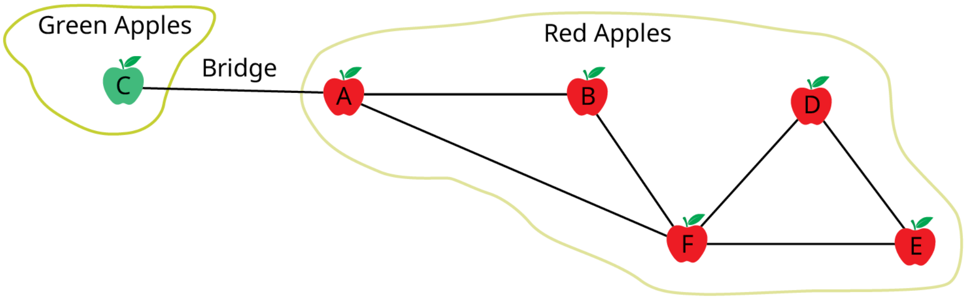


Figure 12.8.14: Bridge between Red and Green Apples

The creation of a Hamilton path requires a visit to each vertex, just as picking up all the apples requires a visit to each apple. A and E are both red apples; so, a path from A to E would both start and end at a red apple, just as you were asked to do. And you wouldn't be able to cross the bridge twice because that would mean visiting A twice, which is not allowed in a Hamilton path. So, it is impossible to find a Hamilton path from A to E just as it was impossible to pick up all the apples without crossing the bridge more than once. By the same reasoning, if a graph has a bridge, there will never be a Hamilton path that begins and ends on the same side of that bridge, meaning beginning and ending at vertices that would be in the same component if the bridge were removed from the graph.

✓ Example 12.8.3: Finding a Hamilton Path If One Exists

Find a Hamilton path from vertex s to vertex v for each graph in Figure ✓ 12.8.15 or indicate that there is none.

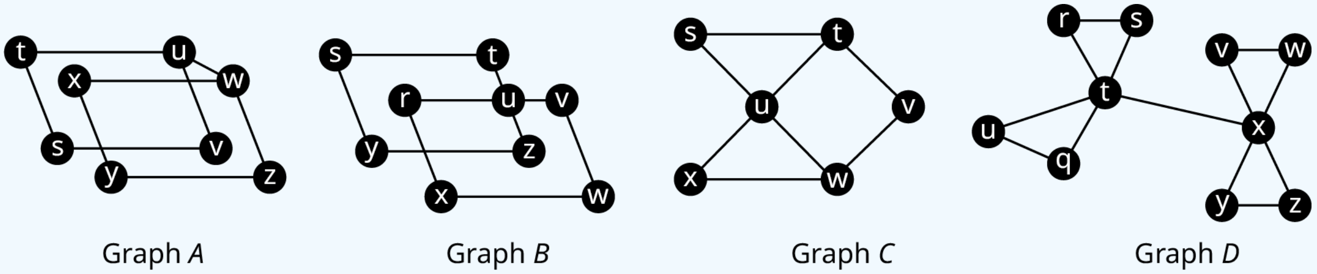


Figure ✓ 12.8.15 Graphs A, B, C, and D

Answer

Graph A: Edge uw is a bridge connecting component $\{s, t, u, v\}$ to the component $\{w, x, y, z\}$. There is no Hamilton path from vertex s to vertex v because they would be part of the same component if the bridge uw were removed.

Graph B: There are no bridges in Graph B. The only method we have to determine if a Hamilton path from vertex s to vertex v exists is to try every possibility. From vertex s, we can visit either vertex y or vertex t. We will try vertex y first and then come back to see what happens with vertex t. After visiting y, we must visit z and then u, but then we have to decide between vertices r, t, and v next as shown in Figure ✓ 12.8.16.

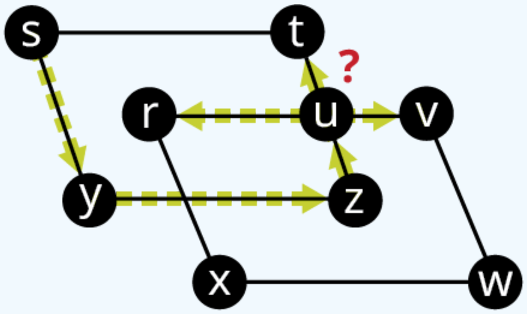


Figure ✓ 12.8.16 Choose Vertex r, t, or v

Vertex *v* is not an option since we want to end at *v*. Vertex *t* is not an option since that would force us to go to visit *s* a second time. So, we must go to vertex *r* next. After vertex *r*, we must visit *x*, then *w*, then *v*, but we missed vertex *t* as shown in Figure ✓ 12.8.17.

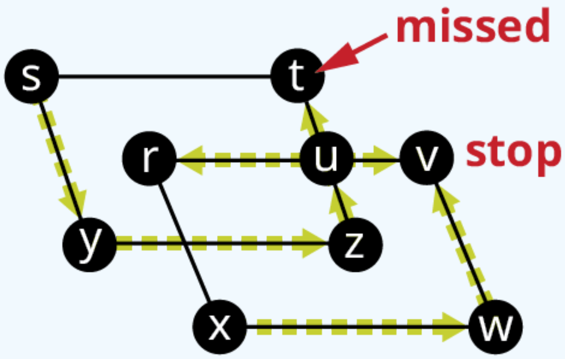


Figure ✓ 12.8.17: Missed Vertex *t*

Let's go back to the beginning and choose *t* instead of *y*. After *t*, we must go to *u* and then we have a choice to make between *r*, *v*, and *z* as shown in Figure ✓ 12.8.18.

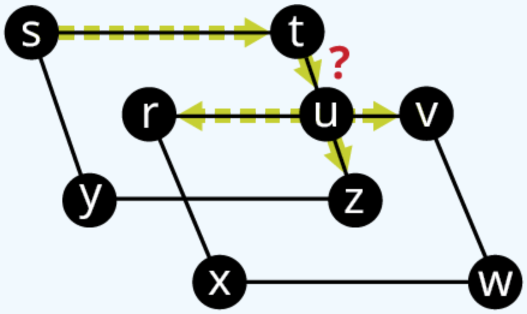


Figure ✓ 12.8.18 Choose Vertex *v*, *r*, or *z*

Vertex *v* is not an option since we want to end at *v*. Vertex *z* is not an option since that would force us to go *y* and then to visit *s* a second time. So, we must go to vertex *r* next. After *r*, we must go to *x* then *w* then *v*, where we have to stop even though we have missed vertices *y* and *z*, as shown in Figure ✓ 12.8.19.

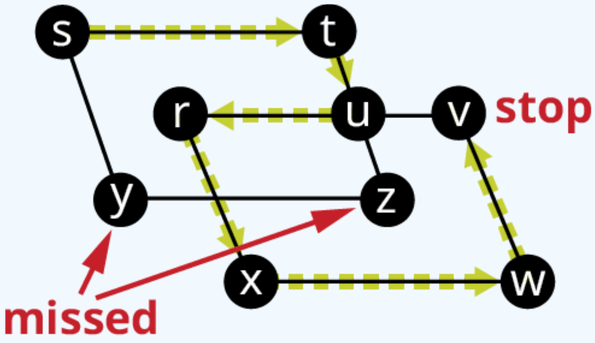
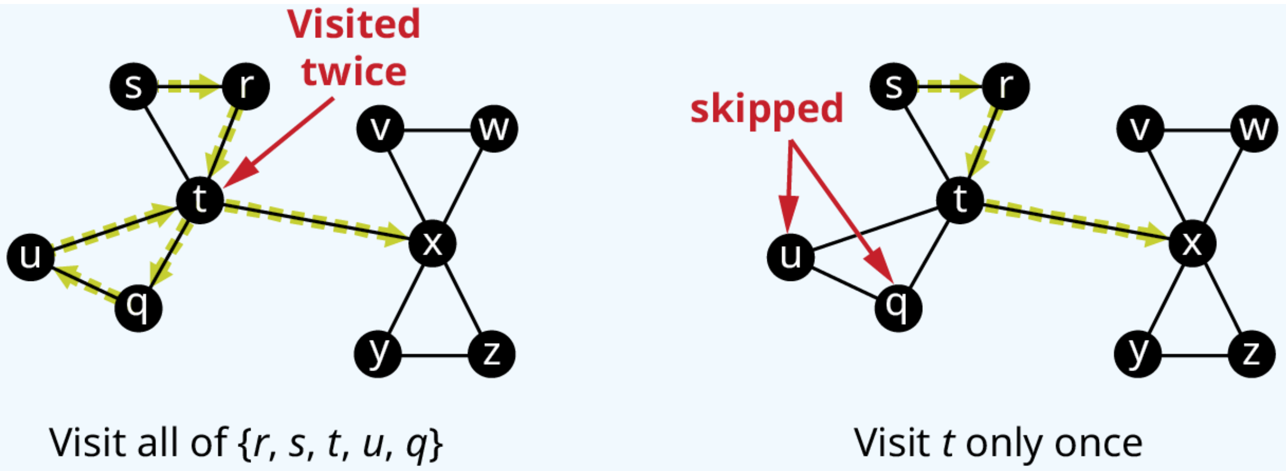


Figure ✓ 12.8.19 Missed Vertices *y* and *z*

So, we have tried every possible route and there are no Hamilton paths between *s* and *v* in Graph *B*.

Graph *C*: In Graph *C*, there is a Hamilton path, $s \rightarrow t \rightarrow u \rightarrow x \rightarrow w \rightarrow v$.

Graph *D*: In Graph *D*, there is a bridge, *tx*, which would form components $\{r, s, t, u, q\}$ and $\{v, w, x, y, z\}$ if it were removed. Since *s* and *v* would be in different components, it is possible there is a Hamilton path between them. The only way to know is to try all possibilities. If we begin at *s*, we can go to *r* then *t*, or we can go directly to *t*, either way, we have a problem as you can see in Figure ✓ 12.8.20.



Visit all of $\{r, s, t, u, q\}$ Visit t only once

Figure 12.8.20: Vertices Visited Twice or Skipped

If we visit all the vertices in the component $\{r, s, t, u, q\}$, we will have to visit t a second time in order to cross the bridge. If we visit t only once, we have to skip some of the vertices. So, there is no Hamilton path between s and v.

Your Turn 12.8.3

Use Graph L to find a Hamilton path between each pair of vertices or indicate that there isn't one.

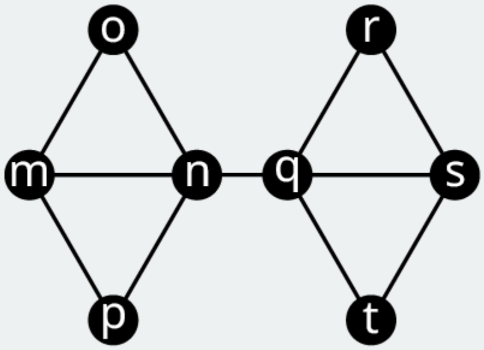


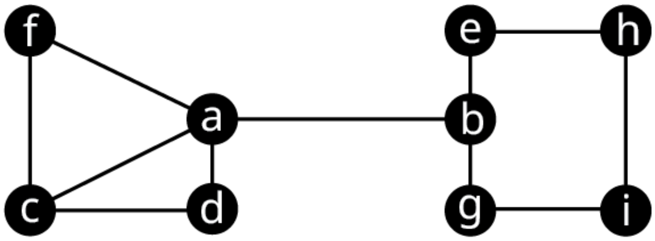
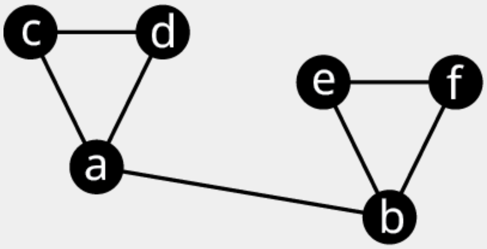
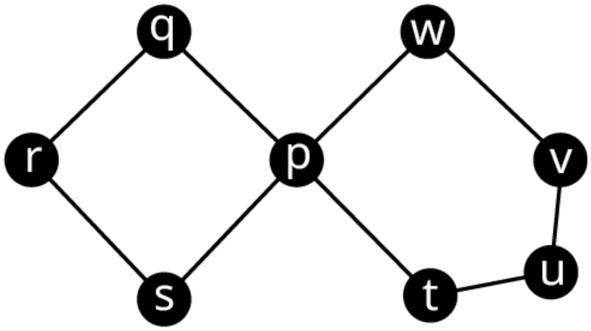
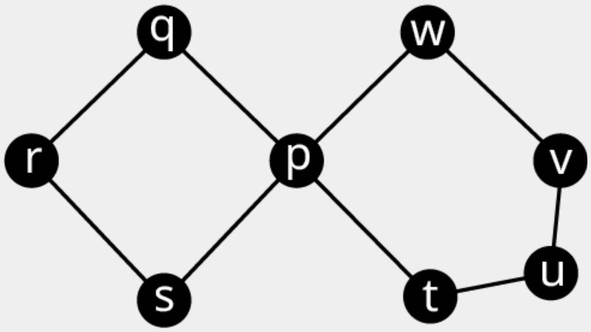
Figure 12.8.11: Graph L

1. p to r
2. m to p
3. o to q

There is not a short way to determine if there is a Hamilton path between two vertices on a graph that works in every situation. However, there are a few common situations that can help us to quickly determine that there is no Hamilton path. Some of these are listed in Table 12.8.1.

Table 12.8.1 Some Impossible Hamilton Paths

Scenario	Diagram

Scenario	Diagram
Scenario 1 If an edge *ab* is a bridge, then there is no Hamilton path between a pair of vertices that are on the same side of edge *ab*. We saw this in Graph *A* of Example 12.40.	 No Hamilton path between any two vertices in the component {*a*, *c*, *d*, *f*}. No Hamilton path between any two vertices in {*b*, *e*, *h*, *g*, *i*}.
Scenario 2 If an edge *ab* is a bridge with at least three components on each side, then there is no Hamilton path beginning or ending at *a* or *b*. We saw this in Graph *D* of Example 12.40.	 No Hamilton path beginning or ending at *a* or *b*.
	
Scenario 3 If a graph is composed of two cycles joined only at a single vertex *p*, and *v* is any vertex that is NOT adjacent to *p*, then there are no Hamilton paths beginning or ending at *p*. We saw this in Graph *B* of Example 12.40.	 No Hamilton path can be formed starting or ending at vertices, *r*, *v*, or *u* because they are not adjacent to *p*.

Checkpoint

There are also many other situations in which a Hamilton path is not possible. These are just a few that you will encounter.

Hamilton Path or Euler Trail?

We learned in Euler Trails that an Euler trail visits each edge exactly once, whereas a Hamilton path visits each vertex exactly once. Let’s practice distinguishing between the two.

Example 12.8.4: Distinguishing between Hamilton Path and Euler Trail

Use Figure ✓ 12.8.12 to determine if the given sequence of vertices is a Hamilton path, an Euler trail, both, or neither.

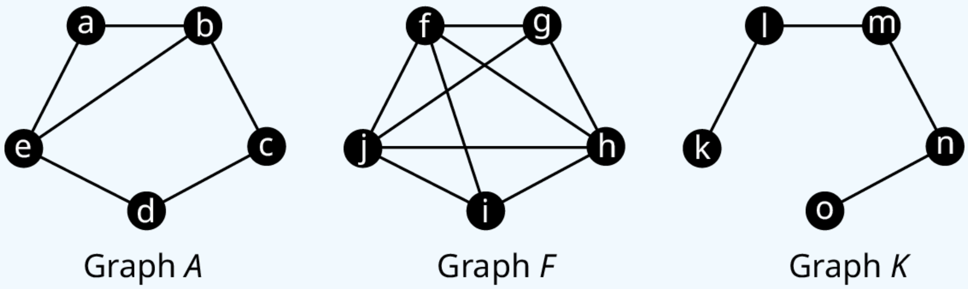


Figure ✓ 12.8.12 Graphs A, F, and K

1. Graph A, $e \rightarrow b \rightarrow a \rightarrow e \rightarrow d \rightarrow c \rightarrow b$
2. Graph F, $f \rightarrow g \rightarrow j \rightarrow h \rightarrow i$
3. Graph K, $k \rightarrow l \rightarrow m \rightarrow n \rightarrow o\backslash$

Answer

1. Since the sequence covers every edge once but visits vertices more than once, it is only an Euler trail.
2. Since the sequence visits every vertex exactly once but skips some edges, it is only a Hamilton path.
3. Since the sequence visits each edge and each vertex exactly once, it is both an Euler trail and a Hamilton path.

Your Turn 12.8.4

Use Figure 12.262 to determine if the given sequence of vertices is a Hamilton path, an Euler trail, both, or neither.

Graph A, $a \rightarrow b \rightarrow e \rightarrow d \rightarrow c$

Graph F, $g \rightarrow h \rightarrow i \rightarrow j \rightarrow h \rightarrow f \rightarrow g \rightarrow j \rightarrow f \rightarrow i$

Graph K, $o \rightarrow n \rightarrow m \rightarrow l$

WORK IT OUT

As we saw the Emerald City lies at the center of the Magical Land of Oz, with Gillikin Country to the north, Winkie Country to the east, Munchkin Country to the west, and Quadling Country to the south. Munchkin Country and Winkie Country each shares a border with Gillikin Country and Quadling Country. Let’s apply graph theory to Dorothy’s famous journey through Oz one more time!

Draw a graph in which each vertex is one of the regions of Oz. Is there a Hamilton path that Dorothy could follow, instead of the yellow brick road, to lead her from the land of the Munchkins, through all the regions of Oz exactly once, and end in the Emerald City? If so, what might it be? Compare your results with those of a classmate.

Check Your Understanding

1. For the following exercises, fill in the blank with *the same as* or *different from* to make the statement true.
2. Unlike in a Hamilton cycle, the vertex where the Hamilton path begins is __________ the vertex where the Hamilton path ends.

3. If a sequence of vertices represents a Hamilton path, the number of vertices listed should be _______ the number of vertices in the whole graph.
4. To determine if a graph has a Hamilton path, use a method that is _________ the method used to determine if a graph has an Euler trail.
5. If a graph with a bridge has a Hamilton path, the starting vertex should be on the side of the bridge that is ________ the side of the bridge with the ending vertex.

6. A path between two vertices of a graph that visits each vertex of the graph exactly once is called an Euler path

a. True
b. False

7. Any graph that has exactly two vertices of odd degree has a Hamilton path.

a. True
b. False

8. If a graph is composed of two cycles joined only at a single vertex p, then no Hamilton path can be formed starting or ending at any vertex that is adjacent to p.

a. True
b. False

9. If an edge ab is a bridge with at least three components on each side, then there is no Hamilton path between vertex a and any vertex on the other side of edge ab.

a. True
b. False

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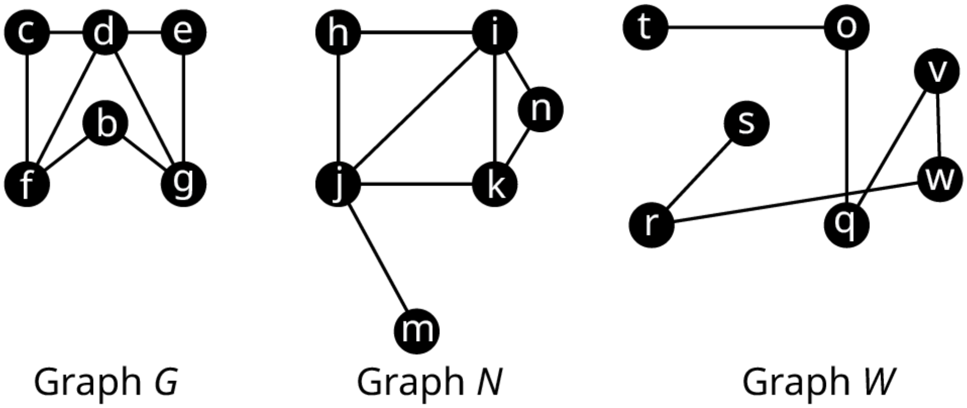
12.8.1: Exercises

For the following exercises, use the figure to determine whether the sequence of vertices in the given graph is a Hamilton path, an Euler trail, both, or neither.



❓ Exercise 12.8.1.1

Graph G: $f \to b \to g \to e \to d \to c$

❓ Exercise 12.8.1.2

Graph G: $g \to b \to f \to c \to d \to e$

❓ Exercise 12.8.1.3

Graph G: $f \to b \to g \to d \to f \to c \to d \to e \to g$

❓ Exercise 12.8.1.4

Graph W: $v \to w \to r \to s \to t \to o \to q$

❓ Exercise 12.8.1.5

Graph W: $s \to r \to w \to v \to q \to o \to t$

❓ Exercise 12.8.1.6

Graph N: $h \to i \to k \to n \to j \to h$

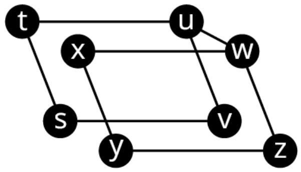
❓ Exercise 12.8.1.7

Graph N: $n \to i \to h \to j \to m$

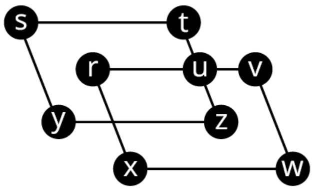
❓ Exercise 12.8.1.8

Graph N: $m \to j \to h \to i \to k \to n \to i \to j \to k$

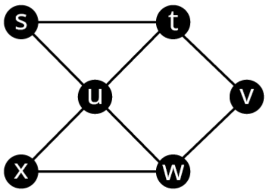
For the following exercises, use the figure to explain why the given sequence of vertices does not represent a Hamilton path.



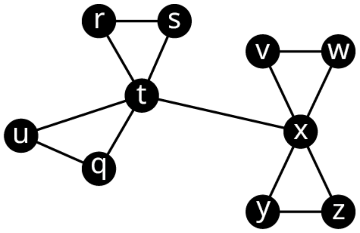
Graph A



Graph B



Graph C



Graph D

Exercise 12.8.1.9

Graph A: $t \to s \to v \to u \to x \to w \to y \to z$

Exercise 12.8.1.10

Graph B: $w \to x \to r \to u \to z \to y \to s \to t \to u \to v$

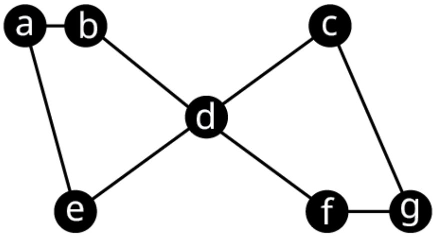
Exercise 12.8.1.11

Graph C: $s \to u \to w \to v \to t$

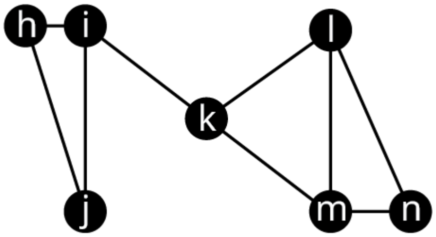
Exercise 12.8.1.12

Graph D: $r \to \ \to t \to q \to u \to t \to x \to v \to w \to x \to z \to y$

For the following exercises, use the figure to find a path that fits the description or indicate which scenario from the figure makes it impossible.



Graph H



Graph Q

13.

A Hamilton path in Graph H that begins at vertex c and ends at vertex e.

Exercise 12.8.1.14

A Hamilton path in Graph Q that begins at vertex n and ends at vertex h.

Exercise 12.8.1.15

A Hamilton path in Graph H that begins at vertex c and ends at vertex g.

Exercise 12.8.1.16

A Hamilton path in Graph Q that begins at vertex m and ends at vertex j.

? Exercise 12.8.1.17

A Hamilton path in Graph H that begins at vertex g.

? Exercise 12.8.1.18

A Hamilton path in Graph Q that begins at vertex i.

? Exercise 12.8.1.19

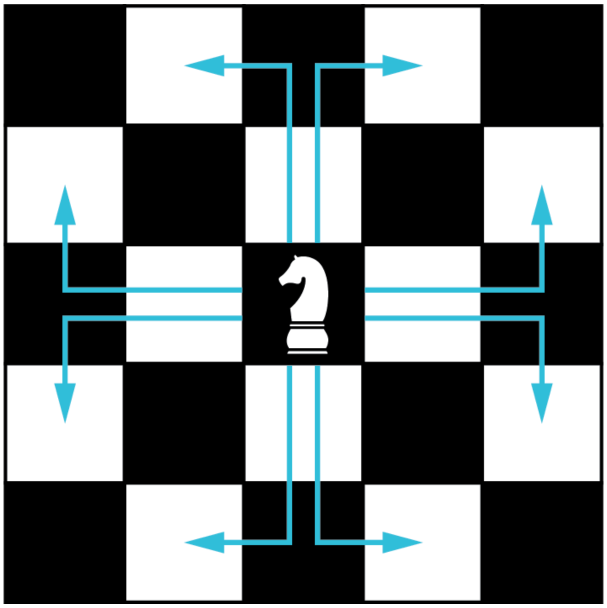
A path between n and j in Graph Q that is NOT a Hamilton path, and explain why it is not a Hamilton path.

? Exercise 12.8.1.20

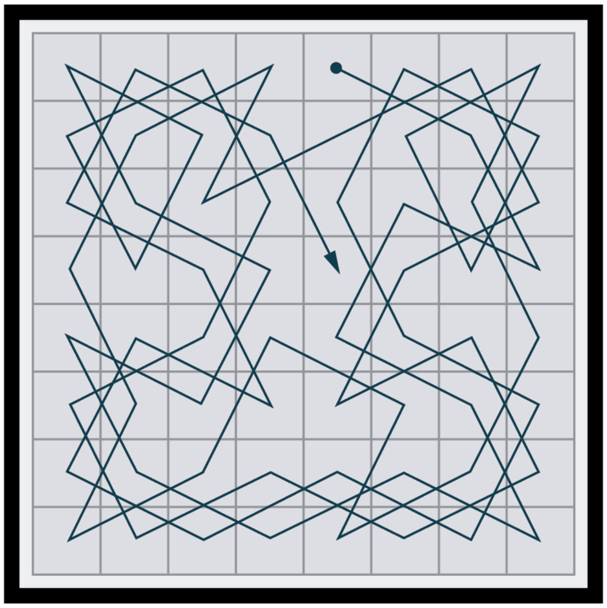
A path between a and c in Graph H that is NOT a Hamilton path, and explain why it is not a Hamilton path.

21.

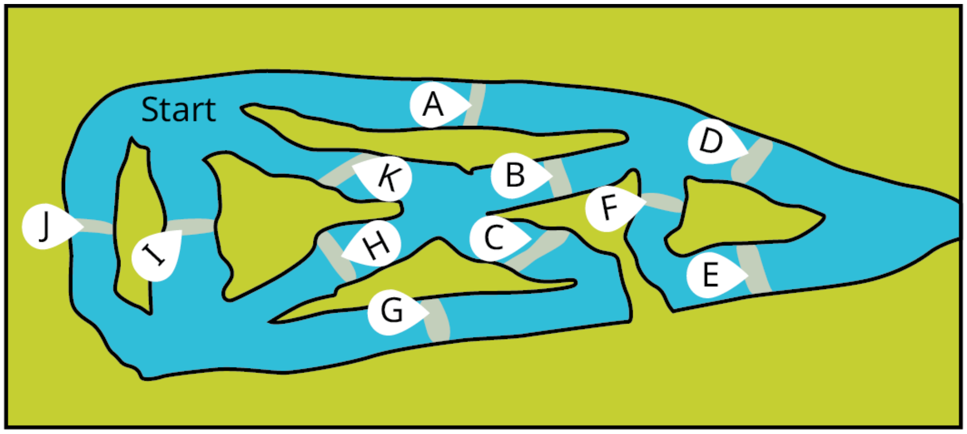
In chess, a knight can move in any direction, but it must move two spaces then turn and move one more space. The eight possible moves a knight can make from a given space are shown in the figure.



A knight's tour is a sequence of moves by a knight on a chessboard (of any size) such that the knight visits every square exactly once. If the knight's tour brings the knight back to its starting position on the board, it is called a closed knight's tour. Otherwise, it is called an open knight's tour. Determine if the Knight's tour shown in the figure is a Hamilton path, an Euler trail, or both, for the graph of all possible knight moves on an eight-by-eight chess board in which the vertices are spaces on the board and the edges indicate that the knight can move directly from one space to the other. Explain your reasoning.



Recall from the section Euler Circuits, as part of the Camp Woebegone Olympics, there is a canoeing race with a checkpoint on each of the 11 different streams as shown in the figure. The contestants must visit each checkpoint.



Draw a graph in which the vertices represent checkpoints, and an edge indicates that it is possible to travel from one checkpoint to the next without passing through another checkpoint.

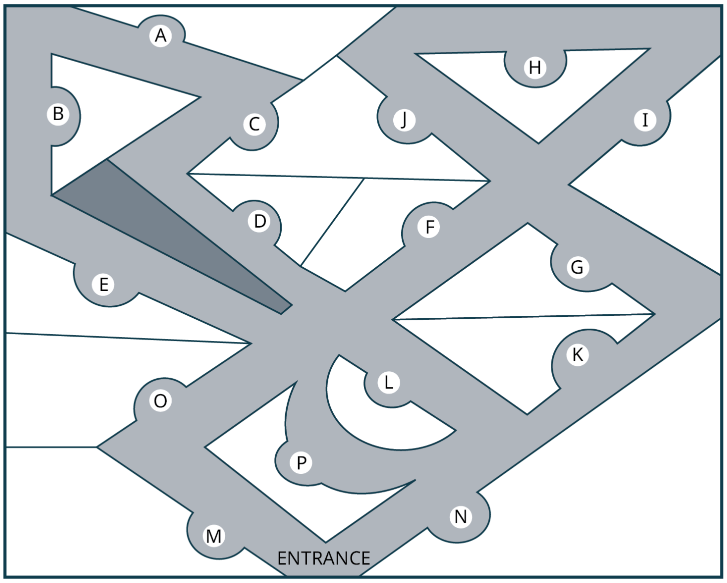
Exercise 12.8.1.23

Find a Hamilton path beginning at vertex A and ending at vertex E.

Exercise 12.8.1.24

What does this Hamilton path represent in the context of the race?

The figure shows a map of zoo exhibits A through P. Use it to answer each question.



25.

Draw a graph to represent the routes through the zoo in which the edges represent walkways and the vertices represent exhibits. Two vertices are connected if a person can walk between the exhibits they represent without passing another exhibit.

? Exercise 12.8.1.26

Use the graph you created to find a route that begins at exhibit M, ends at exhibit J, and visits each exhibit exactly once.

12.8.1: Exercises is shared under a not declared license and was authored, remixed, and/or curated by LibreTexts.

12.9: Traveling Salesperson Problem

Figure 12.9.1: Each door on the route of a traveling salesperson can be represented as a vertex on a graph. (credit: "Three in a row, Heriot Row" by Jason Mason/Flickr, CC BY 2.1)

Learning Objectives

1. Distinguish between brute force algorithms and greedy algorithms.
2. List all distinct Hamilton cycles of a complete graph.
3. Apply brute force method to solve traveling salesperson applications.
4. Apply nearest neighbor method to solve traveling salesperson applications.

We looked at Hamilton cycles and paths in the previous sections Hamilton Cycles and Hamilton Paths. In this section, we will analyze Hamilton cycles in complete weighted graphs to find the shortest route to visit a number of locations and return to the starting point. Besides the many routing applications in which we need the shortest distance, there are also applications in which we search for the route that is least expensive or takes the least time. Here are a few less common applications that you can read about on a website set up by the mathematics department at the University of Waterloo in Ontario, Canada:

- Design of fiber optic networks
- Minimizing fuel expenses for repositioning satellites
- Development of semi-conductors for microchips
- A technique for mapping mammalian chromosomes in genome sequencing

Before we look at approaches to solving applications like these, let's discuss the two types of algorithms we will use.

Brute Force and Greedy Algorithms

An algorithm is a sequence of steps that can be used to solve a particular problem. We have solved many problems in this chapter, and the procedures that we used were different types of algorithms. In this section, we will use two common types of algorithms, a brute force algorithm and a greedy algorithm. A brute force algorithm begins by listing every possible solution and applying each one until the best solution is found. A greedy algorithm approaches a problem in stages, making the apparent best choice at each stage, then linking the choices together into an overall solution which may or may not be the best solution.

To understand the difference between these two algorithms, consider the tree diagram in Figure 12.214. Suppose we want to find the path from left to right with the largest total sum. For example, branch A in the tree diagram has a sum of $10 + 2 + 11 + 13 = 36$.

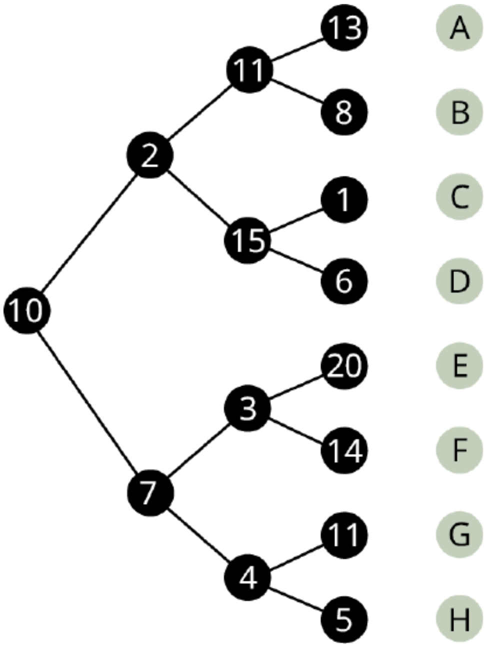


Figure 12.9.2: Points Along Different Paths

To be certain that you pick the branch with greatest sum, you could list each sum from each of the different branches:

A: $10+2+11+13=36$

B: $10+2+11+8=31$

C: $10+2+15+1=28$

D: $10+2+15+6=33$

E: $10+7+3+20=40$

F: $10+7+3+14=34$

G: $10+7+4+11=32$

H: $10+7+4+5=26$

Then we know with certainty that branch E has the greatest sum.

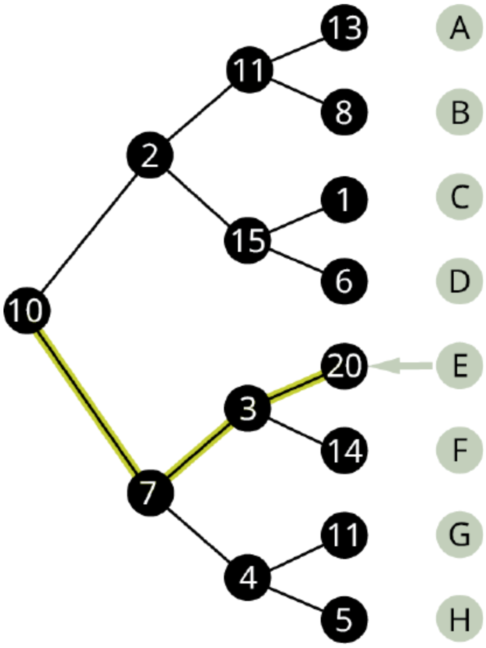

13
11
8
2
1
15
6
10
20
3
14
7
11
4
5
A
B
C
D
E
F
G
H

Figure 12.9.3: Branch E

Now suppose that you wanted to find the branch with the highest value, but you only were shown the tree diagram in phases, one step at a time.

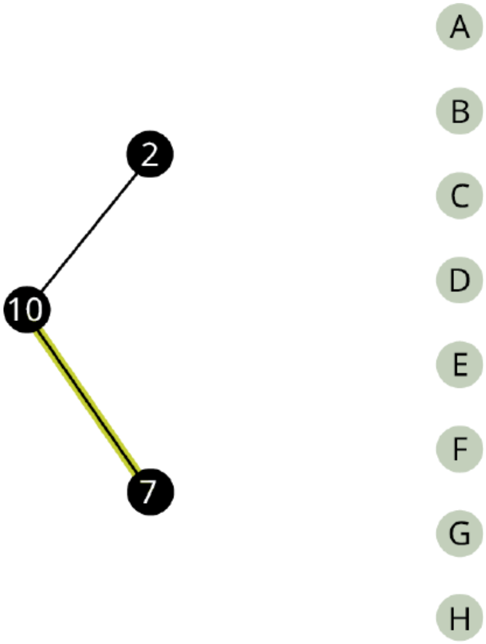

A
B
2
C
10
D
E
F
7
G
H

Figure 12.9.4: Tree Diagram Phase 1

After phase 1, you would have chosen the branch with 10 and 7. So far, you are following the same branch. Let’s look at the next phase.

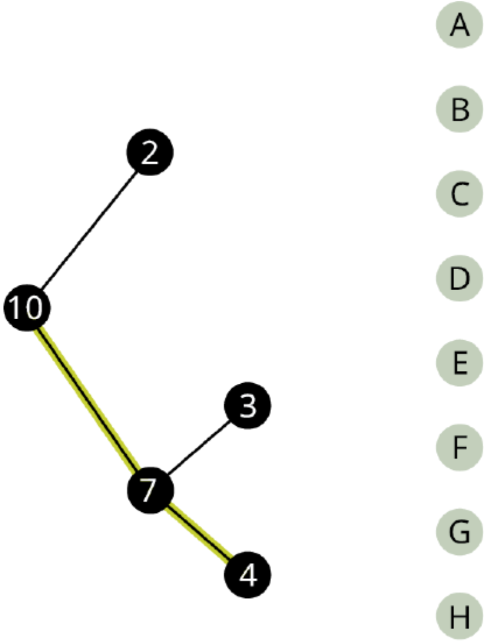


Figure $12.9.5$: Tree Diagram Phase 2

After phase 2, based on the information you have, you will choose the branch with 10, 7 and 4. Now, you are following a different branch than before, but it is the best choice based on the information you have. Let’s look at the last phase.

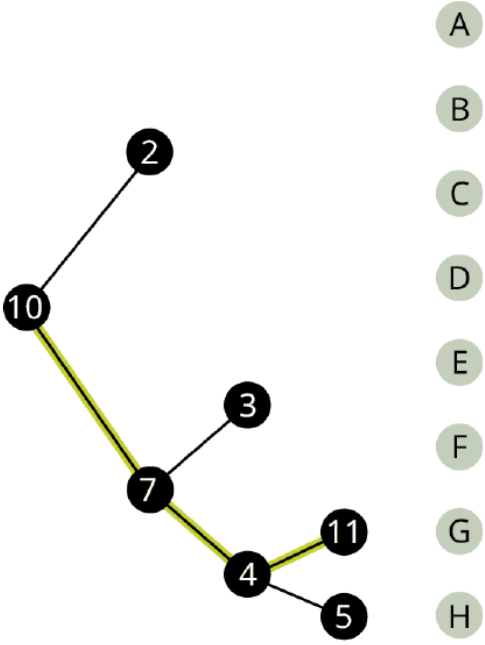


Figure $12.9.6$: Tree Diagram Phase 3

After phase 3, you will choose branch G which has a sum of 32.

The process of adding the values on each branch and selecting the highest sum is an example of a brute force algorithm because all options were explored in detail. The process of choosing the branch in phases, based on the best choice at each phase is a greedy algorithm. Although a brute force algorithm gives us the ideal solution, it can take a very long time to implement. Imagine a tree diagram with thousands or even millions of branches. It might not be possible to check all the sums. A greedy algorithm, on the other hand, can be completed in a relatively short time, and generally leads to good solutions, but not necessarily the ideal solution.

✓ Example 12.9.1: Distinguishing between Brute Force and Greedy Algorithms

A cashier rings up a sale for $4.63 cents in U.S. currency. The customer pays with a $5 bill. The cashier would like to give the customer $0.37 in change using the fewest coins possible. The coins that can be used are quarters ($0.25), dimes ($0.10), nickels ($0.05), and pennies ($0.01). The cashier starts by selecting the coin of highest value less than or equal to $0.37, which is a quarter. This leaves $\$0.37-\$0.25=\$0.12$. The cashier selects the coin of highest value less than or equal to $0.12, which is a dime. This leaves $\$0.12-\$0.10=\$0.02$. The cashier selects the coin of highest value less than or equal to $0.02, which is a penny. This leaves $\$0.02-\$0.01=\$0.01$. The cashier selects the coin of highest value less than or equal to $0.01, which is a penny. This leaves no remainder. The cashier used one quarter, one dime, and two pennies, which is four coins. Use this information to answer the following questions.

1. Is the cashier’s approach an example of a greedy algorithm or a brute force algorithm? Explain how you know.
2. The cashier’s solution is the best solution. In other words, four is the fewest number of coins possible. Is this consistent with the results of an algorithm of this kind? Explain your reasoning.

Answer

1. The approach the cashier used is an example of a greedy algorithm, because the problem was approached in phases and the best choice was made at each phase. Also, it is not a brute force algorithm, because the cashier did not attempt to list out all possible combinations of coins to reach this conclusion.
2. Yes, it is consistent. A greedy algorithm does not always yield the best result, but sometimes it does.

Your Turn 12.9.1

Suppose that you lost the combination to a combination lock that consisted of three digits, and each was selected from the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9. You desperately need to open the lock without breaking it. You decide to check all possible combinations methodically, 000, then 001, then 002, and so on until you find the right combination. Is this an example of a brute force algorithm or a greedy algorithm?

The Traveling Salesperson Problem

Now let’s focus our attention on the graph theory application known as the traveling salesperson problem (TSP) in which we must find the shortest route to visit a number of locations and return to the starting point.

Recall from Hamilton Cycles, the officer in the U.S. Air Force who is stationed at Vandenberg Air Force base and must drive to visit three other California Air Force bases before returning to Vandenberg. The officer needed to visit each base once. We looked at the weighted graph in Figure 12.9.7 representing the four U.S. Air Force bases: Vandenberg, Edwards, Los Angeles, and Beal and the distances between them.

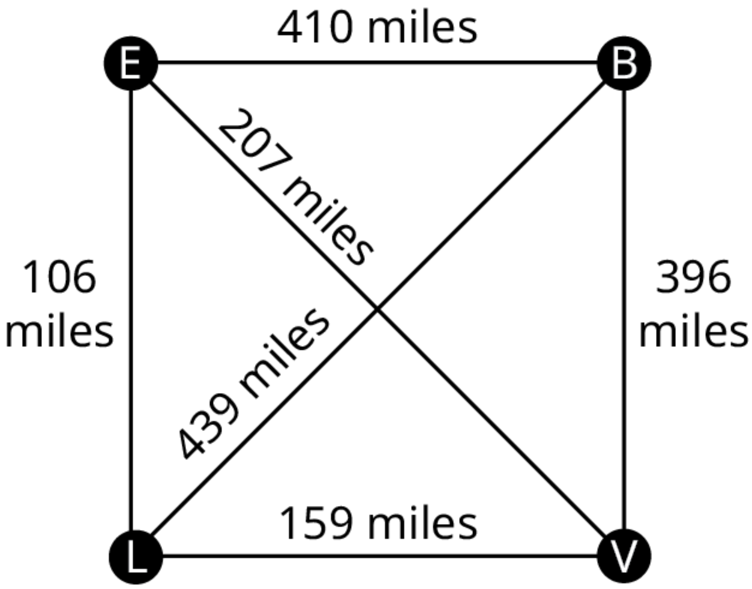
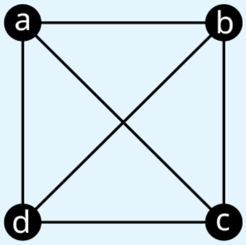
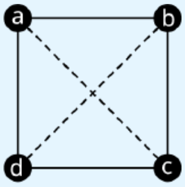
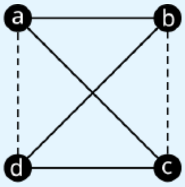
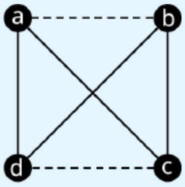
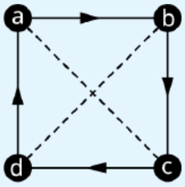
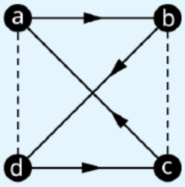
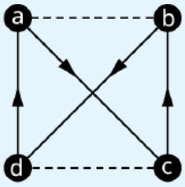
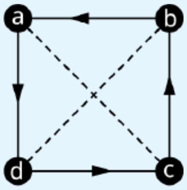
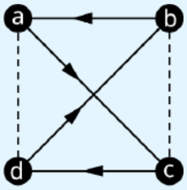
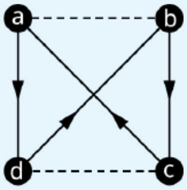


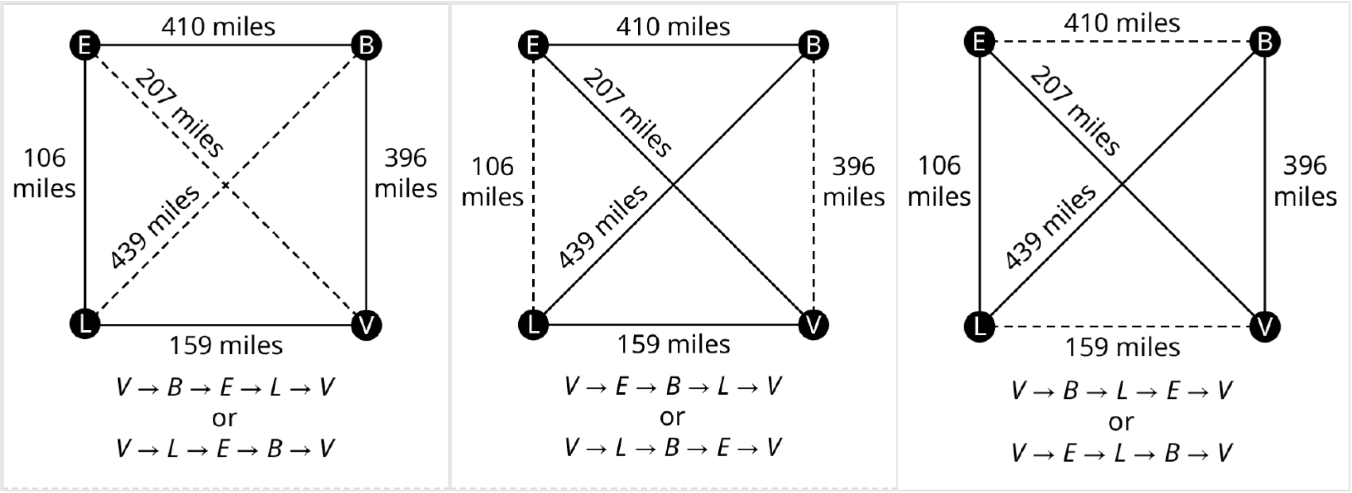
Figure 12.9.7: Graph of Four California Air Force Bases

Any route that visits each base and returns to the start would be a Hamilton cycle on the graph. If the officer wants to travel the shortest distance, this will correspond to a Hamilton cycle of lowest weight. We see in Table 12.9.1 that there are six distinct Hamilton cycles (directed cycles) in a complete graph with four vertices, but some lie on the same cycle (undirected cycle) in the graph.

Table 12.9.1 Hamilton Cycles in a Complete Graph with Four Vertices

Complete Graph	Cycle	Cycle	Cycle
			
Clockwise Hamilton Cycle	 $a \to b \to c \to d \to a$	 $a \to b \to d \to c \to a$	 $a \to c \to b \to d \to a$
Counterclockwise Hamilton Cycle	 $a \to d \to c \to b \to a$	 $a \to c \to d \to b \to a$	 $a \to d \to b \to c \to a$

Since the distance between bases is the same in either direction, it does not matter if the officer travels clockwise or counterclockwise. So, there are really only three possible distances as shown in Figure 12.9.8



$V \rightarrow B \rightarrow E \rightarrow L \rightarrow V$
or
$V \rightarrow L \rightarrow E \rightarrow B \rightarrow V$

$V \rightarrow E \rightarrow B \rightarrow L \rightarrow V$
or
$V \rightarrow L \rightarrow B \rightarrow E \rightarrow V$

$V \rightarrow B \rightarrow L \rightarrow E \rightarrow V$
or
$V \rightarrow E \rightarrow L \rightarrow B \rightarrow V$

Figure $12.9.8$: Three Possible Distances

The possible distances are:

$$\begin{gathered}396+410+106+159=1071 \\ 207+410+439+159=1215 \\ 396+439+106+207=1148\end{gathered}$$

So, a Hamilton cycle of least weight is $V \rightarrow B \rightarrow E \rightarrow L \rightarrow V$ (or the reverse direction). The officer should travel from Vandenberg to Beal to Edwards, to Los Angeles, and back to Vandenberg.

Finding Weights of All Hamilton Cycles in Complete Graphs

Notice that we listed all of the Hamilton cycles and found their weights when we solved the TSP about the officer from Vandenberg. This is a skill you will need to practice. To make sure you don't miss any, you can calculate the number of possible Hamilton cycles in a complete graph. It is also helpful to know that half of the directed cycles in a complete graph are the same cycle in reverse direction, so, you only have to calculate half the number of possible weights, and the rest are duplicates.

FORMULA

In a complete graph with n vertices,

- The number of distinct Hamilton cycles is $(n-1)!$.
- There are at most $\frac{(n-1)!}{2}$ different weights of Hamilton cycles.

Checkpoint

TIP! When listing all the distinct Hamilton cycles in a complete graph, you can start them all at any vertex you choose. Remember, the cycle $a \rightarrow b \rightarrow c \rightarrow a$ *is the same cycle as* $b \rightarrow c \rightarrow a \rightarrow b$ *so there is no need to list both.*

Example $12.9.2$: Calculating Possible Weights of Hamilton Cycles

Suppose you have a complete weighted graph with vertices *N*, *M*, *O*, and *P*.

1. Use the formula $(n-1)!$ to calculate the number of distinct Hamilton cycles in the graph.
2. Use the formula $\frac{(n-1)!}{2}$ to calculate the greatest number of different weights possible for the Hamilton cycles.
3. Are all of the distinct Hamilton cycles listed here? How do you know?
   Cycle 1: $N \rightarrow M \rightarrow O \rightarrow P \rightarrow N$
   Cycle 2: $N \rightarrow M \rightarrow P \rightarrow O \rightarrow N$
   Cycle 3: $N \rightarrow O \rightarrow M \rightarrow P \rightarrow N$
   Cycle 4: $N \rightarrow O \rightarrow P \rightarrow M \rightarrow N$

   Cycle 5: $N \rightarrow P \rightarrow M \rightarrow O \rightarrow N$
   Cycle 6: $N \rightarrow P \rightarrow O \rightarrow M \rightarrow N$
4. Which pairs of cycles must have the same weights? How do you know?

Answer

1. There are 4 vertices; so, $n = 4$. This means there are $(n-1)! = (4-1)! = 3 \cdot 2 \cdot 1 = 6$ distinct Hamilton cycles beginning at any given vertex.
2. Since $n = 4$, there are $\frac{(n-1)!}{2} = \frac{(4-1)!}{2} = \frac{6}{2} = 3$ possible weights.
3. Yes, they are all distinct cycles and there are 6 of them.
4. Cycles 1 and 6 have the same weight, Cycles 2 and 4 have the same weight, and Cycles 3 and 5 have the same weight, because these pairs follow the same route through the graph but in reverse.

Checkpoint

TIP! When listing the possible cycles, ignore the vertex where the cycle begins and ends and focus on the ways to arrange the letters that represent the vertices in the middle. Using a systematic approach is best; for example, if you must arrange the letters M, O, and P, first list all those arrangements beginning with M, then beginning with O, and then beginning with P, as we did in Example 12.42.

Your Turn $12.9.2$

A complete weighted graph has vertices *V*, *W*, *X*, *Y*, and *Z*. List all the distinct Hamilton cycles. Use *V* as the starting vertex. Identify the pairs of cycles that have the same weight because they are reverses of each other.

The Brute Force Method

The method we have been using to find a Hamilton cycle of least weight in a complete graph is a brute force algorithm, so it is called the brute force method. The steps in the brute force method are:

Step 1: Calculate the number of distinct Hamilton cycles and the number of possible weights.

Step 2: List all possible Hamilton cycles.

Step 3: Find the weight of each cycle.

Step 4: Identify the Hamilton cycle of lowest weight.

✓ Example $12.9.3$: Applying the Brute Force Method

On the next assignment, the air force officer must leave from Travis Air Force base, visit Beal, Edwards, and Vandenberg Air Force bases each exactly once and return to Travis Air Force base. There is no need to visit Los Angeles Air Force base. Use Figure ✓ $12.9.9$ to find the shortest route.

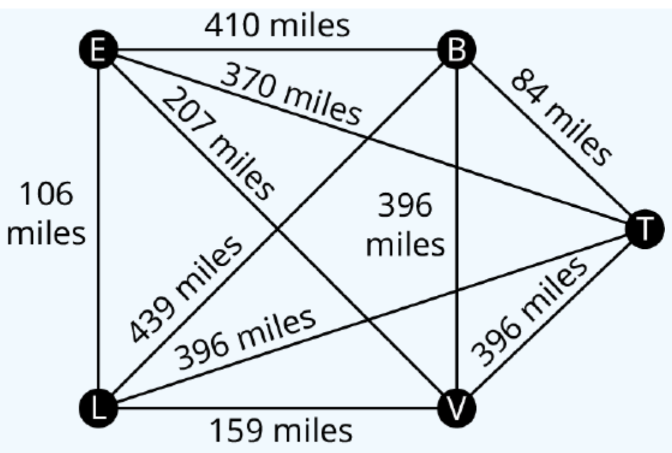

410 miles
E
B
370 miles
84 miles
207 miles
106 miles
396 miles
T
439 miles
396 miles
396 miles
L
V
159 miles

Figure ✓ 12.9.9 Distances between Five California Air Force Bases

Answer

Step 1: Since there are 4 vertices, there will be $(4-1)!=3!=6$ cycles, but half of them will be the reverse of the others; so, there will be $\frac{(4-1)!}{2}=\frac{6}{2}=3$ possible distances.

Step 2: List all the Hamilton cycles in the subgraph of the graph in Figure 12.222.

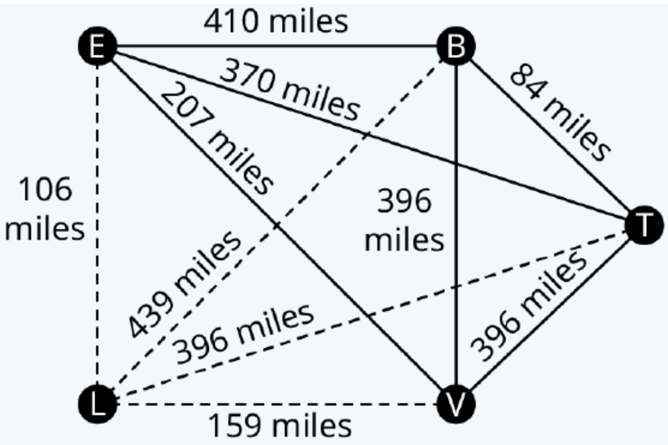


Figure ✓ 12.9.10 Subgraph with Cities *B*, *E*, *T*, and *V*

To find the 6 cycles, focus on the three vertices in the middle, $B, E,$ and V. The arrangements of these vertices are BEV, BVE, EBV, EVB, VBE, and VEB. These would correspond to the 6 cycles:

$$1: T \to B \to E \to V \to T$$

$$2: T \to B \to V \to E \to T$$

$$3: T \to E \to B \to V \to T$$

$$4: T \to E \to V \to B \to T$$

$$5: T \to V \to B \to E \to T$$

$$6: T \to V \to E \to B \to T$$

Step 3: Find the weight of each path. You can reduce your work by observing the cycles that are reverses of each other.

$$1: 84 + 410 + 207 + 396 = 1097$$

$$2: 84 + 396 + 207 + 370 = 1071$$

$$3: 370 + 410 + 396 + 396 = 1572$$

4: Reverse of cycle 2, 1071

5: Reverse of cycle 3, 1572

6: Reverse of cycle 1, 1097

Step 4: Identify a Hamilton cycle of least weight.

The second path, $T \to B \to V \to E \to T$, and its reverse, $T \to E \to V \to B \to T$, have the least weight. The solution is that the officer should travel from Travis Air Force base to Beal Air Force Base, to Vandenberg Air Force base, to Edwards Air Force base, and return to Travis Air Force base, or the same route in reverse.

Your Turn 12.9.3

Suppose that the Air Force officer needed to leave from Travis Air Force base, visit each of Beal, Edwards, and Los Angeles Air Force bases exactly once and return to Travis. Use Figure 12.278 to find the shortest route.

Now suppose that the officer needed a cycle that visited all 5 of the Air Force bases in Figure 12.221. There would be $(5-1)! = 4! = 4 \times 3 \times 2 \times 1 = 24$ different arrangements of vertices and $\frac{(5-1)!}{2} = \frac{4!}{2} = \frac{24}{2} = 12$ distances to compare using the brute force method. If you consider 10 Air Force bases, there would be $(10-1)! = 9! = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 362,880$ different arrangements and $\frac{(10-1)!}{2} = \frac{9!}{2} = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2} = 181,440$ distances to consider. There must be another way!

The Nearest Neighbor Method

When the brute force method is impractical for solving a traveling salesperson problem, an alternative is a greedy algorithm known as the nearest neighbor method, which always visit the closest or least costly place first. This method finds a Hamilton cycle of relatively low weight in a complete graph in which, at each phase, the next vertex is chosen by comparing the edges between the current vertex and the remaining vertices to find the lowest weight. Since the nearest neighbor method is a greedy algorithm, it usually doesn't give the best solution, but it usually gives a solution that is "good enough." Most importantly, the number of steps will be the number of vertices. That's right! A problem with 10 vertices requires 10 steps, not 362,880. Let's look at an example to see how it works.

Suppose that a candidate for governor wants to hold rallies around the state. They plan to leave their home in city *A*, visit cities *B*, *C*, *D*, *E*, and *F* each once, and return home. The airfare between cities is indicated in the graph in Figure 12.9.11.

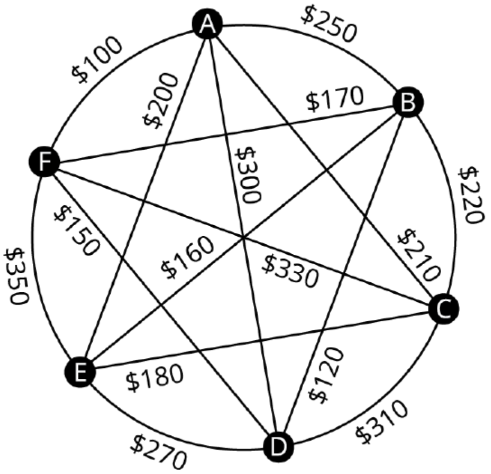
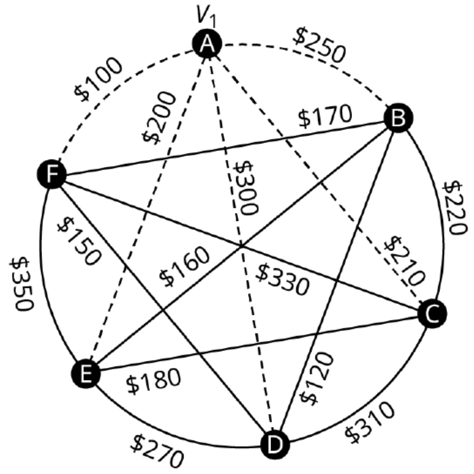


Figure 12.9.11: Airfares between Cities *A*, *B*, *C*, *D*, *E*, and *F*

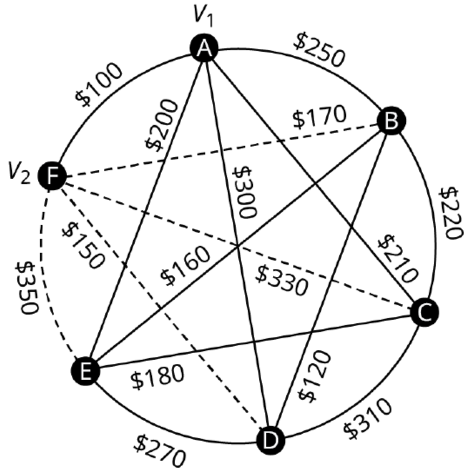
Let's help the candidate keep costs of travel down by applying the nearest neighbor method to find a Hamilton cycle that has a reasonably low weight. Begin by marking starting vertex as V_1 for "visited 1st." Then to compare the weights of the edges between *A* and vertices adjacent to A : $\$250, \$210, \$300, \$200$, and $\$100$ as shown in Figure 12.9.12. The lowest of these is $\$100$, which is the edge between A and F.



V_1
A
B
C
D
E
F
$250
$100
$200
$170
$300
$220
$150
$160
$330
$210
$350
$180
$120
$310
$270

Figure 12.9.12: Finding the Second Vertex

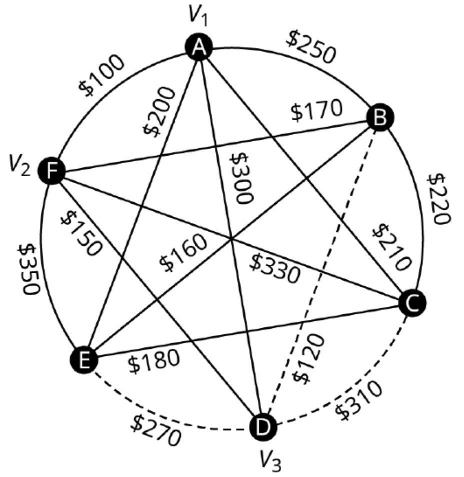
Mark F as V_2 for "visited 2nd" then compare the weights of the edges between F and the remaining vertices adjacent to F : $\$170, \$330, \$150$ and $\$350$ as shown in Figure 12.9.13. The lowest of these is $\$150$, which is the edge between F and D.



V_1
V_2
A
B
C
D
E
F
$250
$100
$200
$170
$300
$220
$150
$160
$330
$210
$350
$180
$120
$310
$270

Figure 12.9.13: Finding the Third Vertex

Mark D as V_3 for "visited 3rd." Next, compare the weights of the edges between D and the remaining vertices adjacent to D : $\$120, \$310$, and $\$270$ as shown in Figure 12.9.14. The lowest of these is $\$120$, which is the edge between D and B.



V_1
V_2
V_3
A
B
C
D
E
F
$250
$100
$200
$170
$300
$220
$150
$160
$330
$210
$350
$180
$120
$310
$270

Figure 12.9.14: Finding the Fourth Vertex

So, mark B as V_4 for "visited 4th." Finally, compare the weights of the edges between B and the remaining vertices adjacent to B : $\$160$ and $\$220$ as shown in Figure $12.9.15$. The lower amount is $\$160$, which is the edge between B and E

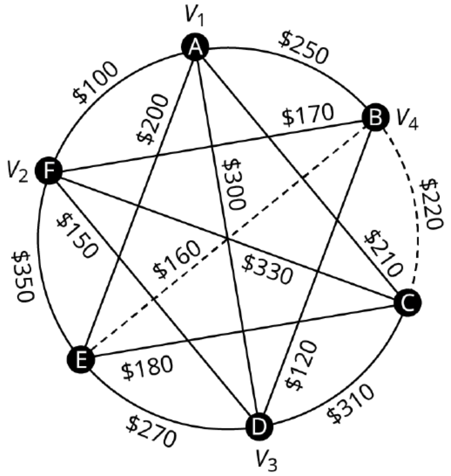


Figure $12.9.15$: Finding the Fifth Vertex

Now you can mark E as V_5 and mark the only remaining vertex, which is C, as V_6. This is shown in Figure $12.9.16$. Make a note of the weight of the edge from E to C, which is $\$180$, and from C back to A, which is $\$210$.

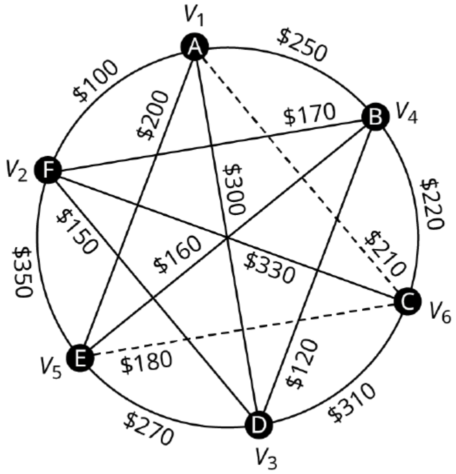


Figure $12.9.16$: Finding the Sixth Vertex

The Hamilton cycle we found is $A \rightarrow F \rightarrow D \rightarrow B \rightarrow E \rightarrow C \rightarrow A$. The weight of the circuit is $\$100+\$150+\$120+\$160+\$180+\$210=\$920$. This may or may not be the route with the lowest cost, but there is a good chance it is very close since the weights are most of the lowest weights on the graph and we found it in six steps instead of finding 120 different Hamilton cycles and calculating 60 weights. Let's summarize the procedure that we used.

Step 1: Select the starting vertex and label semanticsV1V1 for "visited 1st." Identify the edge of lowest weight between semanticsV1V1 and the remaining vertices.

Step 2: Label the vertex at the end of the edge of lowest weight that you found in previous step as semanticsVnVn where the subscript n indicates the order the vertex is visited. Identify the edge of lowest weight between semanticsVnVn and the vertices that remain to be visited.

Step 3: If vertices remain that have not been visited, repeat Step 2. Otherwise, a Hamilton cycle of low weight is $V_1 \rightarrow V_2 \rightarrow \cdots \rightarrow V_n$ →V1V1→V2→⋯→Vn→V1.

✓ Example 12.9.4: Using the Nearest Neighbor Method

Suppose that the candidate for governor wants to hold rallies around the state but time before the election is very limited. They would like to leave their home in city *A*, visit cities *B*, *C*, *D*, *E*, and *F* each once, and return home. The airfare between cities is not as important as the time of travel, which is indicated in Figure ✓ 12.9.17. Use the nearest neighbor method to find a route with relatively low travel time. What is the total travel time of the route that you found?

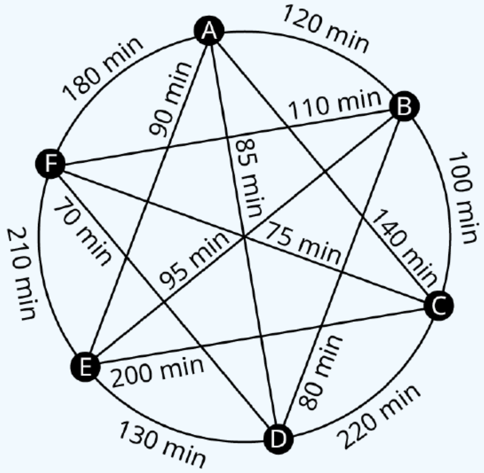


Figure ✓ 12.9.17 Travel Times between Cities *A*, *B*, *C*, *D*, *E* and *F*

Answer

Step 1: Label vertex *A* as semanticsV1V1. The edge of lowest weight between *A* and the remaining vertices is 85 min between *A* and *D*.

Step 2: Label vertex *D* as semanticsV2V2. The edge of lowest weight between *D* and the vertices that remain to be visited, *B*, *C*, *E*, and *F*, is 70 min between *D* and *F*.

Repeat Step 2: Label vertex *F* as semanticsV3V3. The edge of lowest weight between *F* and the vertices that remain to be visited, *B*, *C*, and *E*, is 75 min between *F* and *C*.

Repeat Step 2: Label vertex *C* as semanticsV4V4. The edge of lowest weight between *C* and the vertices that remain to be visited, *B* and *E*, is 100 min between *C* and *B*.

Repeat Step 2: Label vertex *B* as semanticsV5V5. The only vertex that remains to be visited is *E*. The weight of the edge between *B* and *E* is 95 min.

Step 3: A Hamilton cycle of low weight is $A \to D \to F \to C \to B \to E \to A$. So, a route of relatively low travel time is *A* to *D* to *F* to *C* to *B* to *E* and back to *A*. The total travel time of this route is: 85min+70min+75min+100min+95min+90min=515minor8hrs35min85min+70min+75min+100min+95min+90min=515minor8hrs35min

Your Turn 12.9.4

Use the nearest neighbor method to find a Hamilton cycle of relatively low weight beginning and ending at vertex *D* in Figure 12.240 and find its total weight. Give your answer in hours and minutes.

Check Your Understanding

1. The advantage of a greedy algorithm is that it is more efficient.

a. True
b. False

2. The disadvantage of a brute force algorithm is that it does not always give the ideal solution.

a. True
b. False

3. The nearest neighbor method is an example of a brute force algorithm.

a. True
b. False

4. The brute force method is an example of a greedy algorithm.

a. True
b. False

5. The brute force method is used to find a Hamilton cycle of least weight in a complete graph.

a. True
b. False

6. The nearest neighbor method is used to find the ideal solution to the traveling salesperson problem.

a. True
b. False

7. The traveling salesman problem involves finding the shortest route to travel between two points.

a. True
b. False

8. The traveling salesman problem can be represented as finding a Hamilton cycle of least weight on a weighted graph.

a. True
b. False

9. There is always more than one Hamilton cycle of least weight, a given Hamilton cycle and the reverse of that Hamilton cycle.

a. True
b. False

10. The greatest possible number of distinct weights for the Hamilton cycles of a complete graph with n vertices is (n-1)!

a. True
b. False

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12.9.1: Exercises

For the following exercises, determine whether the algorithm described is a greedy algorithm or a brute force algorithm.

Exercise 12.9.1.1

The algorithm for creating graph colorings in the section Navigating Graphs involved coloring the vertex of highest degree first, coloring as many other vertices as possible each color from highest to lowest degree, then repeating this process for the remaining vertices.

Exercise 12.9.1.2

Pallets of goods are to be transported on 10 flatbed trucks which have weight limits. To determine which goods will be shipped together, all the possible ways to divide the goods into 10 groups is listed and the total weight of each group is calculated.

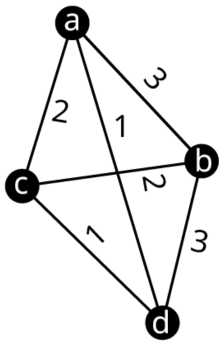
Exercise 12.9.1.3

A wedding planner is creating a seating arrangement for the reception dinner. The couple has provided a list of which guests must be seated together. The wedding planner prefers to use the fewest tables possible so that there is more space to mingle at the reception. The planner creates a list of all possible seating arrangements and selects one that meets these criteria.

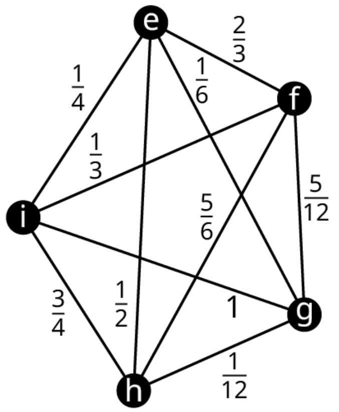
Exercise 12.9.1.4

Packages must be loaded into freight cars to be transported by train. It is preferred to use the fewest freight cars possible to keep the costs down. As each freight car is packed, the package with the largest girth that will fit in the freight car is loaded and this is repeated until the freight car can hold no more packages. When a freight car can hold no more packages, the next freight car is loaded.

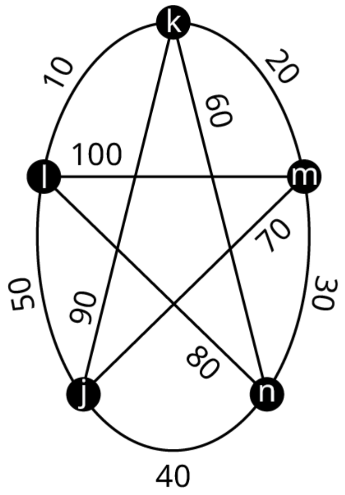
For the following exercises, use the figure to calculate the number of distinct Hamilton cycles beginning at the given vertex in the given graph. How many of those could possibly result in a different weight?



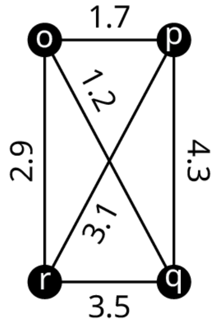
Graph A



Graph B



Graph C



Graph D

Exercise 12.9.1.5

Graph A, vertex a

? Exercise 12.9.1.6

Graph *B*, vertex *e*

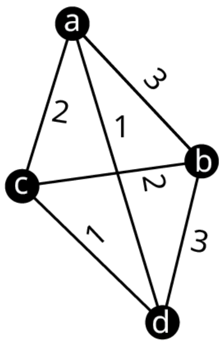
? Exercise 12.9.1.7

Graph *C*, vertex *k*

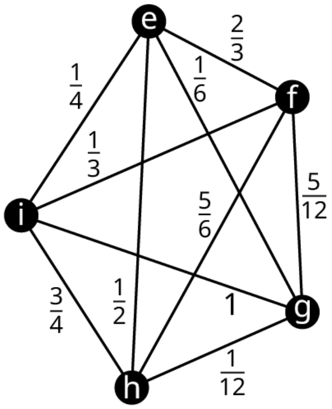
? Exercise 12.9.1.8

Graph *D*, vertex *o*

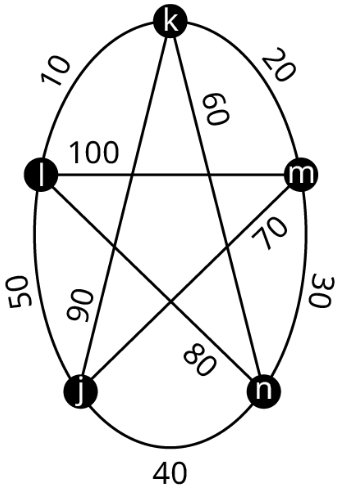
For the following exercises, use the figure to list all the distinct Hamilton cycles beginning at the given vertex in the given graph. Indicate which pairs of Hamilton cycles are reverses of each other.


a
3
2
1
b
c
2
1
3
d

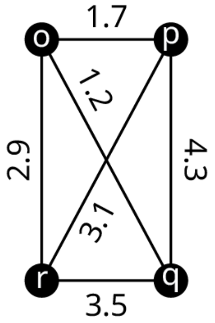
Graph A


e
2/3
1/4
1/6
f
1/3
5/12
i
5/6
3/4
1/2
1
g
1/12
h

Graph B


k
10
20
60
100
l
m
70
50
90
30
80
j
n
40

Graph C


1.7
o
p
1.2
2.9
4.3
3.1
r
q
3.5

Graph D

? Exercise 12.9.1.9

Graph *A*, vertex *a*

? Exercise 12.9.1.10

Graph *B*, vertex *e*

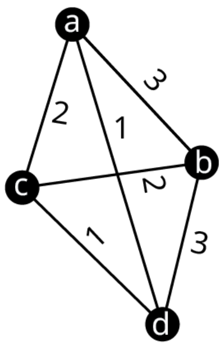
? Exercise 12.9.1.11

Graph *C*, vertex *k*

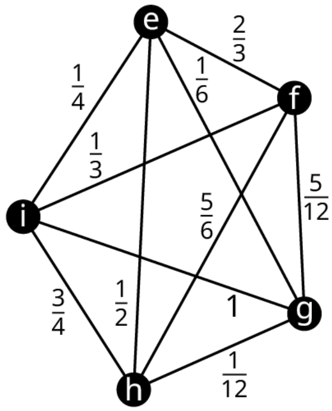
? Exercise 12.9.1.12

Graph *D*, vertex *o*

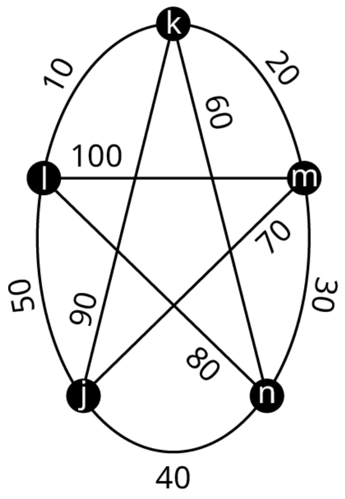
For the following exercises, use the figure to find a Hamilton cycle of least weight for the given graph, beginning at the given vertex, and using the brute force method. What is the weight of the cycle?


a
3
2
1
b
c
2
1
3
d

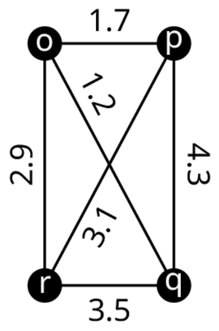
Graph A


e
$\frac{2}{3}$
$\frac{1}{4}$
$\frac{1}{6}$
f
$\frac{1}{3}$
$\frac{5}{12}$
i
$\frac{5}{6}$
$\frac{3}{4}$
$\frac{1}{2}$
1
g
h
$\frac{1}{12}$

Graph B


k
10
20
60
100
l
m
70
50
90
30
80
j
n
40

Graph C


o
1.7
p
1.2
2.9
4.3
3.1
r
q
3.5

Graph D

? Exercise 12.9.1.13

Graph *A*, vertex *a*

? Exercise 12.9.1.14

Graph *B*, vertex *e*

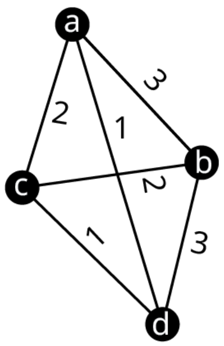
? Exercise 12.9.1.15

Graph *C*, vertex *k*

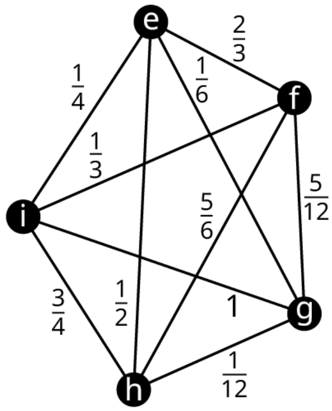
? Exercise 12.9.1.16

Graph *D*, vertex *o*

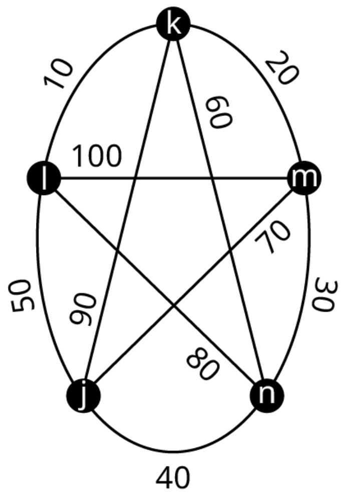
For the following exercises, use the figure to find a Hamilton cycle of low weight for the given graph, beginning at the given vertex, and using the nearest neighbor method. What is the weight of the cycle?


a
3
2
1
b
c
2
1
3
d

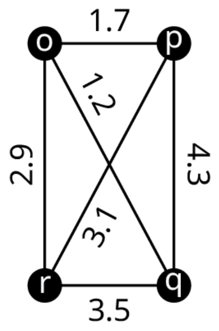
Graph A


e
$\frac{2}{3}$
$\frac{1}{4}$
$\frac{1}{6}$
f
$\frac{1}{3}$
$\frac{5}{12}$
i
$\frac{5}{6}$
$\frac{3}{4}$
$\frac{1}{2}$
1
g
h
$\frac{1}{12}$

Graph B


k
10
20
60
100
l
m
70
50
90
30
80
j
n
40

Graph C


o
1.7
p
1.2
2.9
4.3
3.1
r
q
3.5

Graph D

Exercise 12.9.1.17

Graph *A*, vertex *a*

Exercise 12.9.1.18

Graph *B*, vertex *e*

Exercise 12.9.1.19

Graph *C*, vertex *k*

Exercise 12.9.1.20

Graph *D*, vertex *o*

For the following exercises, use your solutions to Exercises 13–20 to compare the results of the brute force method to the results of the nearest neighbor method for each graph. Indicate whether the Hamilton cycle was the same or different and whether the weights were the same or different. If the weights are different, indicate which method gave the lower weight. Are your observations consistent with the characteristics of brute force algorithms and greedy algorithms? Explain your reasoning.

Exercise 12.9.1.21

Graph *A*, vertex *a*

Exercise 12.9.1.22

Graph *B*, vertex *e*

Exercise 12.9.1.23

Graph *C*, vertex *k*

Exercise 12.9.1.24

Graph *D*, vertex *o*

For the following exercises, use the table to create a complete weighted graph in which the vertices are the given cities, and the weights are the distances between them.

Cities	*U*	*V*	*W*	*X*	*Y*	*Z*
U	0	89	37	49	54	28
V	89	0	76	68	92	112
W	37	76	0	45	52	49
X	49	68	45	0	66	47
Y	54	92	52	66	0	29
Z	28	112	49	47	29	0

Distances between Cities *U*, *V*, *W*, *X*, *Y*, and *Z* in kilometers

Exercise 12.9.1.25

U, V, W, X

Exercise 12.9.1.26

U, W, Y, Z

Exercise 12.9.1.27

U, X, Y, Z

Exercise 12.9.1.28

U, V, W, X, Y

Exercise 12.9.1.29

U, W, X, Y, Z

Exercise 12.9.1.30

U, V, W, X, Y, Z

For the following exercises, use your solutions to Exercises 25–30 and the nearest neighbor method to find a Hamilton cycle to solve the traveling salesperson problem of finding a reasonably short route to leave from city U, visit each of the other cities listed and return to city U. Indicate the distance required to travel the route you found.

Exercise 12.9.1.31

U, V, W, X

Exercise 12.9.1.32

U, W, Y, Z

Exercise 12.9.1.33

U, X, Y, Z

Exercise 12.9.1.34

U, V, W, X, Y

Exercise 12.9.1.35

U, W, X, Y, Z

Exercise 12.9.1.36

U, V, W, X, Y, Z

For the following exercises, use your solutions to Exercises 25–30 and the brute force method to find a Hamilton cycle of lowest weight to solve the traveling salesperson problem of finding a shortest route to leave from city U, visit each of the other cities listed and return to city U. Indicate the distance required to travel the route you found.

? Exercise 12.9.1.37

U, V, W, X

? Exercise 12.9.1.38

U, W, Y, Z

? Exercise 12.9.1.39

U, X, Y, Z

? Exercise 12.9.1.40

U, V, W, X, Y

? Exercise 12.9.1.41

U, W, X, Y, Z

For the following exercises, use your solutions to the indicated exercises to compare the results of the brute force method to the results of the nearest neighbor method for each traveling salesman problem of finding a reasonably short route to leave from city U, visit each of the other cities listed and return to city U. Indicate whether the greedy algorithm resulted in a Hamilton cycle of the same weight, lower weight, or higher weight. Is this consistent with the characteristics of brute force algorithms and greedy algorithms? Explain your reasoning

? Exercise 12.9.1.42

Exercises 32 and 38: U, W, Y, Z

? Exercise 12.9.1.43

Exercises 31 and 37: U, V, W, X

? Exercise 12.9.1.44

Exercises 34 and 40: U, V, W, X, Y

? Exercise 12.9.1.45

Exercises 35 and 41: U, W, X, Y, Z

? Exercise 12.9.1.46

Exercises 33 and 39: U, X, Y, Z

The products at a particular factory are manufactured in phases. The same equipment is utilized for each phase, but it must be formatted differently to accomplish different tasks. The transition time to convert between a format for one task and another task varies. The times are given in Table 12.13. In the following exercises, use the table and the nearest neighbor method to find an order in which to complete the tasks, which keeps the transition times down and ends with the same set up as it began so that the factory is ready to start the next batch. Assume that there are no restrictions on which tasks can be completed in which order. Hint:

The nearest neighbor algorithm may give a different result depending on which vertex is the starting vertex, so, you must check all possibilities.

Task	A	B	C	D	E	F	G
A	0	75	130	45	120	70	100
B	75	0	140	65	115	25	60
C	130	140	0	35	55	20	125
D	45	65	35	0	50	30	40
E	120	115	55	50	0	95	145
F	70	25	20	30	95	0	15
G	100	60	125	40	145	15	0

Manufacturing Equipment Transition Times in Minutes

? Exercise $12.9.1.47$

A, C, D, F, G

? Exercise $12.9.1.48$

B, D, E, F, G

? Exercise $12.9.1.49$

A, B, C, D, E, F

? Exercise $12.9.1.50$

B, C, D, E, F, G

? Exercise $12.9.1.51$

A, B, C, D, E, F, G

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12.10: Trees

Figure 12.10.1: In graph theory, graphs known as trees have structures in common with live trees. (credit: “Row of trees in Roslev” by AKA CJ/Flickr, Public Domain)

Learning Objectives

1. Describe and identify trees.
2. Determine a spanning tree for a connected graph.
3. Find the minimum spanning tree for a weighted graph.
4. Solve application problems involving trees.

We saved the best for last! In this last section, we will discuss arguably the most fun kinds of graphs, trees. Have you every researched your family tree? Family trees are a perfect example of the kind of trees we study in graph theory. One of the characteristics of a family tree graph is that it never loops back around, because no one is their own grandparent!

What Is A Tree?

Whether we are talking about a family tree or a tree in a forest, none of the branches ever loops back around and rejoins the trunk. This means that a tree has no cyclic subgraphs, or is acyclic. A tree also has only one component. So, a tree is a connected acyclic graph. Here are some graphs that have the same characteristic. Each of the graphs in Figure 12.10.2 is a tree.

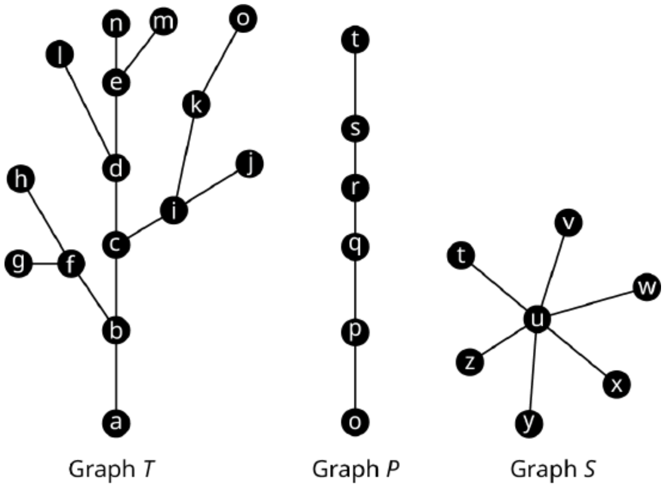


Figure 12.10.2: Graphs T, P, and S

Let’s practice determining whether a graph is a tree. To do this, check if a graph is connected and has no cycles.

✓ Example 12.10.1: Identifying Trees

Identify any trees in Figure ✓ 12.10.3. If a graph is not a tree, explain how you know.

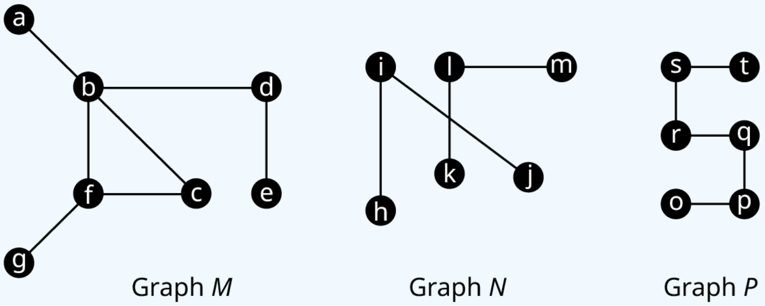


Figure ✓ 12.10.3: Graphs M, N, and P

Answer

- Graph M is not a tree because it contains the cycle (b, c, f).
- Graph N is not a tree because it is not connected. It has two components, one with vertices h, i, j, and another with vertices k, l, m.
- Graph P is a tree. It has no cycles and it is connected.

Your Turn 12.10.1

There are some configurations that are commonly used when setting up computer networks. Several of them are shown in the given figure. Which of the configurations in the figure appear to have the characteristics of a tree graph? If a configuration does not appear to have the characteristics of a tree graph, explain how you know.

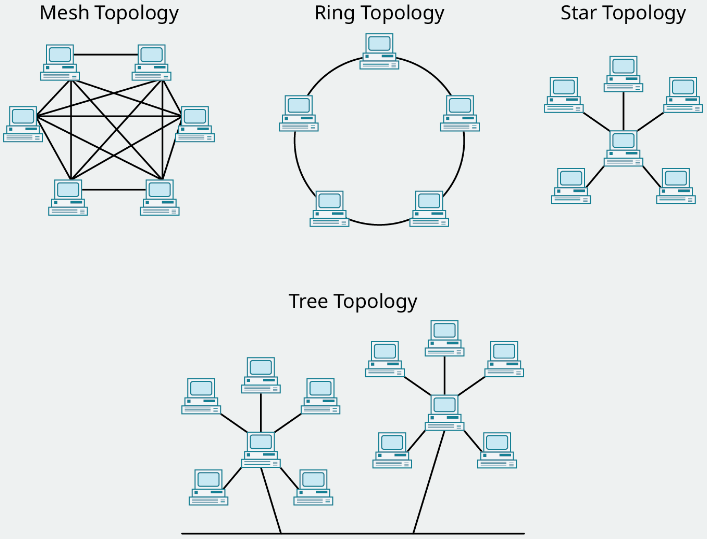


Figure 12.10.4: Common Network Configurations

Types of Trees

Mathematicians have had a lot of fun naming graphs that are trees or that contain trees. For example, the graph in Figure 12.10.5 is not a tree, but it contains two components, one containing vertices a through d, and the other containing vertices e through g, each of which would be a tree on its own. This type of structure is called a forest. There are also interesting names for trees with certain characteristics.

- A **path graph or linear graph** is a tree graph that has exactly two vertices of degree 1 such that the only other vertices form a single path between them, which means that it can be drawn as a straight line.
- A **star tree** is a tree that has exactly one vertex of degree greater than 1 called a **root,** and all other vertices are adjacent to it.

- A **starlike tree** is a tree that has a single **root** and several paths attached to it.
- A **caterpillar tree** is a tree that has a central path that can have vertices of any degree, with each vertex not on the central path being adjacent to a vertex on the central path and having a degree of one.
- A **lobster tree** is a tree that has a central path that can have vertices of any degree, with paths consisting of either one or two edges attached to the central path.

Examples of each of these types of structures are given in Figure $12.10.6$.

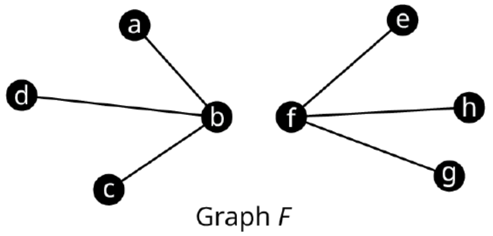

a
d
b
c
e
f
h
g
Graph *F*

Figure $12.10.5$: Forest Graph F

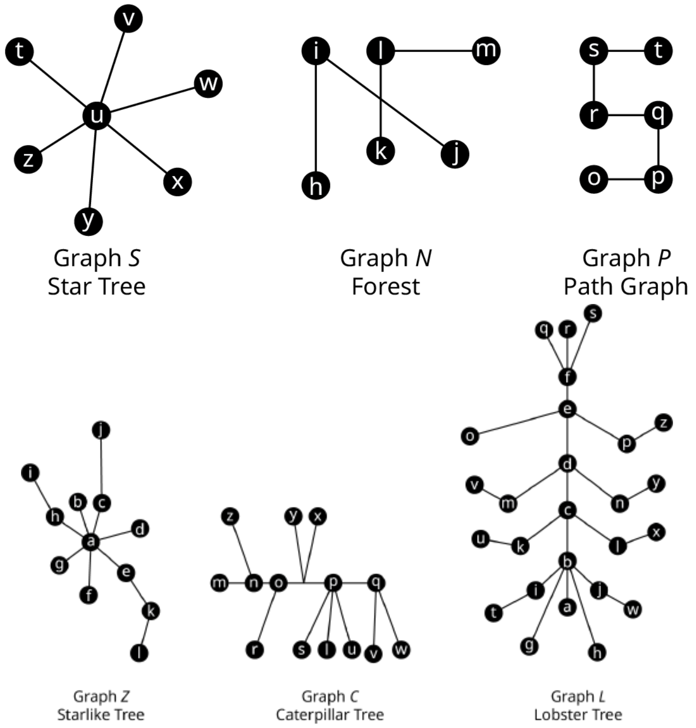

Graph *S*
Star Tree
Graph *N*
Forest
Graph *P*
Path Graph
Graph *Z*
Starlike Tree
Graph *C*
Caterpillar Tree
Graph *L*
Lobster Tree

Figure $12.10.6$: Six Types of Trees

✓ Example $12.10.2$: Identifying Types of Trees

Each graph in Figure ✓ $12.10.7$ is one of the special types of trees we have been discussing. Identify the type of tree.

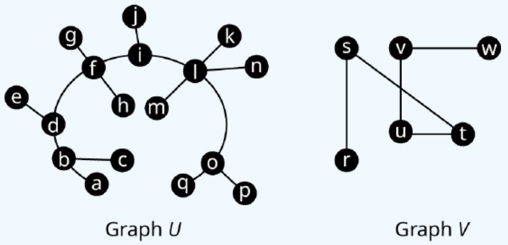

Graph *U*
Graph *V*

Figure ✓ $12.10.7$ Graphs U and V

Answer

Graph U has a central path $a \to b \to d \to f \to i \to l \to o \to q$. Each vertex that is not on the path has degree 1 and is adjacent to a vertex that is on the path. So, U is a caterpillar tree.

Graph V is a path graph because it is a single path connecting exactly two vertices of degree one, $r \to s \to u \to v \to w$.

Your Turn 12.10.2

Of the network configurations from Figure 12.10.7, which, if any, has the characteristics of a

1. Star tree?
2. Caterpillar tree?
3. Path graph?

Characteristics of Trees

As we study trees, it is helpful to be familiar with some of their characteristics. For example, if you add an edge to a tree graph between any two existing vertices, you will create a cycle, and the resulting graph is no longer a tree. Some examples are shown in Figure 12.10.8. Adding edge bj to Graph T creates cycle (b, c, i, j). Adding edge rt to Graph P creates cycle (r, s, t). Adding edge tv to Graph S creates cycle (t, u, v).

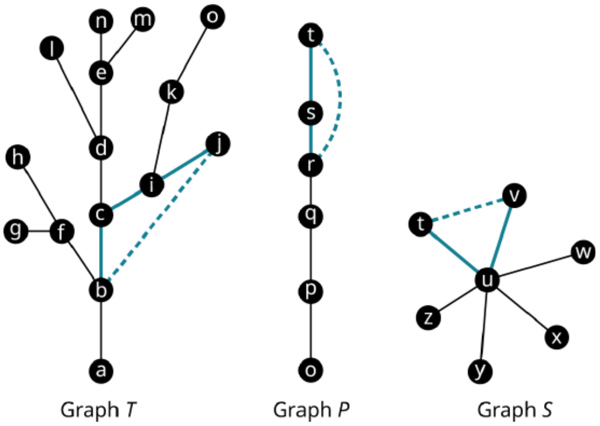


Figure 12.10.8: Adding Edges to Trees

It is also true that removing an edge from a tree graph will increase the number of components and the graph will no longer be connected. In fact, you can see in Figure \(\PageIndex{8}\ that removing one or more edges can create a forest. Removing edge qr from Graph P creates a graph with two components, one with vertices o, p and q, and the other with vertices r, s, and t. Removing edge uw from Graph S creates two components, one with just vertex w and the other with the rest of the vertices. When two edges were removed from Graph T, edge bf and edge cd, creates a graph with three components as shown in Figure 12.10.9.

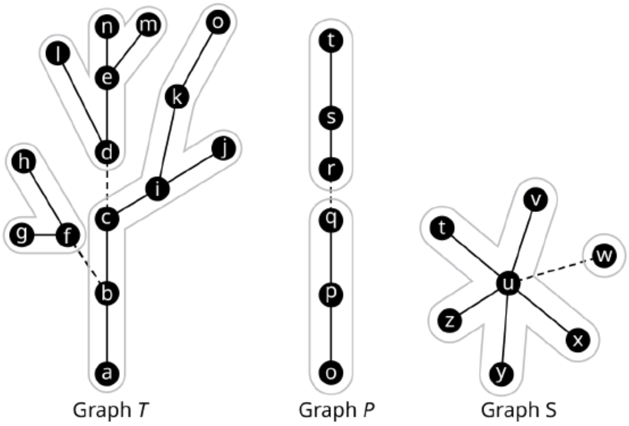


Figure 12.10.9: Removing Edges from Trees

A very useful characteristic of tree graphs is that the number of edges is always one less than the number of vertices. In fact, any connected graph in which the number of edges is one less than the number of vertices is guaranteed to be a tree. Some examples are given in Figure 12.10.10.

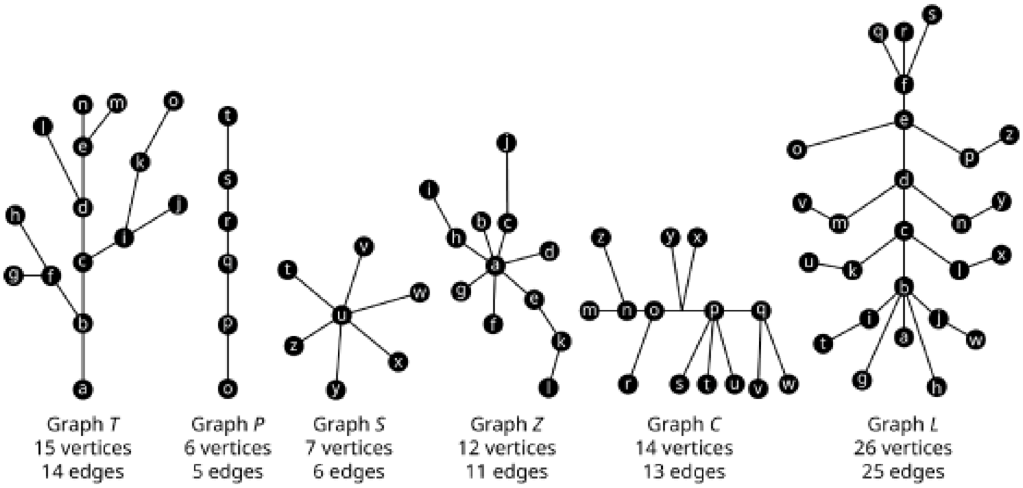

Graph T
15 vertices
14 edges
Graph P
6 vertices
5 edges
Graph S
7 vertices
6 edges
Graph Z
12 vertices
11 edges
Graph C
14 vertices
13 edges
Graph L
26 vertices
25 edges

Figure 12.10.10: Number of Vertices and Edges in Trees vs. Other Graphs

FORMULA

The number of edges in a tree graph with n vertices is $n-1$.

A connected graph with n vertices and $n-1$ edges is a tree graph.

Example 12.10.3: Exploring Characteristics of Trees

Use Graphs *I* and *J* in Figure ✓ 12.10.11 to answer each question.

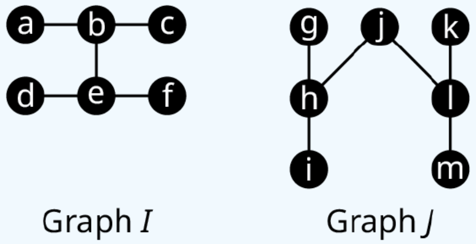

a
b
c
d
e
f
g
j
k
h
l
i
m
Graph I
Graph J

Figure ✓ 12.10.11 Graphs *I* and *J*

1. Which vertices are in each of the components that remain when edge *be* is removed from Graph *I*?
2. Determine the number of edges and the number of vertices in Graph *J*. Explain how this confirms that Graph *J* is a tree.
3. What kind of cycle is created if edge *im* is added to Graph *J*?

Answer

1. When edge *be* is removed, there are two components that remain. One component includes vertices *a*, *b*, and *c*. The other component includes vertices *d*, *e*, and *f*.
2. There are seven vertices and six edges in Graph *J*. This confirms that Graph *J* is a tree because the number of edges is one less than the number of vertices.
3. The pentagon (*i*, *h*, *j*, *l*, *m*) is created when edge *im* is added to Graph *J*.

Your Turn 12.10.3

Use Graphs *I* and *J* in Figure 12.10.11 to answer each question.

Which vertices are in each of the components that remain when edge *jl* is removed from Graph *J*?

Determine the number of edges and the number of vertices in Graph I. Explain how this confirms that Graph I is a tree.

What kind of cycle is created if edge cf is added to Graph I?

Who Knew?: Graph Theory in the Movies

In the 1997 film *Good Will Hunting*, the main character, Will, played by Matt Damon, solves what is supposed to be an exceptionally difficult graph theory problem, “Draw all the homeomorphically irreducible trees of size $n = 10$n=10.” That sounds terrifying! But don’t panic. Watch this great Numberphile video to see why this is actually a problem you can do at home!

Video

e problem in Good Will Hunting - Nu
nberphile
Numbe
Watch on YouTube

The Problem in *Good Will Hunting* by Numberphile

Spanning Trees

Suppose that you planned to set up your own computer network with four devices. One option is to use a “mesh topology” like the one in Figure 12.10.12, in which each device is connected directly to every other device in the network.

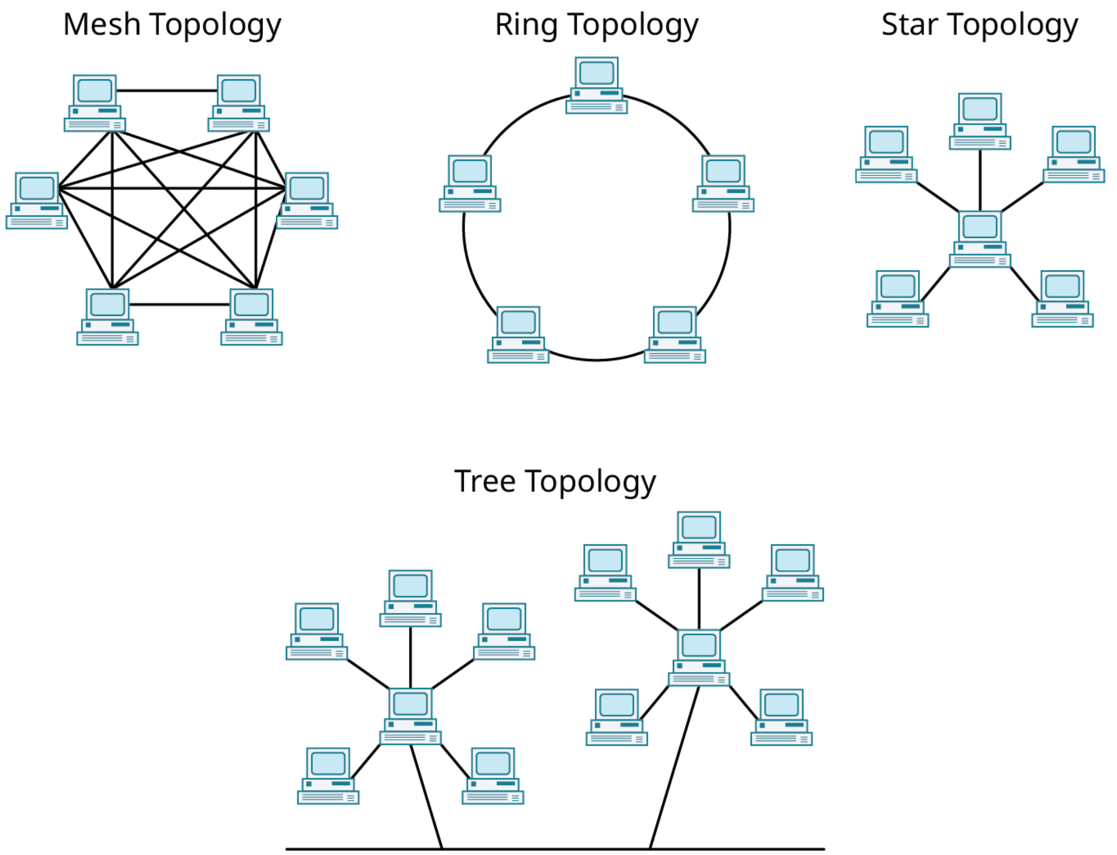


Figure 12.10.12: Common Network Configurations

The mesh topology for four devices could be represented by the complete Graph A_1 in Figure 12.10.13 where the vertices represent the devices, and the edges represent network connections. However, the devices could be networked using fewer connections. Graphs A_2, A_3, and A_4 of Figure 12.10.13 show configurations in which three of the six edges have been removed. Each of the Graphs A_2, A_3 and A_4 in Figure 12.242 is a tree because it is connected and contains no cycles. Since Graphs A_2, A_3 and A_4 are also subgraphs of Graph A_1 that include every vertex of the original graph, they are also known as spanning trees.

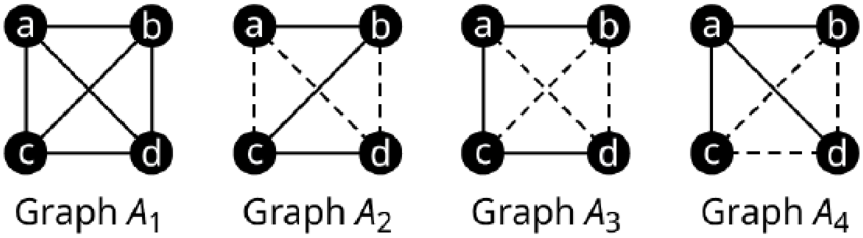


Figure 12.10.13: Network Configurations for Four Devices

By definition, spanning trees must span the whole graph by visiting all the vertices. Since spanning trees are subgraphs, they may only have edges between vertices that were adjacent in the original graph. Since spanning trees are trees, they are connected and they are acyclic. So, when deciding whether a graph is a spanning tree, check the following characteristics:

- All vertices are included.
- No vertices are adjacent that were not adjacent in the original graph.
- The graph is connected.
- There are no cycles.

✓ Example 12.10.49: Identifying Spanning Trees

Use Figure ✓ 12.10.14 to determine which of graphs M_1, M_2, M_3, and M_4, are spanning trees of Q.

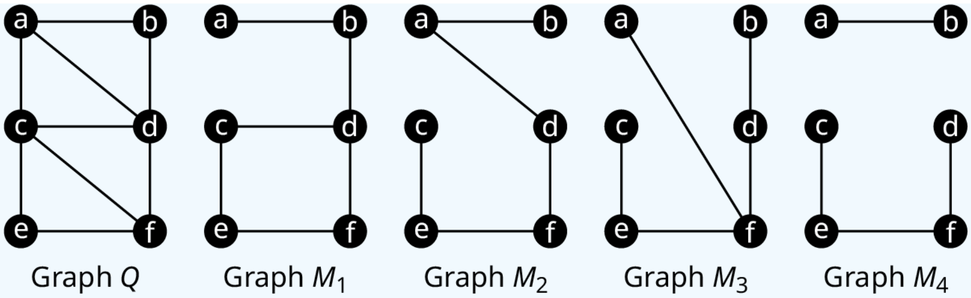


Figure ✓ 12.10.14 Graphs Q, M_1, M_2, $M_{3,}$ and M_4

Answer

1. Graph M_1 is not a spanning tree of Graph Q because it has a cycle (c, d, f, e).
2. Graph M_2 is a spanning tree of Graph Q because it has all the original vertices, no vertices are adjacent in M_2 that weren't adjacent in Graph Q, Graph M_2 is connected and it contains no cycles.
3. Graph M_3 is not a spanning tree of Graph Q because vertices a and f are adjacent in Graph M_3 but not in Graph Q.
4. Graph M_4 is not a spanning tree of Graph Q because it is not connected.

So, only graph M_2 is a spanning tree of Graph Q.

Your Turn 12.10.4

Use the given figure for the following exercises.

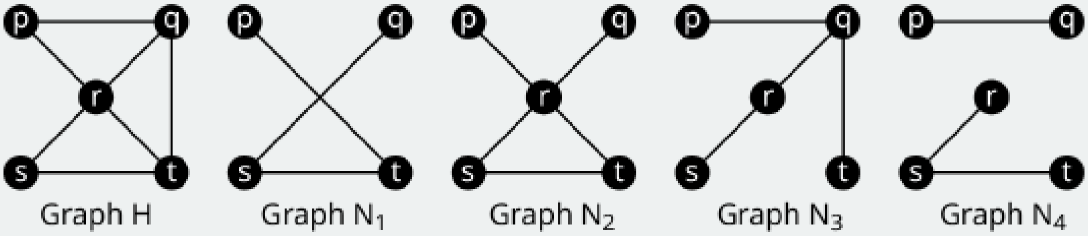


Figure 12.10.15

1. Since sq is not an edge in Graph H, Graph N_1 cannot be a spanning tree of H.
   a. True
   b. False
2. Graph N_2 is a spanning tree of Graph H.
   a. True
   b. False
3. Graph N_3 is a spanning tree of Graph H.
   a. True
   b. False
4. Since there is no path between p and t in Graph N_4, it cannot be a spanning tree of any graph.
   a. True
   b. False

Constructing a Spanning Tree Using Paths

Suppose that you wanted to find a spanning tree within a graph. One approach is to find paths within the graph. You can start at any vertex, go any direction, and create a path through the graph stopping only when you can't continue without backtracking as shown in Figure 12.10.16.

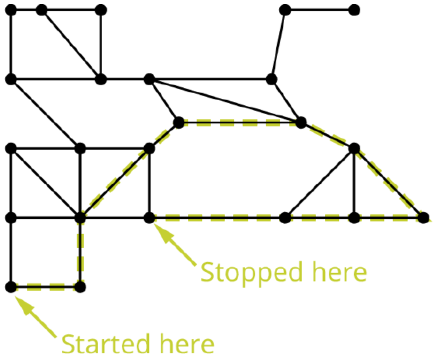

Stopped here
Started here

Figure 12.10.16: First Phase to Construct a Spanning Tree

Once you have stopped, pick a vertex along the path you drew as a starting point for another path. Make sure to visit only vertices you have not visited before as shown in Figure 12.10.17.

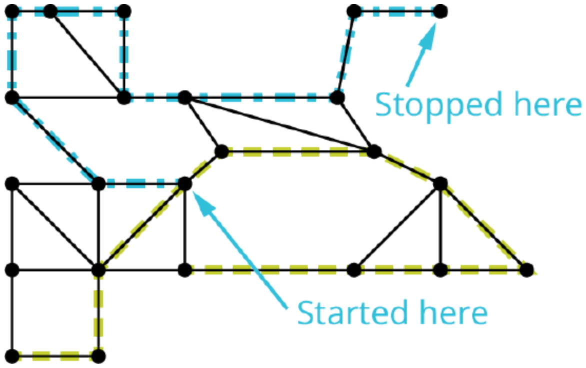

Stopped here
Started here

Figure 12.10.17: Intermediate Phase to Construct a Spanning Tree

Repeat this process until all vertices have been visited as shown in Figure 12.10.18.

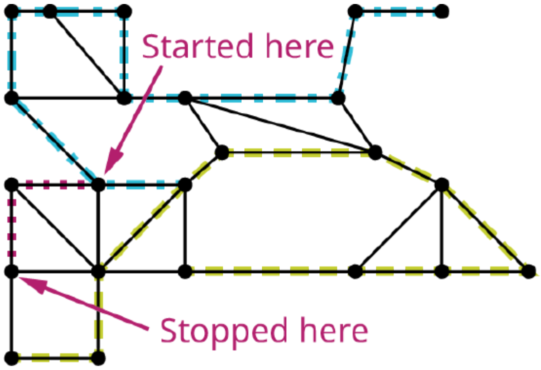

Started here
Stopped here

Figure 12.10.18: Final Phase to Construct a Spanning Tree

The end result is a tree that spans the entire graph as shown in Figure 12.10.19.

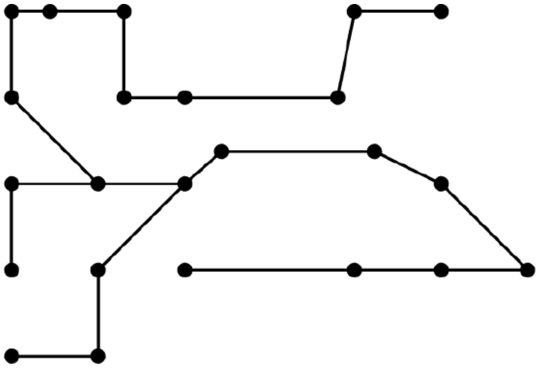


Figure 12.10.19: The Resulting Spanning Tree

Notice that this subgraph is a tree because it is connected and acyclic. It also visits every vertex of the original graph, so it is a spanning tree. However, it is not the only spanning tree for this graph. By making different turns, we could create any number of distinct spanning trees.

✓ Example 12.10.5: Constructing Spanning Trees

Construct two distinct spanning trees for the graph in Figure ✓ 12.10.20.

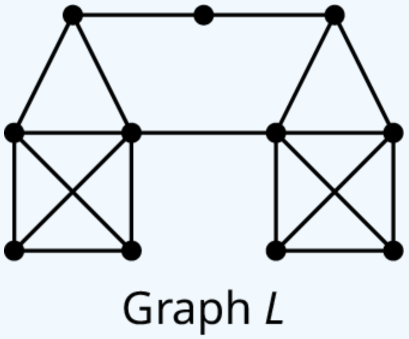


Figure ✓ 12.10.20 Graph *L*

Answer

Two possible solutions are given in Figure ✓ 12.10.21 and Figure ✓ 12.10.22.

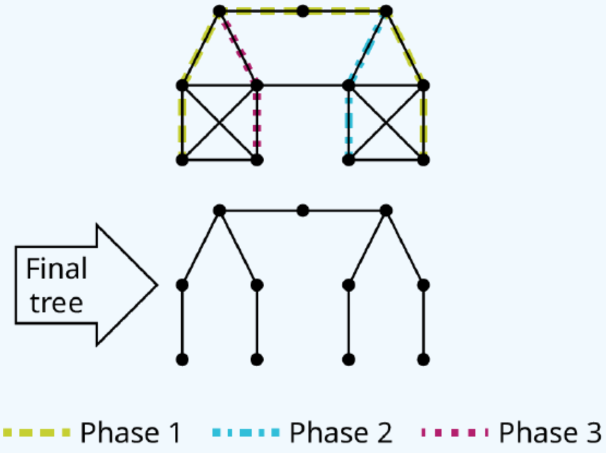

Final tree
Phase 1
Phase 2
Phase 3

Figure ✓ 12.10.21 First Spanning Tree for Graph *L*

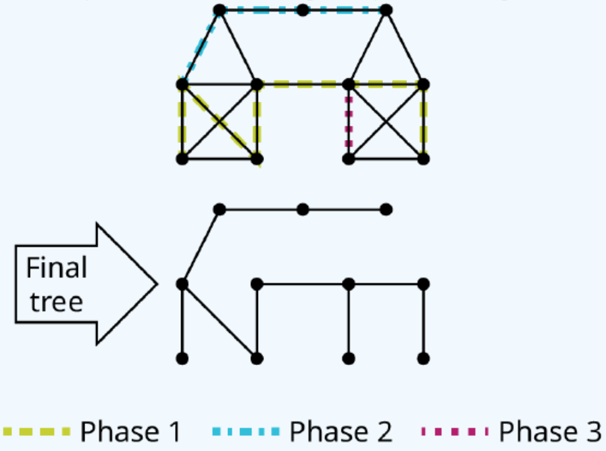

Final tree
Phase 1
Phase 2
Phase 3

Figure ✓ 12.10.22: Second Spanning Tree for Graph *L*

Your Turn 12.10.5

Construct three distinct spanning trees for Graph *J*.

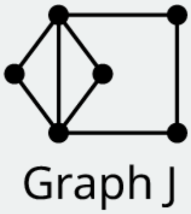


Figure 12.10.23: Graph *J*

Revealing Spanning Trees

Another approach to finding a spanning tree in a connected graph involves removing unwanted edges to reveal a spanning tree. Consider Graph *D* in Figure 12.10.24.

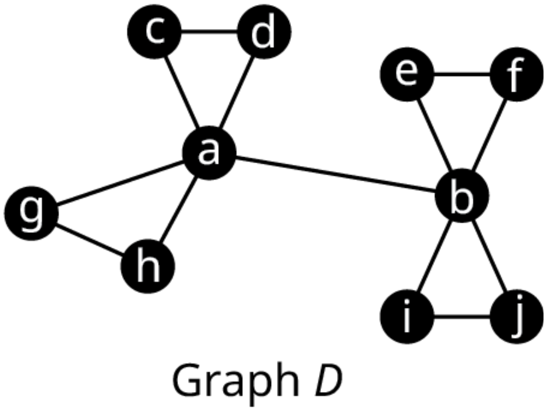


Figure 12.10.24: Graph *D*

Graph *D* has 10 vertices. A spanning tree of Graph *D* must have 9 edges, because the number of edges is one less than the number of vertices in any tree. Graph *D* has 13 edges so 4 need to be removed. To determine which 4 edges to remove, remember that trees do not have cycles. There are four triangles in Graph *D* that we need to break up. We can accomplish this by removing 1 edge from each of the triangles. There are many ways this can be done. Two of these ways are shown in Figure 12.10.25.

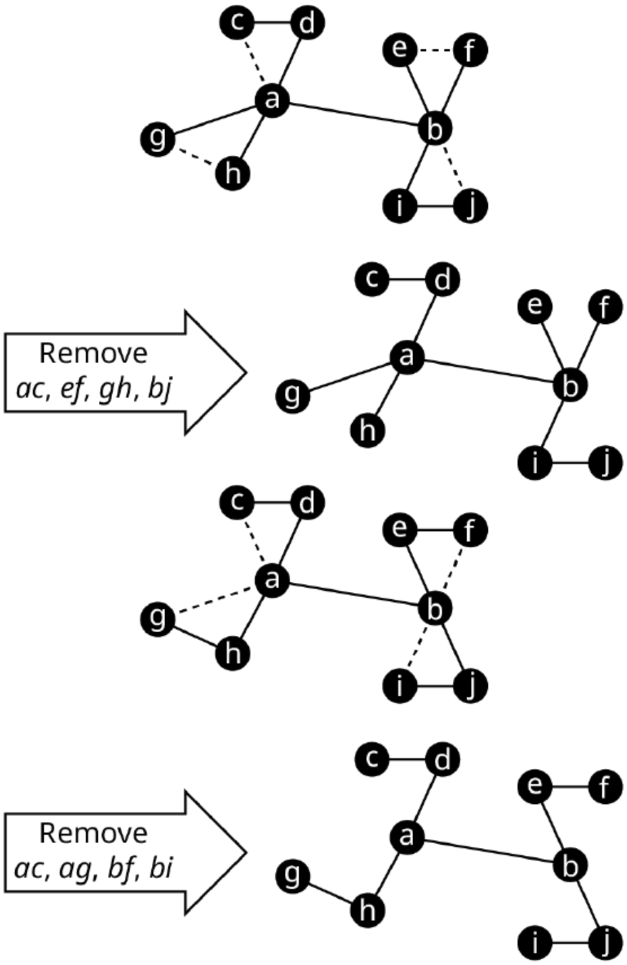
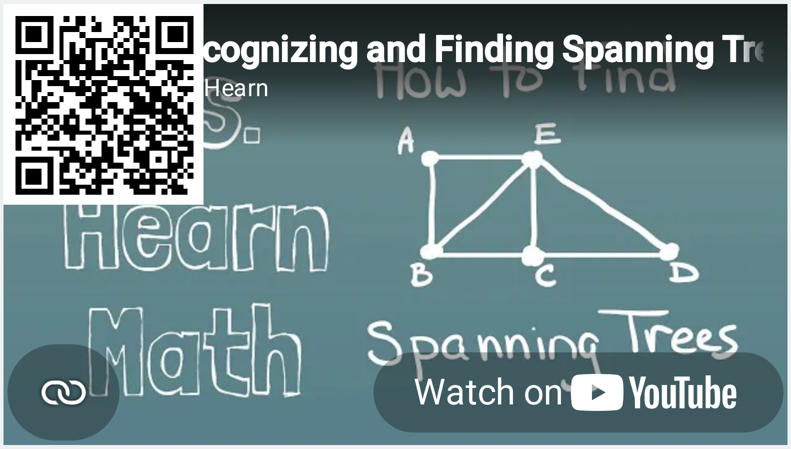

c
d
e
f
a
b
g
h
i
j
Remove
ac, ef, gh, bj
Remove
ac, ag, bf, bi

Figure $12.10.25$: Removing Four Edges from Graph D

Video


cognizing and Finding Spanning Tre
Hearn
How to find
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Math
A
E
B
C
D
Spanning Trees
Watch on YouTube

Spanning Trees in Graph Theory

Example $12.10.6$: Removing Edges to Find Spanning Trees

Use the graph in Figure 12.255 to answer each question.

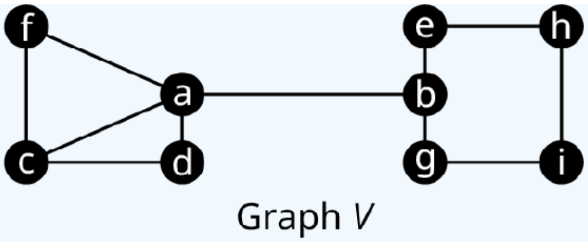


Figure ✓ 12.10.26 Graph *V*

1. Determine the number of edges that must be removed to reveal a spanning tree.
2. Name all the undirected cycles in Graph *V*.
3. Find two distinct spanning trees of Graph *V*.

Answer

1. Graph *V* has nine vertices so a spanning tree for the graph must have 8 edges. Since Graph *V* has 11 edges, 3 edges must be removed to reveal a spanning tree.
2. (a, c, d), (a, c, f), (a, d, c, f), and (b, e, h, i, g)
3. To find the first spanning tree, remove edge *ac*, which will break up both of the triangles, remove edge *cf* , which will break up the quadrilateral, and remove *be*, which will break up the pentagon, to give us the spanning tree shown in Figure ✓ 12.10.27.

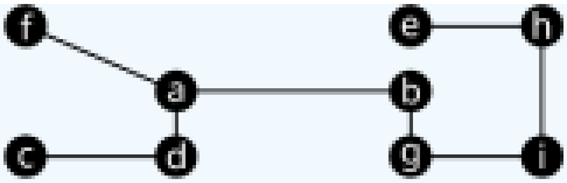


Figure ✓ 12.10.27 Spanning Tree Formed Removing ac, cf, and be

To find another spanning tree, remove ad, which will break up (a, c, d) and (a, d, c, f), remove af to break up (a, c, f), and remove hi to break up (b, e, h, i, g). This will give us the spanning tree in Figure ✓ 12.10.28.

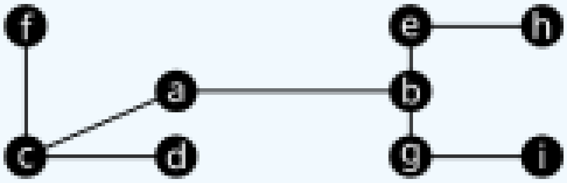


Figure ✓ 12.10.28: Spanning Tree Formed Removing ad, *af*, and *hi*

Your Turn 12.10.6

Name three edges that you could remove from Graph V in Figure 12.316 to form a third spanning tree, different from those in the solution to Example 12.50 Exercise 3.

Who Knew?: Chains of Affection

Here is a strange question to ask in a math class: Have you ever dated your ex's new partner's ex? Research suggests that your answer is probably no. When researchers Peter S. Bearman, James Moody, and Katherine Stovel attempted to compare the structure of heterosexual romantic networks at a typical midwestern high school to simulated networks, they found something surprising. The actual social networks were more like spanning trees than other possible models because there were very few short cycles. In particular, there were almost no four-cycles.

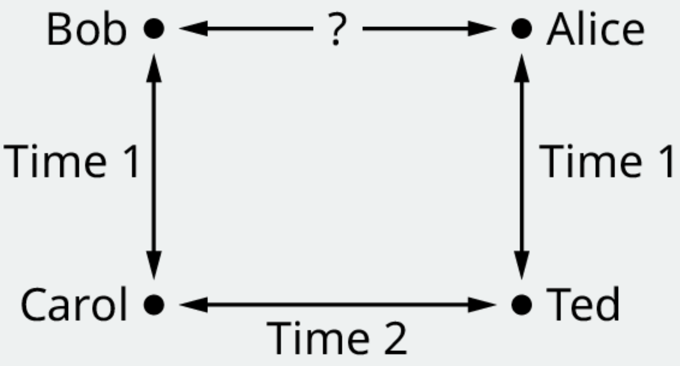


Figure 12.10.29: Chains of Affection

"…the prohibition against dating (from a female perspective) one's old boyfriend's current girlfriend's old boyfriend – accounts for the structure of the romantic network at [the highschool]."

In their article "Chains of Affection: The Structure of Adolescent Romantic and Sexual Networks," the researchers went on to explain the implications for the transmission of sexually transmitted diseases. In particular, social structures based on tree graphs are less dense and more likely to fragment. This information can impact social policies on disease prevention. (Peter S. Bearman, James Moody, and Katherine Stovel, "Chains of Affection: The Structure of Adolescent Romantic and Sexual Networks," *American Journal of Sociology* Volume 110, Number 1, pp. 44-91, 2004)

Kruskal's Algorithm

In many applications of spanning trees, the graphs are weighted and we want to find the spanning tree of least possible weight. For example, the graph might represent a computer network, and the weights might represent the cost involved in connecting two devices. So, finding a spanning tree with the lowest possible total weight, or minimum spanning tree, means saving money! The method that we will use to find a minimum spanning tree of a weighted graph is called Kruskal's algorithm. The steps for Kruskal's algorithm are:

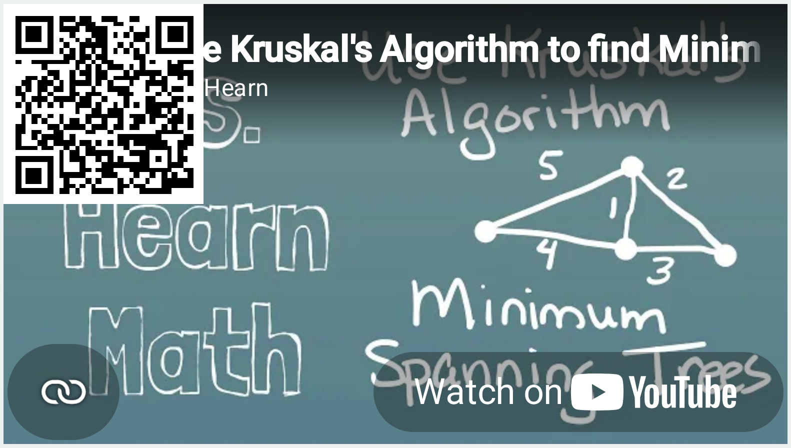
Step 1: Choose any edge with the minimum weight of all edges.

Step 2: Choose another edge of minimum weight from the remaining edges. The second edge does not have to be connected to the first edge.

Step 3: Choose another edge of minimum weight from the remaining edges, but do not select any edge that creates a cycle in the subgraph you are creating.

Step 4: Repeat step 3 until all the vertices of the original graph are included and you have a spanning tree.

Video


e Kruskal's Algorithm to find Minim
Hearn
Algorithm
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Math
Minimum
Spanning Trees
Watch on YouTube

Use Kruskal's Algorithm to Find Minimum Spanning Trees in Graph Theory

✓ Example 12.10.7: Using Kruskal's Algorithm

A computer network will be set up with six devices. The vertices in the graph in Figure ✓ 12.10.30 represent the devices, and the edges represent the cost of a connection. Find the network configuration that will cost the least. What is the total cost?

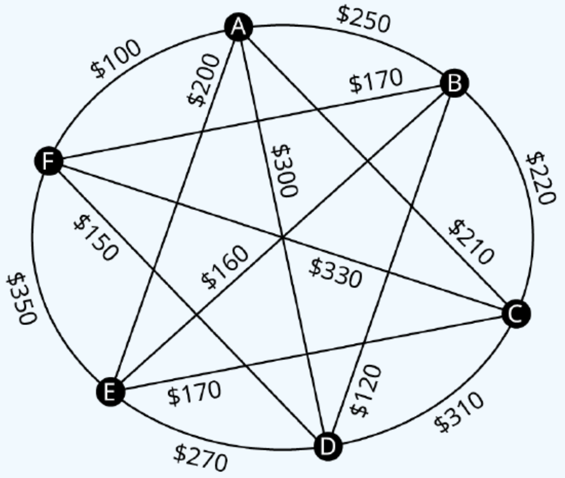
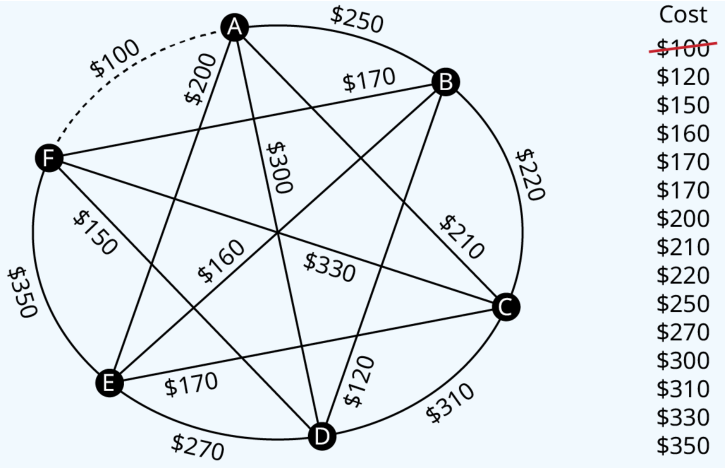

A
B
C
D
E
F
$250
$100
$200
$170
$220
$300
$210
$150
$160
$330
$350
$170
$120
$310
$270

Figure ✓ 12.10.30 Graph of Network Connection Costs

Answer

A minimum spanning tree will correspond to the network configuration of least cost. We will use Kruskal's algorithm to find one. Since the graph has six vertices, the spanning tree will have six vertices and five edges.

Step 1: Choose an edge of least weight. We have sorted the weights into numerical order. The least is $100. The only edge of this weight is edge *AF* as shown in Figure ✓ 12.10.31


A
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F
$250
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$200
$170
$300
$220
$150
$160
$330
$210
$350
$170
$120
$310
$270
Cost
~~$100~~
$120
$150
$160
$170
$170
$200
$210
$220
$250
$270
$300
$310
$330
$350

Figure ✓ 12.10.31 Step 1 Select Edge *AF*

Step 2:

Choose the edge of least weight of the remaining edges, which is *BD* with $120. Notice that the two selected edges do not need to be adjacent to each other as shown in Figure ✓ 12.10.32.

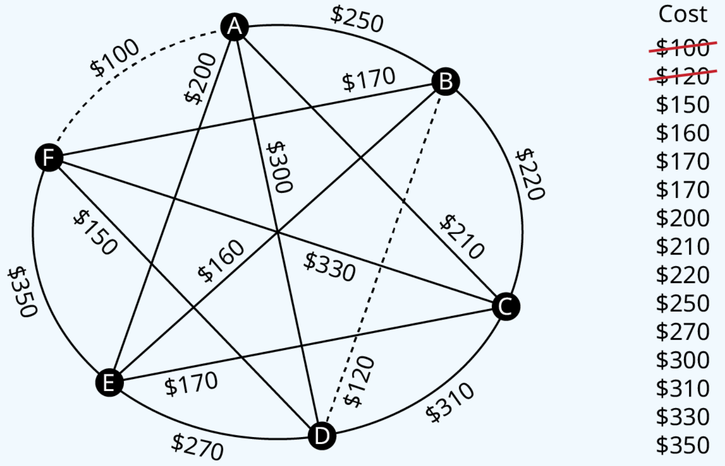
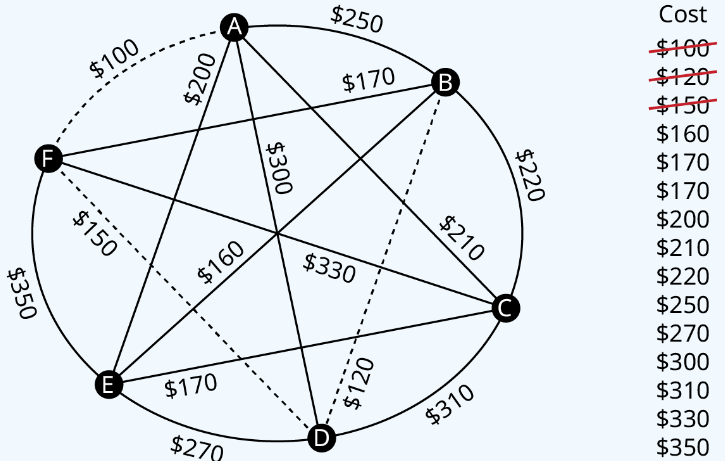

A
B
C
D
E
F
$250
$100
$200
$170
$300
$220
$150
$160
$330
$210
$350
$170
$120
$310
$270
Cost
~~$100~~
~~$120~~
$150
$160
$170
$170
$200
$210
$220
$250
$270
$300
$310
$330
$350

Figure ✓ 12.10.32: Step 2 Select Edge *BD*

Step 3: Select the lowest weight edge of the remaining edges, as long as it does not result in a cycle. We select *DF* with $150 since it does not form a cycle as shown in Figure ✓ 12.10.33.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$350
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$120
$310
$270
Cost
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~~$120~~
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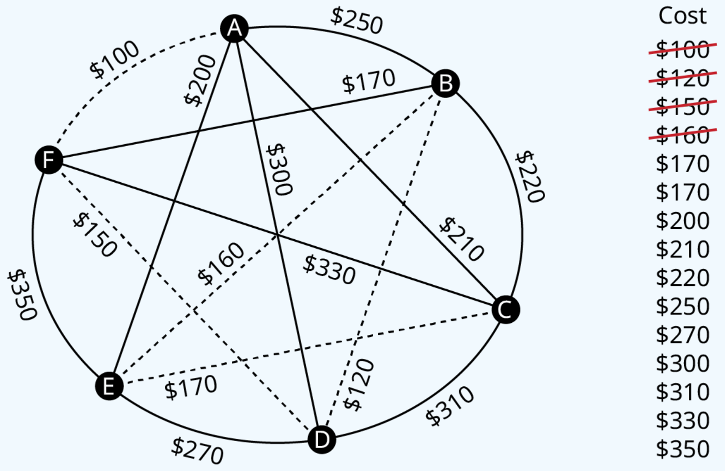
$160
$170
$170
$200
$210
$220
$250
$270
$300
$310
$330
$350

Figure ✓ 12.10.33: Step 3 Select Edge *DF*

Repeat Step 3: Select the lowest weight edge of the remaining edges, which is *BE* with $160 and it does not form a cycle as shown in Figure ✓ 12.10.34. This gives us four edges so we only need to repeat step 3 once more to get the fifth edge.


A
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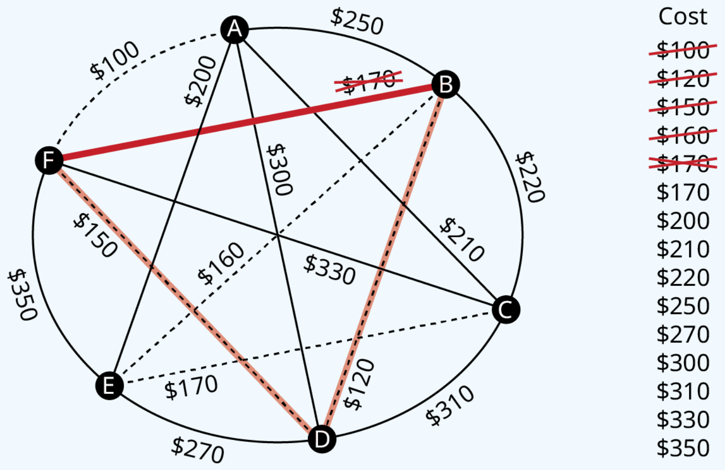
$200
$170
$220
$300
$150
$210
$160
$330
$350
$170
$120
$310
$270
Cost
~~$100~~
~~$120~~
~~$150~~
~~$160~~
$170
$170
$200
$210
$220
$250
$270
$300
$310
$330
$350

Figure ✓ 12.10.34: Repeat Step 3 Select Edge *DF*

Repeat Step 3: The lowest weight of the remaining edges is $170. Both *BF* and *CE* have a weight of $170, but *BF* would create cycle (*b*, *d*, *f*) and there cannot be a cycle in a spanning tree as shown in Figure ✓ 12.10.35.


A
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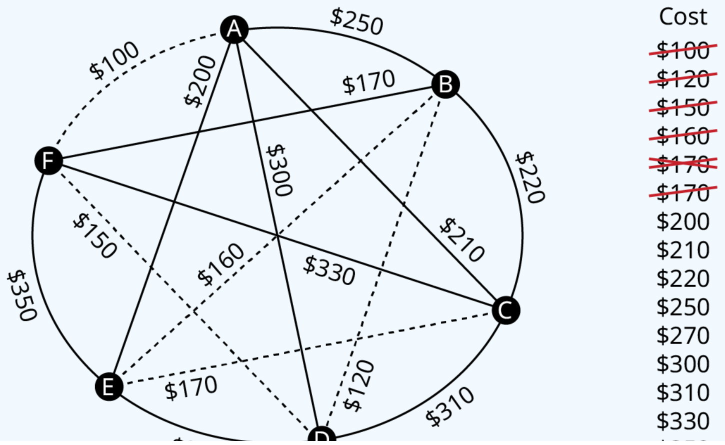
D
E
F
$250
$100
$200
~~$170~~
$220
$300
$150
$210
$160
$330
$350
$170
$120
$310
$270
Cost
~~$100~~
~~$120~~
~~$150~~
~~$160~~
~~$170~~
$170
$200
$210
$220
$250
$270
$300
$310
$330
$350

Figure ✓ 12.10.35: Repeat Step 3 Do Not Select Edge *BF*

So, we will select *CE*, which will complete the spanning tree as shown in Figure ✓ 12.10.36.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F
$250
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$170
$220
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$210
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$350
$170
$120
$310
Cost
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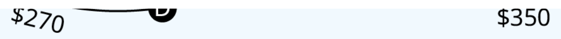

Figure ✓ 12.10.36: Repeat Step 3 Select Edge *CE*

The minimum spanning tree is shown in Figure ✓ 12.10.37. This is the configuration of the network of least cost. The spanning tree has a total weight of $\$100+\$120+\$150+\$160+\$170=\$700$, which is the total cost of this network configuration.

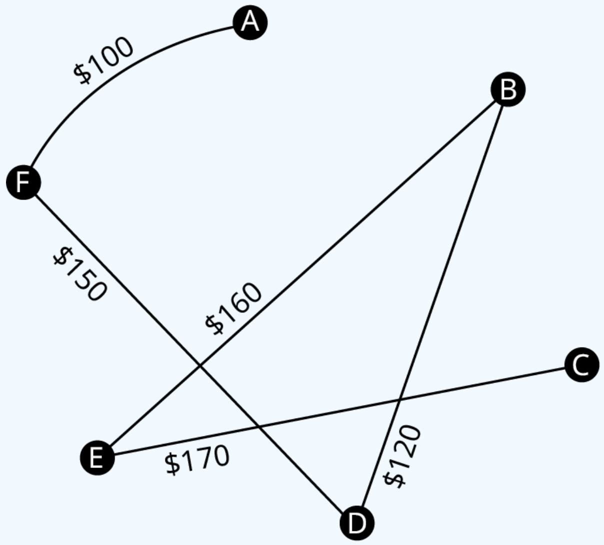

Figure ✓ 12.10.37: Final Minimum Spanning Tree

📌 Your Turn 12.10.7

Find a minimum spanning tree for the weighted graph. Give its total weight.

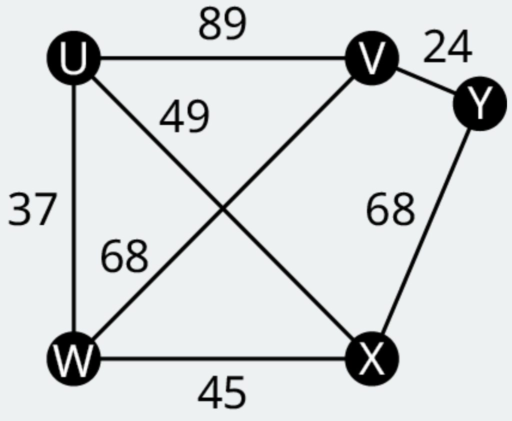

Figure 📌 12.10.38: Weighted Graph

Check Your Understanding

Which of the following statements are true and which are false

1. The number of cycles in a spanning tree is one less than the number of vertices.
2. A spanning tree contains no triangles.
3. A spanning tree includes every vertex of the original graph.There is a unique path between each pair of vertices in a spanning tree.A spanning tree must be connected.Kruskal’s algorithm is a method for finding all the different spanning trees in a given graph.

4. Only graphs that are trees have spanning trees.A minimum spanning tree of a given graph can be found using Kruskal’s algorithm.
5. A minimum spanning tree of a given graph is the subgraph, which is a tree, includes every vertex of the original graph, and which has the least weight of all spanning trees.
6. If a graph contains any cut edges, they must be included in any spanning tree.

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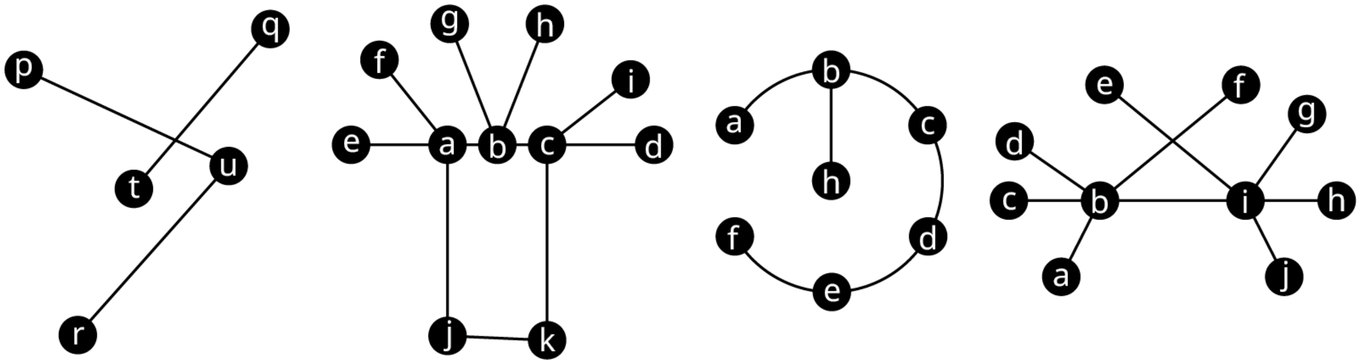
12.10.1: Exercises

For the following exercises, refer to the figure shown.



? Exercise 12.10.1.1

Which graphs, if any, are trees?

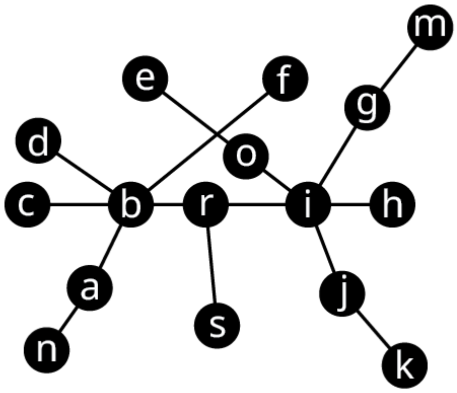
? Exercise 12.10.1.2

Which graphs, if any, are not trees because they are not connected?

? Exercise 12.10.1.3

Which graphs, if any, are not trees because they contain a cycle?

For the following exercises, refer to the figure shown. Identify any graphs that fit the given description.


m
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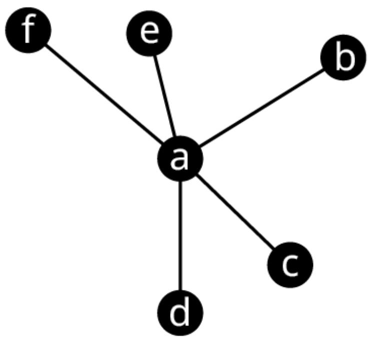
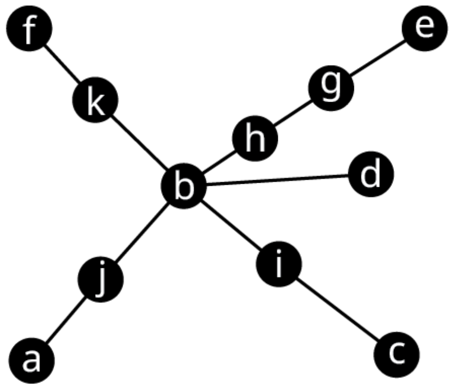
d
o
c
b
r
i
h
a
s
j
n
k

Graph T_1



Graph T_2


f
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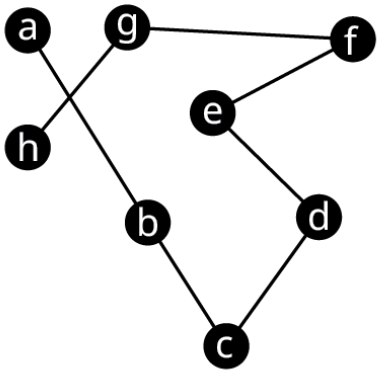
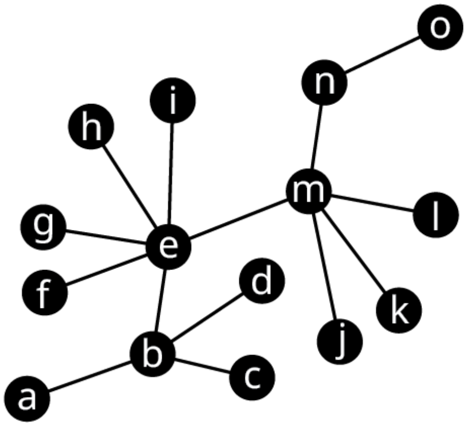
k
g
h
b
d
j
i
a
c

Graph T_3



Graph T_4


o
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i
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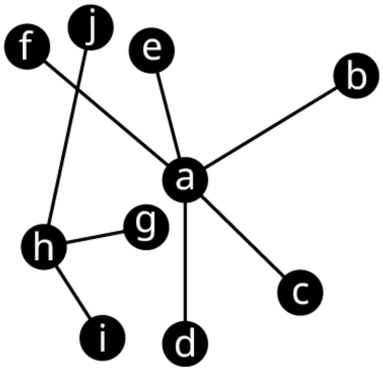
m
l
g
e
d
f
k
j
b
c
a

Graph T_5


f
j
e
b
a
g
h
c
i
d

Graph T_6

Exercise 12.10.1.4

Tree graph

Exercise 12.10.1.5

Star graph

Exercise 12.10.1.6

Star like graph

Exercise 12.10.1.7

Line graph (or path graph)

Exercise 12.10.1.8

Lobster graph

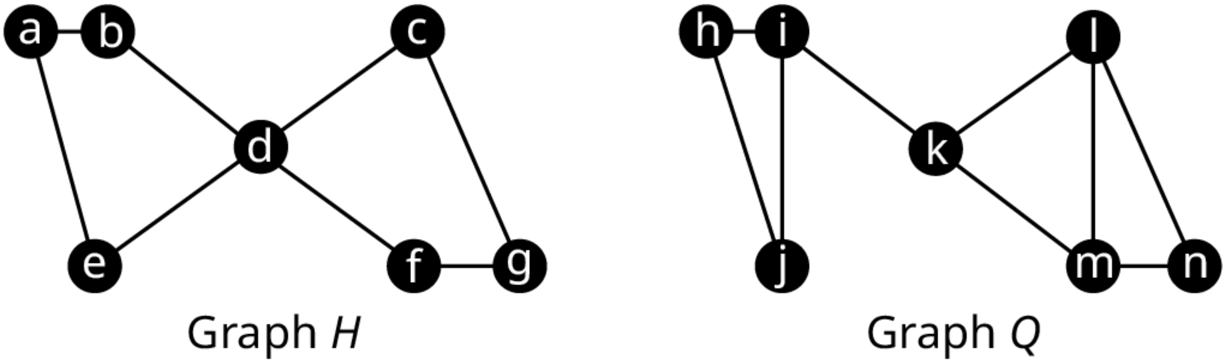
? Exercise 12.10.1.9

Caterpillar graph

? Exercise 12.10.1.10

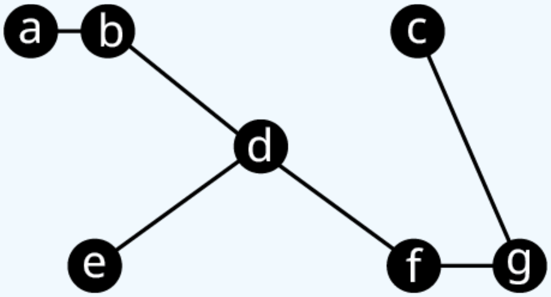
Forest graph

For the following exercises, use the figure shown to answer the questions.



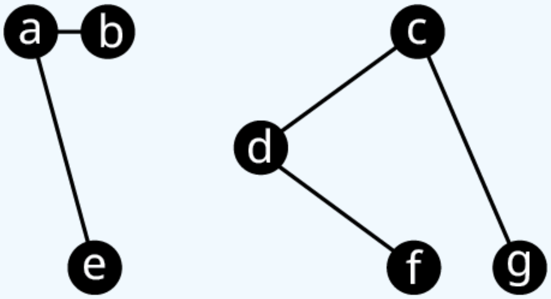
? Exercise 12.10.1.11

Determine whether Graph H_1 is a spanning tree of Graph H. If not, explain how you know.



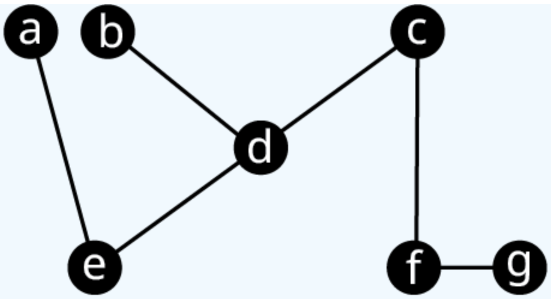
? Exercise 12.10.1.12

Determine whether Graph H_2 is a spanning tree of Graph H. If not, explain how you know.



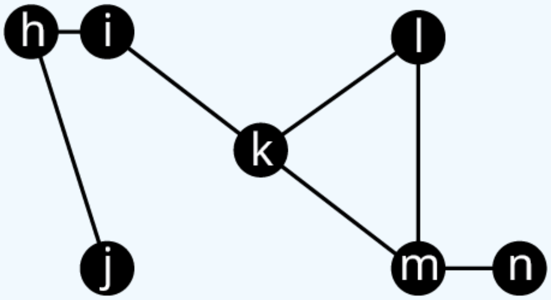
? Exercise 12.10.1.13

Determine whether Graph H_3 is a spanning tree of Graph H. If not, explain how you know.



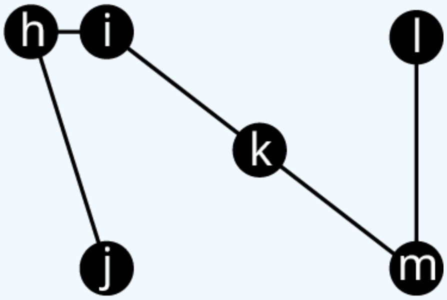
Exercise 12.10.1.14

Determine whether Graph Q_1 is a spanning tree of Graph Q. If not, explain how you know.



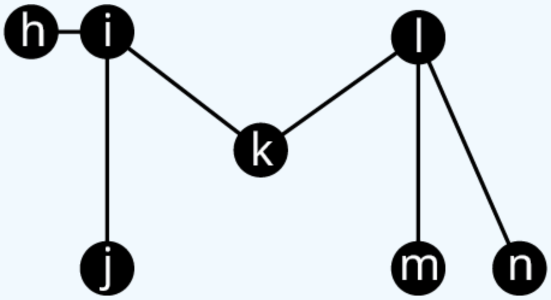
Exercise 12.10.1.15

Determine whether Graph Q_2 is a spanning tree of Graph Q. If not, explain how you know.

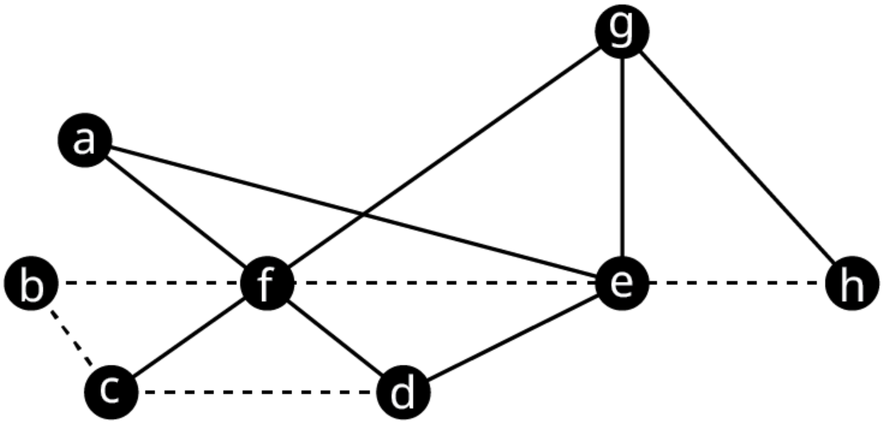


Exercise 12.10.1.16

Determine whether Graph Q_3 is a spanning tree of Graph Q. If not, explain how you know.



For the following exercises, a student has been asked to construct a spanning tree for Graph O, as shown in the figure. The dashed lines show the first step that the student took, creating a path from vertex h to vertex d.



Exercise 12.10.1.17

How many more edges must be included with the dashed edges to create a spanning tree?

Exercise 12.10.1.18

List three unused (solid) edges from Graph *O* that cannot be used to complete the spanning tree.

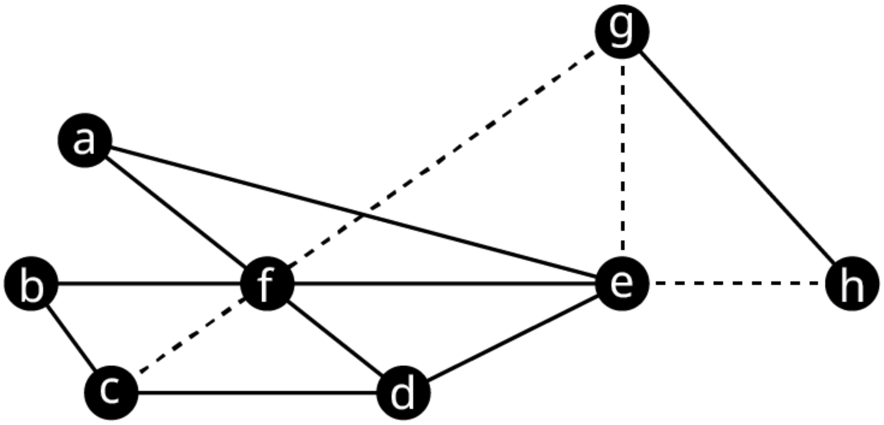
Exercise 12.10.1.19

Give an example of a set of edges that do not have *e* as an endpoint, which would complete the spanning tree.

Exercise 12.10.1.20

Give an example of a set of edges that do not have *f* as an endpoint, which would complete the spanning tree.

For the following exercises, a student has been asked to construct a spanning tree for Graph *O*, as shown in the figure. The dashed lines show the first step that the student took, creating a path from vertex *c* to vertex *h*.



Exercise 12.10.1.21

How many more edges must be included with the dashed edges to create a spanning tree?

Exercise 12.10.1.22

List two unused edges from Graph *O* that cannot be used to complete the spanning tree.

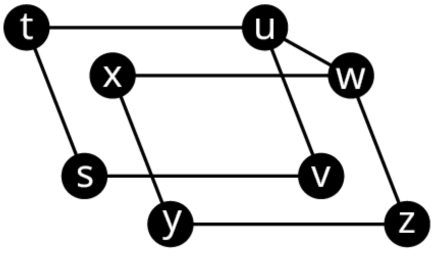
Exercise 12.10.1.23

Give an example of a set of edges that do not have *f* as an endpoint, which would complete the spanning tree.

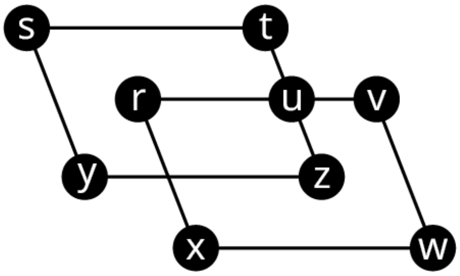
Exercise 12.10.1.24

Give an example of a set of edges that have neither *c* nor *e* as an endpoint, which would complete the spanning tree.

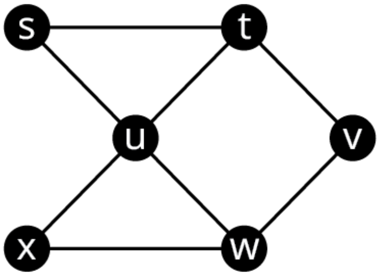
For the following exercises, use Graphs *A*, *B*, and *C*.



Graph *A*



Graph *B*



Graph *C*

Exercise 12.10.1.25

How many edges must be removed from Graph *A* to create a spanning tree?

Exercise 12.10.1.26

How many edges must be removed from Graph *B* to create a spanning tree?

Exercise 12.10.1.27

How many edges must be removed from Graph *C* to create a spanning tree?

Exercise 12.10.1.28

Identify all the distinct cyclic subgraphs of Graph *A*.

Exercise 12.10.1.29

Identify all the cyclic subgraphs of Graph *B*.

Exercise 12.10.1.30

Identify all the cyclic subgraphs of Graph *C*.

Exercise 12.10.1.31

Draw four spanning trees of Graph *A* each of which includes edges *vs*, *uv*, *wz* and *xy*.

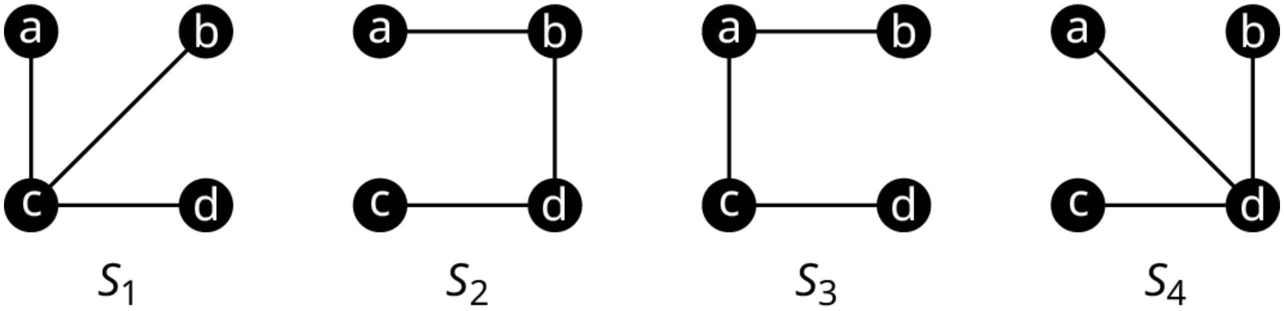
Exercise 12.10.1.32

Draw four spanning trees of Graph B which includes edge ut, but not ur.

Exercise 12.10.1.33

Draw four spanning trees of Graph C that each have only one edge with an endpoint at vertex u.

For the following exercises, use the figure shown. Draw a graph that fits the given description.



Exercise 12.10.1.34

S_1, S_2, S_3 and S_4 are all spanning trees.

Exercise 12.10.1.35

S_1 and S_2 are spanning trees, but S_4 is not.

Exercise 12.10.1.36

S_3 and S_4 are spanning trees but S_1 is not.

Exercise 12.10.1.37

S_2 and S_3 are spanning trees but S_1 is not.

Exercise 12.10.1.38

S_2 and S_3 are spanning trees but S_4 is not.

Exercise 12.10.1.39

S_1, S_2, and S_3 are spanning trees but S_4 is not.

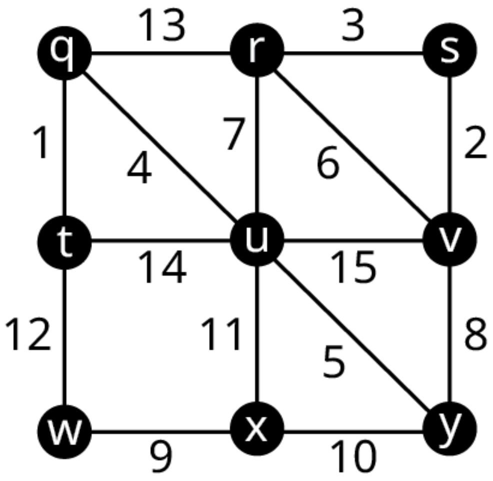
Exercise 12.10.1.40

S_2, S_3, and S_4 are spanning trees but S_1 is not.

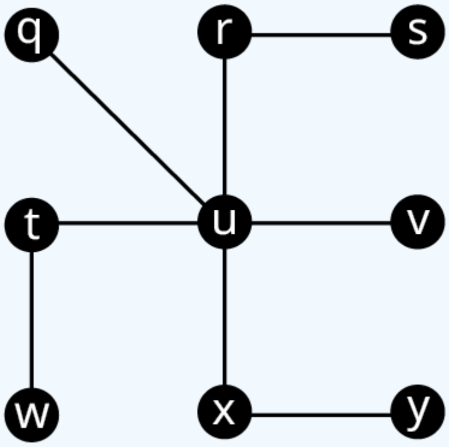
Exercise 12.10.1.41

S_1 and S_4 are spanning trees but S_2 and S_3 are not.

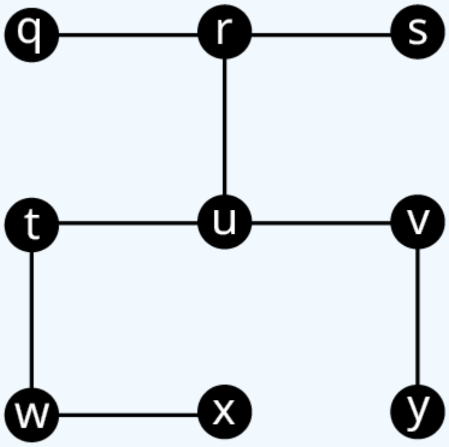
For the following exercises, use the figure shown to find the weight of the given spanning tree.


q
r
s
t
u
v
w
x
y
13
3
1
4
7
6
2
14
15
12
11
5
8
9
10

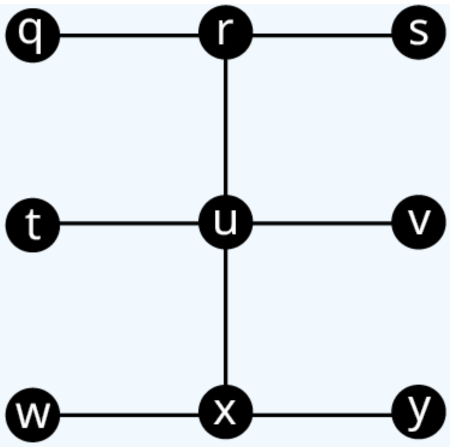
Exercise 12.10.1.42



Exercise 12.10.1.43



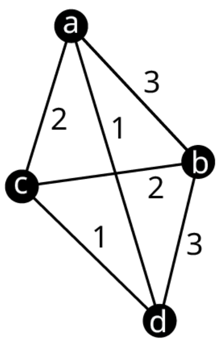
Exercise 12.10.1.44



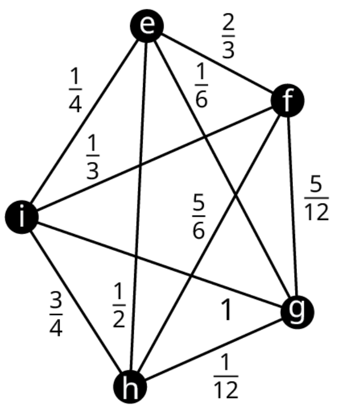
Exercise 12.10.1.45

Use Kruskal’s algorithm to draw a minimum spanning tree for Graph Z in the provided figure. Find its weight.

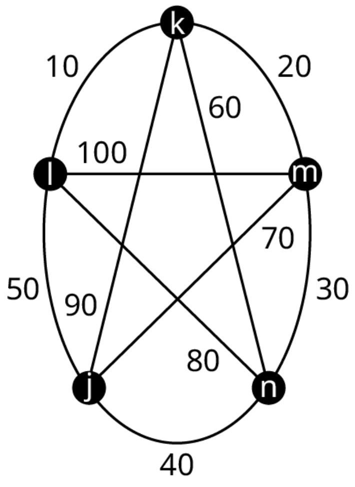
For the following exercises, draw a minimum spanning tree for the given graph, and calculate its weight.


a
3
2
1
b
c
2
1
3
d

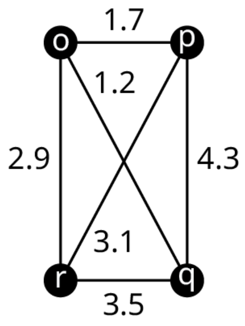
Graph A


e
2/3
1/4
1/6
f
1/3
i
5/6
5/12
3/4
1/2
1
g
h
1/12

Graph B


k
10
20
60
100
l
m
70
50
90
30
80
j
n
40

Graph C


1.7
o
p
1.2
2.9
4.3
3.1
r
q
3.5

Graph D

Exercise 12.10.1.46

Graph A

Exercise 12.10.1.47

Graph C

Exercise 12.10.1.48

Graph B

Exercise 12.10.1.49

Graph D

For the following exercises, draw a weighted graph to represent the given information. Then use the graph to find a minimum spanning tree and give its weight. Explain what the weight represents in the given scenario.

Exercise 12.10.1.50

City planners are tasked with building roadways to connect locations A, B, C, and D. The cost to build the roadways between any given pair of locations is given in the table.

	A	B	C	D
A	-	125	320	275
B	125	-	110	540
C	320	110	-	1,010
D	275	540	1,010	-

Exercise 12.10.1.51

In a video game, the goal is to visit five different lands, V, W, X, Y and Z, without losing all your lives. The paths between the lands are rated for danger, 1 being lowest and 10 being highest. Once a path has been traversed successfully, it is free from danger. The ratings are given in the table.

	V	W	X	Y	Z
V	-	2	4	9	10
W	2	-	6	8	No Path
X	4	6	-	7	No Path
Y	9	8	7	-	5
Z	10	No Path	No Path	5	-

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SECTION OVERVIEW

12.11: Chapter Summary

12.11.0: Key Terms

12.11.1: Key Concepts

12.11.2: Videos

12.11.3: Formula Review

12.11.4: Projects

12.11.5: Chapter Review

12.11.6: Chapter Test

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12.11.0: Key Terms

12.1 Graph Basics

- vertex
- edge
- loop
- graph (simple graph)
- multigraph
- adjacent (neighboring)
- degree

12.2 Graph Structures

- complete
- subgraph
- cycle
- cyclic subgraph
- clique

12.3 Comparing Graphs

- isomorphic
- isomorphism
- planar
- nonplanar
- complement
- complementary

12.4 Navigating Graphs

- walk (directed walk)
- trail (directed trail)
- path (directed path)
- closed
- open
- closed walk
- circuit (closed trail)
- directed cycle (closed path)
- coloring (graph coloring)
- n-coloring
- chromatic number

12.5 Euler Circuits

- connected
- component
- disconnected
- Euler circuit
- Eulerian graph
- Chinese postman problem
- Eulerization

12.6 Euler Trails

- algorithm
- Fleury’s algorithm
- Euler trail

- bridge
- local bridge

12.7 Hamilton Cycles

- Hamilton cycle, or Hamilton circuit
- n factorial
- weighted graph
- total weight

12.8 Hamilton Paths

- Hamilton path

12.9 Traveling Salesperson Problem

- brute force algorithm
- greedy algorithm
- traveling salesperson problem (TSP)
- brute force method
- nearest neighbor method

12.10 Trees

- acyclic
- tree
- forest
- path graph or linear graph
- star tree
- root
- starlike tree
- caterpillar tree
- lobster tree
- spanning tree
- minimum spanning tree

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12.11.1: Key Concepts

12.1 Graph Basics

- Graphs and multigraphs represent objects as vertices and the relationships between the objects as edges.
- The degree of a vertex is the number of edges that meet it and the degree can be zero.
- An edge must have a vertex at each end.
- Multigraphs may contain loops and double edges, but simple graphs may not.

12.2 Graph Structures

- The sum of the degrees of the vertices in a graph is twice the number of edges.
- In a complete graph every pair of vertices is adjacent.
- A subgraph is part of a larger graph.
- Cycles are a sequence of connected vertices that begin and end at the same vertex but never visit any vertex twice.

12.3 Comparing Graphs

- Two graphs are isomorphic if they have the same structure.
- When graphs are relatively small, we can use visual inspection to identify an isomorphism by transforming one graph into another without breaking connections or adding new ones.
- An isomorphism between two graphs preserves adjacency.
- If two graphs differ in number of vertices, number of edges, degrees of vertices, or types of subgraphs, they cannot be isomorphic.
- When the complements of two graphs are isomorphic, so are the graphs themselves.

12.4 Navigating Graphs

- Walks, trails, and paths are ways to navigate through a graph using a sequence of connected vertices and edges.
- Closed walks, circuits, and directed cycles are ways to navigate from a vertex on a graph and return to the same vertex.
- Colorings are a way to organize the vertices of a graph into groups so that no two members of a group are adjacent.
- Maps can be represented with planar graphs, which can always be colored using four colors or fewer.

12.5 Euler Circuits

- A connected graph has only one component.
- The Euler circuit theorem states that an Euler circuit exists in every connected graph in which all vertices have even degree, but not in disconnected graphs or any graph with one or more vertices of odd degree.
- The Chinese postman problem asks how to find the shortest closed trail that visits all edges at least once.
- If an Euler circuit exists, it is always the best solution to the Chinese postman problem.
- Eulerization is the process of adding duplicate edges to a graph so that the new multigraph has an Euler circuit.
- The minimum number of duplicated edges needed to eulerize a graph is half the number of odd vertices or more.

12.6 Euler Trails

- An Euler trail exists whenever a graph has exactly two vertices of odd degree.
- When a bridge is removed from a graph, the number of components increases.
- A bridge is never part of a circuit.
- When a local bridge is removed from a graph, the distance between vertices increases.
- An edge that is part of a triangle is never a local bridge.

12.7 Hamilton Cycles

- A Hamilton cycle is a directed cycle, or circuit, that visits each vertex exactly once.
- Some Hamilton cycles are also Euler circuits, but some are not.
- Hamilton cycles that follow the same undirected cycle in the same direction are considered the same cycle even if they begin at a different vertex.
- The number of unique Hamilton cycles in a complete graph with n vertices is the same as the number of ways to arrange $n-1$ distinct objects.

- Weighted graphs have a value assigned to each edge, which can represent distance, time, money and other quantities.

12.8 Hamilton Paths

- A Hamilton path visits every vertex exactly once.
- Some Hamilton paths are also Euler trails, but some are not.

12.9 Traveling Salesperson Problem

- A brute force algorithm always finds the ideal solution but can be impractical whereas a greedy algorithm is efficient but usually does not lead to the ideal solution.
- A Hamilton cycle of lowest weight is a solution to the traveling salesperson problem.
- The brute force method finds a Hamilton cycle of lowest weight in a complete graph.
- The nearest neighbor method is a greedy algorithm that finds a Hamilton cycle of relatively low weight in a complete graph.

12.10 Trees

- A brute force algorithm always finds the ideal solution but can be impractical whereas a greedy algorithm is efficient but usually does not lead to the ideal solution.
- A Hamilton cycle of lowest weight is a solution to the traveling salesperson problem.
- The brute force method finds a Hamilton cycle of lowest weight in a complete graph.
- The nearest neighbor method is a greedy algorithm that finds a Hamilton cycle of relatively low weight in a complete graph.

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12.11.2: Videos

12.1 Graph Basics

- Graph Theory: Create a Graph to Represent Common Boundaries on a Map

12.2 Graph Structures

- The Mathematical Secrets of Pascal's Triangle by Wajdi Mohamed Ratemi

12.3 Comparing Graphs

- Determine If Two Graphs Are Isomorphic and Identify the Isomorphism

12.4 Navigating Graphs

- Walks, Trails, and Paths in Graph Theory
- Closed Walks, Closed Trails (Circuits), and Closed Paths (Directed Cycles) in Graph Theory
- Coloring Graphs Part 1: Coloring and Identifying Chromatic Number
- The Four Color Map Theorem – Numberphile
- Coloring Graphs Part 2: Coloring Maps and the Four Color Problem
- Neil deGrasse Tyson Explains the Möbius Strip

12.5 Euler Circuits

- Connected and Disconnected Graphs in Graph Theory
- Recognizing Euler Trails and Euler Circuits
- Existence of Euler Circuits in Graph Theory

12.6 Euler Trails

- Bridges and Local Bridges in Graph Theory
- Fluery's Algorithm to Find an Euler Circuit

12.10 Trees

- The Problem in *Good Will Hunting* by Numberphile
- Spanning Trees in Graph Theory
- Use Kruskal's Algorithm to Find Minimum Spanning Trees in Graph Theory

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12.11.3: Formula Review

12.2 Graph Structures

For the Sum of Degrees Theorem, $\text{sum of the degrees} = 2 \times \text{number of}$ or $\text{number of edges} =$ mfrac

The number of edges in a complete graph with n vertices is the sum of the whole numbers from 1 to $n-1$, $1+2+3+\cdots+(n-1)$.

The number of edges in a complete graph with n vertices is $1+2+3+\cdots+(n-1) = \frac{n(n-1)}{2}$.

12.7 Hamilton Cycles

The number of ways to arrange n distinct objects is $n!$.

The number of distinct Hamilton cycles in a complete graph with n vertices is $(n-1)!$.

12.9 Traveling Salesperson Problem

- In a complete graph with n vertices, the number of distinct Hamilton cycles is $(n-1)!$.
- In a complete graph with n vertices, there are at most $\frac{(n-1)!}{2}$ different weights of Hamilton cycles.

12.10 Trees

- The number of edges in a tree graph with n vertices is $n-1$. A connected graph with n vertices and $n-1$ edges is a tree graph.

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12.11.4: Projects

Everyone Gets a Turn! – Graph Colorings

Let’s put your knowledge of graph colorings to work! Your task is to plan a field day following these steps.

1. Select between seven and ten activities for your field day. You can look online for ideas.
2. Create a survey asking for the participants to select the three to five events in which they would most like to participate. Survey between seven and ten people.
3. Use the results of your survey to create a graph in which each vertex represents one of the events. A pair of vertices will be adjacent if there is at least one participant who would like to participate in both events.
4. Find a minimum coloring for the graph. Explain how you found it and how you know the chromatic number of the graph.
5. Use your solution to part d to determine the minimum number of timeslots you must use to ensure that everyone has the opportunity to participate in their top three events.
6. Find the complement of the graph you created. Explain what the edges in this graph represent.

A Beautiful Day in the Neighborhood – Euler Circuits

Let’s apply what you have learned to the community in which you live. Using resources such as your county’s property appraiser’s website, create a detailed graph of your neighborhood in which vertices represent turns and intersections. Represent a large enough part of your community to include no fewer than 10 intersections or turns. Then use your graph to answer the following questions.

1. Label the edges of your graph.
2. Determine if your graph is Eulerian. Explain how you know. If it is not, eulerize it.
3. Find an Euler circuit for your graph. Give the sequence of vertices that you found.
4. What does the Euler circuit you found in part c represent for your community?
5. Describe an application for which this Euler circuit might be used.

Dream Vacation – Hamilton Cycles and Paths

Where in the world would you like to travel most: the Eiffel Tower in Paris, a Broadway musical in New York city, a bike tour of Amsterdam, the Tenerife whale and dolphin cruises in the Canary Islands, the Giza Pyramid in Cairo, or maybe the Jokhang Temple in Tibet? Let's plan your dream vacation!

1. Which four destinations are at the top of your bucket list?
2. Draw a complete weighted graph with five vertices representing the four destinations and your home city, and the weights representing the cost of travel between cities.
3. Use a website (such as Travelocity) to find the best airfare between each pair of cities. List the airlines and flight numbers along with the prices. Include cost for ground transportation from the nearest airport if there is no airport at the destination you want to visit.
4. Use the nearest neighbor algorithm to find a Hamilton cycle of low weight beginning and ending in your hometown. What is the weight of this circuit and what does it represent?
5. Use the brute force method to find a Hamilton cycle of lowest weight beginning and ending in your hometown. What is the weight of this circuit? Is it the same or different from the weight of the Hamilton cycle you found in Exercise 4?
6. Suppose that instead of returning home, you planned to move to your favorite location on the list, but you wanted to stop at the other three destinations once along the way. Where would you move? List all Hamilton paths between your hometown and your favorite location.
7. Find the weights of all the Hamilton paths you found in Exercise 6.

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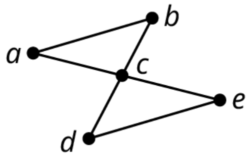
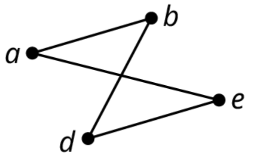
12.11.5: Chapter Review

Graph Basics

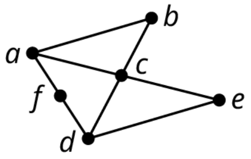
For the following exercises, use the given figure.



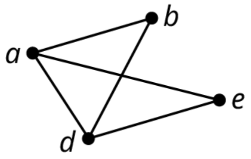
Graph S



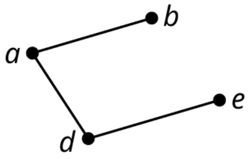
Graph T



Graph U



Graph V



Graph W

Exercise 12.11.5.1

Determine the number of vertices in Graph *S*.

Exercise 12.11.5.2

Identify the graph with the fewest edges.

Exercise 12.11.5.3

Name the vertices in Graph *U*.

Exercise 12.11.5.4

Identify any pairs of vertices in Graph *S* that are not adjacent.

Exercise 12.11.5.5

Which graphs only has vertices of degree 2?

Exercise 12.11.5.6

Identify the graph in which the sum of the degrees of the vertices is 16.

Exercise 12.11.5.7

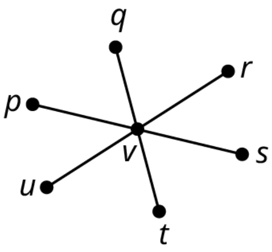
Amazon.com has a network of warehouses that are used to move packages around the United States. Delivery trucks from warehouse deliver packages to other locations. These mail trucks also pick up packages to bring back to their home warehouse. Explain how a graph or multigraph might be drawn to model this scenario by identifying the objects that could be represented by vertices and the connections that could be represented by edges. Indicate whether a graph or a multigraph would be a better model.

Exercise 12.11.5.8

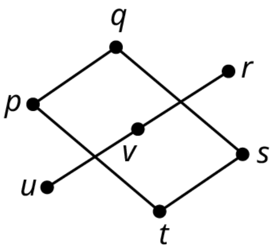
Eduardo has two groups of four friends, group A and group B. Within each group, each of the members of the group are friends with each other but not with those in the other group. The individuals in group A have no other friends, but the individuals in group B each has their own group of four friends, and the individuals within those groups are all friends with each other, but with no one outside their own group. Draw a graph to represent the given scenario.

Graph Structures

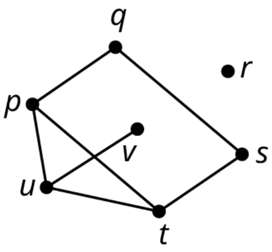
For the following exercises, use the given figure.



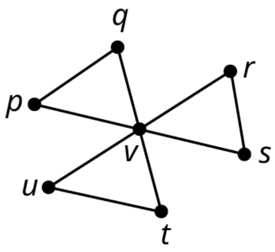
Graph A



Graph B



Graph C



Graph D

Exercise 12.11.5.9

Explain why Graph *B* is not a subgraph of Graph *C*.

Exercise 12.11.5.10

Identify any graphs that have a quadrilateral cyclic subgraph and name the vertices.

Exercise 12.11.5.11

Identify a clique in Graph *C* by listing its vertices.

Exercise 12.11.5.12

Draw a graph with the following characteristics: largest clique has 4 vertices, a pentagonal cyclic subgraph, exactly two vertices of degree 4.

Exercise 12.11.5.13

How many edges are in a complete graph with 12 vertices?

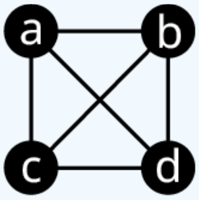
Exercise 12.11.5.14

How many triangles are in a complete graph with 11 vertices?

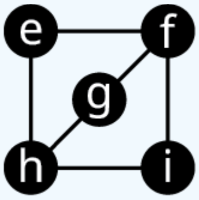
Comparing Graphs

Exercise 12.11.5.15

Identify three differences between Graph 1 and Graph 2 that demonstrate the graphs are not isomorphic.

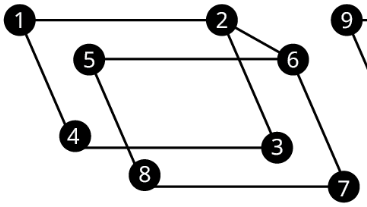


Graph 1

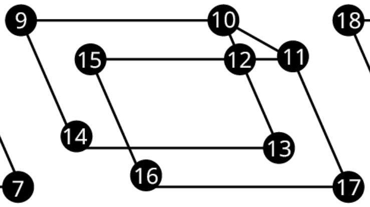


Graph 2

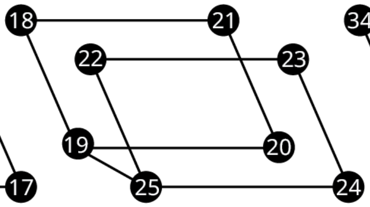
For the following exercises, use the given figure.



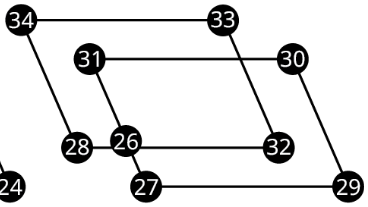
Graph 3



Graph 4



Graph 5



Graph 6

Exercise 12.11.5.16

Determine if Graph 3 is isomorphic to Graph 4. If so, identify a correspondence between the vertices which demonstrates the isomorphism. If not, identify at least two characteristics that verifies this.

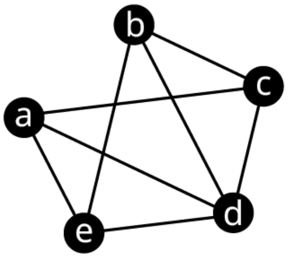
Exercise 12.11.5.17

Determine if Graph 3 is isomorphic to Graph 5. If so, identify a correspondence between the vertices that demonstrates the isomorphism. If not, identify a characteristic that verifies this.

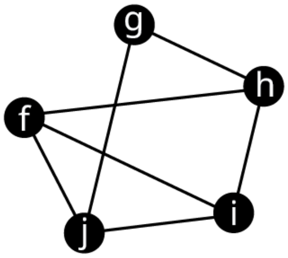
Exercise 12.11.5.18

Consider Graph 4 and Graph 5. Determine if one graph is a subgraph of the other. If so, give a correspondence between vertices that demonstrates this relationship. If not, identify conflicting characteristics.

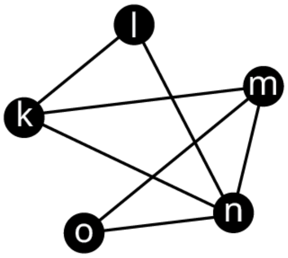
For the following exercises, use the given figure.



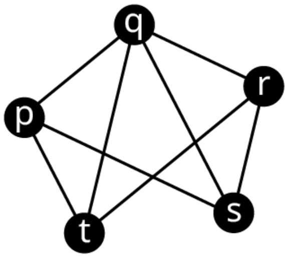
Graph 7



Graph 8



Graph 9



Graph 10

Exercise 12.11.5.19

Draw the complements of Graphs 7 and 9. Determine whether the graphs you drew are isomorphic to each other and explain how you know. Use this information to determine whether Graphs 7 and 8 are isomorphic.

Exercise 12.11.5.20

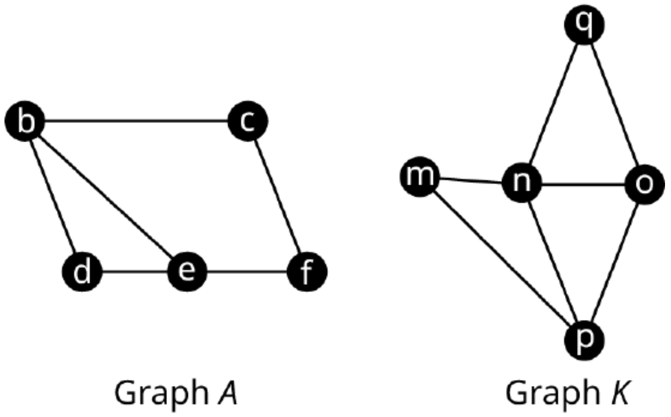
Draw the complements of Graphs 7 and 10. Determine whether the graphs you drew are isomorphic to each other and explain how you know. Use this information to determine whether Graphs 7 and 10 are isomorphic.

Exercise 12.11.5.21

Draw the complements of Graphs 8 and 9. Determine whether the graphs you drew are isomorphic to each other and explain how you know. Use this information to determine whether Graphs 8 and 9 are isomorphic.

Navigating Graphs

For the following exercises, use Graph *A* in the given figure. Consider each sequence of vertices. Determine if it is only a walk, both a walk and a path, both a walk and a trail, all three, or none of these.



? Exercise 12.11.5.22

$e \rightarrow d \rightarrow b \rightarrow e \rightarrow f$

? Exercise 12.11.5.23

$e \rightarrow b \rightarrow d \rightarrow e \rightarrow f \rightarrow c \rightarrow b \rightarrow d$

? Exercise 12.11.5.24

$e \rightarrow f \rightarrow b \rightarrow d$

For the following exercises, use Graph K in Figure 12.352. Identify each sequence of vertices as a closed walk, circuit (closed trail), and/or directed cycle (closed path). Indicate all that apply.

? Exercise 12.11.5.25

$n \rightarrow o \rightarrow q \rightarrow n \rightarrow p \rightarrow m \rightarrow n$

? Exercise 12.11.5.26

$m \rightarrow n \rightarrow o \rightarrow q \rightarrow n \rightarrow p \rightarrow o \rightarrow n \rightarrow m$

? Exercise 12.11.5.27

$p \rightarrow o \rightarrow q \rightarrow n \rightarrow m \rightarrow p$

For the following exercises, use Graphs A and K in Figure 12.352.

? Exercise 12.11.5.28

Are Graphs A and K planar graphs? What does your answer tell you about the chromatic number of each graph?

? Exercise 12.11.5.29

How many vertices are in the largest complete subgraph of Graph A? What does this tell you about the chromatic number of Graph A?

Exercise 12.11.5.30

Create a coloring of Graph A, which uses exactly four colors, or explain why it is not possible.

Exercise 12.11.5.31

Create a coloring of Graph K, which uses exactly two colors, or explain why it is not possible.

Exercise 12.11.5.32

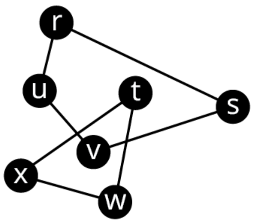
Determine the chromatic number of Graph A. Give a coloring that supports your conclusion.

Exercise 12.11.5.33

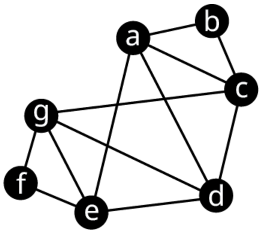
Determine the chromatic number of Graph K. Give a coloring that supports your conclusion.

Euler Circuits

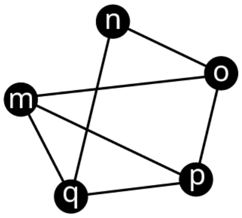
For the following exercises, use the graphs and multigraphs shown. Identify any graphs and/or multigraphs with the given characteristics. If there are none, state so.



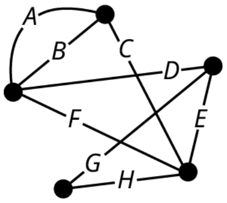
Graph 11



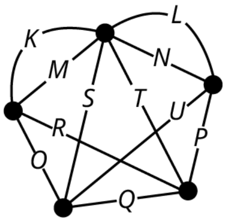
Graph 12



Graph 13



Multigraph 14



Multigraph 15

Exercise 12.11.5.34

connected

Exercise 12.11.5.35

disconnected

Exercise 12.11.5.36

Eulerian

Euler Trails

For the following exercises, use the graphs and multigraphs in Figure 12.354.

Exercise 12.11.5.37

List the set of vertices for each component in Graph 13.

Exercise 12.11.5.38

Determine whether the sequence of edges represents an Euler circuit in Multigraph 15:
$K \to L \to N \to M \to O \to S \to T \to Q \to U \to P \to R$

Exercise 12.11.5.39

Find an Euler circuit beginning and ending at vertex g in Graph 12 if one exists. Otherwise, explain how you know such an Euler circuit does not exist.

Exercise 12.11.5.40

Give an example of a pair of edges that could be duplicated to eulerize Multigraph 14.

For the following exercises, use the graphs and multigraphs in Figure 12.354. Identify any graphs and/or multigraphs with the given characteristics. If there are none, state so.

Exercise 12.11.5.41

Exactly two vertices of odd degree

Exercise 12.11.5.42

Has an Euler trail

Exercise 12.11.5.43

Has exactly one local bridge

For the following exercises, use the graphs and multigraphs in Figure 12.354. In each exercise a graph and a sequence of vertices are given.

Exercise 12.11.5.44

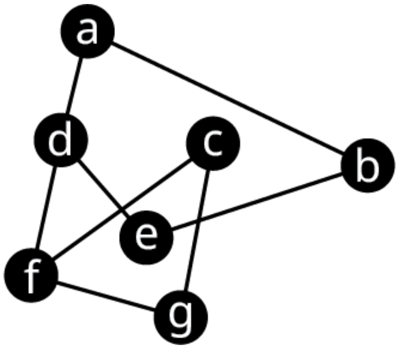
Determine whether the sequence of edges, $A \to B \to C \to H \to G \to D \to F \to E$, is an Euler trail, an Euler circuit, or neither for the graph. If it is neither, explain why.

Exercise 12.11.5.45

Suppose that an edge were added to Graph 11 between vertices s and w. Determine if the graph would have an Euler trail or an Euler circuit, and find one.

Hamilton Cycles

For the following exercises, refer to the graph in the given figure.



? Exercise 12.11.5.46

A student has been asked to use Fleury’s algorithm to construct an Euler trail in the given graph. The student decides to begin the trail at vertex d. Is this a good choice, why or why not?

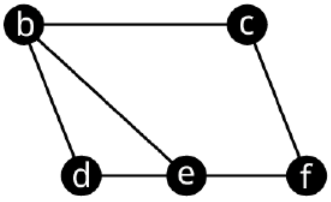
? Exercise 12.11.5.47

A student who is using Fleury’s algorithm to construct an Euler trail has decided to begin with $f \to d \to a \to b \to \ldots$. If the student is off to a good start, help the student by completing the Euler trail. If the student has made an error, explain the error.

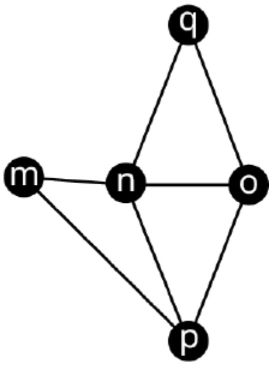
? Exercise 12.11.5.48

Use Fleury’s algorithm to construct an Euler trail for Graph 16 beginning at the vertex of your choice.

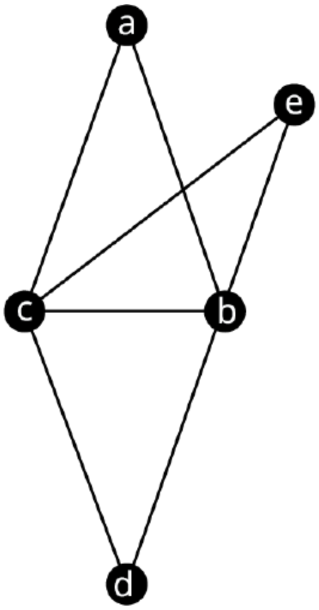
For the following exercises, use the graphs from the given figures to determine whether the sequence of vertices in the given graph is a Hamilton cycle, an Euler circuit, both, or neither.



Graph A



Graph K



Graph F

Exercise 12.11.5.49

Graph F. $a \to b \to e \to c \to b \to d \to c \to a$

Exercise 12.11.5.50

Graph K. $m \to n \to q \to o \to p \to m$

Exercise 12.11.5.51

Graph A. $b \to d \to e \to f \to c \to b \to e$

For the following exercises, evaluate the factorial expression for the given value of n.

Exercise 12.11.5.52

$(n-1)!, n = 12$

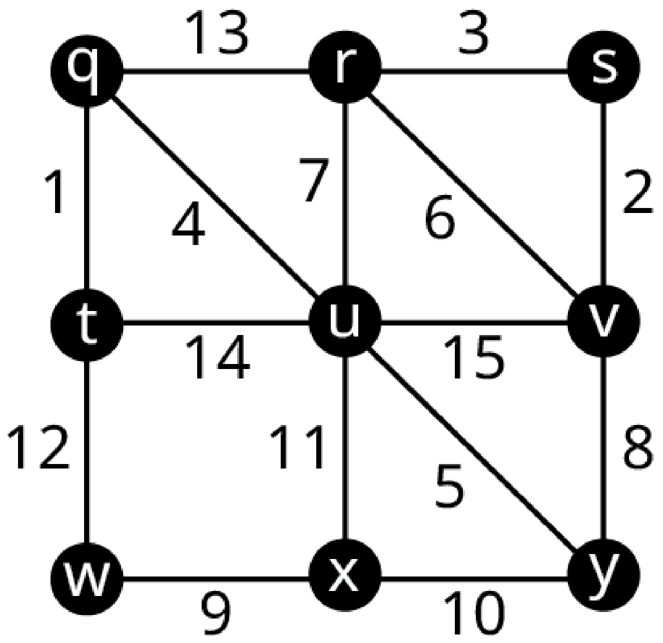
Exercise 12.11.5.53

$(n-1)!, n = 14$

Exercise 12.11.5.54

Calculate the number of distinct Hamilton cycles in a complete graph with 13 vertices.

For the following exercises, use the figure shown to find the weight of each Hamilton cycle.



Exercise 12.11.5.55

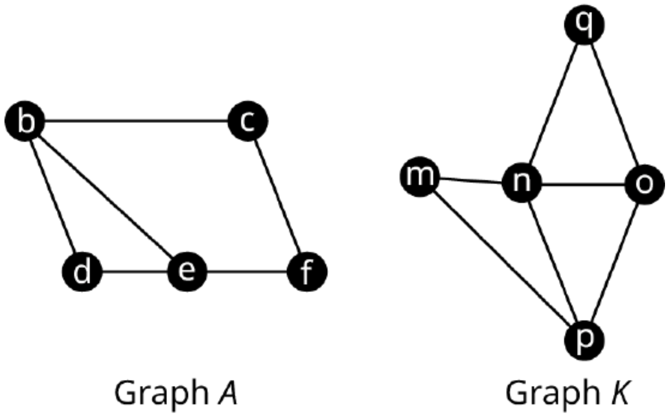
$q \to t \to w \to x \to u \to y \to v \to s \to r \to q$

? Exercise 12.11.5.56

$w \to x \to y \to u \to v \to s \to r \to q \to t \to w$

Hamilton Paths

For the following exercises, use the figure shown to determine whether the sequence of vertices in the given graph is a Hamilton path, an Euler trail, both, or neither.



? Exercise 12.11.5.57

Graph A. $e \to b \to c \to f \to e \to b \to e$

? Exercise 12.11.5.58

Graph A. $b \to c \to f \to e \to d \to b \to e$

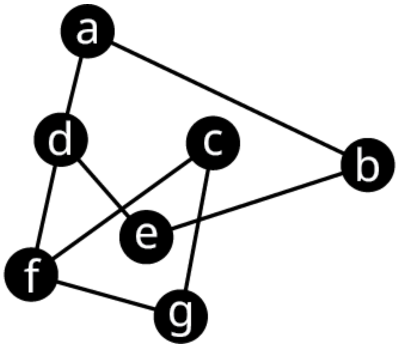
? Exercise 12.11.5.59

Graph K. $n \to q \to o \to p \to m$

? Exercise 12.11.5.60

Graph K. $o \to q \to m \to n \to p$

Recall the three common scenarios in which it is not possible to have a Hamilton between two vertices.
Scenario 1: If an edge ab is a bridge, then there is no Hamilton path between a pair of vertices that are on the same side of edge ab.
Scenario 2: If an edge ab is a bridge with at least three components on each side, then there is no Hamilton path beginning or ending at a or b.
Scenario 3: If a graph is composrd of two cycles joined only at a single vertex p, and v is any vertex that is *not* adjacent to p, then there are no Hamilton paths beginning or ending at p.
For the following exercises, identify the scenario that guarantees there is no Hamilton path between the given pair of vertices in the given graph.



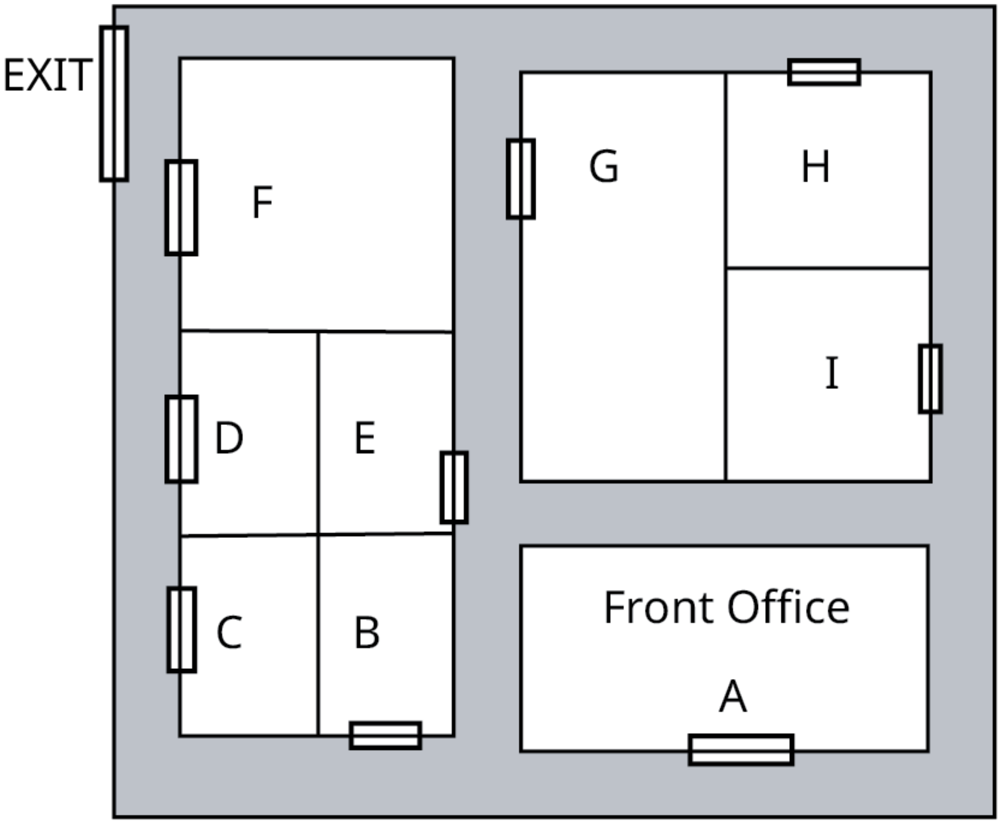
? Exercise 12.11.5.61

Vertices a and e.

? Exercise 12.11.5.62

Vertices d and f.

The principal of an elementary school plans to visit each classroom exactly once before leaving for the day. The floor plan of the school is given. Use this information for the following exercises.



? Exercise 12.11.5.63

Draw a graph to represent the floor plan for the elementary school in which each vertex is a room and an edge between two vertices indicates that there is a path between the two rooms that does not pass the door to another room.

Exercise 12.11.5.64

Use your graph to find a route that the principal can take beginning at the Front Office in room A, visiting each room exactly once (without passing by another room), and ending at room F, which is next to the exit.

Exercise 12.11.5.65

Would the route you found be best to trace out a Hamilton path, an Euler trail, both, or neither in the graph you drew? Explain how you know.

Traveling Salesperson Problem

For the following exercises, determine whether the algorithm described is a greedy algorithm or a brute force algorithm.

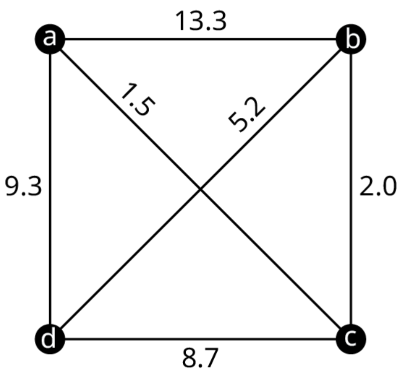
Exercise 12.11.5.66

A lumber distributor is loading pallets onto trucks with the intention of using the fewest trucks possible to send a shipment. The distributor loads the pallets with the greatest length that will fit on the truck first and continues loading until no more pallets will fit. Then the next truck is loaded using the same approach.

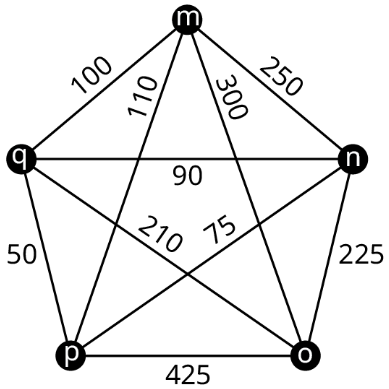
Exercise 12.11.5.67

A traveler wants to visit five cities by airplane. The traveler lists all the possible orders in which the cities can be visited then calculates the best airfare for each itinerary and selects the least expensive option.

For the following exercises, list all the distinct Hamilton cycles beginning at the given vertex in the given graph. Indicate which pairs of Hamilton cycles are reverses of each other.



Graph 17



Graph 18

Exercise 12.11.5.68

Graph 17, vertex a

Exercise 12.11.5.69

Graph 18, vertex m

For the following exercises, find a Hamilton cycle of least weight for the given graph in Figure 12.362, beginning at the given vertex, and using the brute force method. What is the weight of the cycle?

Exercise 12.11.5.70

Graph 18; vertex m

Exercise 12.11.5.71

Graph 17; vertex a

For the following exercises, find a Hamilton cycle of low weight for the given graph in Figure 12.362, beginning at the given vertex, and using the nearest neighbor method. What is the weight of the cycle?

Exercise 12.11.5.72

Graph 18 vertex m

Exercise 12.11.5.73

Graph 17, vertex a

Exercise 12.11.5.74

The products at a particular factory are manufactured in phases. The same equipment is utilized for each phase, but it must be formatted differently to accomplish different tasks. The transition time to convert between a format for one task and another task varies. The times are given in the table below.

Task	*A*	*B*	*C*	*D*	*E*	*F*	*G*
A	0	75	130	45	120	70	100
B	75	0	140	65	115	25	60
C	130	140	0	35	55	20	125
D	45	65	35	0	50	30	40
E	120	115	55	50	0	95	145
F	70	25	20	30	95	0	15
G	100	60	125	40	145	15	0

Use the table to draw a graph in which the vertices represent tasks A, C, D, F, and G, and the weights of the edges indicate the times in minutes to transition between pairs of tasks. Use the nearest neighbor method to find an order in which to complete tasks A, C, D, F, and G, which keeps the transition times down and ends with the same set up as it began so that the factory is ready start the next batch. Assume that there are no restrictions on which tasks can be completed in which order. Hint: The nearest neighbor algorithm may give a different result depending on which vertex is the starting vertex; so, you must check all possibilities.

Trees

For the following exercises, draw a graph with the given characteristics.

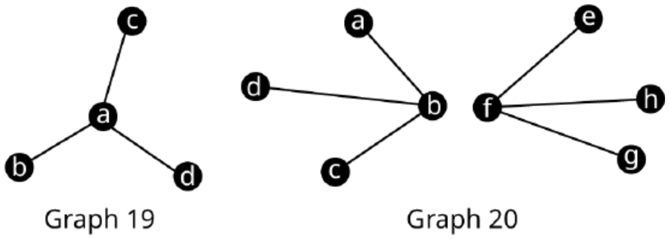
Exercise 12.11.5.75

A tree with eight vertices, exactly two of degree three.

? Exercise 12.11.5.76

A connected graph with eight vertices, exactly two of degree 3, which is not a tree.

For the following exercises, identify each type of graph in the given figure: tree graph, star graph, starlike graph, line graph, lobster graph, caterpillar graph, and/or forest graph.



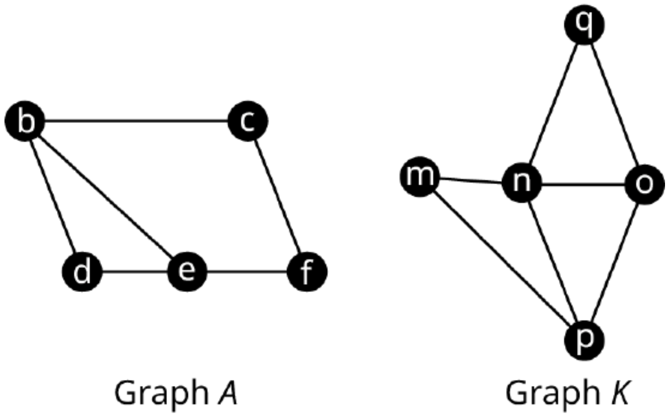
? Exercise 12.11.5.77

Graph 19

? Exercise 12.11.5.78

Graph 20

For the following exercises, use the figure shown to draw three possible spanning trees that fit the given description.



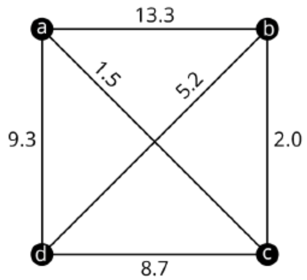
? Exercise 12.11.5.79

Three spanning trees of Graph *A*, which include both edges *be* and *de*.

? Exercise 12.11.5.80

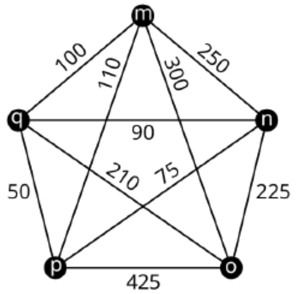
Three spanning trees of Graph *K*, which include edges *mn* and *oq*, but do not include *no*.

For the following exercises, use Kruskal’s Algorithm to find a minimum spanning tree of the given graph. Graph it and calculate its weight.



13.3
a
b
1.5
5.2
9.3
2.0
d
c
8.7

Graph 17



m
100
110
300
250
q
n
90
50
210
75
225
p
o
425

Graph 18

? Exercise 12.11.5.81

Graph 17

? Exercise 12.11.5.82

Graph 18

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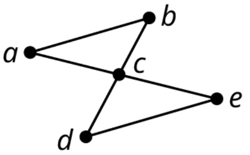
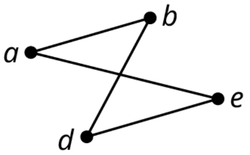
12.11.6: Chapter Test

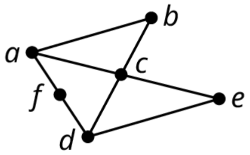
For the following Exercises, use the figure shown.



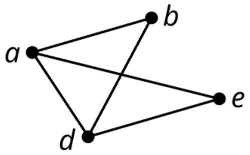
Graph S



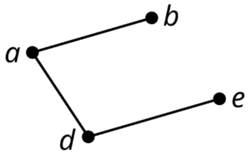
Graph T



Graph U



Graph V



Graph W

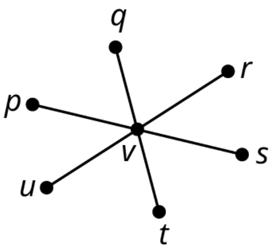
Exercise 12.11.6.1

Name the edges in Graph *T*.

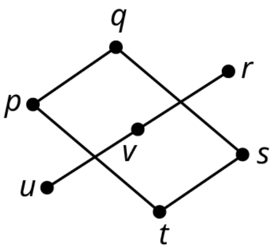
Exercise 12.11.6.2

Identify the graph(s) with six edges.

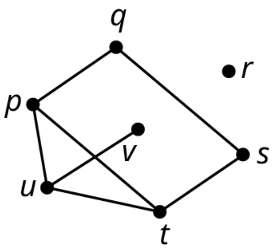
For the following exercises, use the figure shown.



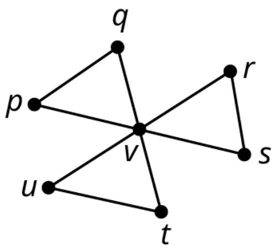
Graph A



Graph B



Graph C



Graph D

Exercise 12.11.6.3

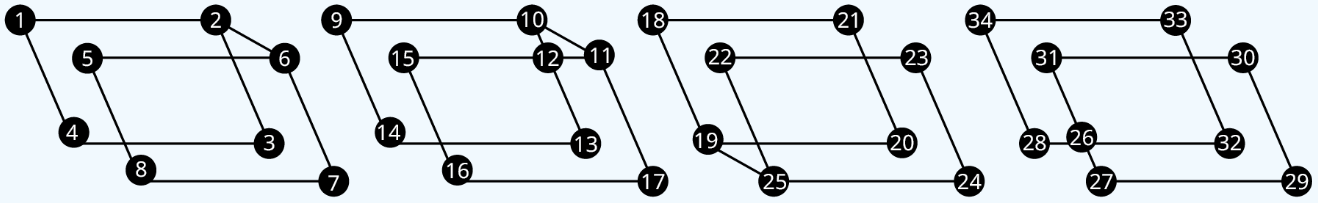
Identify any of the graphs that is a subgraph of Graph *D*.

Exercise 12.11.6.4

Identify any graphs with no cyclic subgraphs of any size.

Exercise 12.11.6.5

Consider Graph 4 and Graph 6 in the given figure. Determine if one graph is a subgraph of the other. If so, give a correspondence between vertices that demonstrates this relationship. If not, identify conflicting characteristics.


1
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6
4
3
8
7
9
10
15
12
11
14
13
16
17
18
21
22
23
19
20
25
24
34
33
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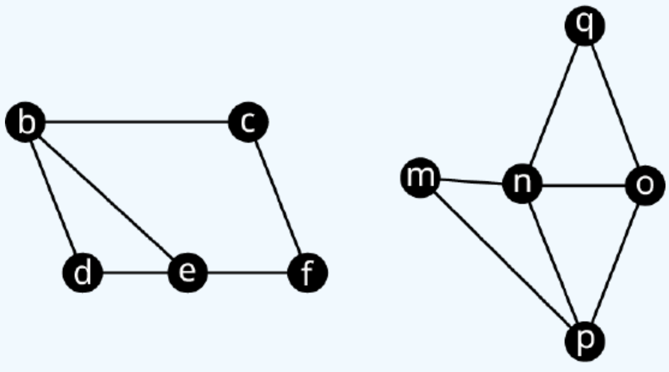
30
28
26
32
27
29

Graph 3 Graph 4 Graph 5 Graph 6

Exercise 12.11.6.6

For the following exercise, use Graph *A* in the given figure. Consider each sequence of vertices. Determine if it is only a walk, both a walk and a path, both a walk and a trail, all three, or none of these.



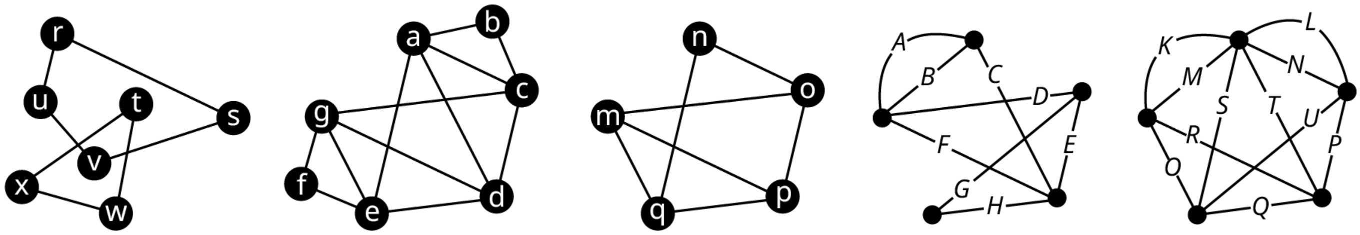
Graph *A* Graph *K*

$d \to e \to b \to c \to f$

Exercise 12.11.6.7

For the following exercise, use Graphs *A* and *K* in Figure 12.369. Determine the chromatic number of Graph *K*. Give a coloring that supports your conclusion.

For the following exercise, use the graphs and multigraphs in the given figure.



Graph 11 Graph 12 Graph 13 Multigraph 14 Multigraph 15

Exercise 12.11.6.8

Identify any graphs and/or multigraphs that are not Eulerian. If there are none, state so.

Exercise 12.11.6.9

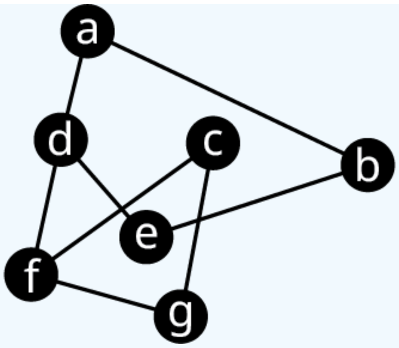
List the set of vertices for each component in Graph 11.

Exercise 12.11.6.10

Find an Euler circuit beginning and ending at vertex *b* in Graph 12 if one exists.

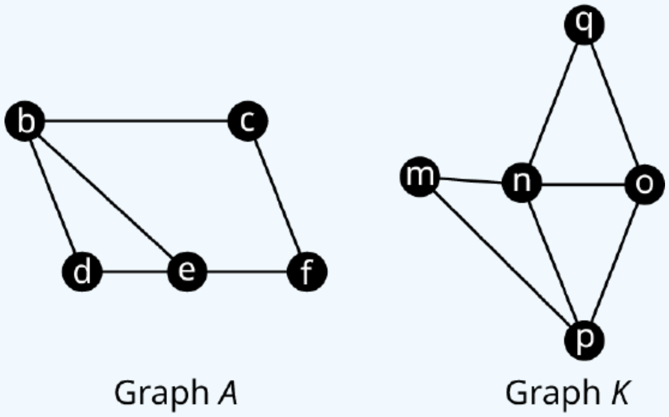
Exercise 12.11.6.11

Use Fleury’s algorithm to construct an Euler trail for the given graph beginning at vertex *f* of your choice.



Exercise 12.11.6.12

Use the graphs shown to determine whether the sequence of vertices $d \to b \to c \to f \to e \to d$ is a Hamilton cycle, an Euler circuit, both, or neither.

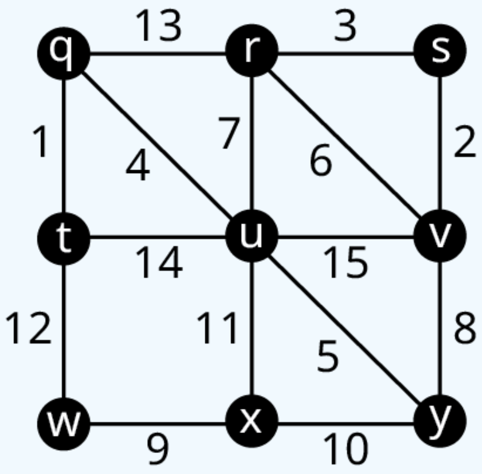

b
c
d
e
f
Graph A
q
m
n
o
p
Graph K

Exercise 12.11.6.13

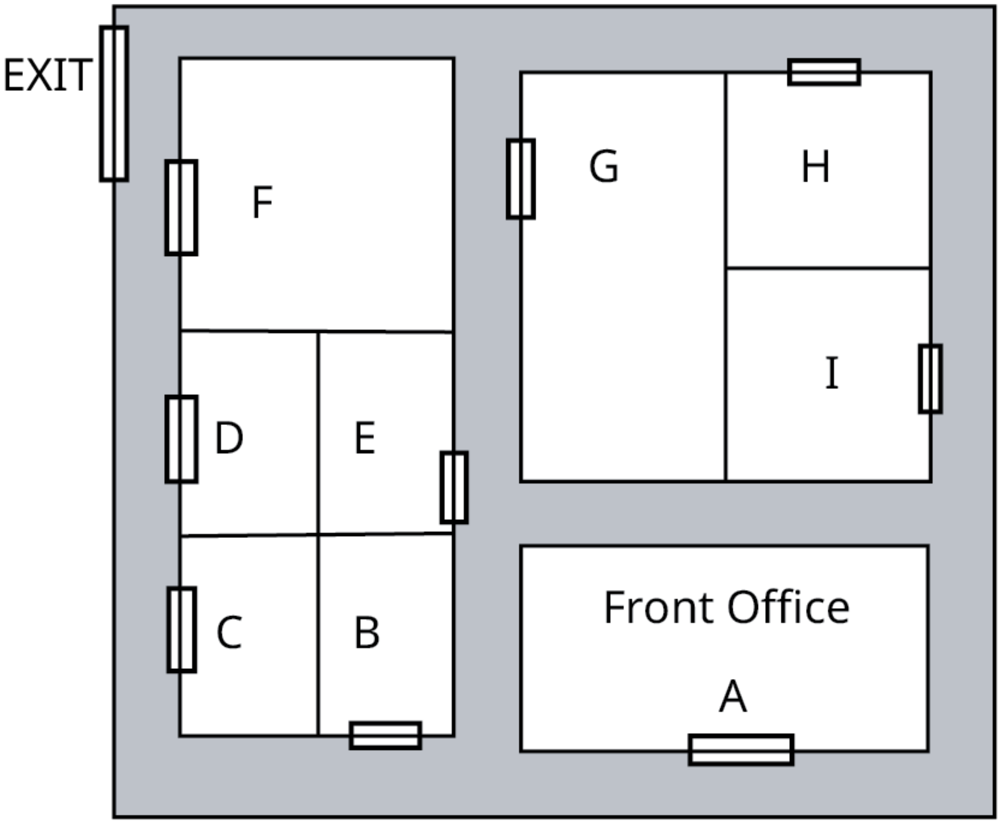
Calculate the number of distinct Hamilton cycles in a complete graph with 15 vertices.

Exercise 12.11.6.14

Use the figure shown to find the weight of the given Hamilton cycle:
$t \to w \to x \to u \to y \to v \to s \to r \to q \to t$


q
r
s
t
u
v
w
x
y
13
3
1
4
7
6
2
14
15
12
11
5
8
9
10

For the following exercises, use this information: For the Halloween celebration at an elementary school, the students from Classroom *I* will visit every classroom once and return to their own classroom. The floor plan of the school is given.



Exercise 12.11.6.15

Draw a graph to represent the classrooms of the elementary school in which each vertex is a classroom and an edge between two vertices indicates that there is a path between the two rooms that does not pass the door to another classroom.

Exercise 12.11.6.16

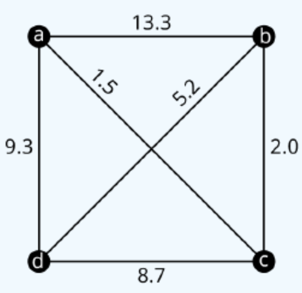
Use your graph to find a Hamilton circuit beginning and ending at *I*.

Exercise 12.11.6.17

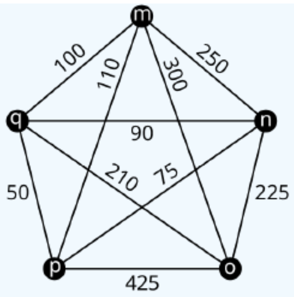
Explain what the Hamilton circuit you found represents for the students in Classroom *I*.

Exercise 12.11.6.18

Find a Hamilton cycle of low weight for Graph 18, beginning at vertex q, and using the nearest neighborhood method. What is the weight of the cycle?


13.3
a
b
1.5
5.2
9.3
2.0
d
c
8.7

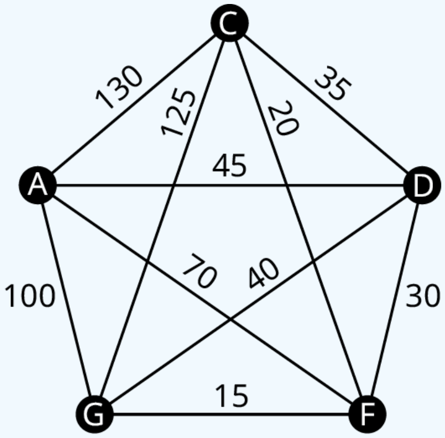
Graph 17


m
100
110
300
250
q
n
90
210
75
50
225
p
o
425

Graph 18

? Exercise 12.11.6.19

Use Kruskal’s Algorithm to find a minimum spanning tree for the below graph. Graph the tree and give its weight.


C
130
125
35
20
A
45
D
70
40
100
30
G
15
F

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CHAPTER OVERVIEW

13: Math and...

13.0: Introduction
13.1: Math and Art
  13.1.0: Exercises
13.2: Math and the Environment
  13.2.0: Exercises
13.3: Math and Medicine
  13.3.0: Exercises
13.4: Math and Music
  13.4.0: Exercises
13.5: Math and Sports
  13.5.0: Exercises
13.6: Chapter Summary
  13.6.0: Key Terms
  13.6.1: Key Concepts
  13.6.2: Formula Review
  13.6.3: Projects
  13.6.4: Chapter Review
  13.6.5: Chapter Test

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13.0: Introduction

Where do we find math around us? Math can be found in areas that are expected and sometimes in areas that are surprising. There are many ways that mathematical concepts, such as those in this text, are infused in the world around us. In this chapter, we will explore a sampling of five distinct areas from everyday life where math's impact plays a meaningful role.

Figure $13.0.1$ Math can be found in many different areas and subjects. (credit: modification of work "Chemistry/Physics Library" by University Libraries/Flickr, CC BY 2.0)

Math's impact on art can be found in numerical relationships that are known to create or enhance beauty. The Fibonacci numbers are one mathematical example that can be found in nature such as in petal count of a rose. On a different note, a mathematical exploration can aid in making a convincing argument on how we can positively impact our environment. Whether looking at the choices of a single individual or the larger impact offered from a collaborative effort, there are measurable responses to positively address climate change. Turning to medicine, which has been a topic of global importance in recent years, we will explore how math is used to determine drug dosage rates and test the validity of a vaccine. Switching back to items of aesthetic nature, we will examine some foundational components of music which, like art, brings beauty and joy to our lives. Finally, we will explore some ways that math is used in sports to predict future performance and analyze tournaments styles.

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13.1: Math and Art

Learning Objectives

1. Identify and describe the golden ratio.
2. Identify and describe the Fibonacci sequence and its application to nature.
3. Apply the golden ratio and the Fibonacci sequence relationship.
4. Identify and compute golden rectangles.

> *Art is the expression or application of human creative skill and imagination, typically in a visual form such as painting or sculpture, producing works to be appreciated primarily for their beauty or emotional power. -Oxford Dictionary*

Art, like other disciplines, is an area that combines talent and experience with education. While not everyone considers themself skilled at creating art, there are mathematical relationships commonly found in artistic masterpieces that drive what is considered attractive to the eye. Nature is full of examples of these mathematical relationships.

Figure $13.1.1$: Sunflower seeds appear in a pattern that involves the Fibonacci sequence. (credit: “Sunflower Surprise” by frankieleon/Flickr, CC BY 2.0)

Enroll in a cake decorating class and, when you learn how to create flowers out of icing, you will likely be directed as to the number of petals to use. Depending on the desired size of a rose flower, the recommendation for the number of petals to use is commonly 5, 8, or 13 petals. If learning to draw portraits, you may be surprised to learn that eyes are approximately halfway between the top of a person’s head and their chin. Studying architecture, we find examples of buildings that contain golden rectangles and ratios that add to the beautifying of the design. The Parthenon (Figure $13.1.2$), which was built around 400 BC, as well as modern-day structures such the Washington Monument are two examples containing these relationships. These seemingly unrelated examples and many more highlight mathematical relationships that we associate with beauty in artistic form.

Figure $13.1.2$: The Parthenon in Greece demonstrates the golden ratio. (credit: “Parthénon” by Julien Maury/Flickr, Public Domain Mark 1.0)

Golden Ratio

The golden ratio, also known as the golden proportion, is a ratio aspect that can be found in beauty from nature to human anatomy as well as in golden rectangles that are commonly found in building structures. The golden ratio is expressed in nature from plants to creatures such as the starfish, honeybees, seashells, and more. It is commonly noted by the Greek letter ϕ (pronounced “fee”). $\phi =$ mfrac52ϕ=1+52, which has a decimal value approximately equal to 1.618.

Consider Figure 13.1.2: Note how the building is balanced in dimension and has a natural shape. The overall structure does not appear as if it is too wide or too tall in comparison to the other dimensions.

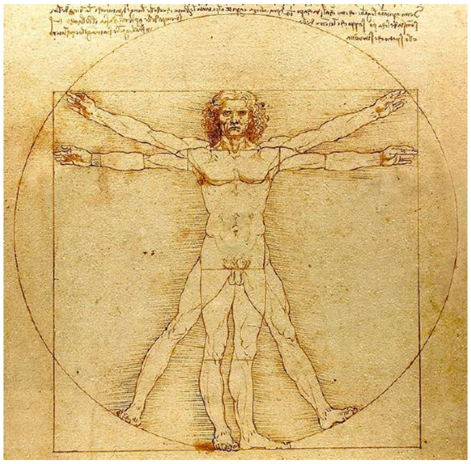

Figure 13.1.3: *Vitruvian Man* by Leonardo da Vinci (credit: "Vitruvian Man" by Leonardo da Vinci/Wikimedia Commons, Public Domain)

The golden ratio has been used by artists through the years and can be found in art dating back to 3000 BC. Leonardo da Vinci is considered one of the artists who mastered the mathematics of the golden ratio, which is prevalent in his artwork such as *Virtuvian Man* (Figure 13.1.4). This famous masterpiece highlights the golden ratio in the proportions of an ideal body shape.

The golden ratio is approximated in several physical measurements of the human body and parts exhibiting the golden ratio are simply called golden. The ratio of a person’s height to the length from their belly button to the floor is ϕ or approximately 1.618. The bones in our fingers (excluding the thumb), are golden as they form a ratio that approximates ϕ. The human face also includes several ratios and those faces that are considered attractive commonly exhibit golden ratios.

✓ Example 13.1.1: Using Golden Ratio and a Person’s Height

If a person’s height is 5 ft 6 in, what is the approximate length from their belly button to the floor rounded to the nearest inch, assuming the ratio is golden?

Answer

Step 1: Convert the height to inches

$5 \text{ ft } 6 \text{ in} = 66 \text{ in}$

Step 2: Calculate the length from the belly button to the floor, L.

$66/L = 1.618$
$L = 40.8 \text{ in}$

The length from the person’s belly button to the floor would be approximately 41 in.

Your Turn 13.1.1

If a person’s height is 6 ft 2 in, what is the approximate length from their belly button to the floor rounded to the nearest inch if the ratio is golden?

Fibonacci Sequence and Application to Nature

Figure 13.1.4: Rose petals appear in a Fibonacci spiral. (credit: “rilke4” by monchoohcnom/Flickr, Public Domain Mark 1.0)

The Fibonacci sequence can be found occurring naturally in a wide array of elements in our environment from the number of petals on a rose flower to the spirals on a pine cone to the spines on a head of lettuce and more. The Fibonacci sequence can be found in artistic renderings of nature to develop aesthetically pleasing and realistic artistic creations such as in sculptures, paintings, landscape, building design, and more. It is the sequence of numbers beginning with 1, 1, and each subsequent term is the sum of the previous two terms in the sequence (1, 1, 2, 3, 5, 8, 13, …).

The petal counts on some flowers are represented in the Fibonacci sequence. A daisy is sometimes associated with plucking petals to answer the question “They love me, they love me not.” Interestingly, a daisy found growing wild typically contains 13, 21, or 34 petals and it is noted that these numbers are part of the Fibonacci sequence. The number of petals aligns with the spirals in the flower family.

✓ Example 13.1.2: Applying the Fibonacci Sequence to Rose Petals

Suppose you were creating a rose out of icing, assuming a Fibonacci sequence in the petals, how many petals would be in the row following a row containing 13 petals?

Answer

The number of petals on a rose is often modeled with the numbers in the Fibonacci sequence, which is 1, 1, 2, 3, 5, 8, 13,…, where the next number in the sequence is the sum of $8+13=21$. There would be 21 petals on the next row of the icing rose.

Your Turn 13.1.2

If a circular row on a pinecone contains 21 scales and models the Fibonacci sequence, approximately how many scales would be found on the next circular row?

Golden Ratio and the Fibonacci Sequence Relationship

Mathematicians for years have explored patterns and applications to the world around us and continue to do so today. One such pattern can be found in ratios of two adjacent terms of the Fibonacci sequence.

Recall that the Fibonacci sequence = 1, 1, 3, 5, 8, 13,… with 5 and 8 being one example of adjacent terms. When computing the ratio of the larger number to the preceding number such as 8/5 or 13/8, it is fascinating to find the golden ratio emerge. As larger numbers from the Fibonacci sequence are utilized in the ratio, the value more closely approaches ϕ, the golden ratio.

✓ Example 13.1.3: Finding Golden Ratio in Adjacent Fibonacci Terms

The 24th Fibonacci number is 46,368 and the 25th is 75,025. Show that the ratio of the 25th and 24th Fibonacci numbers is approximately ϕ. Round your answer to the nearest thousandth.

Answer

$$\frac{75,025}{46,368} = 1.618$$

The ratio of the 25th and 24th term is approximately equal to the value of ϕ rounded to the nearest thousandth, 1.618.

Your Turn 13.1.3

The 23rd Fibonacci number is 28,657 and the 24th is 46,368. Show that the ratio of the 24th and 23rd Fibonacci numbers is approximately ϕ. Round your answer to the nearest thousandth.

Figure 13.1.5: The pyramids of Giza in Egypt (credit: “Giza Pyramids” by Vincent Brown/Flickr, CC BY 2.0)

Golden Rectangles

Turning our attention to man-made elements, the golden ratio can be found in architecture and artwork dating back to the ancient pyramids in Egypt (Figure 13.6) to modern-day buildings such as the UN headquarters. The ancient Greeks used golden rectangles—any rectangles where the ratio of the length to the width is the golden ratio—to create aesthetically pleasing as well as solid structures, with examples of the golden rectangle often being used multiple times in the same building such as the Parthenon, which is shown in Figure 13.1.1. Golden rectangles can be found in twentieth-century buildings as well, such as the Washington Monument.

Looking at another man-made element, artists paintings often contain golden rectangles. Well-known paintings such as Leonardo da Vinci’s *The Last Supper* and the *Vitruvian Man* contain multiple golden rectangles as do many of da Vinci’s masterpieces.

Whether framing a painting or designing a building, the golden rectangle has been widely utilized by artists and are considered to be the most visually pleasing rectangles.

✓ Example 13.1.4: Finding Golden Rectangle in Frames

A frame has dimensions of 8 in by 6 in. Calculate the ratio of the sides rounded to the nearest thousandth and determine if the size approximates a golden rectangle.

Answer

8/6 = 1.333; A golden rectangle’s ratio is approximately 1.618. The frame dimensions are close to a golden rectangle.

Your Turn 13.1.4

A frame has dimensions of 10 in by 8 in. Calculate the ratio of the sides rounded to the nearest thousandth and determine if the size approximates a golden rectangle.

People in Mathematics: M.C. Escher

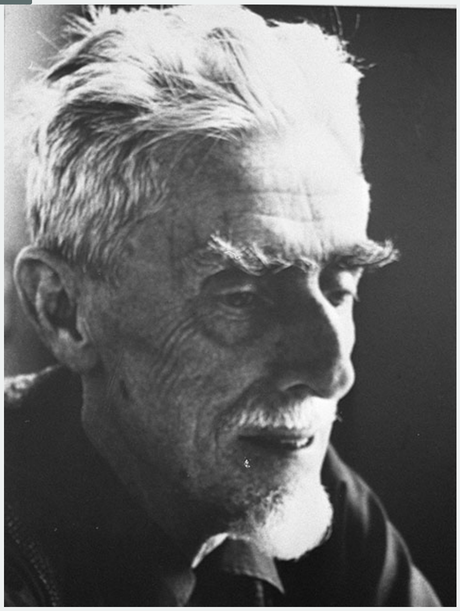

Figure 13.1.6: M.C. Escher (credit: "M.C. Escher" by Hans Peters (ANEFO)/Dutch National Archives, CC0 1.0 Public Domain)

Mauritis Cornelis Escher was a Dutch-born world-famous graphic artist and his work can be found in murals, stamps, wallpaper designs, illustrations in books, and even carpets. Over his lifetime, M.C. Escher created hundreds of lithographs and wood engravings as well as more than 2,000 sketches.

Escher's work is characterized with the infusion of geometric designs that obey most of the mathematical rules. If you study his work closely, you can see where he breaks a mathematical relationship to create famous illusions such as soldiers marching around the top of a square turret where the soldiers appear to be always going uphill but are contained on a single set of stairs in a square. Look closely and the golden ratio as well as golden rectangles abound in Escher's work.

Like many famous people, M.C. Escher did not find success in his early school years. Before finding success, Escher failed his final school exam and quit a short stint in architecture. Finding a graphic arts teacher who recognized Escher's talent, Escher completed art school and enjoyed traveling through Italy, where he found much of his inspiration for his work.

Check Your Understanding

1. What is the value of the golden ratio to the nearest thousandth?
2. What are the first 10 terms of the Fibonacci sequence?
3. What is a golden rectangle?

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13.1.0: Exercises

Exercise 13.1.0.1

A person’s height is 5 ft 2 in. What is the approximate length from their belly button to the floor rounded to the nearest inch?

Exercise 13.1.0.2

A person’s height is 6 ft 3 in. What is the approximate length from their belly button to the floor rounded to the nearest inch?

Exercise 13.1.0.3

A person’s length from their belly button to the floor is 3 ft 11 in. What is the person’s approximate height rounded to the nearest inch?

Exercise 13.1.0.4

A person’s length from their belly button to the floor is 58 in. What is the person’s approximate height rounded to the nearest inch?

Exercise 13.1.0.5

The spikes on a pineapple mirror the Fibonacci sequence. If a row on a pineapple contains five spikes, approximately how many spikes would be found on the next larger row of spikes?

Exercise 13.1.0.6

The leaves on a plant mirror the Fibonacci sequence. If a set of leaves on the plant contains 5 leaves, how many leaves would be found on the previous smaller set of leaves?

Exercise 13.1.0.7

The spines on a head of lettuce mirror the Fibonacci sequence. If a head of lettuce contains 13 spines, approximately how many spines would be found on the next inside layer?

Exercise 13.1.0.8

The seeds on a sunflower mirror the Fibonacci sequence. If a circular layer on the sunflower contains 55 seeds, approximately how many seeds would be found on the next larger circular layer?

Exercise 13.1.0.9

The segments on a palm frond mirror the Fibonacci sequence. If a palm frond contains 89 segments, approximately how many segments would be found on the next larger palm frond?

Exercise 13.1.0.10

The 19th term of the Fibonacci sequence is 4,181 and the 20th term is 6,765. What is the 21st term of the sequence?

? Exercise 13.1.0.11

The 23rd term of the Fibonacci sequence is 28,657 and the 24th term is 46,368. What is the 22nd term of the sequence?

? Exercise 13.1.0.12

The 18th term of the Fibonacci sequence is 2,584 and the 20th term is 6,765. What is the 19th term of the sequence?

? Exercise 13.1.0.13

The 25th term of the Fibonacci sequence is 75,025 and the 20th term is 6,765. What is the 24th term of the sequence?

? Exercise 13.1.0.14

The 10th Fibonacci number is 55 and the 11th is 89. Show that the ratio of the 11th and 10th Fibonacci numbers is approximately ϕ. Round your answer to the nearest thousandth.

? Exercise 13.1.0.15

The 23rd Fibonacci number is 28,657 and the 24th is 46,368. Show that the ratio of the 24th and 23rd Fibonacci numbers is approximately ϕ. Round your answer to the nearest ten-thousandth.

? Exercise 13.1.0.16

The 22nd Fibonacci number is 17,711 and the 21st is 10,946. Show that the ratio of the 22nd and 21st Fibonacci numbers is approximately ϕ. Round your answer to the nearest ten-thousandth.

? Exercise 13.1.0.17

The 16th term of the Fibonacci sequence is 987. Use the approximate value of ϕ of 1.618 to estimate the 15th term. Round your answer to the nearest whole number.

? Exercise 13.1.0.18

The 26th term of the Fibonacci sequence is 121,393. Use the approximate value of ϕ of 1.618 to estimate the 25th term. Round your answer to the nearest whole number.

? Exercise 13.1.0.19

A frame has dimensions of 20 in by 24 in. Calculate the ratio of the sides rounded to the nearest tenth and determine if the size approximates a golden rectangle.

? Exercise 13.1.0.20

A fence has dimensions of 75 in by 45 in. Calculate the ratio of the sides rounded to the nearest tenth and determine if the size approximates a golden rectangle.

? Exercise 13.1.0.21

A frame has a length of 50 in. Calculate the width rounded to the nearest inch if the frame is to be a golden rectangle.

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13.2: Math and the Environment

Figure $13.2.1$: Solar panels harness the sun's energy to power homes, businesses, and various methods of transportation. (credit: modification of work "Craters of the Moon solar array" by NPS Climate Change Response/Flickr, Public Domain Mark 1.0)

Learning Objectives

1. Compute how conserving water can positively impact climate change.
2. Discuss the history of solar energy.
3. Compute power needs for common devices in a home.
4. Explore advantages of solar power as it applies to home use.

Climate change and emissions management have been debated topics in recent years. However, more and more people are recognizing the impacts that have resulted in temperature changes and are seeking timely and effective action. The World Meteorological Organization shared in a June 2021 publication that "2021 is a make-or-break year for climate action, with the window to prevent the worst impacts of climate change—which include ever more frequent more intense droughts, floods and storms—closing rapidly." The problem no longer belongs to a few countries or regions but rather is a worldwide concern measured with increasing temperatures leading to decreased glacier coverage and resulting rise in sea levels.

The good news is, there are small steps that each of us can do that collectively can positively impact climate change.

Making a Positive Impact on Climate Change—Water Usage

Our use of water is one element that impacts climate change. Having access to clean, potable water is critical for not only our health but also for the health of our ecosystem. About 1 out of 10 people on our planet do not have easy access to clean water to drink. As each of us conserves water, we prolong the life span of fresh water from our lakes and rivers and also reduce the impact on sewer systems and drainage in our communities. Additionally, as we conserve water, we also conserve electricity that is used to bring water to and in our homes. So, what can we do to help conserve water?

✓ Example $13.2.1$: Brushing Your Teeth (One Person's Contribution)

Brushing your teeth with the water running continually uses about 4 gal of water. Turning the faucet off when you are not rinsing uses less than one-fourth of a gallon of water. Considering the recommendation to brush your teeth twice a day, how much water would be saved in a week if the faucet was off when not rinsing?

Answer

Leaving the water running continually:

Step 1: Calculate gallons used not with water running continually:

Brushing twice a day for 7 days using 4 gal of water for each brushing

$(2 \text{ times a day})(7 \text{ days})(4 \text{ gal}) = 56 \text{ gal}$

Step 2: Calculate gallons used turning the faucet off when you are not rinsing:

Brushing twice a day for 7 days using 0.25 gal of water for each brushing

$(2 \text{ times a day})(7 \text{ days})(0.25 \text{ gal}) = 3.5 \text{ gal}$

Step 3: Calculate savings:

$$\begin{aligned}\text{Savings} &= 56 \text{ gal} - 3.5 \text{ gal} \\ &= 52.5 \text{ gal}\end{aligned}$$

During one week, 52.5 gal of water would be saved if one person turned the faucet off except when rinsing when brushing your teeth.

Your Turn 13.2.1

Brushing your teeth with the water running continually uses about 4 gal of water. Turning the faucet off when you are not rinsing uses less than one-fourth of a gallon of water. Consider the recommendation to brush your teeth twice a day, how much water would be saved in a month (4 weeks) if the faucet was off when not rinsing?

✓ Example 13.2.2: Brushing Your Teeth (Multiple People's Contribution – Town)

Using the data in Example ✓ 13.2.1, how much water would be saved in a month if one-fifth of a town's population of 15,000 turned the faucet off when brushing their teeth except when rinsing?

Answer

Step 1: From Example 13.5, we found that 1 person saves 52.5 gal per week.

Step 2: Calculate the population to save water:

$\text{One-fifth of } 15{,}000 \text{ people} = 3{,}000 \text{ people}$

Step 3: One-fifth of a town's population turning off the faucet when brushing their teeth for a month:

$(3{,}000)(52.5 \text{ gal per week})(4 \text{ weeks}) = 630{,}000 \text{ gal}$

During one month, 630,000 gallons of water would be saved if one-fifth of a town of 15,000 people turned the faucet off except when rinsing when brushing their teeth.

Your Turn 13.2.2

Using the data in Your Turn 📌 13.2.1, how much water would be saved in a month if one-third of a town's population of 15,000 turned the faucet off when brushing their teeth except when rinsing?

✓ Example 13.2.3: Brushing Your Teeth (Multiple People's Contribution – State)

Using the data in Example 13.5, how much water would be saved in a year if one-fourth of the population of the state of Minnesota, which is approximately 5.6 million people, turned the faucet off when brushing their teeth except when rinsing for a year (52 weeks)?

Answer

Step 1: From Example 13.5, we found that 1 person saves 52.5 gal per week.

Step 2: Calculate the population to save water:

$\text{One-fourth of } 5.6 \text{ million people} = 1.4 \text{ million people}$

Step 3: One-fourth of a town's population turning off the faucet when brushing their teeth for a month:

$(1.4 \text{ million})(52.5 \text{ gal per week})(52 \text{ weeks}) = 3{,}822 \text{ million gal}$

During one year, 3,822 million gal of water would be saved if one-fourth of the state of Minnesota turned the faucet off except when rinsing when brushing their teeth.

Your Turn 13.2.3

Using the data in Example 13.5, how much water would be saved in a year if one-sixth of the population of the state of Florida, which is approximately 21.6 million people, turned the faucet off when brushing their teeth except when rinsing for a year (52 weeks)?

History of Solar Energy

In the mid-1800s, Willoughby Smith discovered photoconductive responsiveness in selenium. Shortly thereafter, William Grylls Adams and Richard Evans Day discovery that selenium can produce electricity if exposed to the sun was a major breakthrough. Less than 10 years later, Charles Fritts invented the first solar cells using selenium. Jumping a mere 100 years later, Bell Labs in the United States produced the first practical photovoltaic cells in the mid-1950s and developed versions used to power satellites in the same decade.

Solar panel use has exploded in recent decades and is now used by residences, organizations, businesses, and government buildings such as the White House, space to power satellites, and various methods of transportation. One reason for the expansion is a continuing drop in cost combined with an increase in performance and durability. In the mid-1950s, the cost of a solar panel was around $300 per watt capability. Twenty years later, the cost was a third of the 1950s' cost. Currently, solar panel cost has dropped to less than $1 per watt while decreasing in size as well as increasing in longevity. The dropping price and improved performance has moved solar to a modest investment that can pay for itself in less than half the time of systems from 15 years ago.

Who Knew?: Solar Power's Age

The sun has been harnessed by humans for centuries. The earliest recorded use of tapping the sun's energy for power dates back to the seventh century BC when man focused the sun's rays through a magnifying glass to create fire. Four thousand years later, we find historical record of using mirrors to focus the sun and light torches, often for ceremonial proceedings. Use of the sun to light torches continued through the centuries and has been recorded by various cultures including the Chinese civilization in 20 AD and beyond.

In more recent years, the sun was harnessed to power ovens on ships traversing to oceans in the 1700s. At the same time, the power of the sun was utilized to power steamboats through the 1800s. Mária Telkes, a Hungarian-born American scientist, invented a widely deployed solar seawater distiller used on World War II life rafts. Soon after, she partnered with architect Eleanor Raymond to design the first modern home to be completely heated by solar power. Air warmed on rooftop collectors transferred heat to salts, which stored the heat for later use.

Although solar panels as we know them today are relatively new in history, use of the sun to harness power is much older.

Compute Power Needs for Common Home Devices

A kilowatt (kW) is 1,000 watts (W). A kilowatt-hour (kWh) is a measurement of energy use, which is the amount of energy used by a 1,000-watt device to run for an hour. Using the definition of a kilowatt-hour, to calculate how long it would take to consume 1 kWh of power, we divide 1,000 by the watts use of a device.

FORMULA

1,000/watts=time needed to use1 kW

For example, a 75 W bulb would take $1{,}000 \div 75 = 13.3 \text{ hours}$ to use 1 kW of power.

FORMULA

watts/1,000=kilowatt hours

Example 13.2.4: Calculating the Kilowatt-Hours Needed to Run a Television

A 48 in plasma television uses about 200 W. How many kilowatt-hours are needed to run the television in a month if the television is one for an average of 2.5 hours a day?

Answer

Step 1: $1,000/(200 \text{ watts }) = 5$ hours to use 1 kW

Step 2: (2.5 hours a day)(30 days) $= 75$ hours of use

Step 3: 75 hour / 5 hours per \(\mathrm{kW}=15 \mathr**m{~kW}\)**

The television will consume about 15 kW in a month.

Your Turn 13.2.4

A 40 in plasma television uses about 175 W. How many kilowatt-hours are needed to run the television in a month if the television is on for an average of 3 hours a day?

Example 13.2.5: Calculating the Cost to Run a Refrigerator

A medium-sized Energy Star–rated refrigerator uses about 575 W and runs for about 8 hours per day. What is the monthly (30 days) cost of running the refrigerator if the electric rate is 12 cents per kilowatt-hour?

Answer

Step 1: Calculate the watts per day:

$(575 \text{ W})(8 \text{ hours}) = 4{,}600 \text{ W per day}$

Step 2: Calculate the kilowatt-hours.

$(4{,}600)/(1{,}000) = 4.6 \text{ kWh}$

Step 3: Calculate the daily cost.

$(4.6 \text{ kWh})(12 \text{ cents}) = 55 \text{ cents} = \$0.55$

Step 4: Calculate the monthly cost.

$(\$0.55)(30) = \$16.50$

It would cost about $16.50 to run the refrigerator for a month.

Your Turn 13.2.5

A dorm-sized Energy Star–rated refrigerator uses about 375 W and runs for about 9 hours per day. What is the monthly cost of running the refrigerator if the electric rate is 14 cents per kilowatt-hour?

Example 13.2.6: Calculating the Kilowatt-Hours to Run an Oven

An electric oven is labeled as 4,000 W. How much would it cost to bake a cake for 30 minutes if the electric rate is 14 cents per kilowatt-hour?

Answer

Step 1: Determine the time it takes to use 1 kW of power:

$1{,}000/(4{,}000 \text{ watts}) = 0.25 \text{ hours to use } 1 \text{ kW}$

For every 15 minutes, the oven uses 1 kW of power.

Step 2: Determine how many kilowatt-hours are needed to bake the cake for 30 minutes:

$(30 \text{ minutes})/(15 \text{ minutes per kW}) = 2 \text{ kW}$

Step 3: Calculate the cost of the oven usage:

$(2 \text{ kW})(14 \text{ cents per kWh}) = 28 \text{ cents}$

It would cost about 28 cents to bake the cake.

Your Turn 13.2.6

A toaster oven is labeled as 2,000 W. How much would it cost to warm leftovers from a meal for 15 minutes if the electric rate is 12 cents per kilowatt-hour?

Solar Advantages

There are multiple advantages that solar power can offer us today including reducing greenhouse gas and CO_2 emissions, powering vehicles, reducing water pollution, reducing strain on limited supply of other power options such as fossil fuels. We will look further at reducing greenhouse gas and CO_2 emissions.

Any gas that prevents infrared radiation from escaping Earth's atmosphere is a greenhouse gas. There are 24 currently identified greenhouse gases of which carbon dioxide is one. When measuring the impact of any of the greenhouse gases, the measurements are given in units of carbon dioxide emissions. For this reason, greenhouse gas and carbon dioxide have become interchangeable in discussions.

People in Mathematics: Charles Fritts and Mohammad M. Atalla

Charles Fritts, a New York inventor, is credited with creating the first solar cell, which he installed on a rooftop in New York City in 1884. While the solar cell was not very efficient, having a rate of conversion between 1 to 2%, this was a major step early in solar power energy. Today's solar cells have an efficiency on average of 15 to 20%, which yields a notably higher impact. Nonetheless, the work that Fritts successfully completed marked the start of solar energy through the use of photovoltaic solar panels in the United States.

Mohamed M. Atalla was an Egyptian-born scientist who moved to the United States to complete his studies, and undertook research and development at Bell Laboratories in New Jersey. Many of the early efficiency gains in solar cells were due to his development of processes for using silicon within electronic devices. Atalla's work led to the invention of silicon transistors and microchips (including his own invention of the MOSFET, the most widely used transistor in the world), and quickly increased the efficiency of solar cells.

✓ Example 13.2.7: Calculating the Solar Power for Average Home Use in Kilowatts

If a home uses approximately 30 kW of electricity per day, what size solar system would be needed to fuel 80% of a home's needs for a month (30 days)?

Answer

$(30 \text{ kW hours})(30 \text{ days})(0.80) = 720 \text{ kW}$

A solar system capable of producing 720 kW a month would be needed.

Your Turn 13.2.7

A tiny home uses approximately 12 kW of electricity per day. What size solar system would be needed to fuel 80% of a home's needs for a month (30 days)?

Check Your Understanding

1. What is the relationship between 1 kW and watts?

2. An average bath uses about 35 gal of water. A water-saving showerhead uses approximately 2 gal of water per minute. If a person typically takes one bath a week, how much water is saved by replacing the baths with showers lasting 5 minutes over the course of a month?
3. How long does it take for a 30 W Google Nest audio to use 1 kW of power?
4. How long would it take for a 60 W bulb to consume 1 kW of power? Round your answer to the nearest hour.

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13.2.0: Exercises

? Exercise 13.2.0.1

A typical showerhead uses 5 gal of water per minute. A water-saving showerhead uses approximately 2 gal of water per minute. How much water would one person save in a month if they take a 6-minute shower 4 times a week?

? Exercise 13.2.0.2

An average toilet uses 5 gal of water per flush. A high-efficiency toilet uses about 1.25 gal per flush. How much water would a household save in a week if the toilet was flushed 8 times a day?

? Exercise 13.2.0.3

When washing dishes, leaving the faucet running utilizes about 15 gal every 5 minutes, where filling the sink and turning the faucet off except to rinse the dishes uses about 5 gal for washing a dishpan-sized load of dishes. How much water is saved by not leaving the faucet running if it takes 10 minutes to wash a dishpan-sized load of dishes each day for a month containing 30 days?

? Exercise 13.2.0.4

Leaving the water running when washing your hands consumes about 4 gal of water, whereas turning the water off when lathering reduces the water used to 1 gal. How much water is saved in an apartment of 6 people in an academic term of 16 weeks if each person averages washing their hands 3 times a day?

? Exercise 13.2.0.5

How long would it take for an energy-saving light bulb to consume 1 kW of power if the bulb is rated at 7.5 W? Round your answer to the nearest hour.

? Exercise 13.2.0.6

How long would it take for a 120 W light bulb to consume 1 kW of power? Round your answer to the nearest hour.

? Exercise 13.2.0.7

A portable television uses about 80 W per hour. How many kilowatt-hours are needed to run the television during a 3-day trip if the television is run for an average of 5.5 hours a day?

? Exercise 13.2.0.8

A flat iron to straighten hair is rated at 331 W. If it is used for 15 minutes a day, 5 times a week, how much would it cost a user over the course of a month if the electric rate is 13 cents per kilowatt-hour? Round your answer to the nearest cent.

? Exercise 13.2.0.9

You purchase a window air conditioner for your apartment living room rated at 1,000 W. If you run the air conditioner for an average of 3 hours a day for a week, how much would it cost if the electric rate is 12 cents per kilowatt-hour?

Exercise 13.2.0.10

A cabin uses approximately 25 kW of electricity per day. What size solar system would be needed to fuel 90% of the cabin's needs for a month if the cabin is used 2 days a week?

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13.3: Math and Medicine

Figure 13.9: Shoppers wear masks during the Covid-19 pandemic. (credit: "True Covid Scene - Mask Buying" by Joey Zanotti/Flickr, CC BY 2.0)

Learning Objectives

1. Compute the mathematical factors utilized in concentrations/dosages of drugs.
2. Describe the history of validating effectiveness of a new drug.
3. Describe how mathematical modeling is used to track the spread of a virus.

The pandemic that rocked the world starting in 2020 turned attention to finding a cure for the Covid-19 strain into a world race and dominated conversations from major news channels to households around the globe. News reports decreeing the number of new cases and deaths locally as well as around the world were part of the daily news for over a year and progress on vaccines soon followed. How was a vaccine able to be found so quickly? Is the vaccine safe? Is the vaccine effective? These and other questions have been raised through communities near and far and some remain debatable. However, we can educate ourselves on the foundations of these discussions and be more equipped to analyze new information related to these questions as it becomes available.

Concentrations and Dosages of Drugs

Consider any drug and the recommended dosage varies based on several factors such as age, weight, and degree of illness of a person. Hospitals and medical dispensaries do not stock every possible needed concentration of medicines. Drugs that are delivered in liquid form for intravenous (IV) methods in particular can be easily adjusted to meet the needs of a patient. Whether administering anesthesia prior to an operation or administering a vaccine, calculation of the concentration of a drug is needed to ensure the desired amount of medicine is delivered.

The formula to determine the volume needed of a drug in liquid form is a relatively simple formula. The volume needed is calculated based on the required dosage of the drug with respect to the concentration of the drug. For drugs in liquid form, the concentration is noted as the amount of the drug per the volume of the solution that the drug is suspended in which is commonly measured in g/mL or mg/mL.

Suppose a doctor writes a prescription for 6 mg of a drug, which a nurse calculates when retrieving the needed prescription from their secure pharmaceutical storage space. On the shelves, the drug is available in liquid form as 2 mg per mL. This means that 1 mg of the drug is found in 0.5 mL of the solution. Multiplying 6 mg by 0.5 mL yields 3 mL, which is the volume of the prescription per single dose.

FORMULA

$\text{Volume needed} = (\text{medicine dosage required})\,(\text{weight of drug by volume}).$

A common calculation for the weight of a liquid drug is measured in grams of a drug per 100 mL of solution and is also called the percentage weight by volume measurement and labeled as % w/v or simply w/v.

Checkpoint

Note that the units for a desired dose of a drug and the units for a solution containing the drug or pill form of the drug must be the same. If they are not the same, the units must first be converted to be measured in the same units.

Suppose you visit your doctor with symptoms of an upset stomach and unrelenting heartburn. One possible recourse is sodium bicarbonate, which aids in reducing stomach acid.

✓ Example 13.3.1: Calculating the Quantity in a Mixture

How much sodium bicarbonate is there in a 250 mL solution of 1.58% w/v sodium bicarbonate?

Answer

$1.58\%\ \mathrm{w/v} = 1.58\ \mathrm{g}$ sodium bicarbonate in 100 mL. If there is 250 mL of the solution, we have 2.5 times as much sodium bicarbonate as in 100 mL. Thus, we multiply 1.58 by 2.5 to yield 3.95 g sodium bicarbonate in 250 mL solution.

Your Turn 13.3.1

How many milligrams of sodium chloride are there in 200 mL of a 0.9% w/v normal saline solution?

✓ Example 13.3.2: Calculating the Quantity of Pills Needed

A doctor prescribes 25.5 mg of a drug to take orally per day and pills are available in 8.5 mg. How many pills will be needed each day?

Answer

The prescription and the pills are in the same units which means no conversions are needed. We can divide the units of the drug prescribed by the units in each pill: $25.5/8.5 = 3$. So, 3 pills will be needed each day.

Your Turn 13.3.2

How many pills would be needed for a patient who has been prescribed 25.5 mg of a drug if each pill contains 4.25 mg.

✓ Example 13.3.3: Calculating the Drug Dose in Milligrams, Based on Patient Weight

A patient is prescribed 2 mg/kg of a drug to be delivered intramuscularly, divided into 3 doses per day. If the patient weighs 45 kg, how many milligrams of the drug should be given per dose?

Answer

Step 1: Calculate the total daily dose of the drug based on the patient's weight (measured in kilograms):

$(2\ \mathrm{mg/kg})(45\ \mathrm{kg}) = 90\ \mathrm{mg}$

Step 2: Divide the total daily dose by the number of doses per day:

$90\ \mathrm{mg}/3 = 30\ \mathrm{mg}$

The patient should receive 30 mg of the drug in each dose.

Your Turn 13.3.3

A patient is prescribed 1.4 mg/kg of a drug to be delivered intramuscularly, divided into 2 doses per day. If the patient weighs 60 kg, how many milliliters of the drug should be given per dose?

Checkpoint

Note that the units for a patient's weight must be compatible with the units used in the medicine measurement. If they are not the same, the units must first be converted to be measured in the same units.

✓ Example 13.3.4: Calculating the Drug Dose in Milliliters, Based on Patient Weight

A patient is prescribed 2 mg/kg of a drug to be delivered intramuscularly, divided into 3 doses per day. If the drug is available in 20 mg/mL and the patient weighs 60 kg, how many milliliters of the drug should be given per dose?

Answer

Step 1: Calculate the total daily dose of the drug (measured in milligrams) based on the patient's weight (measured in kilograms):

$(2 \text{ mg/kg})(60 \text{ kg}) = 120 \text{ mg}$

Step 2: Calculate the volume in each dose:

$(120 \text{ mg daily total})/(3 \text{ doses a day}) = 40 \text{ mg per dose}$

Step 3: Calculate the volume based on the strength of the stock:

$(\text{prescribed dose needed})/(\text{stock dose}) = \text{volume}$
$(40 \text{ mg per dose})/(20 \text{ mg/mL}) = 2 \text{ mL}$

The patient should receive 2 mL of the stock drug in each dose.

Your Turn 13.3.4

A patient is prescribed 2.5 mg/kg of a drug to be delivered intramuscularly, divided into 2 doses per day. If the drug is available in 5 mg/mL and the patient weighs 52 kg, how many milligrams of the drug should be given per dose?

Who Knew?: Math Statistics from the CDC

The Centers for Disease Control and Prevention (CDC) states that about half the U.S. population in 2019 used at least one prescription drug each month, and about 25% of people used three or more prescription drugs in a month. The resulting overall collective impact of the pharmaceutical industry in the United States exceeded $1.3 trillion a year prior to the 2020 pandemic.

Validating Effectiveness of a New Vaccine

The process to develop a new vaccine and be able to offer it to the public typically takes 10 to 15 years. In the United States, the system typically involves both public and private participation in a process. During the 1900s, several vaccines were successfully developed, including the following: polio vaccine in the 1950s and chickenpox vaccine in the 1990s. Both of these vaccines took years to be developed, tested, and available to the public. Knowing the typical timeline for a vaccine to move from development to administration, it is not surprising that some people wondered how a vaccine for Covid-19 was released in less than a year's time.

Lesser known is that research on coronavirus vaccines has been in process for approximately 10 years. Back in 2012, concern over the Middle Eastern respiratory syndrome (MERS) broke out and scientists from all over the world began working on researching coronaviruses and how to combat them. It was discovered that the foundation for the virus is a spike protein, which, when delivered as part of a vaccine, causes the human body to generate antibodies and is the platform for coronavirus vaccines.

When the Covid-19 pandemic broke out, Operation Warp Speed, fueled by the U.S. federal government and private sector, poured unprecedented human resources into applying the previous 10 years of research and development into targeting a specific vaccine for the Covid-19 strain.

People in Mathematics: Shibo Jiang

Dr. Shibo Jiang, MD, PhD, is co-director the Center for Vaccine Development at the Texas Children's Hospital and head of a virology group at the New York Blood Center. Together with his colleagues, Jiang has been working on vaccines and treatments for a range of viruses and infections including influenzas, HIV, Sars, HPV and more recently Covid-19. His work has been recognized around the world and is marked with receiving grants amounting to over $20 million from U.S. sources as well as the same from foundations in China, producing patents in the United States and China for his antiviral products to combat world concerns.

Jiang has been a voice for caution in the search for a vaccine for Covid-19, emphasizing the need for caution to ensure safety in the development and deployment of a vaccine. His work and that of his colleagues for over 10 years on other coronaviruses paved the way for the vaccines that have been shared to combat the Covid-19 pandemic.

Mathematical Modeling to Track the Spread of a Vaccine

With a large number of people receiving a Covid-19 vaccine, the concern at this time is how to create an affordable vaccine to reach people all over the world. If a world solution is not found, those without access to a vaccine will serve as incubators to variants that might be resistant to the existing vaccines.

As we work to vaccinate the world, attention continues with tracking the spread of the Covid-19 and its multiple variants. Mathematical modeling is the process of creating a representation of the behavior of a system using mathematical language. Digital mathematical modeling plays a key role in analyzing the vast amounts of data reported from a variety of sources such as hospitals and apps on cell phones.

When attempting to represent an observed quantitative data set, mathematical models can aid in finding patterns and concentrations as well as aid in predicting growth or decline of the system. Mathematical models can also be useful to determine strengths and vulnerabilities of a system, which can be helpful in arresting the spread of a virus.

The chapter on Graph Theory explores one such method of mathematical modeling using paths and circuits. Cell phones have been helpful in tracking the spread of the Covid-19 virus using apps regulated by regional government public health authorities to collect data on the network of people exposed to an individual who tests positive for the Covid-19 virus.

People in Mathematics: Gladys West

Dr. Gladys West is a mathematician and hidden figure with a rich résumé of accomplishments spanning Air Force applications and work at NASA. Born in 1930, West rose and excelled both academically and in her professional life at a time when Black women were not embraced in STEM positions. One of her many accomplishments is the Global Positioning System (GPS) used on cell phones for driving directions.

Figure 13.3.2: Gladys West (credit: "Dr. Gladys West Hall" by The US Air Force/Wikimedia Commons, Public Domain)

West began work as a human computer, someone who computes mathematical computations by hand. Considering the time and complexity of some calculations, she became involved in programming computers to crunch computations. Eventually, West created a mathematical model of Earth with detail and precision that made GPS possible, which is utilized in an array of

> devices from satellites to cell phones. The next time you tag a photo or obtain driving directions, you are tapping into the mathematical modeling of Earth that West developed.

Consider the following graph (Figure 13.3.3):

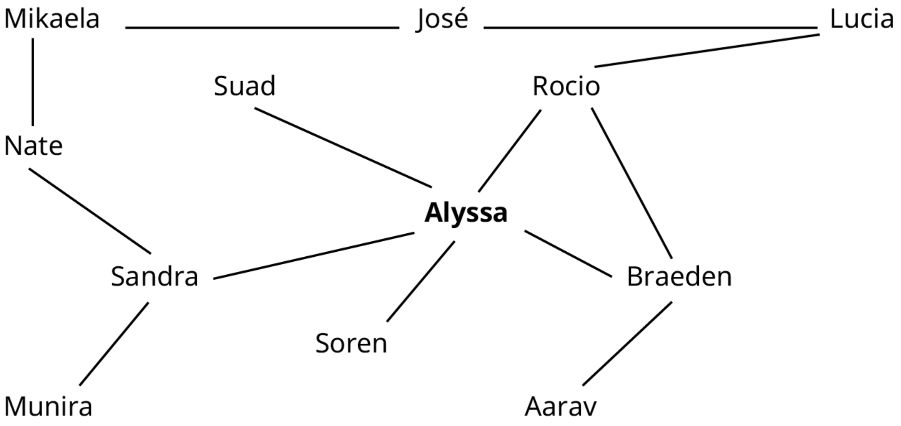

Figure 13.3.3: Contact Tracing for Math 111 Section 1

At the center of the graph, we find Alyssa, whom we will consider positive for a virus. Utilizing the technology of phone apps voluntarily installed on each phone of the individuals in the graph, tracking of the spread of the virus among the 6 individuals that Alyssa had direct contact with can be implemented, namely Suad, Rocio, Braeden, Soren, and Sandra.

Let's look at José's exposure risk as it relates to Alyssa. There are multiple paths connecting José with Alyssa. One path includes the following individuals: José to Mikaela to Nate to Sandra to Alyssa. This path contains a length of 4 units, or people, in the contact tracing line. There are 2 more paths connecting José to Alyssa. A second path of the same length consists of José to Lucia to Rocio to Braeden to Alyssa. Path 3 is the shortest and consists of José to Lucia to Rocio to Alyssa. Tracking the spread of positive cases in the line between Alyssa and José aids in monitoring the spread of the infection.

Now consider the complexity of tracking a pandemic across the nation. Graphs such as the one above are not practical to be drawn on paper but can be managed by computer programs capable of computing large volumes of data. In fact, a computer-generated mathematical model of contact tracing would look more like a sphere with paths on the exterior as well as on the interior. Mathematical modeling of contact tracing is complex and feasible through the use of technology.

✓ Example 13.3.5: Using Mathematical Modeling

For the following exercises, use the sample contact tracing graph to identify paths (Figure ✓ 13.3.4).

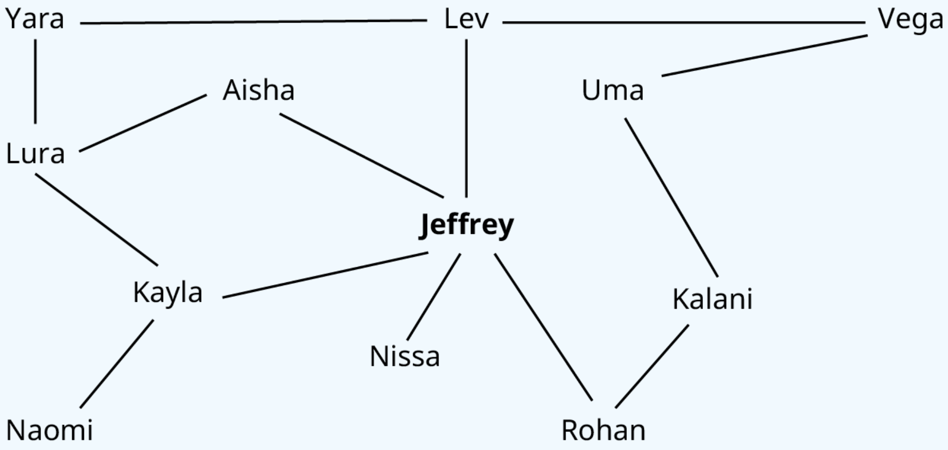

Figure ✓ 13.3.4 Contact Tracing for ECON 250 Section 1

1. How many people have a path of length 2 from Jeffrey?
2. Find 2 paths between Kayla and Rohan.
3. Find the shortest path between Yara and Kalani. State the length and people in the path.

Answer

1. 5 (Lura, Naomi, Kalani, Vega, Yara)
2. Answers will vary. Two possible answers are as follows:
    a. Kayla, Jeffrey, Rohan
    b. Kayla, Lura, Yara, Lev, Vega, Uma, Kalani, Rohan
3. Length is 4. People in path = Yara, Lev, Vega, Uma, Kalani

Your Turn 13.3.5

For the following exercises, use Figure 13.3.5.

List all the names of those who have a path length of three from Uma.

What is the length of the shortest path from Naomi to Vega?

Check Your Understanding

Exercise 13.3.1

What two pieces of information are needed to calculate the volume of a prescription drug to be dispensed?

Exercise 13.3.2

Research on coronavirus vaccines began in 2020.

a. True
b. False

Exercise 13.3.3

What is mathematical modeling, and how is it used in the world of medicine with pandemics?

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13.3.0: Exercises

Exercise 13.3.0.1

How many grams of sodium bicarbonate are contained in a 300 mL solution of 1.35% w/v sodium bicarbonate?

Exercise 13.3.0.2

How many grams of sodium bicarbonate are contained in a 175 mL solution of 1.85% w/v sodium bicarbonate?

Exercise 13.3.0.3

Using a saline solution that is 0.75% w/v, how many milligrams of sodium chloride are in 150 mL?

Exercise 13.3.0.4

Using a saline solution that is 1.25% w/v, how many milligrams of sodium chloride are in 200 mL?

Exercise 13.3.0.5

A prescription calls for a patient to receive 23 mg daily of a drug to be taken in pill form for 5 days. If the pills are available in 5.75 mg, how many pills will the patient need for the full prescription run?

Exercise 13.3.0.6

A prescription calls for a patient to receive 21 mg daily of a drug to be taken in pill form daily. If the pills are available in 3.5 mg, how many pills will the patient need each day?

Exercise 13.3.0.7

A patient is prescribed 4mg/kg of a drug to be delivered daily intramuscularly, divided into 2 doses. If the patient weighs 30 kg, how many milligrams of the drug would be needed for each dose?

Exercise 13.3.0.8

A patient is prescribed 1.5 mg/kg of a drug to be delivered intramuscularly, divided into 3 doses per day. If the drug is available in 12.5 mg/mL and the patient weighs 50 kg, how many milliliters of the drug would be given per dose?

Exercise 13.3.0.9

A patient is prescribed 0.5 mg/kg of a drug to be delivered intramuscularly, divided into 2 doses per day. If the drug is available in 2.5 mg/mL and the patient weighs 45 kg, how many milliliters of the drug would be given per dose?

Exercise 13.3.0.10

A patient is prescribed 1.5 mg/kg of a drug to be delivered intramuscularly, divided into 3 doses per day. If the drug is available in 30 mg/mL and the patient weighs 54 kg, how many milliliters of the drug would be given per dose?

For the following exercises, use the mathematical modeling graph showing contact tracing for students in a particular class.

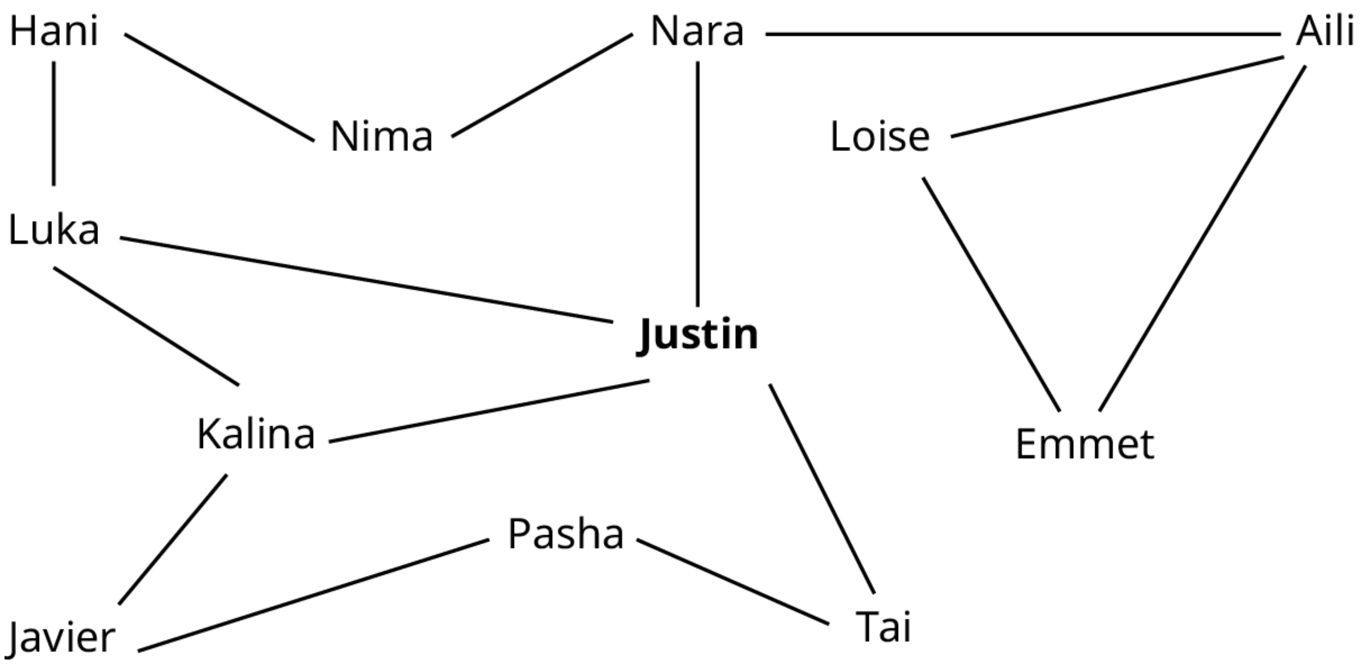

Figure 13.13: Contract Tracing for Students in a Class

? Exercise 13.3.0.11

List the people who have a length of 2 from Justin.

? Exercise 13.3.0.12

Find 2 paths of with a length of 3 from Emmet.

? Exercise 13.3.0.13

Find the shortest path from Aili to Kalina.

? Exercise 13.3.0.14

Which people does the model show as directly in contact with Nara?

? Exercise 13.3.0.15

Find the shortest path from Tai to Hani.

? Exercise 13.3.0.16

Of the 12 people in the model, how many have a path of 2 or less from Justin?

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13.4: Math and Music

Learning Objectives

1. Describe the basics of frequency related to sound.
2. Describe the basics of pitch as it relates to music.
3. Describe and evaluate musical notes, half-steps, whole steps, and octaves.
4. Describe and find frequencies of octaves.

> *“The world’s most famous and popular language is music.”*
>
> *- Psy, South Korean singer, rapper, songwriter, and record producer*

Imagine a world without music and many of us would struggle to fill the void. Music uplifts, inspires, heals, and generally adds dimension to virtually every aspect of our lives. But what is music? For some it is a song; for others it may be the sounds of birds or the rhythmic sound of drumming or a myriad of other sounds. Whatever you consider music, it is all around us and is an integral part of our lives. “Music can raise someone’s mood, get them excited, or make them calm and relaxed. Music also—and this is important—allows us to feel nearly or possibly all emotions that we experience in our lives. The possibilities are endless” (Galindo, 2009).

Figure Figure 13.4.1: Contact Tracing for ECON 250 Section 1 Friends sing music together around a campfire. (credit: modification of work “Fire is hot! (2)” by Chetan Sarva/Flickr, CC BY 2.0)

What music you listen to can impact your mood and emotions. In similar fashion, the music we choose can often tell those around us something about our current moods and emotions. Consider the music you may have been listening to as you today or even as you are reading this text. What cues to your mood do your music selections share? Albert Einstein is quoted as saying, “If I were not a physicist, I would probably be a musician. I often think in music. I live my daydreams in music. I see my life in terms of music.” What clues do your recent music choices say about your mood or how your day is going?

Basics of Frequency as It Relates to Sound

Every sound is created by an object vibrating and these vibrations travel in waves that are captured by our ears. Some vibrations we may be able to see, such as a plucked guitar string moving, whereas other vibrations we may not be able to see, such as the sound created when we hold our breath when accidentally dropping our cell phone on a hard floor. We don’t see the vibrations of our cell phone hitting the floor; however, any audible sound created in the fall is the result of vibrations in the form of sound waves, which can be pictured similarly to waves moving through the ocean.

The waves of sounds each have a frequency, or rate of vibration of sound waves, that measures the number of waves completed in a single second and are measured in hertz (Hz; one Hz is one cycle per second). Louder sounds have stronger vibrations or are created closer to our ear. The further an ear is from the source of the sound, the quieter the sound will appear.

Sounds range in frequency from 16 Hz to ultrasonic values, with humans able to hear sounds in a frequency range of about 20 Hz to 20,000 Hz. Adults lose the ability to hear the upper end of the range and typically top out in the ability to hear in a frequency of 15,000–17,000 Hz. Sounds with a frequency above 17,000 Hz are less likely to be heard by adults while still being audible to children.

While frequency plays a key role in audible sounds, so too does the sound level, which can be measured in decibels (dB), which are the units of measure for the intensity of a sound or the degree of loudness. As a sound level increases, the decibel level increases.

A person with average hearing can hear sounds down to 0 dB. Those with exceptionally good hearing can hear even quieter sounds, down to approximately –5 dB. The following table includes sample sounds with their related decibel values.

Sound	Decibels (dB)
Firecrackers	140
Take-off of military jet from aircraft carrier	130
Clap of thunder	120
Auto horn standing next to the vehicle	110
Outboard motor	100
Motorcycle	90
Noise inside a car in city traffic	80
Typical washing machine	70
Public conversation, such as at a restaurant	60
Private conversation	50
Hum of a computer with fan blowing	40
Quiet whisper	30
Swishing leaves	20
Regular breathing	10
Lowest typical sound audible by teenagers	0

✓ Example 13.4.1: Selecting Decibel Value of Sounds

Select the most representative decibel value for each of the following sounds:

1. car wash: 25 dB, 55 dB, 85 dB
2. vacuum cleaner: 15 dB, 70 dB, 90 dB
3. ship’s engine room: 30 dB, 65 dB, 95 dB
4. approaching subway car: 70 dB, 100 dB, 120 dB

Answer

1. 85 dB
2. 70 dB
3. 95 dB
4. 100 dB

Your Turn 13.4.1

Match each of the following sounds with a corresponding decibel level: live outdoor concert, birds chirping, window air conditioner, garbage disposal.

a. 40 dB
b. 60 dB
c. 80 dB
d. 100 dB

Basics of Pitch

When considering the various sound levels the human ear can hear, the ear perceives sound both from the frequency level and the pitch of a sound. The quality of the sound is referred to as pitch, the tonal quality of a sound and how high or low the tone. Sounds with a high frequency have a high pitch, such as 900 Hz, and sounds with a low pitch have a low frequency, such as 50 Hz.

Let's take a look at frequency and pitch using a string instrument such as a guitar or piano. When a string is plucked on a guitar or a key is played on a piano, the related string vibrates at a frequency that is related to the length and thickness of the string. The frequency is measurable and has a singular value. The pitch of the note played is open for interpretation, as the pitch is a function of personal opinion.

Who Knew?: Graphing Calculators and Music

It may be interesting to note that a TI-84 and TI-Nspire graphing calculators can be utilized to tune a musical instrument by measuring the frequency of a note using a small plug-in accessory that captures the sound waves from a note and displays the corresponding frequency. Using the displayed frequency, an instrumentalist can then make the needed adjustment to perfectly tune an instrument. This method of tuning an instrument can be helpful whether a novice player or a seasoned instrumentalist because the instrument can be tuned precisely to the correct frequency.

Note Values, Half-Steps, Whole Steps, and Octaves

> *"There are not more than five musical notes, yet the combinations of these five give rise to more melodies than can ever be heard."*
>
> *- Sun Tzu, Chinese strategist*

Moving our exploration to note values, the frequency of all notes is well defined by a specific and unique frequency for each note that is measurable. We will explore keys on a keyboard to discuss notes that have the same relationships with any instrument or musical piece.

Let's look at Figure 13.4.2. The white keys are labeled with the letters A–G and the photo begins with middle C, which can be found in the middle of a keyboard. This labeling of the keys repeats across an entire keyboard and keys to the right have a higher pitch and frequency than keys to the left. Each of the keys correlates to a musical note.

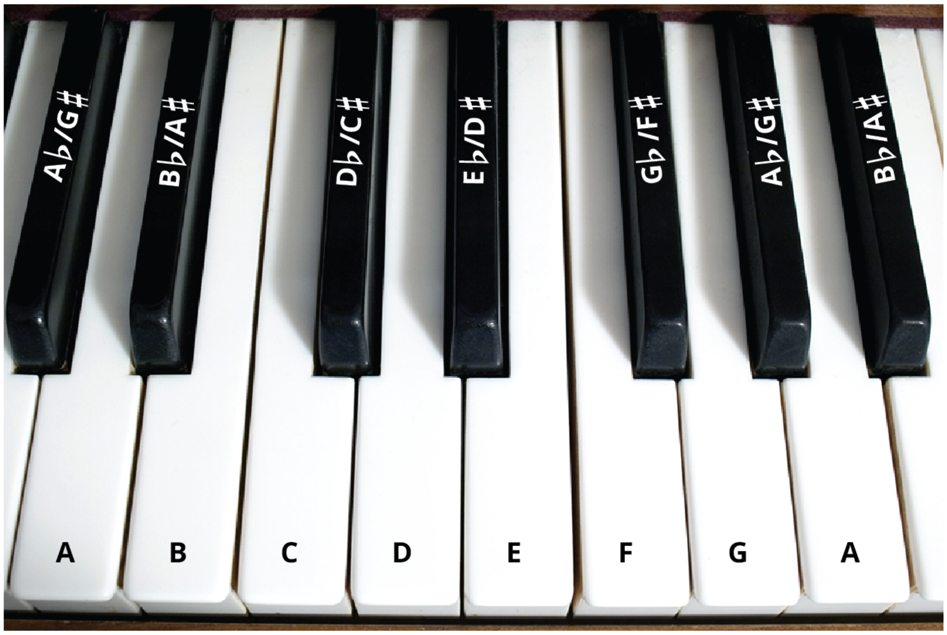

Figure 13.4.2 Piano Keys and Notes (credit: modification of work "Contemplate" by Walt Stoneburner/Flickr, CC BY 2.0)

Movement up or down between any two consecutive keys (black and white) or notes constitutes a half-step. Movement of one half-step sometimes involves a sharp (#) or a flat (♭) symbol. For example, $D^{\#}$ is one half-step above D and $D^{\flat}$ is one half-step below D.

Note that this is not always true as one half-step above B is C, and one half-step below F is E. In similar fashion, a whole step is movement up or down between any two half-steps on a keyboard.

Example 13.4.2: Identifying Half-Steps

Name which keys are one half-step up and one half-step down from the following:

1. D
2. E
3. G#

Answer

1. up $D^{\#}$, down $D^{\flat}$
2. up F, down $E^{\flat}$
3. up A, down G

Your Turn 13.4.2

Name which keys are one half-step up and one half-step down from the following:

a. $F^{\#}$
b. B
c. $G^{\flat}$

Example 13.4.3: Identifying Whole Steps

Name which keys are one whole step up and one whole step down from the following:

1. $F^{\#}$
2. E
3. $A^{\flat}$

Answer

1. Up $G^{\#}$, down E
2. Up $F^{\#}$, down D
3. Up $B^{\flat}$, down $G^{\flat}$

Your Turn 13.4.3

Name which keys are one whole step up and one whole step down from the following:

1. $D^{\flat}$
2. $C^{\#}$
3. E

You may have noticed that there are eight letters of the alphabet used to label notes. Selecting any one note and counting up 12 half-steps you will find that the numbering for notes begins at the same value as you started from. This collection of 12 consecutive half-notes is called an octave and is a basic foundational component in music theory.

Example 13.4.4: Listing All Notes in an Octave

List the 12 notes forming an octave, beginning with the note C.

Answer

C, $C^{\#}$, D, $D^{\#}$, E, F, $F^{\#}$, G, $G^{\#}$, A, $A^{\#}$, B

Your Turn 13.4.4

List the 12 notes forming an octave, beginning with the note G.

Frequencies of Octaves

Notes that are one octave apart have the same name and are related in frequency values. Given the frequency of any note, the frequency of same note one octave higher is doubled and this pattern continues as you move up and down the notes on a keyboard or any other musical instrument. Song writers and singers use this knowledge to change the pitch of a note up or down to align with a person's vocal range. Regardless of which C is played or sung, the pitch is the same and the frequency is related by a power or two.

Labeled keys on a keyboard are numbered for ease in identification. For example, middle C is labeled as C_4 on a full keyboard as it is the fourth C from the left in a set of eight notes. The frequency of C_4 is 262 Hz, rounded to the nearest whole number.

✓ Example 13.4.5: Calculating the Frequency Values of Octaves

Given that the frequency of C_4 is 262 Hz, find the approximate frequency of C_6.

Answer

> The frequency of each consecutive higher octave doubles. Given that the frequency of C_4 is 262 Hz, the frequency of C_5 is found by doubling the frequency of C_4, which is 524 Hz. In similar fashion, the frequency of C_6 is found by doubling the frequency of C_5, which yields 1,048 Hz.

Your Turn 13.4.5

Given that the frequency of E_4 is 330 Hz, find the approximate frequency of E_2 rounded to the nearest whole number.

People in Mathematics: David Cope

David Cope has a somewhat eclectic list of job titles ranging from author and music professor to scientist and artificial intelligence researcher. Cope combined his interests when he developed software that can analyze a piece of music and create a new and unique musical piece in the same style as the original. Some of his well-known products have been based off of the classical music of Mozart, creating what has been called Mozart's "42nd Symphony," as well as other genres including opera and a range of current music styles. Cope has also composed original musical pieces in collaboration with a computer.

We have explored some basics components of frequency, pitch, note relationships, and octaves, which are building blocks of music. It may be exciting to learn that the mathematical relationships found in music are vast and grow in complexity beyond the math commonly studied in high school.

Who Knew?: Spotify Royalty Payments

Streaming services have grown exponentially in popularity thanks in large part to customized music listening through cell phone use and devices such as Google as well as Amazon Echo and Alexa devices for home and vehicles, adding to ways that artists are paid royalties. Spotify, which was launched in 2008, typically plays artists \$0.06 per time a song is streamed, with some artists receiving up to \$0.84 per play amounting to over \$9 billion in revenue for Spotify in 2020. Since 2014, Spotify's revenue has grown over a billion dollars a year, with roughly half of their revenue being paid out in royalties, which was good news for artists during the Covid-19 pandemic when in-person concerts and shopping were hindered.

Check Your Understanding

? Exercise 13.4.1

What is the difference between frequency and pitch of a note

? Exercise 13.4.2

Are higher frequencies associated with higher or lower pitch notes

? Exercise 13.4.3

What frequency range can humans hear

? Exercise 13.4.4

As a sound appears louder, does the decibel value increase or decrease

? Exercise 13.4.5

What is the lowest decibel value that most people can hear

? Exercise 13.4.6

What is the typical audible frequency range that adults can hear?

? Exercise 13.4.7

How does the frequency of a note change when increased by one octave?

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13.4.0: Exercises

For the following exercises, select the most representative decibel level for each sound.

Exercise 13.4.0.1

Gas lawn mower: 70 dB, 100 dB, 120 dB

Exercise 13.4.0.2

Radio playing loud enough to sing along to at home: 20 dB, 50 dB, 70 dB

Exercise 13.4.0.3

Office noise on a floor of people working at desks: 40 dB, 60, dB, 80 dB

Exercise 13.4.0.4

Conversation in a quiet room: 60 dB, 80 dB, 100 dB

Exercise 13.4.0.5

Fast-moving train passing by when sitting in a car about 20 ft from the tracks with windows down: 40 dB, 60 dB, 80 dB

For the following exercises, state the requested note.

Exercise 13.4.0.6

What is one half-step above $G^{\#}$?

Exercise 13.4.0.7

What is one whole step down from $G^{\flat}$?

Exercise 13.4.0.8

What is three half-steps above D?

Exercise 13.4.0.9

What is two whole steps down from $A^{\#}$?

Exercise 13.4.0.10

What is two half-steps down from E?

Exercise 13.4.0.11

What is an octave above B_4?

Exercise 13.4.0.12

What is two octaves below F_5?

Exercise 13.4.0.13

What is three octaves above G_2?

Exercise 13.4.0.14

What is an octave below G_2?

Exercise 13.4.0.15

Given that the frequency of $A_4 = 440\text{ Hz}$, what is the frequency of A_1 ?

Exercise 13.4.0.16

Given that the approximate frequency of $\mathrm{F}_4 = 349\text{ Hz}$, what is the approximate frequency of F_5, rounded to the nearest whole number?

Exercise 13.4.0.17

Given that the approximate frequency of $A_4\# = 233\text{ Hz}$, what is the approximate frequency of $A_6\#$, rounded to the nearest whole number?

Exercise 13.4.0.18

Given that the approximate frequency of $\mathrm{D}_4 = 294\text{ Hz}$, what is the approximate frequency of D_2, rounded to the nearest whole number?

Exercise 13.4.0.19

Given that the approximate frequency of $\mathrm{G}_4\# = 349\text{ Hz}$, what is the approximate frequency of $\mathrm{G}_3\#$, rounded to the nearest whole number?

Exercise 13.4.0.20

Given that the frequency of $A_3 = 220\text{ Hz}$, what is the frequency of A_5 ?

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13.5: Math and Sports

Learning Objectives

1. Describe why data analytics (statistics) is crucial to advance a team’s success.
2. Describe single round-robin method of tournaments.
3. Describe single-elimination method of tournaments.
4. Explore math in baseball, fantasy football, hockey, and soccer (projects at the end of the section).

Sports are big business and entertainment around the world. In the United States alone, the revenue from professional sports is projected to bring in over $77 billion, which includes admission ticket costs, merchandise, media coverage access rights, and advertising. So, whether or not you enjoy watching professional sports, you probably know someone who does. Some celebrities compete to be part of half-time shows and large companies vie for commercial spots that are costly but reach a staggering number of viewers, some who only watch the half-time shows and advertisements.

Figure 13.5.1: Fans support their team by attending games and wearing team gear. (credit: “Cheering Touchdowns” by Steven Miller/Flickr, CC BY 2.0)

Data Analytics (Statistics) Is Crucial to Advance a Team’s Success

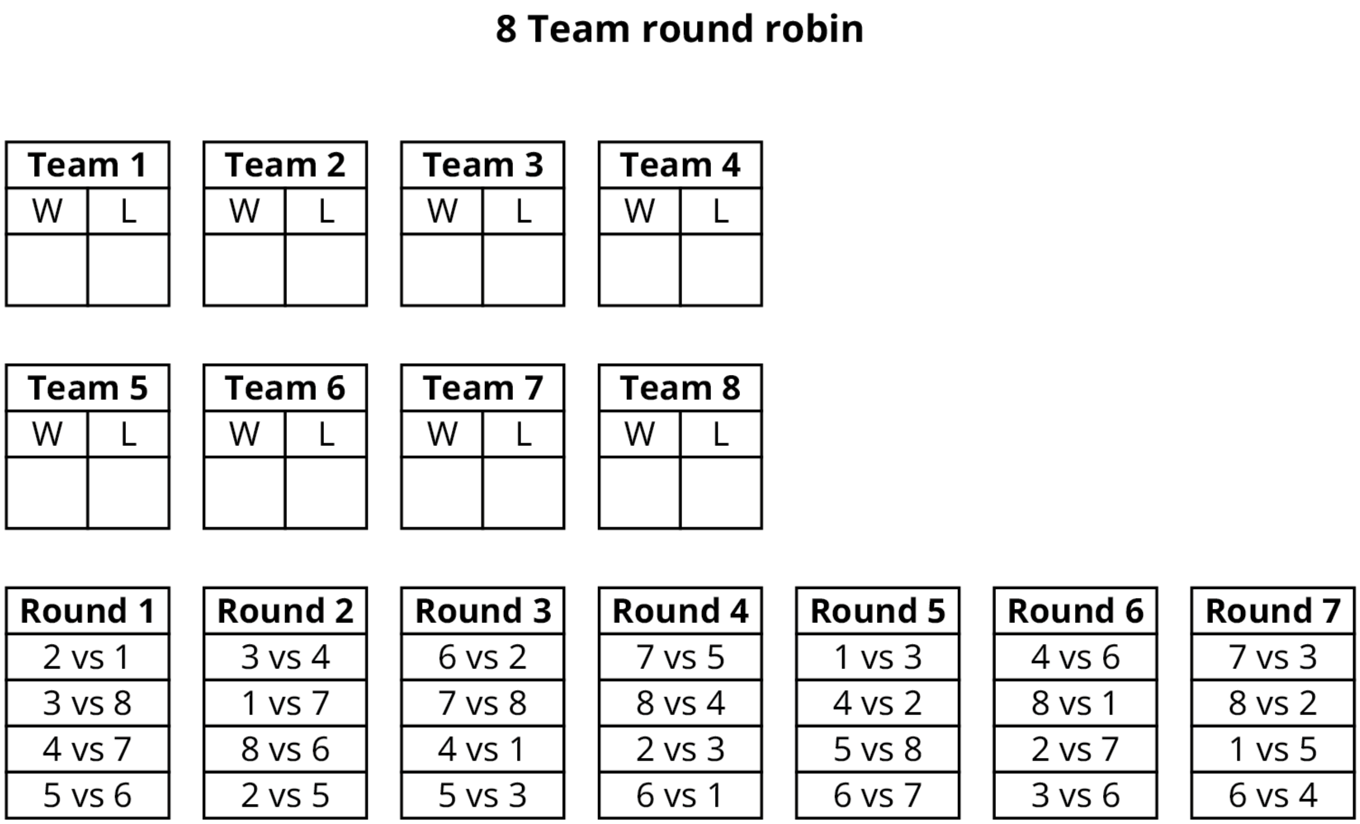
Analyzing the vast data that today’s world has amassed to find patterns and to make predictions for future results has created a degree field for data analytics at many colleges, which is in high demand in places that might surprise you. One such place is in sports, where being able to analyze the available data on your team’s players, potential recruits, opposing team strategies, and opposing players can be paramount to your team’s success.

Hollywood turned the notion of using data analytics into a major motion picture back in 2011 with the release of *Moneyball*, starring Brad Pitt, which grossed over $110 million. The critically acclaimed movie, based on a true story as shared in a book by Michael Lewis, follows the story of a general manager for the Oakland Athletics who used data analytics to take a team comprised of relatively unheard of players to ultimately win the American League West title in a year’s time. The win caught the eye of other team managers and owners, which started an avalanche of other teams digging into the data of players and teams.

In today’s world of sports, a team has multiple positions utilizing data analytics from road scouts who evaluate a potential recruit’s skills and potential to the ultimate position of general manager who is typically the highest-paid (non-player) employee with the exception of the coaches. Being able to understand and evaluate the available data is big business and is a highly sought after skill set. In college and professional sports, it is no longer sufficient to have a strong playbook and great players. The science to winning is in understanding the math of the data and using it to propel your team to excelling.

Single Round-Robin Tournaments

A common tournament style is single round-robin tournaments (Figure 13.5.2), where each team or opponent plays every other team or opponent, and the champion is determined by the team that wins the most games. Ties are possible and are resolved based on league rules.



8 Team round robin

Team 1		Team 2		Team 3		Team 4	
W	L	W	L	W	L	W	L

Team 5		Team 6		Team 7		Team 8	
W	L	W	L	W	L	W	L

Round 1	Round 2	Round 3	Round 4	Round 5	Round 6	Round 7
2 vs 1	3 vs 4	6 vs 2	7 vs 5	1 vs 3	4 vs 6	7 vs 3
3 vs 8	1 vs 7	7 vs 8	8 vs 4	4 vs 2	8 vs 1	8 vs 2
4 vs 7	8 vs 6	4 vs 1	2 vs 3	5 vs 8	2 vs 7	1 vs 5
5 vs 6	2 vs 5	5 vs 3	6 vs 1	6 vs 7	3 vs 6	6 vs 4

Figure $13.5.2$: Single-Round Robin Tournament

An advantage of the round-robin tournament style is that no one team has the advantage of seeding, which eliminates some teams from playing against each other based on rank of their prior performance. Rather, each team plays every other team, providing equal opportunity to triumph over each team. In this sense, round-robin tournaments are deemed the fairest tournament style.

One hindrance to employing a round-robin-style tournament is the potential for the number of games involved in tournament play to determine a winner. Determining the number of games can be found easily using a formula which, as we will see, can quickly grow in the number of games required for a single round-robin tournament.

FORMULA

The number of games in a single round-robin tournament with n teams is $n(n-1)/2$.

✓ Example $13.5.1$: Calculating the Number of Games in Single Round-Robin Tournaments

Find the number of games in a single round-robin tournament for each of the following numbers of teams:

1. 4 teams
2. 8 teams
3. 20 teams

Answer

1. Using the formula with 4 teams yields $4(4-1)/2=6$ tournament games.
2. Using the formula with 8 teams yields $8(8-1)/2=28$ tournament games.
3. Using the formula with 20 teams yields $20(20-1)/2=190$ tournament games.

Your Turn 13.5.1

Find the number of games in a single round-robin tournament for each of the following numbers of teams:

a. 5 teams
b. 12 teams
c. 25 teams

As the examples show, single round-robin tournament play can quickly grow in the number of games required to determine a champion. As such, some tournaments elect to employ variations of single round-robin tournament play as well as other tournament styles such as elimination tournaments.

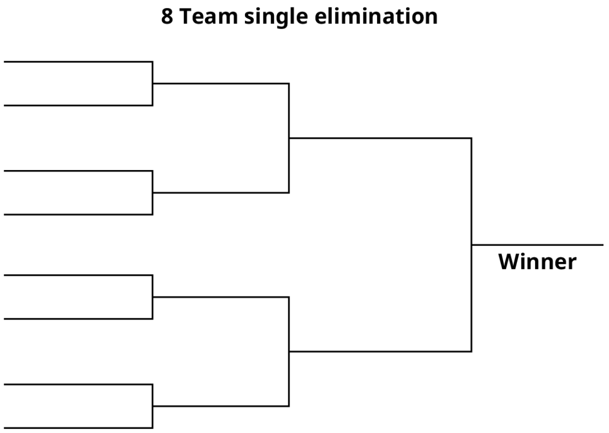


Figure 13.5.3: Single-Elimination Tournament (credit: final TK)

Single-Elimination Tournaments

When desiring a more efficient tournament style to determine a champion, one option is single-elimination tournaments (Figure 13.18), where teams are paired up and the winner advances to the next round of play. The losing team is defeated from tournament play and does not advance in the tournament, although some leagues offer consolation matches.

FORMULA

The number of games in a single-elimination tournament with n teams is $(n-1)$.

Example 13.5.2: Calculating the Number of Games in Single-Elimination Tournaments

Find the number of games in a single-elimination tournament for each of the following numbers of teams:

a. 4 teams
b. 8 teams
c. 20 teams

Answer

1. Using the formula with 4 teams yields $(4-1)=3$ tournament games.
2. Using the formula with 8 teams yields $(8-1)=7$ tournament games.
3. Using the formula with 20 teams yields $(20-1)=19$ tournament games.

Your Turn 13.5.2

Find the number of games in a single-elimination tournament for each of the following numbers of teams:

a. 5 teams
b. 12 teams
c. 25 teams

A single-elimination tournament offers an advantage over single round-robin tournament style of play in the number of games needed to complete the tournament. As you can see, in comparing the number of games in a single round-robin tournament in Example 13.22 with the number of games in single-elimination tournament as shown in Example $13.5.2$, the number of games required for single round-robin can quickly become unmanageable to schedule.

There are modifications to both the round-robin and elimination tournament styles such as double round-robin and double-elimination tournaments. Next time you observe a college or professional sporting event, see if you can determine the tournament style of play.

Who Knew?: Sports Popularity Shifts

Sport has been a popular entertainment venue for hundreds of years and the popularity of various sports shifts over time as well as in different regions of the world today. In today’s world, the most popular sport is, overwhelmingly, soccer, with over 4 billion fans followed by cricket with 2.5 billion fans. It may surprise you to learn that American football doesn’t rank in the top 10 most popular sports in the world today, yet table tennis ranks in sixth place and golf fills the last spot in the top 10 world sports.

In the early 1930s, baseball ranked a close second, with basketball virtually tying for third place. By the mid-1940s, hockey slightly led in first place over basketball. Ten years later, hockey remained in the number one sport, but cricket pushed basketball to third place. In the 1960s, soccer dominated in popularity. Over the next 30 years, basketball dropped in popularity and there was much movement in the popularity of sports. By the late 1990s, soccer swept to first place, where it has since continued to grow in popularity.

Check Your Understanding

1. How is data analytics used in sports today?
2. What is the most popular sport in today’s world?
3. What is the formula to compute the number of games played in a single round-robin tournament with n teams assuming single round?
4. What is the formula to compute the number of games played in a single-elimination tournament?

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13.5.0: Exercises

? Exercise 13.5.0.1

Briefly describe data analytics.

? Exercise 13.5.0.2

Name a sport’s career title that relies on analysis of data.

For the following exercises, compute how many games would be played in the style of tournament and number of teams given in each question. Assume all tournaments are single round-robin or single-elimination.

? Exercise 13.5.0.3

How many games would be played with 4 teams using a round-robin tournament?

? Exercise 13.5.0.4

How many games would be played with 8 teams using a round-robin tournament?

? Exercise 13.5.0.5

How many games would be played with 10 teams using a round-robin tournament?

? Exercise 13.5.0.6

How many games would be played with 25 teams using a round-robin tournament?

? Exercise 13.5.0.7

How many games would be played with 12 teams using a round-robin tournament?

? Exercise 13.5.0.8

How many games would be played with 4 teams using a single-elimination tournament?

? Exercise 13.5.0.9

How many games would be played with 9 teams using a single-elimination tournament?

? Exercise 13.5.0.10

How many games would be played with 15 teams using a single-elimination tournament?

? Exercise 13.5.0.11

How many games would be played with 50 teams using a single-elimination tournament?

? Exercise 13.5.0.12

How many games would be played with 24 teams using a single-elimination tournament?

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SECTION OVERVIEW

13.6: Chapter Summary

13.6.0: Key Terms

13.6.1: Key Concepts

13.6.2: Formula Review

13.6.3: Projects

13.6.4: Chapter Review

13.6.5: Chapter Test

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13.6.0: Key Terms

13.1 Math and Art

- golden ratio
- ϕ
- Fibonacci sequence
- golden rectangle

13.2 Math and the Environment

- greenhouse gas
- CO_2 emissions
- watt
- kilowatt (kW)

13.3 Math and Medicine

- concentrations/dosages of drugs
- mathematical modeling

13.4 Math and Music

- frequency
- pitch
- Hertz
- decibel
- half-step
- whole step
- flat
- sharp
- octave

13.5 Math and Sports

- data analytics
- seeding

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13.6.1: Key Concepts

13.1 Math and Art

- The golden ratio, ϕ, can be found in nature, and the relationship is often associated with beauty and balance.
- The Fibonacci sequence reflects a pattern of numbers that can be found in various places in nature. The sequence can be used to predict other values that follow the Fibonacci pattern.
- State some naturally occurring applications of the Fibonacci sequence.
- State some naturally occurring applications of the golden ratio.
- Determine if a rectangle is golden.
- State some artistic applications of the golden rectangle.

13.2 Math and the Environment

- Recognize how water conservation by one person, family, community, or nation can positively impact the world’s freshwater supply.
- Recall components from the history of solar use by mankind.
- Calculate electrical demand given watts.
- Recognize advantages of residential solar power.

13.3 Math and Medicine

- Compute volumes of prescription drugs in liquid and pill form.
- Validate the effectiveness of a new drug.
- Mathematical modeling can be used to describe and track the spread of a virus.

13.4 Math and Music

- As frequency of a sound increases, the pitch of the sound increases.
- Hertz is a unit of measurement for frequency.
- Decibel is a unit of measurement for the intensity of sound.
- Half-steps and whole-steps describe one type of movement between two notes on a keyboard.
- Octaves are a collection of any 12 consecutive notes, which is a foundation in music theory.

13.5 Math and Sports

- Describe how a round-robin tournament is organized.
- Compute the number of games played in a round-robin tournament.
- Describe how a single-elimination tournament is organized.
- Compute the number of games played in a single-elimination tournament.

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13.6.2: Formula Review

13.2 Math and the Environment

$1{,}000/\text{watts} = \text{time needed to}$

$\text{watts}/1,000 = \text{kilowatt}$

13.3 Math and Medicine

$\text{Volume needed} = (\text{medicine dosage required})\ (\text{weight of drug by volume}).$

13.5 Math and Sports

Number of games in a single round-robin tournament with n teams is $n(n-1)/2$.

Number of games in a single-elimination tournament with n teams is $(n-1)$.

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13.6.3: Projects

Lucas Sequence and Fibonacci Sequence

The Lucas numbers bear some similarity to the Fibonacci numbers and exhibit a stronger link to the golden ratio. Edouard Lucas is credited with naming the Fibonacci numbers and the Lucas numbers were so named in his honor. The Lucas numbers play a role in finding prime numbers that are utilized in encrypting data for actions such as using your debit card to obtain money at a cash machine or when making a credit card purchase for point of sale as well as when shopping online.

Complete the following questions to explore numbers in the Lucas sequence as well as their relationships to the numbers in the Fibonacci sequence.

1. Conduct an Internet search to find out what a Lucas number is and how the Lucas numbers are related to the Fibonacci numbers.
2. What are the first two numbers in the Lucas sequence?
3. Describe how the next number in the Lucas sequence is determined and compare this to how the next number is determined in the Fibonacci sequence.
4. Complete the following table listing the first 10 terms in the Fibonacci and Lucas sequence:

Term	0	1	2	3	4	5	6	7	8	9
Fibonacci Numbers	0	1	1	2		5		13		34
Lucas Numbers			3		7		18		47	

- Interestingly, many patterns can be found in looking at the relationships in the Fibonacci and Lucas numbers. Look closely at the chart in question 3 to discover one such pattern. Observe the Fibonacci numbers in the third and fifth terms and compare with the Lucas number in the fourth term of the sequence. Describe the pattern found. Does this pattern continue in the table?
- Research the Fibonacci or the Lucas numbers to find an application in our world distinct from what has been shared in this project and section. Write a paragraph sharing the findings of your research.

Solar Array for a Residence

One of the first steps in adding solar to a residence is determining the size of a system to achieve the desired output. In this project, we will explore the solar needs of a residence and estimate needs of a solar array to supply electrical output to meet various percentages of electrical need.

Step 1: Obtain an electric bill from your apartment/home. Find the average monthly or yearly usage if listed or call the electric company to inquire. If an electric bill is not available, use the Internet to find an average monthly or yearly electric usage for your area.

Step 2: Determine a daily and hourly usage. Divide the average monthly usage by 30 or the yearly average by 365. Divide again by 24 to calculate an average hourly electric usage, which will yield the average kilowatt-hours for how much electrical power your is being utilized in an hour.

Step 3: Multiply your average hourly use (kilowatts) by 1,000 to convert to watts.

Step 4: Use the Internet to determine the average daily peak hours of sunlight where you live.

Step 5: Divide your average hourly watts (Step 3) by the average daily peak hours (Step 4) to calculate the average energy needed for a solar array to produce every hour.

Step 6: Determine the average energy needed in a solar array per hour to meet each of the following:

a. 100% coverage of average energy (Step 5)

b. 80% coverage of average energy
c. 50% coverage of average energy

Step 7: Using the values computed in Step 6, compute the residential savings based on an average cost of 12 cents per watt.

Vaccine Validation

Validation of vaccines is a topic that exploded in the news when the Covid-19 pandemic spread across the world. As governments and organizations looked for a vaccine to curb the spread and minimize the severity of infection, concern was expressed by some for what appeared to be a quick discovery for a Covid-19 vaccine.

Conduct an Internet search to explore the following questions. Pay special attention to the sources you select to ensure that they are credible sources.

1. Research the term *efficacy rates*. Express what efficacy means in your own words.
2. Using a minimum of two sources, compose a well-developed paragraph describing what validation of a vaccine means.
3. Using a minimum of two sources, compose a well-developed paragraph sharing the steps in validating a vaccine.
4. Using a minimum of two sources, compose a well-developed paragraph describing how a vaccine is determined to be validated.
5. Using your research for Questions 1–3, write a summary paragraph sharing your reflections on the validation of the Covid-19 vaccine or on validating a vaccine in general. What key components did you learn? What would you like to learn more about related to validating a vaccine?

Frequency and Ultrasonic Sounds

Ultrasonic sounds have been utilized for a variety of reasons, from purportedly repelling rodents and other animals as well as a variety of other applications. Using an Internet search, complete the following questions to explore some of these applications and examine the validity of various claims.

Repelling Insects, Rodents, and Small Animals

Some radio stations purport to play a high pitch sound dually with their music to aid in deterring insects and other annoying bugs to aid in providing a bug-reduced listening environment. To deter small rodents, some products claim to emit ultrasonic sounds that drive away mice and other similar pests.

1. Research the science behind ultrasonic pest controls, paying attention to the source of the information that you find. Compare the information found on advertisements, reviews, and scientific articles.
    a. What frequency ranges do ultrasonic pest deterrent devices utilize and how do these frequencies compare to the audible range that humans can hear?
    b. Write a short paragraph comparing the claims in the advertisements with independent reviews and scientific articles.
    c. What do you conclude about ultrasonic pest controls and why?

Disbursing Teenagers

Some business owners and communities have turned to products such as the “mosquito” sonic deterrent device to discourage groups of teenagers from loitering around storefronts and community landmarks, citing a public nuisance issue and public safety concerns.

2. Research the science behind ultrasonic deterrent devices as they apply to dispensing teenagers. Compare the information found on advertisements, reviews, and scientific articles.
    a. What frequency ranges do ultrasonic teenager deterrent devices utilize and how do these frequencies compare to the audible range that adults can hear?
    b. Write a short paragraph comparing the claims in advertisements with independent reviews and scientific articles.
    c. What are some of the ethical debates surrounding the use of ultrasonic teenager deterrent devices?
    d. What do you conclude about business owners or communities using ultrasonic teenager deterrent devices and why?

Jewelry Cleaner

Use of pastes and liquid chemicals to clean jewelry can be harsh on stones as well as metals. So how can we safely obtain the sparkling clean look at home that jewelry stores provide? Some would say the answer is to use an ultrasonic jewelry cleaner, but do these really work?

3. Research the science behind ultrasonic jewelry cleaners, including the phrase “cavitation process” in your search. Compare the information found on advertisements, reviews, and scientific articles.
    a. Write a short paragraph detailing how ultrasonic jewelry cleaners work and what role the cavitation process plays in the claims for ultrasonic cleaning.
    b. Do ultrasonic jewelry cleaners utilize low or high frequencies? How do these frequencies compare to the audible range that humans can hear?
    c. Write a short paragraph comparing the claims in the advertisements with independent reviews and scientific articles.
    d. What do you conclude about ultrasonic jewelry cleaners and why?

Specialized Ringtones

As the use of cell phones has become commonplace and families grow towards each member having their own cell phone, specialized ringtones have become popular and can aid in identifying who is calling just by the ringtone.

Ever hear of ringtones that can be heard by teens but often not their teachers? The banning of cell phone use by K–12 students during class time as been implemented across a wide array of schools and some students have purportedly found ways to get around teachers hearing a cell phone ring through the use of ultrasonic ringtones.

4. Research the science behind ultrasonic ringtones. Compare the information found on advertisements, reviews, and scientific articles.
    a. Do ultrasonic ringtones utilize low or high frequencies?
    b. Write a short paragraph detailing how ultrasonic ringtones work. How do these frequencies compare to the audible range that adults can hear? Include ages that are purported to hear and not hear the ringtones as well as the frequency ranges utilized.
    c. Write a short paragraph comparing the claims in the advertisements with independent reviews and scientific articles.
    d. What do you conclude about ultrasonic ringtones and why?

Streaming Services and Math

With the ability to stream music virtually anywhere you are, it is not surprising that Google Play Music, Apple Music, and a slew of other companies such as Spotify, Amazon Music, YouTube Music, Sound Cloud, Pandora, Deezer Music, Tidal, Napster, and Bandcamp have invested heavily to bring streaming service to users worldwide. Streaming services have expanded to offer virtually every genre of music with vast libraries to meet diverse user requests.

Considering all of the choices available for streaming music, there is a wide array of options for subscribing. Conduct an Internet research to review your current streaming choices, if any, and evaluate competitors’ products.

1. In this project, you will explore options for music streaming service subscriptions. Select a minimum of five streaming services listed above that you are not currently utilizing and determine the below components. Format your findings in an easy-to-read format such as a table similar to the one shown below.
    a. Monthly subscription cost
    b. Available features

Available Features	Service Choice 1:	Service Choice 2:	Service Choice 3:	Service Choice 4:	Service Choice 5:
Able to stream unlimited music					
Able to purchase individual songs					
Able to purchase individual whole albums					
Ability to create personalized play lists					

Available Features	Service Choice 1:	Service Choice 2:	Service Choice 3:	Service Choice 4:	Service Choice 5:
Ability to select specific songs to play					
Ability to listen					
Other =					

2. Premium cost per month, if available = ______
   a. Feature(s) offered with premium monthly subscription = ______
3. What is (are) your current music streaming services, if any?
   a. What is your current monthly service charge(s)?
   b. What features does your current streaming service offer?
4. Write a paragraph summarizing your findings and include if your current streaming service(s) meets your needs or if research for this project has you considering changing streaming services. Support your rationale.

Math and Baseball

Baseball is known to have one of the largest pools of statistics related to the game and its players. Managers, coaches, and pitchers study the statistics of the players on opposing teams to give their team an edge by knowing what pitches to throw for the best probability to be missed by a batter. In similar fashion, batters study pitchers' statistics to learn a pitcher's strength and how to predict what a pitcher will throw and how to best hit against a pitcher.

The three primary baseball statistics are batting average, home runs, and runs batted in (RBIs), which are the components of the title of Triple Crown winner that is awarded to players who dominate in these three areas. However, there is a wealth of other statistics to evaluate when studying the performance of a player.

Conduct an Internet search to research statistics and how they are calculated in the following categories:

Batting Statistics

1. There are about 30 batting statistics. Select a minimum of 10 batting statistics. Compose an organized list including the name of the statistic, abbreviation, explanation of what it represents, as well as how it is calculated.
   As an example: AB/HR represents *at bats per home run* and is calculated by the number of times a player is at bat divided by home runs.

Pitching Statistics

2. There are about 40 pitching statistics. Select a minimum of 10 pitching statistics. Compose an organized list including the name of the statistic, abbreviation, explanation of what it represents, as well as how it is calculated.
   As an example: K/9 represents *strikeouts per nine innings* and is calculated by the number of strikeouts times nine divided by the number of innings pitched.

Fielding Statistics

3. There are around 10 fielding statistics. Select a minimum of five fielding statistics. Compose an organized list including the name of the statistic, abbreviation, explanation of what it represents, as well as how it is calculated.
   As an example: FP represents *fielding percentage* is calculated by the number of total plays divided by the number of total chances.

Overall

4. Select a player from recent years to evaluate. The player can be one that you have followed, one from a favorite team, or any current player. Find the statistics shared in Questions 1–3 to use in evaluating the potential strengths and weaknesses of the player. Write a short paragraph analyzing your selected player, supported by the statistics from the answers to Questions 1–3.

Math and Fantasy Football

Fantasy football offers spectators an added dimension to football season with a competitive math-based game where the active components are real-life players in the current season. For clarity in this exercise, the actual fantasy football players will be denoted as FFP and actual professional team members will be denoted as players.

While some fantasy football leagues have slightly different setups or scoring systems, most share some common elements.

Often using a lottery system to determine who picks first, second, and so on, FFPs select 15 current players to comprise their personal fantasy football team. The players selected can be from any professional teams and a FFP can utilize any team recognized by their league. FFPs can elect to keep the same players on their team for the whole league play or trade for any player not selected by another FFP in their league.

At the start of each week during football season, each fantasy football player selects their roster of actual players to comprise their roster of starting players. Typically, a starting roster consists of the following players:

Number to Select	Position Abbreviation	Position Title
1	QB	Quarterback
1	K	Kicker
1	TE	Tight End
2	RB	Running Back
1	D/ST	Defense
2	WR	Wide receiver
1	RB or WR	Flex

As actual professional games are played, points are tallied based on your league's scoring system. The points the team members on your starting roster make during the week are computed and whichever FFP has the highest score for the week wins that week.

The FFPs with the best records of wins versus losses enters fantasy football playoffs to determine the ultimate league champion and collects the pot.

The above overview of fantasy football describes the basic game play. The fun comes in understanding and analyzing the math behind the scoring.

1. Talk to people you know who have played fantasy football in the past and interview them. Inquire about how they select players and how league(s) they have played in have computed the weekly scoring.
   a. Was a league manager recruited to manage the record keeping of scores?
   b. Was an online scoring system utilized or did the fantasy football use a streamlined or specialized scoring system?
   c. What strategies were used to select the team of 15 players and who would be on the weekly starting roster?
   d. Were any online resources utilized to aid in choosing which players to select?
   e. How was math utilized to select team members or who to place as a starter each week? Be as specific as possible, citing examples when available.
2. Conduct an Internet research to find two different resources that a FFP could utilize when selecting players for their team or for their starting roster. Describe the math involved in the resources.
3. Conduct an Internet research to learn more about the scoring utilized by fantasy football leagues. Write a short paragraph sharing how the use of math and knowledge of football statistics is used and how they are calculated to aid a FFP in selecting their team and starting roster.

Math and Hockey

Hockey is full of math from obvious components such as scoring and statistics to the shape of the rink and the angles involved in puck movement.

Collegiate and professional hockey games are 60 minutes long and are divided into three periods of 20 (60/3) minutes each. At any one time, there are five players and one goalie on the ice for each team. If a player is called on a penalty and is placed in the penalty box, that team now has four players, which is 20% less players on the ice competing against five opponents. In some instances, a team may have two players in the penalty box at one time, resulting in three players, or 40% less players on the ice compared to a full team. Being one player down for a 2-minute penalty or potentially 5 minutes for a major penalty leads to an imbalance on the ice and calls for a quick change of offense and defense strategy.

Rink Composition—North American

An ice rink is comprised of various geometrical shapes, each with precise dimensions.

1. Research the dimensions of a hockey rink and draw an accurate scale model on graph paper. Be sure to include the scale for your model and indicate all units.
2. List the shapes and the numbers of each kind of hockey rink. You should find four different shapes, some with multiple dimensions.
3. Describe the shape and dimensions of a hockey puck using geometric vocabulary.

Statistics

Using an Internet search, select two top hockey players from the same league to answer the following questions:

4. Write a paragraph sharing a minimum of five statistics for each player you have selected. Describe how each of the statistics are calculated and what each statistic means.
5. Write a paragraph comparing the two players and determine who you believe is the better player. Support your choice.

Scoring

The basics of scoring in ice hockey is simple, the team with the most goals is the winner. But, how to score the most goals involves much math!

6. Research one of the following components involved in hockey puck movement on the ice and write a paragraph summarizing your findings. Be specific and detailed in your summary.
   a. Angles
   b. Velocity vectors
   c. Angle of incident and angle of return
   d. Speed and acceleration (player as well as puck movement)

Math and Soccer

As the world's most popular sport, you'll be excited to confirm that soccer is full of mathematics ranging from scoring and statistics to footwork, angles, and field shape.

Soccer requires understanding of mathematical concepts and equations as well as skill, fitness, and game knowledge. One such example is angles, which you all will remember from your geometry class. While players are not carrying protractors and measuring angles during play, mental calculation of angles is a constant in any successful player's thinking. A goalie is not physically able to cover the entire open net region and a player must calculate an angle to kick the ball consistent with the net opening while predicting the ability of the goalie to stop the ball from entering the net.

1. Research angles as they apply to soccer play. Provide two examples of different plays indicating angle of a player's body, angle of foot striking the ball, and angle to the net. Include relevant dimensions. Adding a diagram that may aid in clarity is an option.
2. Draw a scale model of a soccer field including dimensions with labels.

Using an Internet search, select two top soccer players from the same league to answer the following questions:

3. Write a paragraph sharing a minimum of five statistics for each player you have selected. Describe how each of the statistics are calculated and what each statistic means.
4. Write a paragraph comparing the two players and determine who you believe is the better player. Support your choice.

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13.6.4: Chapter Review

Section 13.1 Math and Art

? Exercise 13.6.4.1

A person's length from their belly button to the floor is 25 in. What is the person's approximate height rounded to the nearest inch?

? Exercise 13.6.4.2

The spiral seeds on a fruit mirror the Fibonacci sequence. If a row of the spiral contains 34 leaves, how many leaves would be found on the next larger spiral of seeds?

? Exercise 13.6.4.3

The seeds on a sunflower mirror the Fibonacci sequence. If a circular layer on the sunflower contains 89 seeds, approximately how many seeds would be found on the next larger circular layer?

? Exercise 13.6.4.4

A term of the Fibonacci sequence is 17,711 and the previous term is 10,946. What is the following term of the sequence?

? Exercise 13.6.4.5

The 35th term of the Fibonacci sequence is 9,227,465. Use the approximate value of ϕ of 1.618 to estimate the 34th term. Round your answer to the nearest whole number.

? Exercise 13.6.4.6

A frame has dimensions of 16 in by 24 in. Calculate the ratio of the sides rounded to the nearest tenth and determine if the size approximates a golden rectangle.

Section 13.2 Math and the Environment

? Exercise 13.6.4.7

An average toilet uses 5 gal of water per flush. A high-efficiency toilet uses about 1.25 gal per flush. How much water would a household save in a month (30 days) if the toilet was flushed 6 times a day?

? Exercise 13.6.4.8

When washing dishes, leaving the faucet running utilizes about 15 gal every 5 minutes, whereas filling the sink and turning the faucet off except to rinse the dishes uses about 6 gal for washing a dishpan-sized load of dishes. How much water is saved by not leaving the faucet running if it takes 10 minutes to wash a dishpan-sized load of dishes each day for a year containing 30 days in each month?

? Exercise 13.6.4.9

Leaving the water running when washing your hands consumes about 4 gal of water, whereas turning the water off when lathering reduces the water used to 1 gal. How much water is saved in an apartment of 4 people in 6 months (180 days) if each person averages washing their hands twice a day?

? Exercise 13.6.4.10

How long would it take for an energy-saving light bulb to consume 1 kW of power if the bulb is rated at 2.5 W? Round your answer to the nearest hour.

? Exercise 13.6.4.11

How long would it take for a 60 W light bulb to consume 1 kW of power? Round your answer to the nearest hour.

? Exercise 13.6.4.12

A portable fan uses about 40 W. How many kilowatt-hours are needed to run the fan during a 10-day trip if the fan is run for an average of 3.5 hours a day?

Section 13.3 Math and Medicine

? Exercise 13.6.4.13

In a 2% solution of lidocaine, how many milligrams are there per milliliter?

? Exercise 13.6.4.14

You are prescribed a nebulizer solution to aid with asthma. If the solution is 0.7% w/v, what volume is needed to deliver a 7 mg dose?

? Exercise 13.6.4.15

A prescription calls for a patient to receive 17 mg of a drug to be taken in oral form. If the pills are available in 3.4 mg, how many pills will the patient need daily?

? Exercise 13.6.4.16

A prescription calls for a patient to receive 9 mg daily of a drug to be taken daily for 10 days. If the pills are available in 2.25 mg, how many pills are needed to fill the prescription?

? Exercise 13.6.4.17

A patient is prescribed 2 mg/kg of a drug to be delivered daily intramuscularly, divided into three doses. If the patient weighs 30 kg, how many milligrams of the drug is needed for each dose?

? Exercise 13.6.4.18

A patient is prescribed 0.75 mg/kg of a drug to be delivered intramuscularly once per day. If the drug is available in 2.5 mg/mL and the patient weighs 63 kg, how many milliliters of the drug should be given per day?

Section 13.4 Math and Music

? Exercise 13.6.4.19

Select the most representative sound registering 50 dB: airport gate noise, light rain on a car, dirt bike racing.

? Exercise 13.6.4.20

Select the most representative sound registering 100 dB: city traffic when in a car with windows rolled up, children playing on a school playground, professional sporting event such as football.

? Exercise 13.6.4.21

What frequency range can teenagers typically hear but adults over 20 are often not able to hear?

? Exercise 13.6.4.22

What is three whole steps up from $G^{\#}$?

? Exercise 13.6.4.23

What is one half-step down from F?

? Exercise 13.6.4.24

What is an octave above D_3?

? Exercise 13.6.4.25

Given that the frequency of $\mathrm{A}_4 = 440\ \mathrm{Hz}$, what is the frequency of A_6 ?

Section 13.5 Math and Sports

For the following exercises, compute how many games would be played in the style of tournament and number of teams given in each question. Assume all tournaments are single round-round or single-elimination.

? Exercise 13.6.4.26

How many games would be played with 10 teams using a round-robin tournament?

? Exercise 13.6.4.27

How many games would be played with 25 teams using a round-robin tournament?

? Exercise 13.6.4.28

How many games would be played with 12 teams using a round-robin tournament?

? Exercise 13.6.4.29

How many games would be played with 4 teams using a single-elimination tournament?

? Exercise 13.6.4.30

How many games would be played with 9 teams using a single-elimination tournament?

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13.6.5: Chapter Test

? Exercise 13.6.5.1

A person’s length from their belly button to the floor is 3 ft 8 in, assuming the ratio is golden. What is the person’s approximate height rounded to the nearest inch?

? Exercise 13.6.5.2

The leaves on an invasive weed mirror the Fibonacci sequence. If a set of leaves on the plant contains 13 leaves, how many leaves would be found on the previous smaller set of leaves?

? Exercise 13.6.5.3

The 14th term of the Fibonacci sequence is 377 and the 15th term is 610. What is the 13th term of the sequence?

? Exercise 13.6.5.4

A fence has dimensions of 40 in by 25 in. Calculate the ratio of the sides rounded to the nearest tenth and determine if the size approximates a golden rectangle.

? Exercise 13.6.5.5

An average toilet uses 5 gal of water per flush. A high-efficiency toilet uses about 1.25 gal per flush. How much water would a dorm save in an academic year (32 weeks) if the toilet was flushed 22 times a day?

? Exercise 13.6.5.6

When washing dishes, leaving the faucet running utilizes about 15 gal every 5 minutes, whereas filling the sink and turning the faucet off except to rinse the dishes uses about 5 gal for washing a dishpan-sized load of dishes. How much water is saved by not leaving the faucet running if it takes 10 minutes to wash a dishpan-sized load of dishes each day for a week?

? Exercise 13.6.5.7

How long would it take for a 75 W light bulb to consume 1 kW of power? Round your answer to the nearest hour.

? Exercise 13.6.5.8

A phone charger uses about 20 W. How many kilowatt-hours are needed to use the charger during a month if the charger is used for an average of 1.5 hours a day?

? Exercise 13.6.5.9

A pharmacy has a 1 L bottle of 0.9% solution of sodium chloride. How many grams of sodium chloride are contained in the bottle?

? Exercise 13.6.5.10

A prescription calls for a patient to receive 5.1 mg of a drug to be taken in oral form. If the pills are available in 1.7 mg, how many pills will the patient need daily?

? Exercise 13.6.5.11

A prescription calls for a patient to receive 7 mg daily of a drug to be taken daily for 7 days. If the pills are available in 1.75 mg, how many pills are needed to fill the prescription?

? Exercise 13.6.5.12

A patient is prescribed 3.8 mg/kg of a drug to be delivered daily intramuscularly, divided into two doses. If the patient weighs 20 kg, how many milligrams of the drug are needed for each dose?

? Exercise 13.6.5.13

Select the most representative decibel value for a college classroom with students working in groups: 10 dB, 30 dB, 50 dB.

? Exercise 13.6.5.14

What is typically the top frequency that adults over 20 can hear: 17 Hz, 170 Hz, 1,700 Hz, 17,000 Hz?

? Exercise 13.6.5.15

What note is three whole steps down from $F^{\#}$?

? Exercise 13.6.5.16

Given that the approximate frequency of $\mathrm{C_4}^{\#} = 277\ \mathrm{Hz}$, what is the approximate frequency of $\mathrm{C_2}^{\#}$ rounded to the nearest whole number?

For the following exercises, compute how many games would be played in the style of tournament and number of teams given in each question. Assume all tournaments are single round-robin or single-elimination.

? Exercise 13.6.5.17

How many games would be played with 22 teams using a round-robin tournament?

? Exercise 13.6.5.18

How many games would be played with 7 teams using a round-robin tournament?

? Exercise 13.6.5.19

How many games would be played with 8 teams using a single-elimination tournament?

? Exercise 13.6.5.20

How many games would be played with 17 teams using a single-elimination tournament?

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CHAPTER OVERVIEW

14: Appendix

14.0: Integer Powers of 10

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14.0: Integer Powers of 10

Nonnegative Integer Powers of 10

The phrase **nonnegative integers** refers to the set containing 0, 1, 2, 3, … and so on. In the expression 10^5, 10 is called the **base**, and 5 is called the **exponent**, or **power**. The exponent 5 is telling us to multiply the base 10 by itself 5 times. So, $10^5 = 10 \times 10 \times 10 \times 10 \times 10 = 100{,}000$. By definition, any number raised to the 0 power is 1. So, $10^0 = 1$.

In the following table, there are several nonnegative integer powers of 10 that have been written as a product. Notice that higher exponents result in larger products. What do you notice about the number of zeros in the resulting product?

Exponential Form	Product	Number of Zeros in Product
10^0	1	0
10^1	10	1
10^2	$10 \times 10 = 100$	2
10^3	$10 \times 10 \times 10 = 1{,}000$	3
10^4	$10 \times 10 \times 10 \times 10 = 10{,}000$	4
10^5	$10 \times 10 \times 10 \times 10 \times 10 = 100{,}000$	5

That’s right! The number of zeros is the same as the power each time!

Negative Integer Powers of 10

The **reciprocal** of a number is 1 divided by that number. For example, the reciprocal of 10 is $\frac{1}{10}$. We use negative exponents to indicate a reciprocal. For example, $10^{-1} = \frac{1}{10^1} = \frac{1}{10}$. Similarly, any expression with a negative exponent can be written with a positive exponent by taking the reciprocal. Several negative powers of 10 have been simplified in the table that follows. What do you notice about the number of zeros in the denominator (bottom) of each fraction?

Exponential Form	Equivalent Simplified Expression	Number of Zeros in Denominator
10^{-1}	$\frac{1}{10^1} = \frac{1}{10}$	1
10^{-2}	$\frac{1}{10^2} = \frac{1}{10 \times 10} = \frac{1}{100}$	2
10^{-3}	$\frac{1}{10^3} = \frac{1}{10 \times 10 \times 10} = \frac{1}{1{,}000}$	3
10^{-4}	$\frac{1}{10^4} = \frac{1}{10 \times 10 \times 10 \times 10} = \frac{1}{10{,}000}$	4

That’s right! The number of zeros is the same as the positive version of the power each time.

In the following table, we will write the same powers of 10 as decimals. Count the number of decimal places to the right of the decimal point. What do you notice?

Exponential Form	Equivalent Simplified Expression	Number of Decimal Places to Right of Decimal
10^{-1}	$\frac{1}{10^1} = 1 \div 10 = 0.1$	1
10^{-2}	$\frac{1}{10^2} = 1 \div 100 = 0.01$	2
10^{-3}	$\frac{1}{10^3} = 1 \div 1{,}000 = 0.001$	3
10^{-4}	$\frac{1}{10^4} = 1 \div 10{,}000 = 0.0001$	4

That’s right! The number of decimal places to the right of the decimal point is the same as the positive version of the power each time.

Multiplying Integers by Positive Powers of 10

Did you know that the distance from the sun to Earth is over 90 million miles? This value can be represented as 90,000,000, or we can write it as a product: $9 \times 10{,}000{,}000 = 9 \times 10^7$, which is actually a more compact way of writing 90 million. Notice that the power of 7 reflects the number of zeros in 90 million. Several products of positive integers and powers of 10 are given in the table that follows. Notice that the number of zeros is the same as the exponent except in one case.

Exponential Form	Product	Number of Zeros in Product
5×10^1	$5 \times 10 = 50$	1
13×10^2	$13 \times 100 = 1{,}300$	2
8×10^3	$8 \times 1{,}000 = 8{,}000$	3
15×10^4	$15 \times 10{,}000 = 150{,}000$	4
70×10^5	$70 \times 100{,}000 = 7{,}000{,}000$	6

The only case in which the number of zeros didn't equal the exponent was the last case. Why do you think that happened? That's right! We multiplied by 70 which also had a zero. So, the product had a zero from the 70 and 5 zeros from 10^5 for a total of 6 zeros in 7,000,000.

Multiplying by Negative Powers of 10

As we have seen, negative powers of 10 are decimals. Several products of positive integers and powers of 10 are given in the table below. Notice that multiplying an integer by 10 raised to a negative integer power results in a smaller number than you started with. Also, the number of decimal places to the right of the decimal point is the same as the exponent except in one case.

Exponential Form	Product	Number of Decimal Places to Right of Decimal
3×10^{-1}	$3 \times 0.1 = 0.3$	1
13×10^{-2}	$13 \times 0.01 = 0.13$	2
9×10^{-3}	$9 \times 0.001 = 0.009$	3
15×10^{-4}	$15 \times 0.0001 = 0.0015$	4
70×10^{-5}	$70 \times 0.00001 = 0.00070 \text{ or } 0.0007$	5 (6 if we leave on the extra 0)

The only case in which the number of decimal places to the right of the decimal point didn't equal the positive version of the exponent was the last case. Why do you think that happened? That's right! We multiplied by 70, which ended in zero.

Moving the Decimal Place

A helpful shortcut when multiplying a number by a power of 10 is to "move the decimal point." The following table shows several powers of 10, both positive and negative. Compare the location of the decimal point in the original number to the location of the decimal point in the product. How has it changed?

Exponential Form	Product	How the Position of the Decimal Point Changed
5×10^1	$5. \times 10 = 5_{\wedge}\underset{\smile}{0}. = 50$	1 place to the right
13×10^2	$13. \times 100 = 13_{\wedge}\underset{\smile}{0}\underset{\smile}{0}. = 1{,}300$	2 places to the right
8×10^3	$8. \times 1{,}000 = 8_{\wedge}\underset{\smile}{0}\underset{\smile}{0}\underset{\smile}{0}. = 8{,}000$	3 places to the right
15×10^4	$15. \times 10000 = 15_{\wedge}\underset{\smile}{0}\underset{\smile}{0}\underset{\smile}{0}\underset{\smile}{0}. = 150{,}000$	4 places to the right
70×10^5	$70. \times 100{,}000 = 70_{\wedge}\underset{\smile}{0}\underset{\smile}{0}\underset{\smile}{0}\underset{\smile}{0}\underset{\smile}{0}. = 7{,}000{,}000$	5 places to the right

Exponential Form	Product	How the Position of the Decimal Point Changed
3×10^{-1}	$3. \times 0.1 = .\underset{\smile}{3}{}_{\wedge} = 0.3$	1 place to the left
13×10^{-2}	$13. \times 0.01 = .\underset{\smile}{1}\underset{\smile}{3}{}_{\wedge} = 0.13$	2 places to the left
9×10^{-3}	$9. \times 0.001 = .\underset{\smile}{0}\underset{\smile}{0}\underset{\smile}{9}{}_{\wedge} = 0.009$	3 places to the left
15×10^{-4}	$15. \times 0.0001 = .\underset{\smile}{0}\underset{\smile}{0}\underset{\smile}{1}\underset{\smile}{5}{}_{\wedge} = 0.0015$	4 places to the left
70×10^{-5}	$70 \times 0.00001 = .\underset{\smile}{0}\underset{\smile}{0}\underset{\smile}{0}\underset{\smile}{7}\underset{\smile}{0}{}_{\wedge} = 0.0007$	5 places to the left

Notice that multiplying by a positive power of 10 moves the decimal point to the right, making the value larger, while multiplying by a negative power of 10 moves the decimal point to the left, making the value smaller. Also, the number of decimal places that the decimal point moves is exactly the positive version of the exponent.

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15: Answer Key

15.0: Chapter 1
15.1: Chapter 2
15.2: Chapter 3
15.3: Chapter 4
15.4: Chapter 5
15.5: Chapter 6
15.6: Chapter 7
15.7: Chapter 8
15.8: Chapter 9
15.9: Chapter 10
15.10: Chapter 11
15.11: Chapter 12
15.12: Chapter 13

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15.0: Chapter 1

Your Turn

1.1.1

1. Set notation lists the elements inside curly braces, separated by commas. Your list may include different tools. Sets are usually designated with capital letters.
$T = \{$ wrench, screwdriver, hammer, pliers $\}$
One possible solution: $T = \{$ wrench, screwdriver, hammer, pliers $\}$.

1.1.2

1. A well-defined set clearly communicates whether the element is a member of the set or not. "A collection of medium-sized potatoes" is not well defined. What makes a potato medium-sized?
This is not a well-defined set.

2. A well-defined set clearly communicates whether the element is a member of the set or not. You can determine which musical artists were members of the group when it formed in 1995.
$B = \{$ will.i.am, apl. de. ap, taboo $\}$
This is a well-defined set.

1.1.3

1. When there are no elements that satisfy the set conditions, you represent the set with either version of the empty set notation: a zero with a slash through it or empty curly braces.
$\emptyset$ or $\}$
$\emptyset$ or $\}$

1.1.4

1. Be sure to include zero because the requirements include "greater than or equal to zero." You stop before 10 because 10 is not a single-digit number.

$$D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$
$$D = \{0, 1, 2, \ldots, 9\}$$

1.1.5

1. This set is infinite. You represent large sets by giving enough numbers to define the pattern, and then using three dots, the ellipsis, to show that the set continues. When no number follows the ellipsis, you know the set continues infinitely.

$$M = \{1, 3, 5, \cdots\}$$
$$M = \{1, 3, 5, \ldots\}$$

1.1.6

1. The roster method lists the elements of the set. Set builder notation uses a letter to represent a sample member of the set, a vertical bar, and then a description of a sample member. In this case, because it is not practical to list all types of cars, it is much easier to use set builder notation.

$$C = \{c \mid c \text{ is a type of car }\}$$
$$C = \{c \mid c \text{ is a car }\}$$

1.1.7

1. The roster method lists the elements of the set. Set builder notation uses a letter to represent a sample member of the set, a vertical bar, and then a description of a sample member. In this case, because it is not practical to list all musical instruments, it is much easier to use set builder notation.
$I = \{i \mid i$ is a musical instrument $\}$
$I = \{i \mid i$ is a musical instrument $\}$

1.1.8

1. The cardinal value is the number of elements in the set. There are no prime numbers less than 2.

$$n(\mathrm{P})=0$$
$$n(P)=0$$

2. The cardinal value is the number of elements in the set. There are 26 lowercase letters of the English alphabet.

$$n(\mathrm{P})=26$$
$$n(A)=26$$

1.1.9

1. There are four elements in set B, so this set is finite. finite

2. There is no largest or smallest real number, so this set is infinite. infinite

1.1.10

1. Equal sets have the same elements, without regard to order. These sets have the same elements, although in a different order.
Set B is equal to set $A, B=A$

2. Equal sets have the same elements, without regard to order. These sets are not equal because only one has the element f. Equivalent sets have the same number of elements. One set has four elements, while the other set has five elements, so they are not equivalent. These sets are neither equal nor equivalent. neither

3. Equal sets have the same elements, without regard to order. Equivalent sets have the same number of elements. These sets are equivalent but not equal because they have the same number of elements (four), but not the same elements. One has a b, while the other has a c.
Set B is equivalent to set $C, B \sim C$

1.2.1

1. Step 1: Begin with the set itself, as every set is a subset of itself. The original set has two elements, so its cardinality is 2.
\{head, tails \}
Step 2: Next, list all the proper subsets containing one less element than the original set. That is, list the sets with one element.
\{heads\}
\{tails \}
Step 3: Continue this process, reducing the number of elements in the subset by one each step. On this step, list the subsets with zero elements, or the empty set.
\{\}
\{heads, tails $\};\{$ heads $\},\{$ tails $\};$ and $\emptyset$

1.2.2

1. A set is a proper subset of L if every member of the subset is a member of L and the subset does not contain an element that is in L. The proper subsets of L with one member include the following.
\{Articuno \}
\{Zapdos \}
\{Moltres \}
\{Mewtwo\}
A set with one member could contain any one of the following:
\{Articuno $\},\{$ Zapdos $\},\{$ Moltres $\}$, or $\{$ Mewtwo $\}$.

2. A set is a proper subset of L if every member of the subset is a member of L and the subset does not contain an element that is in L. The proper subsets of L with three members will include three elements and leave out one.
\{Articuno, Zapdos, Moltres\}
\{Articuno, Zapdos, Mewtwo\}
\{Articuno, Moltres, Mewtwo\}
\{Zapdos, Moltres, Mewtwo \}
Any of the following combinations of three members would work:
\{Articuno, Zapdos, Mewtwo\}, \{Articuno, Moltres, Mewtwo\}, or \{Zapdos, Moltres, Mewtwo\}.

3. A set is a proper subset of L if every member of the subset is a member of L and the subset does not contain an element that is in L. The only proper subset of L with no members is the empty set, represented by $\}$ or $\emptyset$ The empty set is represented as $\}$ or $\emptyset$.

1.2.3

1. The set of even numbers given in this question is a subset of the set of natural numbers. The set of even numbers is a proper subset because there are members of the natural numbers (1 and 3 for example) that are not even numbers. The notation for "the even numbers are a proper subset of the natural numbers":

$$E \subset \mathbb{N}$$

Note that there are also negative even numbers that were not a part of this question.

$$E \subset \mathbb{N}$$

1.2.4

1. Use the formula for the number of subsets of set A with n elements: $2^{n(A)}$.

Set T has nine elements.

$$2^{n(T)} = 2^9 = 512$$

1.2.5

1. Set builder notation uses a letter to represent a sample member of the set, a vertical bar, and then a description of a sample member.

$$\{m \mid m = 5n \text{ where } n \in \mathbb{N}\}$$
$$\{m \mid m = 5n \text{ where } n \in \mathbb{N}\}$$

1.2.6

1. Serena also ordered a fish sandwich and chicken nuggets, because for the two sets to be equal they must contain the exact same items: \{fish sandwich, chicken nuggets\} = \{fish sandwich, chicken nuggets\}. Equal sets have the same elements. For the choices to be equal, they needed to order the same food. Serena ordered a fish sandwich and chicken nuggets.

1.2.7

1. There are multiple possible solutions. Each set must contain two players, but both players cannot be the same, otherwise the two sets would be equal, not equivalent. For example, \{Maria, Shantelle } and \{Angie, Maria\}.

1.3.1

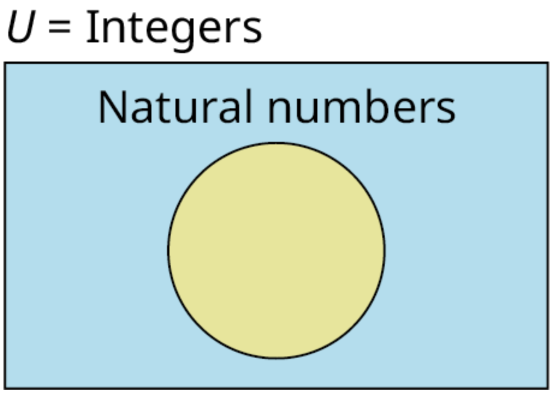
1. L is a subset of U if every element of L is a member of U. Because all lions are cats, every member of the set of Lions is a member of the set of Cats.
$L \subset C$
The set of lions is a subset of the universal set of cats. In other words, the Venn diagram depicts the relationship that all lions are cats. This is expressed symbolically as $L \subset U$.

1.3.2

1. Sets are disjoint if they have no members in common. No member of the set of Eagles is a member of the set of Canaries. No member of the set of Canaries is a member of the set of Eagles. These sets are disjoint. The set of eagles and the set of canaries are two disjoint subsets of the universal set of all birds. No eagle is a canary, and no canary is an eagle.

1.3.3

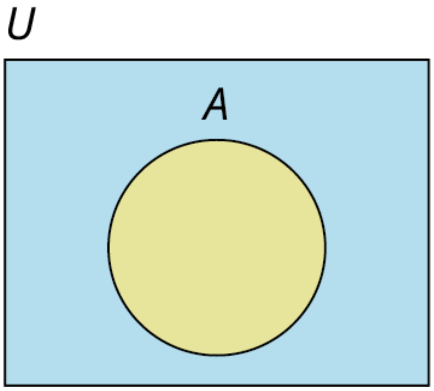
1. The universal set is the set of integers. Draw a rectangle and label it with $U =$ Integers. Next, draw a circle in the rectangle and label with Natural numbers.
$U =$ Integers
Natural numbers


U = Integers
Natural numbers

Venn diagram with universal set, $U =$ Integers, and subset
$\mathbb{N} =$ Natural numbers.

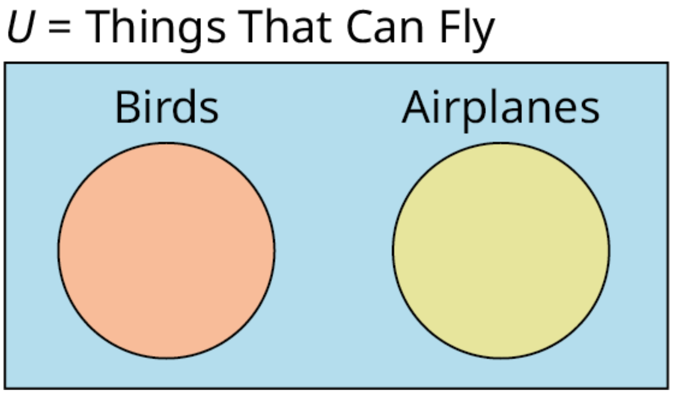
2.

U



Venn Diagram with universal set, U and subset A.

1.3.4


U = Things That Can Fly
Birds
Airplanes

1.

$U =$ Things That Can Fly

Venn Diagram with universal set, $U =$ Things that can fly with disjoint subsets Airplanes and Birds.

1.3.5

1. A' represents the complement of set A. The complement of $\text{set } A$ is the set of all elements in the universal set U that are not in set A. List the elements of U that are not in A.
\{orange, green, indigo, violet\}
$A' = \{$ orange, green, indigo, violet $\}$

2. A' represents the complement of set A. The complement of set A is the set of all elements in the universal set U that are not in set A. List the elements of U that are not in A. Because A includes the cats that are not lions, the complement will include the cats that are lions.

$$A' = \{c \mid c \text{ is a lion }\}$$
$$A' = \{c \in U \mid c \text{ is a lion }\} \text{ or } A' = \{c \in U \mid c \notin A\}$$

1.4.1

1. The members that two sets have in common are included in the intersection of the sets. Only element a is in both sets.

$$A \cap B = \{a\}$$
$$A \cap B = \{a\}$$

1.4.2

1. The members that two sets have in common are included in the intersection of the sets. The sets have no elements in common, so their intersection is the empty set. You can represent the empty set by either the zero with a slash or empty curly braces.

$$A \cap B = \emptyset \text{ or } A \cap B = \{\}$$
$$A \cap B = \{\}$$

1.4.3

1. The members that two sets have in common are included in the intersection of the sets. The two sets have the vowels in common.

$$A \cap B = \{a, e, i, o, u\}$$
$$A \cap B = B = \{a, e, i, o, u\}$$

1.4.4

1. The union of two sets includes all the members of the first set and all the members of the second set. Do not list duplicates twice.

$$A \cup B = \{a, d, h, p, s, y\}$$
$$A \cup B = \{a, d, h, p, s, y\}$$

1.4.5

1. The union of two sets includes all the members of the first set and all the members of the second set. Do not list duplicates twice.
$A \cup B = \{$ red, yellow, blue, orange, green, purple $\}$
$A \cup B = \{$ red, yellow, blue, orange, green, purple $\}$

A set with one member could contain any one of the following:

$\{\text{Articuno}\}, \{\text{Zapdos}\}, \{\text{Moltres}\}, \text{ or } \{\text{Mewtwo}\}.$

1.4.6

1. The union of two sets includes all the members of the first set and all the members of the second set. Do not list duplicates twice. The vowels in set B are a subset of set A, so the union is equal to set A.

$$A \cup B = \{a, b, c, d, \ldots, z\}$$
$$A \cup B = \{a, b, c, \ldots, z\} = A$$

1.4.7

1. 33

Use the formula: $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$$n(A \cup B) = n(A) + n(B) - n(A \cap B) = 23 + 17 - 7 = 33$$

1.4.8

1. 113

Use the formula: $n(A\cup B)=n(A)+n(B)-n(A\cap B)$

$$n(A\cup B)=n(A)+n(B)-n(A\cap B)=35+78-0=113$$

1.4.9

1. To find A or B, find $A\cup B$. The union of two sets includes all the members of the first set and all the members of the second set. Do not list duplicates twice.
A or $B=A\cup B=\{h,a,p,y,w,e,s,o,m\}$
A or $B=A\cup B=\{h,a,p,y,w,e,s,o,m\}$.

2. To find A and C, find $A\cap C$. The intersection of two sets includes the elements that are in both sets.
A and $C=A\cap C=\{a,h\}$
A and $C=A\cap C=\{a,h\}$.

3. To find B or C, find $B\cup C$. The union of two sets includes all the members of the first set and all the members of the second set. Do not list duplicates twice.
B or $C=B\cup C=\{a,w,e,s,o,m,t,h\}$
B or $C=B\cup C=\{a,w,e,s,o,m,t,h\}$.

4. To find $(A$ and $C)$ and B, you need to follow the order of operations and evaluate the parentheses first.

First, evaluate inside the parentheses. Find A and $C=A\cap C$. The intersection of two sets includes the elements that are in both sets.
A and $C=A\cap C=\{h,a,p,y\}\cap\{m,a,t,h\}=\{a,h\}$

To find $(A$ and $C)$ and $B=(A\cap C)\cap B$, find $\{a,h\}\cap B$.
$\{a,h\}\cap\{a,w,e,s,o,m\}=\{a\}$
$(A$ and $C)$ and $B=(A\cap C)\cap B=\{a,h\}\cap\{a,w,e,s,o,m\}=\{a\}$.

1.4.10

1. 127

Total number of guests $=150$
$n($ neither soup nor salad $)=23$
To find the number who had soup, salad, or both, subtract 23 from 150.

$$150-23=127$$

2. 50

Total number of guests $=150$
n (neither soup nor salad $)=23$
To find the number who had soup, salad, or both, subtract 23 from 150.

$$150-23=127$$

This represents the number of elements in the union of the sets of Soup and Salad eaters.

Substitute.	$127=92+85-n(A\cap B)$
Simplify.	$127=177-n(A\cap B)$
Subtract 127.	$127-127=177-n(A\cap B)-127$
Simplify.	$0=50-n(A\cap B)$
Add the variable.	$0+n(A\cap B)=50-n(A\cap B)+n(A\cap B)$
Simplify.	$n(A\cap B)=50$

1.4.11

1. $A\cap B=\{3,5,7\}$.

The members that two sets have in common are included in the intersection of the sets. Look for the elements where the circles overlap.

$$A \cap B=\{3,5,7\}$$

2. $A \cup B=\{1,2,3,5,7,9\}$.

The union of two sets includes all the members of the first set and all the members of the second set. Do not list duplicates twice. List the elements that are inside the two circles.

$$A \cup B=\{1,2,3,5,7,9\}$$

3. $A \cap B^{\prime}=\{1,9\}$.

First, find $B^{\prime}$, the complement of B. The complement of set B is the set of all elements in the universal set U that are not in set B. List the elements of U that are not in B.

$$B^{\prime}=\{0,1,4,6,8,9\}$$

Now find

$$A \cap B^{\prime}=\{1,3,5,7,9\} \cap\{0,1,4,6,8,9\}$$

The intersection of two sets are the elements that are in both sets.

$$A \cap B^{\prime}=\{1,9\}$$

4. $n\left(A \cap B^{\prime}\right)=2$.

First, find $B^{\prime}$, the complement of B. The complement of set B is the set of all elements in the universal set U that are not in set B. List the elements of U that are not in B.

$$B^{\prime}=\{0,1,4,6,8,9\}$$

Now find

$$A \cap B^{\prime}=\{1,3,5,7,9\} \cap\{0,1,4,6,8,9\}$$

The intersection of two sets are the elements that are in both sets.
$A \cap B^{\prime}=\{1,9\}$
$n\left(A \cap B^{\prime}\right)$ is the number of elements in the set, which is two.

1.4.12

1. $A \cap B=\{3,5,7\}$.

The members that two sets have in common are included in the intersection of the sets. Look for the elements where the circles overlap.

$$A \cap B=\{3,5,7\}$$

2. $A \cup B=\{1,2,3,5,7,9\}$.

The union of two sets includes all the members of the first set and all the members of the second set. Do not list duplicates twice. List the elements that are inside the two circles.

$$A \cup B=\{1,2,3,5,7,9\}$$

3. $A \cap B^{\prime}=\{1,9\}$.

First, find $B^{\prime}$, the complement of B. The complement of set B is the set of all elements in the universal set U that are not in set B. List the elements of U that are not in B.

$$B^{\prime}=\{0,1,4,6,8,9\}$$

Now find

$$A \cap B^{\prime}=\{1,3,5,7,9\} \cap\{0,1,4,6,8,9\}$$

The intersection of two sets are the elements that are in both sets.

$$A \cap B' = \{1, 9\}$$

4. $n(A \cap B') = 2$.

First, find B', the complement of B. The complement of set B is the set of all elements in the universal set U that are not in set B. List the elements of U that are not in B.

$$B' = \{0, 1, 4, 6, 8, 9\}$$

Now find

$$A \cap B' = \{1, 3, 5, 7, 9\} \cap \{0, 1, 4, 6, 8, 9\}$$

The intersection of two sets are the elements that are in both sets.
$A \cap B' = \{1, 9\}$
$n(A \cap B')$ is the number of elements in the set, which is two.

1.5.1

1. To find A or B, find $A \cup B$. The union of two sets includes all the members of the first set and all the members of the second set. Do not list duplicates twice.

Because the two sets are disjoint, the number of elements in the union of the two sets is the sum of the number of elements in the two sets.

$$23 + 17 = 40$$

2. To find A or B, find $A \cap B$. The intersection of two sets includes the elements they have in common. They have no elements in common, so the number of elements in the intersection is zero.

3. A' is the complement of A. The complement of $\text{set } A$ is the set of all elements in the universal set U that are not in set A. The number of elements that are not in set A :

$$17 + 10 = 27$$

1.5.2

1. $n(B) = n(AB^+) + n(AB^-) + n(B^+) + n(B^-) = 14$

Count every blood type that has any type of B as part of the blood type.

$$n(B) = n(AB^+) + n(AB^-) + n(B^+) + n(B^-) = 3 + 1 + 8 + 2 = 14$$

2. $n(B') = n(U) - n(B) = 86$

You could count every type that does not have a B as part of the blood type. However, if you did the previous exercise, you already know the complement of B.

$$n(B') = n(U) - n(B) = 100 - 14 = 86$$

3. $n(B \cup Rh^+) = 87$

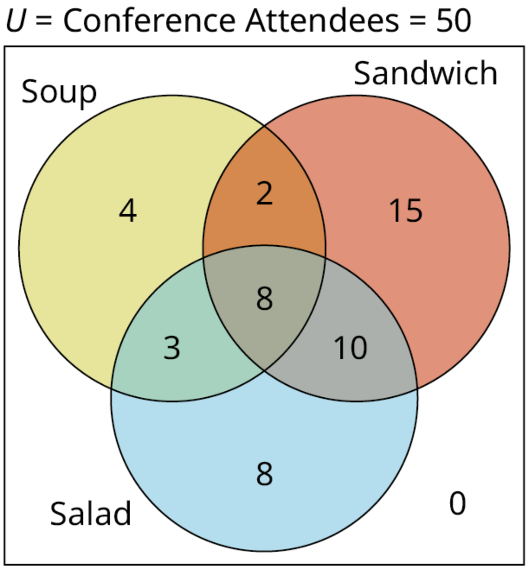
Count the number of people who are in either the B circle or the RH^+circles.

$$1 + 3 + 2 + 8 + 36 + 37 = 87$$

1.5.3

1. Draw a rectangle to represent the universal set. Label it with " $U =$ Conference Attendees $= 50$."

Inside the rectangle, draw three overlapping circles. Label one circle Soup, one Sandwich, and one Salad. In the Soup circle, write 4 in the part that does not overlap any other circle, write 2 in the part that only overlaps the Sandwich circle, write 8 in the part that overlaps all three circles, and write 3 in the part that overlaps only the Salad circle. In the Sandwich circle, write 15 in the part that does not overlap any other circle and write 10 in the part that only overlaps the Salad circle. In the Salad circle, write 8 in the part that does not overlap any other circle. Write 0 outside all the circles.



Venn diagram – Attendees at a conference with sets: Soup, Sandwich, Salad – Complete Solution

1.5.4

1. $A\cap(B\cap C)$

First, evaluate the operation inside the parentheses.

$$B\cap C=\{0,2,4,6,8,10,12\}\cap\{0,3,6,9,12\}$$

The intersection of two sets includes the elements that are common to both sets.

$$\begin{aligned}&B\cap C=\{0,6,12\}\\&A\cap(B\cap C)=\{0,1,2,3,4,5,6\}\cap\{0,6,12\}\\&A\cap(B\cap C)=\{0,6\}\\&A\cap(B\cap C)=\{0,1,2,3,4,5,6\}\cap\{0,6,12\}=\{0,6\}\end{aligned}$$

2. $(A\cap B)\cup(A\cap C)$

Evaluate the expression in the left parentheses.
$A\cap B=\{0,1,2,3,4,5,6\}\cap\{0,2,4,6,8,10,12\}$
The intersection of two sets includes the elements that are common to both sets.
$A\cap B=\{0,2,4,6\}$
Evaluate the expression in the right parentheses.
$A\cap C=\{0,1,2,3,4,5,6\}\cap\{0,3,6,9,12\}$
$A\cap C=\{0,3,6\}$
Now, you can find $(A\cap B)\cup(A\cap C)$.

$$\begin{aligned}&(A\cap B)\cup(A\cap C)=\{0,2,4,6\}\cup\{0,3,6\}\\&(A\cap B)\cup(A\cap C)=\{0,2,3,4,6\}\\&(A\cap B)\cup(B\cap C)=\{0,2,4,6\}\cup\{0,3,6\}=\{0,2,3,4,6\}\end{aligned}$$

3. $(A\cup C')\cap(B\cup C')$
C', the complement of C, includes the elements of U that are not in C.

$$\begin{aligned}&C=\{0,3,6,9,12\}\text{ and }U=\{0,1,2,3,4,5,6,7,8,9,10,11,12\}\\&C'=\{1,2,4,5,7,8,10,11\}\end{aligned}$$

Evaluate the expression in the left parentheses.

$$A\cup C'=\{0,1,2,3,4,5,6\}\cap\{1,2,4,5,7,8,10,11\}$$

The intersection of two sets includes the elements that are common to both sets.

$$A \cup C' = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 10, 11\}$$

Evaluate the expression in the right parentheses.

$$\begin{aligned} B \cup C' &= \{0, 2, 4, 6, 8, 10, 12\} \cap \{1, 2, 4, 5, 7, 8, 10, 11\} \\ B \cup C' &= \{0, 1, 2, 4, 5, 6, 7, 8, 10, 11, 12\} \end{aligned}$$

Now, you can find $(A \cup C') \cap (A \cup C')$.

$$\begin{aligned} &(A \cup C') \cap (B \cup C') = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 10, 11\} \cap \{0, 1, 2, 4, 5, 6, 7, 8, 10, 11, 12\} \\ &(A \cup C') \cap (B \cup C') = \{0, 1, 2, 4, 5, 6, 7, 8, 10, 11\} \\ &(A \cup C') \cap (B \cup C') \\ &= \{0, 1, 2, 3, 4, 5, 6, 7, 8, 10, 11\} \cap \{0, 1, 2, 4, 5, 6, 7, 8, 10, 11, 12\} = \{0, 1, 2, 4, 5, 6, 7, 8 \end{aligned}$$

1.5.5

1. First, draw the left side of the equation.

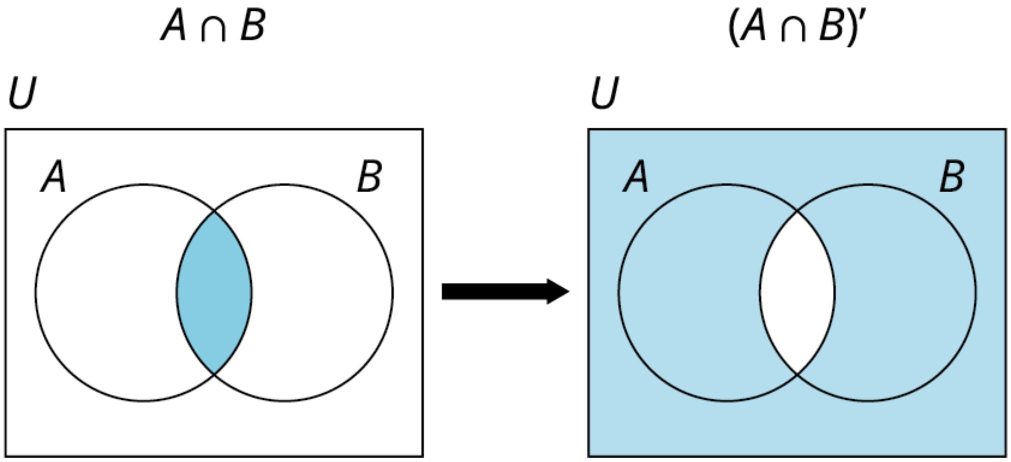
For $A \cap B$, draw a rectangle to represent the universal set U. Label the rectangle " U." Inside the rectangle, draw overlapping circles to represent two sets. Label one circle " A " and the other circle " B." Shade the overlap.

Now, draw the complement of the intersection. Draw the same Venn diagram, but shade everything except the overlap to represent $(A \cap B)'$.
Now, draw the right side of the equation: $A' \cup B'$.
First, draw A'. Draw a rectangle to represent the universal set U. Label the rectangle " U." Inside the rectangle, draw a circle and label the circle " A." Shade everything outside the circle.

To the right of the rectangle, draw the U symbol.
Now, draw B'. Draw a rectangle to represent the universal set U. Label the rectangle " U." Inside the rectangle, draw a circle and label the circle " B." Shade everything outside the circle.

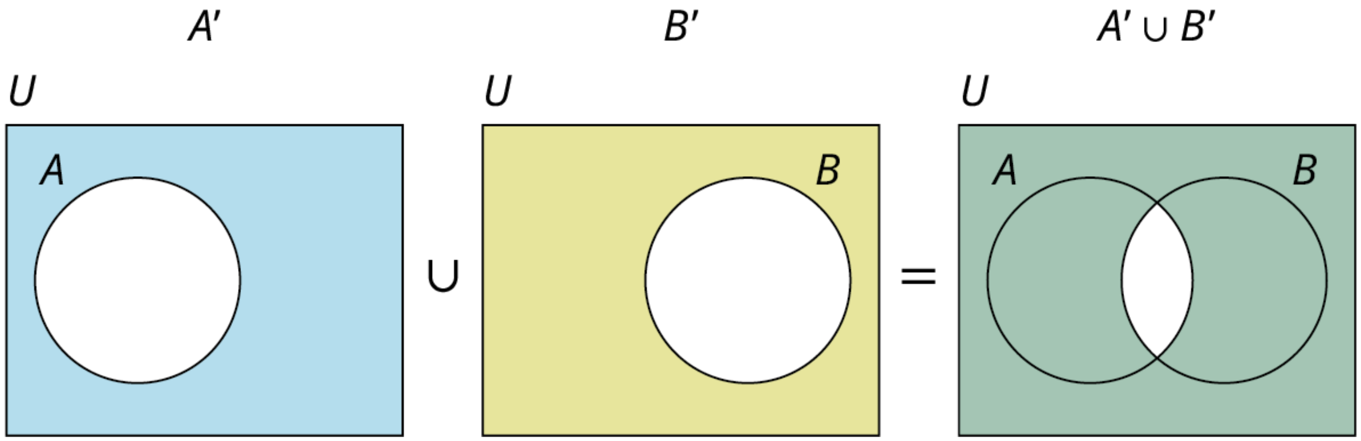
To the right of the second rectangle, draw the equal symbol: =.
For $A' \cup B'$, draw a rectangle to represent the universal set U. Label the rectangle " U." Inside the rectangle, draw overlapping circles to represent two sets. Label one circle " A " and the other circle " B." Shade everything except for the overlap.

The left side of the equation is:



Venn diagram of intersection of two sets and its complement.

The right side of the equation is given by:



Venn diagram of union of the complement of two sets.

Check Your Understanding

1. set

2. cardinality

3. not a well-defined set

4. 12

5. equivalent, but not equal 6. finite

7. The roster method lists the individual members of the set inside curly braces, separated by commas. Once you have established the pattern, you can use the ellipsis to avoid needing to list them all. Be sure to list the final member of the set (Z) to avoid giving the appearance of an infinite set.

$$A = \{A, B, C, D, \cdots, Z\}$$

Set builder notation uses a letter to represent a sample member of the set, a vertical bar, and then a description of a sample member.
$A = \{a \mid a$ is a capital letter of the English alphabet $\}$
Roster method: $\{\mathrm{A}, \mathrm{B}, \mathrm{C}, \ldots, \mathrm{Z}\}$, and set builder notation:
$\{x \mid x$ is a capital letter of the English alphabet $\}$

8. subset

9. To be a subset of a set, every member of the subset must also be a member of the set. To be a proper subset, there must be at least one member of the set that is not also in the subset.

10. empty

11. true

12. Use the formula for the number of subsets of set A with n elements: $2^{n(A)}$. Set A has ten elements.

$$2^{n(A)} = 2^{10} = 1024$$
$$2^{10} = 1024$$

13. Equivalent sets have the same number of elements. Because $n(A) = n(B)$, the sets are equivalent. equivalent

14. equal. Equal sets have the same elements. For each set, there can be no element that is not an element of the other set. Sets A and B are equal because every member of A is in B and every member of B is in A.

15. relationship

16. universal

17. disjoint or non-overlapping

18. complement

19. disjoint

20. To help you remember, think of the intersection of two streets. The intersection is the pavement that the two streets have in common.
intersection

21. To help you remember, think of the union of the United States. A union joins things together. union

22. $A \cup B$

It is very helpful that the union symbol $(\cup)$ looks somewhat like a U.

23. $A \cap B$

24. The intersection of two sets includes the elements they have in common. Because A is a subset of B, all the elements of A are also in B. There can be no other elements in B that are in A. Thus, $A \cap B = A$.

25. The union of two sets includes all the members of the first set and all the members of the second set. Do not list duplicates twice.

Because A is a subset of B, all the elements of A are also in B. There can be no elements in A that are not in B. Thus, $A \cup B = B$.

26. Disjoint sets have no elements in common. The intersection of two sets includes the elements they have in common. Because set A and set B are disjoint, their intersection is the empty set. empty

27. $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

28. overlap

By beginning with the region involving the most overlap, you start with the regions involving the most sets and the most complexity. Work toward the simpler parts of the diagram.

29. central

30. intersection of all three sets, $A \cap B \cap C$

31. parentheses, complement

32. equation, true

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15.1: Chapter 2

Your Turn

2.1

1. A logical statement is a sentence that can be identified as being true or false. This sentence is a logical statement because it can be identified as false.
Logical statement, false.

2. A logical statement is a sentence that can be identified as being true or false. This sentence is a logical statement because it can be identified as true.
Logical statement, true.

3. A logical statement is a sentence that can be identified as being true or false. This sentence is a not a logical statement because questions cannot be identified as either true or false.
Not a logical statement, questions cannot be determined to be either true or false.

2.2

1. Choose a lowercase letter. The most common choices are p, q, and r. You are not being asked to determine the truth value.
p: The movie Gandhi won the Academy Award for Best Picture in 1982.
p: The movie Gandhi won the Academy Award for Best Picture in 1982.

2. Choose a lowercase letter. The most common choices are p, q, and r. You are not being asked to determine the truth value.
q : Soccer is the most popular sport in the world.
q : Soccer is the most popular sport in the world.

3. Choose a lowercase letter. The most common choices are p, q, and r. You are not being asked to determine the truth value.
r : All oranges are citrus fruits.
r : All oranges are citrus fruits.

2.3
1. The negation of a logical statement has the opposite truth value of the original statement.

Original statement: Ted Cruz was not born in Texas.
Remove "not."
Negation: Ted Cruz was born in Texas.
Ted Cruz was born in Texas.

2. The negation of a logical statement has the opposite truth value of the original statement.

Original statement: Adele has a lovely singing voice.
Add "not."
Negation: Adele does not have a lovely singing voice.
Adele does not have a beautiful voice.

3. The negation of a logical statement has the opposite truth value of the original statement.

Original statement: Leaves convert carbon dioxide to oxygen during the process of photosynthesis.
Add "not."
Negation: Leaves do not convert carbon dioxide to oxygen during the process of photosynthesis.
Leaves do not convert carbon dioxide to oxygen during the process of photosynthesis.

2.4

1. The negation of a logical statement has the opposite truth value of the original statement.

Original statement: $\sim p$
To negate a negative statement, remove the tilde, since the opposite of a negative is a positive.
Negation: $\sim(\sim p)=p$
p

2. The negation of a logical statement has the opposite truth value of the original statement.

Original statement: q
To negate an affirmative statement, add a tilde.
Negation: $\sim q$
$\sim q$

3. The negation of a logical statement has the opposite truth value of the original statement.

Original statement: r
To negate an affirmative statement, add a tilde.
Negation: $\sim r$
$\sim r$

2.5

1. "Wonder Woman is stronger than Captain Marvel" is the negation of "Wonder Woman is not stronger than Captain Marvel."

Thus, symbolically you write $\sim(\sim p) = p$.
p
2. The statement r represents "Woody and Buzz Lightyear are best friends."

The negation $\sim r$ is translated "Woody and Buzz Lightyear are not best friends."
Woody and Buzz Lightyear are not best friends.

2.6

1. It would be incorrect to say that the sum of all consecutive integers results in a prime number because 0 and $1 = 1$, which is not prime. You can say that "The sum of some consecutive integers results in a prime number."
The sum of some consecutive integers results in a prime number.

2. Based on the premises provided, no birds give live birth to their young. This statement, however, is false, as some birds do give birth to live young.

No birds give live birth to their young.

3. All squares are rectangles. All rectangles are parallelograms, so a square is a parallelogram. All parallelograms have four sides, so a square has four sides. Thus, all squares have four sides.

You could draw a Venn diagram that shows this. Squares are a subset of rectangles. Rectangles are a subset of parallelograms. Parallelograms are a subset of things with four sides. Thus, squares are a subset of things with four sides.

All squares are parallelograms and have four sides.

2.7

1. You negate a "Some...not" statement by replacing "Some...not" with "All."

Original statement: Some apples are not sweet.
Negation: All apples are sweet.
All apples are sweet.

2. You negate a "No" statement by replacing "No" with "Some."

Original statement: No triangles are squares.
Negation: Some triangles are squares.
Some triangles are squares.

3. You negate a "some" statement by replacing "Some" with "No."

Original statement: Some vegetables are green.
Negation: No vegetables are green.
No vegetables are green.

2.8

1. Negation; $\sim$.
2. Conjunction; $\wedge$.
3. Biconditional; $\leftrightarrow$.

2.9

1. Today we will go skiing, but last night it did not snow.

Both "and" and "but" use the conjunction connective.
The given statements:
p : Last night it snowed.
q : Today we will go skiing.
Break down the compound statement into its components.

Today we will go skiing,	but	not	last night it snowed
q	$\wedge$	$\sim$	p

2. $q \leftrightarrow p$

Today we will go skiing if and only if it snowed last night.
The given statements:
p : Last night it snowed.
q : Today we will go skiing.
Break down the compound statement into its components.

Unknown environment 'tabular'

$$q \leftrightarrow p$$

3. $p \vee \sim q$

Last night it snowed or today we will not go skiing.
The given statements:
p : Last night it snowed.
q : Today we will go skiing.
Break down the compound statement into its components.

4. $p \rightarrow q$

If it snowed last night, then today we will go skiing.
The given statements:
p : Last night it snowed.
q : Today we will go skiing.

Last night it snowed	or	not	today we will go skiing,
p	$\vee$	$\sim$	q

$p \vee \sim q$

2.10

1. $\sim r \rightarrow (p \vee q)$

Given statements:
p : My roommates ordered pizza.
q : I ordered wings.
r : Our friends came over to watch the game.
Translate $\sim r$ as :Our friends did not come over to watch the game.

$\rightarrow$ is the conditional, which will be translated "if...then."
V is the inclusive or.
The comma before "then" shows that the statements in parentheses belong together.
If our friends did not come over to watch the game, then my roommates ordered pizza or I ordered wings. If our friends did not come over to watch the game, then my roommates ordered pizza or I ordered wings.

2. $(p \wedge q) \rightarrow r$
Λ is the conjunction, which will be translated "and."
$\rightarrow$ is the conditional, which will be translated "if...then."
The comma before "then" shows that the statements in parentheses belong together.
Given statements:
p : My roommates ordered pizza.
q : I ordered wings.
r : Our friends came over to watch the game.
If my roommates ordered pizza and I ordered wings, then our friends came over to watch the game.
If my roommates ordered pizza and I ordered wings, then our friends came over to watch the game.

3. $\sim(p \vee r)$
$\sim$ is the negation symbol. Replace the symbol with "It is not the case that..."
V is the inclusive or.
Given statements:
p : My roommates ordered pizza.
q : I ordered wings.
r : Our friends came over to watch the game.
It is not the case that my roommates ordered pizza or our friends came over to watch the game.
It is not the case that my roommates ordered pizza or our friends came over to watch the game.

2.11

1. $((p \vee q) \wedge(\sim r))$

The order of logical operations is
1. Parentheses
2. Negation
3. Disjunction, conjunction
4. Conditional
5. Biconditional

For same order operations, work left to right.
The highest order operation in this expression is negation, so add parentheses around the negation.

$$p \vee q \wedge(\sim r)$$

Disjunction and conjunction are the same order, so work left to right. First evaluate the exclusive or.

$$(p \vee q) \wedge(\sim r)$$

Then finish by evaluating the conjunction.

$$((p \vee q) \wedge(\sim r))$$

2. $((\sim p) \rightarrow(q \vee r))$

The order of logical operations is
1. Parentheses
2. Negation
3. Disjunction, conjunction
4. Conditional
5. Biconditional

For same order operations, work left to right.

$$\sim p \rightarrow q \vee r$$

The highest order operation in this expression is negation, so add parentheses around the negation.

$$(\sim p) \rightarrow q \vee r$$

The next highest order operation is the disjunction. Add parentheses around the disjunction.

$$(\sim p) \rightarrow (q \vee r)$$

Finally, you can evaluate the conditional.

$$((\sim p) \rightarrow (q \vee r))$$

3. $((\sim p) \vee (\sim q)) \leftrightarrow (\sim (p \wedge q))$; this is another example of De Morgan's Laws and it is always true. The order of logical operations is

1. Parentheses
2. Negation
3. Disjunction, conjunction
4. Conditional
5. Biconditional

For same order operations, work left to right.

$\sim p \vee \sim q \leftrightarrow \sim (p \wedge q)$
The right side of the biconditional already has parentheses. The next step in the order of operation is negation, so add parentheses around the three negations.

$$(\sim p) \vee (\sim q) \leftrightarrow (\sim (p \wedge q))$$

The next step in the order of operations is disjunction/conjunction. Add parentheses around the disjunction.

$$((\sim p) \vee (\sim q)) \leftrightarrow (\sim (p \wedge q))$$

The lowest order operation is the biconditional. Finish by evaluating the biconditional.

$$(((\sim p) \vee (\sim q)) \leftrightarrow (\sim (p \wedge q)))$$

2.12

1. The negation is $p : 3 \times 5 \neq 14$

Since $3 \times 5 = 15$, the negation is true.
$p : 3 \times 5 \neq 14$, true

2. The negation is q : No house are built with bricks.

The negation is false since some houses are built of bricks.
q : No houses are built with bricks; false

3. The negation is $\sim r$: Abuja is not the capital of Nigeria.

The negation is a false statement since Abuja is the capital of Nigeria.
$\sim r$: Abuja is not the capital of Nigeria; false

2.13

1. Given statements:
p : Yellow is a primary color.
p is true.
q : Blue is a primary color.
q is true.
The $\wedge$ symbol represents the conjunction and is translated "and" or "but."
A conjunction is only true when both statements are true.

$p \wedge q$
True ∧ True = True
True

2. Given statements:
p : Yellow is a primary color.
p is true.
r : Green is a primary color.
r is false.
$\sim r$ is true.
$\sim r \wedge p$
A conjunction is only true when both statements are true.
True ∧ True = True
False

3. Given statements:
q : Blue is a primary color.
q is true.
r : Green is a primary color.
r is false.
$q \wedge r$
A conjunction is only true when both statements are true.
True ∧ False = False
True

2.14

1. Given statements:
p : Yellow is a primary color.
p is true.
q : Blue is a primary color.
q is true.

$$p \vee q$$

A disjunction is only false when both statements are false.
True ∨ True = True
True

2. Given statements:
p : Yellow is a primary color.
p is true.
$\sim p$ is false.
r : Green is a primary color.
r is false.
$\sim p \vee r$
A disjunction is only false when both statements are false.
False ∨ False = False
False

3. Given statements:
q : Blue is a primary color.
q is true.
r : Green is a primary color.
r is false.
$q \vee r$
A disjunction is only false when both statements are false.

True $\vee$ False $=$ True
True

2.15

1. Given statements:
p : Yellow is a primary color.
p is true.
q : Blue is a primary color.
q is true.
r : Green is a primary color.
r is false.

p	q	r	q	$q \wedge p$	$(q \wedge p) \vee r$
T	T	F	F	F	F

false

2. Given statements:
p : Yellow is a primary color. p is true.
q : Blue is a primary color.
q is true.
r : Green is a primary color.
r is false.

p	q	r	r	$p \vee q$	$(p \vee q) \wedge (r)$
T	T	F	T	T	T

true

3. Given statements:
p : Yellow is a primary color. p is true.
q : Blue is a primary color.
q is true.
r : Green is a primary color.
r is false.

p	q	r	$(p \wedge r)$	$(p \wedge r)$	$(p \wedge r) \wedge q$
T	T	F	F	T	T

true

2.16

1.

p	q	q	$p \wedge \ q$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

2.

p	q	$p \vee q$	$(p \vee q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

3.

p	q	r	q	$p \wedge \ q$	$(p \wedge \ q) \vee r$
T	T	T	F	F	T
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	F	F	T
F	T	F	F	F	F
F	F	T	T	F	T
F	F	F	T	F	F

2.17

1. Valid

p	p	$p \vee \ p$
T	F	T
F	T	T

2. Not valid

p	q	p	q	$p \vee \ q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

2.18

1. False

p	q	$p \rightarrow q$
T	F	**F**

2. True

p	q	q	$p \rightarrow \ q$
T	F	T	**T**

3. True

p	q	p	$p \rightarrow q$
T	F	F	**T**

2.19
1. Valid

p	q	p	$p \vee q$	$q \rightarrow (p \vee q)$
T	T	F	T	**T**
T	F	F	F	**T**
F	T	T	T	**T**
F	F	T	T	**T**

2. Not valid

p	q	p	$q \wedge p$	$\sim p \rightarrow (q \wedge p)$
T	T	F	T	**T**
T	F	F	F	**T**
F	T	T	F	**F**
F	F	T	F	**F**

2.20
1. False

p	q	$p \leftrightarrow q$
T	F	**F**

2. True

p	q	q	$p \leftrightarrow q$
T	F	T	**T**

3. True

p	q	p	$p \leftrightarrow q$
T	F	F	**T**

2.21
1. Valid

p	q	$p \wedge q$	$\sim (p \wedge q)$	p	q	$\sim p \vee \sim q$	$\sim (p \wedge q) \leftrightarrow (\sim p \vee \sim q)$
T	T	T	F	F	F	F	**T**
T	F	F	T	F	T	T	**T**
F	T	F	T	T	F	T	**T**
F	F	F	T	T	T	T	**T**

2. Not Valid

p	q	p	$q \wedge p$	$\sim p \leftrightarrow (q \wedge p)$
T	T	F	T	**F**
T	F	F	F	**T**
F	T	T	F	**F**
F	F	T	F	**F**

3. Valid

p	q	$p \to q$	p	$p \vee q$	$(p \to q) \leftrightarrow (\sim p \vee q)$
T	T	T	F	T	**T**
T	F	F	F	F	**T**
F	T	T	T	T	**T**
F	F	T	T	T	**T**

4. Valid

p	q	r	p	q	$p \wedge q$	\(\left(\right) \to {r}\)	$p \vee\ q$	$(\sim p \vee \sim q) \vee r$	$(p \wedge q \to r) \leftrightarrow (\sim p \vee \sim q \vee r)$
T	T	T	F	F	T	T	F	T	**T**
T	T	F	F	F	T	F	F	F	**T**
T	F	T	F	T	F	T	T	T	**T**
T	F	F	F	T	F	T	T	T	**T**
F	T	T	T	F	F	T	T	T	**T**
F	T	F	T	F	F	T	T	T	**T**
F	F	T	T	T	F	T	T	T	**T**
F	F	F	T	T	F	T	T	T	**T**

2.22

1. $p \to q$ is logically equivalent to $q \to\ p$.

p	q	$p \to q$	q	p	$q \to\ p$	$(p \to q) \leftrightarrow (\sim q \to \sim p)$
T	T	T	F	F	T	**T**
T	F	F	T	F	F	**T**
F	T	T	F	T	T	**T**
F	F	T	T	T	T	**T**

2. $p \to q$ is not logically equivalent to $p \vee\ q$.

p	q	$p \to q$	q	$p \vee$ q	$(p \to q) \leftrightarrow (p \vee \sim q)$
T	T	T	F	T	**T**
T	F	F	T	T	**F**
F	T	T	F	F	**F**

p	q	$p \rightarrow q$	q	$p \vee$ q	$(p \rightarrow q) \leftrightarrow (p \vee \sim q)$
F	F	T	T	T	**T**

2.23
1.

If Elvis Presley wore capes, then some superheroes wear capes.

2.

If some superheroes wear capes, then Elvis Presley wore capes.

3.

If Elvis Presley did not wear capes, then no superheroes wear capes.

4.

If no superheroes wear capes, then Elvis Presley did not wear capes.

2.24
1. p: Dora is an explorer.
2. q: Boots is a monkey.
3. Inverse
4. Converse
5. Converse
2.25
1. If my friend does not live in California, then my friend lives in San Francisco. True.
2. If my friend does not live in San Francisco, then my friend lives in California. True.
3. If my friend lives in California, then my friend does not live in San Francisco. False.
2.26
1. Jackie did not play softball and she did not run track.
2.

Brandon did not study for his certification exam, or he did not pass his exam.

2.27
1. Edna Mode made a new superhero costume, and it includes a cape.
2.

I had pancakes for breakfast, and I did not use maple syrup.

2.28
1. Some people like ice cream, but ice cream makers will not make a profit.
2. Raquel cannot play video games, but somebody will play video games.
2.29
1. Eric needs to replace the light bulb, and Marcos did not leave the light bulb on all night, and Dan did not break the light bulb.
2. Trenton went to school, and Regina went to work, and Merika did not clean the house.
2.30
1.

p	q	$p \vee q$	\(\text{~}\left(\right)\)	p	q	$p \wedge q$	$(p \vee q) \leftrightarrow (p \wedge q)$
T	T	T	F	F	F	F	**T**
T	F	T	F	F	T	F	**T**
F	T	T	F	T	F	F	**T**
F	F	F	T	T	T	T	**T**

2.31
1. Some people like history.
2.

Some people do not like reading.

3.

The polygon is not an octagon.

2.32
1. My classmate does not like history.
2. Homer likes to read.
3. The polygon does not have five sides.
2.33
1. If my roommate does not go to work, then they will not be able to pay their bills.
2. If penguins cannot fly, then we will watch the news.
3. If Marcy goes to the movies, then she will buy water.

Check Your Understanding

1. logical statement
2.

negation

3. p
4.

p

5.

premises

6.

Inductive

7.

quantifiers

8. Some giraffes are not tall.
9. compound statement
10.

connective

11.

biconditional, $\leftrightarrow$

12.

Parentheses, $(\,)$

13.

Conjunction, $\wedge$; disjunction, $\vee$ (in any order)

14.

valid

15.

true

16.

truth table

17.

four

18.

two

19. one-way contract
20. conclusion
21. If the hypothesis of a conditional statement is false, the conditional statement is true. A false hypothesis can have either a true or a false conclusion. In both situations, the conditional is true. hypothesis
22. biconditional
23. biconditional
24. true
25. always true, valid, or a tautology.
26. conditional
27. logically equivalent
28. inverse
29. converse, inverse
30. $\sim p \vee \sim q$
31. $\sim p \wedge \sim q$
32. $p \wedge \sim q$
33. De Morgan's Laws
34. premise
35. A logical argument is valid if its conclusion follows from the premises. The validity is not affected by whether the premises are true or not. valid
36. inductive
37. deductive
38. fallacy
39. sound

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15.2: Chapter 3

Your Turn

3.1

1. Yes. When 54 is divided by 9, the result is 6 with no remainder. Also, 54 can be written as the product of 9 and 6.

3.2

1. The last digit is 0, so 45,730 is divisible by 5, since the rule states that if the last digit is 0 or 5, the original number is divisible by 5.

3.3

1. The sum of the digits is 32. Since 32 is not divisible by 9, neither is 342,887.

3.4

1. The last digit is even, so 2 divides 43,568. The sum of the digits is 26. Since 26 is not divisible by 3, neither is 43,568. The rule for divisibility by 6 is that the number be divisible by both 2 and 3. Since 43,568 is not divisible by 3, it is not divisible by 6.

3.5

1. Since the last digit of 87,762 is not 0, it is not divisible by 10.

3.6

1. The number formed by the last two digits of 43,568 is 68 and 68 is divisible by 4. Since the number formed by the last two digits of 43,568 is divisible by 4, so is 43,568.

3.7

1. Yes, 1,429 is prime.

3.8

1. Yes, 859 is a prime number.

3.9

1. 5,067,322 is a composite number.

3.10

1. No, 1,477 is composite.

3.11

1. $90 = 2 \times 3^2 \times 5$

3.12

1. $85 = 5 \times 17$

3.13

1. $2^3 \times 5 \times 7$

3.14

1. The number 180 has three prime factors.

3.15

1. The GCD is 9.

3.16

1. The GCD of 36 and 128 is 4.

3.17

1. 40

3.18

1. The largest square bricks that can be used are 20 cm by 20 cm.

3.19

1. The largest team size that can be formed is 7 students.

3.20

1. 60

3.21

1. 140

3.22

1. 92,400

3.23

1. 360

3.24

1. The sun, Venus, and Jupiter will line up again in 220,830 days.

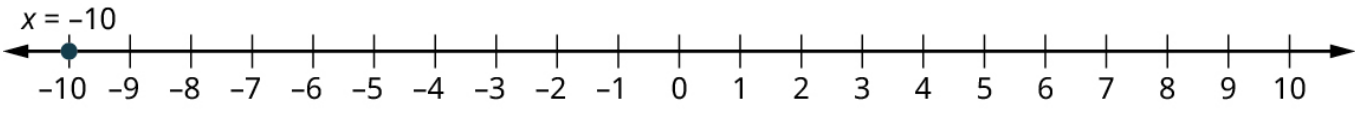
3.25

1. The first person to receive both giveaways would be the person who submits the 11,700th submission.

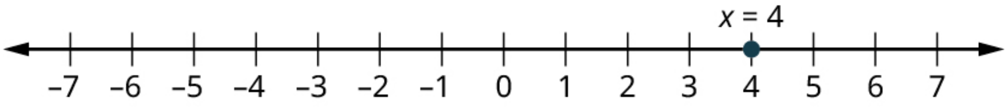
3.26

1. integer

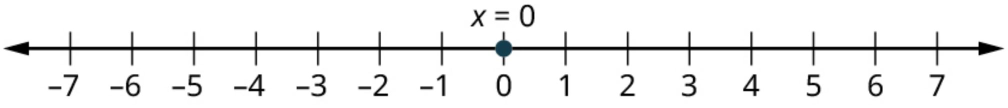
2. not an integer
3. not an integer
4. integer
5. integer
3.27
1.


x = –10
–10 –9 –8 –7 –6 –5 –4 –3 –2 –1 0 1 2 3 4 5 6 7 8 9 10

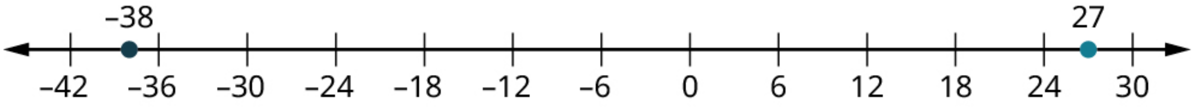
2.


x = 4
–7 –6 –5 –4 –3 –2 –1 0 1 2 3 4 5 6 7

3.

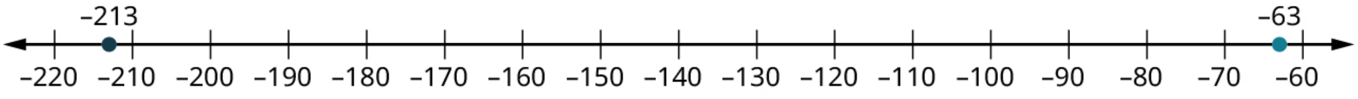

x = 0
–7 –6 –5 –4 –3 –2 –1 0 1 2 3 4 5 6 7

3.28
1.


–38
27
–42 –36 –30 –24 –18 –12 –6 0 6 12 18 24 30

$27 > -38$ and $-38 < 27$

3.29
1.


–213
–63
–220 –210 –200 –190 –180 –170 –160 –150 –140 –130 –120 –110 –100 –90 –80 –70 –60

$-63 > -213$ and $-213 < -63$

3.30
1. 101 is larger. $101 > 98$ and $98 < 101$.
3.31
1. 38
3.32
1. 81
3.33
1. −7. Since |−18| > |11|, the answer matches the sign of −18.
3.34
1. −62. Since a larger positive number was subtracted from a smaller positive number, a negative result was expected.
3.35
1. 71. Subtracting a negative number is the same as adding a positive number.
3.36
1. −17. Since |19| < |−36|, the sign of the answer matches the sign of −36, which is negative.
3.37
1. $89
3.38
1. 2,106. Since both numbers are positive, the product is positive.
3.39
1. −234. Since the numbers have opposite signs, the product is negative.
3.40
1. −29. The numbers have opposite signs, so the division will result in a negative number.

3.41

1. 7. Since the signs of the numbers match, the division results in a positive number.

3.42

1. The average daily balance was $529.

3.43

1. 313

3.44

1. 72

3.45

1. 630

3.46

1. 2,701

3.47

1. −15

3.48

1. 1,516

3.49

1. 5

3.50

1. 403

3.51

1. 94 is not a perfect square.
2. 441 is a perfect square.

3.52

1. not a rational number
2. rational number
3. rational number
4. rational number
5. rational number

3.53

1. $a \times b = 8 \times 26 = 208$ and $b \times c = 14 \times 12 = 168$. The fractions are not equivalent.

3.54

1. $\frac{3}{1}0$

3.55

1. \(\frac\)

3.56

1. $\frac{4}{5}73$

3.57

1. $\frac{1}{3}40$

3.58

1. $\frac{3}{7}36$

3.59

1. \(\frac439\)

3.60

1. $3\frac{1}{7}26$

3.61

1. $\frac{1}{3}114$

3.62

1. 5.108

3.63

1. 18.63

3.64

1. 1.6

3.65

1. \(\frac\)

3.66

1. $\frac{2}{1}110$

3.67

1. 2.664

3.68

1. $\frac{3}{8}35$
2. $\frac{1}{1}112$

3.69

1. $\frac{2}{0}3192$

3.70

1. \(\frac351\), or in decimal form, $0.319\overline{09}$

3.71

1. The process used above yields $\frac{5}{4}3260$.

3.72

1.

Studying math: 5 hours
Studying history: 2.5 hours
Studying writing: 1.25 hours
Studying physics: 1.25 hours

3.73

1. 720 calories of protein

3.74

1. 321.868 km

3.75

1. 23.656 liters

3.76

1. $\frac{4}{1}00$
2. $\frac{5}{0}100$

3.77

1. 0.14
2. 0.07

3.78

1. 300
2. 841.64

3.79

1. 120
2. 800

3.80

1. 7%
2. 85%

3.81

1. 440 calories of protein

3.82

1. 12% of registered voters in the small town voted in the primaries.

3.83

1. We want the original price of the item, which is the total. We know the percent, 40, and the percentage of the total, $30. To find the original cost, use \(\frac\), with $x = 30$ and $n = 40$. Calculating with those values yields \(\frac40\, = \,75\). So, the original was $75.

3.84

1. perfect square
2. not a perfect square
3. perfect square
4. not a perfect square

3.85

1. rational
2. irrational
3. irrational
4. irrational

3.86

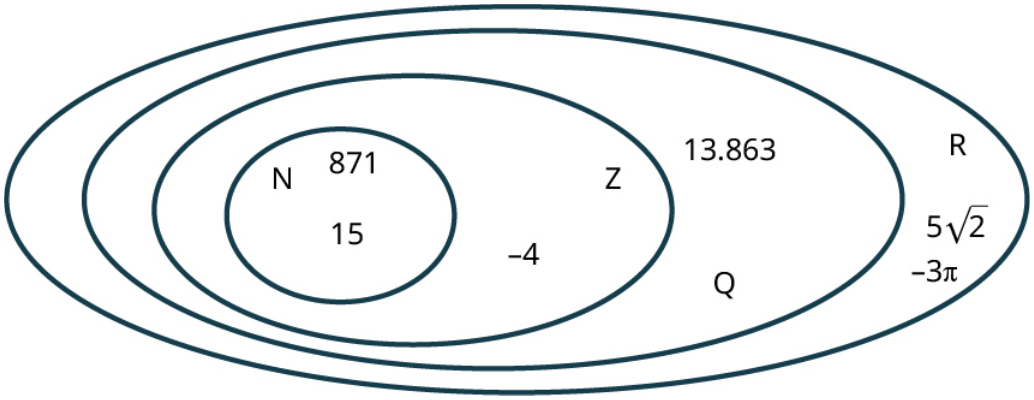
1. $5\sqrt{22}$. The rational part is 5, and the irrational part is $\sqrt{22}$.

3.87

1. $\sqrt{733}$. The rational part is 1, and the irrational part is $\sqrt{733}$.

3.88

1. $11\sqrt{15}$. The rational part is 11, and the irrational part is $\sqrt{15}$.
3.89
1. $18\sqrt{15}$
3.90
1. 7.3π
3.91
1. The two numbers being subtracted do not have the same irrational part, so the operation cannot be performed without a calculator.
3.92
1. $12\sqrt{3}$
2. $\frac{3}{4}225\sqrt{11}$
3.93
1. $37.8\sqrt{6}$
2. 19
3.94
1. \(\frac{5}\)
2. \(\frac18\)
3.95
1. \(\frac25{4} + \frac{4}\)
3.96
1. real
2. not real
3. real
3.97
1. irrational number
2. integer, rational number
3. rational number
3.98
1.



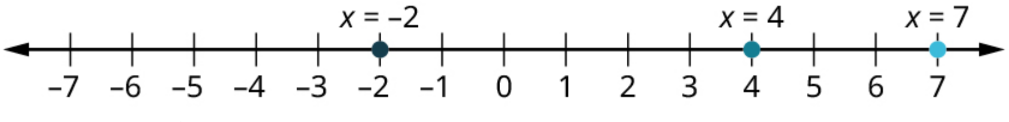
Venn diagram showing −4, 13.863, 15, 871, $5\sqrt{2}$, and -3π

3.99
1. dstributive property
2. additive inverse property
3.100
1. $9 \times 8 = 99$. Using that, the problem can be changed to 99×8. Change to $99 = (100 - 1)$. Using the distributive property, $99 \times 8 = (100 - 1) \times 8 = 100 \times 8 - 1 \times 8 = 800 - 8 = 792$.
3.101
1. 93 = 9 (mod 12)
2. 387 = 3 (mod 12)
3.102
1. 4:00
3.103
1. 9:00
3.104
1. 4

3.105
1. 5:00
3.106
1. Thursday
3.107
1. 12^{21}
2. Since the bases are not the same (one is 3, the other 4), this cannot be simplified using the product rule for exponents.
3.108
1. b^9
3.109
1. b^2
3.110
1. $2^{14} \times 19^{14}$
3.111
1. $a^6 \times b^6$
3.112
1. \(\frac\)
2. \(\frac\)
3.113
1. 11^{48}
2. a^{42}
3.114
1. \(\frac\)
2. \(\frac\)
3.115
1. 6×13^{-8}
2. $c^5 \times 2^{-9}$
3.116
1. \(\frac\)
2. \(\frac16\)
3.117
1. Is not written in scientific notation; 42.67 is not at least 1 and less than 10.
2. Is written in scientific notation
3. Is not written in scientific notation; The absolute value of –80.91 is not at least 1 and less than 10.
3.118
1. -3.38×10^4
2. 4.5×10^{-3}
3. 1×10^0
3.119
1. 0.0046113×10^{12}
3.120
1. $14,911.0 \times 10^{-6}$
3.121
1. 1,020,000
2. 0.0000409
3.122
1. 9.601×10^{13}
2. 1.53×10^{-6}
3.123
1. 1.198×10^4
2. 2.07×10^{-39}
3.124
1. 1.14256×10^{-43}
3.125
1. 9.0777×10^{28}
3.126
1. 6.87×10^7
2. 6.2881×10^3
3.127

1. 2.4×10^{-5}
2. 3.75×10^{0}
3.128
1. The transistor is 1.38×10^{-8} m larger than the diameter of an atom.
3.129
1. Neptune is 8.930×10^{1}, or 89.3, times further from the sun that Mercury.
3.130
1. 7.5×10^{9} cubic meters
3.131
1. A person exhales, on average, 8.4×10^{2}, or 840 pounds of carbon dioxide per year.
3.132
1. This is an arithmetic sequence. Every term is the previous term minus 2.2.
2. This is not an arithmetic sequence. The difference between terms 1 and 2 is 2, but between terms 3 and 4 the difference is 4. The differences are not the same.
3. This is an (infinite) arithmetic sequence. Every term is the previous term plus 6. The ellipsis indicates the pattern continues.
3.133
1. $a_1 = 4.5$, $d = 3.6$, $a_{36} = 310.5$
3.134
1. $d = 5$, $a_1 = -24$, and $a_{151} = 726$
3.135
1. 12,675.5
3.136
1. Christina will save $265 in week 52.
3.137
1. There are 2,520 seats in the theater.
3.138
1. It is a geometric sequence; common ratio is 5.
2. It is not a common ratio; term 2 is the first term multiplied by −2, but the sixth term is the fifth term multiplied by 3.
3. It is a geometric sequence; common ratio is $-\frac{1}{1}0$.
3.139
1. $\frac{3}{2}$
2. 2,048
3.140
1. 84,652,645
2. 40.444444
3.141
1. The amount in the account was $11,671.03 (rounded to two decimal places).
3.142
1. There are 1.6493×10^{13} organisms after 20 hours.
3.143
1. 0. 99996948242188

Check Your Understanding

1. 31 and 701 are prime. 56, 213 and 48 are composite.
2. $2 \times 5 \times 457$
3. 2
4. 630
5. The maximum number of bags that can be filled in this way is 10.
6. −4, 430
7. 

8. −13, −7, −2, 4, 10
9. 7
10. 13
11. 36
12. parentheses
13. exponents
14. −22

15. parentheses
16. 49
17. $-41,\ \frac{4}{3},\ 2.75,\ 0.2\overline{13}$ are rational; $\sqrt{13}$ is not.
18. $\frac{3}{5}$
19. $\frac{1}{9}24$
20. $\frac{3}{4}100$
21. $3\frac{1}{1}12$
22. $\frac{7}{3}3$
23. $\ \frac{3}{2}$
24. Using the process from the chapter, $\frac{3}{0}7336$, and there are other answers.
25. $3\frac{1}{1}12$
26. $110.25
27. 228
28. 7 new employees will be hired.
29. $10\sqrt{5}$
30. $-7\sqrt{7}$
31. $48\sqrt{5}$
32. \(\frac{7}\)
33. $\sqrt{77},\ -19,\ 38.902$
34.

$\mathbb{N} \subset \mathbb{Z}$	$\mathbb{N} \subset \mathbb{Q}$	$\mathbb{N} \subset \mathbb{R}$	$\mathbb{Z} \subset \mathbb{Q}$	$\mathbb{Z} \subset \mathbb{R}$	$\mathbb{Q} \subset \mathbb{R}$

35. distributive property
36. 2
37. 1
38. 5
39. Friday
40. a^8
41. 5^{-4} or \(\frac{1}\)
42. 6^9b^9
43. \(\frac\)
44. \(\frac\)
45. 4.56×10^{-3}
46. $567,000,000$
47. 5.48×10^3
48. -9.55×10^6
49. 3.552×10^8
50. 5.75×10^3
51. A pile of dollar bills that reaches the moon would contain 3.521×10^{12} bills.
52. No. The difference from term 1 to term 2 is different than the difference from term 4 to term 5.
53. 8
54. 613
55. $d = 3,\ a_1 = 14$
56. 35050
57. There will be 426 people in their survey group after 100 days.
58. Yes, each term is the previous term multiplied by 2.
59. The common ratio is −10.
60. 2,919.293 (rounded off to three decimal places)
61. 5.714 (rounded to three decimal places)
62. $30,188.57 (rounded off to two decimal places)

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15.3: Chapter 4

Your Turn

4.1

1. 416
2. 1,851
3. 17,488

4.2

1. $9 \times 10^2 + 2 \times 10^1 + 4 \times 10^0$
2. $1 \times 10^3 + 2 \times 10^2 + 7 \times 10^1 + 9 \times 10^0$
3. $4 \times 10^6 + 1 \times 10^5 + 3 \times 10^4 + 0 \times 10^3 + 0 \times 10^2 + 4 \times 10^1 + 5 \times 10^0$

4.3

1. 621
2. 3,203
3. 40,630,891

4.4

1. 1269

4.5

1. 42,136

4.6

1. 6,105,643

4.7

1. 257

4.8

1. 6,054

4.9

1. 1,248,073

4.10

1. 77
2. 240
3. 3,447

4.11

1. XXVII
2. XLIX
3. DCCXXXIX
4. MMMDCXLVII

4.12

1. 0, 1, 2, 3

4.13

1. 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B

4.14

1. 157

4.15

1. 2,014

4.16

1. 851

4.17

1. 27

4.18

1. 0, 1, 2, 3
10, 11, 12, 13
20, 21, 22, 23
30, 31, 32, 33
100

4.19

1.

0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B

10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 1A, 1B

20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 2A, 2B

30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 3A, 3B

40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 4A, 4B

50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 5A, 5B

60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 6A, 6B

70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 7A, 7B

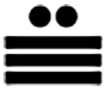
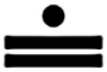
80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 8A, 8B

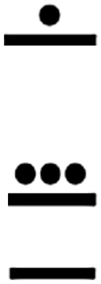
90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 9A, 9B

A0, A1, A2, A3, A4, A5, A6, A7, A8, A9, AA, AB

B0, B1, B2, B3, B4, B5, B6, B7, B8, B9, BA, BB

100

4.20
1. 0, 1, 2, 10, 11, 12, 20, 21, 22, 100
4.21
1. 2010_7
4.22
1. $554B_{12}$
4.23
1. 10001001_2
4.24
1. 
2. 



4.25
1. The result has the digit 7 in it. In base 4, the 7 is an illegal symbol.
4.26
1. The remainders include 10, which in base 6 is an illegal symbol.
4.27
1. Since 12 is larger than 10, the base 10 number cannot have less digits than the base 12 number. Since it did, we know an error has been made.
4.28
1. 1242_6
4.29
1.

+	0	1	2	3
0	0	1	2	3
1	1	2	3	10
2	2	3	10	11

3	3	10	11	12

Base 4 Addition Table

4.30

1. 633_7

4.31

1.

+	0	1	2	3	4	5	6	7	8	9	A	B	C	D
0	0	1	2	3	4	5	6	7	8	9	A	B	C	D
1	1	2	3	4	5	6	7	8	9	A	B	C	D	10
2	2	3	4	5	6	7	8	9	A	B	C	D	10	11
3	3	4	5	6	7	8	9	A	B	C	D	10	11	12
4	4	5	6	7	8	9	A	B	C	D	10	11	12	13
5	5	6	7	8	9	A	B	C	D	10	11	12	13	14
6	6	7	8	9	A	B	C	D	10	11	12	13	14	15
7	7	8	9	A	B	C	D	10	11	12	13	14	15	16
8	8	9	A	B	C	D	10	11	12	13	14	15	16	17
9	9	A	B	C	D	10	11	12	13	14	15	16	17	18
A	A	B	C	D	10	11	12	13	14	15	16	17	18	19
B	B	C	D	10	11	12	13	14	15	16	17	18	19	1A
C	C	D	10	11	12	13	14	15	16	17	18	19	1A	1B
D	D	10	11	12	13	14	15	16	17	18	19	1A	1B	1C

Base 14 Addition Table

4.32

1. $13B9_{12}$

4.33

1. 10010010_2

4.34

1. 32_6

4.35

1. 227_{12}

4.36

1. The symbols 4 and 5 are not legal symbols in base 4. Careful use of the base 4 addition table would correct this error.

	1	3	3
+	1	1	2
	3	1	1

The correct answer is 311_4.

4.37

1. This is correct if the numbers are base 10 numbers, but these numbers are base 14 numbers. In base 14, 9 + 9 is not 18, but instead is 13. Careful use of the base 14 addition table generates the correct answer, 163_{14}.

4.38

1.

*	0	1	2	3

0	0	0	0	0
1	0	1	2	3
2	0	2	10	12
3	0	3	12	21

Base 4 Multiplication Table
4.39
1.

*	0	1	2	3	4	5	6	7	8	9	A	B	C	D
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	A	B	C	D
2	0	2	4	6	8	A	C	10	12	14	16	18	1A	1C
3	0	3	6	9	C	11	14	17	1A	1D	22	25	28	2B
4	0	4	8	C	12	16	1A	20	24	28	2C	32	36	3A
5	0	5	A	11	16	1B	22	27	2C	33	38	3D	44	49
6	0	6	C	14	1A	22	28	30	36	3C	44	4A	52	58
7	0	7	10	17	20	27	30	37	40	47	50	57	60	67
8	0	8	12	1A	24	2C	36	40	48	52	5A	64	6C	76
9	0	9	14	1D	28	33	3C	47	52	5B	66	71	7A	85
A	0	A	16	22	2C	38	44	50	5A	66	72	7C	88	94
B	0	B	18	25	32	3D	4A	57	64	71	7C	89	96	A3
C	0	C	1A	28	36	44	52	60	6C	7A	88	96	A4	B2
D	0	D	1C	2B	3A	49	58	67	76	85	94	A3	B2	C1

4.40
1. 4000_6
2. 1010111_2
4.41
1. 4369_{12}
4.42
1. $10_6 \div 3_6 = 2_6$
2. \({50_{12}} \div = {6_{12}}\)
4.43
1. The symbols 4 and 5 are not legal symbols in base 4. Careful use of the base 4 multiplication table would correct this error. The correct answer is 333_4.
4.44
1. This is correct if the numbers are base 10 numbers, but these numbers are base 14 numbers. In base 14, \({49_{14}} \times {9_{14}} = 2{\text{D}}\) is not 81, but instead is 5B. Careful use of the base 14 addition table (Table 4.9) generates the correct answer, \({49_{14}} \times {9_{14}} = 2{\text{D}}\).

Check Your Understanding

1. A system in which the position of a numeral determines the value associated with that numeral.
2. 279
3. $4 \times 10^4 + 5 \times 10^3 + 2 \times 10^2 + 0 \times 10^1 + 9 \times 10^0$
4. 10
5. A numeral is a symbol representing a number. A number is a quantity or amount.

6. 601,947
7. 60
8. 20
9. Roman numerals do not use place value.
10. 341
11. 209
12. 247
13. CDLXXIX
14. A base 25 system would require 25 symbols.
15. Since the 4 is the second digit, its place value is 18^1 times 4, or 72.
16.

329

17. 40_9
18. 511_8
19. 2,126
20. In base 4, 5 is not a valid symbol. So, a mistake has been made.
21.

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	10
2	2	3	4	5	6	7	10	11
3	3	4	5	6	7	10	11	12
4	4	5	6	7	10	11	12	13
5	5	6	7	10	11	12	13	14
6	6	7	10	11	12	13	14	15
7	7	10	11	12	13	14	15	16

22. 1216_6
23. 7_8
24. 82_{14}
25. 5_{12}
26. In base 8, 8 is not a valid symbol. So, a mistake has been made.
27. A common base 14 error is performing the operation in base 10.
28. the addition table
29. The process is the same, except the multiplication table for the base is used instead of the familiar base 10 rules.
30. the multiplication table for the base
31. 2240_6
32. B_{14}
33. The 8 is not a symbol used in base 5.
34. A symbol that is not used in that base is present, or a base 10 rule is used.

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15.4: Chapter 5

Your Turn

5.1

1. 18 plus 11 OR the sum of 18 and 11
2. 27 times 9 OR the product of 27 and 9
3. 84 divided by 7 OR the quotient of 84 and 7
4. p minus q OR the difference of p and q

5.2

1. $43+67$
2. $(45)(3)$
3. $45 \div 3$
4. $89-42$

5.3

1. $n-20$
2. $(n+2) \bullet 6$ OR $6(n+2)$
3. n^3-5
4. $60h+40$

5.4

1. $5y=50$
2. $\frac{1}{2}n=30$
3. $3n-7=2$
4. $2x+7=21$

5.5

1. $24-(17-6)=13$
2. $(3 \bullet 6)+13=31$
3. $(12-6) \div (5-3)=3$
4. $5 \bullet (3^2+5)=70$

5.6

1. 9
2. -6

5.7

1. $5x^2-3x-7$

5.8

1. $-9x+4$

5.9

1. $2y+10$
2. $-2a-2b+8$
3. $112+16x$
4. $6x+18$
5. $3a+4$

5.10

1.

$$2x^2-11x+12$$

5.11

1.

$$4x^2+x-2$$

5.12

1. $x=3$

5.13

1. $y=28$

5.14

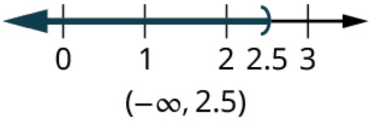
1. Henry has 42 books.

5.15

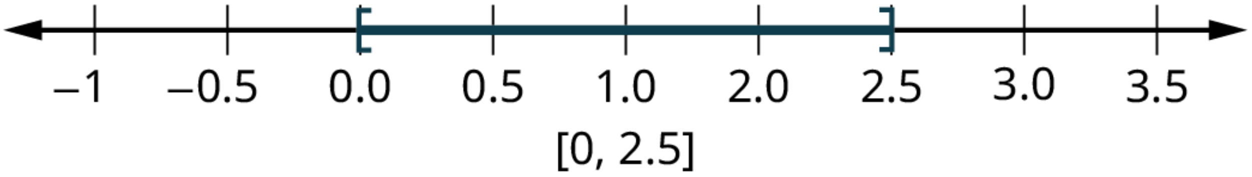
1. Total cost is \$48.85

5.16

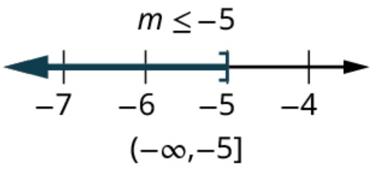
1. Answers will vary. For example: You can rent a paddleboard for \$25 per hour with a water shoe purchase of \$75. If you spent \$200, how many hours did you rent the paddle board for?
You rented the paddle board for 5 hours.
5.17
1. $12 = -1$, which is false; therefore, this is a false statement, and the equation has no solution.
5.18
1. $-12 = -12$, which is true; therefore, this is a true statement, and there are infinitely many solutions.
5.19
1. $t = \frac{I}{\mathrm{P}r}$
5.20
1. \(h = \frac\)
5.21



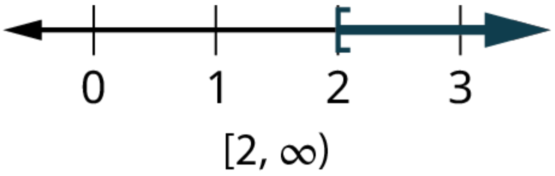
1.
5.22
1. $0 \le x \le 2.5$
Graph:



5.23

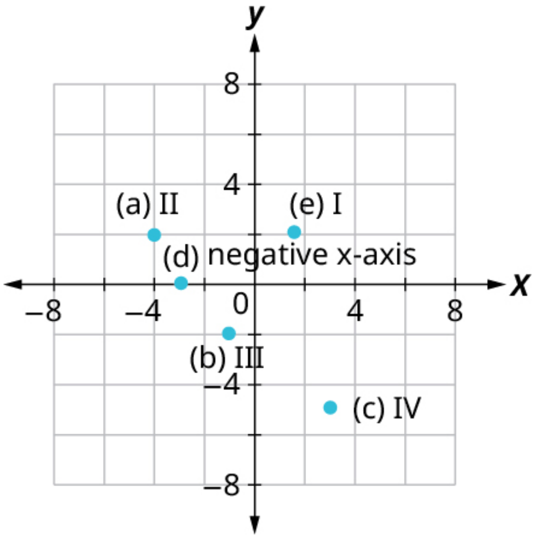


1.
5.24
1.



$p \geq 2$
5.25
1. Taleisha can send/receive 106 or fewer text messages and keep her monthly bill no more than \$50.
5.26
1. You could take up to 17 credit hours and stay under \$2,000.
5.27
1. Malik must tutor at least 23 hours.
5.28
1. a = 1 U.S. dollar, and b = 1.21 Canadian dollars, the ratio is 1 to 1.21; or 1:1.21; or $\frac{1}{1}.21$.
5.29
1. With a = 170 pounds on Earth, and b = 64 pounds on Mars, the ratio is 170 to 64; or 170:64; or $\frac{1}{7}064$.
5.30
1. \$219.51
5.31
1. 501.6 pounds
5.32
1. 720 cookies
5.33
1. The constant of proportionality (centimeters divided by inches) is 2.54. This tells you that there are 2.54 centimeters in one inch.

5.34
1. 30.5 hours (or 30½ hours, or 30 hours and 30 minutes)
5.35
1. \$207.50
5.36
1. 125.9 miles
5.37
1. The scale is $1 \text{ inch} = 91.25 \text{ miles}$. The other borders would calculate as: eastern and western borders are 273.75 miles, and northern border is 365 miles.
5.38
1. 184 inches, or 15 feet, 4 inches.
5.39



1.
5.40
1.

I. Yes
II. Yes

2.

I. No
II. No

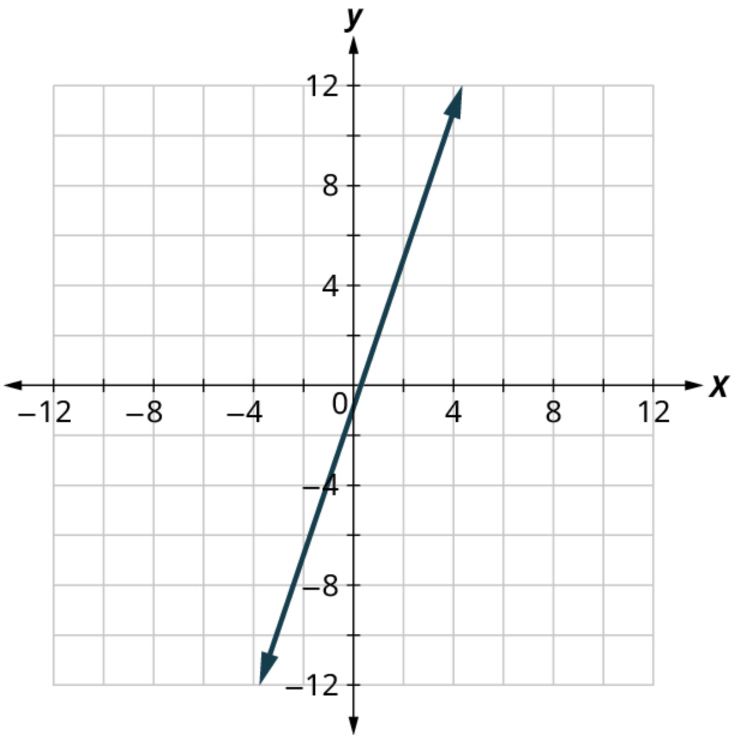
3.

I. Yes
II. Yes

4.

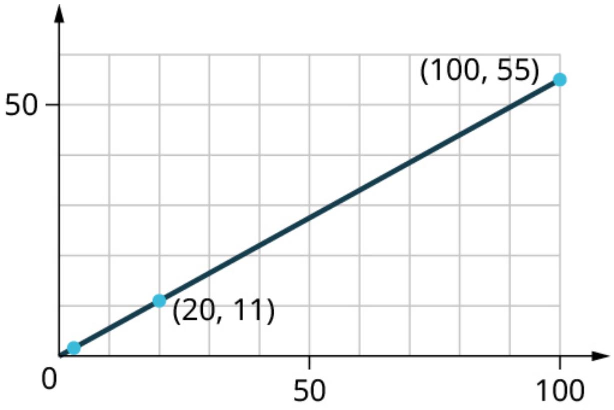
I. Yes
II. Yes

5.41



1.

5.42



1.

Your friend will pay $1.65.

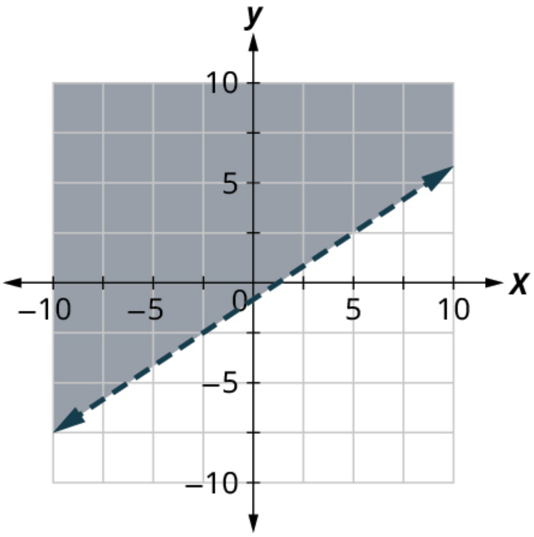
5.43

1. Yes
2. Yes
3. No
4. No
5. No

5.44

1. $y \geq -2x + 3$

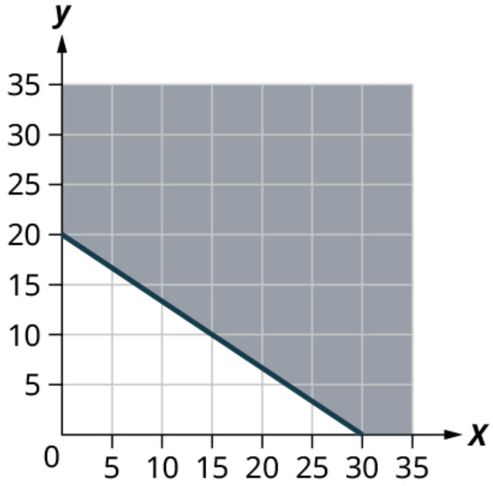
5.45



1.

5.46

1. $11x + 16.5y \geq 330$



2.

3. Answers will vary.

5.47

1. $x^2 + 4x + 3$

5.48

1. $2x^2 - 5x - 3$

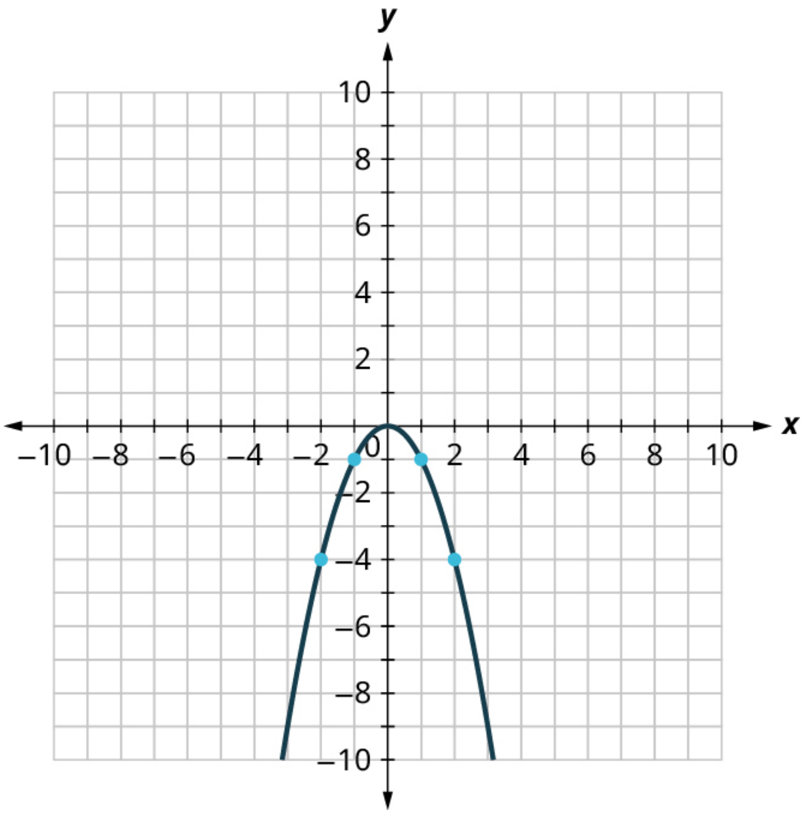
5.49

1. $(x + 2)(x + 4)$

5.50

1. $(x - 7)(x - 9)$

5.51



1.

5.52

1. $x=-3,\ x=2$

5.53

1. $x=2$ or $x=-5$

5.54

1. $x=3$ or $x=-5$

5.55

1. $x=5,\ x=-5$

5.56

1. $p=\frac{7}{5},\ p=-\frac{7}{5}$

5.57

1. $a=5,\ a=-3$

5.58

1. $y=1,\ y=\frac{2}{3}$

5.59

1. 16, 15 and –16, –15

5.60

1. $\text{width } = 5 \text{ feet, length} = 6 \text{ feet}$

5.61

1. 22
2. 6
3. $3t^2-2t+1$

5.62

1. There 205 unread emails after 7 days.

5.63

1. This relation is a function.
2. This relation is not a function.

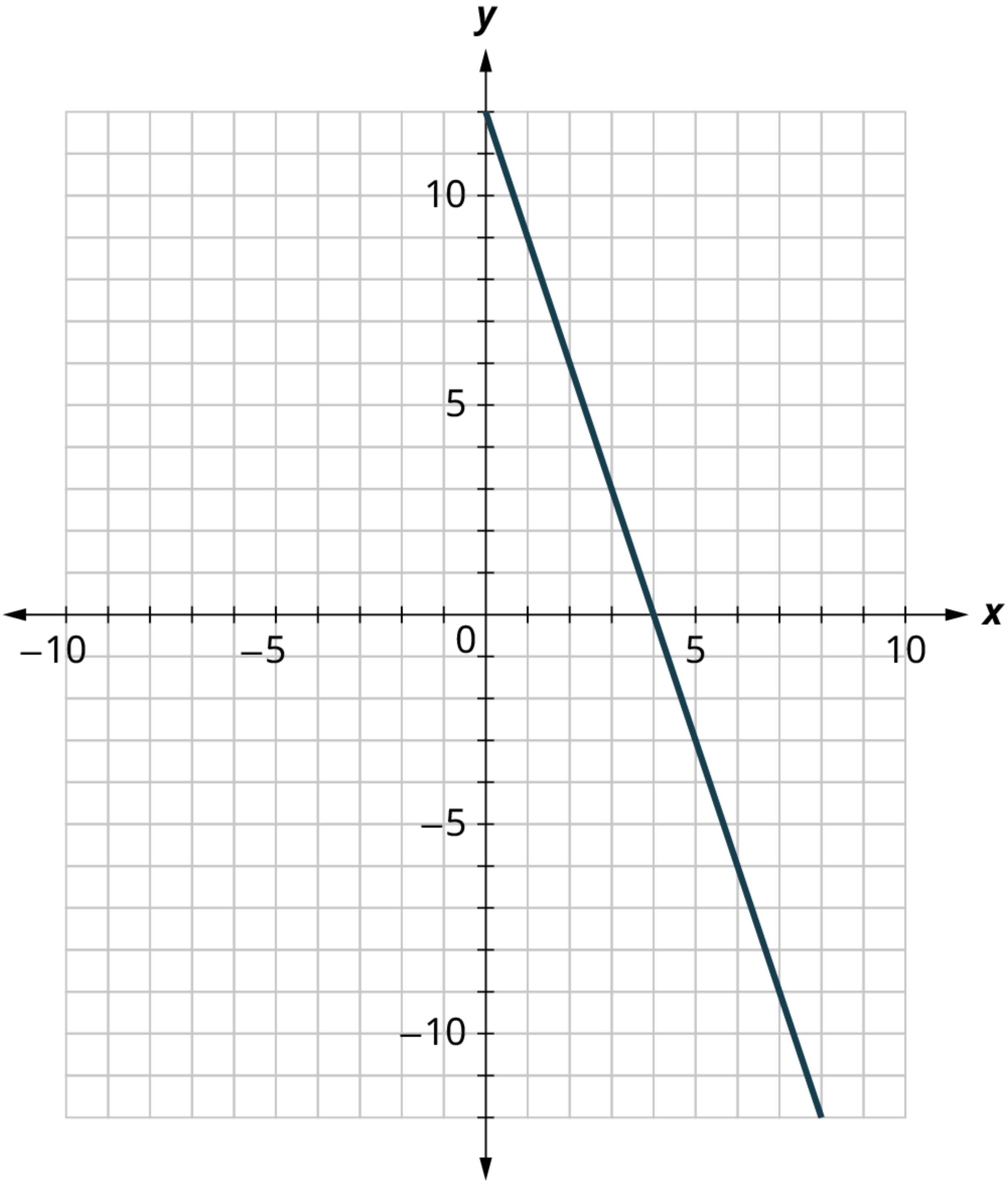
5.64

1. Both George and Mike have two phone numbers. Each x-value is not matched with only one y-value. This relation is not a function.

5.65

1. function

2. not a function
3. function
5.66
1. This graph represents a function.
5.67
1. This graph does not represent a function.
5.68
1. The domain is the set of all x-values of the relation: $\{1,2,3,4,5\}$.
2. The range is the set of all y-values of the relation: $\{1,8,27,64,125\}$.
5.69
1. The ordered pairs of the relation are: $\{(-3,\ 3),(-2,\ 2),(-1,\ 0),(0,\ -1),(2,\ -2),(4,\ -4)\}$.
2. The domain is the set of all x-values of the relation: $\{-3,-2,-1,0,2,4\}$.
3. The range is the set of all y-values of the relation: $\{-4,-2,-1,0,2,3\}$.
5.70
1. The graph crosses the x-axis at the point (2, 0). The x-intercept is (2, 0). The graph crosses the y-axis at the point (0, −2). The y-intercept is (0, −2).
5.71
1. The x-intercept is (4, 0) and the y-intercept is (0, 12).



5.72

1. $m = \frac{-}{4}3$

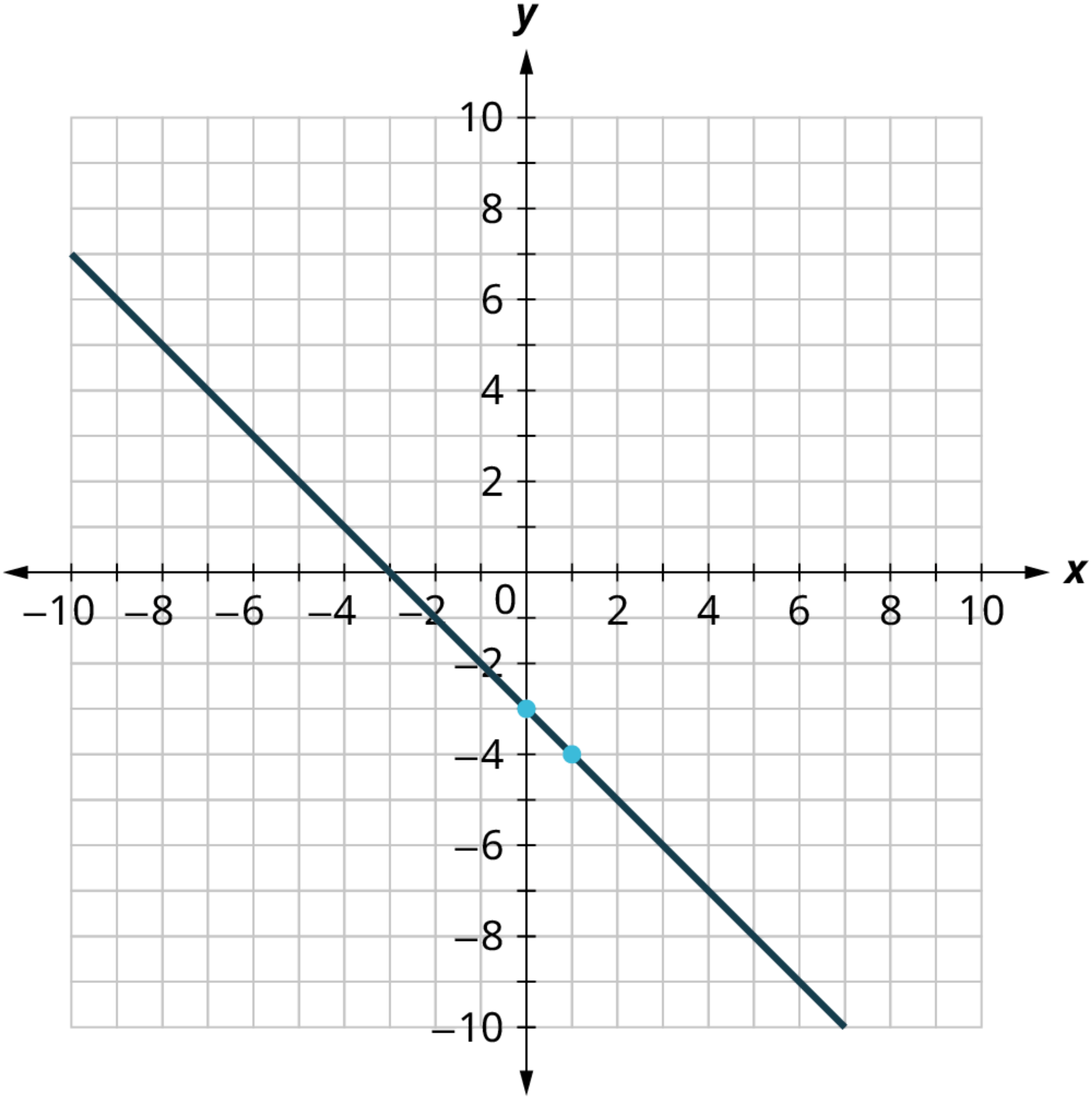
5.73

1. slope $= -1$

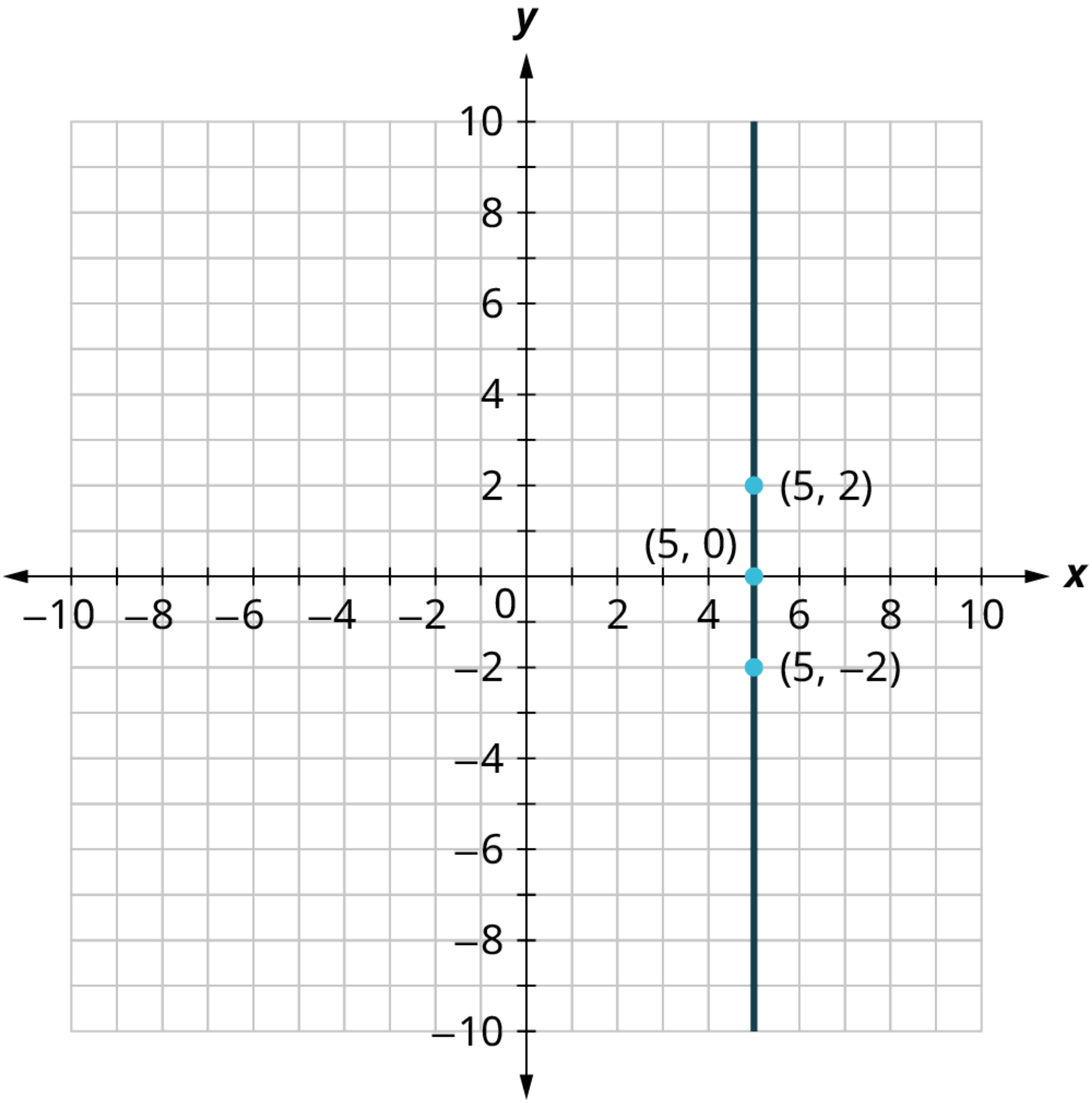
5.74

1. slope $m = 2$ and y-intercept $= (0, -1)$
2. slope $m = -\frac{1}{4}$ and y-intercept $= (0, 4)$

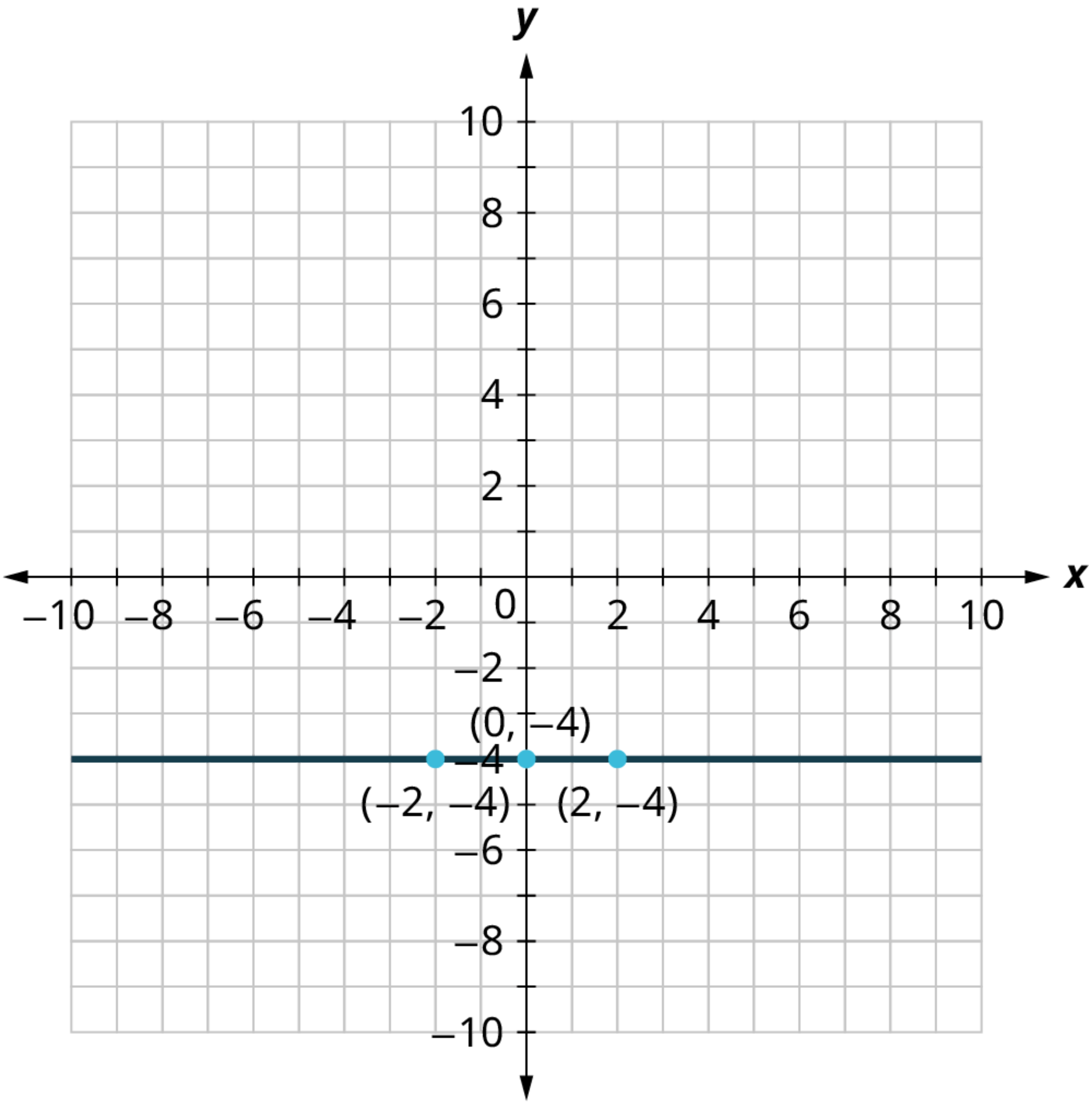
5.75



1.
5.76



1.
5.77



1.

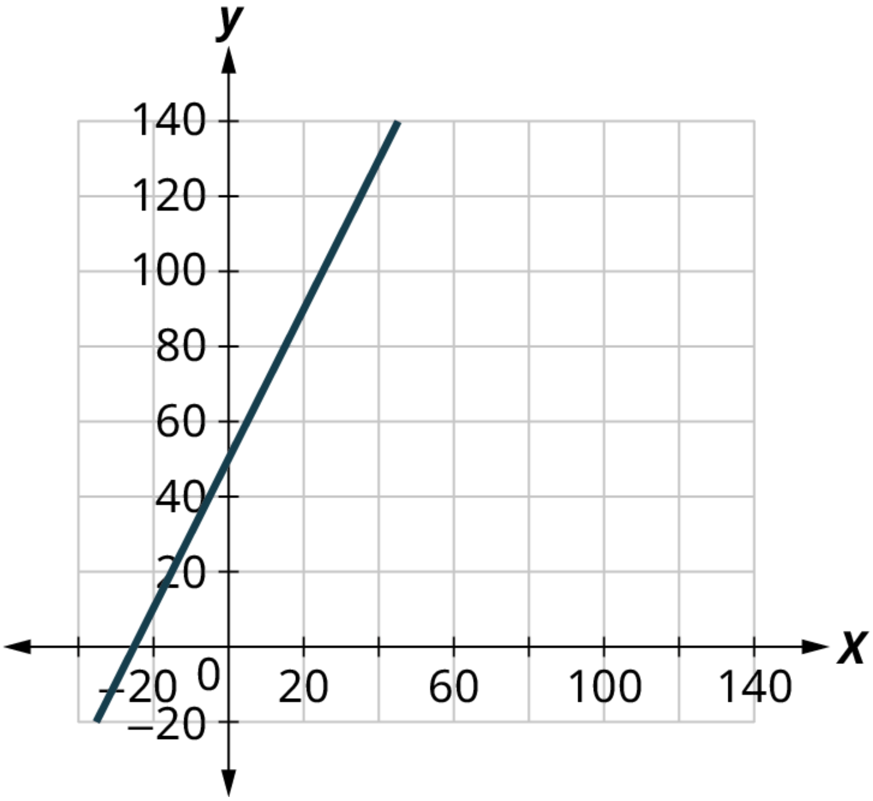
5.78

1. (0, 20) is the y-intercept and represents that there were 20 teachers at Jones High School in 1990. There is no x-intercept.
2. In the first 5 years the slope is 2; this means that on average, the school gained 2 teachers every year between 1990 and 1995. Between 1995 and 2000, the slope is 4; on average the school gained 4 teachers every year. Then the slope is 0 between 2000 and 2005 meaning the number of teachers remained the same. There was a decrease in teachers between 2005 and 2010, represented by a slope of –2. Finally, the slope is 4 between 2010 and 2020, which indicates that on average the school gained 4 teachers every year.
3. Answers will vary. Jones High School was founded in 1990 and hired 2 teachers per year until 1995, when they had an increase in students and they hired 4 teachers per year for the next 5 years. Then there was a hiring freeze, and no teachers were hired between 2000 and 2005. After the hiring freeze, the student population decreased, and they lost 2 teachers per year until 2010. Another surge in student population meant Jones High School hired 4 new teachers per year until 2020 when they had 80 teachers at the school.

5.79

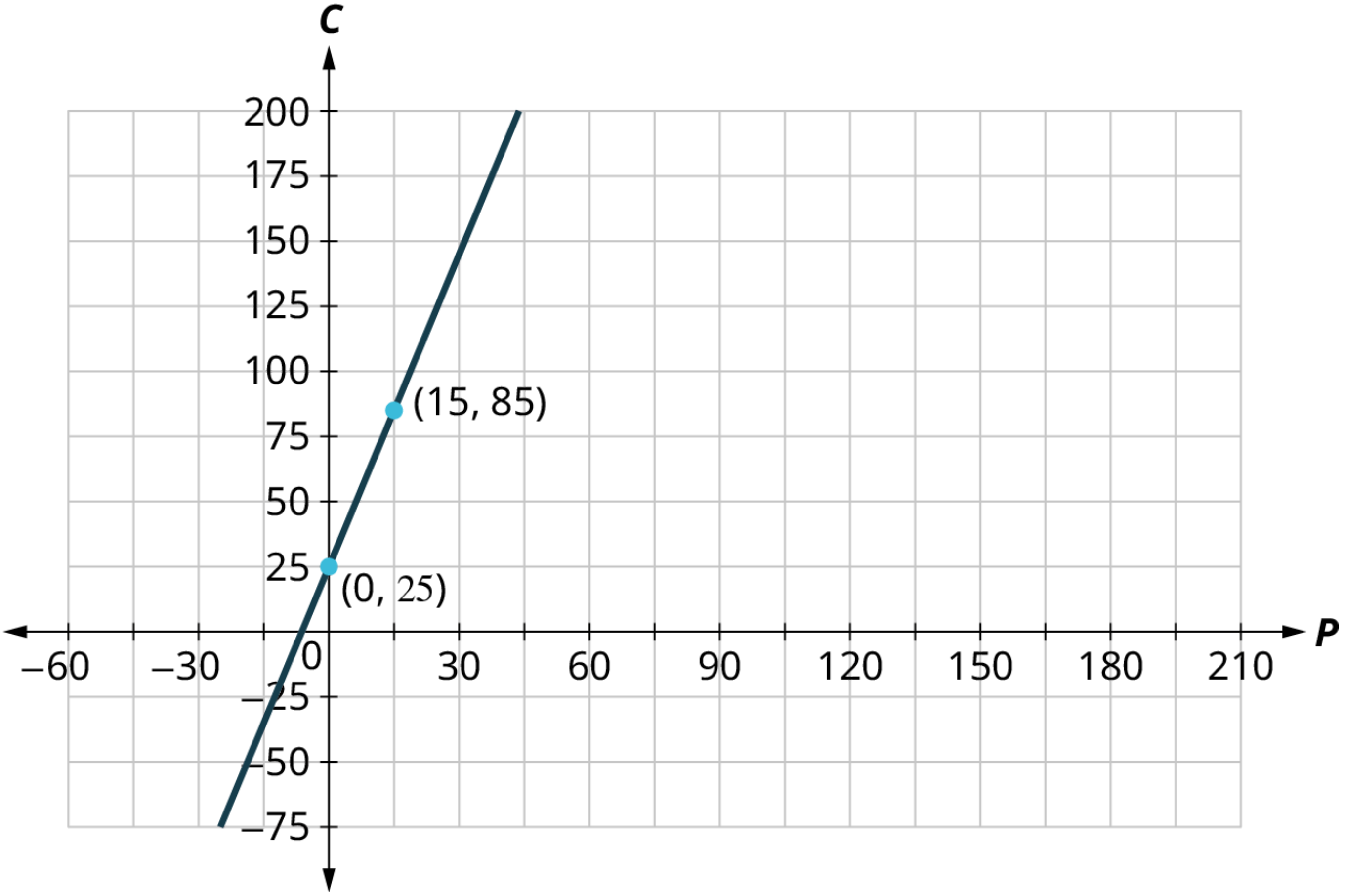
1. 50 inches
2. 66 inches
3. The slope, 2, means that the height h increases 2 inches when the shoe size(s) increases 1 size.

4. The h-intercept means that when the shoe size is 0, the woman's height is 50 inches.

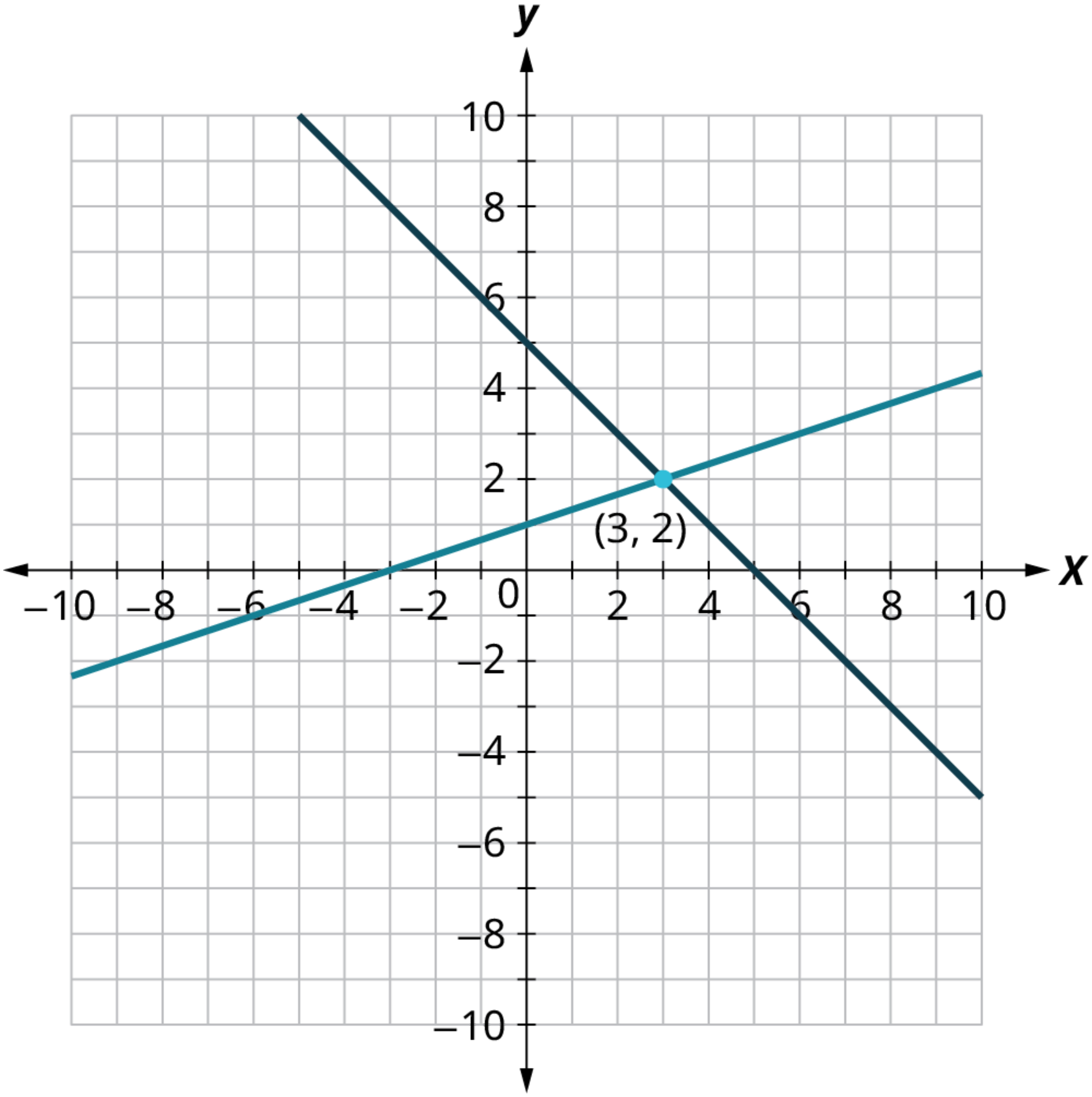


5.80
1. $25
2. $85
3. The slope, 4, means that the weekly cost, C, increases by $4 when the number of pizzas sold, p, increases by 1. The C-intercept means that when the number of pizzas sold is 0, the weekly cost is $25.

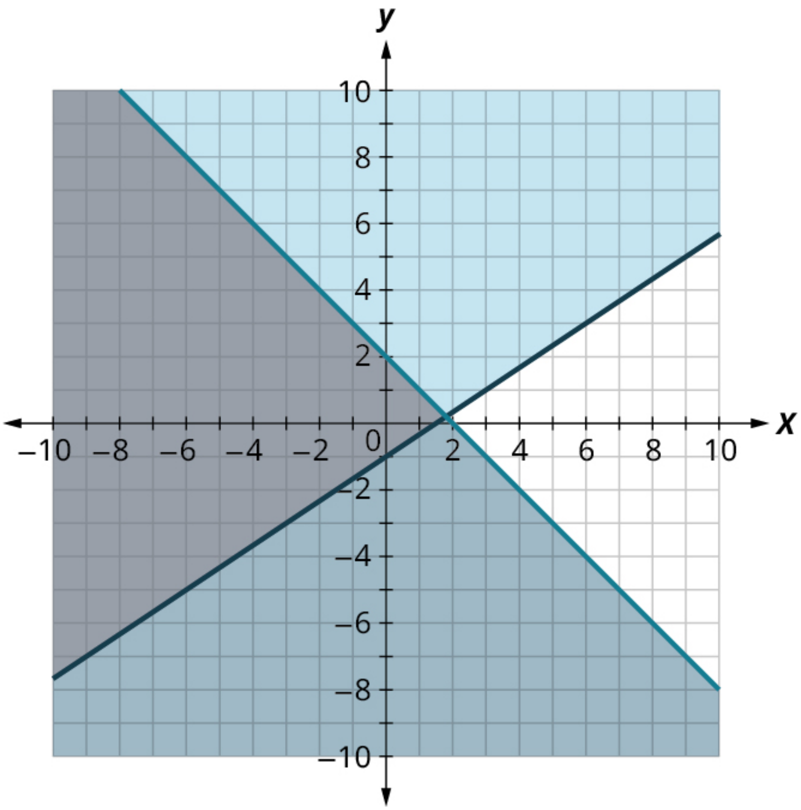
4. Graph:



5.81
1. No
2. Yes
5.82
1. No
2. No
5.83

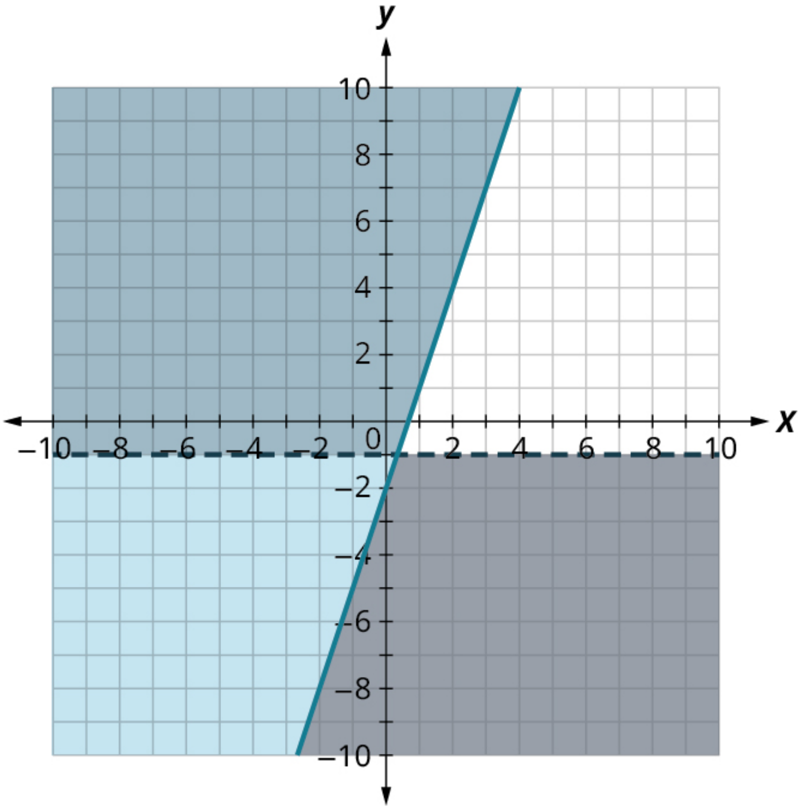


1.
5.84
1. (3, 2)
5.85
1. $(3, 2)$
5.86
1. There is no solution to this system.
5.87
1. There are infinitely many solutions to this system.
5.88
1. Jenna burns 8.3 calories per minute circuit training and 11.2 calories per minute while on the elliptical trainer.
5.89
1. not a solution
2. a solution
5.90
1. The region containing (0, 0) is the solution to the system of linear inequalities.
5.91
1. The solution is the darkest shaded region.



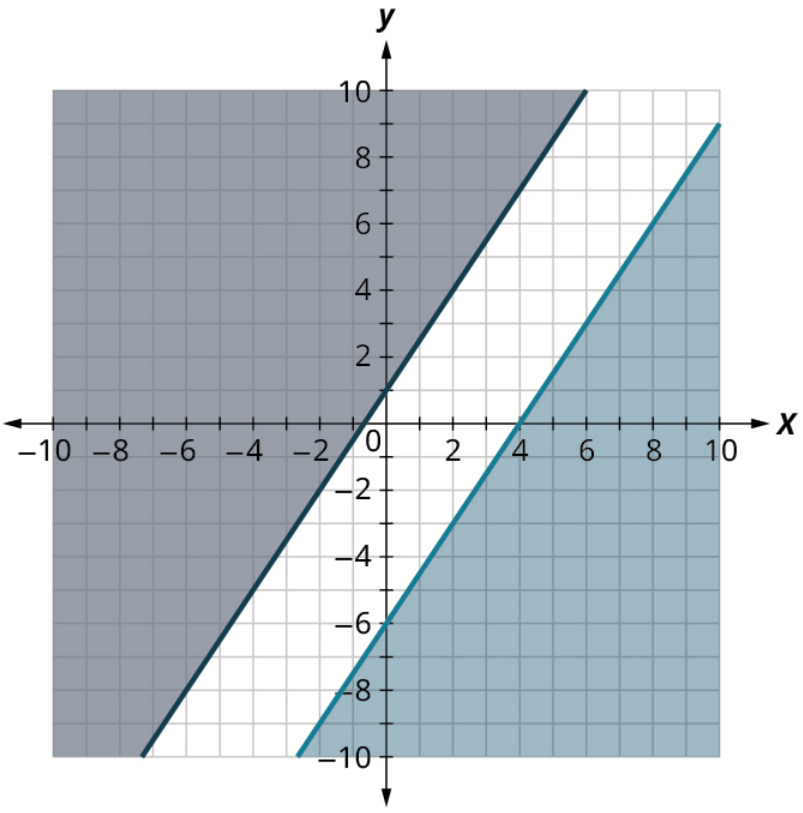
5.92

1. The solution is the lighter shaded region.



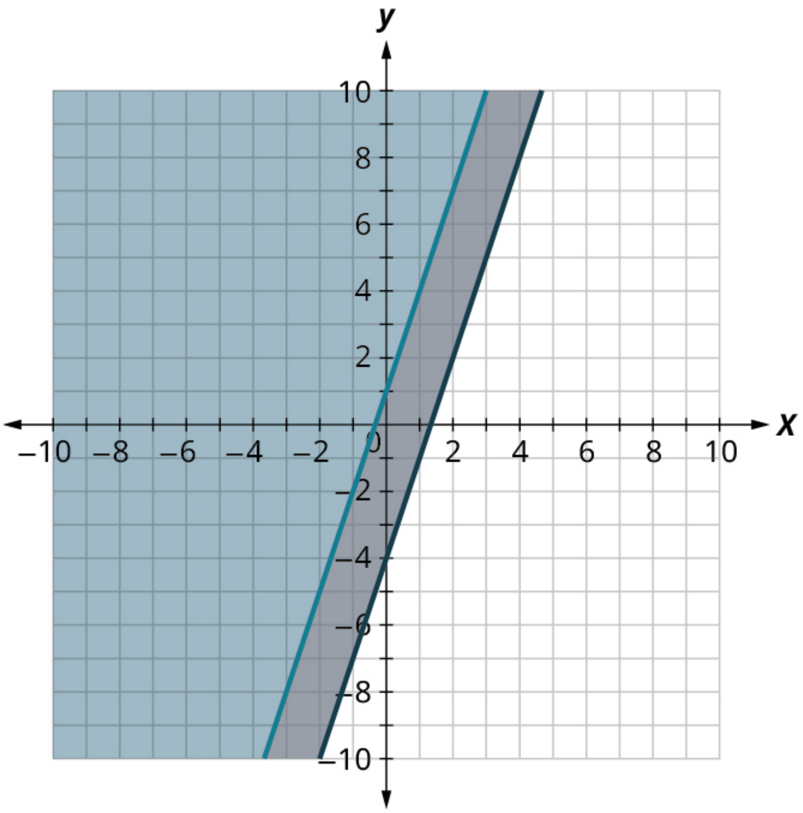
5.93

1. No solution.



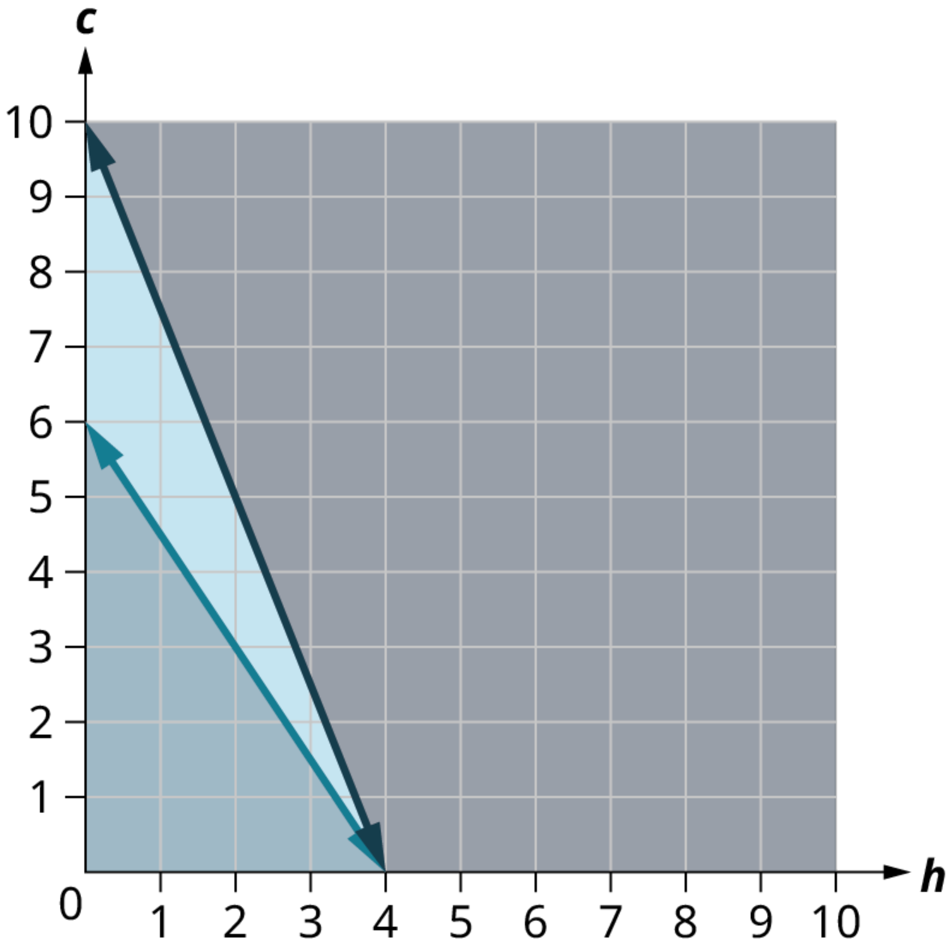
5.94

1. The solution is the lighter shaded region.



5.95

1. $\begin{cases} 240h + 160C \geq 800 \\ 1.40h + 0.50C \leq 5 \\ h \geq 0 \\ C \geq 0 \end{cases}$



2.

3. The point $(3, 2)$ is not in the solution region. Omar would not choose to eat 3 hamburgers and 2 cookies.

4. The point $(2, 4)$ is in the solution region. Omar might choose to eat 2 hamburgers and 4 cookies.

5.96

1. With a = the number of bags of apples sold, and b = the number of bunches of bananas sold, the objective function is $P = 4a + 6b$.

5.97

1. $T = 20x + 28y$

5.98

1. The constraints are $a + b \leq 20$ and $3a + 5b \leq 70$. The summary is: $P = 4a + 6b$, $a + b \leq 20$, and $3a + 5b \leq 70$.

5.99

1. The constraints are:

$15 \leq x \leq 22$
$13 \leq y \leq 19$

So the system is:

$T = 20x + 28y$
$15 \leq x \leq 22$
$13 \leq y \leq 19$

5.100

1. The maximum value for the profit P occurs when $x = 15$ and $y = 5$. This means that to maximize their profit, the Robotics Club should sell 15 bags of apples and 5 bunches of bananas every day.

Check Your Understanding

1. $J = V + 2$, $V = J - 2$
2. $5x + 8$, $2n + 3m$
3. $(8x + 12x) \div (4x - 2x)$
4. -3
5. $x^2 - 2xy + y^2$
6. $3x\left(3x^2 + x - 2\right)$
7. multiplication
8. division
9. It is a correct solution strategy.

Let

$$\begin{aligned} x &= 38 \\ 8(38-2) &\overset{?}{=} 6(38+10) \\ 8(36) &\overset{?}{=} 6(48) \\ 288 &= 288\checkmark \end{aligned}$$

10. It is a correct solution strategy.

Let

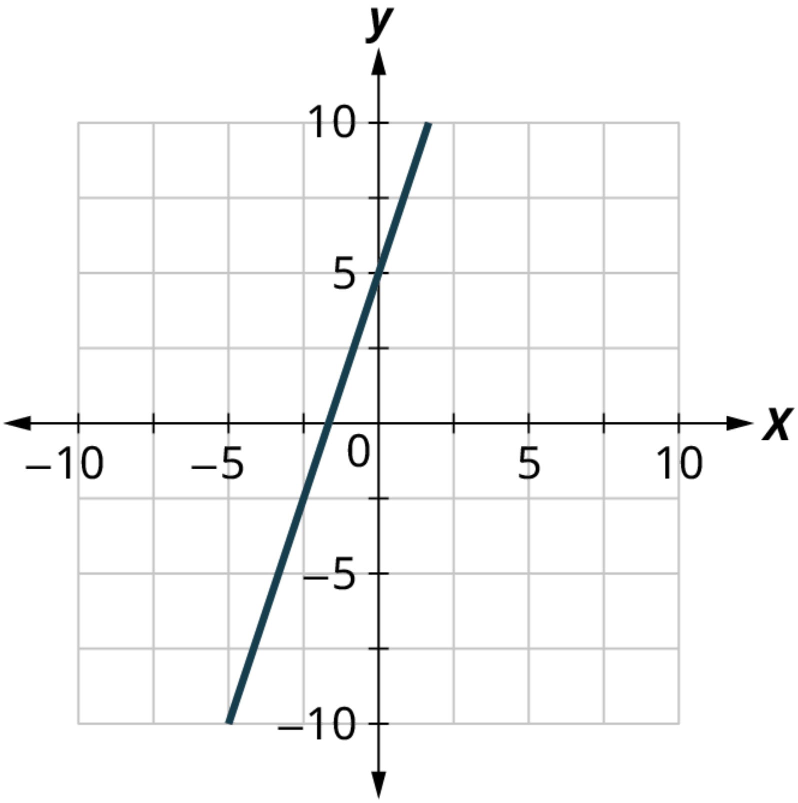
$$\begin{aligned} x &= -2 \\ 7+4(2+5(-2)) &\overset{?}{=} 3(6(-2)+7)-(13(-2)+36) \\ 7+4(2-10) &\overset{?}{=} 3(-12+7)-(-26+36) \\ 7+4(-8) &\overset{?}{=} 3(-5)-(10) \\ 7-32 &\overset{?}{=} -15-10 \\ -25 &= -25\checkmark \end{aligned}$$

11. This is not a correct solution strategy. The negative sign is not distributed correctly in the second line of the solution strategy. The second line should read $8x + 7 - 2x + 9 = 22 - 4x + 4$.
12. \(= 1.7x + 3\)
13. $40.40
14. \(= 1.6y + 5\)
15. $40.20
16. The Enjoyable Cab Company, because the cab fare will be $ 0.20 less than what it would cost to take a taxi from the Nice Cab Company.
17. Luis is; there are infinitely many solutions. If this is solved using the general strategy, it simplifies to $0 = 0$. This is a true statement, so there are infinitely many solutions.
18. d $F = \frac{9}{5}C + 32$
19. b $K = C + 273$, although d $C = K - 273$ is an equivalent formula.
20. a $K = \frac{5}{9}(F - 32) + 273$
21. b $R = \frac{9}{5}C + 492$
22. d $(-\infty, -1)$
23. b $[-5, \infty)$
24. c $\left[\frac{3}{2}, \infty\right)$
25. a $(-4, 3)$
26. e $6x \geq 24$
27. b $-6x > 18$
28. c $4x - 3 > -11$
29. c $-3x + 14 > -13$
30. b $8x < 764$
31. d $50x > 8{,}120$
32. a True
33. a True
34. a True
35. b False
36. a True
37. $12{:}16$
38. $16{:}28$
39. 324.00 British pounds (None of these.)
40. 10 inches

41. Yes he can, but barely. At 37 miles per gallon, Albert can drive 499.5 miles. While in theory he can make it, he probably should fill up with gasoline somewhere along the way!

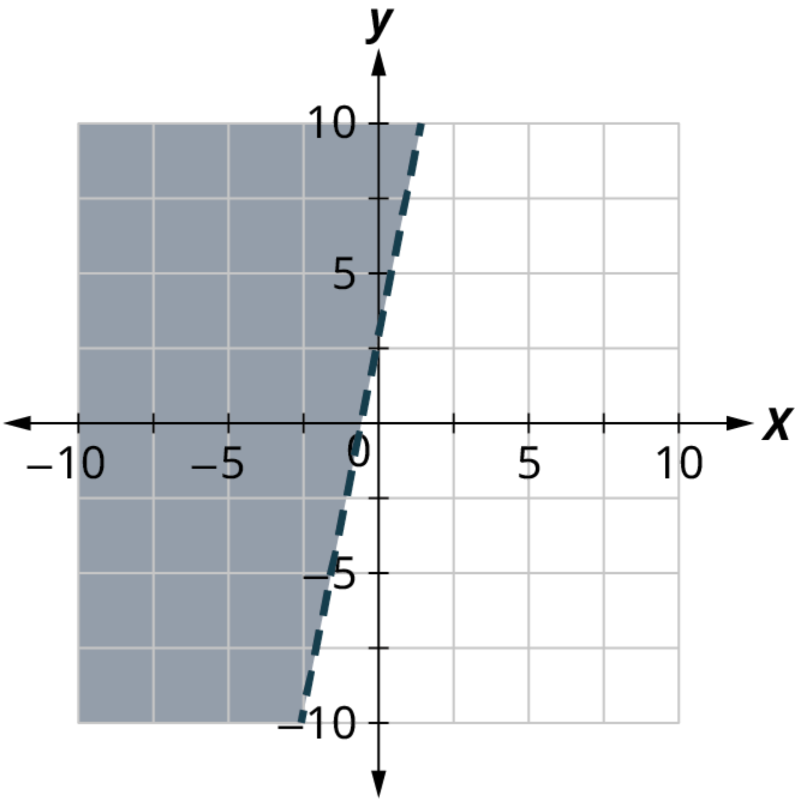
42. $\$40.66$

43. d $y = \frac{5}{3}$



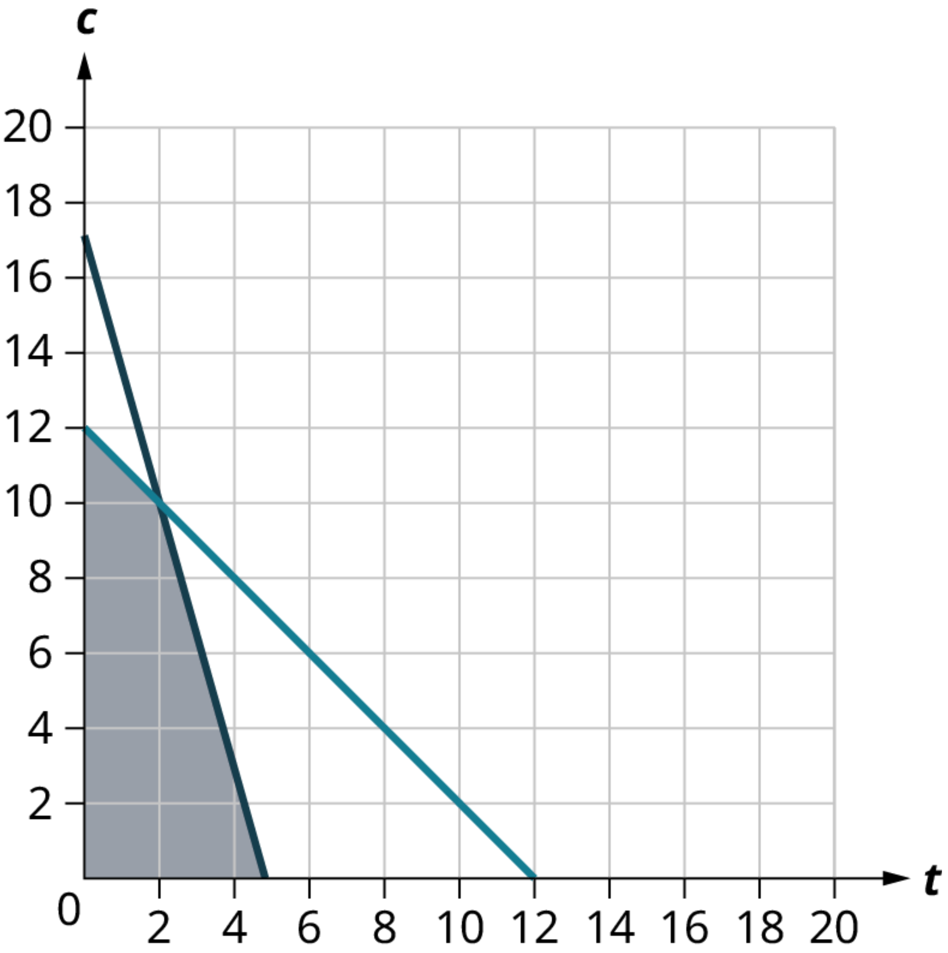
44.

45. b $y = \frac{1}{2}x + 4$



46.

47. b $y \leq -2x + 5$
48. d
49. b
50. c
51. d
52. b False
53. b False
54. a True
55. b False
56. \(x = \frac{5}{2} \pm \frac{2}\)
57. a True
58. b False
59. b False
60. a True
61. b False
62. a True
63. b False
64. b False
65. a True
66. b False
67. Elimination
68. Substitution
69. Substitution
70. Elimination
71. Substitution
72. Elimination
73. Elimination
74. Substitution
75. c
76. e
77. d
78. b
79. a
80. c
81. a
82. b
83. d
84. a
85. c
86. d



87.
88. b
89. $140

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15.5: Chapter 6

Your Turn

6.1
1. $\frac{3}{1}00$
2. $\frac{9}{4}100$
3. $\frac{6}{7}.2100$
4. $\frac{6}{7}0100$

6.2
1. 0.09
2. 0.24
3. 0.0218

6.3
1. 41%
2. 2%
3. 924.81%

6.4
1. 338
2. 2058.75

6.5
1. 250
2. 6,000

6.6
1. 62.5%
2. 40%

6.7
1. $529.50

6.8
1. 66.31%

6.9
1. 80 bulbs.

6.10
1. Discount is $496; sale price is $1,054.00
2. Discount is $2.75; sale price is $24.75

6.11
1. 35%
2. 12%

6.12
1. $13.00
2. $220.00

6.13
1. Discount is $330; sale price of the bed is $220.

6.14
1. 15%

6.15
1. $59.00

6.16
1. Markup is $396; retail price is $2,196.00.
2. Markup is $1.05; retail price is $11.55.

6.17
1. 58.33%
2. 200%

6.18
1. $35.00
2. $8.99

6.19
1. The retail price of the bed is $1,260.00.

6.20

1. 24%

6.21

1. $38.00

6.22

1. Sales tax is $104.93; total price is $1,603.93.
2. Sales tax is $1.88; total price is $28.77.

6.23

1. Sales tax rate = 5.25%
2. Sales tax rate = 6.75%

6.24

1. $145.78
2. $489.00

6.25

1. Sales tax is $111.22. The total price Daryl pays is $1,260.22.

6.26

1. The sales tax is $9.

6.27

1. Interest = $2,409.99, the loan payoff is $9,109.99
2. Interest = $8,901, the loan payoff is $34,701.00

6.28

1. She will pay $878.75 in interest, total repayment is $10,378.75.

6.29

1. He will pay $727.86 in interest, payoff is $9,127.86.

6.30

1. The cost to borrow is $115.51 and their payoff is $3,815.51.

6.31

1. Interest = \$253.8, FV = \$4,753.80
2. Interest = \$406.00, FV = \$2,406.00
3. Interest= \$1019.18, FV = \$121,019.18
4. Interest = \$253.89, FV = \$4,933.89

6.32

1. Mia will receive $6,254.85.

6.33

1. $846.58
2. $172.91

6.34

1. $709.59
2. $13,182.74

6.35

1. $2,313.74

6.36

1. $443.90

6.37

1. $14,285.72. $14,285.72 needs to be invested so that, after 10 years at 7.5% interest, the investment will be worth $25,000.
2. $97,709.93. $97,709.93 needs to be invested so that, after 35 years at 6.5% interest, the investment will be worth $320,000.
3. $48,813.56. $48,813.56 needs to be invested so that, after 270 months at 3.75% interest, the investment will be worth $90,000.

6.38

1. $10,549.46

6.39

1. The CD is worth $5,627.54 after 4 years.

6.40

1. CD would have been worth $5,600. Oksana earned $27.54 more with compound interest.

6.41

1. $11,443.81
2. $26,551.23

6.42

1. $5,813.64

6.43

1. $3,555.88

6.44
1. $100,811.07
2. $82,135.75
6.45
1. $96,271.23
6.46
1. A rate of 7% compounded quarterly is equivalent to a simple interest rate of 7.19%.
6.47
1. The effective annual yield is 2.53%.
6.48
1. The effective annual yields for the banks are 3.1158% for Smith Bank, 3.11% for Park Bank, 3.1381% for Town Bank, and 3.144% for Community Bank. Community Bank has the best yield.
6.49
1. Mateo’s budget is below.

Income Source	Amount	Expense	Amount
Electrician	$3,375.00	Mortgage	$987
Side jobs	$300.00	Truck payment	$589
		Truck insurance	$312
		Utilities	$167
		Clothing	$150
		Entertainment	$400
		Credit Card	$325
		Food	$470
		Gasoline	$375

His total monthly income is $3,675.00, and his monthly expenses are $3,775. Mateo falls $100.00 short each month.
6.50
1. Maddy’s budget is below.

Income Source	Amount	Expense	Amount
Engineer	$6,093.75	Mortgage	$1,452.89
		Car payment	$627.38
		Car insurance	$179.00
		Health Insurance	$265.00
		Utilities	$320.00
		Clothing	$150.00
		Entertainment	$400.00
		Credit Card	$450.00
		Food	$370.00
		Gasoline	$175.00

Her total monthly income is $6,093.75, and her monthly expenses are $4,389.27. Maddy has $1,704.48 in extra income per month. This is her cushion in the budget.
6.51
1. Heather’s budget is now

Income Source	Amount	Expense	Amount
Nursing	$3,765.40	Mortgage	$1,240
Part-time	$672.00	Car Payment	$489
		Student Loan	$728
		Car Insurance	$139
		Utilities	$295
		Clothing	$150
		Entertainment	$300
		Credit Card	$200
		Food	$400
		Gasoline	$250

The changes have added $355.00 to her budget. As her extra income is $601.40, she can afford the changes.
6.52
1. Heather’s total income is $4,437.40.
For the necessities, Heather should budget $2,218.70.
For her wants, she should budget $1,331.22.
For savings and extra debt service, she should budget $887.48.
Her necessities total $3,536.00, which exceeds the suggested budget amount of $2,218.70.
Her wants total $300.00, which is below the suggested budget amount of $1,331.22.
Her excess income is $601.40, which is below the suggested budget amount of $887.48.
Heather should make some changes.
6.53
1. Elijah should budget $1,631.72 for needs, $979.03 for wants, and $652.69 for savings and debt service. When choosing where to live, what to eat, and what to drive, he should make choices that keep those costs, combined with debt service costs, gasoline, and utilities, below $1,631.72. This means he will have to make decisions about what his priorities are. Elijah should then figure out his wants, and stay within the limits here, that is, keep those costs below $979.03. Finally, he can begin to build his savings with the remaining $652.69.
6.54
1. Her monthly income is $3,641.66.
Needs (50%): $1,820.83, Wants (30%): $1,092.50, Savings (20%): $728.33
2. $1,061.83
3. $1,092.50, $728.33
4. She only has budgeted $1,061.83 for other necessities. It is difficult to imagine Fran being able to afford to change jobs and move, unless she reallocates money that she would want to save or use for entertainment or takes on another job. If Fran uses all the money that the 50-30-20 budget sets aside for savings, she then would have $1,790.16 to spend on those other necessities. If she moves away from the 50-30-20 philosophy, she may be able to afford the move. However, that means changing her priorities.
6.55
1. $5,952.97
6.56
1. $3,741.46
6.57
1. $8,978.57
6.58
1. 17.31%
2. 13.2%
6.59
1. $42,499.63
6.60
1. $93,037.59
6.61
1. $1,227.30

6.62
1. $237.50 per year, $5,237.50 on maturity date.
6.63
1. $108,501.30
6.64
1. 4.28%
2. 3.29%
6.65
1. Looking at the table, the current price of a share is $30.68.
2. The high was $56.28, and the low was $30.05.
3. August 4, 2022
4. 4.77%
5. $4.66
6.66
1. $336.00
6.67
1. $52
2. 2.6%
6.68
1. $108,352.43
6.69
1. $1,815.83
6.70
1. 23.75%
6.71
1. 4.35%
6.72
1. 371.74%
2. 13.8%
6.73
1. $3,336.13
6.74
1. Traditional
6.75
1. $5,460
2. $10,920
3. 100% return on the $5,460 deposit
6.76
1. $813,128.60
2. $457,384.83
6.77
1. $355.91
2. $62.48
6.78
1. $213.07
2. $209.96
6.79
1. $34,700, 3.79%, 6 years
2. $539.57
3. $25,837.42
4. $104.14
5. $2,246.62
6. The amount paid to principal increases each month
6.80
1. $6,437.18
6.81
1. $9,745
6.82
1. Tiana can take out $12,500, the maximum in year 6.

6.83
1. Gap = \$10,850, Federal loans = \$7,500, Remaining \$3,350
6.84
1. \$2,987.63
6.85
1. \$82.70
2. \$137.42
6.86
1. \$259.69
6.87
1. There is no impact on Ryann’s interest rate. The total Ryann pays back increases since the interest is extended to more periods.
6.88
1. \$188.03
6.89
1. \$5,310
6.90
1. \$11,735
2. \$97.80
6.91
1. Store-issued credit cards
2. Fees (both annual and penalty fees), reward programs, credit limits
3. Travel and entertainment cards
6.92
1. April 01 2015 to April 30 2015
2. \$9,000.00
3. \$5.99
6.93
1. \$567.32
6.94
1. \$22.64
2. \$13.49
3. \$77.77
6.95
1. \$566.44
2. \$3,267.17
6.96
1. \$1,152.35
6.97
1. \$47.47
2. \$84.00
3. \$19.90
6.98
1. \$33,838.48
6.99
1. \$32,777.75
6.100
1. \$436.70
2. \$649.47
6.101
1. \$1,516.05
6.102
1. \$10,567.00
6.103
1. \$401.05
6.104
1. 6.25%
2. $0.00003291\overline{6}$
6.105
1. \$74.49

2. $1,015.25
6.106
1. Purchase
2. Purchase
6.107
1. Personal injury protection
2. Liability insurance
6.108
1. $593.36
6.109
1. $42.02
6.110
1. $949.72
6.111
1. $319,620.60
6.112
1. $93,741.60
6.113
1. 6.1%
2. $1,383.61
3. $730.38
4. $723.66
5. $133,513.05
6. Payment 225
6.114
1. $1,521.20
6.115
1. $47,280
6.116
1. $80,300
6.117
1. $39,700
6.118
1. $25
6.119
1. Refund of $1,920
6.120
1. FICA is $244.80, SSI is $198.40
6.121
1. $9,114
6.122
1. $11,787
6.123
1. His credits exceed his taxes owed, so he receives a refund, which is for $1,673.94.

Check Your Understanding

1. 100
2. 0.387
3. 190.4
4. 834.15 (rounded to two decimal places)
5. 24.07%
6. 16 new drivers will be hired.
7. a reduction in price
8. the amount a retailer adds to the cost of goods to cover overhead and profit
9. $30.60
10. $497.60
11. Yes
12. $42.67
13. Interest is money paid by a borrower to a lender for the privilege of borrowing money.

14. The amount borrowed is the principal.
15. $1,169.10
16. $11,374.87
17. $266.82
18. $21,818.19
19. interest that is earned on both the principal and the interest from previous periods
20. compound
21. $15,698.69
22. $405.62
23. $30,266.48
24. Compare their effective annual yield.
25. 4.98%
26. A budget is an estimation of income and expenses over some period of time.
27. Expenses that represent basic living requirement and debt service.
28.

Income	Amount	Expenses	Amount
Job	$3,450.00	Rent	$925.00
		Car payment	$178.54
		Car insurance	$129.49
		Credit card	$117.00
		Gas	$195.00
		Food	$290.00
		Amazon Prime	$21.99
		Internet	$49.99
		Going out	$400.00
		Total	$2,307.01

29. $1,142.99
30. David should allocate $1,735 to necessary expenses, $1,035 to expenses that are wants, and $690 to savings and extra debt service.
31. David spends $1,835.03 on necessary expenses, which is fairly close to the 50-30-20 budget philosophy guidelines. He spends $471.98 on expenses that are wants, which is well below the guideline of $1,035. He has $1,149.99 in extra income, which is well above the savings and debt service guidelines.
32. A deposit account, held at a bank or other financial institution, which bears some interest on the deposited money.
33. Once money is deposited in a CD, the money cannot be withdrawn until the end of the CD period.
34. A money market account is more flexible. Transactions can be made after creating the money market account.
35. $8,549.93
36. 25.6%
37. $21,441.03
38. $2,030.65
39. Stocks
40. Income must be below a threshold for a person to use a Roth IRA.
41. They invest in different investment vehicles so that no single investment can have a large impact on the value of the mutual fund.
42. Each year, the holder of the bond receives $200. For 10 years, that’s $2,000 total earned with the bond.
43. Yes
44. $3,500, which is 4% of her income
45. $114.00
46. 9.93%
47. $5,207.59
48. Varies
49. Loan with a fixed period, and the borrower pays a fixed amount per period until the loan is paid off

50. More is applied to the principal.
51. $59.11
52. $2,233.12
53. $42,962.51
54. $39,544.50
55. Spring (March) before the start of the fall term
56. 9 years
57. The difference between the cost of college (including books, fees, room and board, etc.) and all non-loan financial aid they receive
58. In subsidized loans, the interest does not accrue while the student is still in school. In unsubsidized student loans, the interest begins accruing as soon as the money is disbursed.
59. $31,000
60. $13,028.61
61. 10 years
62. 10%
63. Bank issued, store issued, charge (Travel/entertainment) cards
64. Store issued
65. Due date, minimum payment due, and balance due
66. $19.30
67. $4,104.27
68. $696.21
69. Cost of delivering the vehicle to the dealer
70. Registers your car with the state, gets the license plate, and assigns the title of the car to the lender
71. $628.89
72. Interest rate higher loan term shorter
73. Title and registration, documentation fee, sales tax
74. $333.34
75. 3.75%
76. Purchasing
77. Cover the damage caused to your car if involved in an accident with another vehicle
78. Buying
79. Renting
80. Buying
81. Mortgage
82. $663.32
83. $129,802.20
84. $554.06
85. Escrow
86. Any and all income, including wages, salary, interest earned, gifts, and winnings
87. After deductions are subtracted
88. $37,000
89. After taxes are calculated, credits are subtracted from taxes owed.
90. No, only for up to $147,000
91. $4,813
92. $1,600

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15.6: Chapter 7

Your Turn

7.1
1. $4 \times 15 = 60$

7.2
1. $4 \times 15 \times 3 \times 2 = 360$

7.3
1. $3 \times 10 \times 10 \times 10 \times 10 \times 26 \times 26 \times 26 = 5{,}272{,}800{,}000$

7.4
1. 120

7.5
1. 720
2. 132
3. 70

7.6
1. 30
2. 24,024
3. 5,814

7.7
1. ${}_{15}P_3 = 2{,}730$

7.8
1. combination
2. combination

7.9
1. 15
2. 45
3. 2,002

7.10
1. 30,856
2. 111,930

7.11
1. 290,004

7.12
1. $\{\heartsuit, \spadesuit, \clubsuit, \diamondsuit\}$
2. {A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K}
3. {1, 2, 3, 4}
4. {3, 4, 5, 6, 7, 8, 9, 10, 11, 12}

7.13
1. Independent
2. Dependent

7.14
1. {H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6}

7.15
1. {J♡ Q♡, J♡ K♡, Q♡ J♡, Q♡ K♡, K♡ J♡, K♡ Q♡}

7.16
1. $P(\text{H} < 5) = 1$
2. $0 < P(\text{H} < 4) < 1$
3. $P(\text{H} \geq 5) = 0$

7.17
1. $\frac{1}{6}$
2. $\frac{2}{3}$
3. $\frac{1}{2}$

7.18
1. $\frac{5}{1}6$

7.19
1. $\frac{1}{3}$
2. $\frac{1}{2}$

3. $\frac{2}{3}$
7.20
1. \(\frac13 = 1.3\% \)
7.21
1. theoretical
2. subjective
3. empirical
7.22
1. $\frac{1}{3}16$
7.23
1. $\frac{1}{4}443680 = \frac{3}{9}10$
7.24
1. \(1 - \frac \approx 10.7\%\)
2. \(\frac \approx 0.86\%\)
7.25
1. $13{:}3$
2. $12{:}4 = 3{:}1$
7.26
1. The odds for E are $4{:}1$ and the odds against E are $1{:}4$.
7.27
1. $P(E) = \frac{1}{1}6$
2. $P(E) = \frac{2}{.}53.5 \approx 0.714$
7.28
1. Mutually exclusive
2. Not mutually exclusive
3. Mutually exclusive
7.29
1. $\frac{4}{1}0 + \frac{2}{1}0 = \frac{3}{5}$
2. Not appropriate; the events are not mutually exclusive.
3. $\frac{3}{1}0 + \frac{2}{1}0 = \frac{1}{2}$
7.30
1. $\frac{1}{2}$
2. $\frac{1}{2}$
3. $\frac{2}{3}$
7.31
1. $\frac{1}{3}$
2. 1
3. $\frac{2}{3}$
7.32
1. $\frac{3}{2}8$
2. $\frac{3}{5}6$
3. $\frac{3}{2}8$
7.33
1.

Roll	Probability
1	$\frac{1}{3}$
2	$\frac{1}{4}$
3	$\frac{1}{6}$
4	$\frac{1}{1}2$
5	$\frac{1}{1}2$
6	$\frac{1}{1}2$

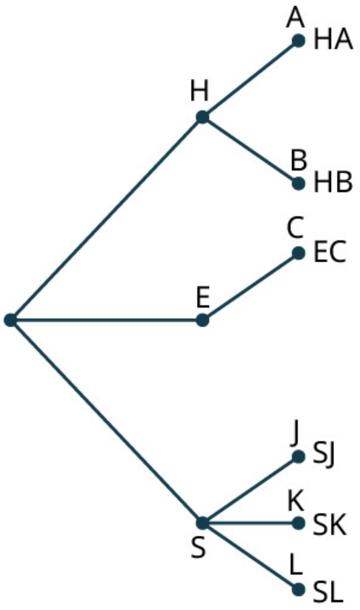
7.34
1. Not binomial (more than two outcomes)

2. Not binomial (not independent)
3. Binomial
4. Not binomial (number of trials isn’t fixed)
7.35
1. 0.146
2. 0.190
3. 0.060
7.36
1. 0.2972
2. 0.6615
3. 0.0919
4. 0.5207
5. 0.3643
7.37
1. $\frac{1}{0}3$
2. $\frac{3}{2}$
3. $\frac{2}{5}4 = \$6.25$
7.38
1. If you roll the special die many times, the mean of the numbers showing will be around 3.33.
2. If you repeat the coin-flipping experiment many times, the mean of the number of heads you get will be around 1.5.
3. If you play this game many times, the mean of your winnings will be around $10.
7.39
1. If the player bets on 7, the expected value is $-\$0.17$.
2. If the player bets on 12, the expected value is $-\$0.14$.
3. If the player bets on any craps, the expected value is $-\$0.11$.
The best bet for the player is any craps; the best bet for the casino is the bet on 7.

Check Your Understanding

1. 540
2. 14
3. 1,024
4. 800
5. 25,920
6. 120
7. 120
8. 1,320
9. 1,680
10. ${}_{15}P_4 = 32{,}760$
11. permutations
12. combinations
13. 66
14. 560
15. 20
16. 560
17. {0, 1, 2, 3, 4, 5, 6}
18.

	Crinkle Fries	Curly Fries	Onion Rings
8 Nuggets	8 nuggets with crinkle fries	8 nuggets with curly fries	8 nuggets with onion rings
12 Nuggets	12 nuggets with crinkle fries	12 nuggets with curly fries	12 nuggets with onion rings



19.
20. {8 nuggets with crinkle fries, 8 nuggets with curly fries, 8 nuggets with onion rings, 12 nuggets with crinkle fries, 12 nuggets with curly fries, 12 nuggets with onion rings}
21. {history with Anderson, history with Burr, English with Carter, sociology with Johnson, sociology with Kirk, sociology with Lambert}
22. $\frac{1}{4}$
23. $\frac{1}{2}$
24. $\frac{1}{2}$
25. 0
26. theoretically
27. subjectively
28. empirically
29. $\text{number of heads} > 20$
30. 89.9%
31. \(\frac \approx 0.278\%\)
32. \(\frac \approx 0.079\%\)
33. \(\frac \approx 0.16\%\)
34. \(\frac \approx 1.67\%\)
35. \(\frac \approx 1.9\%\)
36. \(\frac \approx 1.9\%\)
37. $1{:}2$
38. $5{:}1$
39. $1{:}1$
40. $3{:}5$
41. $11{:}2$
42. $\frac{4}{1}3$
43. $\frac{5}{1}2$
44. $\frac{3}{5}$
45. $\frac{3}{5}$
46. $\frac{3}{5}$
47. $\frac{2}{5}$
48. $\frac{2}{5}$
49. $\frac{3}{1}0$
50. $\frac{1}{4}$
51. $\frac{1}{9}$
52. $\frac{1}{6}$
53. $\frac{2}{1}5$
54. $\frac{4}{4}5$
55. $\frac{8}{4}5$
56. No (more than two outcomes)

57. Yes
58. No (number of trials is not fixed)
59. 0.9453
60. 0.1366
61. 0.8230
62. 4.1
63. If you roll this die many times, the mean of the numbers rolled will be around 4.1.
64. $0.417
65. If you play this game many times, the mean of the amount won/lost each time will be about 42 cents.
66. Yes; the expected value is positive.

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15.7: Chapter 8

Your Turn

8.1

1. None of the above (there’s no sample being selected here; the entire population is being surveyed)
2. Stratified random sample (the strata are the different majors)
3. Simple random sample

8.2

1.

Major	Frequency
Biology	6
Education	1
Political Science	3
Sociology	2
Undecided	4

8.3

1.

Number of people in the residence	Frequency
1	12
2	13
3	8
4	6
5	1

8.4

1.

Age range	Frequency
25-29	2
30-34	6
35-39	2
40-44	4
45-49	1
50-54	2
55-59	2
60-64	0
65-70	1

(Answers may vary depending on bin boundary decisions)

8.5

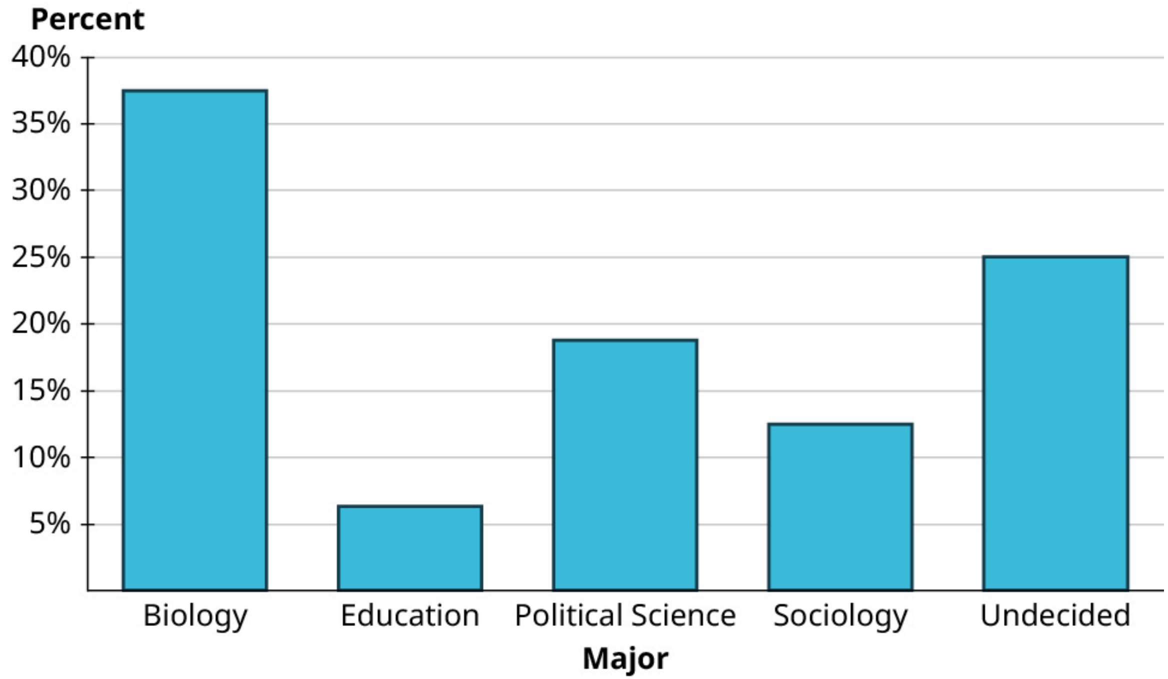
1.

Major	Frequency	Proportion

Major	Frequency	Proportion
Biology	6	37.5%
Education	1	6.3%
Political Science	3	18.8%
Sociology	2	12.5%
Undecided	4	25%

Note that these percentages add up to 100.1%, due to the rounding.

8.6

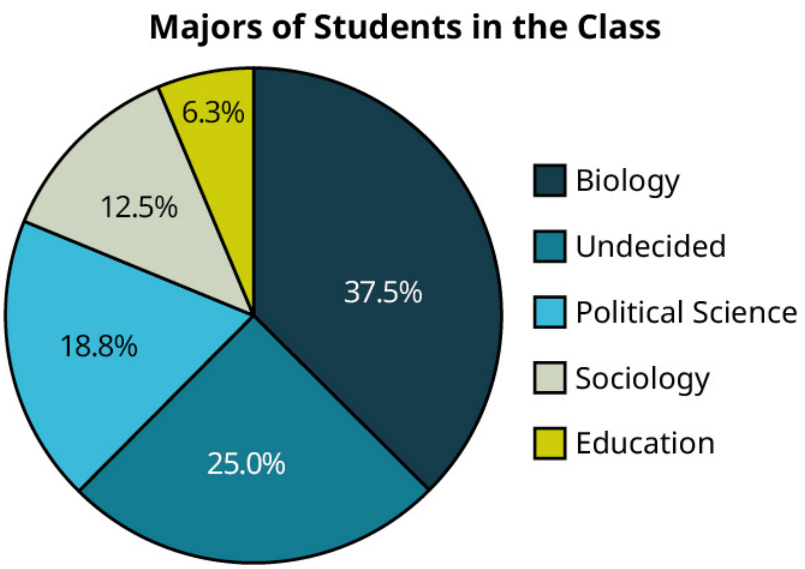


1.

8.7

1. Southeast
2. Just over 10%
3. Outlying Areas and Rocky Mtns.

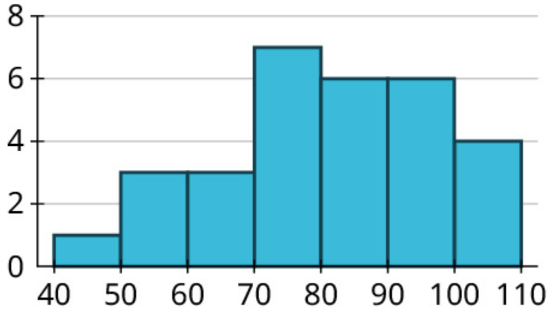
8.8



1.
8.9
1. Twenty-four
2. The longest commutes are 60, 50, and 36 miles; the shortest are 4, 6, and 7 miles.
3. 4, 6, 7, 10, 10, 10, 12, 12, 12, 14, 15, 18, 18, 20, 25, 25, 25, 30, 30, 35, 35, 36, 50, 60
8.10
1.

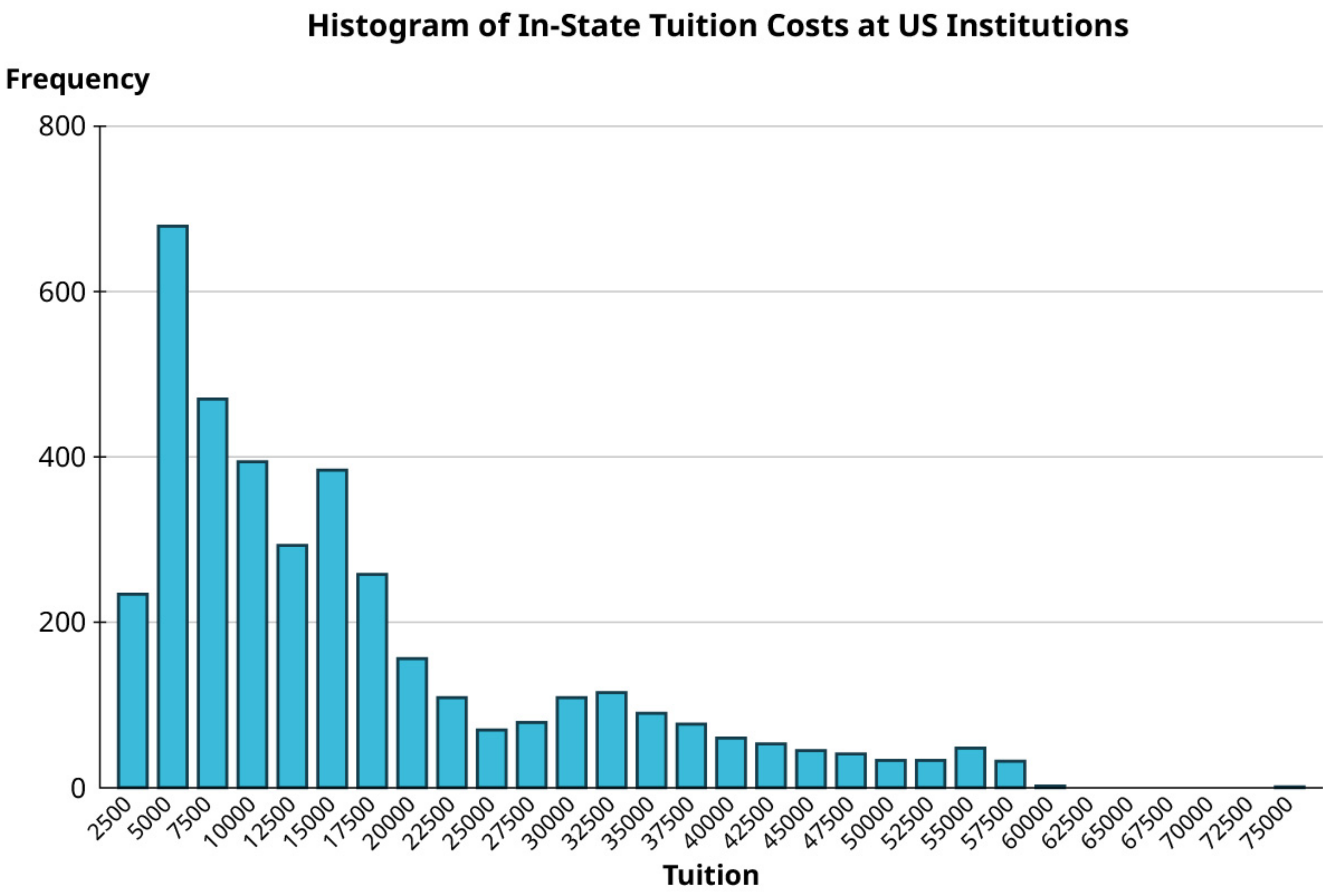
4	7
5	9 7 4
6	9 8 7
7	8 7 5 2 2 1 0
8	9 6 5 4 4 1
9	7 7 6 3 3 1
10	7 6 3 1

8.11



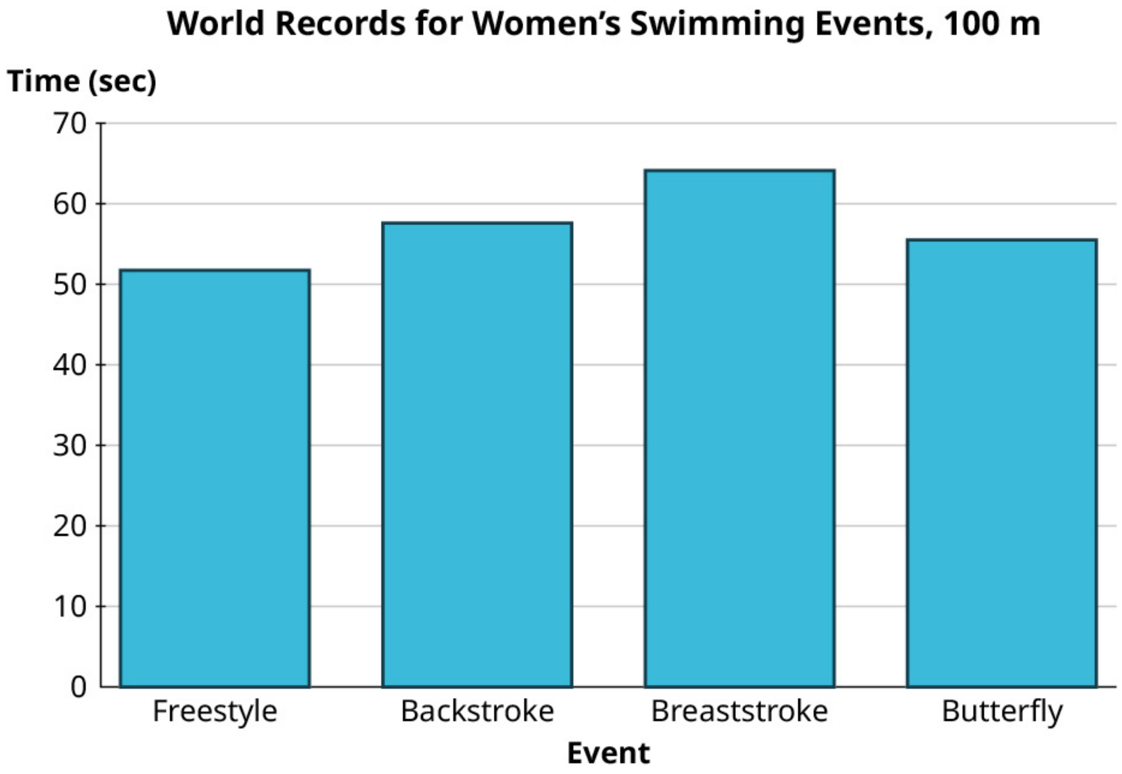
1.
8.12
1.

Answers may vary based on bin choices. Here’s the result for bins of width 2,500:



The data are strongly right-skewed.

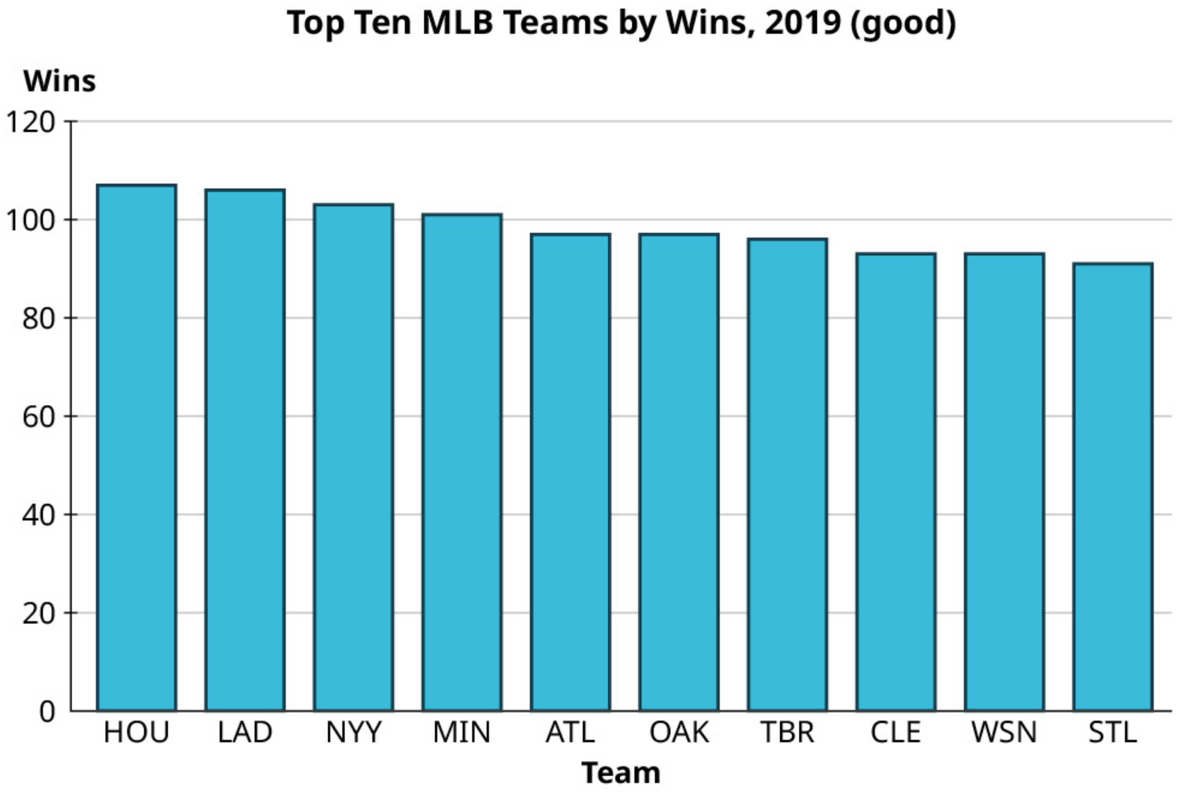
8.13

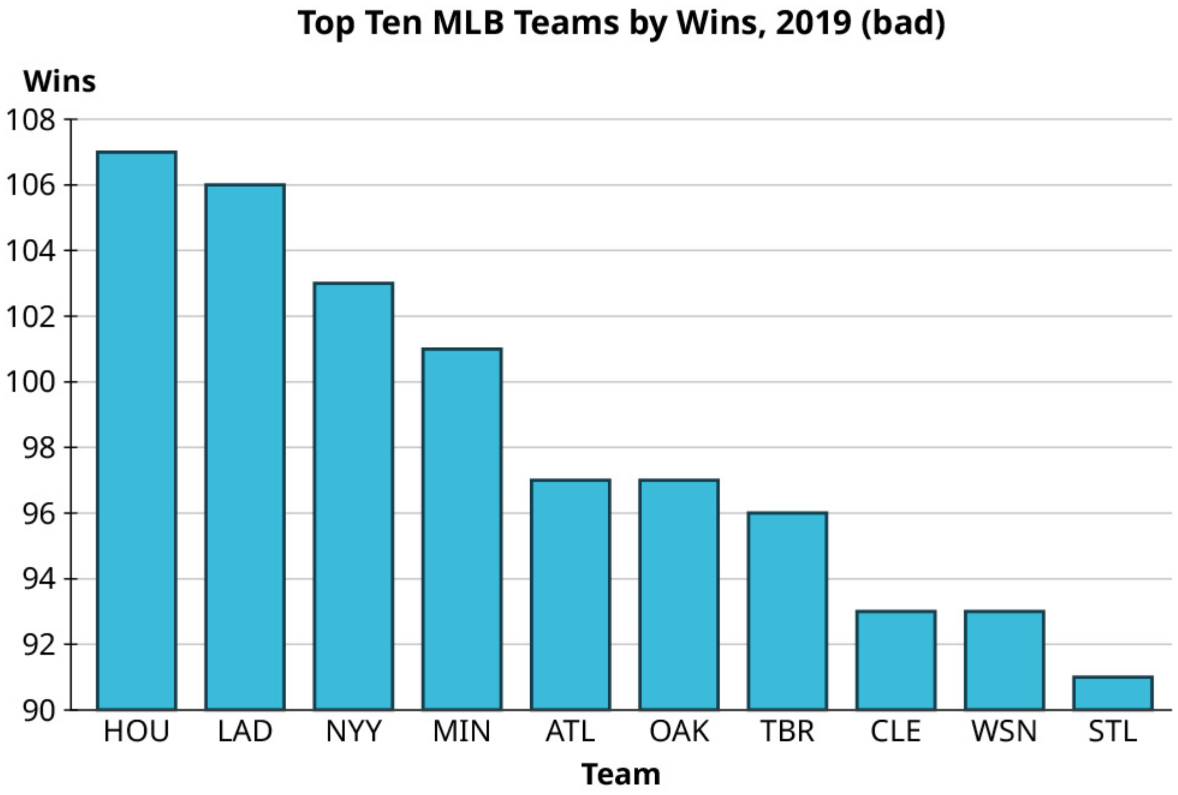


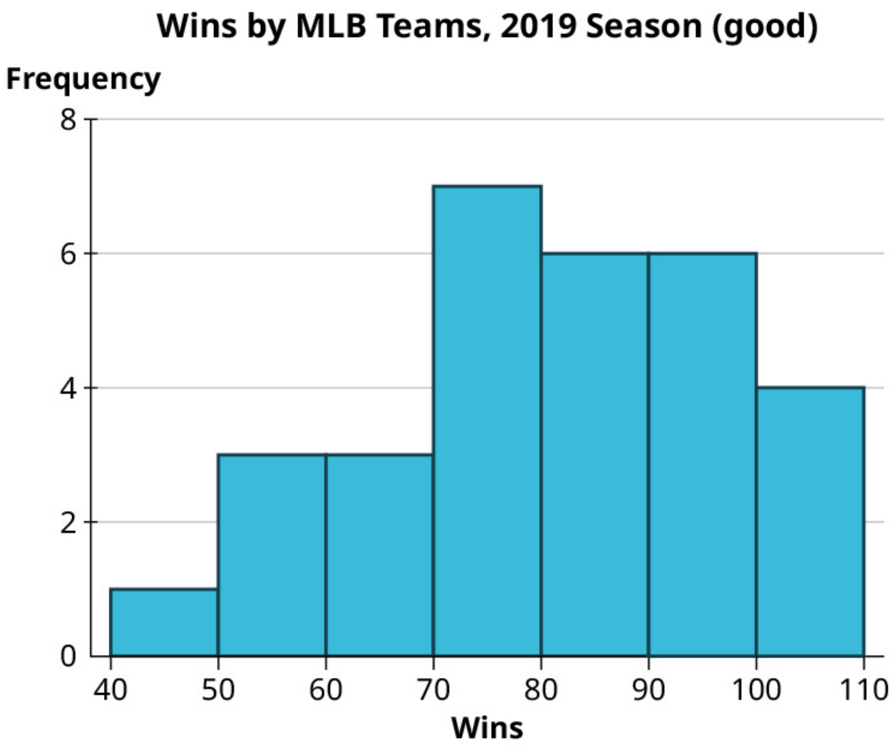
1.

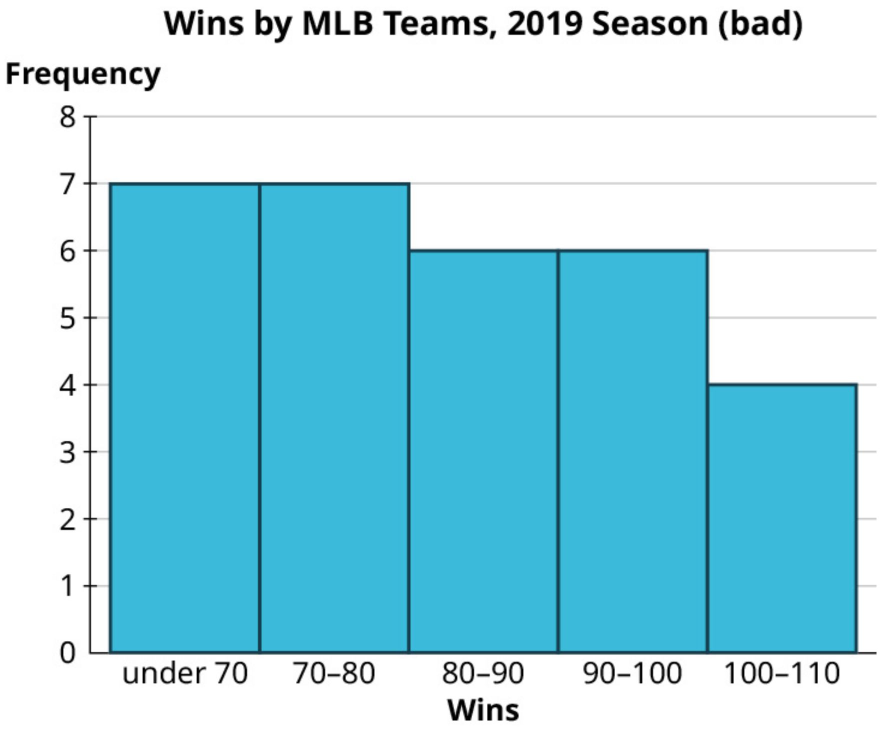
8.14
1.

Top ten teams by wins:









8.15
1. There are two modes: 89 and 104, each of which appears three times.
8.16
1.

2

8.17
1. 136
8.18

1. 82.5
8.19
1. 2
8.20
1. 12.556
8.21
1. $\frac{6}{2}33 = 1.879$
8.22
1. Modes: 72, 84, 93, 97
Median: 82.5
Mean: 80.967 (rounded to three decimal places)
8.23
1. Mode or median
2. Median or mean
3. Median or mean
8.24
1. 67
8.25
1. $\$74,514 - \$480 = \$74,034$
8.26
1.

$$s = \sqrt{752.5} \approx 27.432$$

8.27
1. $13,333.77.
8.28
1. 15
2. 70th percentile
8.29
1. 1026
2. 1318
3. 82.7th
4. 99.2nd
8.30
1. $3,120
2. $26,465.20
3. 32.3rd
4. 98.5th
8.31
1.

The red (leftmost) distribution has mean 11, the blue (middle) has mean 13, and the yellow (rightmost) has mean 14.

8.32
1. 6
8.33
1. Mean: 150; standard deviation: 20
8.34
1. 99.7%
2. 95%
3. 68%
8.35
1.

65 and 75

2.

26 and 54

3.

110 and 290

8.36
1.

47.5%

2.

15.85%

3.

81.5%

8.37
1.
-1.8
1.6
-0.6
8.38
1. –44
30
6
8.39
1.

80th

9th

97th

8.40
1.

3.9

6.3

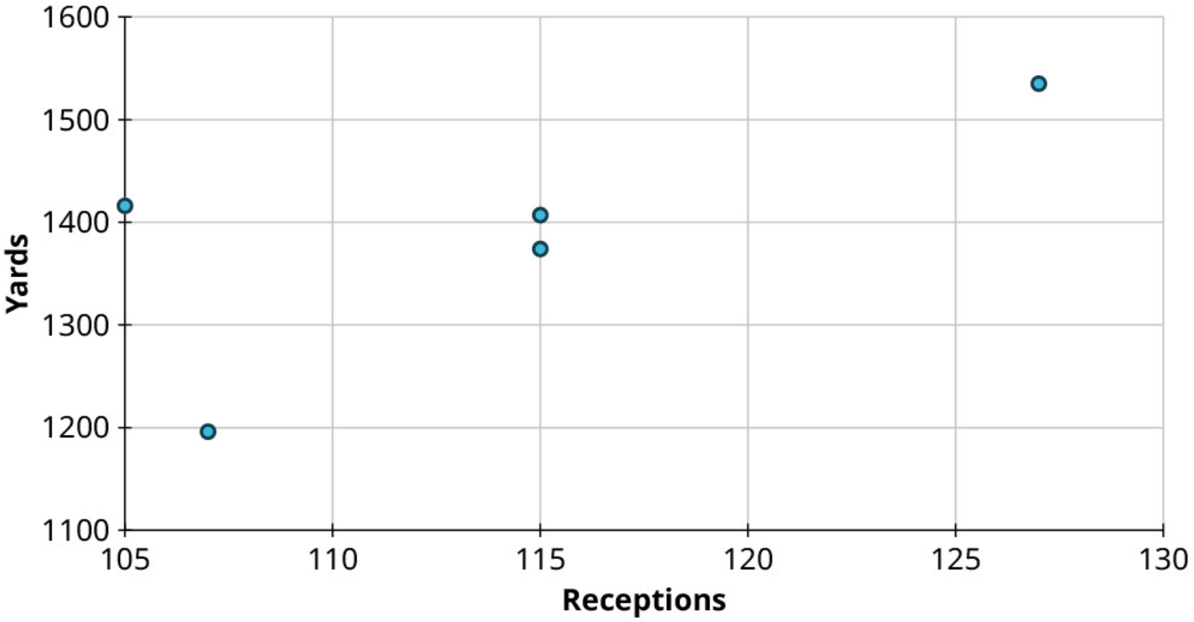
2.9

8.41
1. 1.29
2. 94.5th
3. An LSAT score of 161 is better
8.42
1. 72
2. 6
3. 93rd
8.43
1.

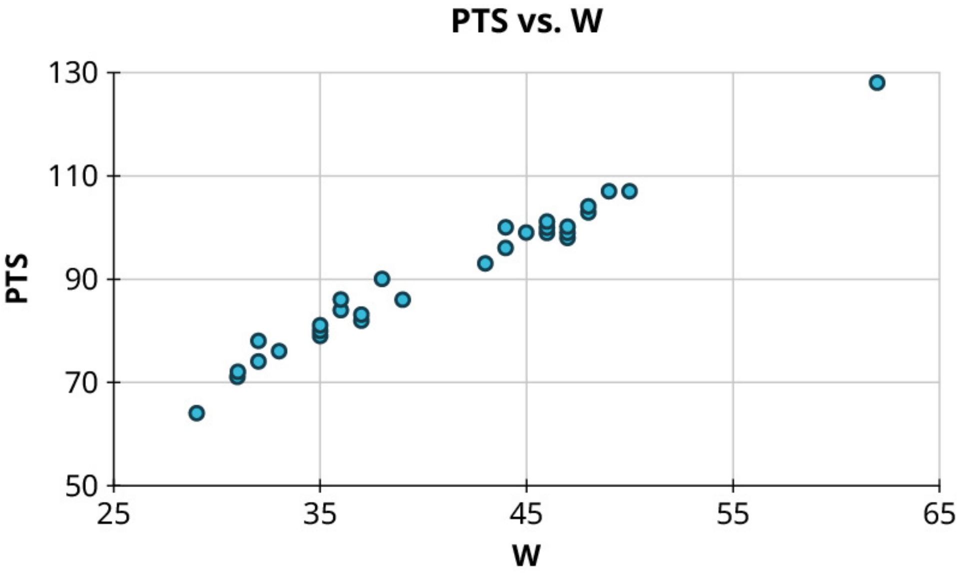
Using NORM.INV: 1092.8

Using PERCENTILE: 1085

8.44
1. Income
2. Either; neither one seems to directly influence the other (they’re both influenced by the student’s academic ability)
3. GPA
8.45



1.
8.46



1.
8.47
1.

  a. No curved pattern
  b. Strong negative relationship, $r \approx -0.9$

2.

  a. Curved pattern

3.

  a. No curved pattern
  b. No apparent relationship, $r \approx 0$

4.

  a. No curved pattern
  b. Weak positive relationship, $r \approx 0.6$

8.48
1.

1. $r=-0.91$
2. Not appropriate
3. $r=-0.01$
4. $r=0.62$

8.49
1. $y=0.75\left(\frac{20}{5}\right)(x-100)+200=3x-100$

8.50
1.

$y=1.8x+16.9$

8.51
1.

$y=0.174x+14.5$

2.

26.68

3.

Predicted: 28; actual: 18. The Phillies were caught around 10 fewer times than expected.

4.

Every 10 additional steal attempts will result in getting caught about 1.7 times on average.

8.52
1. $y=0.5481x+0.1055$, where x is the proportion of made field goals and y is the proportion of made three-point field goals
2. 0.347
3. Predicted: 0.340; actual: 0.368. The Aces made about 2.8% more of their three-point shots than expected.
4. An increase of 1% in made field goal attempts will result in an expected increase of 0.55% in made three-point field goal attempts.

Check Your Understanding

1. Randomization is being used; cluster random sample.
2. No randomization is being used.
3. Randomization is being used; stratified random sample.
4.

Genre	Frequency
Cooking	1
Non-fiction	3
Romance	4
Thriller	3
True Crime	3
Young Adult	6

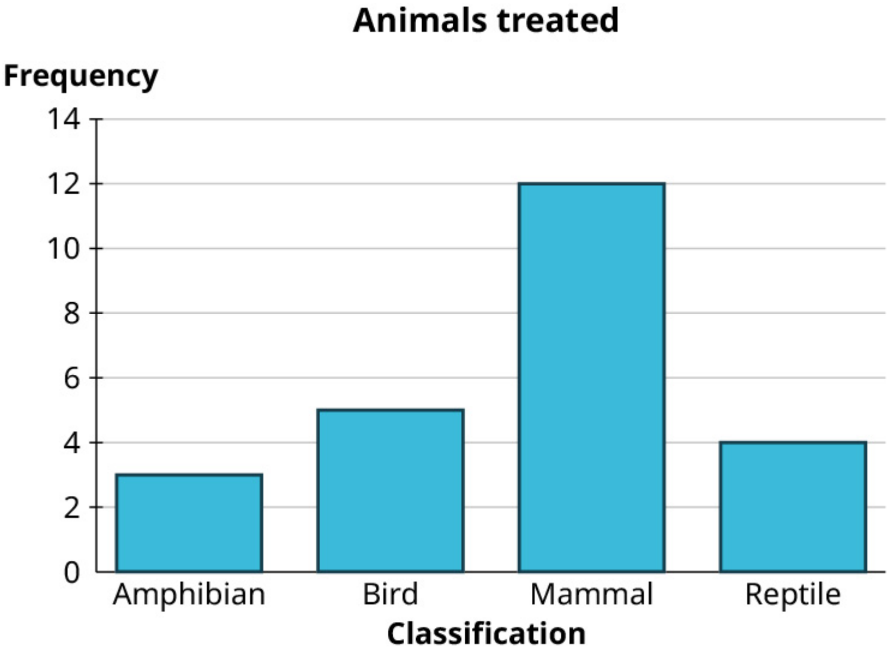
5.

Number of classes	Frequency
1	1
2	3
3	16
4	8
5	4

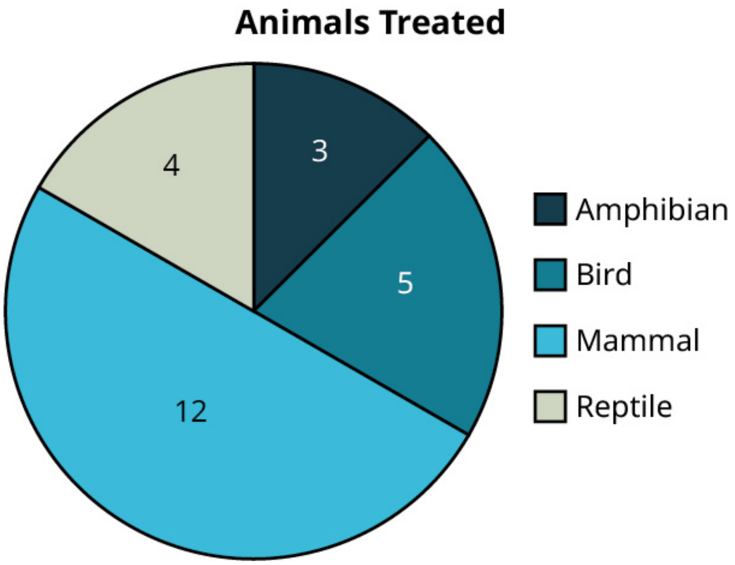
6.

Range of Cell Phone Subscriptions Per Hundred People	Frequency
0.0 – 24.9	1
25.0 – 49.9	3
50.0 – 74.9	1
75.0 – 99.9	6
100.0 – 124.9	7
125.0 – 149.9	3
150.0 – 174.9	3
175.0 – 199.9	1

(Note: Answers may vary based on choices made about bins.)



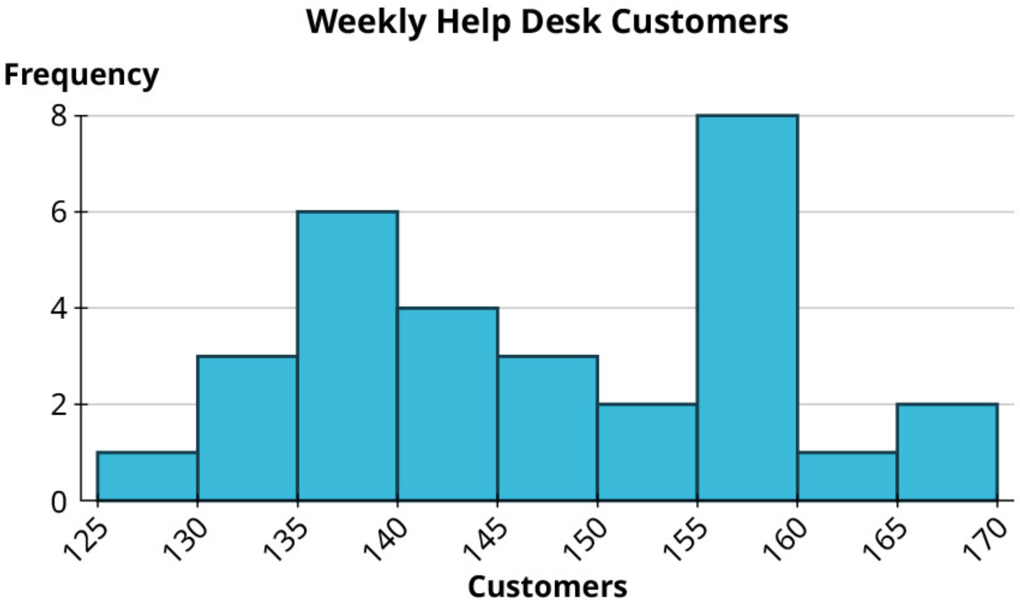
7.



8.

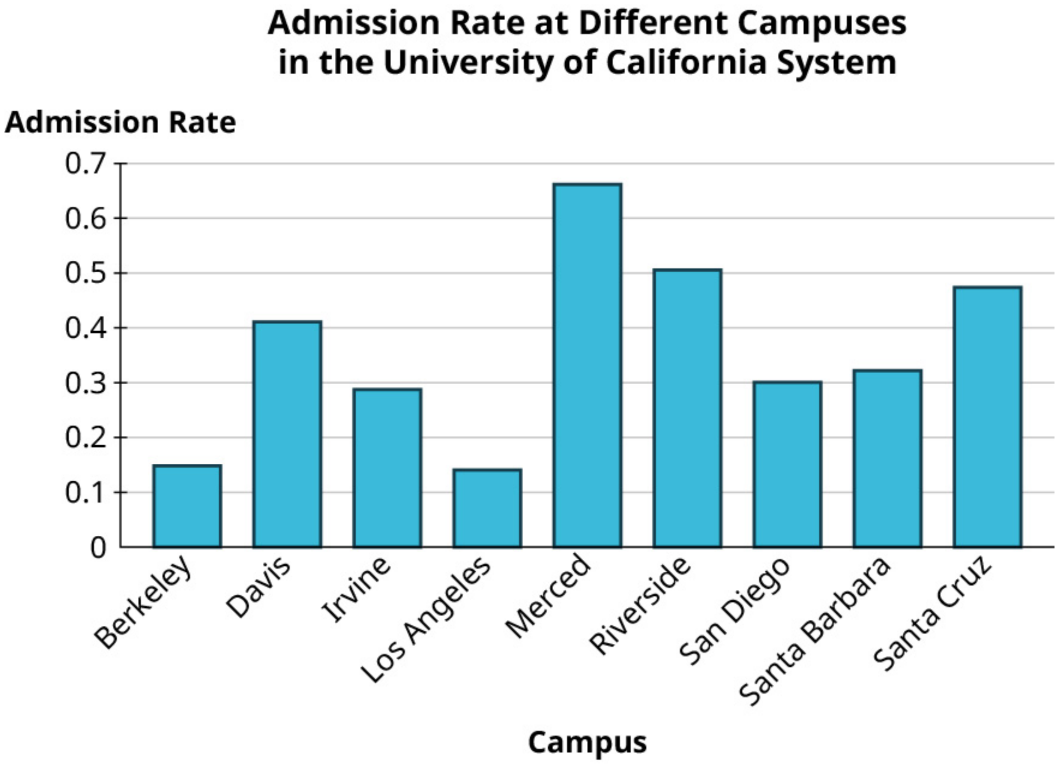
9.

12	7
13	0 2 3 6 6 7 8 9 9
14	1 2 3 3 6 8 8
15	3 3 5 6 6 6 7 7 8 8
16	4 7 8



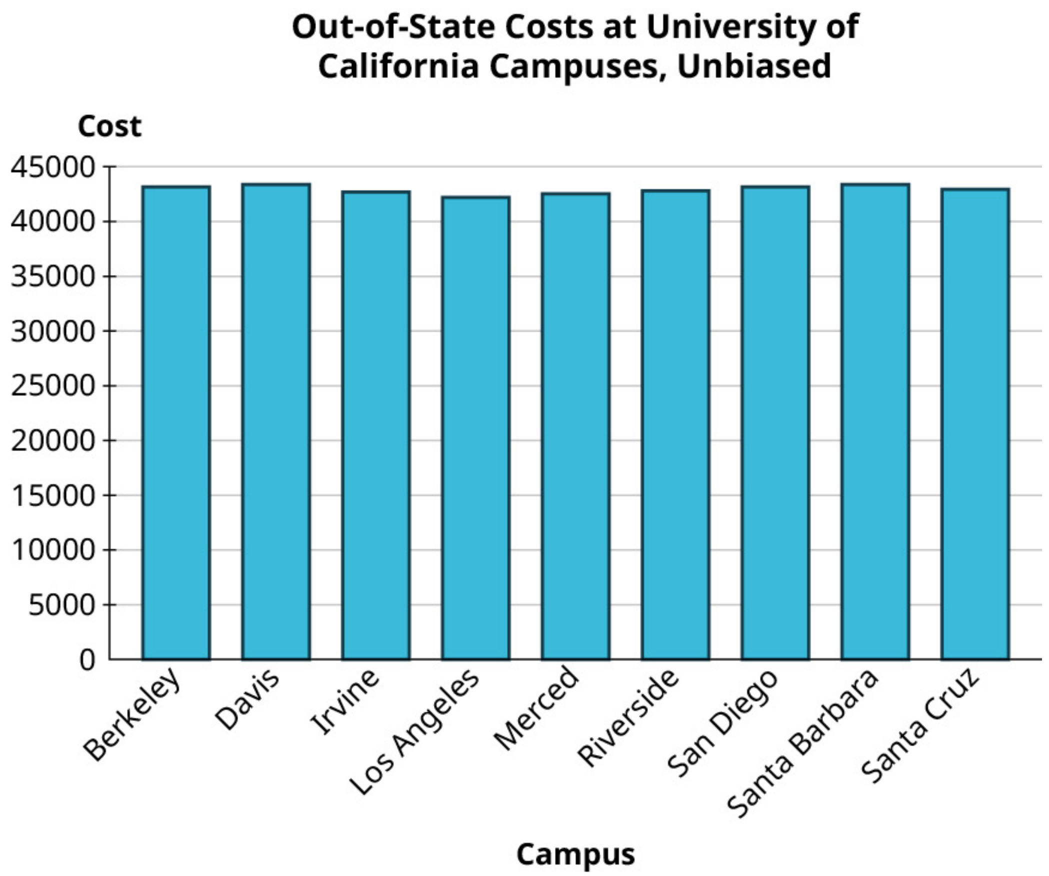
10.

11.

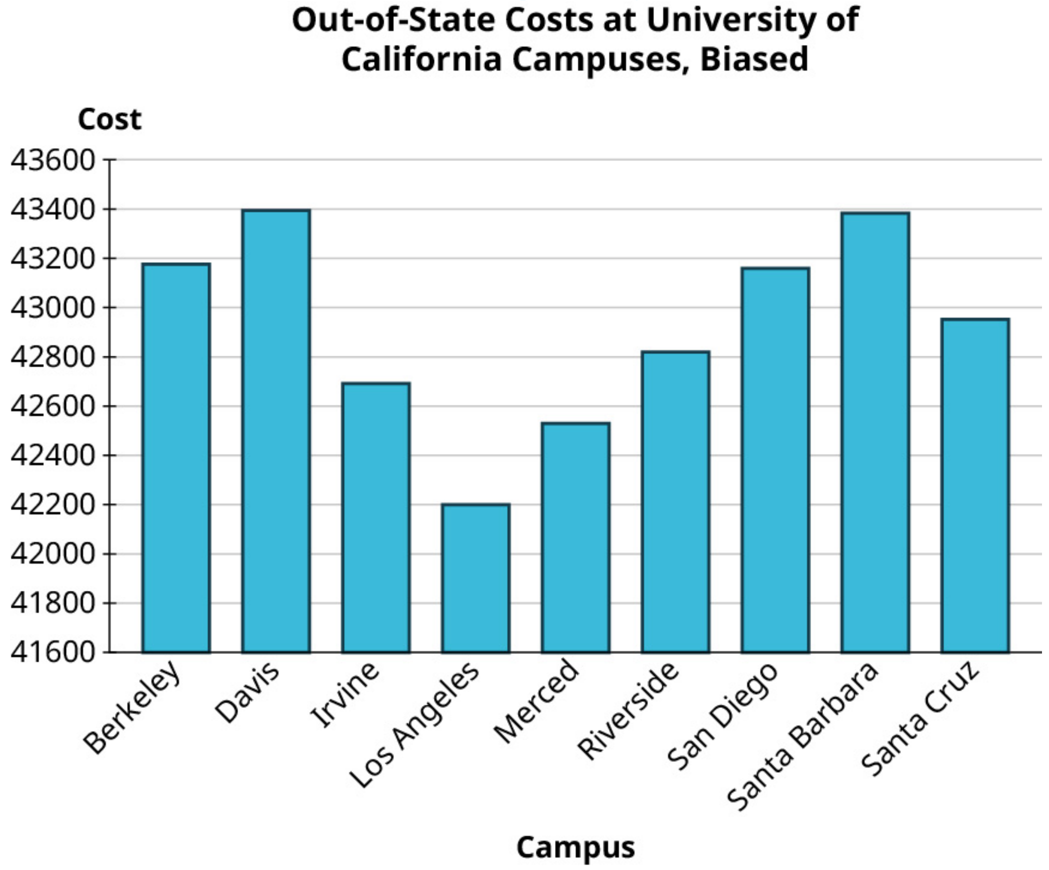


(data source:https://data.ed.gov/)

12.



(data source: https://data.ed.gov/)



(data source: https://data.ed.gov/)

13.

Mode: 112

Median: 113

Mean: 112.64

14. 3
15. 3
16. $\frac{107}{32} \approx 3.344$
17.

156

18.

147

19.

147.2

20.

Mode: not useful; every value appears only once

Median: 0.322

Mean: 0.3612

21.

Mode: not useful; every value appears only once

Median: $42,952

Mean: $42924.78

22. Mode: $13,380
Median: $11,207
Mean: $15,476.79
23. Since the data are right skewed, the mean will be bigger than the mean. Thus, the workers would rather use the median, while the management will prefer the mean.
24. 9
25. $\sqrt{10.5} \approx 3.240$
26. $168 - 127 = 41$
27. 11.306
28. Range: $0.6617 - 0.1404 = 0.5213$
29. Standard deviation: 0.170
30. Range: $\$43,394 - 42,218 = \$1,176$
31. Standard deviation: $398.37
32. 18
33. 14
34. 80th
35. 60th
36. 71.7
37. 101.2
38. 70.25
39. 51.7th
40. 79.3rd
41. 13.8th
42. 95%
43. 34%
44. 84th
45. 0.583
46. 71.2

47. 72nd
48. 115.38
49. 27th
50.

23.2

51.

1300 on the SAT

52. $\mu = 60$; $\sigma = \frac{1}{2}\sqrt{120} \approx 5.477$
53. 96.6th
54. 57
55.

PERCENTILE gives 20.86; NORM.INV gives 21.11.

56.

PERCENTILE gives 11.82; NORM.INV gives 11.68.

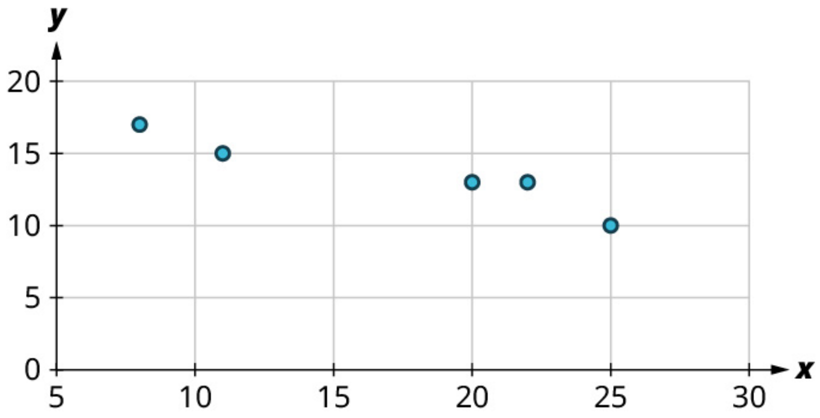
57.

PERCENTRANK gives 92.6th; NORM.DIST gives 91.9th.

58.

PERCENTRANK gives 77th; NORM.DIST gives 79.2nd.

59.



60.

a. Yes

61.

a. No
b. Weak positive relationship; $r \approx 0.5$

62.

0.96

63. $y = 2900x - 478$
64.

$7,475

65.

Less than expected by $1,495.68

66.

For every $1,000 increase in out-of-state tuition, we expect average monthly salary to increase by $161.

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15.8: Chapter 9

Your Turn

9.1

1. gram (g)
2. meter (m)
3. liter (L)

9.2

1. kilogram, hectogram, decagram

9.3

1. 2.5 m

9.4

1. 245 cm

9.5

1. 0.76 daL

9.6

1. 145.7 cg

9.7

1. 1.25 L

9.8

1. Yes, 352 mg of caffeine is equal to 0.352 g, which is less than 0.5 g.

9.9

1. Since 3,810 m is 3.81 km, Kyrie's skydive was at a greater altitude.

9.10

1. km^2

9.11

1. 1,800 cm^2

9.12

1. Yes. The unit of value, cm^2, is appropriate given the size of a sheet of paper. Because the total area of 2,412.89 cm^2 represents four sheets of paper, each sheet is approximately 600 cm^2, which seems reasonable for a sheet of letter-size paper.

9.13

1. 153.29 m^2

9.14

1. 1,365,000 cm^2

9.15

1. 1.06 m^2

9.16

1. 34 m^2

9.17

1. 1,250 m^2

9.18

1. 70 m^2

9.19

1. 8 m^3

9.20

1. 40,000 cm^2

9.21

1. Yes. The unit of value, mm^3, may be appropriate given the size of the paperweight. The volume of 125,000 mm^3 is equivalent to a volume of 125 cm^3, which would represent a cubic box with dimensions of $5\,\text{cm} \times 5\,\text{cm} \times 5\,\text{cm}$.

9.22

1. 8,000 mm^3

9.23

1. 37,854,000 cm^3

9.24

1. 465 m^3

9.25

1. 19 L

9.26

1. 37,854 L

9.27

1. 800 mL

9.28

1. 100,000 boxes

9.29

1. 6,000 boxes

9.30

1. 18.75 L

9.31

1. 2.5 g

9.32

1. 1,300 kg

9.33

1. A reasonable value for the weight of an Etruscan shrew is 2 g. A 2 L bottle of water weighs 2 kg, and since it is one of the smallest mammals, the shrew must weigh less than that, so 2 g seems reasonable.

9.34

1. 0.175 kg

9.35

1. 4,869 g

9.36

1. 1,230,000 mg

9.37

1. 7.5 mL

9.38

1. 5 kg

9.39

1. 111,111 potatoes

9.40

1. 250 °C

9.41

1. 167 °F

9.42

1. Since Toronto has a temperature of 14.1 °C, Madrid has the higher temperature by 2.6 °C.

9.43

1. Recall that water boils at 100 °C. The more reasonable value for the temperature of the mixture is 148 °C. So, 48 °C is not sufficient to bring the mixture to a boil.

9.44

1. 37 °C

9.45

1. 60 °C

9.46

1. –2 °C

9.47

1. 63 °C

9.48

1. 180 minutes, or 1 hour and 30 minutes

Check Your Understanding

1. meter
2. grams
3. 12 hectoliters
4. 15.20 m or 15.2 m
5. 134 dam
6. 1.27 hg
7. 750,000,000 mm
8. They are equal in size.
9. 12 m^2
10. 1,200 m^2

11. 2.5 m^2
12. 2
13. 57,000
14. 21,700
16. 50,000 cm^3
17. 236 mL
18. 1,500 cm^3
19. 42.5
20. 1,500
21. 6,750
22. 800
23. 10.2
24. 163.52
25. 50 g
26. 180 kg
27. 624 g
28. 8.9
29. 17,000
30. 70
31. 0.144
32. 196,000
33. 0.09
34. 28.5 °F
35. 20 °C
36. 71.1 °C
37. 210
38. 100
39. 35
40. 107
41. 81
42. 399

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15.9: Chapter 10

Your Turn

10.1

1. $\overrightarrow{BD}$ is the ray that starts at point B and extends infinitely in the direction of point D.
2. $\overline{AB}$ represents the line segment that starts at point A and ends at point B.
3. $\overleftarrow{BA}$ represents the ray that starts at point A and extends infinitely in the direction of point B.
4. $\overleftrightarrow{AD}$ represents a line that contains the points A and D. Notice the arrowheads on both ends of the line above AD, which means that the line continues infinitely in both directions.

10.2

1. Answers will vary. One way $\overline{AE}$, $\overline{EI}$, $\overline{IE}$, $\overline{EB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DG}$; Second way $\overline{AH}$, $\overline{HI}$, $\overline{IE}$, $\overline{EB}$, $\overline{BC}$, $\overline{CF}$, $\overline{FG}$.

10.3

1. $\overleftrightarrow{RS} \parallel \overleftrightarrow{UV}$
$\overleftrightarrow{RS} \perp \overleftrightarrow{XY}$
$\overleftrightarrow{UV} \perp \overleftrightarrow{XY}$

10.4

1. $\overline{AB} \cap \overline{BC} = B$
2. $\overrightarrow{BC} \cup \overleftarrow{CA} = \overleftrightarrow{AC}$
3. $\overline{BC} \cap \overline{AC} = \overline{BC}$

10.5

1. Point A is located at $(-5, -2)$; Point B is located at $(-3, 4)$; Point C is located at $(3, -6)$; Point D is located at $(5, -2)$; Point E is located at $(3, 2)$; Point F is located at $(5, 5)$.
2. Points A and B are on the straight line $\overleftrightarrow{AB}$.
3. The line that begins at point E in the direction of point F is a ray, $\overrightarrow{EF}$.
4. The line from point C to point D is a line segment $\overline{CD}$.
5. Yes, this represents a plane. One reason is that the figure contains four points that are not on the line $\overleftrightarrow{AB}$.

10.6

1. Plane $MNOP$ intersects with plane $OPQR$, and plane $OPQR$ intersects with plane $QRST$.

10.7

1.

Acute Angles	Obtuse Angles	Right Angles	Straight Angles
$\angle AOB$ $\angle AOC$ $\angle BOC$ $\angle BOD$ $\angle COD$ $\angle DOE$ $\angle FOE$	$\angle AOE$ $\angle BOF$ $\angle COF$	$\angle AOD$ $\angle BOE$ $\angle DOF$	$\angle AOF$

10.8

1. \(\begin{array}{rcl}{5x}&{ = }& \ {2x + 5} & { = } & \end{array}\)

10.9

1. $m\measuredangle(6x) = 24^\circ$, $m\measuredangle(9x) = 36^\circ$, $m\measuredangle(7x+2) = 30^\circ$

10.10

1. $m\measuredangle 4 = 67^\circ$, $m\measuredangle 1 = 113^\circ$, $m\measuredangle 3 = 113^\circ$

10.11

1. 130°

10.12

1. $m\measuredangle 5 = 120^\circ$, $m\measuredangle 4 = 120^\circ$, $m\measuredangle 8 = 120^\circ$

10.13

1. $m\measuredangle 1 = 132^\circ$, $m\measuredangle 5 = 132^\circ$

10.14

1. $m\measuredangle 5 = 56^\circ$
$m\measuredangle 6 = 62^\circ$
$m\measuredangle 7 = 118^\circ$
$m\measuredangle 8 = 62^\circ$
$m\measuredangle 9 = 118^\circ$

10.15

1. $m\measuredangle x = 66^\circ$, $m\measuredangle (x+1) = 67^\circ$, $m\measuredangle (x-19) = 47^\circ$

10.16

1. $m\measuredangle 1 = 35^\circ$; $m\measuredangle 2 = 2 = 85^\circ$; $m\measuredangle 3 = 60^\circ$

10.17

1. Triangle ABC is congruent to triangle DEF.

10.18

1. SAS

10.19

1. ASA

10.20

1. The triangles are similar.

10.21

1. $x = 6$, $y = 24$

10.22

1. The tree is 86 feet high.

10.23

1. 60 ft

10.24

1. rectangle
2. pentagon
3. heptagon
4. parallelogram

10.25

1. Shapes 1, 2, 4, and 6 are triangles; shape 3 is a pentagon; shape 5 is a parallelogram; and shape 7 is a rectangle.

10.26

1. 120 in

10.27

1. 22.4 in

10.28

1. $a = 108^\circ$

The sum of the interior angles is 540°.

10.29

1. We have the sum of interior angles is 360°. Then, $x = -9$. The other angles measure 54°, 111°, 44°.

10.30

1. $7(51.43) \ = 360^\circ$

10.31

1. $C = 2(2.25)\pi = 14.14 \text{ cm}$

10.32

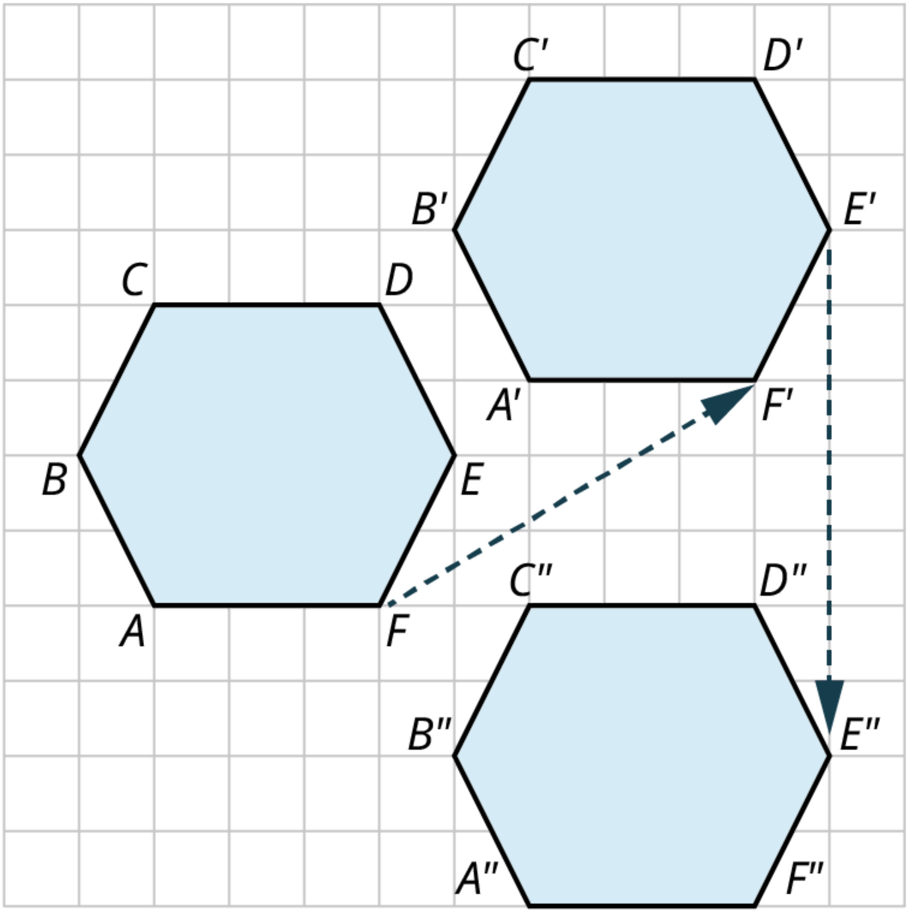
1. $r = 2.5\,\text{cm}$

10.33

1. You need to buy 15.74 feet of trim.

10.34

1. The translated hexagon has labels A'', B'', C'', D'', E'', F''.



10.35

1. Rotate the triangle about the rotation point 90° to the right three times.

10.36

1. This tessellation could be produced with a reflection of the triangle vertically, then each triangle is rotated 180° and translated to the right.

10.37

1. From the first square on the left, rotate the square 30° to the right, or $A1 \rightarrow A2$. Then, reflect the square over the horizontal, or $A2 \rightarrow A3$. Next, reflect all three squares over the vertical line. The lavender triangles comprise another pattern that tessellates, fits in with the squares, and fills the gaps.

10.38

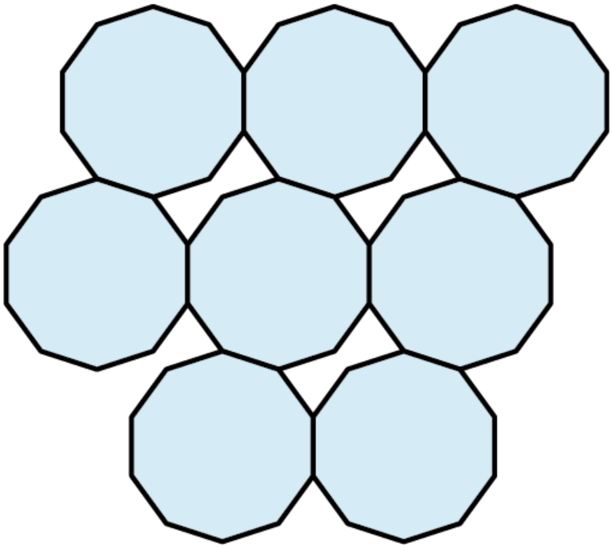
1. No

10.39

1. Not by themselves, but by adding an equilateral triangle, the two regular polygons do tessellate the plane without gaps.

10.40

1. We made a tessellation with a regular decagon (10 sides) and an irregular hexagon. We see that the regular decagon will not fill the plane by itself. The gap is filled, however, with an irregular hexagon. These two shapes together will fill the plane.



10.41
1. $8\,\mathrm{cm}^2$
10.42
1. 108 ft^2
10.43
1. 45 boxes at a cost of $2,025.00
10.44
1. $270\,\mathrm{in}^2$
10.45
1. 13,671 square feet; cost is $19,872.95.
10.46
1. $132\,\mathrm{in}^2$
10.47
1. $d_2 = 10$
10.48
1. 40 in
10.49
1. $A = 96.25\ \mathrm{cm}^2$
10.50
1. $30\ \mathrm{yd}^2$
10.51
1. $28.3\,\mathrm{cm}^2$
10.52
1. the 15-inch pizza
10.53
1. $A = 21.875 + 4.8 = 26.7\,\mathrm{ft}^2$
10.54
1. $21.46\,\mathrm{cm}^2$
10.55
1. $300
10.56
1. $\begin{aligned} SA &= 432\ \mathrm{cm}^2 \\ V &= 540\ \mathrm{cm}^3 \end{aligned}$
10.57
1. $100\,\mathrm{cm}^2$
10.58
1. /**/

$$\begin{aligned} SA &= 336\ \mathrm{cm}^2 \\ V &= 480\ \mathrm{cm}^3 \end{aligned} \tag{15.9.1}$$

/**/
10.59
1. $1{,}192\ \text{ft}^2$
10.60
1. /**/

$$\begin{aligned} SA &= 527.78\ \text{cm}^2 \\ V &= 769.69\ \text{cm}^3 \end{aligned} \tag{15.9.2}$$

/**/
10.61
1. approximately 8 1/3 cans of soup
10.62
1. 6.25 ft by 6.25 ft
10.63
1. $1.26\,\text{ft wide} \times 1.26\,\text{ft long} \times 1.26\,\text{ft high},$ $95.26
10.64
1. $a = 5$
10.65
1. 1,140 ft
10.66
1. The slanted distance will be 120.4 inches.
10.67
1. The side lengths are $15, 15\sqrt{3}, 30.$
10.68
1. The ladder reaches 12 feet up the wall and sits $12\sqrt{3}\ \text{feet}$ from the wall.
10.69
1. Each side x equals $x = 4\sqrt{2} = 5.66.$
10.70
1. /**/

$$\begin{gathered} r = 6.53 \\ y = 4.2 \end{gathered} \tag{15.9.3}$$

/**/
10.71
1. $\alpha = 46.3^\circ$, $\beta = 43.7^\circ$, $x = 5.73$
10.72
1. 2,241 ft
10.73
1. $c = 8.06$, one angle is 60°, and the other angle is 30°.
10.74
1. 11.9°
10.75
1. 46 ft

Check Your Understanding

1. The line containing point D and point A is a line segment from point D to point A, $\overline{DA}$, or from point A to point D, $\overline{AD}$.
2. The line containing points C and B is a straight line that extends infinitely in both directions and contains points C and B.
3. This is a ray that begins at point E, although it does not contain point E, and extends in the direction of point F.
4. $\overline{AB} \cup \overline{BD} = \overline{AD}$. The union of line segment $\overline{AB}$ and the line segment $\overline{BD}$ contains all points in each line segment combined.
5. $\overrightarrow{BD} \cap \overline{BC} = \overline{BC}$. The intersection of the ray $\overrightarrow{BD}$ and the line segment $\overline{BC}$ contains only the points common to each set, $\overline{BC}$.
6. $\overrightarrow{BA} \cup \overrightarrow{BD} = \overleftrightarrow{AD}$. The union of the ray starting at point B and extending infinitely in the direction of A and the ray starting at point B and extending infinitely in the direction of D is the straight line extending infinitely in both directions containing points A, B, C, and D.
7. Two lines are parallel if the distance between the lines is constant implying that the lines cannot intersect.
8. Perpendicular lines intersect forming a 90° angle between them.
9. Yes, because it contains a line and a point not on the line.
10. straight
11. obtuse

12. right
13. acute
14. $m\measuredangle 1 = 149^\circ$ by supplementary angles with $\measuredangle 31^\circ$.
15. $m\measuredangle 3 = 31^\circ$ by vertical angles with the angle measuring 31°.
16. $m\measuredangle 5 = 149^\circ$ by corresponding angles with $\measuredangle 1$.
17. $x = 89^\circ$
18. $x = 67^\circ$
19. $x = 77^\circ$ and $y = 26^\circ$
20. These are similar triangles, so we can solve using proportions.

$\begin{array}{rcl} \frac{3}{2} & = & \frac{7}{x} \\ 3x & = & 14 \\ x & = & \frac{1}{4}3 \end{array}$	\(\begin{array}{*{20}{rcl}}{\frac{4}{2}}&{ = }&{\frac} \ {\frac14{3}(4)}&{ = }&{2(4 + y)} \{\frac56{3}}&{ = }&{8 + 2y} \{56}&{ = }&{3(8 + 2y)} \{56}&{ = }&{24 + 6y} \{32}&{ = }&{6y} \{\frac16{3}}&{ = }&{y}\end{array}\)

Then, $x = \frac{1}{4}3$ and $y = \frac{1}{6}3$.
21. Set up the proportions.

$$\begin{array}{rcl} \frac{6}{a} & = & \frac{1}{2}14 \\ 6(14) & = & 12a \\ 84 & = & 12a \\ 7 & = & a \end{array}$$

Thus, $t = 20$ and $a = 7$.
22. pentagon
23. octagon
24. heptagon
25. $6(5) = 30 \text{ cm}$
26. $S = (6 - 2)\,180^\circ = 720^\circ$
27. 120°

28.
$$\begin{array}{rcl} 360 & = & 152 + 9x + (5x + 1) + (x + 12) \\ & = & 15x + 165 \\ 195 & = & 15x \\ 13 & = & x \\ x + 12 & = & 25^\circ,\ 9x = 117^\circ,\ 5x + 1 = 66^\circ \end{array}$$

29. $C = 2\pi(3) = 6\pi = 18.85 \text{ cm}$
30. The patterns are repeated shapes that can be transformed in such a way as to fill the plane with no gaps or overlaps.
31. Starting with the triangle with the point labeled A, the triangle is translated point by point 3 units to the right and 3 units up to point A'. Then, the triangle labeled A' is translated 3 units to the right and 3 units up to point A''.
32. The triangle is rotated about the rotation point 180° to vertex B'.
33. The dark triangle is reflected about the vertical line showing the light back, and then reflected about the horizontal line. The pattern is repeated leaving a white diamond between the shapes.
34. 3.3.3.3.3.3
35. 7.5 cm^2
36. 25 ft
37. 168 cm^2
38. 64.5 cm^2
39. 201.1 in^2
40. 116.38 in^2
41. 706.86 in^2
42. 386.6 cm^2
43. 416 in^2
44. 640 in^3
45. 942.48 cm^2
46. $1,570 \text{ cm}^3$
47. 511.35 in^2
48. /**/

$$\begin{array}{c} a = 2.5 \\ b = 2.5\sqrt{3} \end{array} \tag{15.9.4}$$

/**/

49. /**/

$$\begin{aligned} b &= 10 \\ c &= 10\sqrt{2} \end{aligned} \tag{15.9.5}$$

/**/

50. $b = 6$

51. 24.2 ft

52. $c = 11.7\text{ cm}$

53. $x = 14.3\text{ m}$

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15.10: Chapter 11

Your Turn

11.1
1. Yes, Joe Biden won the majority.
11.2
1. Tony Cambell won a plurality of votes in the Republican primary.
11.3
1. Hearn and Lim tied. Yes, there must be a third election.
2. Kelly must be removed in the runoff.
11.4
1. Option B.
11.5
1. $4 + 6 + 4 + 7 = 21$
2. None
3. 4
11.6
1. Using ranked-choice voting, Yoda is determined to be the winner.
11.7
1. Blue received 78 points.
11.8
1. Candidate B would be considered the winner using the ranked-choice voting method.
2. Candidate C would be considered the winner using the Borda count method.
3. Different candidates won. It appears that the vote counts were so close that a small shift in either direction could change the results of either method.
11.9
1.

Opponent Runner	Scorsese (S)	Phillips (P)	Mendes (M)	Tarantino (T)	Bong (B)	Points
Scorsese wins	--	SP 5100	SM 5600	ST 5600	~~SB~~ ~~2100~~	3
Phillips wins	~~PS~~ ~~3600~~	--	PM 6800	PT 3600	~~PB~~ ~~2400~~	1
Mendes wins	MS 3100	MP 1900	--	MT 0	~~MB~~ ~~0~~	0
Tarantino wins	~~TS~~ ~~3100~~	TP 5100	TM 8700	--	~~TB~~ ~~4000~~	2
Bong wins	BS 6600	BP 6300	BM 8700	BT 4700	--	4

2. Bong Joon-ho won according to the pairwise comparison method.
3. Yes, the winner is a Condorcet candidate.
11.10
1. Candidate P wins. Candidate P is not a Condorcet candidate because P lost to Candidate Q.

11.11
1. Candidate C won 25 points.
2. The greatest number of points another candidate could win is 24 points.
3. No, Candidate C cannot lose or tie.

11.12
1. Dough Boys Pizza is the winner for dinner.

11.13
1. There is a tie between Rainbow China and Dough Boys Pizza.

11.14
1. Al Gore is the winner.

11.15
1. Animal Kingdom
2. Animal Kingdom
3. Magic Kingdom

11.16
1. Using the plurality voting method, there would be a tie between Option B and Option C.
2. Using the ranked-choice voting method, Option B would be the winner.
3. No, there was not a majority candidate in Round 1.

11.17
1. Using the pairwise comparison voting method, Option B wins.
2. Using the Borda count method, Option A and Option B tie.
3. Yes. The majority criterion fails for Borda count.

11.18
1. Yes, because Option A is a Condorcet candidate.

11.19
1. Option A. Yes, the Condorcet criterion is satisfied.
2. Option A. Yes, the Condorcet criterion is satisfied.
3. Option B. No, the Condorcet criterion is not satisfied.

11.20
1. Standard Poodle.
2. Standard Poodle.
3. This election does not violate the monotonicity criterion.
4. Increasing the ranking for a winner of a Borda count election on a ballot will increase that candidate’s Borda score while decreasing another candidate’s Borda score, but leaving the remaining candidates’ Borda score unchanged. So, a Borda count election will never violate the monotonicity criterion.

11.21
1. Labrador retriever wins the election.
2. Golden retriever wins the election.
3. This election violates the monotonicity criterion.
4. It is possible that the monotonicity criterion would be met in other ranked-choice election scenarios, but overall, the ranked-choice voting method is said to fail the monotonicity criterion even if it failed in only one scenario.

11.22
1. Chihuahua.
2. Yorkshire Terrier.
3. Yes, this election violates IIA.

11.23
1. Jim wins.
2. Pam wins.
3. Yes, this is a violation of IIA.

11.24
1. B $\frac{2}{3}$; 0.67, C $\frac{2}{3}$; 0.67, D $\frac{2}{3}$; 0.67, and E $\frac{2}{3}$; 0.67
2. B $\frac{3}{2}$; 1.5, C $\frac{3}{2}$; 1.5, D $\frac{3}{2}$; 1.5, and E $\frac{3}{2}$; 1.5
3. Yes, the constant ratio is $\frac{3}{2}$ pencils per desk.

11.25
1. 42 pencils would be allocated.
2. 42 pencils would be allocated.
3. 34 desks

11.26
1. IL 700,000; OH 700,000; MI 700,000; GA 800,000; NC 700,000

2. IL 0.0000014; OH 0.0000014; MI 0.0000014; GA 0.0000013; NC 0.0000014
3. IL 0.000001; OH 0.000001; MI 0.000001; GA 0.000001; NC 0.000001
4. The ratio of State Population to Representative Seats seems to be either 700,000 or 800,000 to 1. There does appear to be a constant ratio of about 0.000001 to 1 of Representative Seats to State Population when rounding to six decimal places. This is the same as the top five states.
11.27
1. 60,449.4
11.28
1. The states are the Hernandez family and the Higgins family. The seats are the pieces of candy. The house size is 313. The state populations are three in the Hernandez family and four in the Higgins family. The total population is 7.
2. The standard divisor is the ratio of the number of children to the number of pieces of candy. 0.0224 children per piece of candy.
11.29
1. 0.98
11.30
1.

Family	Family’s Standard Quota
Hernandez	175.6667 candies
Higgins	234.2222 candies
Ho	117.1111 candies
Total	527

The sum is very close to 527.

11.31
1. 6, 5, 6, 7, 5, 6
2. 35
3. No.
11.32
1. It is not possible to give a fractional part of a gift card. Also, traditional rounding to the nearest integer results in 4 gift cards for each student, which leaves one extra gift card.
11.33
1. 30, 18, 20
2. 68
3. 2
11.34
1. The final Hamilton apportionment is Fictionville as follows: 1, Pretendstead 3, Illusionham 5, and Mythbury 26.
11.35
1. Method V violates the quota rule because State F receives 4 seats instead of 5 or 6.
11.36
1. 53.33
2. 480
3. 96
4. Answers may vary. With a modified divisor of 0.100, the modified quota is 120.
5. The modified quota from part 3 was the smallest, because the divisor was the largest of the three. Dividing the same number by a larger value gives a smaller result.
11.37
1.

Each state would receive the following seats: Fictionville 1, Pretendstead 2, Illusionham 4, and Mythbury 28.

11.38
1. Fictionville 2, Pretendstead 3, Illusionham 5, and Mythbury 25.
11.39
1.

The apportionment is Fictionville 2, Pretendstead 2, Illusionham 5, and Mythbury 26.

11.40

1. The largest state is Mythbury. The citizens would likely favor the Jefferson method of apportionment most since they received the most seats by that method. They would likely favor the Adams and Webster methods of apportionment least because they received the least number of seats by those method.
2. As a group the other three states received nine seats by either the Hamilton method, seven seats by the Jefferson Method, and ten seats by either the Adams method or the Webster method. They would likely favor the Adams method and Webster method most and favor the Jefferson methods least.

11.41

1. State A loses 0, seats State B loses 0, and State C loses 3.
2. State A loses 0 seats, State B loses 1, and State C loses 4.
3. State C, the largest state, loses the most representatives each time the divisor is increased.

11.42

1. The standard divisor is 214,079.1236 citizens per seat. The standard quota for Colorado is 2.5210 seats.
2. The standard divisor is 213,479.4622 citizens per seat. The standard quota for Colorado is 2.5281 seats.
3. The standard quota increased.
4. It must have been the case that either the fractional part 0.5281 ranked lower amongst the other fractional parts of the state quotas than the fractional part 0.5210 did, or there were fewer remaining seats to be distributed, or both.

11.43

1. The final apportionment is: A 183, B 124, C 14, and D 2, which sums to 323.
2. The final apportionment is: A 184, B 125, C 13, and D 2, which sums to 324.
3. Yes, this demonstrates the Alabama paradox because State C receives 14 seats if the house size is 323, but only 13 seats if the house size is 324.

11.44

1. Hospital C lost a respirator while hospital A gained a seat, but hospital C has a higher growth rate than hospital A.

11.45

1. The Hamilton reapportionment is: 19 for Mudston, 13 for WallaWalla, and 6 for Dilberta. This is an example of the population paradox because WallaWalla lost a seat to Dilberta, even though WallaWalla's population grew by 2.24 percent while Dilberta's only grew by 1.56 percent.

11.46

1. 90
2. 95
3. The original state of Beaversdam lost a seat to the original state of Beruna when the new state of Chippingford was added.

11.47

1. The reapportionment gives 38 seats to Neverwood, 46 seats to Mermaids Lagoon, and 9 seats to Marooners Rock. This is an example of the new-states paradox because the original state Mermaids Lagoon lost a seat to the original state Neverwood when the new state was added to the union.

11.48

1.

\({\text{Original Standard Divisor}} \approx \frac

391 \approx 194,400\)

$\text{New Population} \approx 76,000,000 + 300,000 = 76,300,000$

\({\text{New House Size}} \approx \frac

\approx 392\)

There are $392 - 391 = 1$ new seats to be apportioned to New Mexico.

11.49

1. Only Hamilton's method violates the population paradox.

Check Your Understanding

1. Answers may vary. Example: Ranked-choice, Borda count, and pairwise comparison.
2. True.
3. True
4. Borda count
5. Pairwise comparison
6. Hare
7. In two-round voting, only the top two candidates from Round 1 move on to Round 2, and there are only two rounds. In ranked-choice voting, all candidates except those in last place move on to the next round, and there can be many rounds of voting.
8. (Independence of) Irrelevant Alternatives Criterion

9. Pairwise comparison
10. Borda count method
11. Pairwise comparison
12. All of them: Plurality, ranked-choice, pairwise comparison, and Borda count
13. True, because a majority candidate is always the Condorcet candidate.
14. False, because the ranked-choice method violates the Condorcet criterion, but it doesn’t violate the majority criterion.
15. No. Arrow’s Impossibility Theorem does not apply to approval voting because it is not a ranked voting system
16. False. Arrow’s Impossibility Theorem says that no ranked voting system is perfect and that voter profiles may arise that will lead to a violation of one or more fairness criteria, but it does not guarantee that those voter profiles will occur or are even likely to occur.
17. True. Candidates are rated as approved or not approved, and voters can give multiple candidates the same rating.
18. Student 1 is correct.
19. Both are correct.
20. Neither are correct.
21. Neither are correct.
22. Student 2 is correct.
23. Student 2 is correct.
24. Both are correct.
25.

The Hamilton method.

26.

The Hamilton method.

27.

The Jefferson method.

28.

The Adams method.

29.

The Adams method.

30.

The Hamilton and Jefferson methods.

31.

Webster’s method.

32.

Larger

33. The Alabama paradox
34. Population paradox
35. Quota rule violation
36. None of these
37. New-states paradox
38. The population paradox could occur.
39. The Alabama paradox could occur.
40. The new-state paradox could occur.

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15.11: Chapter 12

Your Turn

12.1

1. The vertices are *p*, *q*, *r*, *s*, and *t*. The edges are *pq*, *pr*, *pt*, *qr*, *qs*, *qt*, *rs*, and *st*.

12.2

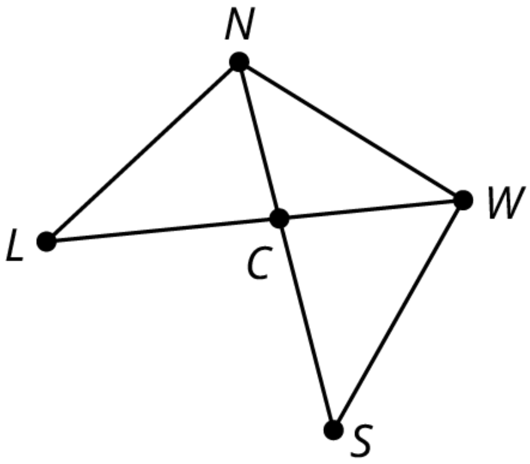
1. The pairs that are not adjacent are *p* and *s*, *p* and *t*, and *r* and *t*.

12.3

1. Vertex *q*

12.4

1.



Graph Representing Common Boundaries Between Regions of Oahu

12.5

1. The objects that are represented with vertices are poker players. An edge between a pair of vertices would indicate the two players competed against each other at table. This is best represented as a graph because players from the same table never meet a second time. Also, a player cannot compete against themself; so, there is no need for a loop.

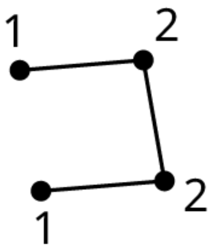
12.6

1. Graph *E*, 34
2. Graph *D*, 18
3. higher

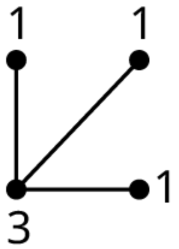
12.7

1. 3
2. Answers may vary. Two possible graphs are shown.

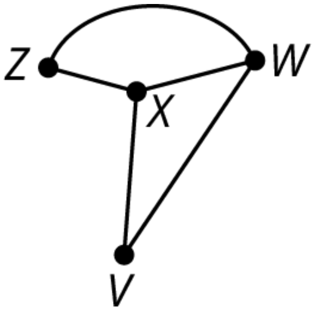
Graph 1:



Graph 2:



12.8
1. 124,750 introductions
12.9
1.



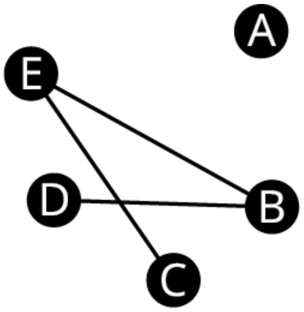
12.10
1. Triangle: (a, b, e or a, d, e)
Quadrilateral: (b, c, d, e or a, b, e, d)
Pentagon: (a, b, c, d, e)
12.11
1. (B, D, C, E, F)
12.12
1. 186
12.13
1. Graph C_1 has 6 edges, but Graph C_2 has 8.
Graph C_1 has 4 vertices and Graph C_2 has 5.
Graph C_1 has no vertex of degree 4, but Graph C_2 has one vertex of degree 4.
They do not have the same cycles. For example, Graph C_2 has a pentagon cycle, but Graph C_1 does not.
12.14
1. Answers may vary.

- The total number of vertices in each graph is different. The Diamonds graph has 17 vertices while Dots graph has only 16.
- The degrees of vertices differ. The Diamonds graph has vertices of degrees 4 while the Dots graph does not.
- The graphs have different sizes of cyclic subgraphs. The Diamonds graph has 4 squares (4-cycles), while the Dots graph has 3 squares. Also, the Dots graph has 8-cycles while the Diamonds graph does not.

12.15
1. Answers may vary. There are four possible isomorphisms:

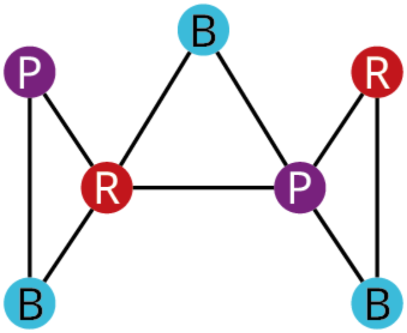
- $a - q$, $d - s$, $c - p$, and $b - r$.
- $a - p$, $d - s$, $c - q$, and $b - r$.
- $a - q$, $d - r$, $c - p$, and $b - s$.
- $a - p$, $d - r$, $c - q$, and $b - s$.

12.16
1. Caden is correct because Graph E could be untangled so that d is to the left of c and the edges bc and ad are no longer crossed. Maubi is incorrect because (a, b, c, d) is a quadrilateral in Graph E. Javier is incorrect because the portion of the graph he highlighted does not have 3 vertices and is not a triangle.
12.17
1.

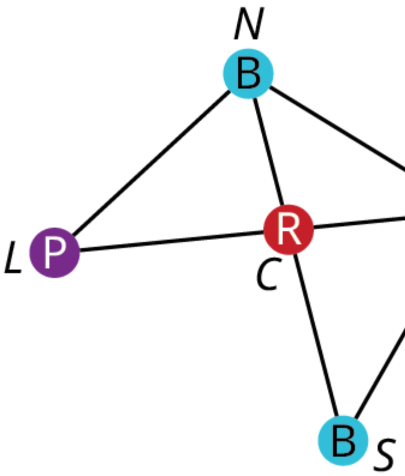


Complement of Graph H

12.18
1. The degrees are 0. There are no adjacent vertices in Graph *N* since all vertices are adjacent in Graph *M*.
12.19
1. c Only $C \to V \to G \to B \to B$ is a walk.
12.20
1. walk, path, and trail
2. walk and a trail, but not a path
3. none of these
12.21
1. To fly from Palm Beach to any other city, you must take the flight from PBI to TPA. To return, you must take the flight from TPA to PBI. This means that the graph of any itinerary that begins and ends at PBI covers the same edge twice and is not a circuit.
2. The degree of PBI is 1 meaning that there is only one edge connecting PBI to other parts of the graph.
12.22
1. Yes. The chromatic number is 4 or less.
2. 3
3. 3 or 4
4. Graphs 1 and 3
5. Answers may vary. Here is an example of a 3-coloring:



12.23
1. The chromatic number is 3. Here is a 3-coloring of the graph:



12.24

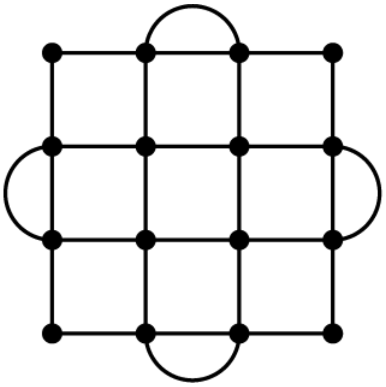
1. Graphs G, H, and I are connected; so, they each have a single component with the vertices $\{a, b, c, d\}$. Graph F is disconnected with two components, $\{a, c, d\}$ and $\{b\}$. Graph J is disconnected with two components, $\{a, d\}$ and $\{b, c\}$. Graph K is disconnected with three components, $\{a\}$, $\{b, d\}$, and $\{c\}$.

12.25

1. Each pair of vertices would be connected by a path that was at most 6 edges long. So, the graph would be connected.

12.26

1. The multigraph of the neighborhood is shown:



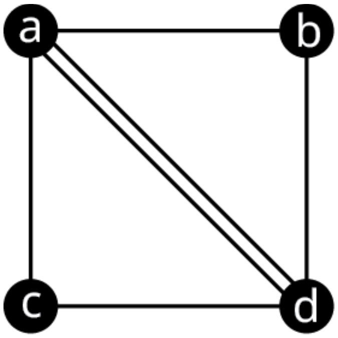
2. Yes, the graph is connected. It has only one component.
3. There are four corner vertices of degree 2 and the remaining twelve vertices are all degree 4.
4. Yes
5. Yes, an Eulerian graph has an Euler circuit; so, it is possible.

12.27

1. Graph X does not have an Euler circuit because it is disconnected.
2. $a \to b \to g \to d \to f \to c \to g \to e \to a$

12.28

1.



12.29
1. Yes.
2. No. It doesn't cover every edge.
3. No. It is not a walk, because the vertices b and e are not adjacent.

12.30
1. In a complete graph with three or more vertices, any three vertices form a triangle which is a cycle. Any edge in a graph that is part of a cycle cannot be a bridge so, there are no bridges in a complete graph. Also, any edge that is part of a triangle cannot be a local bridge because the removal of any edge of a triangle will leave a path of only two edges between the vertices of that edge. So, there are no bridges or local bridges in a complete graph.

12.31
1. m
2. qn, qt; qn bridge
3. rs, st, tq, qn
4. nm, np; nm,, bridge
5. om, mp, pn, nm
6. $s \to q \to r \to s \to t \to q \to n \to o \to m \to p \to n \to m$

12.32
1. The graph has Euler trails. They must begin and end at vertices u and v. An example is $v \to w \to x \to u \to z \to y \to w \to u$

12.33
1. both

12.34
1. $n! = 6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$ and $(n-1)\,! = (6-1)! = 5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

12.35
1. $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

12.36
1. 120

12.37
1. 26

12.38
1. No
2. No
3. Yes

12.39
1. $C \to A \to B \to F \to D \to E$

12.40
1. $p \to m \to o \to n \to q \to t \to s \to r$
2. none
3. none

12.41
1. Hamilton path
2. Euler trail
3. neither

12.42
1. brute force algorithm

12.43
1.
   1. $V \to W \to X \to Y \to Z \to V$
   2. $V \to W \to X \to Z \to Y \to V$
   3. $V \to W \to Y \to X \to Z \to V$
   4. $V \to W \to Y \to Z \to X \to V$
   5. $V \to W \to Z \to X \to Y \to V$
   6. $V \to W \to Z \to Y \to X \to V$
   7. $V \to X \to W \to Y \to Z \to V$
   8. $V \to X \to W \to Z \to Y \to V$
   9. $V \to X \to Y \to W \to Z \to V$
   10. $V \to X \to Y \to Z \to W \to V$ (reverse of 6)
   11. $V \to X \to Z \to W \to Y \to V$

12. $V \to X \to Z \to Y \to W \to V$ (reverse of 4)
13. $V \to Y \to X \to W \to Z \to V$
14. $V \to Y \to X \to Z \to W \to V$ (reverse of 5)
15. $V \to Y \to W \to X \to Z \to V$
16. $V \to Y \to W \to Z \to X \to V$ (reverse of 11)
17. $V \to Y \to Z \to X \to W \to V$ (reverse of 2)
18. $V \to Y \to Z \to W \to X \to V$ (reverse of 8)
19. $V \to Z \to X \to Y \to W \to V$ (reverse of 3)
20. $V \to Z \to X \to W \to Y \to V$ (reverse of 15)
21. $V \to Z \to Y \to X \to W \to V$ (reverse of 1)
22. $V \to Z \to Y \to W \to X \to V$ (reverse of 7)
23. $V \to Z \to W \to X \to Y \to V$ (reverse of 13)
24. $V \to Z \to W \to Y \to X \to V$ (reverse of 9)

12.44
1. The officer should travel from Travis Air Force Base to Beal Air Force Base, to Edwards Air Force Base, to Los Angeles Air Force Base, and return to Travis.

12.45
1. $D \to B \to E \to A \to C \to F \to D$; 550 min or 9 hrs 10 min.

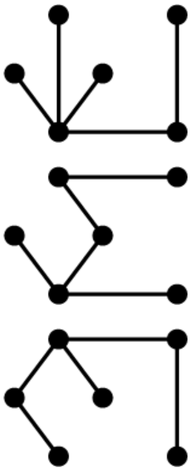
12.46
1. The star topology is a tree, and the tree topology is a tree. The ring topology and the mesh topology are not trees because they contain cycles.

12.47
1. star topology
2. tree topology
3. none

12.48
1. Vertices k, l, and m are in one component, and vertices g, h, i, and j are in the other.
2. There are 6 vertices and 5 edges, which confirms Graph I is a tree, because the number of edges is one less than the number of vertices.
3. a quadrilateral

12.49
1. a True
2. b False
3. a True
4. a True

12.50
1. Answers may vary. Here are three possible spanning trees of Graph J:

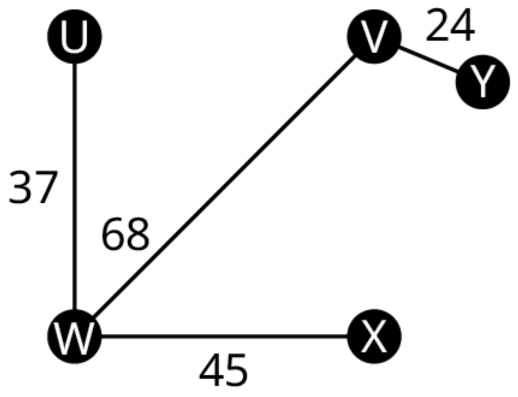


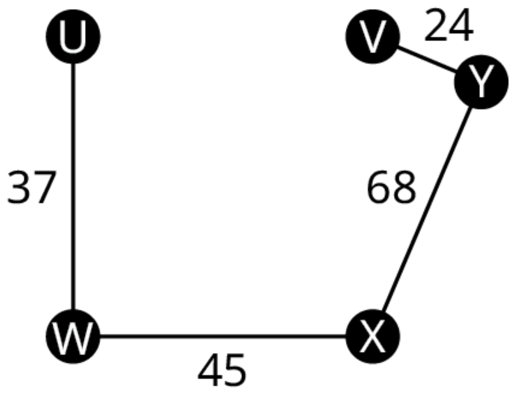
12.51
1. The three edges must include one edge from list A and one pair of edges from list B.

List A: *be*, *eh*, *hi*, *gi*, *bg*

List B: *ac* and *ad*, *ac* and *af*, *ac* and *cd*, *ac* and *cf*, *ad* and *af*, *ad* and *cd*, *ad* and *cf*, *af* and *cd*, *af* and *cf*, or *cd* and *cf*.

12.52
1. There are two possible minimum spanning trees, each with a total weight of 174:





Check Your Understanding

1. a True
2. b False
3. a True
4. b False
5. b False
6. b False
7. a True
8. b False
9. a True
10. b
11. a True
12. The sum of the degrees is always double the number of edges, which means it has to be an even number, but 13 is odd.
13. Yes. If n is the number of vertices in a complete graph, the number of edges is \(\frac{2} = \frac{n}{2}(n - 1)\) which is half the number of vertices times one less than the number of vertices.
14. always true
15. never true
16. sometimes true
17. never true
18. always true
19. sometimes true
20. sometimes true
21. never true
22. sometimes true
23. always true
24. always true
25. sometimes true
26. always true
27. sometimes true
28. always true
29. always true
30. sometimes true
31. always true
32. sometimes true
33. always true
34. always true
35. has no repeated egdes
36. is closed
37. has no repeated vertices
38. has no repeated edges
39. n
40. at least

51. edge
52. Fleury’s
53. circuit, trail
54. components
55. local bridge
56. bridge
57. vertex
58. complete
59. is
60. is
61. is not
62. is
63. is
64. is not
65. is not
66. is not
67. different from
68. the same as
69. different from
70. different from
71. b False
72. b False
73. b False
74. a True
75. a True
76. b False
77. b False
78. b False
79. a True
80. b False
81. b False
82. a True
83. a True
84. b False
85. b False
86. a True
87. a True
88. a True
89. a True
90. b False
91. b False
92. a True
93. a True
94. a True

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15.12: Chapter 13

Your Turn

13.1
1. 46 in

13.2
1. 34 scales

13.3
1. 1.618

13.4
1. 1.25, which is close to a golden rectangle

13.5
1. 210 gal

13.6
1. 1,050,000 gal

13.7
1. 9,828 million gal of water

13.8
1. 15.8 kW

13.9
1. $14.18

13.10
1. $0.06

13.11
1. 288 kW

13.12
1. 1.8 mg in 200 mL

13.13
1. 6 pills

13.14
1. 42 mg

13.15
1. 13 mL

13.16
1. Yara, Jeffrey
2. 5

13.17
1. birds chirping
2. window air conditioner
3. garbage disposal
4. live outdoor concert

13.18
1. up G, down F
2. up C, down $B^{\flat}$
3. up G, down F

13.19
1. up $E^{\flat}$, down B
2. up $D^{\#}$, down B
3. up $F^{\#}$, down D

13.20
1. G, $G^{\#}$, A, $A^{\#}$, B, C, $C^{\#}$, D, $D^{\#}$, E, F, $F^{\#}$

13.21
1. 83 Hz

13.22
1. 10 games
2. 66 games
3. 300 games

13.23

1. 4 games
2. 11 games
3. 24 games

Check Your Understanding

1. 1.618
2. 1, 1, 2, 3, 5, 8, 13, 21, 34, 55
3. A rectangle whose length divided by the width is approximately equal to ϕ or 1.618.
4. $1\,\text{kW} = 1{,}000\,\text{watts}$
5. 100 gal
6. 33 1/3 hours
7. 17 hours
8. medicine dosage required, weight of drug by volume
9. b Research on coronavirus vaccines began approximately 10 years before the COVID-19 pandemic hit the world.
10. Mathematical modeling is a graphical representation of a behavior, such as exposure to a virus. It can be helpful in mapping contract tracing with virus exposure during a pandemic.
11. Frequency is a static measured value, whereas pitch is subjective.
12. higher
13. 20–20,000 Hz
14. increase
15. 0 dB
16. 20–17,000Hz
17. It doubles.
18. Data analytics is used in sports to analyze data on team’s players, potential recruits, opposing teams, and opposing players. This helps to find patterns and make predictions for future results.
19. Soccer (football)
20. $n\,(n-1)\,/2$
21. $n-1$

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Index

A

- adding algebraic expressions
  - 5.1: Algebraic Expressions
- addition
  - 4.3: Addition and Subtraction in Base Systems
- Algebra building blocks
  - 5.1: Algebraic Expressions
- algebra learning
  - 5.1: Algebraic Expressions
- algebra operations
  - 5.1: Algebraic Expressions
- algebraic equations
  - 5.1: Algebraic Expressions
- algebraic exercises
  - 5.1: Algebraic Expressions
- algebraic expressions
  - 5.1: Algebraic Expressions
  - 5.2: Linear Equations in One Variable with Applications
- algebraic translation
  - 5.1: Algebraic Expressions
- alternate bases
  - 4.4: Multiplication and Division in Base Systems
- ancient mathematics
  - 4.1: Early Numeration Systems
- ancient numeration systems
  - 4.2: Converting with Base Systems
- apportionment
  - 11.4: Apportionment Methods
- arithmetic
  - 4.3: Addition and Subtraction in Base Systems
  - 4.4: Multiplication and Division in Base Systems
- arithmetic in different bases
  - 4.2: Converting with Base Systems
- arithmetic systems
  - 4.1: Early Numeration Systems
- Aryabhata
  - 4.0: Hindu-Arabic Positional System

B

- Babylonian numerals
  - 4.1: Early Numeration Systems
- Babylonian system
  - 4.1: Early Numeration Systems
  - 4.2: Converting with Base Systems
- base 10 comparison
  - 4.4: Multiplication and Division in Base Systems
- base 10 system
  - 4.0: Hindu-Arabic Positional System
- base 12
  - 4.3: Addition and Subtraction in Base Systems
  - 4.4: Multiplication and Division in Base Systems
- base 14
  - 4.3: Addition and Subtraction in Base Systems
- base 2
  - 4.3: Addition and Subtraction in Base Systems
- base 6
  - 4.3: Addition and Subtraction in Base Systems
  - 4.4: Multiplication and Division in Base Systems
- base 7
  - 4.3: Addition and Subtraction in Base Systems
  - 4.4: Multiplication and Division in Base Systems
- base conversion
  - 4.2: Converting with Base Systems
  - 4.3: Addition and Subtraction in Base Systems
- base conversion errors
  - 4.2: Converting with Base Systems
- base four
  - 4.2: Converting with Base Systems
- base fourteen
  - 4.2: Converting with Base Systems
- base operations exercises
  - 4.4: Multiplication and Division in Base Systems
- base seven
  - 4.2: Converting with Base Systems
- base six
  - 4.2: Converting with Base Systems
- base systems
  - 4.3: Addition and Subtraction in Base Systems
  - 4.4: Multiplication and Division in Base Systems
- base ten
  - 4.2: Converting with Base Systems
- base ten to base six conversion
  - 4.2: Converting with Base Systems
- base twelve
  - 4.2: Converting with Base Systems
- base two
  - 4.2: Converting with Base Systems
- binary
  - 4.3: Addition and Subtraction in Base Systems
- binary system
  - 4.2: Converting with Base Systems
- Brahmagupta
  - 4.0: Hindu-Arabic Positional System

C

- calculation errors
  - 4.3: Addition and Subtraction in Base Systems
- coefficients
  - 5.1: Algebraic Expressions
- constants
  - 5.1: Algebraic Expressions
- constructing equations
  - 5.2: Linear Equations in One Variable with Applications
- conversion factor
  - 9.1: The Metric System
- credit cards
  - 6.10: Credit Cards
- cultural counting differences
  - 4.1: Early Numeration Systems
- cultural numbering systems
  - 4.2: Converting with Base Systems

D

- digital systems
  - 4.2: Converting with Base Systems
- digits
  - 4.2: Converting with Base Systems
- Distributive Property
  - 5.1: Algebraic Expressions
- dividing expressions
  - 5.1: Algebraic Expressions
- division process
  - 4.2: Converting with Base Systems
- division rules
  - 4.4: Multiplication and Division in Base Systems
- Dozenal Society
  - 4.2: Converting with Base Systems

E

- error detection
  - 4.3: Addition and Subtraction in Base Systems
- error identification
  - 4.4: Multiplication and Division in Base Systems
- evaluating expressions
  - 5.1: Algebraic Expressions
- example problems
  - 4.1: Early Numeration Systems
- expanded form
  - 4.0: Hindu-Arabic Positional System
- exponential expressions
  - 4.0: Hindu-Arabic Positional System
- exponentiation
  - 4.0: Hindu-Arabic Positional System

G

- general strategy for equations
  - 5.2: Linear Equations in One Variable with Applications

H

- historical counting
  - 4.1: Early Numeration Systems
- history of numbers
  - 4.0: Hindu-Arabic Positional System

I

- infinitely many solutions
  - 5.2: Linear Equations in One Variable with Applications
- invention of zero
  - 4.1: Early Numeration Systems

L

- learning objectives
  - 4.3: Addition and Subtraction in Base Systems
- like terms
  - 5.1: Algebraic Expressions
- linear equations
  - 5.2: Linear Equations in One Variable with Applications
- linear equations in one variable
  - 5.2: Linear Equations in One Variable with Applications

M

- mathematical bases
  - 4.2: Converting with Base Systems
- mathematical expressions
  - 4.0: Hindu-Arabic Positional System
- mathematical operations
  - 4.4: Multiplication and Division in Base Systems
  - 5.1: Algebraic Expressions
- mathematics history
  - 4.0: Hindu-Arabic Positional System
- Mayan numerals
  - 4.1: Early Numeration Systems
  - 4.2: Converting with Base Systems
- Mayan system
  - 4.1: Early Numeration Systems
- multiplication tables
  - 4.4: Multiplication and Division in Base Systems

multiplying expressions
5.1: Algebraic Expressions

N
no solution
5.2: Linear Equations in One Variable with Applications
number evaluation
4.0: Hindu-Arabic Positional System
number representation
4.1: Early Numeration Systems
number systems
4.2: Converting with Base Systems
4.3: Addition and Subtraction in Base Systems
numeral conversion
4.0: Hindu-Arabic Positional System
4.1: Early Numeration Systems
numeral conversion exercises
4.0: Hindu-Arabic Positional System
numeral examples
4.0: Hindu-Arabic Positional System
numeral exercises
4.1: Early Numeration Systems
numeral legacy
4.1: Early Numeration Systems
numeral representation
4.2: Converting with Base Systems
numeral systems
4.0: Hindu-Arabic Positional System
numerals vs numbers
4.0: Hindu-Arabic Positional System
numeration systems
4.0: Hindu-Arabic Positional System
numeric history
4.1: Early Numeration Systems
numeric systems
4.1: Early Numeration Systems

O
order of operations
5.1: Algebraic Expressions

P
place value
4.1: Early Numeration Systems
4.2: Converting with Base Systems
place value notation
4.0: Hindu-Arabic Positional System
place value system
4.0: Hindu-Arabic Positional System
positional value
4.1: Early Numeration Systems
properties of equations
5.2: Linear Equations in One Variable with Applications

R
repeated addition
4.4: Multiplication and Division in Base Systems
Roman numerals
4.1: Early Numeration Systems
Roman system
4.1: Early Numeration Systems

S
simplifying expressions
5.1: Algebraic Expressions
solving equations
5.2: Linear Equations in One Variable with Applications
solving formulas
5.2: Linear Equations in One Variable with Applications
solving linear equations
5.2: Linear Equations in One Variable with Applications
subtracting algebraic expressions
5.1: Algebraic Expressions
subtraction
4.3: Addition and Subtraction in Base Systems
symbolic notation
5.1: Algebraic Expressions
symbolic numerals
4.1: Early Numeration Systems

T
translating expressions
5.1: Algebraic Expressions

V
variable isolation
5.2: Linear Equations in One Variable with Applications
variables
5.1: Algebraic Expressions
Venn diagrams
1.3: Understanding Venn Diagrams

Glossary

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